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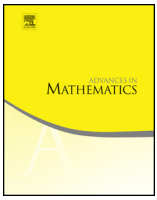

When a system of real quadratic equations has a solution ☆

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ABSTRACT

We provide a sufficient condition for solvability of a system of real quadratic equations $p_i(x) = y_i$, $i = 1, \ldots, m$, where $p_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ are quadratic forms. By solving a positive semidefinite program, one can reduce it to another system of the type $q_i(x) = \alpha_i$, $i = 1, \ldots, m$, where $q_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ are quadratic forms and $\alpha_i = \operatorname{trace} q_i$. We prove that the latter system has solution $x \in \mathbb{R}^n$ if for some (equivalently, for any) orthonormal basis $A_1, \ldots, A_m$ in the space spanned by the matrices of the forms q_i, the operator norm of $A_1^2 + \ldots + A_m^2$ does not exceed η/m for some absolute constant $\eta > 0$. The condition can be checked in polynomial time and is satisfied, for example, for random q_i provided $m \leq \gamma\sqrt{n}$ for an absolute constant $\gamma > 0$. We prove a similar sufficient condition for a system of homogeneous quadratic equations to have a non-trivial solution. While the condition we obtain is of an algebraic nature, the proof relies on analytic tools including Fourier analysis and measure concentration.

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1. Introduction and main results

1.1. Systems of real quadratic equations

Let $q_1, \ldots, q_m : \mathbb{R}^n \longrightarrow \mathbb{R}$ be quadratic forms,

$$q_i(x) = \langle Q_i x, x \rangle \quad \text{for} \quad i = 1, \ldots, m,$$

where Q_i are $n \times n$ symmetric matrices and

$$\langle x, y \rangle = \sum_{i=1}^{n} \xi_i \eta_i \quad \text{for} \quad x = (\xi_1, \ldots, \xi_n) \quad \text{and} \quad y = (\eta_1, \ldots, \eta_n)$$

is the standard scalar product in $\mathbb{R}^n$.

Let $\alpha_1, \ldots, \alpha_m$ be real numbers. We want to find out when the system of equations

$$q_i(x) = \alpha_i \quad \text{for} \quad i = 1, \ldots, m \tag{1.1}$$

has a solution $x \in \mathbb{R}^n$. Such systems of equations appear in various contexts, see, for example, [6], [12], [13]. If the number m of equations is fixed in advance, one can decide in polynomial time whether the system has a solution [1], [8], [4]. The same is true if the number n of variables is fixed in advance, in which case a polynomial time algorithm to test feasibility exists even if q_i are polynomials of an arbitrary degree, see, for example, [5].

If m and n are both allowed to grow, the problem becomes computationally hard. Unless the computational complexity hierarchy collapses, there is no polynomial time algorithm to test the feasibility of (1.1). Furthermore, it is not known whether the feasibility problem belongs to the complexity class **NP**. In other words, it is not known whether one can present a polynomial size certificate for the system (1.1) to have a solution when it is indeed feasible (note that using repeated squaring of the type $x_{n+1} = x_n^2$, one can construct examples of feasible systems for which no solution has a polynomial size description).

In fact, testing the feasibility of an arbitrary system of real polynomial equations can be easily reduced to testing the feasibility of a system (1.1). First, we gradually reduce the degree of polynomials by repeatedly introducing new variables and equations of the type $\xi_{ij} - \xi_i \xi_j = 0$, which allows us to replace the product $\xi_i \xi_j$ of old variables by a single new variable ξ_{ij}, and hence eventually reduce a given polynomial system to a system

$$q_i(x) = 0 \quad \text{for} \quad i = 1, \ldots, m,$$

where q_i are quadratic, not necessarily homogeneous, polynomials. Then we introduce another variable τ and replace the above system by a system of homogeneous quadratic equations

$$\tau^2 q_i\left(\tau^{-1}x\right) = 0 \quad \text{for} \quad i = 1, \ldots, m$$

with one more quadratic constraint $\tau^2 = 1$.

We are also interested in systems of homogeneous equations

$$q_i(x) = 0 \quad \text{for} \quad i = 1, \ldots, m, \tag{1.2}$$

in which case we want to find out whether the system has a non-trivial solution $x \neq 0$. The problem is also computationally hard. We briefly sketch how an efficient algorithm for testing the existence of a non-trivial solution in (1.2) would produce an efficient algorithm for testing the feasibility of (1.1). Given a system (1.1), by introducing a new variable τ, as above we replace (1.1) by a system of homogeneous quadratic equations, where we want to enforce $\tau \neq 0$. This is done by introducing yet another variable σ and the equation

$$R^2\tau^2 - \left(\xi_1^2 + \ldots + \xi_n^2\right) = \sigma^2$$

binding all variables together, so that if $\tau = 0$ then all other variables are necessarily 0. Here R is meant to be a very large constant and in fact, it can be treated as infinitely large, with computations in the ordered field of rational functions in R, the trick first introduced in [9].

In this paper, we present computationally simple sufficient criteria for (1.1), respectively (1.2), to have a solution, respectively a non-trivial solution. We start with by now a standard procedure of semidefinite relaxation.

1.2. Positive semidefinite relaxation

For an $n \times n$ real symmetric matrix X, we write $X \succeq 0$ to say that X is positive semidefinite.

Given (1.1), we consider the following system of linear equations

$$\operatorname{trace}(Q_i X) = \alpha_i \quad \text{for} \quad i = 1, \ldots, m \qquad \text{where} \qquad X \succeq 0 \tag{1.3}$$

in $n \times n$ positive semidefinite matrices X. Unlike (1.1), the system (1.3) is convex and efficient algorithms are available to test its feasibility, see [13] for a survey. Clearly, if $x = (\xi_1, \ldots, \xi_n)$ is a solution to (1.1) then the matrix $X = (x_{ij})$ defined by $x_{ij} = \xi_i\xi_j$ is a positive semidefinite solution to (1.3). If $m \leq 2$, then the converse is true: if the system (1.3) has a solution then so does (1.1), see, for example, Section II.13 of [2]. For $m \geq 3$ the system (1.3) may have solutions while (1.1) may be infeasible. For example, the system of quadratic equations

$$\xi_1^2 = 1, \quad \xi_2^2 = 1 \quad \text{and} \quad \xi_1\xi_2 = 0$$

does not have a solution, whereas the 2×2 identity matrix I is the solution to its positive semidefinite relaxation. One corollary of our results is that such examples are, in some sense, "atypical".

Our goal is to find a computationally simple criterion when a solution to (1.3) implies the existence of a solution to (1.1).

Let X be a solution to (1.3). Since $X \succeq 0$, we can write $X = TT^*$ for an $n \times n$ matrix T. Then

$$\operatorname{trace}(Q_i X) = \operatorname{trace}(Q_i T T^*) = \operatorname{trace}(T^* Q_i T).$$

Let us define matrices

$$\widehat{Q_i} = T^* Q_i T \quad \text{for} \quad i = 1, \ldots, m \tag{1.4}$$

and the corresponding quadratic forms $\widehat{q_i} : \mathbb{R}^n \longrightarrow \mathbb{R}$,

$$\widehat{q_i}(x) = \langle \widehat{Q_i} x, x \rangle = q_i(Tx) \quad \text{for} \quad i = 1, \ldots, m. \tag{1.5}$$

If $x \in \mathbb{R}^n$ is a solution to the system

$$\widehat{q_i}(x) = \alpha_i \quad \text{for} \quad i = 1, \ldots, m \tag{1.6}$$

then $y = Tx$ is a solution to (1.1). We note that

$$\alpha_i = \operatorname{trace} \widehat{Q_i} \quad \text{for} \quad i = 1, \ldots, m. \tag{1.7}$$

It may happen that the system (1.1) has a solution while (1.6) does not, but if X and hence T are invertible, the systems (1.1) and (1.6) are equivalent. Furthermore, if there are no invertible $X \succeq 0$ satisfying (1.3), then the affine subspace defined by the equations $\operatorname{trace}(Q_i X) = \alpha_i$ intersects the cone of positive semidefinite matrices at a proper face, and the system (1.1) can be effectively reduced to a system of quadratic equations in fewer variables, cf., for example, Section II.12 of [2]. Summarizing, a solution X to (1.3) allows us to replace (1.1) by a similar system, where the right hand sides α_i are the traces of the quadratic forms in the left hand side.

Ultimately, we are interested in finding out when the system (1.6) of quadratic equations with additional conditions (1.7) has a solution $x \in \mathbb{R}^n$.

1.3. Reduction to an orthonormal basis and the main result

Before we state our main result, some remarks are in order. As agreed, we consider the system (1.1) where $\alpha_i = \operatorname{trace} q_i$. Without loss of generality, we assume that the quadratic forms q_i and hence their matrices Q_i are linearly independent. For an invertible $m \times m$ matrix $M = (\mu_{ij})$, let us define new forms

$$\tilde{q}_i = \sum_{j=1}^m \mu_{ij} q_j \quad \text{for} \quad i = 1, \ldots, m$$

and new right hand sides

$$\tilde{\alpha}_i = \sum_{j=1}^m \mu_{ij} \alpha_j \quad \text{for} \quad i = 1, \ldots, m.$$

Then the system (1.1) has a solution if and only if the system

$$\tilde{q}_i(x) = \tilde{\alpha}_i \quad \text{for} \quad i = 1, \ldots, m$$

has a solution. Hence, ideally, a criterion for the system (1.1) to have a solution should depend not on the forms $q_1, \ldots, q_m$ per se (or their matrices $Q_1, \ldots, Q_m$) but on the subspace span $(q_1, \ldots, q_m)$ in the space of quadratic forms (equivalently, on the subspace span $(Q_1, \ldots, Q_m)$ in the space of $n \times n$ real symmetric matrices).

We consider the standard inner product in space of $n \times n$ real matrices:

$$\langle X, Y \rangle = \operatorname{trace} X^* Y.$$

In particular, for symmetric matrices $X = (\xi_{ij})$ and $Y = (\eta_{ij})$ we have

$$\langle X, Y \rangle = \operatorname{trace} XY = \sum_{1 \le i,j \le n} \xi_{ij} \eta_{ij}$$

and the space of $n \times n$ symmetric matrices becomes a Euclidean space.

We will be using the following observation. Let $\mathcal{L}$ be a subspace in the space of $n \times n$ symmetric matrices and let $A_1, \ldots, A_m$ be an orthonormal basis of $\mathcal{L}$, so that

$$\langle A_i, A_j \rangle = \operatorname{trace} A_i A_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \ne j. \end{cases}$$

Then the matrix $A_1^2 + \ldots + A_m^2$ does not depend on a choice of an orthonormal basis and hence is an invariant of the subspace $\mathcal{L}$. Indeed, if $B_1, \ldots, B_m$ is another orthonormal basis of $\mathcal{L}$, then

$$B_i = \sum_{j=1}^m \mu_{ij} A_j \quad \text{for} \quad i = 1, \ldots, m$$

and some orthogonal matrix $M = (\mu_{ij})$ and hence

$$\sum_{i=1}^m B_i^2 = \sum_{i=1}^m \left(\sum_{1 \le j_1, j_2 \le m} \mu_{ij_1} \mu_{ij_2} A_{j_1} A_{j_2} \right) = \sum_{1 \le j_1, j_2 \le m} \left(\sum_{i=1}^m \mu_{ij_1} \mu_{ij_2} \right) A_{j_1} A_{j_2} = \sum_{j=1}^m A_j^2.$$

For an $n \times n$ real symmetric matrix Q, we denote by $\|Q\|_{\mathrm{op}}$ the operator norm of Q, that is, the largest absolute value of an eigenvalue of Q.

We prove the following main result.

Theorem 1.1. *There is an absolute constant $\eta > 0$ such that the following holds. Let $Q_1, \ldots, Q_m$, $m \geq 3$, be linearly independent $n \times n$ symmetric matrices and let $q_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ for $i = 1, \ldots, m$ be the corresponding quadratic forms,*

$$q_i(x) = \langle Q_i x, x \rangle \quad \text{for} \quad i = 1, \ldots, m.$$

Suppose that

$$\left\| \sum_{i=1}^m A_i^2 \right\|_{\mathrm{op}} \leq \frac{\eta}{m}$$

for some (equivalently, for any) orthonormal basis $A_1, \ldots, A_m$ of the subspace $\mathrm{span}(Q_1, \ldots, Q_m)$. *Then the system of quadratic equations*

$$q_i(x) = \mathrm{trace}\, Q_i \quad \text{for} \quad i = 1, \ldots, m$$

has a solution $x \in \mathbb{R}^n$.

We prove a similar result for systems of homogeneous quadratic equations, where we are interested in finding a non-trivial solution.

Theorem 1.2. *There is an absolute constant $\eta > 0$ such that the following holds. Let $Q_1, \ldots, Q_m$, $m \geq 3$, be $n \times n$ real symmetric matrices such that*

$$\mathrm{trace}\, Q_i = 0 \quad \text{for} \quad i = 1, \ldots, m,$$

and let $q_i : \mathbb{R}^n \longrightarrow \mathbb{R}$,

$$q_i(x) = \langle Q_i x, x \rangle \quad \text{for} \quad i = 1, \ldots, m,$$

be the corresponding quadratic forms. Suppose that

$$\left\| \sum_{i=1}^m A_i^2 \right\|_{\mathrm{op}} \leq \frac{\eta}{m}$$

for some (equivalently, for any) orthonormal basis $A_1, \ldots, A_m$ of the subspace $\mathrm{span}(Q_1, \ldots, Q_m)$. *Then the system (1.2) of equations has a solution $x \neq 0$.*

Remark 1.3. Our proofs of Theorems 1.1 and 1.2 work for $\eta = 10^{-6}$, however, we made no effort to optimize this constant.

Note that the operator norm of the matrix $\sum_{i=1}^m A_i^2$ is its largest eigenvalue. Thus, the criterion appearing in Theorems 1.1 and 1.2 is algebraic like the problem itself. Despite that, the proofs of these theorems rely on analytic tools: introduction of the Gaussian measure, the Fourier transform asymptotic, and the measure concentration. We discuss this in more detail in Section 2.

1.4. Discussion

1.4.1. Computational complexity

Given matrices $Q_1, \ldots, Q_m$, one can compute an orthonormal basis $A_1, \ldots, A_m$ of $\operatorname{span}(Q_1, \ldots, Q_m)$, using, for example, the Gram-Schmidt orthogonalization process. Then one can check the inequality for the operator norm of $A_1^2 + \ldots + A_m^2$. These are standard linear algebra problems that can be solved in polynomial time. However, we don't know how to find a solution x in polynomial time or whether a solution x with a *polynomial size description* even exists when the conditions of Theorems 1.1 and 1.2 are satisfied.

1.4.2. The case of random matrices

Let $Q_1, \ldots, Q_m$ be independent symmetric random matrices with entries above the diagonal being independent normal random variables of expectation 0 and variance 1 and the diagonal entries being normal of expectation 0 and variance 2. In other words, up to the scaling factor of $\sqrt{n}$, the matrices $Q_1, \ldots, Q_m$ are sampled independently from the Gaussian Orthogonal Ensemble (GOE).

We assume that $m \leq n$. As n grows, with high probability we have (we ignore low-order terms)

$$\|Q_i\|_{\mathrm{op}} \approx 2\sqrt{n} \quad \text{and} \quad \langle Q_i, Q_i \rangle \approx n^2 \quad \text{for} \quad i = 1, \ldots, m,$$

see, for example, Section 2.3 of [14].

Let $A_1, \ldots, A_m$ be the orthonormal basis of $\operatorname{span}(Q_1, \ldots, Q_m)$ obtained by the Gram - Schmidt orthogonalization from $Q_1, \ldots, Q_m$. Then, up to a normalizing factor, each A_i is also sampled from GOE, so we have

$$\|A_i\|_{\mathrm{op}} \approx \frac{2}{\sqrt{n}} \quad \text{for} \quad i = 1, \ldots, m,$$

with high probability. Hence

$$\left\| \sum_{i=1}^m A_i^2 \right\|_{\mathrm{op}} \leq \sum_{i=1}^m \|A_i\|_{\mathrm{op}}^2 \approx \frac{4m}{n}.$$

Hence if $m \leq \sqrt{\eta n}/2$, with high probability the conditions of Theorems 1.1 and 1.2 are satisfied. Similar behavior can be observed for other models of random symmetric matrices with independent entries sampled from a distribution with expectation 0, variance

1 and sub-Gaussian tail. Informally, for the conditions of Theorems 1.1 and 1.2 to hold, we want n to be substantially larger than m and the subspace span $(Q_1, \ldots, Q_m)$ to be sufficiently generic.

1.4.3. The metric geometry of the cone of positive semidefinite matrices

As before, we consider the space Sym_n of $n \times n$ symmetric matrices as a Euclidean space. Let $\mathcal{S}_+ \subset \mathrm{Sym}_n$ be the convex cone of positive semidefinite matrices. From Section 1.4.2, we deduce the following metric property of $\mathcal{S}_+$: There is an absolute constant $\gamma > 0$ such that if $\mathcal{A} \subset \mathrm{Sym}_n$ is an affine subspace of codimension $m \leq \gamma\sqrt{n}$ sampled at random from the uniform (Haar) probability measure on the Grassmannian of codimension m subspaces containing the identity matrix I_n, then $\mathcal{A}$ contains a positive semidefinite matrix of rank 1 with probability approaching 1 as n grows.

We don't know if the estimates of Theorem 1.1 and Sections 1.4.2 and 1.4.3 are optimal, or, for example, whether we can make m in Section 1.4.2 and $\operatorname{codim} \mathcal{A}$ in Section 1.4.3 proportional to n instead of $\sqrt{n}$. There is a vast literature on the *average* characteristics of the set of solutions for systems of real polynomial equations, see, for example, [7], [11] and reference therein, but much less appears to be known regarding solvability of such systems with high probability.

1.4.4. Solving positive semidefinite relaxation

Suppose we want to apply Theorem 1.1 to test the solvability of the original system (1.1), where we do not necessarily have $\alpha_i = \operatorname{trace} q_i$. We begin by looking for a solution X to the positive semidefinite program (1.3). If there is no solution X, we conclude that the system (1.1) has no solutions. If there is a solution $X \succeq 0$ with $\operatorname{rank} X \leq 1$, we conclude that the system (1.1) has a solution. The difficulty arises when we find a solution $X \succeq 0$ but with $\operatorname{rank} X > 1$. It is known that if there is a solution $X \succeq 0$, then there is a solution $X \succeq 0$ with an additional constraint

$$\operatorname{rank} X \ \leq \ \left\lfloor \frac{\sqrt{8m+1}-1}{2} \right\rfloor.$$

Any extreme point of the set of solutions to (1.3) satisfies this condition, see, for example, Section II.13 of [2]. Curiously, if we are to use Theorem 1.1 to ascertain the existence of a solution, it makes sense to try to find an $X \succeq 0$ not on the boundary, but as close as possible to the "middle" of the set of solutions of (1.3) because we want the transformed matrices $\widehat{Q}_i$ given by (1.4) to be as generic as possible. For example, one can look for X with the maximum von Neumann entropy

$$\sum_{j=1}^{n} \lambda_j \ln \frac{1}{\lambda_j},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of X, see, for example, [15]. Finding such an X is a convex optimization problem and hence can be solved efficiently. Informally, if the

number m of equations is rather small compared to the number n of variables, if the matrices $Q_1, \ldots, Q_m$ of equations in (1.1) are sufficiently generic, and if the solutions X to the positive semidefinite relaxation (1.3) can be found deep enough the cone of $\mathcal{S}_+$ positive semidefinite matrices, then the system (1.1) will have a solution.

We note that in the homogeneous case one should also be careful about working with the positive semidefinite relaxation. Namely, if $X \succeq 0$ is a solution to the system of equations

$$\operatorname{trace}(Q_i X) = 0 \quad \text{for} \quad i = 1, \ldots, m, \tag{1.8}$$

we factor $X = TT^*$, define $\widehat{Q_i}$ by (1.4) and define $\widehat{q}_i$ by (1.5), then to deduce the existence of a non-trivial solution to the system (1.2) from the existence of a non-trivial solution to the system

$$\widehat{q}_i(x) = 0 \quad \text{for} \quad i = 1, \ldots, m,$$

we must require T and hence X to be invertible. If there are no invertible $X \succeq 0$ satisfying (1.8), we reduce (1.2) to a system of homogeneous quadratic equations in fewer variables, see Section 1.2.

In the rest of the paper, we prove Theorems 1.1 and 1.2. Although the statements are real algebraic, our proofs use analytic methods, in particular, the Fourier transform.

2. Outline of the proof

In what follows, we denote the imaginary unit by $\sqrt{-1}$, so as to use i for indices.

Let $Q_1, \ldots, Q_m$ be $n \times n$ real symmetric matrices and let I be the $n \times n$ identity matrix. For real $\tau_1, \ldots, \tau_m$, we consider the matrix

$$Q(t) = I - \sqrt{-1} \sum_{i=1}^{m} \tau_i Q_i \quad \text{for} \quad t = (\tau_1, \ldots, \tau_m).$$

Since the eigenvalues $\lambda_1(t), \ldots, \lambda_n(t)$ of the linear combination $\sum_{i=1}^{m} \tau_i Q_i$ are real, we have

$$\det Q(t) = \prod_{i=1}^{n} \bigl(1 - \sqrt{-1}\lambda_i(t)\bigr) \neq 0 \quad \text{for all} \quad t \in \mathbb{R}^m.$$

Therefore, we can pick a branch of

$$\det{}^{-\frac{1}{2}} Q(t),$$

which we select so that at $t = 0$ we get 1.

It is also more convenient to rescale and define quadratic forms by

$$q(x) = \frac{1}{2}\langle Qx, x\rangle.$$

Our proof of Theorem 1.1 hinges on the analysis of the Fourier transform of the function $F(t) := \det{}^{-\frac{1}{2}} Q(t),\ t \in \mathbb{R}^m$. Namely, we prove the following result.

Theorem 2.1. *Let $Q_1, \ldots, Q_m$ be $n \times n$ real symmetric matrices, let*

$$q_i(x) = \frac{1}{2}\langle Q_i x, x\rangle \quad \text{for} \quad i = 1, \ldots, m,$$

be the corresponding quadratic forms and let $\alpha_1, \ldots, \alpha_m$ be real numbers. Suppose that

$$\int\limits_{\mathbb{R}^m} \left| \det{}^{-\frac{1}{2}} \left(I - \sqrt{-1} \sum_{i=1}^m \tau_i Q_i \right) \right| \, dt \ < \ +\infty \tag{2.1}$$

and that

$$\int\limits_{\mathbb{R}^m} \det{}^{-\frac{1}{2}} \left(I - \sqrt{-1} \sum_{i=1}^m \tau_i Q_i \right) \exp\left\{ -\sqrt{-1} \sum_{i=1}^m \alpha_i \tau_i \right\} \, dt \ \neq 0. \tag{2.2}$$

Then the system (1.1) of equations has a solution $x \in \mathbb{R}^n$.

We prove a similar result for homogeneous systems.

Theorem 2.2. *Let $Q_1, \ldots, Q_m$ and $q_1, \ldots, q_m$ be as in Theorem 2.1 and assume, additionally, that $m < n$. Suppose that*

$$\int\limits_{\mathbb{R}^m} \det{}^{-\frac{1}{2}} \left(I - \sqrt{-1} \sum_{i=1}^m \tau_i Q_i \right) \, dt \ \neq 0,$$

where the integral converges absolutely. Then the system (1.2) of equations has a solution $x \neq 0$.

We prove Theorems 2.1 and 2.2 in Section 3. Theorems 1.1 and 1.2 are deduced from Theorems 2.1 and 2.2 respectively. Since the proofs are very similar, below we discuss the plan of the proof of Theorem 1.1 only.

First, we note that we can replace matrices $Q_1, \ldots, Q_m$ by an orthonormal set of matrices $A_1, \ldots, A_m$ and quadratic forms q_i by quadratic forms

$$a_i(x) = \frac{1}{2}\langle A_i x, x\rangle \quad \text{for} \quad i = 1, \ldots, m.$$

We let

$$\alpha_i = \frac{1}{2} \operatorname{trace} A_i$$

and consider an equivalent system

$$a_i(x) = \alpha_i \quad \text{for} \quad i = 1, \ldots, m$$

of quadratic equations, see Section 1.3.

Using Theorem 2.1, we conclude that it suffices to prove that

$$\int\limits_{\mathbb{R}^m} \det{}^{-\frac{1}{2}} \left(I - \sqrt{-1} \sum_{i=1}^m \tau_i A_i \right) \exp\left\{ -\sqrt{-1} \sum_{i=1}^m \alpha_i \tau_i \right\} \, dt \neq 0, \tag{2.3}$$

where the integral converges absolutely. Up to a scaling normalization factor, we rewrite the integral in polar coordinates as follows.

Let $\mathbb{S}^{m-1} \subset \mathbb{R}^m$ be the unit sphere endowed with the Haar probability measure. For $w \in \mathbb{S}^{m-1}$, $w = (\omega_1, \ldots, \omega_m)$, we define the matrix

$$A(w) = \sum_{i=1}^m \omega_i A_i.$$

Up to a non-zero scaling factor, in polar coordinates the integral (2.3) can be written as

$$\int\limits_{\mathbb{S}^{m-1}} \left(\int\limits_0^{+\infty} \tau^{m-1} \det{}^{-\frac{1}{2}} \left(I - \sqrt{-1} \tau A(w) \right) \exp\left\{ -\frac{\sqrt{-1}\tau}{2} \operatorname{trace} A(w) \right\} \, d\tau \right) dw. \tag{2.4}$$

The rest of the proof relies on an analysis of this integral. As a first step, we show that the contribution of the tail of the inside integral in (2.4) is negligible. Namely, we prove in Lemma 5.1 that for any $w \in \mathbb{S}^{m-1}$, we have

$$\int\limits_{5\sqrt{m}}^{+\infty} \tau^{m-1} \left| \det{}^{-\frac{1}{2}} \left(I - \sqrt{-1} \tau A(w) \right) \right| \, d\tau \ \leq \ \frac{1}{20m} m^{m/2} e^{-3m}. \tag{2.5}$$

In particular, this proves that the integral (2.4) converges absolutely and that the integrals (2.4) and (2.3) are equal, up to a scaling factor that is the surface area of the unit sphere $\mathbb{S}^{m-1} \subset \mathbb{R}^m$.

This allows us to consider the integration over the interval $[0, 5\sqrt{m}]$ in the inner integral in (2.4). To analyze this integral, denote by $\lambda_1(w), \ldots, \lambda_n(w)$ the eigenvalues of $A(w)$. A simple calculation yields

$$\det{}^{-\frac{1}{2}}\left(I-\sqrt{-1}\tau A(w)\right)\exp\left\{-\frac{\sqrt{-1}\tau}{2}\operatorname{trace}A(w)\right\}$$
$$=\exp\left\{\frac{1}{2}\sum_{k=2}^{\infty}\frac{(\tau\sqrt{-1})^k}{k}\sum_{j=1}^{n}\lambda_j^k(w)\right\},$$

see the derivation in (5.1). Note that the summation starts from $k=2$. This is achieved due to the first step in the argument allowing us to set $\alpha_i=\frac{1}{2}\operatorname{trace}A_i$. Moreover, $\sum_{j=1}^n\lambda_j^2(w)=1$ for all $w\in\mathbb{S}^{m-1}$ due to orthonormality of the matrices $A_1,\ldots,A_m$.

Next, we divide the points $w\in\mathbb{S}^{m-1}$ into tame and wild. For a tame point, we show that the term corresponding to $k=2$ in the expression above is dominating which would mean that the expression above is close to $\exp\left\{-\frac{\tau^2}{4}\right\}$. To prove it, we need to control $\sum_{j=1}^n\lambda_j^k(w)$ for all $k\geq 3$. However, as we show below, a control for $k=3,4$ turns out to be sufficient. More precisely, we classify a point $w\in\mathbb{S}^{m-1}$ as tame if

$$\left|\sum_{j=1}^{n}\lambda_j^3(w)\right|\ \leq\ \frac{1}{25m^{3/2}}\quad\text{and}\quad\sum_{j=1}^{n}\lambda_j^4(w)\ \leq\ \frac{1}{625m^2}.$$

The second inequality here is a bound on the 4-Schatten norm of $A(w)$: $\|A(w)\|_{\mathrm{S}_4}\leq 1/(625m^2)$. In contrast to it, the first inequality bounds the third moment of the eigenvalues, and not the 3-Schatten norm, as we have to exploit the cancellation of positive and negative eigenvalues.

In Lemma 5.2, we prove that if $w\in\mathbb{S}^{m-1}$ is tame, then

$$\begin{aligned}&\Re\int_0^{5\sqrt{m}}\tau^{m-1}\ \det{}^{-\frac{1}{2}}\left(I-\sqrt{-1}\tau A(w)\right)\exp\left\{-\frac{\sqrt{-1}\tau}{2}\operatorname{trace}A(w)\right\}\,d\tau\\ &\geq\frac{1}{2}\int_0^{5\sqrt{m}}\tau^{m-1}\exp\left\{-\frac{\tau^2}{4}\right\}\,d\tau\approx 2^{m-2}\Gamma\left(\frac{m}{2}\right).\end{aligned}\tag{2.6}$$

We note that the value of (2.6) is much larger than the tail estimate (2.5). Moreover, in Lemmas 4.1 and 4.2 we bound the expectations

$$\mathbb{E}\left(\sum_{j=1}^{n}\lambda_j^3(w)\right)^2\ \leq\ \frac{120\eta}{m(m+2)(m+4)}\quad\text{and}\quad\mathbb{E}\sum_{j=1}^{n}\lambda_j^4(w)\ \leq\ \frac{3\eta}{(m+2)m}.\tag{2.7}$$

This is the point where the quantity $\left\|\sum_{i=1}^m A_i^2\right\|_{\mathrm{op}}$ reveals itself. It turns out that both expectations above can be controlled in terms of this operator norm alone.

It follows then by the Markov inequality that a random $w\in\mathbb{S}^{m-1}$ is tame with probability at least $7/8$, and hence tame points $w\in\mathbb{S}^{m-1}$ contribute significantly to the integral (2.4).

It remains to show that the contribution of wild points $w \in \mathbb{S}^{m-1}$ cannot offset the contribution of tame points.

This relies on a concentration inequality for the 4-Schatten norm of matrices $A(w)$ on the unit sphere $\mathbb{S}^{m-1}$, which we derive in Lemma 4.4. This inequality is leveraged against the deterioration of the bounds on the eigenvalues of $A(w)$ which occurs for the wild points. To this end, we partition the set of wild points into a number of subsets according to the size of $\|A(w)\|_{\mathrm{S}_4}$, and apply the concentration inequality to prove that the contribution of the points in each layer to the integral (2.4) is negligible.

This argument is carried out in Section 6.

In Sections 4 and 5, we do some preliminary work: we prove bounds (2.7) as well as some other useful bounds on the eigenvalues of $A(w)$ in Section 4. In Section 5 we derive (2.5) and show that a similar integral over the interval $[0, 5\sqrt{m}]$ can be controlled by $\|A(w)\|_{\mathrm{op}}$, which is in turn bounded in terms of $\|A(w)\|_{\mathrm{S}_4}$.

3. Proofs of Theorems 2.1 and 2.2

3.1. Enter Gaussian measure

We consider the standard Gaussian measure in $\mathbb{R}^n$ with density

$$\frac{1}{(2\pi)^{n/2}} e^{-\|x\|^2/2} \quad \text{where} \quad \|x\| = \sqrt{\xi_1^2 + \ldots + \xi_n^2} \quad \text{for} \quad x = (\xi_1, \ldots, \xi_n).$$

Considering a quadratic form $q(x) = \langle Qx, x \rangle$ as a random variable, we observe that

$$\mathbb{E}\, q = \operatorname{trace} Q,$$

so that the equation $q(x) = \operatorname{trace} Q$ "holds on average".

The proof of Theorems 2.1 and 2.2 is based on a Fourier transform formula.

Lemma 3.1. *Let $Q_1, \ldots, Q_m$ be $n \times n$ real symmetric matrices and let*

$$q_i(x) = \frac{1}{2}\langle Q_i x, x \rangle \quad \textit{for} \quad i = 1, \ldots, m,$$

be the corresponding quadratic forms. Then for any real $\alpha_1, \ldots, \alpha_m$ and any real $\sigma > 0$, we have

$$\begin{aligned} &\frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} \exp\left\{ -\frac{\sigma^2}{2} \sum_{i=1}^m (q_i(x) - \alpha_i)^2 \right\} e^{-\|x\|^2/2}\, dx \\ =&\frac{1}{\sigma^m (2\pi)^{m/2}} \int\limits_{\mathbb{R}^m} \det{}^{-\frac{1}{2}} \left(I - \sqrt{-1} \sum_{i=1}^m \tau_i Q_i \right) \exp\left\{ -\sqrt{-1} \sum_{i=1}^m \alpha_i \tau_i \right\} e^{-\|t\|^2/2\sigma^2}\, dt. \end{aligned}$$

Proof. As is well-known, for a positive definite matrix Q and the corresponding form

$$q(x) = \frac{1}{2}\langle Qx, x\rangle$$

we have

$$\frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} e^{-q(x)}\, dx = \det{}^{-\frac{1}{2}} Q.$$

Consequently, for $t \in \mathbb{R}^m$, $t = (\tau_1, \ldots, \tau_m)$, in a sufficiently small neighborhood of 0, we have

$$\frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} \exp\left\{\sum_{i=1}^m \tau_i q_i(x)\right\} e^{-\|x\|^2/2}\, dx = \det{}^{-\frac{1}{2}} \left(I - \sum_{i=1}^m \tau_i Q_i\right).$$

Since both sides of the formula are analytic in $\tau_1, \ldots, \tau_m \in \mathbb{C}$ for $\Re\tau_1, \ldots, \Re\tau_m$ in a small neighborhood of 0, we conclude that the above formula holds for all such $\tau_1, \ldots, \tau_m$ and that, in particular,

$$\frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} \exp\left\{\sqrt{-1}\sum_{i=1}^m \tau_i q_i(x)\right\} e^{-\|x\|^2/2}\, dx = \det{}^{-\frac{1}{2}} \left(I - \sqrt{-1}\sum_{i=1}^m \tau_i Q_i\right)$$

for all real $\tau_1, \ldots, \tau_m$.

Therefore,

$$\begin{aligned} &\frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} \exp\left\{\sqrt{-1}\sum_{i=1}^m \tau_i \left(q_i(x) - \alpha_i\right)\right\} e^{-\|x\|^2/2}\, dx \\ &= \det{}^{-\frac{1}{2}} \left(I - \sqrt{-1}\sum_{i=1}^m \tau_i Q_i\right) \exp\left\{-\sqrt{-1}\sum_{i=1}^m \alpha_i \tau_i\right\} \end{aligned} \tag{3.1}$$

for all real $\tau_1, \ldots, \tau_m$.

Next, we use a well-known formula: for $\sigma > 0$ and any real (or complex) α, we have

$$\frac{1}{\sigma\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \exp\left\{\sqrt{-1}\alpha\tau\right\} \exp\left\{-\frac{\tau^2}{2\sigma^2}\right\}\, d\tau = \exp\left\{-\frac{\alpha^2\sigma^2}{2}\right\}.$$

Integrating both sides of (3.1) for $i = 1, \ldots, m$ over $\tau_i \in \mathbb{R}$ with density

$$\frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{\tau_i^2}{2\sigma^2}\right\},$$

we get the desired formula. □

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. By Lemma 3.1, for all $\sigma > 0$, we have

$$\begin{aligned}\sigma^m \int\limits_{\mathbb{R}^n} \exp\left\{-\frac{\sigma^2}{2}\sum_{i=1}^m (q_i(x)-\alpha_i)^2\right\} e^{-\|x\|^2/2}\,dx = \\ (2\pi)^{\frac{n-m}{2}} \int\limits_{\mathbb{R}^m} \det{}^{-\frac{1}{2}}\left(I - \sqrt{-1}\sum_{i=1}^m \tau_i Q_i\right) \exp\left\{-\sqrt{-1}\sum_{i=1}^m \alpha_i\tau_i\right\} e^{-\frac{\|t\|^2}{2\sigma^2}}\,dt.\end{aligned} \tag{3.2}$$

As $\sigma \longrightarrow +\infty$, the right hand side of (3.2) converges to

$$(2\pi)^{\frac{n-m}{2}} \int\limits_{\mathbb{R}^m} \det{}^{-\frac{1}{2}}\left(I - \sqrt{-1}\sum_{i=1}^m \tau_i Q_i\right) \exp\left\{-\sqrt{-1}\sum_{i=1}^m \alpha_i\tau_i\right\} \neq 0.$$

Suppose that the system (1.1) has no solutions $x \in \mathbb{R}^n$. We intend to obtain a contradiction by showing that the left hand side of (3.2) converges to 0 as $\sigma \longrightarrow +\infty$.

Let

$$\gamma = (2\pi)^{\frac{n-m}{2}} \int\limits_{\mathbb{R}^m} \left| \det{}^{-\frac{1}{2}}\left(I - \sqrt{-1}\sum_{i=1}^m \tau_i Q_i\right)\right| dt \ < \ +\infty.$$

Let us choose a $\rho > 0$, to be adjusted later. Then

$$\begin{aligned}&\sigma^m \int\limits_{x\in\mathbb{R}^n:\ \|x\|>\rho} \exp\left\{-\frac{\sigma^2}{2}\sum_{i=1}^m (q_i(x)-\alpha_i)^2\right\} e^{-\|x\|^2/2}\,dx \\ &\leq e^{-\rho^2/4}\sigma^m \int\limits_{x\in\mathbb{R}^n:\ \|x\|>\rho} \exp\left\{-\frac{\sigma^2}{2}\sum_{i=1}^m (q_i(x)-\alpha_i)^2\right\} e^{-\|x\|^2/4}\,dx \\ &\leq e^{-\rho^2/4}\sigma^m \int\limits_{\mathbb{R}^n} \exp\left\{-\frac{\sigma^2}{2}\sum_{i=1}^m (q_i(x)-\alpha_i)^2\right\} e^{-\|x\|^2/4}\,dx \\ &= e^{-\rho^2/4}2^{n/2}\sigma^m \int\limits_{\mathbb{R}^n} \exp\left\{-\frac{\sigma^2}{2}\sum_{i=1}^m (2q_i(x)-\alpha_i)^2\right\} e^{-\|x\|^2/2}\,dx \\ &= e^{-\rho^2/4}2^{n/2}\sigma^m \int\limits_{\mathbb{R}^n} \exp\left\{-\frac{(2\sigma)^2}{2}\sum_{i=1}^m \left(q_i(x)-\frac{\alpha_i}{2}\right)^2\right\} e^{-\|x\|^2/2}\,dx \\ &= e^{-\rho^2/4}2^{n/2}2^{-m}(2\sigma)^m \int\limits_{\mathbb{R}^n} \exp\left\{-\frac{(2\sigma)^2}{2}\sum_{i=1}^m \left(q_i(x)-\frac{\alpha_i}{2}\right)^2\right\} e^{-\|x\|^2/2}\,dx.\end{aligned}$$

From Lemma 3.1,

$$(2\sigma)^m \int\limits_{\mathbb{R}^n} \exp\left\{-\frac{(2\sigma)^2}{2}\sum_{i=1}^m \left(q_i(x)-\frac{\alpha_i}{2}\right)^2\right\} e^{-\|x\|^2/2}\,dx$$
$$\leq (2\pi)^{\frac{n-m}{2}} \int\limits_{\mathbb{R}^m} \left|\det{}^{-\frac{1}{2}}\left(I-\sqrt{-1}\sum_{i=1}^m \tau_i Q_i\right)\right|\,dt = \gamma.$$

Summarizing,

$$\sigma^m \int\limits_{x\in\mathbb{R}^n:\ \|x\|>\rho} \exp\left\{-\frac{\sigma^2}{2}\sum_{i=1}^m (q_i(x)-\alpha_i)^2\right\} e^{-\|x\|^2/2}\,dx$$
$$\leq e^{-\rho^2/4} 2^{n/2} 2^{-m}\gamma.$$

Given $\epsilon > 0$, we choose $\rho(\epsilon) > 0$ such that

$$e^{-\rho^2(\epsilon)/4} 2^{n/2} 2^{-m}\gamma \ \leq\ \frac{\epsilon}{2},$$

so that for all $\sigma > 0$ we have

$$\sigma^m \int\limits_{x\in\mathbb{R}^n:\ \|x\|>\rho(\epsilon)} \exp\left\{-\frac{\sigma^2}{2}\sum_{i=1}^m (q_i(x)-\alpha_i)^2\right\} e^{-\|x\|^2/2}\,dx \ \leq\ \frac{\epsilon}{2}. \tag{3.3}$$

If the system (1.1) has no solution then for some $\delta(\epsilon) > 0$, we have

$$\sum_{i=1}^m (q_i(x)-\alpha_i)^2 \ \geq\ \delta(\epsilon) \quad \text{provided} \quad \|x\| \leq \rho(\epsilon)$$

and hence

$$\sigma^m \int\limits_{x\in\mathbb{R}^n:\ \|x\|\leq\rho(\epsilon)} \exp\left\{-\frac{\sigma^2}{2}\sum_{i=1}^m (q_i(x)-\alpha_i)^2\right\}\,dx$$
$$\leq \sigma^m \rho^n(\epsilon)\nu_n \exp\left\{-\frac{\sigma^2\delta(\epsilon)}{2}\right\},$$

where ν_n is the volume of the unit ball in $\mathbb{R}^n$. Therefore, there is $\sigma_0(\epsilon) > 0$ such that for all $\sigma > \sigma_0(\epsilon)$, we have

$$\sigma^m \int\limits_{x\in\mathbb{R}^n:\ \|x\|\leq\rho(\epsilon)} \exp\left\{-\frac{\sigma^2}{2}\sum_{i=1}^m (q_i(x)-\alpha_i)^2\right\}\,dx \ \leq\ \frac{\epsilon}{2}. \tag{3.4}$$

Combining (3.3) and (3.4), we conclude that the limit of the left hand side of (3.2) is 0 as $\sigma \longrightarrow +\infty$, which is the desired contradiction. □

The proof of Theorem 2.2 is similar.

Proof of Theorem 2.2. Seeking a contradiction, suppose that the only solution to the system is $x = 0$. Then for some $\delta > 0$ we have

$$\sum_{i=1}^{m} q_i^2(x) \ \geq \ \delta \quad \text{for all} \quad x \in \mathbb{R}^n \quad \text{such that} \quad \|x\| = 1. \tag{3.5}$$

From Lemma 3.1, for any $\sigma > 0$, we have

$$\begin{aligned} &\sigma^m \int\limits_{\mathbb{R}^n} \exp\left\{-\frac{\sigma^2}{2}\sum_{i=1}^{m} q_i^2(x)\right\} e^{-\|x\|^2/2}\, dx \\ =&(2\pi)^{\frac{n-m}{2}} \int\limits_{\mathbb{R}^m} \det{}^{-\frac{1}{2}}\left(I - \sqrt{-1}\sum_{i=1}^{m}\tau_i Q_i\right) e^{-\frac{\|t\|^2}{2\sigma^2}}\, dt. \end{aligned} \tag{3.6}$$

From (3.5), the left hand side of (3.6) is bounded above (we use polar coordinates) by

$$\omega_n \sigma^m \int\limits_0^{+\infty} \exp\left\{-\frac{\delta\sigma^2\tau^2}{2}\right\} \tau^{n-1} e^{-\tau^2/2}\, d\tau,$$

where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$. Using the substitution $\xi = \sigma\tau$, we rewrite the integral as

$$\omega_n \sigma^{m-n} \int\limits_0^{+\infty} \exp\left\{-\frac{\delta\xi^2}{2}\right\} \xi^{n-1} e^{-\xi^2/2\sigma^2}\, d\xi$$

and observe that it converges to 0 as $\sigma \longrightarrow +\infty$ (recall that $m < n$). On the other hand, the right hand side of (3.6) converges to

$$(2\pi)^{\frac{n-m}{2}} \int\limits_{\mathbb{R}^m} \det{}^{-\frac{1}{2}}\left(I - \sqrt{-1}\sum_{i=1}^{m}\tau_i Q_i\right) \neq 0,$$

which is the desired contradiction. □

In the rest of the paper, we deduce Theorem 1.1 from Theorem 2.1 and Theorem 1.2 from Theorem 2.2.

4. Controlling eigenvalues

4.1. Preliminaries

In the space of $n \times n$ real matrices we consider the standard inner product, see Section 1.3. The corresponding Euclidean norm is called the *Hilbert-Schmidt* or *Frobenius* norm:

$$\|A\|_{\mathrm{HS}} = \sqrt{\langle A, A\rangle} = \sqrt{\mathrm{trace}(A^*A)}.$$

If, in addition, A is symmetric with eigenvalues $\lambda_1, \ldots, \lambda_n$, we have

$$\|A\|_{\mathrm{HS}} = \sqrt{\sum_{j=1}^n \lambda_j^2}$$

while for the operator norm we have

$$\|A\|_{\mathrm{op}} = \max_{j=1,\ldots,n} |\lambda_j|.$$

We will also consider the 4-Schatten norm defined by

$$\|A\|_{\mathrm{S}_4} = \left(\sum_{j=1}^n \lambda_j^4\right)^{1/4}.$$

This is indeed a norm in the space of $n \times n$ symmetric matrices, see, for example, Chapter 1 of [14]. In particular, we will use that

$$\left| \, \|A\|_{\mathrm{S}_4} - \|B\|_{\mathrm{S}_4} \, \right| \ \leq \ \|A - B\|_{\mathrm{S}_4}. \tag{4.1}$$

Also, we observe that for a symmetric matrix A with eigenvalues $\lambda_1, \ldots, \lambda_n$, we have

$$\sum_{j=1}^n \lambda_j^4 \ \leq \ \left(\max_{j=1,\ldots,n} \lambda_j^2\right) \sum_{j=1}^n \lambda_j^2,$$

from which it follows that

$$\|A\|_{\mathrm{S}_4} \ \leq \ \|A\|_{\mathrm{op}}^{1/2} \|A\|_{\mathrm{HS}}^{1/2}. \tag{4.2}$$

Suppose that B is a positive semidefinite symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then

$$\|B\|_{\mathrm{HS}}^2 = \sum_{j=1}^n \lambda_j^2 \ \leq \ \left(\max_{j=1,\ldots,n} \lambda_j\right) \sum_{j=1}^n \lambda_j = \|B\|_{\mathrm{op}} \left(\operatorname{trace} B\right).$$

We will apply the inequality in the following situation: Let $A_1, \ldots, A_m$ be an orthonormal set of symmetric matrices, so that

$$\langle A_i, A_j \rangle = \operatorname{trace}(A_i A_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Then the matrix

$$B = \sum_{i=1}^m A_i^2$$

is symmetric positive semidefinite and hence we have

$$\left\| \sum_{i=1}^m A_i^2 \right\|_{\mathrm{HS}}^2 \ \leq \ m \left\| \sum_{i=1}^m A_i^2 \right\|_{\mathrm{op}}. \tag{4.3}$$

We also remark that $\langle A, B \rangle \geq 0$ for any two $n \times n$ symmetric positive semidefinite matrices.

We will use the following inequality. Let $A_1, \ldots, A_m$ be an orthonormal set of $n \times n$ symmetric matrices and let B be another $n \times n$, not necessarily symmetric, real matrix. Then

$$\langle A_i, B \rangle = \operatorname{trace}(A_i B) \quad \text{for} \quad i = 1, \ldots, m$$

are the coordinates of the orthogonal projection of B onto $\operatorname{span}(A_1, \ldots, A_m)$ and hence

$$\sum_{i=1}^m \operatorname{trace}^2(A_i B) \ \leq \ \|B\|_{\mathrm{HS}}^2. \tag{4.4}$$

Finally, we will need moments of a random vector $w \in \mathbb{S}^{m-1}$, $w = (\omega_1, \ldots, \omega_m)$. Namely, for integer $\alpha_1, \ldots, \alpha_m \geq 0$, we have

$$\mathbb{E}\, \omega_1^{\alpha_1} \cdots \omega_m^{\alpha_m} = 0 \quad \text{provided at least one } \alpha_i \text{ is odd} \tag{4.5}$$

and

$$\mathbb{E}\, \omega_1^{\alpha_1} \cdots \omega_m^{\alpha_m} = \frac{\Gamma\left(\frac{m}{2}\right) \prod_{i=1}^m \Gamma\left(\beta_i + \frac{1}{2}\right)}{\Gamma^m\left(\frac{1}{2}\right) \Gamma\left(\beta_1 + \ldots + \beta_m + \frac{m}{2}\right)} \quad \text{provided} \quad \alpha_i = 2\beta_i \quad \text{are even},$$

see, for example, [3]. In particular, we will use the following values:

$$\begin{aligned}
\mathbb{E}\,\omega_i^2\omega_j^2 &= \frac{1}{m(m+2)} \quad \text{for} \quad 1 \le i \ne j \le m,\\
\mathbb{E}\,\omega_i^4 &= \frac{3}{m(m+2)} \quad \text{for} \quad i = 1,\ldots,m,\\
\mathbb{E}\,\omega_i^2\omega_j^2\omega_k^2 &= \frac{1}{m(m+2)(m+4)} \quad \text{for distinct} \quad 1 \le i,j,k \le m,\\
\mathbb{E}\,\omega_i^2\omega_j^4 &= \frac{3}{m(m+2)(m+4)} \quad \text{for} \quad 1 \le i \ne j \le m \quad \text{and}\\
\mathbb{E}\,\omega_i^6 &= \frac{15}{m(m+2)(m+4)} \quad \text{for} \quad i = 1,\ldots,m.
\end{aligned} \tag{4.6}$$

In what follows, we fix an orthonormal set $A_1,\ldots,A_m$ of $n\times n$ symmetric matrices. For a random $w \in \mathbb{S}^{m-1}$, $w = (\omega_1,\ldots,\omega_m)$, sampled from the Haar probability measure in $\mathbb{S}^{m-1}$, we define

$$A(w) = \sum_{i=1}^m \omega_i A_i$$

and let $\lambda_1(w),\ldots,\lambda_n(w)$ be the eigenvalues of $A(w)$. Here is our first estimate.

Lemma 4.1. *We have*

$$\mathbb{E}\left(\sum_{j=1}^n \lambda_j^3(w)\right)^2 \le \frac{120}{(m+2)(m+4)}\left\|\sum_{i=1}^m A_i^2\right\|_{\mathrm{op}}.$$

Proof. We have

$$\begin{aligned}
\sum_{j=1}^n \lambda_j^3(w) = \operatorname{trace}\left(\sum_{i=1}^m \omega_i A_i\right)^3 &= \sum_{(i,j,k)\ \text{distinct}} \omega_i\omega_j\omega_k \operatorname{trace}(A_iA_jA_k)\\
&+ \sum_{(i,j):\ i\ne j} \omega_i^2\omega_j \operatorname{trace}(A_i^2A_j) + \sum_{(i,j):\ i\ne j} \omega_i\omega_j^2 \operatorname{trace}(A_iA_j^2)\\
&+ \sum_{(i,j):\ i\ne j} \omega_i^2\omega_j \operatorname{trace}(A_iA_jA_i) + \sum_{i=1}^m \omega_i^3 \operatorname{trace} A_i^3\\
= \sum_{(i,j,k)\ \text{distinct}} \omega_i\omega_j\omega_k \operatorname{trace}(A_iA_jA_k) &+ 3\sum_{(i,j):\ i\ne j} \omega_i^2\omega_j \operatorname{trace}(A_i^2A_j) + \sum_{i=1}^m \omega_i^3 \operatorname{trace} A_i^3.
\end{aligned}$$

Using (4.5) and (4.6), we write

$$\mathbb{E}\left(\sum_{j=1}^n \lambda^3(w)\right)^2 = \frac{T_1 + 27T_2 + 15T_3 + 18T_4 + 9T_5}{m(m+2)(m+4)},$$

where

$$T_1 = \sum_{\substack{(i,j,k) \text{ distinct} \\ (i_1,j_1,k_1) \text{ is a permutation of } (i,j,k)}} \operatorname{trace}(A_i A_j A_k) \operatorname{trace}(A_{i_1} A_{j_1} A_{k_1})$$

$$T_2 = \sum_{(i,j):\ i \neq j} \operatorname{trace}^2(A_i^2 A_j)$$

$$T_3 = \sum_{i=1}^m \operatorname{trace}^2(A_i^3)$$

$$T_4 = \sum_{(i,j):\ i \neq j} \operatorname{trace}(A_i^2 A_j) \operatorname{trace}(A_j^3) \quad \text{and}$$

$$T_5 = \sum_{(i,j,k) \text{ distinct}} \operatorname{trace}(A_i^2 A_j) \operatorname{trace}(A_k^2 A_j).$$

Next, we bound T_1, T_2, T_3, T_4 and T_5.

Applying (4.4) with $B = A_j A_k$, we obtain

$$\sum_{i=1}^m \operatorname{trace}^2(A_i A_j A_k) \ \leq \ \|A_j A_k\|_{\mathrm{HS}}^2 = \operatorname{trace}(A_k A_j^2 A_k) = \operatorname{trace}(A_j^2 A_k^2)$$

and hence

$$\sum_{(i,j,k) \text{ distinct}} \operatorname{trace}^2(A_i A_j A_k) \ \leq \ \sum_{(j,k):\ j \neq k} \operatorname{trace}(A_j^2 A_k^2) \ \leq \ \left\| \sum_{i=1}^m A_i^2 \right\|_{\mathrm{HS}}^2.$$

By the Cauchy - Schwarz inequality, for every permutation σ of $\{1,2,3\}$, we obtain

$$\left| \sum_{(i_1,i_2,i_3) \text{ distinct}} \operatorname{trace}\left(A_{i_1} A_{i_2} A_{i_3}\right) \operatorname{trace}\left(A_{i_{\sigma(1)}} A_{i_{\sigma(2)}} A_{i_{\sigma(3)}}\right) \right|$$

$$\leq \sum_{(i,j,k) \text{ distinct}} \operatorname{trace}^2(A_i A_j A_k) \leq \left\| \sum_{i=1}^m A_i^2 \right\|_{\mathrm{HS}}^2$$

and hence

$$|T_1| \ \leq \ 6 \left\| \sum_{i=1}^m A_i^2 \right\|_{\mathrm{HS}}^2.$$

Applying (4.4) with $B = A_i^2$, we conclude that

$$\sum_{j=1}^m \operatorname{trace}^2(A_i^2 A_j) = \sum_{j=1}^m \operatorname{trace}^2(A_j A_i^2) \ \leq \ \|A_i^2\|_{\mathrm{HS}}^2$$

and hence

$$|T_2| \ \leq \ \sum_{i=1}^m \|A_i^2\|_{\mathrm{HS}}^2 \ \leq \ \left\| \sum_{i=1}^m A_i^2 \right\|_{\mathrm{HS}}^2,$$

where the last inequality follows since the matrices $A_1^2, \ldots, A_m^2$ are symmetric positive semidefinite and hence

$$\langle A_i^2, A_j^2 \rangle \geq 0 \quad \text{for all} \quad i, j.$$

Applying the Cauchy - Schwarz inequality, we obtain

$$|\operatorname{trace} A_i^3| = |\langle A_i, A_i^2 \rangle| \ \leq \ \|A_i\|_{\mathrm{HS}} \|A_i^2\|_{\mathrm{HS}} = \|A_i^2\|_{\mathrm{HS}} \tag{4.7}$$

and hence

$$|T_3| \ \leq \ \sum_{i=1}^m \|A_i\|_{\mathrm{HS}}^2 \ \leq \ \left\| \sum_{i=1}^m A_i^2 \right\|_{\mathrm{HS}}^2.$$

To bound T_4 and T_5 we combine some of the previously obtained estimates.

Applying the Cauchy - Schwarz inequality, (4.4) with $B = \sum_{i=1}^m A_i^2$ and (4.7), we obtain

$$\begin{aligned}
\left| \sum_{j=1}^m \sum_{i=1}^m \operatorname{trace}(A_i^2 A_j) \operatorname{trace}(A_j^3) \right| &= \left| \sum_{j=1}^m \operatorname{trace} \left(A_j \sum_{i=1}^m A_i^2 \right) \operatorname{trace}(A_j^3) \right| \\
&\leq \left| \sum_{j=1}^m \operatorname{trace}^2 \left(A_j \sum_{i=1}^m A_i^2 \right) \right|^{1/2} \left| \sum_{j=1}^m \operatorname{trace}^2 (A_j^3) \right|^{1/2} \\
&\leq \left\| \sum_{i=1}^m A_i^2 \right\|_{\mathrm{HS}} \left(\sum_{j=1}^m \|A_j^2\|_{\mathrm{HS}}^2 \right)^{1/2} \ \leq \ \left\| \sum_{i=1}^m A_i^2 \right\|_{\mathrm{HS}}^2.
\end{aligned}$$

Therefore, using (4.7), we get

$$|T_4| = \left| \sum_{j=1}^m \sum_{i=1}^m \operatorname{trace}(A_i^2 A_j) \operatorname{trace}(A_j^3) - \sum_{i=1}^m \operatorname{trace}^2 (A_i^3) \right| \ \leq \ 2 \left\| \sum_{i=1}^m A_i^2 \right\|_{\mathrm{HS}}^2.$$

It remains to bound T_5. We have

$$T_5 = \sum_{j=1}^{m} \sum_{\substack{(i,k):\\ i\neq j, k\neq j}} \operatorname{trace}(A_i^2 A_j)\operatorname{trace}(A_k^2 A_j) - \sum_{j=1}^{m} \sum_{i:\ i\neq j} \operatorname{trace}^2(A_i^2 A_j)$$
$$= \sum_{j=1}^{m} \left(\sum_{i:\ i\neq j} \operatorname{trace}(A_i^2 A_j) \right)^2 - T_2.$$

Since

$$0 \ \leq \ T_2 \ \leq \ \left\| \sum_{i=1}^{m} A_i^2 \right\|_{\mathrm{HS}}^2,$$

we have

$$|T_5| \ \leq \ \max \left\{ \sum_{j=1}^{m} \left(\sum_{i:i\neq j} \operatorname{trace}(A_i^2 A_j) \right)^2, \quad \left\| \sum_{i=1}^{m} A_i^2 \right\|_{\mathrm{HS}}^2 \right\}.$$

Now,

$$\left(\sum_{i:\ i\neq j} \operatorname{trace}(A_i^2 A_j) \right)^2 = \left(-\operatorname{trace}(A_j^3) + \sum_{i=1}^{m} \operatorname{trace}(A_i^2 A_j) \right)^2$$
$$= \operatorname{trace}^2(A_j^3) - 2\operatorname{trace}(A_j^3) \sum_{i=1}^{m} \operatorname{trace}(A_i^2 A_j) + \left(\sum_{i=1}^{m} \operatorname{trace}(A_i^2 A_j) \right)^2$$
$$= \operatorname{trace}^2(A_j^3) - 2\operatorname{trace}(A_j^3) \operatorname{trace}\left(A_j \sum_{i=1}^{m} A_i^2 \right) + \operatorname{trace}^2\left(A_j \sum_{i=1}^{m} A_i^2 \right)$$

and hence

$$\sum_{j=1}^{m} \left(\sum_{i:i\neq j} \operatorname{trace}(A_i^2 A_j) \right)^2$$
$$= \sum_{j=1}^{m} \operatorname{trace}^2(A_j^3) - 2\sum_{j=1}^{m} \operatorname{trace}(A_j^3) \operatorname{trace}\left(A_j \sum_{i=1}^{m} A_i^2 \right) + \sum_{j=1}^{m} \operatorname{trace}^2\left(A_j \sum_{i=1}^{m} A_i^2 \right)$$

By (4.7), we get

$$\sum_{j=1}^{m} \operatorname{trace}^2(A_j^3) \ \leq \ \sum_{j=1}^{m} \|A_j^2\|_{\mathrm{HS}}^2 \ \leq \ \left\| \sum_{i=1}^{m} A_i^2 \right\|_{\mathrm{HS}}^2.$$

Then, from the Cauchy - Schwarz inequality, (4.7) and (4.4) with $B = \sum_{i=1}^m A_i^2$, we get

$$\left|\sum_{j=1}^m \operatorname{trace}(A_j^3)\operatorname{trace}\left(A_j\sum_{i=1}^m A_i^2\right)\right|$$
$$\leq \left(\sum_{j=1}^m \operatorname{trace}^2(A_j^3)\right)^{1/2}\left(\sum_{j=1}^m \operatorname{trace}^2\left(A_j\sum_{i=1}^m A_i^2\right)\right)^{1/2} \leq \left\|\sum_{i=1}^m A_i^2\right\|_{\mathrm{HS}}^2$$

and from (4.4)

$$\sum_{j=1}^m \operatorname{trace}^2\left(A_j\sum_{i=1}^m A_i^2\right) \;\leq\; \left\|\sum_{i=1}^m A_i^2\right\|_{\mathrm{HS}}^2.$$

Thus

$$|T_5| \;\leq\; 4\left\|\sum_{i=1}^m A_i^2\right\|_{\mathrm{HS}}^2.$$

Summarizing,

$$\mathbb{E}\left(\sum_{j=1}^n \lambda_j^3(w)\right)^2 \;\leq\; \frac{120}{m(m+2)(m+4)}\left\|\sum_{i=1}^m A_i^2\right\|_{\mathrm{HS}}^2 \;\leq\; \frac{120}{(m+2)(m+4)}\left\|\sum_{i=1}^m A_i^2\right\|_{\mathrm{op}},$$

where the last inequality follows by (4.3). □

Next, we bound the 4th moment of the eigenvalues.

Lemma 4.2. *We have*

$$\mathbb{E}\left(\sum_{j=1}^n \lambda_j^4(w)\right) \;\leq\; \frac{3}{m+2}\left\|\sum_{i=1}^m A_i^2\right\|_{\mathrm{op}}.$$

Proof. Using (4.5) and (4.6), we write

$$\mathbb{E}\left(\sum_{j=1}^n \lambda_j^4(w)\right) = \mathbb{E}\operatorname{trace}\left(\sum_{i=1}^m \omega_i A_i\right)^4$$
$$= \mathbb{E}\left(\sum_{(i,j):\ i\neq j} \omega_i^2\omega_j^2\operatorname{trace}(A_iA_jA_iA_j)\right) + \mathbb{E}\left(\sum_{(i,j):\ i\neq j} \omega_i^2\omega_j^2\operatorname{trace}(A_i^2A_j^2)\right)$$

$$+\mathbb{E}\left(\sum_{(i,j):\ i\neq j}\omega_i^2\omega_j^2\operatorname{trace}(A_iA_j^2A_i)\right)+\mathbb{E}\left(\sum_{i=1}^m\omega_i^4\operatorname{trace}(A_i^4)\right)$$
$$=\frac{T_1+2T_2+3T_3}{m(m+2)},$$

where

$$T_1=\sum_{(i,j):\ i\neq j}\operatorname{trace}(A_iA_jA_iA_j),$$
$$T_2=\sum_{(i,j):\ i\neq j}\operatorname{trace}(A_i^2A_j^2)\quad\text{and}$$
$$T_3=\sum_{i=1}^m\operatorname{trace}(A_i^4).$$

We bound T_1, T_2 and T_3.

Applying the Cauchy - Schwarz inequality, we get

$$|T_1|=\left|\sum_{(i,j):\ i\neq j}\operatorname{trace}(A_iA_jA_iA_j)\right|=\left|\sum_{(i,j):\ i\neq j}\langle A_jA_i,A_iA_j\rangle\right|$$
$$\leq\sum_{(i,j):\ i\neq j}\|A_jA_i\|_{\mathrm{HS}}\|A_iA_j\|_{\mathrm{HS}}=\sum_{(i,j):\ i\neq j}\operatorname{trace}(A_i^2A_j^2)=T_2.$$

On the other hand,

$$T_2=\sum_{(i,j):\ i\neq j}\operatorname{trace}(A_i^2A_j^2)=\operatorname{trace}\left(\sum_{i=1}^mA_i^2\right)^2-\sum_{i=1}^m\operatorname{trace}(A_i^4)=\left\|\sum_{i=1}^mA_i^2\right\|_{\mathrm{HS}}^2-T_3.$$

Therefore,

$$|T_1+2T_2+3T_3|\leq|T_1|+2T_2+3T_3\ \leq\ 3T_2+3T_3=3\left\|\sum_{i=1}^mA_i^2\right\|_{\mathrm{HS}}^2$$

The proof now follows by (4.3). □

Next, we prove some uniform bounds.

Lemma 4.3. *For all $w\in\mathbb{S}^{m-1}$, we have*
(1)

$$\|A(w)\|_{\mathrm{op}}\ \leq\ \left\|\sum_{i=1}^mA_i^2\right\|_{\mathrm{op}}^{1/2}\quad\textit{and}$$

(2)
$$\sum_{j=1}^n \lambda_j^4(w) \ \le \ \left\| \sum_{i=1}^m A_i^2 \right\|_{\rm op}.$$

Proof. Repeatedly applying the Cauchy - Schwarz inequality, for any vector $x \in \mathbb{R}^n$ such that $\|x\| = 1$, we obtain

$$\begin{aligned} |\langle A(w)x, x\rangle| = \left| \sum_{i=1}^m \omega_i \langle A_i x, x\rangle \right| \ &\le \ \left(\sum_{i=1}^m \langle A_i x, x\rangle^2 \right)^{1/2} \ \le \ \left(\sum_{i=1}^m \langle A_i x, A_i x\rangle \right)^{1/2} \\ &= \left\langle \left(\sum_{i=1}^m A_i^2 \right) x, x \right\rangle^{1/2} \ \le \ \left\| \sum_{i=1}^m A_i^2 \right\|_{\rm op}^{1/2}, \end{aligned}$$

and Part (1) follows. Note that here we did not use that $A_1, \ldots, A_m$ is an orthonormal set.

To prove Part (2), we bound

$$\sum_{j=1}^n \lambda_j^4(w) \ \le \ \left(\max_{j=1,\ldots,n} \lambda_j^2(w) \right) \sum_{j=1}^n \lambda_j^2(w) = \|A(w)\|_{\rm op}^2 \, \|A(w)\|_{\rm HS}^2.$$

Using that $A_1, \ldots, A_m$ is an orthonormal set, we obtain

$$\|A(w)\|_{\rm HS}^2 = \operatorname{trace}(A^2(w)) = \sum_{i,j=1}^m \omega_i \omega_j \operatorname{trace}(A_i A_j) = \sum_{i=1}^m \omega_i^2 = 1. \tag{4.8}$$

The proof now follows by Part (1). $\square$

Finally, we need a concentration inequality on the unit sphere $\mathbb{S}^{m-1}$ for the 4-Schatten norm of $A(w)$.

Lemma 4.4. *For $\delta \ge 0$, we have*

$$\begin{aligned} &\mathbb{P} \left\{ w \in \mathbb{S}^{m-1} : \ \|A(w)\|_{\rm S_4} \ \ge \ \left(\frac{3}{m+2} \left\| \sum_{i=1}^m A_i^2 \right\|_{\rm op} \right)^{1/4} + \delta \right\} \\ &\qquad \le \ \exp\left\{ -\frac{\delta^2 (m-1)}{2 \left\| \sum_{i=1}^m A_i^2 \right\|_{\rm op}^{1/2}} \right\}. \end{aligned}$$

Proof. We apply a measure concentration inequality on the sphere $\mathbb{S}^{m-1}$. Let

$$\operatorname{dist}(x, y) = \arccos\langle x, y\rangle$$

be the geodesic distance between two points $x, y \in \mathbb{S}^{m-1}$ and let $f : \mathbb{S}^{m-1} \longrightarrow \mathbb{R}$ be a 1-Lipschitz function, so that

$$|f(x) - f(y)| \ \leq \ \operatorname{dist}(x, y) \quad \text{for all} \quad x, y \in \mathbb{S}^{m-1}.$$

Then for $c = \mathbb{E}\, f$ and $\delta > 0$ we have

$$\mathbb{P}\left\{w \in \mathbb{S}^{m-1} : \ f(w) \ \geq \ c + \delta\right\} \ \leq \ \exp\left\{-\frac{\delta^2(m-1)}{2}\right\},$$

see, for example, Section 5.1 of [10].

Let us define a function $g : \mathbb{R}^m \longrightarrow \mathbb{R}$ by

$$g(x) = \|A(x)\|_{\mathrm{S}_4}, \quad \text{where} \quad A(x) = \sum_{i=1}^m \xi_i A_i \quad \text{for} \quad x = (\xi_1, \ldots, \xi_m).$$

Then from (4.1) and Part 2 of Lemma 4.3, for all $x, y \in \mathbb{S}^{m-1}$, we have

$$\begin{aligned} |g(x) - g(y)| \ \leq \|A(x) - A(y)\|_{\mathrm{S}_4} = \|A(x-y)\|_{\mathrm{S}_4} \ &\leq \ \left\|\sum_{i=1}^m A_i^2\right\|_{\mathrm{op}}^{1/4} \|x - y\| \\ &\leq \ \left\|\sum_{i=1}^m A_i^2\right\|_{\mathrm{op}}^{1/4} \operatorname{dist}(x, y). \end{aligned}$$

Therefore, for the expectation $c = \mathbb{E}\, g$ on the unit sphere $\mathbb{S}^{m-1}$, we have

$$\mathbb{P}\left\{w \in \mathbb{S}^{m-1} : \ g(w) \ \geq c + \delta\right\} \ \leq \ \exp\left\{-\frac{\delta^2(m-1)}{2\left\|\sum_{i=1}^m A_i^2\right\|_{\mathrm{op}}^{1/2}}\right\} \quad \text{for} \quad \delta \geq 0.$$

By Lemma 4.2 and the Hölder inequality, we get

$$c = \mathbb{E}\left(\sum_{j=1}^n \lambda_j^4(w)\right)^{1/4} \ \leq \ \left(\mathbb{E}\sum_{j=1}^n \lambda_j^4(w)\right)^{1/4} \ \leq \ \left(\frac{3}{m+2}\left\|\sum_{i=1}^m A_i^2\right\|_{\mathrm{op}}\right)^{1/4},$$

and the proof follows. □

5. Estimating integrals

Recall that we have an orthonormal set $A_1, \ldots, A_m$ of $n \times n$ symmetric real matrices. For $w \in \mathbb{S}^{m-1}$, $w = (\omega_1, \ldots, \omega_m)$, we define the matrix

$$A(w) = \sum_{i=1}^{m} \omega_i A_i.$$

As follows by (4.8), we have

$$\|A(w)\|_{\mathrm{HS}} = 1.$$

In this section, we consider the integral

$$\int_0^{+\infty} \tau^{m-1} \det{}^{-\frac{1}{2}} \left(I - \sqrt{-1}\tau A\right) \exp\left\{-\frac{\sqrt{-1}\tau}{2} \operatorname{trace} A\right\} d\tau,$$

where A is an $n \times n$ symmetric matrix satisfying $\|A\|_{\mathrm{HS}} = 1$ and possibly some other constraints. In particular, we will be interested in the situation when

$$\|A\|_{\mathrm{op}} = O\left(\frac{1}{\sqrt{m}}\right).$$

We will be comparing this integral with

$$\int_0^{+\infty} \tau^{m-1} \exp\left\{-\frac{\tau^2}{4}\right\} d\tau = 2^{m-1}\Gamma\left(\frac{m}{2}\right) \sim \left(\frac{2}{e}\right)^{m/2} m^{m/2}.$$

First, we bound the tail.

Lemma 5.1. *Let A be an $n \times n$ real symmetric matrix such that*

$$\|A\|_{\mathrm{HS}} = 1 \quad \textit{and} \quad \|A\|_{\mathrm{op}} \leq \frac{1}{10\sqrt{m}}.$$

Then for $m \geq 2$,

$$\int_{5\sqrt{m}}^{+\infty} \tau^{m-1} \left| \det{}^{-\frac{1}{2}} \left(I - \sqrt{-1}\tau A\right) \right| d\tau < \frac{1}{20m} m^{m/2} e^{-3m}.$$

Proof. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A, so that

$$\sum_{j=1}^{n} \lambda_j^2 = 1 \quad \text{and} \quad |\lambda_j| \leq \frac{\alpha}{\sqrt{m}} \quad \text{for} \quad j = 1, \ldots, n$$

(we will choose $\alpha = 0.1$ at the end). Then

$$\left| \det{}^{-\frac{1}{2}} \left(I - \sqrt{-1}\tau A\right) \right| = \prod_{j=1}^{n} \left|1 - \sqrt{-1}\tau\lambda_j\right|^{-\frac{1}{2}} = \prod_{j=1}^{n} \left(1 + \lambda_j^2\tau^2\right)^{-\frac{1}{4}}.$$

Let

$$\xi_j = \lambda_j^2\tau^2 \quad \text{for} \quad j = 1, \ldots, n.$$

Since the minimum of the log-concave function $\prod_{j=1}^{n}(1+\xi_j)$ on the convex polyhedron defined by the equation

$$\sum_{j=1}^{n} \xi_j = \tau^2$$

and inequalities

$$0 \ \leq \ \xi_j \ \leq \ \frac{\alpha^2\tau^2}{m} \quad \text{and} \quad j = 1, \ldots, n$$

is attained at its vertex where all but possibly one coordinate are either 0 or $\alpha^2\tau^2/m$, we have

$$\prod_{j=1}^{n} \left(1 + \lambda_j^2\tau^2\right)^{-\frac{1}{4}} \ \leq \ \left(1 + \frac{\alpha^2\tau^2}{m}\right)^{\frac{\alpha^2-m}{4\alpha^2}}.$$

Hence

$$\begin{aligned}
&\int\limits_{\frac{\sqrt{m}}{2\alpha}}^{+\infty} \tau^{m-1} \left| \det{}^{-\frac{1}{2}} \left(I - \sqrt{-1}\tau A\right) \right| d\tau \ \leq \ \int\limits_{\frac{\sqrt{m}}{2\alpha}}^{+\infty} \tau^{m-1} \left(1 + \frac{\alpha^2\tau^2}{m}\right)^{\frac{\alpha^2-m}{4\alpha^2}} d\tau \\
&\quad = \frac{m^{m/2}}{\alpha^m} \int\limits_{1/2}^{+\infty} s^{m-1}(1+s^2)^{\frac{\alpha^2-m}{4\alpha^2}} \, ds \ \leq \ \frac{m^{m/2}}{\alpha^m} \int\limits_{1/2}^{+\infty} (1+s^2)^{\frac{m-1}{2}} (1+s^2)^{\frac{\alpha^2-m}{4\alpha^2}} (2s) \, ds \\
&\quad = \frac{m^{m/2}}{\alpha^m} \int\limits_{1/2}^{+\infty} (1+s^2)^{\frac{-m(1-2\alpha^2)-\alpha^2}{4\alpha^2}} (2s) \, ds \\
&\quad = \frac{m^{m/2}}{\alpha^m} \frac{4\alpha^2}{(1-2\alpha^2)m - 3\alpha^2} \left(\frac{4}{5}\right)^{\frac{(1-2\alpha^2)m-3\alpha^2}{4\alpha^2}}
\end{aligned}$$

Substituting $\alpha = 0.1$, we get

$$\int_{5\sqrt{m}}^{+\infty} \tau^{m-1} \left| \det{}^{-\frac{1}{2}} \left(I - \sqrt{-1}\tau A\right) \right| \, d\tau \;\leq\; m^{m/2} 10^m \frac{0.04}{0.98m - 0.03} \left(\frac{4}{5}\right)^{(24.5)m - 0.75}$$

$$< \; \frac{1}{20m} 10^m m^{m/2} \left(\frac{4}{5}\right)^{(24.5)m} \;< \; \frac{1}{20m} m^{m/2} e^{-3m}. \quad \square$$

Next, we estimate the integral on the initial interval.

Lemma 5.2. *Let A be an $n \times n$ real symmetric matrix such that*

$$\|A\|_{\mathrm{HS}} = 1 \quad \textit{and} \quad \|A\|_{\mathrm{op}} \;\leq\; \frac{1}{10\sqrt{m}}$$

and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A. Then, for $m \geq 1$,

(1) *We have*

$$\int_0^{5\sqrt{m}} \tau^{m-1} \left| \det{}^{-\frac{1}{2}} \left(I - \sqrt{-1}\tau A\right) \right| \, d\tau$$

$$\leq \; \exp\left\{ \frac{625m^2}{8} \sum_{j=1}^n \lambda_j^4 \right\} \int_0^{5\sqrt{m}} \tau^{m-1} \exp\left\{ -\frac{\tau^2}{4} \right\} \, d\tau.$$

(2) *Suppose, in addition, that*

$$\left| \sum_{j=1}^n \lambda_j^3 \right| \;\leq\; \frac{1}{25m^{3/2}} \quad \textit{and} \quad \sum_{j=1}^n \lambda_j^4 \;\leq\; \frac{1}{625m^2}.$$

Then

$$\Re \int_0^{5\sqrt{m}} t^{m-1} \; \det{}^{-\frac{1}{2}} (I - \sqrt{-1}\tau A) \exp\left\{ -\frac{\sqrt{-1}\tau}{2} \operatorname{trace} A \right\} \, d\tau$$

$$\geq \; \frac{1}{2} \int_0^{5\sqrt{m}} \tau^{m-1} \exp\left\{ -\frac{\tau^2}{4} \right\} \, d\tau.$$

Proof. Since in the interval $0 \leq \tau \leq 5\sqrt{m}$, we have

$$|\tau\lambda_j| \;\leq\; \frac{1}{2} \quad \text{for} \quad j = 1, \ldots, n,$$

we can expand

$$\begin{aligned}
&\det{}^{-\frac{1}{2}}\left(I-\sqrt{-1}\tau A\right)\exp\left\{-\frac{\sqrt{-1}\tau}{2}\operatorname{trace}A\right\}\\
&=\exp\left\{-\frac{1}{2}\sum_{j=1}^{n}\ln(1-\sqrt{-1}\tau\lambda_j)-\frac{\sqrt{-1}\tau}{2}\sum_{j=1}^{n}\lambda_j\right\}\\
&=\exp\left\{\frac{1}{2}\sum_{k=2}^{\infty}\frac{(\tau\sqrt{-1})^k}{k}\sum_{j=1}^{n}\lambda_j^k\right\}\\
&=\exp\left\{h(\tau)+\sqrt{-1}g(\tau)\right\},
\end{aligned}\tag{5.1}$$

where

$$h(\tau)=\sum_{s=1}^{\infty}(-1)^s\frac{\tau^{2s}}{4s}\sum_{j=1}^{n}\lambda_j^{2s}\quad\text{and}\quad g(\tau)=\sum_{s=1}^{\infty}(-1)^s\frac{\tau^{2s+1}}{4s+2}\sum_{j=1}^{n}\lambda_j^{2s+1}.$$

We have

$$\sum_{j=1}^{n}\lambda_j^2=\|A\|_{\mathrm{HS}}^2=1$$

and for $s\geq 1$, we have

$$\sum_{j=1}^{n}\lambda_j^{2(s+1)}\ \leq\ \left(\max_{j=1,\ldots,n}\lambda_j^2\right)\sum_{j=1}^{n}\lambda_j^{2s}\ \leq\ \frac{1}{100m}\sum_{j=1}^{n}\lambda_j^{2s}.$$

Consequently, for $0\leq\tau\leq 5\sqrt{m}$, we have

$$\sum_{j=1}^{n}(\tau\lambda_j)^{2(s+1)}\ \leq\ \frac{\tau^2}{100m}\sum_{j=1}^{n}(\tau\lambda_j)^{2s}\ \leq\ \frac{1}{4}\sum_{j=1}^{n}(\tau\lambda_j)^{2s}.$$

Hence the terms of $h(\tau)$ alternate in sign and decrease in the absolute value, from which we deduce that

$$-\frac{\tau^2}{4}\ \leq\ h(\tau)\ \leq\ -\frac{\tau^2}{4}+\frac{\tau^4}{8}\sum_{j=1}^{n}\lambda_j^4\quad\text{for}\quad 0\ \leq\ \tau\ \leq\ 5\sqrt{m}.\tag{5.2}$$

Part (1) now follows from the upper bound in (5.2).

To prove Part (2), we bound $g(\tau)$ assuming that

$$\left|\sum_{j=1}^{n}\lambda_j^3\right|\ \leq\ \frac{\alpha}{m^{3/2}}\quad\text{and}\quad\sum_{j=1}^{n}\lambda_j^4\ \leq\ \frac{\beta}{m^2}$$

(we substitute $\alpha = 1/25$ and $\beta = 1/625$ at the end). For $s \geq 2$, we have

$$\sum_{j=1}^n |\lambda_j|^{2s+1} \ \leq \ \left(\max_{j=1,\ldots,n} |\lambda_j|\right)^{2s-3} \cdot \sum_{j=1}^n \lambda_j^4 \ \leq \ \frac{\beta}{10^{2s-3} m^{s+\frac{1}{2}}}.$$

Therefore, in the interval $0 \leq \tau \leq 5\sqrt{m}$, we have

$$\left|\sum_{j=1}^n (\tau\lambda_j)^3\right| \ \leq \ 125\alpha \quad \text{and} \quad \left|\sum_{j=1}^n (\tau\lambda_j)^{2s+1}\right| \ \leq \ \frac{625\beta}{2^{2s-3}} \quad \text{for} \quad s \geq 2.$$

Therefore, in the interval $0 \leq \tau \leq 5\sqrt{m}$, we have

$$|g(\tau)| \ \leq \ \frac{125\alpha}{6} + \sum_{s=2}^{\infty} \frac{625\beta}{(4s+2)2^{2s-3}} \ \leq \ \frac{125\alpha}{6} + \frac{1250\beta}{30}.$$

Substituting

$$\alpha = \frac{1}{25} \quad \text{and} \quad \beta = \frac{1}{625},$$

we conclude that

$$|g(\tau)| \ \leq \ \frac{5}{6} + \frac{1}{15} = \frac{27}{30} \ < \ \frac{\pi}{3} \quad \text{for all} \quad 0 \leq \tau \leq 5\sqrt{m}.$$

The proof now follows from the lower bound in (5.2). □

The last lemma of this section contains some estimates for our benchmark integral.

Lemma 5.3. *For $m \geq 2$, we have*

(1)

$$\int_0^{+\infty} \tau^{m-1} \exp\left\{-\frac{\tau^2}{4}\right\} \, d\tau \ \geq \ m^{m/2} \sqrt{\frac{\pi}{m}} \left(\frac{2}{e}\right)^{m/2} \quad \textit{and}$$

(2)

$$\int_{5\sqrt{m}}^{+\infty} \tau^{m-1} \exp\left\{-\frac{\tau^2}{4}\right\} \, d\tau \leq \ \sqrt{\frac{2\pi}{m-1}} 2^m m^{m/2} \exp\left\{-\frac{25(m-1)}{8}\right\}.$$

Proof. We have

$$\int\limits_0^{+\infty} \tau^{m-1} \exp\left\{-\frac{\tau^2}{4}\right\}\, d\tau = 2^{m-1} \int\limits_0^{+\infty} s^{\frac{m-2}{2}} \exp\{-s\}\, ds = 2^{m-1}\Gamma\left(\frac{m}{2}\right).$$

To prove Part (1), we use the standard inequality

$$\Gamma(x) \ \geq \ \sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x} \quad \text{for} \quad x \geq 1.$$

To prove Part (2), we bound

$$\begin{aligned}
&\int\limits_{5\sqrt{m}}^{+\infty} \tau^{m-1} \exp\left\{-\frac{\tau^2}{4}\right\}\, d\tau = 2^m m^{m/2} \int\limits_{5/2}^{+\infty} s^{m-1} \exp\left\{-ms^2\right\}\, ds \\
&\leq \ 2^m m^{m/2} \int\limits_{5/2}^{+\infty} \exp\left\{-(m-1)\left(s^2 - \ln s\right)\right\}\, ds \\
&\leq \ 2^m m^{m/2} \int\limits_{5/2}^{+\infty} \exp\left\{-\frac{(m-1)s^2}{2}\right\}\, ds = \frac{2^m m^{m/2}}{\sqrt{m-1}} \int\limits_{\frac{5\sqrt{m-1}}{2}}^{+\infty} \exp\left\{-\frac{\tau^2}{2}\right\}\, d\tau \\
&\leq \ \sqrt{\frac{2\pi}{m-1}} 2^m m^{m/2} \exp\left\{-\frac{25(m-1)}{8}\right\},
\end{aligned}$$

where in the last inequality we use the standard Gaussian probability tail estimate

$$\frac{1}{\sqrt{2\pi}} \int\limits_a^{+\infty} e^{-\tau^2/2}\, d\tau \ \leq \ e^{-a^2/2} \quad \text{for} \quad a \geq 0. \quad \square$$

6. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. We choose

$$\eta = 10^{-6}.$$

Let $A_1, \ldots, A_m$ be an orthonormal basis of the subspace $\operatorname{span}(Q_1, \ldots, Q_m)$ in the space of $n \times n$ symmetric matrices and let

$$a_i(x) = \langle A_i x, x \rangle \quad \text{for} \quad i = 1, \ldots, m$$

be the corresponding quadratic forms. Since the quadratic forms $q_1, \ldots, q_m$ are linear combinations of the forms $a_1, \ldots, a_m$ and vice versa, the system

$$q_i(x) = \operatorname{trace} Q_i \quad \text{for} \quad i = 1, \ldots, m$$

has a solution if and only if the system

$$a_i(x) = \operatorname{trace} A_i \quad \text{for} \quad i = 1, \ldots, m$$

has a solution x. To establish the existence of a solution of the latter system, we use Theorem 2.1, for which we consider the integral

$$\int\limits_{\mathbb{R}^m} \det{}^{-\frac{1}{2}} \left(I - \sqrt{-1} \sum_{i=1}^m \tau_i A_i \right) \exp\left\{ -\frac{\sqrt{-1}}{2} \sum_{i=1}^m \tau_i \operatorname{trace} A_i \right\} \, dt. \tag{6.1}$$

Our goal is to prove that the integral (6.1) converges absolutely to a non-zero value.

Let $\mathbb{S}^{m-1} \subset \mathbb{R}^m$ be the unit sphere endowed with the Haar probability measure. For $w \in \mathbb{S}^{m-1}$, $w = (\omega_1, \ldots, \omega_m)$, let

$$A(w) = \sum_{i=1}^m \omega_i A_i.$$

Then by (4.8) and Lemma 4.3 for every $w \in \mathbb{S}^{m-1}$, we have

$$\|A(w)\|_{\mathrm{HS}} = 1 \quad \text{and} \quad \|A\|_{\mathrm{op}} = \sqrt{\frac{\eta}{m}} \ < \ \frac{1}{10\sqrt{m}}.$$

It follows from Lemma (5.1) that

$$\int\limits_0^{+\infty} \tau^{m-1} \left| \det{}^{-\frac{1}{2}} \left(I - \sqrt{-1} \tau A(w) \right) \right| \, d\tau \ < \ +\infty.$$

Hence the integral (6.1) indeed converges absolutely and, up to a non-zero factor (the surface area of the sphere $\mathbb{S}^{m-1}$) can be written as an absolutely converging integral

$$\mathbb{E} \left(\int\limits_0^{+\infty} \det{}^{-\frac{1}{2}} \left(I - \sqrt{-1} \sum_{i=1}^m \tau A(w) \right) \exp\left\{ -\frac{\sqrt{-1}\tau}{2} \operatorname{trace} A(w) \right\} \, d\tau \right) \tag{6.2}$$

where the expectation is taken with respect to the Haar measure on $\mathbb{S}^{m-1}$. Hence our goal is to prove that the integral (6.2) is non-zero. We intend to prove that the real part of the integral is positive.

Let $\lambda_1(w), \ldots, \lambda_n(w)$ be the eigenvalues of $A(w)$. By Lemma 4.1,

$$\mathbb{E} \left(\sum_{j=1}^n \lambda_j^3(w) \right)^2 \ \leq \ \frac{120\eta}{m(m+2)(m+4)} \ < \ \frac{3}{25000m^3}.$$

Therefore, by the Markov inequality,

$$\mathbb{P}\left\{w:\ \left|\sum_{j=1}^n \lambda_j^3(w)\right| \ >\ \frac{1}{25m^{3/2}}\right\} \ \leq\ \frac{3}{40}. \tag{6.3}$$

By Lemma 4.2,

$$\mathbb{E}\left(\sum_{j=1}^n \lambda_j^4(w)\right) \ \leq\ \frac{3\eta}{m(m+2)} \ <\ \frac{3}{10^6 m^2}.$$

Hence, using the Markov inequality again, we get

$$\mathbb{P}\left\{w:\sum_{j=1}^n \lambda_j^4(w) \ >\ \frac{1}{625m^2}\right\} \ \leq\ \frac{3}{1600}. \tag{6.4}$$

We represent $\mathbb{S}^{m-1}$ as a disjoint union

$$\mathbb{S}^{m-1} = \Omega_0 \cup \Omega_1 \cup \Omega_2,$$

where

$$\Omega_0 = \left\{w \in \mathbb{S}^{m-1}:\quad \left|\sum_{j=1}^n \lambda_j^3(w)\right| \ \leq\ \frac{1}{25m^{3/2}} \quad \text{and} \quad \sum_{j=1}^n \lambda_j^4(w) \ \leq\ \frac{1}{625m^2}\right\},$$

$$\Omega_1 = \left\{w \in \mathbb{S}^{m-1}:\quad \left|\sum_{j=1}^n \lambda_j^3(w)\right| \ >\ \frac{1}{25m^{3/2}} \quad \text{and} \quad \sum_{j=1}^n \lambda_j^4(w) \ \leq\ \frac{1}{625m^2}\right\} \quad \text{and}$$

$$\Omega_2 = \left\{w \in \mathbb{S}^{m-1}:\quad \sum_{j=1}^n \lambda_j^4(w) \ >\ \frac{1}{625m^2}\right\}.$$

From (6.3) and (6.4), we have

$$\mathbb{P}\left(\Omega_0\right) \ \geq\ \frac{7}{8}$$

and hence from Part (2) of Lemma 5.2,

$$\begin{aligned}\Re \int\limits_{\Omega_0} \left(\int\limits_0^{5\sqrt{m}} \det{}^{-\frac{1}{2}}\left(I - \sqrt{-1}\sum_{i=1}^m \tau A(w)\right) \exp\left\{-\frac{\sqrt{-1}\tau}{2}\operatorname{trace} A(w)\right\}\, d\tau\right)\, dw \\ \geq\ \frac{7}{16}\int\limits_0^{5\sqrt{m}} \tau^{m-1}\exp\left\{-\frac{\tau^2}{4}\right\}\, d\tau,\end{aligned} \tag{6.5}$$

where dw is the Haar measure in $\mathbb{S}^{m-1}$.

From (6.3), we have

$$P(\Omega_1) \ \leq \ \frac{3}{40}$$

and hence Part (1) of Lemma 5.2 yields

$$\begin{aligned}
&\int\limits_{\Omega_1} \left(\int\limits_0^{5\sqrt{m}} \tau^{m-1} \left| \det{}^{-\frac{1}{2}} \left(I - \sqrt{-1}\tau A(w) \right) \right| \, d\tau \right) \, dw \\
&\qquad \leq \frac{3}{40} \exp\left\{ \frac{1}{8} \right\} \int\limits_0^{5\sqrt{m}} \tau^{m-1} \exp\left\{ -\frac{\tau^2}{4} \right\} \, d\tau \qquad (6.6) \\
&\qquad < 0.1 \int\limits_0^{5\sqrt{m}} \tau^{m-1} \exp\left\{ -\frac{\tau^2}{4} \right\} \, d\tau.
\end{aligned}$$

For integer $k \geq 1$, let

$$\Omega_2^k = \left\{ w \in \mathbb{S}^{m-1} : \quad \frac{k}{5\sqrt{m}} \ < \ \|A(w)\|_{\mathrm{S}_4} \ \leq \ \frac{k+1}{5\sqrt{m}} \right\}.$$

Then from Part (2) of Lemma 4.3, we have

$$\Omega_2 = \bigcup_{k=1}^{5(\eta m)^{1/4}} \Omega_2^k.$$

By Lemma 4.4, taking into account that $\eta = 10^{-6}$, we get

$$\begin{aligned}
\mathbb{P}\left(\Omega_2^k\right) \ &\leq \ \mathbb{P}\left\{ w \in \mathbb{S}^{m-1} : \ \|A(w)\|_{\mathrm{S}_4} \ \geq \ \left(\frac{3\eta}{m(m+2)} \right)^{1/4} + \frac{k}{6\sqrt{m}} \right\} \\
&\leq \ \exp\left\{ -\frac{k^2(m-1)}{72\sqrt{\eta m}} \right\}.
\end{aligned}$$

In view of Part (1) of Lemma 5.2,

$$\begin{aligned}
&\int\limits_{\Omega_2^k} \left(\int\limits_0^{5\sqrt{m}} \tau^{m-1} \left| \det{}^{-\frac{1}{2}} \left(I - \sqrt{-1}\tau A \right) \right| \, d\tau \right) \, dw \\
&\qquad \leq \exp\left\{ \frac{(k+1)^4}{8} - \frac{k^2(m-1)}{72\sqrt{\eta m}} \right\} \int\limits_0^{5\sqrt{m}} \tau^{m-1} \exp\left\{ -\frac{\tau^2}{4} \right\} \, d\tau.
\end{aligned}$$

Since

$$k \ \leq \ 5(\eta m)^{1/4},$$

we have

$$\frac{(k+1)^4}{8} \ \leq \ 2k^4 \ \leq 50k^2(\eta m)^{1/2}$$

and

$$\frac{(k+1)^4}{8} - \frac{k^2(m-1)}{72\sqrt{\eta m}} \ \leq \ \frac{k^2\sqrt{m}}{20} - 6k^2\sqrt{m} \ < \ -5k^2.$$

Hence

$$\begin{aligned}
&\int\limits_{\Omega_2} \left(\int\limits_0^{5\sqrt{m}} \tau^{m-1} \left| \det{}^{-\frac{1}{2}} \left(I - \sqrt{-1}\tau A\right)\right| \, d\tau \right) \, dw \\
&\qquad < \ \left(\sum_{k=1}^{\infty} \exp\left\{-5k^2\right\} \right) \int\limits_0^{5\sqrt{m}} \tau^{m-1} \exp\left\{-\frac{\tau^2}{4}\right\} \, d\tau \qquad (6.7) \\
&\qquad < \ 0.01 \int\limits_0^{5\sqrt{m}} \tau^{m-1} \exp\left\{-\frac{\tau^2}{4}\right\} \, d\tau.
\end{aligned}$$

Summarizing, from (6.5), (6.6) and (6.7), we get

$$\begin{aligned}
&\left| \mathbb{E} \int\limits_0^{5\sqrt{m}} \det{}^{-\frac{1}{2}} \left(I - \sqrt{-1} \sum_{i=1}^{m} \tau A(w) \right) \exp\left\{ -\frac{\sqrt{-1}\tau}{2} \operatorname{trace} A(w) \right\} \, d\tau \right| \\
&> \ \frac{1}{4} \int\limits_0^{5\sqrt{m}} \tau^{m-1} \exp\left\{-\frac{\tau^2}{4}\right\} \, d\tau
\end{aligned}$$

and hence by Lemma 5.1, the absolute value of the expectation (6.2) is at least

$$\frac{1}{4} \int\limits_0^{5\sqrt{m}} \tau^{m-1} \exp\left\{-\frac{\tau^2}{4}\right\} \, d\tau - \frac{1}{20m} m^{m/2} e^{-3m}.$$

Then by Lemma 5.3, the absolute value of the expectation (6.2) is at least

$$m^{m/2}\left(\sqrt{\frac{\pi}{16m}}\left(\frac{2}{e}\right)^{m/2}-\sqrt{\frac{\pi}{m-1}}2^m\exp\left\{-\frac{25(m-1)}{8}\right\}-\frac{1}{20m}e^{-3m}\right),$$

which is positive for $m\geq 3$. $\square$

Proof of Theorem 1.2. The proof is identical, except we use Theorem 2.2 instead of Theorem 2.1. $\square$

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