AROUND QUANTUM ERGODICITY

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ABSTRACT. We discuss Shnirelman's Quantum Ergodicity Theorem, giving an outline of a proof and an overview of some of the recent developments in mathematical Quantum Chaos.

RÉSUMÉ. Nous discuterons le Théorème Ergodicité Quantique de Shnirelman. Nous donnons l'esquisse d'une preuve et un aperçu des résultats plus récents dans le domaine mathématique du Chaos Quantique.

Let (M, g) be a compact smooth Riemannian manifold without boundary. Consider an orthonormal basis of eigenfunctions of the Laplace–Beltrami operator Δ_g:

$$-\Delta_g u_j = \lambda_j^2 u_j, \quad \|u_j\|_{L^2} = 1,$$

where the sequence of square roots of eigenvalues $\lambda_j \geq 0$, counted with multiplicity, goes to infinity as $j \to \infty$.

We say that some subsequence $\{u_{j_k}\}$ *equidistributes in physical space* if the probability measures $|u_{j_k}|^2 \, d\operatorname{vol}_g$ converge weakly to the normalized volume measure, namely for all $a \in C^\infty(M)$

$$\int_M a(x)|u_{j_k}(x)|^2 \, d\operatorname{vol}_g(x) \to \frac{1}{\operatorname{vol}_g(M)} \int_M a(x) \, d\operatorname{vol}_g(x) \quad \text{as} \quad k \to \infty. \tag{1}$$

In [Shn74a] Shnirelman announced the following remarkable theorem (or rather, its more general version similar to Theorem 2 below) which is one of the foundational results in the field of Quantum Chaos:

Theorem 1 (Shnirelman's Theorem/Quantum Ergodicity). *Assume that the geodesic flow on M is ergodic with respect to the Liouville measure (see (8) below). Then there exists a density 1 subsequence $\{u_{j_k}\}$ which equidistributes in physical space.*

Here 'density 1' means that (once again counting eigenvalues with multiplicity)

$$\frac{\#\{k \mid \lambda_{j_k} \leq R\}}{\#\{j \mid \lambda_j \leq R\}} \to 1 \quad \text{as} \quad R \to \infty.$$

Theorem 1 is striking for two reasons. First of all, the equidistribution property gives us strong information on the distribution of mass of eigenfunctions at high frequency, and it has a natural physical interpretation: if we think of u_{j_k} as pure states of a quantum particle on M, then equidistribution means that in the high energy limit the probability of finding the particle in a macroscopic set converges to the volume

of that set. Secondly, the assumption made on the chaotic behavior of the geodesic flow is very weak, and there are plenty of examples of manifolds with ergodic geodesic flows, such as manifolds of negative sectional curvature. There are also cases where Theorem 1 holds but the associated flow is not ergodic, such as rational polygons, see Marklof–Rudnick [MR12]. These examples however do not satisfy the stronger Theorem 2 stated below.

Following the announcement [Shn74a], more details of the proof were provided in the lecture notes [Shn74b]. A detailed proof was given by Shnirelman later in the addendum to [Laz93]. In between [Shn74b] (which was not accessible in the West at the time) and [Laz93] two other proofs were produced by Zelditch [Zel87] and Colin de Verdière [CdV85]. The general principles used in all of these proofs are similar, and a version of the proof is sketched in §2 below.

A semiclassical version of quantum ergodicity, applying to a larger class of operators than $-\Delta_g$, was proved by Helffer–Martinez–Robert [HMR87]. In the case of (Dirichlet or Neumann) eigenfunctions the analog of Theorem 1 was proved by Gérard–Leichtnam [GL93] for convex domains in $\mathbb{R}^n$ with $W^{2,\infty}$ boundaries (such as the Bunimovich stadium in Figure 1) and by Zelditch–Zworski [ZZ96] for compact Riemannian manifolds with piecewise C^∞ boundaries. In that setting the geodesic flow is replaced by the billiard ball flow, which is defined almost everywhere with respect to the Liouville measure. See Figure 1 for a numerical illustration of quantum ergodicity in this setting. For an overview of various related results, see §3.

1.1. **Semiclassical quantization and quantum ergodicity in phase space.** The proofs of Theorem 1 in fact give a stronger statement, Theorem 2 below. To state it, we introduce the notion of *semiclassical quantization*. To each smooth compactly supported function $a(x,\xi) \in C_c^\infty(T^*M)$ on the cotangent bundle T^*M (called a *classical observable*) we associate the family of operators (called the corresponding *quantum observable*)

$$\mathrm{Op}_h(a) = a^{\mathrm{w}}(x, \tfrac{h}{i}\partial_x) : L^2(M) \to L^2(M),$$

depending on the *semiclassical parameter* $0 < h \ll 1$. The notation $a^{\mathrm{w}}(x, \tfrac{h}{i}\partial_x)$ (where 'w' stands for 'Weyl') is formal: the operators x and ∂_x do not commute, and on a manifold the operators ∂_x are coordinate dependent. One way to define $\mathrm{Op}_h(a)$ is to start with the Weyl quantization formula for the case $M = \mathbb{R}^n$

$$\mathrm{Op}_h(a)f(x) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}(x-y)\xi} a\big(\tfrac{x+y}{2}, \xi\big) f(y)\, dy d\xi \tag{2}$$

and use coordinate charts to piece together a (non-canonical) quantization procedure on a general manifold.
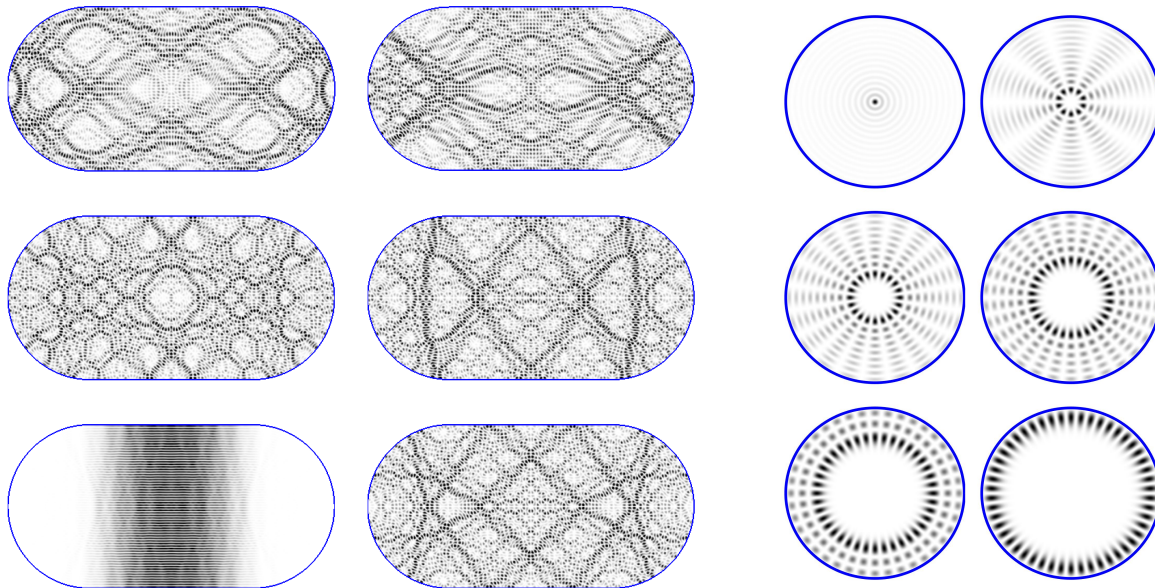
FIGURE 1. Left: several high frequency Dirichlet eigenfunctions of a Bunimovich stadium. The billiard flow is ergodic and most eigenfunctions equidistribute as follows from Quantum Ergodicity. The picture is courtesy of Alex Barnett, see [Bar06] and [BH14] for a description of the method used and for a numerical investigation of Quantum Ergodicity. For a generic stadium there is a sequence which does not equidistribute in phase space as proved by Hassell [Has10], see §3.1 below. Right: eigenfunctions of a disk, where the billiard flow is not ergodic and Quantum Ergodicity fails. An interesting question is what happens for *mixed systems* which have both ergodic and non-ergodic regions; see Schubert [Sch01], Galkowski [Gal14], Rivière [Riv13], and Gomes [Gom18].

The quantization procedure has many useful properties, in particular the following product and adjoint formulas for all $a, b \in C_c^\infty(T^*M)$:

$$\mathrm{Op}_h(a)\,\mathrm{Op}_h(b) = \mathrm{Op}_h(ab) + \mathcal{O}(h)_{L^2 \to L^2}; \tag{3}$$

$$\mathrm{Op}_h(a)^* = \mathrm{Op}_h(\overline{a}) + \mathcal{O}(h)_{L^2 \to L^2}. \tag{4}$$

Moreover, if $a \in C_c^\infty(T^*M)$ then $\|\mathrm{Op}_h(a)\|_{L^2 \to L^2}$ is bounded uniformly as $h \to 0$. We refer the reader to the book of Zworski [Zwo12] for details.

We will choose the semiclassical parameter depending on the eigenvalue λ_j^2 as follows:

$$h_j := \frac{1}{\lambda_j}.$$

This choice is motivated by the observation that the eigenfunction u_j is expected to oscillate at frequency $\sim \lambda_j$ and the normalized differential operators $h_j \partial_{x_\ell}$ then roughly preserve the magnitude of u_j. Under this choice of h_j, the eigenvalue equation $(-\Delta_g - \lambda_j^2)u_j = 0$ becomes

$$(-h_j^2 \Delta_g - 1)u_j = 0. \tag{5}$$

One can define $\mathrm{Op}_h(a)$ for observables a with controlled growth as $\xi \to \infty$ rather than just compactly supported ones (see [Zwo12]). In particular, if $a = a(x)$ is a function of x only, then $\mathrm{Op}_h(a)$ is the multiplication operator by a, which means that the left-hand side of (1) can be written as follows:

$$\int_M a(x)|u_j(x)|^2 \, d\mathrm{vol}_g(x) = \langle \mathrm{Op}_{h_j}(a)u_j, u_j \rangle \quad \text{for all} \quad a \in C^\infty(M)$$

where $\langle \bullet, \bullet \rangle$ denotes the inner product on $L^2(M, d\mathrm{vol}_g)$.

For general classical observables $a(x, \xi)$, the expression $\langle \mathrm{Op}_h(a)u, u \rangle$ can be interpreted as the average value of a for a quantum particle with wave function u; here x denotes the position variables and ξ the momentum variables. This suggests the following generalization of (1): we say that $\{u_{j_k}\}$ *equidistributes in phase space* if for all $a \in C_c^\infty(T^*M)$ we have

$$\langle \mathrm{Op}_{h_{j_k}}(a)u_{j_k}, u_{j_k} \rangle \to \int_{S^*M} a(x, \xi) \, d\mu_L(x, \xi) \quad \text{as} \quad k \to \infty. \tag{6}$$

Here the Liouville measure μ_L on the cosphere bundle $S^*M = \{(x, \xi) \in T^*M : |\xi|_g = 1\}$ is defined by

$$d\mu_L(x, \xi) = c \, d\mathrm{vol}_g(x) dS(\xi)$$

where the densities $d\mathrm{vol}_g$ on M and dS on the fibers of S^*M are induced by the metric g and the constant $c > 0$ is chosen so that μ_L be a probability measure.

The restriction to S^*M in (6) comes from the fact that eigenfunctions 'live' on the cosphere bundle, more precisely

$$a \in C_c^\infty(T^*M), \quad a|_{S^*M} = 0 \quad \Longrightarrow \quad \|\mathrm{Op}_{h_j}(a)u_j\|_{L^2} = \mathcal{O}(h_j). \tag{7}$$

To see (7) we write $-h^2\Delta_g - 1 = \mathrm{Op}_h(|\xi|_g^2 - 1) + \mathcal{O}(h)$, which together with the product formula (3) (or rather, its version for symbols that are not compactly supported) gives $\mathrm{Op}_h(a) = \mathrm{Op}_h(b)(-h^2\Delta_g - 1) + \mathcal{O}(h)_{L^2 \to L^2}$ where $b := (|\xi|_g^2 - 1)^{-1}a \in C_c^\infty(T^*M)$. Applying this to u_j and using the eigenvalue equation (5) we get (7).

The semiclassical version of Theorem 1 is now given by

Theorem 2 (Quantum Ergodicity in phase space)**.** *Assume that the geodesic flow on M is ergodic with respect to the Liouville measure. Then there exists a density 1 subsequence $\{u_{j_k}\}$ which equidistributes in phase space in the sense of* (6).

Here the geodesic flow is considered as a flow on the cosphere bundle, denoted by

$$\varphi_t : S^*M \to S^*M,$$

and ergodicity is defined as follows: for any Borel set $U \subset S^*M$

$$\varphi_t(U) = U \quad \text{for all } t \quad \implies \quad \mu_L(U) = 0 \quad \text{or} \quad \mu_L(U) = 1. \tag{8}$$

We note that equidistribution in phase space is a stronger property than equidistribution in the physical space. A basic example is when $M = \mathbb{R}/2\pi\mathbb{Z}$ is a circle, then the sequence of eigenfunctions $u_j = e^{ijx}$ equidistributes in the physical space but it does not equidistribute in the phase space: instead it is localized on the 'positive half' of the cosphere bundle, given by $\{\xi = 1\}$.

2. Outline of the proof of quantum ergodicity

We now give an outline of the proof of Theorem 2 to illustrate the main ideas used. Our presentation roughly follows the book of Zworski [Zwo12, Chapter 15] and lecture notes by the author [Dya16] and we refer to these sources for the details omitted here. This strategy of the proof is due to Zelditch [Zel96] in the abstract setting of C^*-algebras.

2.1. **Reduction to an averaged statement.** We reduce the 'density 1' type statement of Theorem 2 to an estimate averaged over eigenfunctions, Theorem 3 below. To do this we use the *Weyl law*, which gives the asymptotic growth of eigenvalues:

$$\#\{j \mid \lambda_j \leq R\} = \frac{\omega_n}{(2\pi)^n} \operatorname{vol}_g(M)R^n + o(R^n) \quad \text{as} \quad R \to \infty \tag{9}$$

where $n = \dim M$ and $\omega_n > 0$ is the volume of the unit ball in $\mathbb{R}^n$.

The Weyl law (9) can be proved using two properties of semiclassical quantization:

- *Functional calculus*: If $\chi \in C_c^\infty(\mathbb{R})$ then $\chi(-h^2\Delta_g) = \operatorname{Op}_h(a_\chi)$ is the quantization of a symbol $a_\chi(x, \xi; h) \in C_c^\infty(T^*M)$ which has a full expansion in powers of h and $a_\chi(x, \xi; h) = \chi(|\xi|_g^2) + \mathcal{O}(h)$ as $h \to 0$. See [Zwo12, Theorem 14.9].
- *Trace formula*: If $a \in C_c^\infty(T^*M)$ then the operator $\operatorname{Op}_h(a) : L^2(M) \to L^2(M)$ is trace class and

$$\operatorname{tr} \operatorname{Op}_h(a) = (2\pi h)^{-n} \left(\int_{T^*M} a(x, \xi)\, d\xi dx + \mathcal{O}(h) \right) \quad \text{as} \quad h \to 0 \tag{10}$$

where $d\xi dx$ is the (canonically defined) symplectic volume form on T^*M. Note that in the case of Weyl quantization on $\mathbb{R}^n$ defined by (2) it is easy to see that the trace formula is exact, using that the trace of an operator is the integral of its Schwartz kernel on the diagonal.

Combining these two statements gives for any $\chi \in C_c^\infty(\mathbb{R})$

$$\sum_j \chi\big((h\lambda_j)^2\big) = \operatorname{tr} \chi(-h^2\Delta_g) = (2\pi h)^{-n}\left(\int_{T^*M} \chi(|\xi|_g^2)\,d\xi dx + \mathcal{O}(h)\right) \tag{11}$$

and the Weyl law follows by taking $h := R^{-1}$, approximating the indicator function $\mathbf{1}_{[0,1]}$ above and below by smooth functions χ, and using the monotonicity of the left-hand side of (11) in χ.

We now state the integrated form of Quantum Ergodicity:

Theorem 3 (Integrated Quantum Ergodicity). *Assume that the geodesic flow on M is ergodic with respect to the Liouville measure. Take arbitrary $a \in C_c^\infty(T^*M)$ and put*

$$L_a := \int_{S^*M} a\,d\mu_L, \qquad V_j(a) := \langle \operatorname{Op}_{h_j}(a)u_j, u_j\rangle \tag{12}$$

where μ_L is the Liouville measure on S^*M. Then

$$R^{-n} \sum_{\lambda_j \in [R,2R]} |V_j(a) - L_a|^2 \to 0 \quad as \quad R \to \infty.$$

Theorem 3 states that if we restrict λ_j to a spectral window $[R, 2R]$, where R is large, then $|V_j(a) - L_a|$ is small on average. By the Chebyshev inequality, we then see that there exists $\varepsilon(R)$ which goes to zero as $R \to \infty$ such that $|V_j(a) - L(a)| \leq \varepsilon(R)$ for all $\lambda_j \in [R, 2R]$ except an $\varepsilon(R)$ proportion of these. Taking $R = 2^n$, $n \to \infty$, we get a density 1 sequence of u_j which equidistribute in the phase space for a given classical observable a. Using a diagonal argument one can construct a density 1 sequence which equidistributes with respect to every observable, thus giving Theorem 2.

In the remainder of this section we sketch a proof of Theorem 3. We restrict ourselves to the special case when $L_a = 0$:

$$L_a = 0 \quad \Longrightarrow \quad R^{-n} \sum_{\lambda_j \in [R,2R]} |V_j(a)|^2 \to 0 \quad \text{as} \quad R \to \infty. \tag{13}$$

The general case follows by applying (13) to the shifted observable $a - L_a$ and noting that $V_j(a - L_a) = V_j(a) - L_a$ since $\operatorname{Op}_h(1)$ is the identity operator. (Here $a - L_a$ is not compactly supported but this does not make a difference in the proof.)

2.2. **Replacing by ergodic averages.** We will next use the semiclassical Schrödinger propagator, which is the unitary family of operators

$$U(t) = U(t; h) := \exp(ith\Delta_g/2) : L^2(M) \to L^2(M).$$

For any $a \in C_c^\infty(T^*M)$ we have

$$V_j(a) = \langle \operatorname{Op}_{h_j}(a)u_j, u_j\rangle = \langle U(-t; h_j)\operatorname{Op}_{h_j}(a)U(t; h_j)u_j, u_j\rangle. \tag{14}$$

Here we use that u_j is an eigenfunction of the Laplacian and thus also of $U(t)$, more precisely $U(t; h_j)u_j = e^{-it\lambda_j/2}u_j$. In fact, this is the most important place where one uses the fact that u_j's are eigenfunctions.

The conjugated operator $U(-t; h)\operatorname{Op}_h(a)U(t; h)$ is described by *Egorov's Theorem*:

$$U(-t; h)\operatorname{Op}_h(a)U(t; h) = \operatorname{Op}_h(a \circ \varphi_t) + \mathcal{O}_t(h)_{L^2 \to L^2} \tag{15}$$

where φ_t, defined as the Hamiltonian flow of $|\xi|_g^2/2$, is an extension of the geodesic flow from S^*M to T^*M and the constant in the remainder depends on t but not on j. The statement (15) is what relates classical dynamics (the geodesic flow) to quantum dynamics (the Schrödinger propagator) in the proof. See [Zwo12, Theorem 15.2] for details.

Combining (14) and (15) we see that the quantum observables $V_j(a)$ do not change much if we replace a by its pullback by the geodesic flow:

$$V_j(a) = V_j(a \circ \varphi_t) + \mathcal{O}_t(h_j) \quad \text{for all} \quad t \in \mathbb{R}.$$

Since V_j depends linearly on a, same is then true for *ergodic averages*:

$$V_j(a) = V_j(\langle a \rangle_T) + \mathcal{O}_T(h_j) \quad \text{for all} \quad T > 0 \tag{16}$$

where the ergodic average $\langle a \rangle_T$ is defined by

$$\langle a \rangle_T = \frac{1}{T} \int_0^T a \circ \varphi_t \, dt \ \in \ C_c^\infty(T^*M).$$

Recall that we assumed that the flow φ_t is ergodic on S^*M with respect to the Liouville measure μ_L and that $\int_{S^*M} a \, d\mu_L = 0$. Then by the von Neumann Ergodic Theorem we have

$$\|\langle a \rangle_T\|_{L^2(S^*M; d\mu_L)} \to 0 \quad \text{as} \quad T \to \infty. \tag{17}$$

It remains to convert this bound on $\langle a \rangle_T$ to a bound on $V_j(\langle a \rangle_T)$.

2.3. **Local Weyl bound.** We now bound the right-hand side of (13):

Lemma 2.1. *There exists a constant C such that for all $a \in C_c^\infty(T^*M)$ and all $R > 1$*

$$R^{-n} \sum_{\lambda_j \in [R, 2R]} |V_j(a)|^2 \leq C\|a\|_{L^2(S^*M; d\mu_L)}^2 + \mathcal{O}_a(R^{-1}). \tag{18}$$

Roughly speaking, Lemma 2.1 says that if $a(x, \xi)$ is small after averaging over the points (x, ξ) then $V_j(a)$ is small after averaging over the eigenvalues λ_j.

To show Lemma 2.1, we first bound the sum in (18) by a smoothened out version. Fix a nonnegative cutoff function $\chi \in C_c^\infty((0, \infty))$ such that $\chi = 1$ on $[1, 4]$. Since

$$|V_j(a)| = |\langle \operatorname{Op}_{h_j}(a)u_j, u_j \rangle| \leq \|\operatorname{Op}_{h_j}(a)u_j\|_{L^2},$$

we have

$$
R^{-n} \sum_{\lambda_j \in [R,2R]} |V_j(a)|^2 \leq R^{-n} \sum_j \chi\Big(\frac{\lambda_j^2}{R^2}\Big) \| \operatorname{Op}_{h_j}(a)u_j \|_{L^2}^2
$$

$$
= R^{-n} \sum_j \chi\Big(\frac{\lambda_j^2}{R^2}\Big) \langle \operatorname{Op}_{h_j}(|a|^2)u_j, u_j \rangle + \mathcal{O}(R^{-1}). \tag{19}
$$

Here in the last line we use that $\operatorname{Op}_h(a)^* \operatorname{Op}_h(a) = \operatorname{Op}_h(|a|^2) + \mathcal{O}(h)_{L^2 \to L^2}$ as follows from the algebraic properties (3), (4); we also use the Weyl law (9).

The first term on the right-hand side of (19) looks very much like a trace except that $\operatorname{Op}_{h_j}(|a|^2)$ depends on j. To remove this dependence, we rescale the semiclassical parameter. Put

$$
h := \frac{1}{R}, \quad \tau_j := \frac{h}{h_j} = \frac{\lambda_j}{R}, \quad \tau_j^2 \in \operatorname{supp}\chi.
$$

For any $b \in C_c^\infty(T^*M)$ we have (here we abuse the formal notation $b^{\mathrm{w}}(x, \frac{h}{i}\partial_x)$; see [DZ19, Exercise E.5] for more details)

$$
\operatorname{Op}_h(b) = b^{\mathrm{w}}(x, \tfrac{h}{i}\partial_x) = b^{\mathrm{w}}(x, \tau_j \tfrac{h_j}{i}\partial_x) = \operatorname{Op}_{h_j}(\Lambda_{\tau_j}b) \quad \text{where} \quad \Lambda_\tau b(x, \xi) := b(x, \tau\xi).
$$

This implies that $V_j(\Lambda_{\tau_j}b) = \langle \operatorname{Op}_h(b)u_j, u_j \rangle$. Now, fix b such that $\Lambda_\tau b = |a|^2$ on S^*M for all $\tau > 0$ such that $\tau^2 \in \operatorname{supp}\chi$; for example, one can put $b(x, r\theta) := \psi(r)|a(x,\theta)|^2$ for all $(x,\theta) \in S^*M$, $r > 0$, and an appropriate choice of the cutoff $\psi \in C_c^\infty((0,\infty))$. By (7) we have $\| \operatorname{Op}_{h_j}(|a|^2 - \Lambda_{\tau_j}b)u_j \|_{L^2} = \mathcal{O}(R^{-1})$ and thus

$$
V_j(|a|^2) = V_j(\Lambda_{\tau_j}b) + \mathcal{O}(R^{-1}) = \langle \operatorname{Op}_h(b)u_j, u_j \rangle + \mathcal{O}(R^{-1})
$$

where on the right-hand side the semiclassical parameter $h := R^{-1}$ no longer depends on j. Putting this together with (19) and the Weyl law (9), we get

$$
R^{-n} \sum_{\lambda_j \in [R,2R]} |V_j(a)|^2 \leq R^{-n} \sum_j \chi\Big(\frac{\lambda_j^2}{R^2}\Big) \langle \operatorname{Op}_h(b)u_j, u_j \rangle + \mathcal{O}(R^{-1})
$$

$$
= R^{-n} \operatorname{tr}\big(\chi(-h^2\Delta_g)\operatorname{Op}_h(b)\big) + \mathcal{O}(R^{-1}).
$$

By the functional calculus and the product formula (3) we have $\chi(-h^2\Delta_g)\operatorname{Op}_h(b) = \operatorname{Op}_h(\chi(|\xi|_g^2)b) + \mathcal{O}(h)$. By the trace formula (10) we then have

$$
R^{-n} \sum_{\lambda_j \in [R,2R]} |V_j(a)|^2 \leq (2\pi)^{-n} \int_{T^*M} \chi(|\xi|_g^2)b(x,\xi)\,d\xi dx + \mathcal{O}_a(R^{-1})
$$

$$
\leq C\|a\|_{L^2(S^*M;d\mu_L)}^2 + \mathcal{O}_a(R^{-1})
$$

which finishes the proof of (18).

2.4. **End of the proof.** We are now ready to finish the proof of (13), and thus of Theorem 2. Take some large $T > 0$. We have

$$R^{-n} \sum_{\lambda_j \in [R,2R]} |V_j(a)|^2 = R^{-n} \sum_{\lambda_j \in [R,2R]} |V_j(\langle a \rangle_T)|^2 + \mathcal{O}_T(R^{-1})$$

$$\leq C \|\langle a \rangle_T\|_{L^2(S^*M;d\mu_L)}^2 + \mathcal{O}_T(R^{-1}).$$

Here in the first line we used (16) and the Weyl law (9); in the second line we used (18). The constant in $\mathcal{O}(\bullet)$ depends on T but the constant C does not.

Passing to the limit $R \to \infty$, we get

$$\limsup_{R \to \infty} R^{-n} \sum_{\lambda_j \in [R,2R]} |V_j(a)|^2 \leq C \|\langle a \rangle_T\|_{L^2(S^*M;d\mu_L)}^2.$$

The left-hand side does not depend on T and the right-hand side converges to 0 as $T \to \infty$ by (17). Therefore

$$\lim_{R \to \infty} R^{-n} \sum_{\lambda_j \in [R,2R]} |V_j(a)|^2 = 0$$

which gives (13) and finishes the proof.

3. OVERVIEW OF LITERATURE

Shnirelman's Quantum Ergodicity Theorem inspired a whole new direction of research on concentration of eigenfunctions and related objects. Here we give an overview of some of these developments. The list of topics discussed is by no means complete (achieving this would be difficult), and is somewhat skewed towards the author's own research interests. We refer the reader to the books by Sogge [Sog14] and Zelditch [Zel17], as well as the review articles by Marklof [Mar06], Sarnak [Sar11], and Nonnenmacher [Non13], for more detailed overview of results on quantum ergodicity and related topics.

3.1. **Quantum Unique Ergodicity and semiclassical measures.** Theorem 2 gives equidistribution in phase space for a density one sequence of eigenfunctions under a weak chaotic assumption on the geodesic flow. It is natural to ask if under stronger assumptions one can show that *all* eigenfunctions equidistribute, i.e. for all $a \in C_c^\infty(T^*M)$

$$\langle \mathrm{Op}_{h_j}(a)u_j, u_j \rangle \to \int_{S^*M} a \, d\mu_L \quad \text{as} \quad j \to \infty.$$

This statement, known as *Quantum Unique Ergodicity (QUE)*, was conjectured by Rudnick–Sarnak [RS94] for hyperbolic surfaces (i.e. compact surfaces of constant curvature -1). Since then there has been much progress (some of which is described
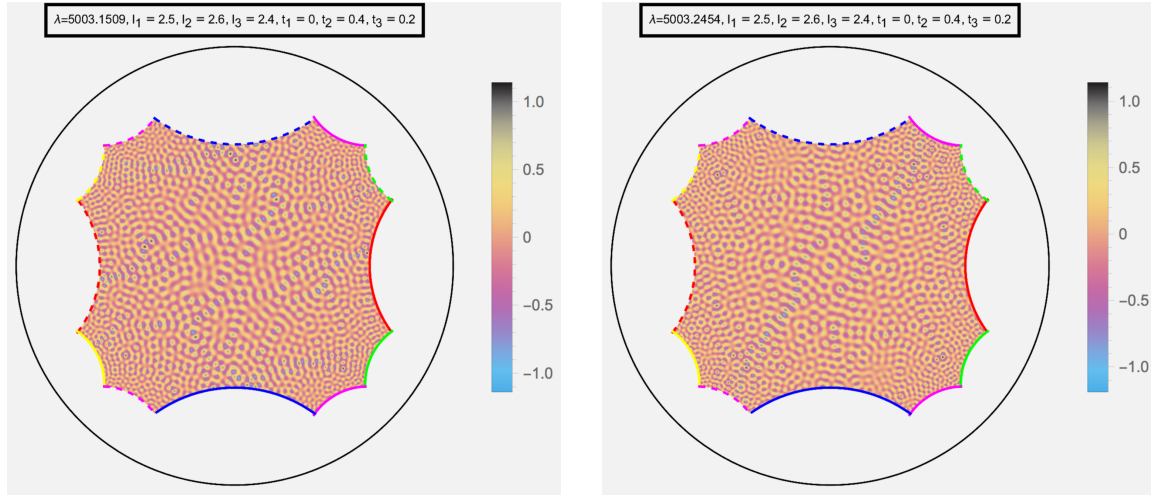
FIGURE 2. Two numerically computed eigenfunctions on a hyperbolic
surface $M = \Gamma \backslash \mathbb{H}^2$, drawn here on a fundamental domain of the surface
in the Poincaré disk model of $\mathbb{H}^2$. While the microscopic features of
the two eigenfunctions are different, on the macroscopic level they both
show equidistribution. Pictures courtesy of Alexander Strohmaier, see
Strohmaier–Uski [SU13] for more details.

below) but the original conjecture is still very much open. See Figure 2 for a numeri-
cal illustration. Quantum Unique Ergodicity can also be formulated in the context of
semiclassical measures, defined as follows:

Definition 3.1. *Let* $\{u_{j_k}\}$ *be a subsequence of eigenfunctions of* $-\Delta_g$ *and* μ *be a*
probability measure on T^*M. *We say* u_{j_k} *converges weakly to* μ *if for all* $a \in C_c^\infty(T^*M)$
we have

$$\langle \mathrm{Op}_{h_{j_k}}(a) u_{j_k}, u_{j_k} \rangle \to \int_{T^*M} a \, d\mu \quad as \quad k \to \infty.$$

We say that μ *is a* semiclassical measure *if it is the weak limit of some subsequence of*
eigenfunctions.

Every semiclassical measure is supported on S^*M and invariant under the geodesic
flow, see for example [Zwo12, Chapter 5]. The Quantum Unique Ergodicity conjecture
can now be restated as follows: *the only semiclassical measure is the Liouville measure.*

3.1.1. *The arithmetic case.* A hyperbolic surface can be represented as the quotient
$M = \Gamma \backslash \mathbb{H}^2$ where $\mathbb{H}^2$ is the hyperbolic plane and Γ is a group of isometries. If we
use the upper half-plane model $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$, with the hyperbolic metric
$g = \frac{|dz|^2}{|\mathrm{Im}\, z|^2}$, then orientation preserving isometries are Möbius maps

$$z \mapsto \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{R}, \quad ad - bc = 1$$

and the total group of isometries is $\mathrm{PSL}(2,\mathbb{R})$, the quotient of $\mathrm{SL}(2,\mathbb{R})$ by the group $\{I, -I\}$. Thus compact hyperbolic surfaces are identified with co-compact discrete subgroups $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$.

A special class of hyperbolic surfaces are the *arithmetic* ones, where the group Γ has certain number theoretic properties. A particularly important example is the *modular surface* $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}^2$, which however is not compact (it has a cusp). For an example of a compact arithmetic hyperbolic surface, see for instance [Mar06, §2].

Arithmetic surfaces have additional symmetries called *Hecke operators* (see for example [Mar06, §6]). Those are a family of operators $M_q : L^2(M) \to L^2(M)$ indexed by positive integers q. The Hecke operators commute with each other and with the Laplacian, so one can form a basis of eigenfunctions for the Laplacian which are also eigenfunctions of all M_q (we call this a *Hecke basis*).

Using these additional symmetries, Lindenstrauss [Lin06] was able to prove the Quantum Unique Ergodicity conjecture for a Hecke basis of eigenfunctions on compact arithmetic surfaces. Brooks–Lindenstrauss [BL14] extended this to any joint basis of the Laplacian and a single Hecke operator. To the author's knowledge it is still an open question whether Quantum Unique Ergodicity holds for every orthonormal basis of eigenfunctions of just the Laplacian (which might not be a Hecke basis if eigenvalues of the Laplacian have multiplicity).

3.1.2. *Ergodic systems where QUE fails.* Hassell [Has10] (following earlier work by Donnelly [Don03]) constructed examples of manifolds where the geodesic flow is ergodic but Quantum Unique Ergodicity fails. These examples include generic Bunimovich stadia, see Figure 1. A key feature of these is the presence of closed trajectories along which the differential of the geodesic/billiard ball flow grows only polynomially in time; for the Bunimovich stadium these are trajectories bouncing between the top and bottom boundary segments. We note that hyperbolic surfaces and, more generally, negatively curved manifolds, do not admit such weakly dispersing trajectories because the geodesic flow has the Anosov property and the differential of the flow grows exponentially fast in time.

Another family of counterexamples to QUE is for toy models of *quantum maps*. These are families of matrices of size $N \to \infty$ (where the effective semiclassical parameter is $h := N^{-1}$) which quantize ergodic symplectic transformations of an even-dimensional torus. (Quantum Ergodicity was proved in this setting by Bouzouina–de Bièvre [BDB96] and Zelditch [Zel97].) In particular, Faure–Nonnenmacher–de Bièvre [FNDB03] showed that for certain two-dimensional quantum cat maps there exists a sequence of eigenstates which converges to the measure

$$\mu = \tfrac{1}{2}\mu_L + \tfrac{1}{2}\delta_\gamma \tag{20}$$

where μ_L is the volume measure and δ_γ is the delta measure on any a priori given closed trajectory γ of the cat map. Anantharaman–Nonnenmacher [AN07a] considered the Walsh-quantized baker's map and constructed semiclassical measures supported on fractal sets.

It is possible to introduce analogs of Hecke operators for quantum cat maps. For two-dimensional quantum cat maps arithmetic Quantum Unique Ergodicity was proved by Kurlberg–Rudnick [KR00]. On the other hand Kelmer [Kel10] constructed semiclassical measures of Hecke eigenfunctions concentrating on proper submanifolds for certain higher dimensional quantum cat maps.

3.1.3. *Entropy and support of semiclassical measures.* In the absense of QUE (in non-arithmetic settings), a natural problem is to restrict as much as possible which flow-invariant probability measures on S^*M can arise as semiclassical measures. In particular, Colin de Verdière conjectured in [CdV85] that for hyperbolic surfaces there cannot be a semiclassical measure supported on a single closed trajectory. This conjecture was solved in the more general setting of manifolds with Anosov geodesic flows by Anantharaman [Ana08]. This was followed by results of Anantharaman–Nonnenmacher [AN07b], Anantharaman–Koch–Nonnenmacher [AKN09], Rivière [Riv10b, Riv10a], and Anantharaman–Silberman [AS13]. In particular, [AN07b] proved a lower bound on the Kolmogorov–Sinai entropy $\mathbf{h}_{\mathrm{KS}}(\mu)$ of any semiclassical measure μ, which for hyperbolic surfaces is

$$\mathbf{h}_{\mathrm{KS}}(\mu) \geq \tfrac{1}{2}. \tag{21}$$

Here the entropy measures how much of the complexity of the geodesic flow is captured by μ. The delta measure δ_γ on a closed geodesic has entropy 0 and the Liouville measure (for hyperbolic surfaces) has entropy 1. The entropy of the measure of the type (20) is exactly $\frac{1}{2}$, so from the point of view of the counterexample of [FNDB03] the bound (21) is sharp.

For hyperbolic surfaces, Dyatlov–Jin [DJ18] showed a different kind of restriction on semiclassical measures μ: each such measure should have full support, i.e. $\mu(U) > 0$ for any nonempty open set $U \subset S^*M$. (See also the expository article [Dya17].) This rules out the fractal counterexamples of the kind found in [AN07a] (which can have entropy close to 1 and thus are not ruled out by the entropy bound (21)). There is no contradiction here since the key new ingredient in [DJ18], the fractal uncertainty principle of Bourgain–Dyatlov [BD18], does not hold for the Walsh quantization used in [AN07a]. On the other hand, linear combinations $c\mu_L + (1-c)\delta_\gamma$ with $0 < c < \frac{1}{2}$ have full support but are ruled out by (21). Dyatlov–Jin–Nonnenmacher [DJN19] recently extended the full support property to general surfaces with Anosov geodesic flows.

3.2. **Quantum ergodicity in other settings.** The natural ideas behind the proof of Theorem 2 make it very tempting to try to adapt this proof to other settings beyond
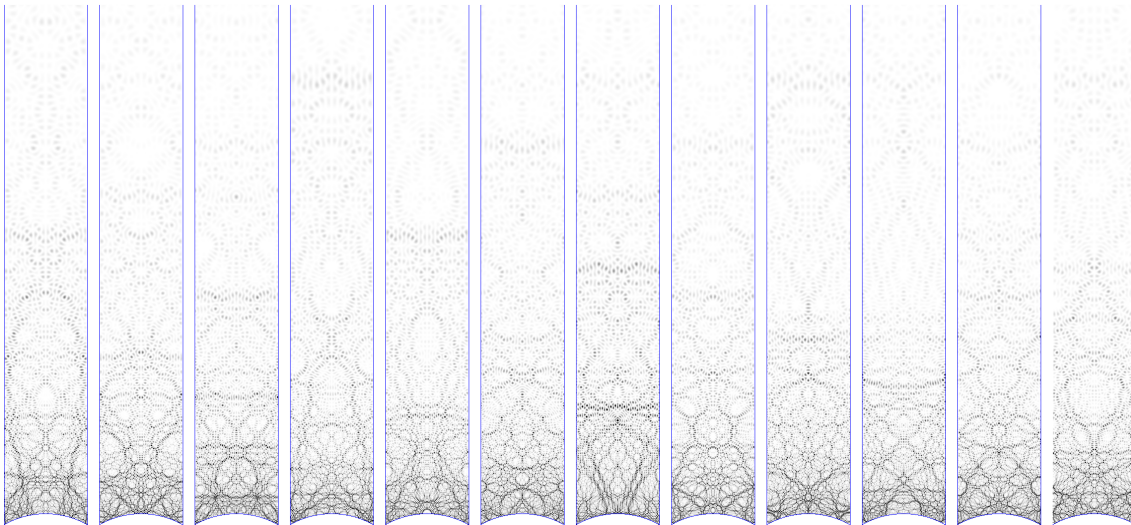
FIGURE 3. A plot of several high-frequency Maass forms on the modular surface, courtesy of Alex Barnett and made using a code of Holger Then. See https://math.dartmouth.edu/~specgeom/maass.php and [The05] for more details.

compact manifolds. While each of these settings presents its own unique challenges, and not all the ideas for the compact case carry well to the more general settings, there have been many Quantum Ergodicity-style equidistribution results, some of which are briefly reviewed below.

3.2.1. *Surfaces with cusps.* The first extension of Quantum Ergodicity was to complete noncompact Riemannian surfaces with cusps, which are infinite ends of the form

$$[r_0, \infty)_r \times \mathbb{S}^1_\theta \quad \text{with the metric} \quad dr^2 + e^{-2r} d\theta^2.$$

An important example is the *modular surface* $\mathrm{PSL}(2, \mathbb{Z})\backslash\mathbb{H}^2$, which has a fundamental domain of the form $\{|\operatorname{Re} z| \leq \frac{1}{2}, \ |z| \geq 1\} \subset \mathbb{H}^2$; here the cusp corresponds to $\operatorname{Im} z \to \infty$. See Figure 3 for a numerical illustration.

The spectrum of the Laplacian on a surface with cusps consists of three parts:

- The 'low' eigenvalues in $[0, \frac{1}{4})$, which are irrelevant for high frequency asymptotics featured in Quantum Ergodicity.
- The continuous spectrum $[\frac{1}{4}, \infty)$. Assuming for simplicity there is only one cusp, it is parametrized by *Eisenstein functions* $E(x; \lambda)$, $\lambda \geq 0$, which are certain solutions to the equation $(-\Delta_g - \lambda^2 - \frac{1}{4})E(x; \lambda) = 0$ generalizing one-dimensional plane waves $e^{-ix\lambda}$, $x \in \mathbb{R}$.

- The embedded eigenvalues, which are L^2 eigenvalues of the Laplacian in $[\frac{1}{4}, \infty)$. Those are abundant on some surfaces, such as the modular surface, but generic surfaces with cusps do not have embedded eigenvalues, see Colin de Verdière [CdV82, CdV83] and Phillips–Sarnak [PS85]. The corresponding eigenfunctions are known as *cusp forms* or *Maass forms*.

Zelditch [Zel91] proved Quantum Ergodicity for hyperbolic surfaces with cusps, of a flavor similar to Theorem 3, featuring both Eisenstein functions and cusp forms. A shorter proof, applying to any surface with cusps which has ergodic geodesic flow, was recently given by Bonthonneau–Zelditch [BZ16]. In the related setting of eigenfunctions of pseudo-Laplacians Quantum Ergodicity was proved by Studnia [Stu19].

In the arithmetic setting of the modular surface, Quantum Unique Ergodicity for Maass–Hecke forms (that is, Maass forms which are eigenfunctions of all the Hecke operators) was proved by Soundararajan [Sou10] following the work of Lindenstrauss [Lin06]. For Eisenstein functions on modular surfaces, equidistribution was proved by Luo–Sarnak [LS95] (in physical space) and Jakobson [Jak94] (in phase space).

3.2.2. *Manifolds with funnel-type ends.* Another noncompact setting is given by complete Riemannian manifolds M with funnel ends. The simplest (2-dimensional) version of a funnel end is

$$[0, \infty)_r \times \mathbb{S}^1_\theta \quad \text{with the metric} \quad dr^2 + \cosh^2 r \, d\theta^2. \tag{22}$$

In constract with cusp ends, which are very narrow, funnel ends are very wide and in particular they have infinite volume. For manifolds with funnels, there are no embedded eigenvalues; the spectrum is purely continuous and parametrized by Eisenstein functions $E(x; \lambda, \omega)$ where λ corresponds to the eigenvalue and ω is a point on the conformal infinity of M (for the case of the basic funnel end (22) we would have $\omega \in \mathbb{S}^1$). Alternatively one can consider Euclidean ends, which have the metric $dr^2 + r^2 d\theta^2$.

Since M has infinite volume, we can no longer talk about the ergodicity of the geodesic flow with respect to the Liouville measure. Instead, one makes assumptions on the set of *trapped geodesics*, which are geodesics which do not escape (forwards or backwards in time) through the infinite ends of M. This set often has fractal structure.

In the setting of hyperbolic manifolds with funnels, Guillarmou–Naud [GN14] showed an 'equidistribution' statement for the Eistenstein functions, under the *pressure condition* which here is equivalent to a certain upper bound on the Hausdorff dimension of the set of trapped geodesics. This was later extended by Ingremeau [Ing17] to the setting where the underlying flow is hyperbolic, assuming again that the pressure condition holds. Ingremeau later removed the pressure condition in [Ing21], relying on the spectral gap proved in [DZ16, BD18]. These results are more similar to Quantum

Unique Ergodicity than Quantum Ergodicity in that they are weak convergence statements for the whole family of Eisenstein functions rather than a density 1 subfamily. The limiting measures depend on ω; integrating in ω one obtains the Liouville measure.

Dyatlov–Guillarmou [DG14] proved a Quantum Ergodicity-type statement in this setting (which applies to most rather than all values of λ, ω), replacing the pressure condition with the much weaker assumption that the set of trapped trajectories has Liouville measure 0.

3.2.3. *Restrictions of eigenfunctions.* Coming back to the setting of compact manifolds M, a natural question to ask is the following: for a submanifold $\Sigma \subset M$, do restrictions of a density 1 sequence of eigenfunctions $u_{j_k}|_\Sigma$ converge weakly to a natural measure? The answer cannot be positive for Σ of arbitrary dimension, for example it is unrealistic to expect equidistribution of restrictions when Σ is a point. Henceforth we restrict to the case when Σ is a hypersurface.

It turns out that assuming that M has ergodic geodesic flow is not enough to ensure equidistribution of restrictions of eigenfunctions. A basic example is when Σ is the fixed point set of some isometric involution $J : M \to M$; then roughly half of the eigenfunctions u_j are odd with respect to J and thus have $u_j|_\Sigma = 0$.

However, if one makes an additional (generically satisfied) assumption that Σ does not have a reflection symmetry with respect to the geodesic flow, then $u_{j_k}|_\Sigma$ equidistributes for a density one sequence. Here the limiting measure for the equidistribution is naturally defined from the Liouville measure and supported on the coball bundle $B^*\Sigma = \{(x, \xi) \in T^*\Sigma : |\xi|_g \leq 1\}$. This result was proved by Toth–Zelditch [TZ12, TZ13], with a different proof of a semiclassical generalization given by Dyatlov–Zworski [DZ13]; see also Christianson–Toth–Zelditch [CTZ13].

In case when M is a surface, Quantum Ergodicity for restrictions has applications to counting nodal domains of eigenfunctions, see Jung–Zelditch [JZ16a, JZ16b].

3.2.4. *Large graphs.* Quantum Ergodicity can be also adapted to models of quantum chaos which are not eigenfunctions of operators on manifolds. In §3.1.2 we briefly discussed one such setting, quantum maps. Here we briefly discuss a different setting, *large regular graphs* where quantum ergodicity was proved by Anantharaman–Le Masson [ALM15]. We refer the reader to the review of Anantharaman–Sabri [AS19] for more information and further results.

Let (G_N) be a sequence of graphs of size N which are k-regular with some fixed $k \geq 3$. We identify the set of vertices of G_N with $\{1, \ldots, N\}$ and functions on this set with vectors in $\mathbb{C}^N$. We will replace the high eigenvalue limit of Theorem 1 by the limit $N \to \infty$ and eigenfunctions of the Laplacian by an orthonormal basis $u_j^{(N)} \in \mathbb{C}^N$,

$1 \leq j \leq N$, of eigenvectors of the graph Laplacian on G_N (for regular graphs, this is same as eigenvectors of the adjacency matrix of the graph).

The graphs G_N for different values of N have no relationship with each other, so it is unclear how to define the quantization of a fixed N-independent observable. Instead one can study the expressions $\langle a^{(N)} u_j^{(N)}, u_j^{(N)} \rangle$ for *any* family of functions $a^{(N)} : \{1, \ldots, N\} \to \mathbb{C}$ with $\max |a^{(N)}| \leq 1$. Under two assumptions discussed below, the paper [ALM15] proves the following version of Integrated Quantum Ergodicity: for any choice of $a^{(N)}$

$$\lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \left| \langle a^{(N)} u_j^{(N)}, u_j^{(N)} \rangle - \langle a^{(N)} \rangle \right|^2 = 0 \quad \text{where} \quad \langle a^{(N)} \rangle := \frac{1}{N} \sum_{\ell=1}^{N} a^{(N)}(\ell).$$

The assumptions on the graphs G_N are as follows:

(1) G_N converges to the regular k-tree in the local weak sense, that is the injectivity radius of a random vertex in G_N converges to infinity in distribution;
(2) G_N is an expander, that is there is an N-independent $\beta > 0$ such that the spectrum of the adjacency matrix of G_N is contained in $[-k + \beta, k - \beta] \cup \{k\}$.

These assumptions are satisfied for random graphs, as well as for certain deterministic examples. For random graphs, Bauerschmidt–Huang–Yau [BHY19] proved stronger equidistribution statements which are the analog of Quantum Unique Ergodicity in this setting.

References

[AKN09]   Nalini Anantharaman, Herbert Koch, and Stéphane Nonnenmacher. Entropy of eigenfunctions. In Vladas Sidoravičius, editor, *New Trends in Mathematical Physics*, pages 1–22, Dordrecht, 2009. Springer Netherlands.

[ALM15]   Nalini Anantharaman and Etienne Le Masson. Quantum ergodicity on large regular graphs. *Duke Math. J.*, 164(4):723–765, 2015.

[AN07a]   Nalini Anantharaman and Stéphane Nonnenmacher. Entropy of semiclassical measures of the Walsh-quantized baker's map. *Ann. Henri Poincaré*, 8(1):37–74, 2007.

[AN07b]   Nalini Anantharaman and Stéphane Nonnenmacher. Half-delocalization of eigenfunctions for the Laplacian on an Anosov manifold. *Ann. Inst. Fourier (Grenoble)*, 57(7):2465–2523, 2007. Festival Yves Colin de Verdière.

[Ana08]   Nalini Anantharaman. Entropy and the localization of eigenfunctions. *Ann. of Math. (2)*, 168(2):435–475, 2008.

[AS13]   Nalini Anantharaman and Lior Silberman. A Haar component for quantum limits on locally symmetric spaces. *Israel J. Math.*, 195(1):393–447, 2013.

[AS19]   Nalini Anantharaman and Mostafa Sabri. Recent results of quantum ergodicity on graphs and further investigation. *Ann. Fac. Sci. Toulouse Math. (6)*, 28(3):559–592, 2019.

[Bar06]   Alexander Barnett. Asymptotic rate of quantum ergodicity in chaotic euclidean billiards. *Communications on Pure and Applied Mathematics*, 59(10):1457–1488, 2006.

[BD18]   Jean Bourgain and Semyon Dyatlov. Spectral gaps without the pressure condition. *Ann. of Math. (2)*, 187(3):825–867, 2018.

[BDB96]   Abdelkader Bouzouina and Stephan De Bièvre. Equipartition of the eigenfunctions of quantized ergodic maps on the torus. *Comm. Math. Phys.*, 178(1):83–105, 1996.

[BH14]   Alex Barnett and Andrew Hassell. Fast computation of high-frequency dirichlet eigenmodes via spectral flow of the interior neumann-to-dirichlet map. *Communications on Pure and Applied Mathematics*, 67(3):351–407, 2014.

[BHY19]   Roland Bauerschmidt, Jiaoyang Huang, and Horng-Tzer Yau. Local Kesten-McKay law for random regular graphs. *Comm. Math. Phys.*, 369(2):523–636, 2019.

[BL14]   Shimon Brooks and Elon Lindenstrauss. Joint quasimodes, positive entropy, and quantum unique ergodicity. *Invent. Math.*, 198(1):219–259, 2014.

[BZ16]   Yannick Bonthonneau and Steve Zelditch. Quantum ergodicity for Eisenstein functions. *C. R. Math. Acad. Sci. Paris*, 354(9):907–911, 2016.

[CdV82]   Yves Colin de Verdière. Pseudo-laplaciens. I. *Ann. Inst. Fourier (Grenoble)*, 32(3):xiii, 275–286, 1982.

[CdV83]   Yves Colin de Verdière. Pseudo-laplaciens. II. *Ann. Inst. Fourier (Grenoble)*, 33(2):87–113, 1983.

[CdV85]   Yves Colin de Verdière. Ergodicité et fonctions propres du laplacien. In *Bony-Sjöstrand-Meyer seminar, 1984–1985*, pages Exp. No. 13, 8. École Polytech., Palaiseau, 1985.

[CTZ13]   Hans Christianson, John A. Toth, and Steve Zelditch. Quantum ergodic restriction for Cauchy data: interior que and restricted que. *Math. Res. Lett.*, 20(3):465–475, 2013.

[DG14]   Semyon Dyatlov and Colin Guillarmou. Microlocal limits of plane waves and Eisenstein functions. *Ann. Sci. Éc. Norm. Supér. (4)*, 47(2):371–448, 2014.

[DJ18]   Semyon Dyatlov and Long Jin. Semiclassical measures on hyperbolic surfaces have full support. *Acta Math.*, 220(2):297–339, 2018.

[DJN19]   Semyon Dyatlov, Long Jin, and Stéphane Nonnenmacher. Control of eigenfunctions on surfaces of variable curvature, 2019. To appear in J. Amer. Math. Soc.; arXiv:1906.08923.

[Don03]   Harold Donnelly. Quantum unique ergodicity. *Proc. Amer. Math. Soc.*, 131(9):2945–2951, 2003.

[Dya16]   Semyon Dyatlov. Lecture notes on quantum chaos. Available online at `http://math.mit.edu/~dyatlov/files/2016/qenotes.pdf`, 2016.

[Dya17]   Semyon Dyatlov. Control of eigenfunctions on hyperbolic surfaces: an application of fractal uncertainty principle. *Journées équations aux dérivées partielles*, 2017.

[DZ13]   Semyon Dyatlov and Maciej Zworski. Quantum ergodicity for restrictions to hypersurfaces. *Nonlinearity*, 26(1):35–52, 2013.

[DZ16]   Semyon Dyatlov and Joshua Zahl. Spectral gaps, additive energy, and a fractal uncertainty principle. *Geom. Funct. Anal.*, 26(4):1011–1094, 2016.

[DZ19]   Semyon Dyatlov and Maciej Zworski. *Mathematical theory of scattering resonances*, volume 200 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2019.

[FNDB03] Frédéric Faure, Stéphane Nonnenmacher, and Stephan De Bièvre. Scarred eigenstates for quantum cat maps of minimal periods. *Comm. Math. Phys.*, 239(3):449–492, 2003.

[Gal14]   Jeffrey Galkowski. Quantum ergodicity for a class of mixed systems. *J. Spectr. Theory*, 4(1):65–85, 2014.

[GL93]   Patrick Gérard and Éric Leichtnam. Ergodic properties of eigenfunctions for the Dirichlet problem. *Duke Math. J.*, 71(2):559–607, 1993.

[GN14]   Colin Guillarmou and Frédéric Naud. Equidistribution of Eisenstein series on convex cocompact hyperbolic manifolds. *Amer. J. Math.*, 136(2):445–479, 2014.

[Gom18]   Sean P. Gomes. Percival's conjecture for the Bunimovich mushroom billiard. *Nonlinearity*, 31(9):4108–4136, 2018.

[Has10]   Andrew Hassell. Ergodic billiards that are not quantum unique ergodic. *Ann. of Math. (2)*, 171(1):605–619, 2010. With an appendix by the author and Luc Hillairet.

[HMR87]   Bernard Helffer, André Martinez, and Didier Robert. Ergodicité et limite semi-classique. *Comm. Math. Phys.*, 109(2):313–326, 1987.

[Ing17]   Maxime Ingremeau. Distorted plane waves in chaotic scattering. *Anal. PDE*, 10(4):765–816, 2017.

[Ing21]   Maxime Ingremeau. Semiclassical limits of distorted plane waves in chaotic scattering without a pressure condition, 2021. To appear in Int. Mat. Res. Not.; arXiv:1706.07691.

[Jak94]   Dmitry Jakobson. Quantum unique ergodicity for Eisenstein series on $\mathrm{PSL}_2(\mathbf{Z})\backslash\mathrm{PSL}_2(\mathbf{R})$. *Ann. Inst. Fourier (Grenoble)*, 44(5):1477–1504, 1994.

[JZ16a]   Junehyuk Jung and Steve Zelditch. Number of nodal domains and singular points of eigenfunctions of negatively curved surfaces with an isometric involution. *J. Differential Geom.*, 102(1):37–66, 2016.

[JZ16b]   Junehyuk Jung and Steve Zelditch. Number of nodal domains of eigenfunctions on nonpositively curved surfaces with concave boundary. *Math. Ann.*, 364(3-4):813–840, 2016.

[Kel10]   Dubi Kelmer. Arithmetic quantum unique ergodicity for symplectic linear maps of the multidimensional torus. *Ann. of Math. (2)*, 171(2):815–879, 2010.

[KR00]   Pär Kurlberg and Zeév Rudnick. Hecke theory and equidistribution for the quantization of linear maps of the torus. *Duke Math. J.*, 103(1):47–77, 2000.

[Laz93]   Vladimir Lazutkin. *KAM theory and semiclassical approximations to eigenfunctions*, volume 24 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1993. With an addendum by A. I. Shnirelman.

[Lin06]   Elon Lindenstrauss. Invariant measures and arithmetic quantum unique ergodicity. *Ann. of Math. (2)*, 163(1):165–219, 2006.

[LS95]   Wen Zhi Luo and Peter Sarnak. Quantum ergodicity of eigenfunctions on $\mathrm{PSL}_2(\mathbf{Z})\backslash\mathbf{H}^2$. *Inst. Hautes Études Sci. Publ. Math.*, 81:207–237, 1995.

[Mar06]   Jens Marklof. Arithmetic quantum chaos. In Jean-Pierre Françoise, Gregory L. Naber, and Tsou Sheung Tsun, editors, *Encyclopedia of mathematical physics. Vol. 1, 2, 3, 4, 5*, pages 212–220. Academic Press/Elsevier Science, Oxford, 2006.

[MR12]   Jens Marklof and Zeév Rudnick. Almost all eigenfunctions of a rational polygon are uniformly distributed. *J. Spectr. Theory*, 2(1):107–113, 2012.

[Non13]   Stéphane Nonnenmacher. Anatomy of quantum chaotic eigenstates. In *Chaos*, volume 66 of *Prog. Math. Phys.*, pages 193–238. Birkhäuser/Springer, Basel, 2013.

[PS85]   Ralph Phillips and Peter Sarnak. On cusp forms for co-finite subgroups of $\mathrm{PSL}(2,\mathbf{R})$. *Invent. Math.*, 80(2):339–364, 1985.

[Riv10a]   Gabriel Rivière. Entropy of semiclassical measures for nonpositively curved surfaces. *Ann. Henri Poincaré*, 11(6):1085–1116, 2010.

[Riv10b]   Gabriel Rivière. Entropy of semiclassical measures in dimension 2. *Duke Math. J.*, 155(2):271–336, 2010.

[Riv13]   Gabriel Rivière. Remarks on quantum ergodicity. *J. Mod. Dyn.*, 7(1):119–133, 2013.

[RS94]   Zeév Rudnick and Peter Sarnak. The behaviour of eigenstates of arithmetic hyperbolic manifolds. *Comm. Math. Phys.*, 161(1):195–213, 1994.

[Sar11]   Peter Sarnak. Recent progress on the quantum unique ergodicity conjecture. *Bull. Amer. Math. Soc. (N.S.)*, 48(2):211–228, 2011.

[Sch01]   Roman Schubert. Semiclassical localization in phase space, 2001. Ph.D. thesis, available at https://people.maths.bris.ac.uk/~marcvs/publications/my_papers/thesis.pdf.

[Shn74a]   Alexander Shnirelman. Ergodic properties of eigenfunctions. *Uspehi Mat. Nauk*, 29(6(180)):181–182, 1974.

[Shn74b]   Alexander Shnirelman. Statistical properties of eigenfunctions. In *Proceedings of the All-USSR School in Differential Equations with Infinite Number of Independent Variables and in Dynamical Systems with Infinitely Many Degrees of Freedom, Dilijan, Armenia, May 21–June 3, 1973.* Armenian Academy of Sciences, Erevan, 1974.

[Sog14]   Christopher D. Sogge. *Hangzhou lectures on eigenfunctions of the Laplacian*, volume 188 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2014.

[Sou10]   Kannan Soundararajan. Quantum unique ergodicity for $SL_2(\mathbb{Z})\backslash\mathbb{H}$. *Ann. of Math. (2)*, 172(2):1529–1538, 2010.

[Stu19]   Elie Studnia. Quantum ergodicity for pseudo-laplacians, 2019. To appear in J. Spect. Th.; arXiv:1905.07761.

[SU13]   Alexander Strohmaier and Ville Uski. An algorithm for the computation of eigenvalues, spectral zeta functions and zeta-determinants on hyperbolic surfaces. *Comm. Math. Phys.*, 317(3):827–869, 2013.

[The05]   Holger Then. Maass cusp forms for large eigenvalues. *Math. Comp.*, 74(249):363–381, 2005.

[TZ12]   John A. Toth and Steve Zelditch. Quantum ergodic restriction theorems. I: Interior hypersurfaces in domains wth ergodic billiards. *Ann. Henri Poincaré*, 13(4):599–670, 2012.

[TZ13]   John A. Toth and Steve Zelditch. Quantum ergodic restriction theorems: manifolds without boundary. *Geom. Funct. Anal.*, 23(2):715–775, 2013.

[Zel87]   Steven Zelditch. Uniform distribution of eigenfunctions on compact hyperbolic surfaces. *Duke Math. J.*, 55(4):919–941, 1987.

[Zel91]   Steven Zelditch. Mean Lindelöf hypothesis and equidistribution of cusp forms and Eisenstein series. *J. Funct. Anal.*, 97(1):1–49, 1991.

[Zel96]   Steven Zelditch. Quantum ergodicity of C^* dynamical systems. *Comm. Math. Phys.*, 177(2):507–528, 1996.

[Zel97]   Steven Zelditch. Index and dynamics of quantized contact transformations. *Ann. Inst. Fourier (Grenoble)*, 47(1):305–363, 1997.

[Zel17]   Steve Zelditch. *Eigenfunctions of the Laplacian on a Riemannian manifold*, volume 125 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2017.

[Zwo12]   Maciej Zworski. *Semiclassical analysis*, volume 138 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.

[ZZ96]   Steven Zelditch and Maciej Zworski. Ergodicity of eigenfunctions for ergodic billiards. *Comm. Math. Phys.*, 175(3):673–682, 1996.

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