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GROWTH OF QUANTUM $6j$-SYMBOLS AND APPLICATIONS TO THE VOLUME CONJECTURE

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Abstract

We prove the Turaev–Viro invariants volume conjecture for a "universal" class of cusped hyperbolic 3-manifolds that produces all 3-manifolds with empty or toroidal boundary by Dehn filling. This leads to two-sided bounds on the volume of any hyperbolic 3-manifold with empty or toroidal boundary in terms of the growth rate of the Turaev–Viro invariants of the complement of an appropriate link contained in the manifold. We also provide evidence for a conjecture of Andersen, Masbaum and Ueno (AMU conjecture) about certain quantum representations of surface mapping class groups.

A key step in our proofs is finding a sharp upper bound on the growth rate of the quantum $6j$-symbol evaluated at $q = e^{\frac{2\pi i}{r}}$.

1. Introduction

The Turaev–Viro invariants $TV_r(M, q)$ of a compact 3-manifold M [**30**] are real numbers depending on an integer $r \geqslant 3$, called level, and a $2r$-th root of unity q. It has been long known that when one chooses $q = e^{\frac{\pi i}{r}}$, the invariants $TV_r(M, q)$ grow at most polynomially in r. In contrast to that, in [**7**], Chen and Yang's extensive computation of the case $q = e^{\frac{2\pi i}{r}}$ suggests that for hyperbolic manifolds the growth is instead exponential and determines the volume. They stated the following.

Conjecture 1. *Let M be a hyperbolic 3-manifold, either closed, with cusps, or compact with totally geodesic boundary. Then as r varies along the odd natural numbers,*

$$\lim_{r\to\infty} \frac{2\pi}{r} \log \left| TV\left(M, e^{\frac{2\pi i}{r}}\right)\right| = Vol(M).$$

The main result in this article is to verify Conjecture 1 for complements of fundamental shadow links. These links, first considered by Costantino and D. Thurston [**10**], are an infinite family of hyperbolic

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links in connected sums $\#^{c+1}(S^1 \times S^2)$, for any $c > 0$, with the following properties.

1) The volume of any fundamental shadow link in $\#^{c+1}(S^1 \times S^2)$ is equal to $2cv_8$, where $v_8 \cong 3.66$ is the volume of the regular ideal hyperbolic octahedron.
2) The Reshetikhin–Turaev invariants on any fundamental shadow link have simple formulae (see Lemma 5.2).
3) The links form a universal class, in the sense that any orientable 3-manifold with empty or toroidal boundary is obtained from a complement of a fundamental shadow by Dehn filling.

The main result of this article is the following.

Theorem 1.1. *For any $c > 0$, Conjecture 1 holds for the complement of any fundamental shadow link L in $\#^{c+1}(S^1 \times S^2)$.*

Detcherry, Kalfagianni and Yang [**15**] verified Conjecture 1 for the complements of the Borromean ring and of the figure eight knot, and Ohtsuki [**22**] verified it for all hyperbolic 3-manifolds obtained by integral Dehn surgeries on the figure eight knot. In [**32**] Wong verified Conjecture 1 for certain octahedral links in S^3 called Whitehead chains and Belletti [**3**] proved it for complements of families of octahedral links in connected sums copies of $S^1 \times S^2$ that are not treated by Theorem 1.

Furthermore, Conjecture 1 has been generalized to assert that the Turaev–Viro invariants determine the Gromov norm of any compact orientable 3-manifold [**12**]. The generalized conjecture was proven for all Gromov norm zero links in 3-manifolds that are connected sums of S^3 with copies of $S^1 \times S^2$ by Detcherry and Kalfagianni [**12**]. An alternative proof for the case of Gromov norm zero knots in S^3 was given in [**15**]. The conjecture was also shown to be closed under certain link cabling and satellite operations [**11**, **12**, **33**].

In [**9**] Costantino proved an extension of Kashaev's original volume conjecture [**19**] for fundamental shadow links. His approach was to consider a version of the colored Jones polynomials of links in manifolds of the form $\#^{c+1}(S^1 \times S^2)$, and for fundamental shadow links relate its asymptotics to the volume of their complement.

The basic building block in the definition of the Turaev–Viro invariants is the quantum $6j$-symbol. We will recall the definitions and basic properties of quantum $6j$-symbols in Section 2. A key ingredient in our proof of Theorem 1.1 is the next theorem that provides an upper bound on the growth of the quantum $6j$-symbol. As we will see in Lemma 3.13 this upper bound is sharp.

Theorem 1.2. *For any integer $r \geq 3$ and any r-admissible 6-tuple $(n_1, n_2, n_3, n_4, n_5, n_6)$, we have*

$$\frac{2\pi}{r}\log\left|\begin{vmatrix} n_1 & n_2 & n_3 \\ n_4 & n_5 & n_6 \end{vmatrix}_{q=e^{\frac{2\pi i}{r}}}\right| \leqslant v_8 + O\left(\frac{\log(r)}{r}\right).$$

Theorem 1.2 should be compared with Costantino's result (Theorem A.1) that under certain constraints, the exponential growth rate of a sequence of quantum $6j$-symbols coincides with the volume of a hyperbolic truncated tetrahedron whose dihedral angles are determined by the sequence of 6-tuples. The inequality of Theorem 1.2 is sharp in the sense that for some sequences the limit as $r \to \infty$ is v_8 (Lemma 3.13).

Combining Theorem 1.1 with a result of Futer, Kalfagianni and Purcell [**17**], we show that the volume of any hyperbolic 3-manifold M, with empty or toroidal boundary, is estimated in terms of the Turaev–Viro invariants of an appropriate link contained in M and that the estimate is asymptotically sharp.

To state our result, given a hyperbolic 3-manifold N containing k embedded horocusps choose a slope s_i on the boundary torus of each of them, and let $l_{\min}$ denote the shortest length of any of the s_i. We write $M = N(s_1, \ldots, s_k)$ for the 3-manifold obtained by Dehn filling N along these k slopes. Also as r varies along the odd natural numbers let

$$lTV(N) = \liminf_{r\to\infty} \frac{2\pi}{r}\log\left|TV\left(M, e^{\frac{2\pi i}{r}}\right)\right|$$

and

$$LTV(N) = \limsup_{r\to\infty} \frac{2\pi}{r}\log\left|TV\left(M, e^{\frac{2\pi i}{r}}\right)\right|.$$

Theorem 1.3. *Let M be a hyperbolic 3-manifold possibly with cusps. There exists a cusped hyperbolic 3-manifold N with $M = N(s_1, \ldots, s_k)$, for some $k \geqslant 1$, and such that $lTV(N) = LTV(N) = \mathrm{vol}(N)$, and*

$$\alpha(\ell_{\min})\ lTV(N) \leqslant \mathrm{vol}(M) < lTV(N).$$

Here $0 \leqslant \alpha(\ell_{\min}) \leqslant 1$ is an explicit function and $\alpha(\ell_{\min})$ approaches 1 as $\ell_{\min} \to \infty$.

Theorem 1.1 also has application to a conjecture of Andersen, Masbaum and Ueno about the quantum representations of surface mapping class groups (AMU conjecture) [**1**]. For a compact orientable surface of genus g and n boundary components $\Sigma_{g,n}$, let $\mathrm{Mod}(\Sigma_{g,n})$ denote its mapping class group. The AMU conjecture asserts that the $SU(2)$ and $SO(3)$ quantum representations of $\mathrm{Mod}(\Sigma_{g,n})$ send mapping classes with non-trivial pseudo-Anosov pieces to elements of infinite order (for large enough level). The reader is referred to Section 6.2 for more details. Mapping classes are realized as monodromies of fibered 3-manifolds and, in particular, mapping classes of surfaces with boundary are realized as monodromies of complements of fibered links in 3-manifolds. It has been long known that fibered links, and in particular hyperbolic fibered links, exist in all orientable 3-manifolds with empty or toroidal boundary. Here we prove the following.

Theorem 1.4. *Let M be the complement of a fundamental shadow link or the double of such a manifold. Given any link L in M there is an additional knot $K \subset M$ such that the complement $M \setminus (K \cup L)$ fibers over S^1 with fiber a surface. Moreover, any monodromy of a fibration of $M \setminus (K \cup L)$ satisfies the AMU conjecture.*

Theorem 1.4 uses a result of Detcherry and Kalfagianni [**12**, **14**] which shows that monodromies of a fibered 3-manifold M satisfy the AMU conjecture provided that $lTV(M) > 0$; that is, provided that the Turaev–Viro invariants grow exponentially with respect to r. In [**14**] the authors used the handful of examples of link complements in S^3 with $lTV(S^3 \setminus L) > 0$ known at the time, to construct the first infinite families of examples that satisfy the AMU conjecture in surfaces $\Sigma_{g,n}$ with $g \geqslant 2$ and $n \geqslant 2$. Since the class of fundamental shadow links is universal, Theorem 1.4 provides an abundance of fibered 3-manifolds with monodromies satisfying the AMU conjecture. Explicit constructions of such manifolds are given in [**13**].

The paper is organized as follows: We recall the quantum $6j$-symbols and preliminaries about Turaev–Viro invariants in Section 2. In Section 3 we prove the upper bound given in Theorem 1.2; the proofs of the technical lemmas used are postponed to Section 4. In Section 5 we introduce fundamental shadow links and prove Theorem 1.1. Applications of the main result on the AMU conjecture and the volume comparison are included in section 6. We also include a proof of Costantino's result, originally proved for the root of unity $q = e^{\frac{\pi i}{r}}$ in [**8**], at a different root of unity $q = e^{\frac{2\pi i}{r}}$ in the Appendix.

All the 3-manifolds we will consider in this paper will be orientable with empty or toroidal boundary.

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2. The quantum $6j$-symbols

In this section we give the basic definitions relating Turaev–Viro invariants and quantum $6j$-symbols. Throughout the rest of the paper $r \geqslant 3$ is an odd integer and $q = e^{\frac{2\pi i}{r}}$. The *quantum integer* $\{n\}$ is defined as $q^n - q^{-n}$, and the quantum factorial $\{n\}!$ is $\prod_{i=1}^{n}\{i\}$. Furthermore, we denote with I_r the set $\{0, 1, \ldots, r-2\}$.

Remark 2.1. In the remainder of the paper, we deal with the $SU(2)$ version of the Turaev–Viro invariants; however, everything remains true for the $SO(3)$ version, with small modifications.

Definition 2.2. We say that a triple (a, b, c) of non-negative integers is *r-admissible* if

- $a, b, c \leqslant r-2$;
- $a+b+c$ is even and $\leqslant 2r-4$;
- $a \leqslant b+c$, $b \leqslant a+c$ and $c \leqslant a+b$.

We say that a 6-tuple $(n_1, n_2, n_3, n_4, n_5, n_6)$ is r-admissible if the 4 triples (n_1, n_2, n_3), (n_1, n_5, n_6), (n_2, n_4, n_6) and (n_3, n_4, n_5) are r-admissible.

Notice that, while in part of the literature, e.g. [**30**] or [**7**], the colors are half integers, we take them to be integers. Our notation will be very similar to that of [**30**], except for the integer colors, and the use of $\{n\}$ instead of $[n] := \frac{\{n\}}{\{1\}}$. This will account for an extra $\{1\}$ factor in some of our formulas. We follow closely the notation of [**12**].

For an r-admissible triple (a, b, c) we can define

$$\Delta(a,b,c) = \left(\sqrt{-1}\zeta_r \frac{\{\frac{a+b-c}{2}\}!\{\frac{a-b+c}{2}\}!\{\frac{-a+b+c}{2}\}!}{\{\frac{a+b+c}{2}+1\}!}\right)^{\frac{1}{2}},$$

where $\zeta_r = 2\sin\left(\frac{2\pi}{r}\right) = -\sqrt{-1}\{1\}_{|q=\exp(2\pi\sqrt{-1}/r)}$. Notice that the number inside the square root is real: each $\{n\}$ is a purely imaginary number, and all the $\sqrt{-1}$ simplify. By convention we take the positive square root of a positive number, and the square root with positive imaginary part of a negative number.

Moreover, for an r-admissible 6-tuple $(n_1, n_2, n_3, n_4, n_5, n_6)$ we can define its quantum $6j$-symbol as

$$(1)\quad \begin{vmatrix} n_1 & n_2 & n_3 \\ n_4 & n_5 & n_6 \end{vmatrix} =$$

$$= \zeta_r^{-1}\left(\sqrt{-1}\right)^{\lambda} \prod_{i=1}^{4} \Delta(v_i) \sum_{z=max(T_i)}^{min(Q_j)} \frac{(-1)^z\{z+1\}!}{\prod_{i=1}^{4}\{z-T_i\}!\prod_{j=1}^{3}\{Q_j-z\}!},$$

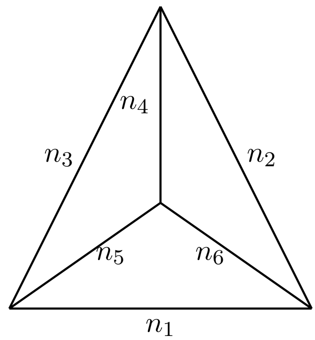

Figure 1. An admissible coloring for a tetrahedron.

where we have the following:

- $\lambda = \sum_{i=1}^{6} n_i$;
- $v_1 = (n_1, n_2, n_3)$, $v_2 = (n_1, n_5, n_6)$, $v_3 = (n_2, n_4, n_6)$ and $v_4 = (n_3, n_4, n_5)$;
- $T_1 = \frac{n_1+n_2+n_3}{2}$, $T_2 = \frac{n_1+n_5+n_6}{2}$, $T_3 = \frac{n_2+n_4+n_6}{2}$ and $T_4 = \frac{n_3+n_4+n_5}{2}$;
- $Q_1 = \frac{n_1+n_2+n_4+n_5}{2}$, $Q_2 = \frac{n_1+n_3+n_4+n_6}{2}$ and $Q_3 = \frac{n_2+n_3+n_5+n_6}{2}$.

Remark 2.3. Notice that if $z \geqslant r-1$, then the summand in (1) corresponding to z is equal to 0.

Definition 2.4. An r-*admissible coloring* for a tetrahedron T is an assignment of an r-admissible 6-tuple $(n_1, n_2, n_3, n_4, n_5, n_6)$ to the set of edges of T, as shown in Figure 1. Similarly, we define an r-*admissible coloring* of a triangulation of a 3-manifold, as an assignment of elements of I_r to each of its edges, in such a way that the 6-tuple assigned to the edges of each tetrahedron is an r-admissible coloring.

Let M be an orientable compact 3-manifold with a partially ideal triangulation τ. By this we mean that some vertices of the triangulation are truncated, and the truncated faces are a triangulation for ∂M.

Denote with $A_r(\tau)$ the set of r-admissible colorings of τ, with V the set of interior vertices of τ and with E the set of interior edges (by which we mean edges that are not contained in the boundary). If $col \in A_r(\tau)$ and $T \in \tau$ we denote with $|T|_{col}$ the quantum $6j$-symbol corresponding to the 6-tuple that col assigns to the edges of T. Similarly, if $e \in E$ we define

$$|e|_{col} = (-1)^{col(e)} \frac{\{col(e)+1\}}{\{1\}}.$$

Define the Turaev–Viro invariant of M at level r in the root q as

$$TV_r(M, \tau, q) := \left(\frac{\sqrt{2}\sin\left(\frac{2\pi}{r}\right)}{\sqrt{r}} \right)^{2|V|} \sum_{col \in A_r(\tau)} \prod_{e \in E} |e|_{col} \prod_{T \in \tau} |T|_{col}.$$

By [30] if τ and τ' are two partially ideal triangulations of M, then $TV_r(M,\tau,q) = TV_r(M,\tau',q)$. Hence we have a topological invariant of M, denoted by $TV_r(M,\ q)$, depending on r and q.

3. The upper bound of the quantum $6j$-symbol

In this section and the next section we complete the proof of Theorem 1.2. In this section we give the proof assuming three technical lemmas the proofs of which occupy Section 4.

Denote with $\Lambda(x)$ the Lobachevski function, defined as

$$\Lambda(x) := -\int_0^x \log|2\sin(t)|dt.$$

It is π-periodic, odd, and real analytic outside of $\{k\pi, k \in \mathbb{Z}\}$.

The tool used to estimate the quantum $6j$-symbol is the following lemma, first appeared in [18, Proposition 8.2] for $q = e^{\frac{i\pi}{r}}$, and then in the other roots of unity in [12, Proposition 4.1].

Lemma 3.1. *For any integer $0 < n < r$ and at $q = e^{\frac{2\pi i}{r}}$,*

$$\log\left|\{n\}!\right| = -\frac{r}{2\pi}\Lambda\left(\frac{2n\pi}{r}\right) + O\left(\log(r)\right),$$

where the term $O(\log(r))$ is such that there exist constants C, r_0 independent of n and r such that $O(\log(r)) \leqslant C\log(r)$ whenever $r > r_0$.

Remark 3.2. If $0 < n < r-1$, we can equally well use the estimate

$$\log\left|\{n+1\}!\right| = -\frac{r}{2\pi}\Lambda\left(\frac{2n\pi}{r}\right) + O(\log(r)),$$

since by applying a Taylor expansion to Λ we find

$$\begin{aligned}\Lambda\left(\frac{2n\pi}{r}+\frac{2\pi}{r}\right) - \Lambda\left(\frac{2n\pi}{r}\right) &= \\ = \frac{2\pi}{r}\Lambda'\left(\frac{2n\pi}{r}\right) + o\left(\frac{1}{r}\right) &= \\ = -\frac{2\pi}{r}\log\left|2\sin\frac{2n\pi}{r}\right| + o\left(\frac{1}{r}\right) &= O\left(\frac{\log(r)}{r}\right),\end{aligned}$$

since $\left|\sin\left(\frac{2\pi n}{r}\right)\right| \geqslant \frac{\pi}{r}$ (because $2n \neq r$). Hence we get

$$-\frac{2\pi}{r}\log(|2\sin\frac{2n\pi}{r}|) \leqslant \frac{\log(r)}{r},$$

since $\log(ax) \leqslant a\log(x)$. Notice again that the constants involved in the $O\left(\frac{\log(r)}{r}\right)$ are independent of n and r.

We need the following lemma.

Lemma 3.3 ([15], Lemma A.3)**.** *For $i \in \{0, \dots, r-2\}$, let $i' = r-2-i$. Then for any admissible 6-tuple $(n_1, n_2, n_3, n_4, n_5, n_6)$, the 6-tuples $(n_1, n_2, n_3, n_4', n_5', n_6')$ and $(n_1', n_2', n_3, n_4', n_5', n_6)$ are admissible and at $q = e^{\frac{2\pi i}{r}}$,*

$$\begin{vmatrix} n_1 & n_2 & n_3 \\ n_4 & n_5 & n_6 \end{vmatrix} = \begin{vmatrix} n_1 & n_2 & n_3 \\ n_4' & n_5' & n_6' \end{vmatrix} = \begin{vmatrix} n_1' & n_2' & n_3 \\ n_4' & n_5' & n_6 \end{vmatrix}.$$

We are now ready to begin the proof of Theorem 1.2 stated in the Introduction. The proof of the theorem will be completed in Section 4.

Theorem 1.2. *For any r, and any admissible 6-tuple $(n_1, n_2, n_3, n_4, n_5, n_6)$, then*

$$\frac{2\pi}{r} \log \left| \begin{vmatrix} n_1 & n_2 & n_3 \\ n_4 & n_5 & n_6 \end{vmatrix}_{q=e^{\frac{2\pi i}{r}}} \right| \leqslant v_8 + O\left(\frac{\log(r)}{r}\right),$$

where $v_8 \cong 3.66$ is the volume of the regular ideal hyperbolic octahedron.

Proof. Applying Lemma 3.1 (together with the subsequent remark) to the formula for the quantum $6j$-symbol (1) we obtain the estimate

$$\frac{2\pi}{r} \log \left| \begin{vmatrix} n_1 & n_2 & n_3 \\ n_4 & n_5 & n_6 \end{vmatrix}_{q=e^{\frac{2\pi i}{r}}} \right| \leqslant \leqslant V(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) + O\left(\frac{\log(r)}{r}\right). \tag{2}$$

Here, $V(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)$ is defined by

$$V(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) := \max_{\max(U_i) \leqslant Z \leqslant \min(V_j, 2\pi)} F(Z, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) + $$
$$+ \nu(\theta_1, \theta_5, \theta_6) + \nu(\theta_2, \theta_4, \theta_6) + \nu(\theta_1, \theta_2, \theta_3) + +\nu(\theta_3, \theta_4, \theta_5),$$

where we have

- $\theta_i = \frac{2\pi n_i}{r}$ and $Z = \frac{2\pi z}{r}$;
- $U_i = \frac{2\pi T_i}{r}$ and similarly $V_j = \frac{2\pi Q_j}{r}$;
- $\nu(\alpha, \beta, \gamma) = \frac{1}{2}(\Lambda(\frac{\alpha+\beta+\gamma}{2}) - \Lambda(\frac{\alpha+\beta-\gamma}{2}) - \Lambda(\frac{\alpha-\beta+\gamma}{2}) - \Lambda(\frac{-\alpha+\beta+\gamma}{2}))$;
- $$F(Z, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) := \sum_{i=1}^{4} \Lambda(Z - U_i) + \sum_{j=1}^{3} \Lambda(V_j - Z) - \Lambda(Z).$$

The admissibility conditions imply that all the variables involved in above formulae take values between 0 and 2π. Notice that the variables θ_i satisfy similar triangular inequalities and admissibility conditions as the variables n_i. In particular $\theta_1 + \theta_2 + \theta_3 \leqslant 4\pi$, $\theta_1 + \theta_5 + \theta_6 \leqslant 4\pi$, $\theta_2 + \theta_4 + \theta_6 \leqslant 4\pi$ and $\theta_3 + \theta_4 + \theta_5 \leqslant 4\pi$.

Next we want to maximize V subject to the admissibility conditions of the variables θ_i. The argument relies on the three technical lemmas, whose proofs will occupy Section 4. The first two lemmas are the following.

Lemma 3.4. *If* $0 \leqslant \alpha, \beta, \gamma \leqslant \pi$*, then* $\nu(\alpha, \beta, \gamma) \leqslant 0$.

Lemma 3.5. *Suppose that we have* $0 \leqslant \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6 \leqslant 2\pi$ *and that* $max(T_i) \leqslant Z \leqslant min(Q_j, 2\pi)$*. Then,*

$$F(Z, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) + 2\nu(\theta_1, \theta_2, \theta_3) \leqslant 8\Lambda\left(\frac{\pi}{4}\right) = v_8.$$

We obtain the following corollary.

Corollary 3.6. *We have*

$$\max_{\max(U_i) \leqslant Z \leqslant \min(V_j, 2\pi)} F(Z, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) + \\ + \nu(\theta_1, \theta_2, \theta_3) + \nu(\theta_1, \theta_5, \theta_6) \leqslant v_8.$$

Proof. Follows immediately by using Lemmas 3.4 and 3.5 and taking averages. q.e.d.

Consider now an admissible 6-tuple $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)$. Using Lemma 3.3, we have that either all the variables θ_i are greater than π or at most one of them is greater than π. The third technical lemma we need for the proof of Theorem 1.2 is the following. It implies that the case where all the θ_i's are larger than π can also be reduced to the case where all of them are less than or equal to π.

Lemma 3.7. *If* $\theta_i > \pi$ *for* $i = 1, \ldots, 6$ *and* $\alpha_i = \theta_i - \pi$*, then*

$$V(\theta_1, \ldots, \theta_6) = V(\alpha_1, \ldots, \alpha_6).$$

Now, assuming Lemma 3.7, we conclude the proof of Theorem 1.2.

In the case where at most one $\theta_i > \pi$, we can assume by symmetry that $\theta_i \leqslant \pi$ for all $i > 1$. Then Lemmas 3.4, 3.5 and Corollary 3.6 imply that

$$V(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) \leqslant \\ \leq \max_{\max(U_i) \leqslant Z \leqslant \min(V_j, 2\pi)} F(Z, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) + \\ + \nu(\theta_1, \theta_2, \theta_3) + \nu(\theta_1, \theta_5, \theta_6) \leqslant v_8.$$

In conclusion, we obtain

$$\frac{2\pi}{r} \log \left| \left| \begin{matrix} n_1 & n_2 & n_3 \\ n_4 & n_5 & n_6 \end{matrix} \right|_{q = e^{\frac{2\pi i}{r}}} \right| \leqslant v_8 + O\left(\frac{\log(r)}{r}\right).$$

q.e.d.

Remark 3.8. In [**12**] a less sharp upper bound on the growth rate of the quantum $6j$-symbol was given, to prove that if a compact 3-manifold M admits a triangulation with t tetrahedra, then

$$lTV(M) \leqslant LTV(M) \leqslant 2.08 v_8 t.$$

The improvement of the upper bound allows us to state a better estimate. We have the following.

Corollary 3.9. *If M is a compact manifold that admits a triangulation with t tetrahedra, then*

$$lTV(M) \leqslant LTV(M) \leqslant v_8 t.$$

Remark 3.10. There is a concept of complexity of a manifold that is related to quantum invariants, the so called shadow complexity. For an overview of shadows and shadow complexity, see for example [**29**, Part 2] or [**10**]. Shadow complexity easily gives a bound on the growth of the Turaev–Viro invariants.

Corollary 3.11. *If M has shadow complexity c, then*

$$lTV(M) \leqslant LTV(M) \leqslant 2cv_8.$$

Furthermore we have equalities for fundamental shadow links.

Proof. The inequality is an immediate consequence of Theorem 1.2 and the shadow formula for the Reshetikhin–Turaev invariants [**29**, Theorem X.3.3]. By [**10**], for fundamental shadow links in $\#^{c+1}(S^2 \times S^1)$ the shadow complexity is c. Hence sharpness follows from Theorem 1.1, which we prove in Section 5. q.e.d.

Moreover, shadow complexity also gives an upper bound on the simplicial volume.

Theorem ([**10**], Theorem 3.37)**.** *Let M be a manifold with (possibly empty) toroidal boundary, simplicial volume $Vol(M)$ and shadow complexity c; then, $Vol(M) \leqslant 2cv_8$. Furthermore this bound is sharp for complements of fundamental shadow links.*

Remark 3.12. The bound in Corollary 3.9 is likely not sharp. However in [**12**] it is used to show that for 3-manifolds M with toroidal or empty boundary $LTV(M)$ is bounded above linearly by the Gromov norm of M. On the other hand, the Gromov norm upper bound of the shadow complexity obtained in [**10**] is quadratic.

Before we move on to prove the volume conjecture for fundamental shadow links, we need to show that the bound of Theorem 1.2 is sharp.

Lemma 3.13. *If the sign is chosen such that $\frac{r\pm 1}{2}$ is even, then*

$$\lim_{r\to\infty} \frac{2\pi}{r} \log \left| \begin{vmatrix} \frac{r\pm1}{2} & \frac{r\pm1}{2} & \frac{r\pm1}{2} \\ \frac{r\pm1}{2} & \frac{r\pm1}{2} & \frac{r\pm1}{2} \end{vmatrix}_{q=e^{\frac{2\pi i}{r}}} \right| = v_8.$$

Proof. Because of the color choice, then $\max T_i > \frac{r}{2}$, hence in the sum defining the quantum $6j$-symbol $\frac{r}{2} < z < r$, and $\{z\} = 2i \sin\big(2\pi z/r\big)$ is an imaginary number with negative sign. Moreover, $0 \leqslant z - T_i < \frac{r}{2}$ and $0 \leqslant Q_j - z < \frac{r}{2}$ for all i, j. Therefore,

$$\frac{(-1)^z \{z+1\}!}{\prod_{i=1}^4 \{z - T_i\}! \prod_{j=1}^3 \{Q_j - z\}!}$$

is an imaginary number, and passing from z to $z+1$ in the sum does not change its sign, since all terms in the denominator do not change sign, and there is a change of sign due to $\{z+2\}$ that gets corrected by $(-1)^{z+1}$. Since there is no change in sign among the summands, the estimate given by (2) is actually an equality. We have $\Delta(\pi, \pi, \pi) = 0$, and

$$F\left(\frac{7\pi}{4}, \pi, \pi, \pi, \pi, \pi, \pi\right) = 8\Lambda\left(\frac{\pi}{4}\right) = v_8.$$

Thus, using Theorem 1.2,

$$\frac{2\pi}{r} \log \left| \left| \begin{matrix} \frac{r\pm1}{2} & \frac{r\pm1}{2} & \frac{r\pm1}{2} \\ \frac{r\pm1}{2} & \frac{r\pm1}{2} & \frac{r\pm1}{2} \end{matrix} \right|_{q=e^{\frac{2\pi i}{r}}} \right| = v_8 + O\left(\frac{\log(r)}{r}\right),$$

which conclude the proof. q.e.d.

4. Proofs of the technical lemmas

We now turn to the proofs of Lemmas 3.4, 3.5 and 3.7 that will complete the proof of Theorem 1.2.

Lemma 3.4. *If* $0 \leqslant \alpha, \beta, \gamma \leqslant \pi$, *then* $\nu(\alpha, \beta, \gamma) \leqslant 0$.

Proof. Put $x = \frac{\alpha+\beta-\gamma}{2}$, $y = \frac{\alpha-\beta+\gamma}{2}$, $z = \frac{-\alpha+\beta+\gamma}{2}$.
Then we need to maximize the function

$$\nu(\alpha, \beta, \gamma) = \vartheta(x, y, z) = \frac{1}{2}(\Lambda(x+y+z) - \Lambda(x) - \Lambda(y) - \Lambda(z)),$$

with the constraints $0 \leqslant x + y \leqslant \pi$, $0 \leqslant x + z \leqslant \pi$ and $0 \leqslant y + z \leqslant \pi$.

To do this, we check first its stationary points in the interior of the domain, then we explore the boundary, and finally the points where ϑ is not smooth.

$$\frac{\partial \vartheta(x,y,z)}{\partial x} = -\frac{1}{2}(\log(2|\sin(x+y+z)|) - \log(2|\sin(x)|)), \tag{3}$$

$$\frac{\partial \vartheta(x,y,z)}{\partial y} = -\frac{1}{2}(\log(2|\sin(x+y+z)|) - \log(2|\sin(y)|)), \tag{4}$$

$$\frac{\partial \vartheta(x,y,z)}{\partial z} = -\frac{1}{2}(\log(2|\sin(x+y+z)|) - \log(2|\sin(z)|)). \tag{5}$$

So by putting them all equal to 0, we first see that $\sin(x) = \pm\sin(y) = \pm\sin(z)$, so either $x = y = z$ modulo π or one of $x + y$, $y + z$ or $x + z$ is equal $k\pi$ for some $k \in \mathbb{Z}$. Suppose $x + y = k\pi$. Then

$$\vartheta(x, y, z) = \Lambda(k\pi + z) - \Lambda(k\pi - y) - \Lambda(y) - \Lambda(z) = 0,$$

because Λ is odd and π-periodic; $y + z = k\pi$ and $x + z = k\pi$ are the same by symmetry.

If instead $x = y = z$ modulo π, substituting $x = y = z$ in (3), we get $\sin(3x) = \pm\sin(x)$. This means that $x = y = z = \frac{k\pi}{4}$ modulo π. In the interior of the domain this implies $x = y = z = \frac{\pi}{4}$. All other possibilities lie outside the domain or on its boundary. In this point $\vartheta = -2\Lambda\left(\frac{\pi}{4}\right) \cong -1.83 < 0$.

The boundary cases $x + y = k\pi$ and permutations were already checked, finding $\vartheta = 0$.

Finally we check the points where ϑ is not smooth. This happens when one of the following holds:

- $x = k\pi$, or $y = k\pi$, or $z = k\pi$; or
- $x + y + z = k\pi$.

Remark 4.1. If P is a point and γ is a direction such that the derivative of ϑ in that direction is $+\infty$, then P cannot be a local maximum of ϑ.

If $x = k\pi$, then $\frac{\partial\vartheta(x,y,z)}{\partial x} = +\infty$ unless $x + y + z = h\pi$, and (x, y, z) cannot be a maximum. If instead $x = k\pi$ and $x + y + z = h\pi$, we have $y + z = (h - k)\pi$ and we are in a case we already checked. $y = k\pi$ and $z = k\pi$ are symmetric.

If instead $x + y + z = k\pi$, we find once again an infinite derivative unless $x = h\pi$, and we reason as before. So in conclusion ϑ is equal to 0 on the boundary of the set $\{0 \leqslant x + y \leqslant \pi, 0 \leqslant x + z \leqslant \pi, 0 \leqslant y + z \leqslant \pi\}$, cannot have a maximum in a non-smooth point and has a unique stationary point in the interior, where it is negative. This concludes the proof. q.e.d.

We will need the following lemma.

Lemma 4.2. *If* $0 \leqslant a, b$ *and* $a + b \leqslant 2\pi$, *then*

$$-v_3 \leqslant \Lambda(a + b) - \Lambda(a) - \Lambda(b) \leqslant v_3,$$

where $v_3 = \Lambda\left(\frac{\pi}{3}\right) \cong 1.01$ *is the volume of the regular ideal tetrahedron.*

Proof. First notice that if $a + b = k\pi$, then because Λ is odd and π-periodic, we have $\Gamma(a, b) = \Lambda(a + b) - \Lambda(a) - \Lambda(b) = 0$. Similarly if $a = 0$ or $b = 0$ then $\Gamma(a, b) = 0$. By calculating the derivatives of Γ and putting them to 0 we obtain, reasoning as before, $a = \pm b$ modulo π. If $a = -b$ modulo π then we have seen that $\Gamma = 0$. Then $a = b$ implies $\sin(2a) = \pm\sin(a)$, and either $a = k\pi$ (in which case $\Gamma = 0$) or $3a = k\pi$.

If $a = \frac{\pi}{3}$ we obtain $\Gamma\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = -3\Lambda(\frac{\pi}{3}) = -v_3$, while $a = \frac{2\pi}{3}$ implies $\Gamma\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right) = 3\Lambda(\frac{\pi}{3}) = v_3$. q.e.d.

We are now ready to prove Lemma 3.5.

Lemma 3.5. *Suppose that we have* $0 \leqslant \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6 \leqslant 2\pi$ *and that* $max(T_i) \leqslant Z \leqslant min(Q_j, 2\pi)$. *Then,*

$$F(Z, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) + 2\nu(\theta_1, \theta_2, \theta_3) \leqslant 8\Lambda\left(\frac{\pi}{4}\right) = v_8.$$

Proof. Put $a_i = Z - U_i$, and $b_j = V_j - Z$. The inverse of this change of variable is as follows.

- $\theta_1 = a_3 + a_4 + b_1 + b_2$;
- $\theta_2 = a_2 + a_4 + b_1 + b_3$;
- $\theta_3 = a_2 + a_3 + b_2 + b_3$;
- $\theta_4 = a_1 + a_2 + b_1 + b_2$;
- $\theta_5 = a_1 + a_3 + b_1 + b_3$;
- $\theta_6 = a_1 + a_4 + b_2 + b_3$, and
- $Z = a_1 + a_2 + a_3 + a_4 + b_1 + b_2 + b_3$.

In these new variables we have,

$$F(Z, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) = \tilde{F}(a_1, a_2, a_3, a_4, b_1, b_2, b_3) =$$
$$= -\Lambda\left(\sum_{i=1}^{4} a_i + \sum_{j=1}^{3} b_j\right) + \sum_{i=1}^{4} \Lambda(a_i) + \sum_{j=1}^{3} \Lambda(b_j),$$

while

$$2\nu(\theta_1, \theta_2, \theta_3) = 2\tilde{\nu}(a_1, a_2, a_3, a_4, b_1, b_2, b_3) =$$
$$= \left(\Lambda\left(\sum_{i=1}^{3}(a_i + b_i)\right) - \sum_{i=1}^{3} \Lambda(a_i + b_i)\right).$$

Let $L = 2\tilde{\nu} + \tilde{F}$, and notice that $\tilde{\nu}$ is independent of a_4, and that L is symmetric under the exchange of a_i with b_i for any $i \neq 4$, and under

$$(a_1, a_2, a_3, a_4, b_1, b_2, b_3) \to (a_{\sigma_1}, a_{\sigma_2}, a_{\sigma_3}, a_4, b_{\sigma_1}, b_{\sigma_2}, b_{\sigma_3}),$$

where σ is any permutation of 3 elements. Also notice that L is periodic of period π in each variable, hence we can assume $0 \leqslant a_i \leqslant \pi$ and $0 \leqslant b_i \leqslant \pi$. Moreover, because of the constraints on the θ_i's and on Z, we have that $0 \leqslant \sum a_i + \sum b_j \leqslant 2\pi$. Denote with Ω the region of $\mathbb{R}^7$ defined by all these inequalities.

We now proceed by first dealing with the points in the boundary of Ω, then with the points where the function L is not differentiable, and finally by finding the stationary points in the interior of Ω. Start by calculating the partial derivatives of L.

$$\frac{\partial L}{\partial a_4} = \log \left| \frac{\sin(a_1+a_2+a_3+a_4+b_1+b_2+b_3)}{\sin(a_4)} \right|, \tag{6}$$

$$\frac{\partial L}{\partial a_1} = \log \left| \frac{\sin(a_1+a_2+a_3+a_4+b_1+b_2+b_3)\sin(a_1+b_1)}{\sin(a_1)\sin(a_1+a_2+a_3+b_1+b_2+b_3)} \right|, \tag{7}$$

$$\frac{\partial L}{\partial a_2} = \log \left| \frac{\sin(a_1+a_2+a_3+a_4+b_1+b_2+b_3)\sin(a_2+b_2)}{\sin(a_2)\sin(a_1+a_2+a_3+b_1+b_2+b_3)} \right|, \tag{8}$$

$$\frac{\partial L}{\partial a_3} = \log \left| \frac{\sin(a_1+a_2+a_3+a_4+b_1+b_2+b_3)\sin(a_3+b_3)}{\sin(a_3)\sin(a_1+a_2+a_3+b_1+b_2+b_3)} \right|, \tag{9}$$

$$\frac{\partial L}{\partial b_1} = \log \left| \frac{\sin(a_1+a_2+a_3+a_4+b_1+b_2+b_3)\sin(a_1+b_1)}{\sin(b_1)\sin(a_1+a_2+a_3+b_1+b_2+b_3)} \right|, \tag{10}$$

$$\frac{\partial L}{\partial b_2} = \log \left| \frac{\sin(a_1+a_2+a_3+a_4+b_1+b_2+b_3)\sin(a_2+b_2)}{\sin(b_2)\sin(a_1+a_2+a_3+b_1+b_2+b_3)} \right|, \tag{11}$$

$$\frac{\partial L}{\partial b_3} = \log \left| \frac{\sin(a_1+a_2+a_3+a_4+b_1+b_2+b_3)\sin(a_3+b_3)}{\sin(b_3)\sin(a_1+a_2+a_3+b_1+b_2+b_3)} \right|. \tag{12}$$

The remaining of the proof is broken into three steps.

Step 1: the boundary points

Suppose we have a maximum for L in a point P in the boundary of Ω. If $a_1 = \pi$, then by periodicity we would have a maximum with $a_1 = 0$, so we study this case instead. The derivative of L (7) with respect to a_1 is $+\infty$ if $a_1+b_1 \neq k\pi$ and $a_1+a_2+a_3+a_4+b_1+b_2+b_3 \neq k\pi$, and we would not get a maximum. Hence, either $a_1+a_2+a_3+a_4+b_1+b_2+b_3 = k\pi$ or $b_1 = k\pi$. In the first case, we have that

$$L = \Lambda(a_2) + \Lambda(b_2) - \Lambda(a_2+b_2) + \Lambda(a_3) + \Lambda(b_3) - \Lambda(a_3+b_3).$$

Then, using Lemma 4.2 we find $L \leqslant 2v_3$.

In the second case,

$$\begin{aligned} L =& \Lambda(a_2) + \Lambda(b_2) - \Lambda(a_2+b_2) + \Lambda(a_3) + \Lambda(b_3) - \Lambda(a_3+b_3) + \\ &+ \Lambda(b_2+b_3+a_2+a_3) + \Lambda(a_4) - \Lambda(b_2+b_3+a_4+a_2+a_3), \end{aligned}$$

and again Lemma 4.2 implies $L \leqslant 3v_3$. If $a_4 = 0$, the same reasoning implies that P cannot be a maximum unless

$$a_1+a_2+a_3+a_4+b_1+b_2+b_3 = k\pi,$$

and in this case

$$\begin{aligned} L =& \Lambda(a_1) + \Lambda(b_1) - \Lambda(a_1+b_1) + \Lambda(a_2) + \Lambda(b_2) - \\ &- \Lambda(a_2+b_2) + \Lambda(a_3) + \Lambda(b_3) - \Lambda(a_3+b_3) \leqslant 3v_3. \end{aligned}$$

If $a_1+a_2+a_3+a_4+b_1+b_2+b_3 = k\pi$, once again we would have $\frac{\partial L}{\partial(-a_4)} = +\infty$ unless $a_4 = 0$ and we would be in the same case as before. The remaining cases are dealt by symmetry.

Step 2: the non-smooth points

First off, notice that L is differentiable at $P = (a_1, a_2, a_3, a_4, b_1, b_2, b_3)$ unless one (or more) of the following equalities (considered modulo π) holds:

1) $a_i = 0$ for some i;
2) $b_j = 0$ for some j;
3) $a_i + b_i = 0$ for some i;
4) $a_1 + a_2 + a_3 + a_4 + b_1 + b_2 + b_3 = 0$;
5) $a_1 + a_2 + a_3 + b_1 + b_2 + b_3 = 0$.

These cases are dealt in a similar fashion as the boundary cases.

Suppose we have a maximum for L in a point P such that $a_1 + a_2 + a_3 + b_1 + b_2 + b_3 = k\pi$. Then, unless $a_1 + b_1 = k\pi$ or $a_1 + a_2 + a_3 + a_4 + b_1 + b_2 + b_3 = k\pi$ the derivative of L with respect to a_1 is $+\infty$, hence P could not be a maximum. Using Lemma 4.2 we obtain that in the first case,

$$L = \Lambda(a_2) + \Lambda(b_2) - \Lambda(a_2 + b_2) + \Lambda(a_3) + \Lambda(b_3) - \Lambda(a_3 + b_3) \leqslant 2v_3,$$

and in the second

$$\begin{aligned} L =&\Lambda(a_2) + \Lambda(b_2) - \Lambda(a_2 + b_2) + \Lambda(a_3) + \Lambda(b_3) - \Lambda(a_3 + b_3) + \\ &+ \Lambda(b_2 + b_3 + a_2 + a_3) + \Lambda(a_4) - \Lambda(b_2 + b_3 + a_4 + a_2 + a_3) \leqslant 3v_3. \end{aligned}$$

The cases $a_i = k\pi$, $b_j = k\pi$, or $a_1 + a_2 + a_3 + a_4 + b_1 + b_2 + b_3 = k\pi$ were already addressed before. If $a_1 + b_1 = 0$, then $a_1 = b_1 = 0$ and it was already addressed. If $a_1 + b_1 = k\pi > 0$, then the derivative of L in the direction $-a_1$ is $+\infty$ unless $a_1 = 0$ or $a_1 + a_2 + a_3 + b_1 + b_2 + b_3 = k\pi$, which are both cases we have dealt with already. The remaining cases are done by the symmetries of L.

Step 3: the interior smooth points

Now we turn to the smooth points in the interior of Ω. By equating (7) and (10) to 0, we find $\sin(a_1) = \pm\sin(b_1)$. Similarly $\sin(a_i) = \pm\sin(b_i)$ for $i = 2, 3$ by equating (8) to (11) and (9) to (12) respectively. Because of the boundary and smoothness conditions, we have that in the interior of the domain this implies $a_i = b_i$ for $i = 1, 2, 3$. By putting equations (7) and (8) to 0, we find

$$\frac{\sin(2a_1)}{\sin a_1} = \pm\frac{\sin(2a_2)}{\sin a_2}.$$

The last equation implies that $\cos(a_1) = \pm\cos(a_2)$ and either $a_1 = a_2$ or $a_1 + a_2 = \pi$. However, if $a_1 + a_2 = \pi$, we would have

$$a_1 + a_2 + a_3 + a_4 + b_1 + b_2 + b_3 = 2a_1 + 2a_2 + 2a_3 + a_4 \geqslant 2\pi,$$

hence, this is not possible in the interior of Ω. Hence, $a_1 = a_2$. Similarly, $a_1 = a_3$ follows from Equations (7) and (9).

Now by putting equation (6) equal to 0 we obtain

$$\sin(6a_1 + a_4) = \pm\sin(a_4).$$

This implies either $6a_1 = k\pi$ or $6a_1 + 2a_4 = k\pi$, but in the first case we would not be in a smooth point (case 5 of the previous step). By plugging everything we obtained in equation (7) we finally find

$$\frac{\sin(a_4)\sin(2a_1)}{\sin(a_1)\sin(2a_4)} = \pm 1.$$

Hence $a_4 = a_1$ or $a_4 = \pi - a_1$. Both cases imply that the stationary points of L must be of the form $\left(\frac{k\pi}{8}, \frac{k\pi}{8}, \frac{k\pi}{8}, \frac{k\pi}{8}, \frac{k\pi}{8}, \frac{k\pi}{8}, \frac{k\pi}{8}\right)$, for $k = 1, 2$. In the first case $L \cong 3.01 < v_8$, while in the second $L = 8\Lambda\left(\frac{\pi}{4}\right) = v_8$.
q.e.d.

We conclude the section with the proof of Lemma 3.7.

Lemma 3.7. *If $\theta_i > \pi$ for $i = 1, \dots, 6$ and $\alpha_i = \theta_i - \pi$, then*

$$V(\theta_1, \dots, \theta_6) = V(\alpha_1, \dots, \alpha_6).$$

Proof. The value of $V(\alpha_1, \dots, \alpha_6)$ is equal, by the Murakami–Yano–Ushijima formula [**21**, Theorems 1 and 2], [**31**, Theorem 1.1], to the volume of the hyperbolic truncated tetrahedron with external dihedral angles $\alpha_1, \dots, \alpha_6$. Thus we need to show that this formula is symmetric under the change $(\theta_1, \dots, \theta_6) \leftrightarrow (\alpha_1, \dots, \alpha_6)$. We now pass to the internal dihedral angles $(\xi_1, \dots, \xi_6)$ with $\xi_i = \pi - \alpha_i$, as these are more natural for the Murakami–Yano–Ushijima formula. In these variables, the formula reads

$$V(a_1, \dots, a_6) := \frac{1}{2}\mathrm{Im}\left(U(z_1, \vec{a}) - U(z_2, \vec{a})\right),$$

where we have

- $a_= e^{\sqrt{-1}\xi_i}$;
-

$$U(z, \vec{a}) = \frac{1}{2}(\mathrm{Li}_2(z) + \mathrm{Li}_2(za_1a_2a_4a_5) + \mathrm{Li}_2(za_1a_3a_4a_6) + \\ + \mathrm{Li}_2(za_2a_3a_5a_6) - \mathrm{Li}_2(-za_1a_2a_3) - \mathrm{Li}_2(-za_1a_5a_6) - \\ - \mathrm{Li}_2(-za_2a_4a_6) - \mathrm{Li}_2(-za_3a_4a_5)), \tag{13}$$

  where Li_2 is the dilogarithm function defined for $z \in \mathbb{C} \setminus [1, \infty)$ by

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-u)}{u} du;$$

- z_1 and z_2 are the solutions of the equation $\alpha + \beta z + \gamma z^2 = 0$, labeled in such a way as to obtain a positive value for V;
-

$$\alpha = 1 + a_1a_2a_4a_5 + a_1a_3a_4a_6 + a_2a_3a_5a_6 + a_1a_2a_3 + a_1a_5a_6 + \\ + a_2a_4a_6 + a_3a_4a_5; \tag{14}$$

•

$$\beta = -\, a_1a_2a_3a_4a_5a_6\big((a_1 - a_1^{-1})(a_4 - a_4^{-1}) + \\ + (a_2 - a_2^{-1})(a_5 - a_5^{-1}) + (a_3 + a_3^{-1})(a_6 - a_6^{-1})\big); \tag{15}$$

•

$$\gamma = a_1a_2a_3a_4a_5a_6(a_1a_2a_3a_4a_5a_6 + a_1a_4 + a_2a_5 + a_3a_6 + \\ + a_1a_2a_6 + a_1a_3a_5 + a_2a_3a_4 + a_4a_5a_6). \tag{16}$$

In these variables, the symmetry we need to explore is

$$(a_1, a_2, a_3, a_4, a_5, a_6) \leftrightarrow (a_1^{-1}, a_2^{-1}, a_3^{-1}, a_4^{-1}, a_5^{-1}, a_6^{-1}).$$

Call α, β, and γ as in formulas (14)–(16), and α', β', γ' the same formulas with $a_i \to a_i^{-1}$ for all i. Let z_1 and z_2 be solutions of $\alpha + \beta z + \gamma z^2 = 0$. Now it is immediate to check that

$$\alpha' = \frac{\gamma}{a_1^2a_2^2a_3^2a_4^2a_5^2a_6^2}, \quad \beta' = \frac{\beta}{a_1^2a_2^2a_3^2a_4^2a_5^2a_6^2}, \quad \text{and} \quad \gamma' = \frac{\alpha}{a_1^2a_2^2a_3^2a_4^2a_5^2a_6^2}.$$

Hence, we need to solve the equation

$$\alpha' + \beta' z + \gamma' z^2 = \frac{1}{a_1^2a_2^2a_3^2a_4^2a_5^2a_6^2}\Big(\gamma + \beta z + \alpha z^2\Big) = 0;$$

call the solutions $\hat{z}_1$ and $\hat{z}_2$. Since it was shown in [**21**, Page 384] that z_1 and z_2 must be complex numbers with absolute value 1, $\hat{z}_1 = \overline{z_1}$ and $\hat{z}_2 = \overline{z_2}$. Now we can compute

$$U(\hat{z}_1, a_1^{-1}, a_2^{-1}, a_3^{-1}, a_4^{-1}, a_5^{-1}, a_6^{-1}) = \frac{1}{2}(\mathrm{Li}_2(\overline{z_1}) + \mathrm{Li}_2(\overline{z_1a_1a_2a_4a_5}) + \\ + \mathrm{Li}_2(\overline{z_1a_1a_3a_4a_6}) + \mathrm{Li}_2(\overline{z_1a_2a_3a_5a_6}) - \mathrm{Li}_2(-\overline{z_1a_1a_2a_3}) - \\ - \mathrm{Li}_2(-\overline{z_1a_1a_5a_6}) - -\mathrm{Li}_2(-\overline{z_1a_2a_4a_6}) - \mathrm{Li}_2(-\overline{z_1a_3a_4a_5})).$$

Because $\mathrm{Li}_2(\overline{a}) = \overline{\mathrm{Li}_2(a)}$, we see that

$$U(\hat{z}_1, a_1^{-1}, a_2^{-1}, a_3^{-1}, a_4^{-1}, a_5^{-1}, a_6^{-1}) = \overline{U(z_1, a_1, a_2, a_3, a_4, a_5, a_6)},$$

and

$$U(\hat{z}_2, a_1^{-1}, a_2, a_3, a_4^{-1}, a_5, a_6) = \overline{U(z_2, a_1, a_2, a_3, a_4, a_5, a_6)}.$$

Since we have to switch the labels as to obtain a positive value of V, we obtain

$$V(a_1^{-1}, a_2^{-1}, a_3^{-1}, a_4^{-1}, a_5^{-1}, a_6^{-1}) = V(a_1, a_2, a_3, a_4, a_5, a_6),$$

concluding the proof of the lemma. q.e.d.

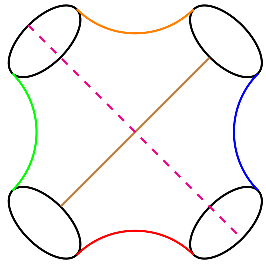

Figure 2. The building block.

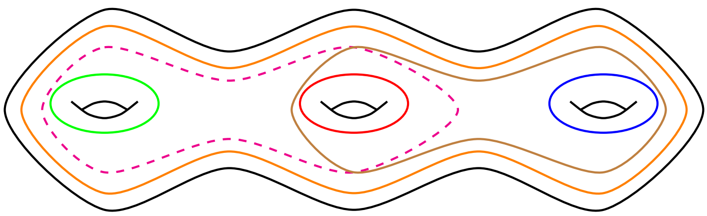

Figure 3. The link in the boundary of the handlebody.

5. The volume conjecture for fundamental shadow links

In this section we define the family of fundamental shadow links and prove the volume conjecture for them. The building block for these links is a 3-ball with 4 disks on its boundary, and 6 arcs connecting them, as in Figure 2. If we take c building blocks $B_1, \ldots, B_c$ and glue them together along the disks, in such a way that each endpoint of each arc is glued to some other endpoint (possibly of the same arc), we obtain a (possibly non-orientable) handlebody of genus $c+1$ with a link in its boundary, such as in Figure 3. By taking the orientable double of this handlebody (the orientable double cover whose boundary is quotiented by the deck involution), we obtain a link inside $M_c := \#^{c+1}(S^1 \times S^2)$. We call a link obtained this way a fundamental shadow link.

The most important features of these links are that their geometry and quantum invariants are well understood.

Lemma 5.1 ([**10**, Proposition 3.33])**.** *If $L \subseteq M_c$ is a fundamental shadow link, then $M_c \setminus L$ is hyperbolic of volume $2cv_8$ and shadow complexity c.*

The next lemma follows from the shadow reformulation of the $SO(3)$ version of the Reshetikhin–Turaev invariants [**28**, **29**]. The proof is given in [**8**, Proposition 4.1] following [**9**].

Lemma 5.2. *Let $J_r = \{0, 2, \ldots, r-3\}$. If $L = L_1 \sqcup \cdots \sqcup L_k \subseteq M_c$ is a fundamental shadow link and $col \in J_r^k$ is a coloring of its components* *with even numbers, then*

$$RT_r\left(M_c, L, col\right) = \left(\frac{2\sin\left(\frac{2\pi}{r}\right)}{\sqrt{r}}\right)^{-c} \prod_{i=1}^{c} \begin{vmatrix} col(i_1) & col(i_2) & col(i_3) \\ col(i_4) & col(i_5) & col(i_6) \end{vmatrix},$$

where i_j is the component of the link L passing through the j-th strand of block i.

Finally, we recall that any compact oriented 3-manifold with toroidal or empty boundary is obtained as a Dehn filling of some of the boundary components of the complement of some fundamental shadow link [**10**, Proposition 3.36].

To relate the Turaev–Viro invariant of $M_c \setminus L$ to the Reshetikhin–Turaev invariant of (M_c, L) of Lemma 5.2 we use the following proposition. It first appeared in [**15**] in a slightly weaker version; we give essentially the same proof, slightly modified when needed.

Proposition 5.3. *For any link $L = L_1 \sqcup \cdots \sqcup L_k$ in a closed oriented 3-manifold M,*

$$TV_r(M \setminus L) = 2^{b_2(M\setminus L)} \sum_{col \in J_r^k} \left|RT_r\left(M, L, col\right)\right|^2,$$

where $b_2\left(M \setminus L\right)$ denotes the rank of $H_2\left(M \setminus L,\ \mathbb{Z}_2\right)$.

Proof. For a compact, oriented 3-manifold X with toroidal boundary let DX denote the double of X along ∂X and let $b_2\left(X\right)$ denote the rank of $H_2\left(X,\ \mathbb{Z}_2\right)$. By [**4**, Theorems 2.9 and 3.2] for the case $q = e^{\frac{\pi i}{r}}$, adapted to other roots of unity in [**15**, Theorems 2.9 and 3.1], we have

$$TV_r(X) = 2^{b_2(M)} RT_r(DX).$$

Now let $X = M \setminus L$. Because of the axioms of the TQFT associated to the Reshetikhin–Turaev invariants, we have

$$RT_r(DX) = \langle Z_r(X), Z_r(X)\rangle,$$

where $Z_r(X)$ is the vector in the $SO(3)$ Reshetikhin–Turaev TQFT hermitian vector space $V_r(\partial X)$.

The boundary of X is a union of connected toroidal components $T_1 \sqcup \cdots \sqcup T_k$, and $V_r(\partial X) = V_r(T_1) \otimes \cdots \otimes V_r(T_k)$. An orthogonal basis for the vector space $V_r(T_i)$ is $(e_j)_{j\in J_r}$ where e_j is the solid torus with boundary T_i and whose core is colored with color j. Therefore, an orthogonal basis for $V_r(\partial X)$ is $(e_{j_1} \otimes \cdots \otimes e_{j_k})_{j\in J_r^k}$. Written in this basis, by the definition of the Reshetikhin–Turaev invariants, we have

$$Z_r(X) = \sum_{col\in J_r^k} RT_r(M, L, col) e_{col_1} \otimes \cdots \otimes e_{col_k}.$$

Hence, we obtain

$$\langle Z_r(X), Z_r(X)\rangle = \sum_{col \in J_r^k} \left|RT_r(M, L, col)\right|^2,$$

which gives the desired result. q.e.d.

We are ready to prove Conjecture 1 for the complements of these links.

Theorem 1.1. *For any fundamental shadow link $L = L_1 \sqcup \cdots \sqcup L_k$ built from c blocks,*

$$\lim_{r\to\infty} \frac{2\pi}{r} \log \left|TV_r(M_c \setminus L)\right| = Vol(M_c \setminus L) = 2cv_8.$$

Proof. If $L = L_1 \sqcup \cdots \sqcup L_k$ we have by Proposition 5.3,

$$TV_r(M_c \setminus L) = 2^{b_2\left(M_c \setminus L\right)} \sum_{col \in J_r^k} \left|RT_r\left(M_c, L, col\right)\right|^2.$$

Because the number of possible colorings is polynomial in r,

$$\begin{aligned} &\frac{2\pi}{r} \log \left|TV_r\left(M_c \setminus L\right)\right| \leqslant \\ &\leqslant \max_{col \in J_r^k} \frac{2\pi}{r} \log \left(\left|RT_r\left(M_c, L, col\right)\right|\right)^2 + O\left(\frac{\log(r)}{r}\right). \end{aligned}$$

By Lemma 5.2, we have that $RT_r(M_c, L, col)$, up to a factor that grows polynomially in r, is equal to

$$\prod_{i=1}^{c} \begin{vmatrix} col(i_1) & col(i_2) & col(i_3) \\ col(i_4) & col(i_5) & col(i_6) \end{vmatrix},$$

where i_j is the component of the link L passing through the j-th strand of block i. Hence, because of Theorem 1.2,

$$\lim_{r\to\infty} \frac{2\pi}{r} \log \left|TV_r\left(M_c \setminus L\right)\right| \leqslant 2cv_8.$$

On the other hand, if we take $col = \left(\frac{r\pm1}{2}, \ldots, \frac{r\pm1}{2}\right)$ to be even colors, we have

$$\begin{aligned} &\lim_{r\to\infty} \frac{2\pi}{r} \log \left|TV_r\left(M_c \setminus L\right)\right| \geqslant \\ &\geqslant \lim_{r\to\infty} \frac{2\pi}{r} \log \left| \begin{vmatrix} \frac{r\pm1}{2} & \frac{r\pm1}{2} & \frac{r\pm1}{2} \\ \frac{r\pm1}{2} & \frac{r\pm1}{2} & \frac{r\pm1}{2} \end{vmatrix}_{q=e^{\frac{2\pi i}{r}}} \right|^{2c} = 2cv_8, \end{aligned}$$

by Lemma 3.13. q.e.d.

6. Applications

In the previous sections we showed that the Turaev–Viro invariants volume conjecture is true for the fundamental shadow links. As recalled in the Introduction, those links are universal in the sense every orientable compact 3-manifold, with empty or toroidal boundary, is obtained by a Dehn surgery along these links. On the other hand, the behavior of Turaev–Viro invariants under Dehn filling was studied in [12]. Here we combine these results with results about estimates of hyperbolic volume change under Dehn filling to derive some interesting applications.

6.1. Dehn filling, volume and Turaev–Viro invariants. Let N be a compact 3-manifold with toroidal boundary whose interior is hyperbolic, and let $T_1, \dots, T_k$ be some components of ∂N. On each T_i, choose a slope s_i, such that the shortest length of any of the s_i is denoted $\ell_{\min}$. If $\ell_{\min} > 2\pi$, then by the Geometrization Theorem, the manifold $M = N(s_1, \dots, s_k)$ obtained by Dehn filling along $s_1, \dots, s_k$ is hyperbolic. Moreover, there is a correlation between its volume and the volume of N. In the theorem below, the upper inequality is by Thurston [26, Theorem 6.5.6] and the lower inequality is by the following result proved in [17].

Theorem 6.1 ([17, Theorem 1.1])**.** *Let N be a cusped hyperbolic 3-manifold, containing embedded horocusps $C_1, \dots, C_k$ (plus possibly others). On each torus $T_i = \partial C_i$, choose a slope s_i, such that the shortest length of any of the s_i is $\ell_{\min} > 2\pi$. Then the manifold $M = N(s_1, \dots, s_k)$ obtained by Dehn filling along $s_1, \dots, s_k$ is hyperbolic, and its volume satisfies*

$$\left(1 - \left(\frac{2\pi}{\ell_{\min}}\right)^2\right)^{3/2} \operatorname{vol}(N) \leqslant \operatorname{vol}(M) < \operatorname{vol}(N).$$

To continue recall that for a compact oriented 3-manifold M, we set

$$lTV(N) = \liminf_{r\to\infty} \frac{2\pi}{r} \log \left| TV\left(M, e^{\frac{2\pi i}{r}}\right)\right|$$

and

$$LTV(N) = \limsup_{r\to\infty} \frac{2\pi}{r} \log \left| TV\left(M, e^{\frac{2\pi i}{r}}\right)\right|,$$

where r runs over all odd integers.

Our results in the previous sections give the following.

Theorem 6.2. *Let M be an orientable, compact 3-manifold with empty or toroidal boundary. There is a hyperbolic link $L_1 \subset M$ such that*

$$lTV(M \setminus L_1) = LTV(M \setminus L_1) = \operatorname{vol}(M \setminus L_1) = 2cv_8,$$

where is $c > 0$ is a constant depending on L_1.

Furthermore, if we set $N = M \setminus L_1$ then for any link L in N we have

$$lTV(N \setminus L) \geqslant 2cv_8 > 0.$$

Proof. By [**10**, Proposition 3.36], there is $c = c(M) > 0$ and a fundamental shadow link $L_1' \subset M_c = \#^{c+1}(S^1 \times S^2)$ such that

(i) The complement of L_1' is hyperbolic with volume $2cv_8$; and
(ii) M is obtained by Dehn filling in M_c along L_1'.

Thus there is $L_1 \subset M$ such that $M \setminus L_1$ is homeomorphic to the complement L_1' in M_c. Now the first part of the theorem follows since $M_c \setminus L_1'$ satisfy the Turaev–Viro invariants volume conjecture.

To see the second part of the claim note that if $L \subset N$ is any link in $N = M \setminus L_1$ then N is obtained from $N \setminus L$ by Dehn filling. Thus by [**12**, Theorem 5.3] we have $lTV(N \setminus L) \geqslant lTV(N)$ and the conclusion follows. q.e.d.

Definition 6.3. We will refer to $N = M \setminus L_1$ in the statement of Theorem 6.2 as a complement of a fundamental shadow link in M.

Combining Theorem 6.2 and Theorem 6.1 gives the following which gives Theorem 1.3 stated in the Introduction.

Theorem 6.4. *Let M be a compact 3-manifold with empty or toroidal boundary. Then there is a hyperbolic link complement $N = M \setminus L_1$ such that M is obtained by Dehn filling on N and we have*

$$\alpha(\ell_{\min})\ lTV(N) \leqslant \mathrm{vol}(M) < lTV(N).$$

Here $\alpha(x) = \left(1 - \left(\frac{2\pi}{x}\right)^2\right)^{3/2}$ *if $x > 2\pi$, and $\alpha(x) = 0$ if $x < 2\pi$.*

Proof. Let M be an orientable, compact 3-manifold with empty or toroidal boundary and such that the interior of M admits a complete hyperbolic structure. By Theorem 6.2 there is a hyperbolic link complement $N \subset M$ such that $lTV(N) = LTV(N) = \mathrm{vol}(N)$ and M is obtained by Dehn filling along some or all the cusps of N, i.e. $M = N(s_1, \dots, s_k)$. The conclusion follows by Theorem 6.1. q.e.d.

Note that $\alpha(\ell_{\min}) > 0$, unless $\ell_{\min} < 2\pi$ and that $\alpha(\ell_{\min})$ approaches 1 as $\ell_{\min} \to \infty$. The theorem says that the volume of M is approximated by the Turaev–Viro invariants of a certain sub-manifold of M. It is known [**26**] that as $\ell_{\min} \to \infty$ we have $\mathrm{vol}(M) \to \mathrm{vol}(N)$, and by Theorem 6.4 as $\ell_{\min} \to \infty$ we also have $\mathrm{vol}(M) \to lTV(N)$, which is consistent with Conjecture 1. In fact, by Conjecture 1 one should expect a 2-sided inequality using the Turaev–Viro invariants of M itself rather than these of a submanifold N. In this direction, we have an one sided inequality given by the following.

Corollary 6.5. *Let $M = N(s_1, \ldots, s_k)$ a 3-manifold obtained by Dehn filing on a fundamental shadow link complement N. If $\ell_{\min} > 2\pi$, then M is hyperbolic and we have*

$$LTV(M) \leqslant B(\ell_{\min}) \operatorname{vol}(M),$$

where $B(\ell_{\min})$ is a function that approaches 1 as $\ell_{\min} \to \infty$.

Proof. By Theorem 1.1, we have $lTV(N) = LTV(N) = \operatorname{vol}(N)$. Since $\ell_{\min} > 2\pi$, Theorem 6.1 applies to give

$$\left(1 - \left(\frac{2\pi}{\ell_{\min}}\right)^2\right)^{3/2} \operatorname{vol}(N) \leqslant \operatorname{vol}(M).$$

By [**12**, Corollary 5.3], we have

$$LTV(M) \leqslant LTV(N) = \operatorname{vol}(N) \leqslant \left(1 - \left(\frac{2\pi}{\ell_{\min}}\right)^2\right)^{-3/2} \operatorname{vol}(M).$$

Setting $B(\ell_{\min}) = \left(1 - \left(\frac{2\pi}{\ell_{\min}}\right)^2\right)^{-3/2}$ we have the desired result. q.e.d.

6.2. Application to the AMU conjecture. Theorem 6.2 says that if N is the complement of a fundamental shadow link in M, then for every link $L \subset N$ the invariants $TV_r(N \setminus L)$ grow exponentially with respect to r. As shown in [**14**] the exponential growth property has applications to the AMU conjecture [**1**]. To give details, for a compact orientable surface of genus g and n boundary components, say $\Sigma_{g,n}$, let $\operatorname{Mod}(\Sigma_{g,n})$ denote its mapping class group.

Definition 6.6. For a mapping class $f \in \operatorname{Mod}(\Sigma_{g,n})$, let $M(f)$ denote the mapping torus of f. We say that f has a non-trivial pseudo-Anosov part if the toroidal decomposition of $M(f)$ contains hyperbolic pieces; or equivalently if the Gromov norm of $M(f)$ is non-zero.

Recall that I_r is the set $\{0, 1, \ldots, r-2\}$. Given a coloring *col* of the components of $\partial\Sigma_{g,n}$ by elements of I_r, by [**5**], there is a finite dimensional $\mathbb{C}$-vector space $V_r(\Sigma_{g,}, col)$ as well as a projective representation

$$\rho_{r,col} : \operatorname{Mod}(\Sigma_{g,n}) \to \mathbb{P}\mathrm{Aut}(V_r(\Sigma_{g,n}, col)).$$

The following statement is known as the AMU conjecture.

Conjecture 2 ([**1**])**.** *Let $f \in \operatorname{Mod}(\Sigma_{g,n})$ be a mapping class. If f contains a pseudo-Anosov part, then for any big enough r there is a choice of colors col of the components of $\partial\Sigma$ such that $\rho_{r,col}(\phi)$ has infinite order.*

In [**1**] Andersen, Masbaum and Ueno verified the conjecture for $\Sigma_{0,4}$. Later, Santharoubane proved it for $\Sigma_{1,1}$ [**23**] and Egsgaard and Jorgensen [**16**] and Santharoubane [**24**] gave partial results for pseudo-Anosov maps on $\Sigma_{0,2n}$. For $g \geqslant 2$, the first examples of mappings classes that satisfy the AMU conjecture, were given by Marché and Santharoubane in [20] and the first construction that leads to infinitely many (independent) examples in each genus was given by Detcherry and Kalfagianni [14].

Let M be a closed orientable 3-manifold with empty or toroidal boundary. Recall that a link J in M is called fibered if the complement $M \setminus J$ is homeomorphic to the mapping torus of a mapping class $f \in \mathrm{Mod}(\Sigma_{g,n})$, for some $n, g \geqslant 0$. The mapping class f is called the monodromy of the fibration. We note that if the first Betti number of $M \setminus J$ is bigger than one then it can fiber over S^1 in infinitely many different ways; that is we have infinitely many non-conjugate mapping classes realized as monodromies of some fibration $M \setminus J \longrightarrow S^1$ [27, §3].

In [14] the authors show that if we have $lTV(M(f)) > 0$ for the mapping torus of a class $f \in \mathrm{Mod}(\Sigma_{g,n})$, then f satisfies the AMU conjecture. On the other hand $lTV(M(f)) > 0$ implies, by [12], that $M(f)$ has non-zero Gromov norm and thus f contains a pseudo-Anosov part. In [14] the authors give explicit constructions of mapping classes that satisfy these conditions. The examples constructed in [14] are all realized as monodromies of fibered links in S^3. Theorem 6.2 provides infinite families of manifolds with toroidal boundary and with Turaev–Viro invariants having exponential growth. By passing to the doubles DN we obtain closed 3-manifolds with $lTV(DN) > 0$. Any mapping class that is realized as a monodromy of a fibered link in some N or DN satisfies the AMU conjecture.

Let $\mathcal{M}$ denote the set of all 3-manifolds N that are complements of fundamental shadow links in a orientable 3-manifolds with empty or toroidal boundary and their doubles DN. We have the following.

Theorem 1.4. *Given $M \in \mathcal{M}$ and a (possibly empty) link $L \subset M$, there is a knot $K \subset M$ such that the link $K \cup L$ is fibered in M. Furthermore, the monodromy of any fibration of $M \setminus (K \cup L)$ is a mapping class that satisfies the AMU conjecture.*

Proof. Let M and L be as above and let L' a link in S^3 so that M is obtained by integral Dehn surgery on all or some of the components of L'. Note, in particular, that if M is a fundamental shadow link complement then L' will contain the link corresponding to the fundamental shadow link L_1 in S^3. Slightly abusing the notation we will also use L to denote the link in L in S^3 corresponding to L.

By Stallings [25, Theorem 2], we can find a knot $K \subset S^3$ so that $J = K \cup L \cup L'$ is a fibered link in S^3. Furthermore, we have the following.

1) The knot K can be chosen so that the linking numbers of K with the components of $L \cup L'$ are arbitrary; that is matching any prechosen collection of integers.

2) The link J is represented as a closed braid (a homogeneous braid in fact) and the fiber, say F_K, of $S^3 \setminus J$ is the natural Seifert surface associated to the closed braid projection. The reader is referred to [**14**] for a refinement of Stallings construction that produces hyperbolic fibered links and for pictures of the fiber surface.

The components of L' are equipped with the framings needed to recover M from $S^3 \setminus J$ by Dehn filling. On the other hand, the Seifert surface F_K also defines a natural framing on J: defined by the linking number of J with a push-off of it on F_K in the direction of the inward normal vector. The surface framing changes as we change the linking numbers of K with the components of J. Since we are allowed to choose these numbers to be arbitrary, by re-choosing K, we can pick K so the framings defined on the components of L' by the fiber F_K agree with the framings of the surgery needed to recover M. Then, the surgery caps off some components of K with disks and also produces a fibered manifold. That is $M \setminus (K \cup L)$ will fiber over S^1 with fiber the surface obtained by F_K by capping the components of ∂F_K corresponding to L'. By the discussion in the paragraph before the statement of the theorem, the monodromy of such a fibration gives a mapping class that contains non-trivial pseudo-Anosov pieces and satisfies the AMU conjecture. q.e.d.

Theorem 1.4 can be used to construct an abundance of mapping classes that satisfy the AMU conjecture. In particular, working with fibered knots in the closed manifolds of $\mathcal{M}$ we can construct classes in $\mathrm{Mod}(\Sigma_{g,1})$. This approach is developed in [**13**] where Detcherry and Kalfagianni show that for every closed oriented 3-manifold M, and g a sufficiently large integer, $\mathrm{Mod}(\Sigma_{g,1})$ contains a coset of an abelian subgroup of rank $\lfloor \frac{g}{2} \rfloor$, consisting of pseudo-Anosov monodromies of fibered knots in M. Furthermore, they prove a similar result for rank two free cosets of $\mathrm{Mod}(\Sigma_{g,1})$.

Note that all the manifolds in $\mathcal{M}$ have Gromov norm at least $4v_8$, but there exist fibered links of smaller Gromov norm. It follows that there are mapping classes in $\mathrm{Mod}(\Sigma_{g,n})$, $n \neq 0$, that do not appear as monodromies of fibered links in any manifold in $\mathcal{M}$. We finish the subsection with the following.

Question 1. *Which mapping classes are realized by Theorem 1.4?*

Appendix A.

The following theorem was originally proved by Costantino in [**8**] for quantum $6j$-symbols evaluated at the root of unity $q = e^{\frac{\pi i}{r}}$, which is different from the one $q = e^{\frac{2\pi i}{r}}$ we considered in this paper. The main difference between the two cases is the following: For certain argument to work, some technical constrains have to be put on the sequence of 6-tuples. It turns out that at $q = e^{\frac{\pi i}{r}}$ as considered in [8] the 6-tuples satisfying these technical constrains never satisfy the admissibility conditions and, as a consequence, the "evaluation" of the quantum $6j$-symbols has to be modified; but at $q = e^{\frac{2\pi i}{r}}$ the set of 6-tuples satisfying both the technical constrains and the admissibility conditions is non-empty, and those 6-tuples are exactly the ones that give dihedral angles of ideal or hyperideal truncated tetrahedra. In this Appendix, we include a proof of the result at the root $q = e^{\frac{2\pi i}{r}}$ for the interested readers. A similar result can also be found in [6].

Theorem A.1 ([8]). *Let $\{(n_1^{(r)}, \dots, n_6^{(r)})\}$ be a sequence of 6-tuples such that*

(1) $0 \leqslant Q_j - T_i \leqslant \frac{r-2}{2}$ for $i = 1, \dots, 4$ and $j = 1, 2, 3$, and
(2) $\frac{r-2}{2} \leqslant T_i \leqslant r - 2$ for $i = 1, \dots, 4$.

Let $\theta_i = \lim_{r\to\infty} \frac{2\pi n_i^{(r)}}{r}$ and let $\alpha_i = |\pi - \theta_i|$. Then

(1) $\alpha_1, \dots, \alpha_6$ are the dihedral angles of an ideal or a hyperideal hyperbolic tetrahedron Δ, and
(2) as r runs over all the odd integers

$$\lim_{r\to\infty} \frac{2\pi}{r} \log \left| \begin{vmatrix} n_1^{(r)} & n_2^{(r)} & n_3^{(r)} \\ n_4^{(r)} & n_5^{(r)} & n_6^{(r)} \end{vmatrix}_{q=e^{\frac{2\pi i}{r}}} \right| = Vol(\Delta).$$

Proof. (1). By Bao and Bonahon [2], six positive numbers $\alpha_1, \dots, \alpha_6$ are the dihedral angles of an ideal or a hyperideal tetrahedron if and only if around each vertex, $\alpha_i + \alpha_j + \alpha_k \leqslant \pi$. The given conditions imply that around each vertex, $2\pi \leqslant \theta_i + \theta_j + \theta_k \leqslant 4\pi$ and $0 \leqslant \theta_i + \theta_j - \theta_k \leqslant 2\pi$. Depending on whether θ_i lies in $[0, \pi]$ or $[\pi, 2\pi]$, these conditions correspond exactly to the Bao–Bonahon conditions.

(2). By Lemma 3.1, we have

$$\begin{aligned} &\lim_{r\to\infty} \frac{2\pi}{r} \log \big|\Delta(n_i, n_j, n_k)\big| = \\ &= -\frac{1}{2}\Lambda\Big(\frac{\theta_1+\theta_2-\theta_3}{2}\Big) - \frac{1}{2}\Lambda\Big(\frac{\theta_2+\theta_3-\theta_1}{2}\Big) - \\ &\quad - \frac{1}{2}\Lambda\Big(\frac{\theta_3+\theta_1-\theta_2}{2}\Big) + \frac{1}{2}\Lambda\Big(\frac{\theta_1+\theta_2+\theta_3}{2}\Big). \end{aligned}$$

Next, we study the asymptotics of

$$\sum_{z=\max(T_i)}^{\min(Q_j)} \frac{(-1)^z [z+1]!}{\prod_{i=1}^4 [z-T_i]! \prod_{j=1}^3 [Q_j - z]!}.$$

Let

$$S_z = \frac{(-1)^z [z+1]!}{\prod_{i=1}^4 [z-T_i]! \prod_{j=1}^3 [Q_j - z]!}.$$

If $\lim_{r\to\infty}\frac{2\pi z}{r}=Z$, then by Lemma 3.1 we have

$$\lim_{r\to\infty}\frac{2\pi}{r}\log|S_z|=\sum_{i=1}^4\Lambda(Z-U_i)+\sum_{j=1}^3\Lambda(V_j-Z)-\Lambda(Z).$$

The strategy is to show that all S_z's for z in between $\max(T_i)$ and $\min(Q_j)$ have the same sign so the growth rate of the sum is determined by that of the largest term.

Since $T_i\leqslant z$ and $z\leqslant Q_j$, and by the assumption that $Q_j-T_i\leqslant\frac{r-2}{2}$, we have $0\leqslant z-T_i\leqslant\frac{r-2}{2}$ and $\leqslant Q_j-z\leqslant\frac{r-2}{2}$ for all $i=1,\dots,4$ and $j=1,2,3$. Hence

$$0\leqslant\frac{2\pi(z-T_i)}{r}\leqslant\pi\quad\text{and}\quad 0\leqslant\frac{2\pi(Q_j-z)}{r}\leqslant\pi.$$

Also, by the assumption that $T_i\leqslant\frac{r-2}{2}$, $z\geqslant T_i$, we have $\frac{r-2}{2}\leqslant z$. Since $[z+1]!=0$ when $z>r-2$, we can assume that $z\leqslant r-2$. Hence

$$\pi\leqslant\frac{2\pi z}{r}\leqslant 2\pi.$$

As a consequence, we have

$$\frac{S_z}{S_{z-1}}=-\frac{\sin\frac{2\pi(z+1)}{r}\sin\frac{2\pi(Q_1-z+1)}{r}\sin\frac{2\pi(Q_2-z+1)}{r}\sin\frac{2\pi(Q_3-z+1)}{r}}{\sin\frac{2\pi(z-T_1)}{r}\sin\frac{2\pi(z-T_2)}{r}\sin\frac{2\pi(z-T_3)}{r}\sin\frac{2\pi(z-T_4)}{r}}>0,$$

and hence all the S_z's have the same sign.

Next we show that the function $F:(\max(U_i),\min(V_j,2\pi))\to\mathbb{R}$ defined by

$$F(Z)=\sum_{i=1}^4\Lambda(Z-U_i)+\sum_{j=1}^3\Lambda(V_j-Z)-\Lambda(Z)$$

has a unique maximum point Z_0. Indeed, by a direct computation, one has

$$F'(Z)=\log\left(\frac{\sin(2\pi-Z)\sin(V_1-Z)\sin(V_2-Z)\sin(V_3-Z)}{\sin(Z-U_1)\sin(Z-U_2)\sin(Z-U_3)\sin(Z-U_4)}\right),$$

and

$$F''(Z)=-\sum_{i=1}^4\cot(Z-U_i)-\sum_{j=1}^3\cot(V_j-Z)-\cot(2\pi-Z).$$

Here we recall the fact that if α and β are two real numbers in $(0,\pi)$ with $\alpha+\beta<\pi$, then $\cot(\alpha)+\cot(\beta)>0$. Now since $(2\pi-Z)+(Z-U_4)=2\pi-U_4\in(0,\pi)$ and $(V_i-Z)+(Z-U_i)=V_i-U_i\in(0,\pi)$ for $i=1,2,3$, we have $F''(Z)<0$, and hence $F'(Z)$ is strictly decreasing and $F(Z)$ is strictly concave down. Together with

$$\lim_{Z\to\max(U_i)}F'(Z)=+\infty\quad\text{and}\quad\lim_{Z\to\min(V_j,2\pi)}F'(V)=-\infty,$$

we conclude that there is a unique $Z_0 \in (\max(U_i), \min(V_j, 2\pi))$ such that $F'(Z_0) = 0$. By the concavity of F, Z_0 is the unique maximum point of F.

Now for each sequence $z^{(r)}$ with $\lim_{r\to\infty} \frac{2\pi z^{(r)}}{r} = Z$, by Lemma 3.1 one has

$$|S_{z^{(r)}}| = \exp\left\{\frac{r}{2\pi}F(Z) + O(\log(r))\right\} \leqslant \exp\left\{\frac{r}{2\pi}F(Z_0) + C\log(r)\right\}.$$

Since all the S_z's have the same sign, we have

$$\left|\sum_{z=\max(T_i)}^{\min(Q_j)} S_z\right| \leqslant \big(\min(Q_j, r-2) - \max(T_i)\big)\exp\left(\frac{r}{2\pi}F(Z_0) + C\log(r)\right),$$

and hence

$$\limsup_{r\to\infty}\frac{1}{r}\log\left|\sum_{z=\max(T_i)}^{\min(Q_j)} S_z\right| \leqslant$$
$$\leqslant \lim_{r\to\infty}\frac{1}{r}\log\left\{\big(\min(Q_j, r-2) - \max(T_i)\big)\exp\left(\frac{r}{2\pi}F(Z_0) + C\log(r)\right)\right\} =$$
$$= \frac{F(Z_0)}{2\pi}.$$

On the other hand, let $z^{(r)}$ be a sequence such that

$$\lim_{r\to\infty}\frac{2\pi z^{(r)}}{r} = Z_0.$$

Then by Lemma 3.1

$$\lim_{r\to\infty}\frac{2\pi}{r}\log|S_{z^{(r)}}| = F(Z_0).$$

Again since all the S_z's have the same sign, we have

$$\left|\sum_{z=\max(T_i)}^{\min(Q_j)} S_z\right| > S_{z^{(r)}},$$

and hence

$$\liminf_{r\to\infty}\frac{2\pi}{r}\log\left|\sum_{z=\max(T_i)}^{\min(Q_j)} S_k\right| \geqslant \lim_{r\to\infty}\frac{2\pi}{r}\log|S_{z^{(r)}}| = F(Z_0).$$

Therefore, we have

$$\lim_{r\to\infty}\frac{2\pi}{r}\log\left|\sum_{z=\max(T_i)}^{\min(Q_j)} S_z\right| = F(Z_0),$$

and

$$\lim_{r\to\infty}\frac{2\pi}{r}\log\left|\left|\begin{matrix} n_1^{(r)} & n_2^{(r)} & n_3^{(r)} \\ n_4^{(r)} & n_5^{(r)} & n_6^{(r)}\end{matrix}\right|_{q=e^{\frac{2\pi i}{r}}}\right| =$$

$$= -\frac{1}{2}\sum_{i,j}\Lambda(V_j-U_i)+\frac{1}{2}\sum_i\Lambda(U_i)+F(Z_0).$$

Then as argued in [8], by the Murakami, Yano and Ushijima formula [21, Theorems 1 and 2], [31, Theorem 1.1] and their symmetry, if $\alpha_1,\dots,\alpha_6$ are the dihedral angles of a hyperideal tetrahedron Δ and $\theta_i=\pi\pm\alpha_i$, then the right hand side is exactly the volume of Δ. q.e.d.

References

[1] Andersen, J. E. and Masbaum, G. and Ueno, K., *Topological quantum field theory and the Nielsen-Thurston classification of M(0,4)*, Math Proc. Cambridge Philos. Soc. **141** (2006), 477–488, MR 2281410, Zbl 1110.57009.

[2] Bao, X. and Bonahon, F., *Hyperideal polyhedra in hyperbolic 3-space*, Bull. Soc. Math. France **130** (2002), no. 3, 457–491, MR 943885, Zbl 1033.52009.

[3] Belletti, G., *A maximum volume conjecture for hyperbolic polyhedra*, Preprint, 2020.

[4] Benedetti, R. and Petronio, C., *On Roberts' proof of the Turaev-Walker theorem*, J. Knot Theory Ramifications **5** (1996), no. 4, 427–439, MR 1406714, Zbl 0890.57029.

[5] Blanchet, C. and Habegger, N. and Masbaum, G. and Vogel, P., *Topological quantum field theories derived from the Kauffman bracket*, Topology **34** (1995), no. 4, 883–927, MR 1362791, Zbl 0887.57009.

[6] Chen, Q. and Murakami J., *Asymptotics of quantum 6j-symbols*, preprint, 2018.

[7] Chen, Q. and Yang, T., *Volume conjectures for the Reshetikhin–Turaev and the Turaev–Viro invariants*, Quantum Topol. **9** (2018), no. 3, 419–460, MR 3827806, Zbl 1405.57020.

[8] Costantino, F., *6j-symbols, hyperbolic structures and the volume conjecture*, Geom. Topol. **11** (2007), 1831–1854, MR 2350469, Zbl 1132.57011.

[9] Costantino, F., *Colored Jones invariants of links in $S^3\#_kS^2\times S^1$ and the volume conjecture*, J. London Math. Soc. **76** (2007), no. 2, 1–15, MR2351605, Zbl 1153.57003.

[10] Costantino, F. and Thurston, D., *3-manifolds efficiently bound 4-manifolds*, J. Topol. **1** (2008), no. 3, 703–745, MR 2417451, Zbl 1166.57016.

[11] Renaud Detcherry, *Growth of Turaev–Viro invariants and cabling*, J. Knot Theory Ramifications **28** (2019), no. 14, 1950041, 8.

[12] Renaud Detcherry and Efstratia Kalfagianni, *Gromov norm and Turaev–Viro invariants of 3-manifolds*, Ann. Sci. Éc. Norm. Supér. (4) **53** (2020), no. 6, 1363–1391.

[13] Detcherry, R. and Kalfagianni, E., *Cosets of monodromies and quantum representations*, *Indiana Univ. Mathematics Journal*, to appear.

[14] Detcherry, R. and Kalfagianni, E., *Quantum representations and monodromies of fibered links*, Adv. Math. **351** (2019), 676–701, MR 3954955, Zbl 1419.57034.

[15] Detcherry, R. and Kalfagianni, E. and Yang, T., *Turaev–Viro invariants, colored Jones polynomials, and volume*, Quantum Topol. **9** (2018), no. 4, 775–813, MR3874003, Zbl 06996468.

[16] Egsgaard, J. K. and Jorgensen, S. F., *The homological content of the Jones representations at* $q = -1$, Journal of Knot Theory and its ramifications. **25** (2016), no. 11, 25pp, MR 3558421, Zbl 1379.57011.

[17] Futer, D. and Kalfagianni, E. and Purcell, J. S., *Dehn filling, volume, and the Jones polynomial*, J. Differential Geom. **78** (2008), no. 3, 429–464, MR2396249, Zbl 1144.57014.

[18] Garoufalidis, S. and Lê, T. T. Q., *Asymptotics of the colored Jones function of a knot*, Geom. Topol. **15** (2011), no. 4, 2135–2180, MR 2860990, Zbl 1239.57029.

[19] Kashaev, R. M., *The hyperbolic volume of knots from the quantum dilogarithm*, Lett. Math. Phys. **39** (1997), no. 3, 269–275, MR 1434238, Zbl 0876.57007.

[20] J. Marché and R. Santharoubane, *Asymptotics of the quantum representations of surface groups*, arXiv:1607.00664.

[21] Murakami, J. and Yano, M., *On the volume of a hyperbolic and spherical tetrahedron*, Comm. Anal. Geom. **13** (2005), no. 2, 379–400, MR2154824, Zbl 1084.51009.

[22] Ohtsuki, T., *On the asymptotic expansion of the quantum* $\mathrm{SU}(2)$ *invariant at* $q = \exp(4\pi\sqrt{-1}/N)$ *for closed hyperbolic 3-manifolds obtained by integral surgery along the figure-eight knot*, Algebr. Geom. Topol. **18** (2018), no. 7, 4187–4274, MR3892244, Zbl 07006390.

[23] Santharoubane, R., *Limits of the quantum SO(3) representations for the one-holed torus*, Journal of Knot Theory and its ramifications **21** (2012), no. 11, 13pp, MR 2969639, Zbl 1260.57034.

[24] Santharoubane, R., *Action of M(0,2n) on some kernel spaces coming from SU(2)-TQFT*, J. Lon. Math. Soc. **95** (2017), no. 3, 785–803, MR 3664518, Zbl 1397.57049.

[25] J. R. Stallings, *Constructions of fibered knots and links*, Proc. of Symposia in Pure Mathematics **32** (1978), 55–60, MR, 0520522.

[26] Thurston, W. P., *The geometry and topology of three-manifolds*, Princeton Univ. Math. Dept. Notes, 1979.

[27] Thurston, W. P., *A norm for the homology of* 3*-manifolds*, Mem. Amer. Math. Soc. **59** (1986), no. 339, i–vi and 99–130, MR 823443, Zbl 0585.57006.

[28] Turaev, V. G., *Shadow links and face models of statistical mechanics*, J. Differential Geom. **36** (1992), no. 1, 35–74, MR1168981, Zbl 0773.57012.

[29] Turaev, V. G., *Quantum invariants of knots and 3-manifolds*, de Gruyter Studies in Mathematics, vol. 18, Walter de Gruyter & Co., Berlin, 1994. MR 1292673, Zbl 0812.57003.

[30] Turaev, V. G. and Viro, O. Y., *State sum invariants of* 3*-manifolds and quantum* $6j$*-symbols*, Topology **4** (1992), 865—902, MR 1191386, Zbl 0779.57009.

[31] Ushijima, A., *A volume formula for generalised hyperbolic tetrahedra*, Non-Euclidean geometries, Math. Appl. (N. Y.), vol. 581, Springer, New York, 2006, pp. 249–265, MR 2191251, Zbl 1096.52006.

[32] Wong, K. H., *Asymptotics of some quantum invariants of the Whitehead chains*, GT.1912.10638.

[33] Wong, K. H., *Volume conjecture, geometric decomposition and deformation of hyperbolic structures*, GT. 1912.11779.

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