FREE SEIFERT PIECES OF PSEUDO-ANOSOV FLOWS

THIERRY BARBOT AND SÉRGIO R. FENLEY

Abstract − We prove a structure theorem for pseudo-Anosov flows restricted to Seifert fibered pieces of three manifolds. The piece is called periodic if there is a Seifert fibration so that a regular fiber is freely homotopic, up to powers, to a closed orbit of the flow. A non periodic Seifert fibered piece is called free. In a previous paper [Ba-Fe1] we described the structure of a pseudo-Anosov flow restricted to a periodic piece up to isotopy along the flow. In the present paper we consider free Seifert pieces. We show that, in a carefully defined neighborhood of the free piece, the pseudo-Anosov flow is orbitally equivalent to a hyperbolic blow up of a geodesic flow piece. A geodesic flow piece is a finite cover of the geodesic flow on a compact hyperbolic surface, usually with boundary. In the proof we introduce almost k-convergence groups and prove a convergence theorem. We also introduce an alternative model for the geodesic flow of a hyperbolic surface that is suitable to prove these results, and we carefully define what is a hyperbolic blow up.

Contents

1. Introduction

The purpose of this article is to prove a structure theorem for pseudo-Anosov flows restricted to Seifert fibered pieces that are called free. This is part of a very broad program to classify all pseudo-Anosov flows in 3-manifolds.

A pseudo-Anosov flow is a flow without stationary orbits that is roughly transversely hyperbolic. This means it has stable and unstable 2-dimensional foliations, that may have singularities along finitely many closed orbits. The singularities are of p-prong type.

These were introduced by Anosov [An] who studied amongst other things geodesic flows in the unit tangent bundle of negatively curved closed manifolds. These flows are now called Anosov flows, and they are smooth.

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In dimension 3 Thurston [Th1, Th2] introduced suspension pseudo-Anosov flows, that are suspensions of pseudo-Anosov homeomorphisms of surfaces. These were used in an essential way to prove the hyperbolization theorem of atoroidal 3-manifolds that fiber over the circle. Mosher [Mo1, Mo2] then generalized this to define general pseudo-Anosov flows in 3-manifolds, that is, flows that are locally like the suspension flows of pseudo-Anosov homeomorphisms. Pseudo-Anosov flows are extremely common due to works of Thurston [Th1, Th3, Th4], Gabai and Mosher [Mo3], Calegari [Cal1, Cal2] and the second author [Fe1, Fe4]. More recently Béguin, Bonatti, and Yu [BBY] introduced extremely general new constructions of Anosov flows in 3-manifolds. These greatly expanded the class of Anosov flows in 3-manifolds. In addition, pseudo-Anosov flows have been used to analyze and understand the topology and geometry of 3-manifolds [Cal1, Fe3, Fe4, Ga-Ka1, Ga-Ka2].

Pseudo-Anosov flows are strongly connected with the topology of 3-manifolds. The existence of a pseudo-Anosov flow implies that the manifold is *irreducible* [Fe3], that is, every embedded sphere bounds a ball [He]. The other important property in 3-manifolds concerns π_1-injective tori. If a closed manifold admits this, one says that the manifold is *toroidal*, otherwise it is called *atoroidal*. The existence of a pseudo-Anosov flow does not imply that the manifold is atoroidal $-$ the simplest counterexample is the unit tangent bundle of a closed surface of negative curvature, in which the geodesic flow is Anosov. But in fact pseudo-Anosov flows in toroidal 3-manifolds are extremely common, see for example [Fr-Wi, Ha-Th, Ba3, Ba-Fe1, BBY]. One very important problem, that will not be addressed in this article is to determine exactly which 3-manifolds admit pseudo-Anosov flows.

The goal of this article is to advance the understanding of pseudo-Anosov flows in relation with the topology of the 3-manifold. A compact, irreducible 3-manifold has a canonical decomposition into Seifert fibered and atoroidal pieces. This is the JSJ decomposition of the manifold [Ja-Sh, Jo]. *Seifert fibered* means that it has a one dimensional foliation by circles. We want to understand how a pseudo-Anosov flow interacts with this decomposition. The atoroidal case is by far the most mysterious and unknown and will not be addressed in this article. We will consider Seifert fibered pieces. Here a lot is already known. First, in a seminal work, Ghys [Gh] proved that an Anosov flow in an $\mathbb{S}^1$ bundle is orbitally equivalent to a finite cover of a geodesic flow. *Orbitally equivalent* means that there is a homeomorphism that sends orbits to orbits. Without explicitly defining it, Ghys introduced the notion of an $\mathbb{R}$-covered Anosov flow: this means that (say) the stable foliation is $\mathbb{R}$-covered, that is, when lifted to the universal cover, it is a foliation with leaf space homeomorphic to the real numbers $\mathbb{R}$. Later the first author [Ba1] extended this result to any Seifert fibered space.

Previously the first author also started the analysis of the structure of an Anosov flow restricted to a Seifert fibered space. He proved [Ba3] that if the flow is $\mathbb{R}$-covered, then the flow restricted to the Seifert piece in an appropriate manner is orbitally equivalent to a finite cover of a geodesic flow of a compact hyperbolic surface with boundary. This is called a *geodesic flow piece*. He also started the study of more general Anosov flows restricted to Seifert fibered pieces. We emphasize that even for Anosov flows, the $\mathbb{R}$-covered property is extremely restrictive, and it does not allow blow ups.

In this article we consider the much more general case of pseudo-Anosov flows and study the relationship with Seifert fibered pieces. Let P be such a piece of the torus decomposition of the manifold M. The fundamental dichotomy here is the following: we say that the piece is *periodic* if there is a Seifert fibration of P so that a regular fiber of the fibration is up to finite powers freely homotopic to a periodic orbit of the flow. Otherwise the piece is called *free*. Geodesic flows have only one piece and it is free. The Handel-Thurston examples are $\mathbb{R}$-covered with two Seifert pieces, both of which are free. The Bonatti-Langevin examples [Bo-La] have periodic Seifert pieces. We previously constructed a very large class of examples in graph manifolds, so that all pieces are periodic [Ba-Fe1], and in [Ba-Fe2], we proved that every pseudo Anosov flow on a graph manifold such that all Seifert pieces are periodic is orbitally equivalent to one of the examples constructed in [Ba-Fe1].

The structure of a pseudo-Anosov flow in a periodic piece is fairly simple and was completely determined in [Ba-Fe1]. A *Birkhoff annulus* is an annulus tangent to the flow in the boundary and transverse to the flow in the interior. For a periodic piece there is a 2-dimensional spine that is a union of a mostly embedded finite collection of Birkhoff annuli. An arbitrarily small neighborhood of this spine is a representative for the Seifert fibered piece. In that way the dynamics of the flow in the piece is extremely simple: there are finitely many closed orbits entirely contained in the piece, their local stable and unstable manifolds and every other orbit piece enters and exits the piece. In particular there are no full non closed orbits contained in the piece.

As we mentioned above there are non-trivial free Seifert pieces that are orbitally equivalent to a geodesic flow piece. In those cases it follows that in the piece the dynamics is extremely rich: there are countably many periodic orbits contained in the piece and uncountably many non periodic full orbits entirely contained in the piece. Given the results mentioned above the natural conjecture is that in a free Seifert piece the flow is orbitally equivalent to a geodesic flow piece. However, the situation is not nearly so simple. We briefly explain one example to illustrate what could happen.

Béguin, Bonatti and Yu introduced new, powerful and vast constructions of Anosov flows. Start with a geodesic flow in a closed hyperbolic surface and consider an orbit that projects to a simple, separating geodesic. The unit tangent bundles M_1, M_2 of the complements S_1, S_2 of the geodesic in the surface are Seifert fibered manifolds. Blow up the orbit using a DA operation, derived from Anosov, and remove a torus neighborhood of this orbit to create a manifold with boundary and an incoming semi-flow in it. Roughly the DA operation transforms a hyperbolic orbit into (in this case) a repelling orbit. This can be achieved by keeping the expansion along the unstable directions and adding an expansion stronger than the contraction in the stable direction. The local unstable leaf splits into two leaves (or one if the leaf is a Möbius band) with a solid torus in between. See the details in [Fr-Wi, BBY] or in section 8. Take the manifold with the boundary a torus and an incoming semiflow. Glue a copy of this manifold with a semiflow which is a time reversal of the flow, so the flow is outgoing from the other manifold. The results of Béguin, Bonatti and Yu [BBY] show that this flow is Anosov. The Seifert manifolds M_1, M_2 survive in the final manifold and are Seifert fibered pieces of the JSJ decomposition of this manifold. They are also free Seifert pieces − see detailed explanation in section 8. But because of the blow up operation, the final flow restricted to M_1 or M_2 cannot be orbitally equivalent to a geodesic flow piece. For example consider M_1: the only possible geodesic flow here would be the geodesic flow of S_1. The problem is that we blew up one orbit corresponding to a boundary geodesic of S_1. In terms of the stable foliation, this means that the stable leaf of this geodesic is blown into an interval of leaves. This behavior is exactly what necessitates the hyperbolic blow up operation in the statement of the Main theorem.

A *hyperbolic blow up* of a geodesic flow piece is obtained by essentially blowing up a geodesic flow piece as above. We explain more in the Sketch of the proof subsection below. The primary goal of this article is to prove the following:

Main theorem − Let (M, Φ) be a pseudo-Anosov flow. Let P be a free Seifert piece in M. Assume that P is not elementary, i.e. that $\pi_1(P)$ does not contain a free abelian group of finite index. Then, in the intermediate cover M_P associated to $\pi_1(P)$ there is a compact submanifold $\hat{P}$ bounded by embedded Birkhoff tori, such that the restriction of the lifted flow $\hat{\Phi}$ to $\hat{P}$ is orbitally equivalent to a hyperbolic blow up of a geodesic flow. This orbital equivalence preserves the restrictions of the weak stable and unstable foliations. Moreover, $\hat{P}$ is almost unique up to isotopy along the lifted flow $\hat{\Phi}$: if $\hat{P}'$ is another compact submanifold bounded by embedded Birkhoff tori, and if $\hat{P}_*$, $\hat{P}'_*$ are the complements in $\hat{P}$, $\hat{P}'$ of the (finitely many) periodic orbits contained in $\partial\hat{P}$, there is a continuous map $t : \hat{P}_* \to \mathbb{R}$ such that the map from $\hat{P}_*$ into M_P mapping x on $\hat{\Phi}^{t(x)}(x)$ is a homeomorphism, with image $\hat{P}'_*$.

See section 7 and Theorem 7.1 for a more detailed statement. The isotopy along the flow does not extend in general to the tangent periodic orbits (see Remark 2.15). A Birkhoff torus is one that is a union of Birkhoff annuli. We need the cover M_P because in M the projection of the embedded Birkhoff tori in M_P may have tangent orbits that collapse together. This is quite common. To get $\hat{P}$ embedded and the structure theorem above, we need to lift to the cover M_P.

The main theorem substantially adds to the understanding of the relationship of pseudo-Anosov flows with the topology of the manifold. We emphasize that to show that the structure of periodic Seifert pieces is given by a spine of Birkhoff annuli is relatively simple given the understanding of the topological structure of the stable and unstable foliations in the universal cover and the analysis of periodic orbits and lozenges (see Background section). Unlike the case of periodic Seifert pieces, the proof of the Main theorem about free Seifert pieces is quite complex and involves several new objects or constructions.

To put the Main theorem in perspective notice that pseudo-Anosov flows are extremely common in 3-manifolds. We already remarked the very general recent constructions of Beguin-Bonatti-Yu of Anosov flows [BBY]. Very roughly they consider "blocks" with smooth semiflows where the non wandering set is hyperbolic and the boundary is a union of transverse tori. Under extremely general conditions they can glue these manifolds to produce Anosov flows. In addition pseudo-Anosov flows are extremely common because of Dehn surgery: they generate pseudo-Anosov flows in the surgered manifolds for the vast majority of Dehn surgeries on a closed orbit of an initial pseudo-Anosov flow. Most of the time the resulting flow is truly pseudo-Anosov, meaning it has p-prong singular orbits. Given this enormous flexibility it is quite remarkable that the structure of a pseudo-Anosov flow in a Seifert fibered piece P is either described by a finite union of Birkhoff annuli (when P is periodic), or by the Main theorem (if P is free). In the course of the proof of the Main theorem, we will prove that if P is free, there is a representative W for P, bounded by Birkhoff annuli, so that it does not have p-prong singularities in the interior. This is in contrast with the case that P is periodic, and this also highlights the remarkable fact that possible singularities do not essentially affect the structure of the pseudo-Anosov flow in a free Seifert piece. The Main theorem implies that a great part of the enormous flexibility of pseudo-Anosov flows resides in the cases that either M is hyperbolic, or in the atoroidal pieces of the JSJ decomposition of M. The study of these is still in its infancy.

Sketch of the proof the main theorem

In order to prove the main theorem we will use a very important result that under very general circumstances a flow is determined up to orbital equivalence by its action on the orbit space [Hae]. The orbit space is the quotient space of the flow in the universal cover. In our situation the orbit space of the flow or a subset of the flow will always be a subset of the plane and it will have induced stable and unstable (possibly singular) foliations.

We introduce a new model to describe geodesic flows in compact hyperbolic surfaces, usually with boundary, that involves the projectivized tangent bundle of an associated orbit space. The fundamental group of the manifold acts in a properly discontinuous cocompact way providing a new model for geodesic flows.

We now consider blow ups. Instead of doing the blow up of the geodesic flow on the 3-manifold level we will do it on the level of actions of the fundamental group on the circle or the line as follows. For the geodesic flow, the stable/unstable foliations are $\mathbb{R}$-covered. The quotient of the stable or unstable leaf space (homeomorphic to $\mathbb{R}$) by the representative h of the regular fiber is a circle $\mathbb{S}^1$ and the orbifold quotient fundamental group $\pi_1(P)/<h>$ acts on these circles. These are convergence group actions. When we lift the flow to a finite cover, the associated actions are not convergence group actions, but are what we call k-convergence group actions. If the unrolling of the fiber direction has order k, then an element of $\pi_1(P)/<h>$ with fixed points in $\mathbb{S}^1$ has $2k$ fixed points that are alternatively attracting and repelling. Recall that for a convergence group action in $\mathbb{S}^1$ we can only have 2 fixed points. We prove a k-convergence group action theorem, extending the convergence group theorem. This is fairly straightforward. Then we define almost k-convergence groups, allowing modifications of the actions in some periodic intervals.

To prove our theorem we will only allow hyperbolic blow ups, where the modifications in the intervals introduce an arbitrary finite number of points that are attracting or repelling. The reason is that the stable and unstable foliations of pseudo-Anosov flows have attracting or repelling holonomy along periodic orbits. We show that such an action is semiconjugate to a k-convergence action.

We build model flows via such actions. We start with two hyperbolic blow ups of Fuchsian actions. Here a Fuchsian action is one associated with a finite cover of the geodesic flow. Using a combination of the two actions we then construct an "orbit" space which is a subset of the plane, and then a model flow with a compact "core" associated with this blow up. In particular the hyperbolic blow up is a new technique to construct flows that are later proved to be a part of pseudo-Anosov flows.

Now consider an <u>arbitrary</u> pseudo-Anosov Φ, and P a free Seifert piece of Φ. Then the fiber h in $\pi_1(P)$ acts on the leaf space of the stable/unstable foliations freely, generating two axes $\mathcal{A}^s, \mathcal{A}^u$ that are homeomorphic to the reals. The fundamental group of the piece $\pi_1(P)$ acts on $\mathcal{A}^s, \mathcal{A}^u$ and their quotients by h, producing two actions on the circle $\mathbb{S}^1$. After a lot of work we show that these two actions are hyperbolic blow ups of Fuchsian actions. Using the hyperbolic blow ups of the two Fuchsian actions, we construct the associated model flow as described above. Finally we show that in the cover M_P and in the carefully defined subset $\hat{P}$ of M_P, the restriction of the pseudo-Anosov flow Φ is orbitally equivalent to the model flow we constructed, finishing the proof of the main theorem.

The reader may feel at first glance that our construction of a hyperbolic blow up through actions on the line is unnecessarily sophisticated, and may consider that a definition involving DA operations in dimension three as explained in this introduction would have been more natural. But if one uses the DA operations, then it is for instance extremely difficult to analyze the flow or establish the structure of the flow up to orbital equivalence. In other words, we have to prove that <u>every</u> free piece of an arbitrary pseudo-Anosov flow has the structure we are proposing and that is very difficult if one considers just DA operations on the flow level. Our approach makes a much more direct connection between the construction of examples and the proof that every free Seifert piece has this form. In the last section of this article (called Examples) we provide several ways to exhibit free Seifert pieces of pseudo-Anosov flows with relatively simple constructions.

What was previously known

As we mentioned before, the first author [Ba3] proved a similar result in the case that P is a free Seifert piece of an $\mathbb{R}$-covered Anosov flow. In [Ba3] the first author also started the study of free Seifert pieces of general Anosov flows. In particular in this article we use some of the constructions and proofs of [Ba3] or [Ba-Fe1]. However, even in the case of $\mathbb{R}$-covered Anosov flows, our Main Theorem is a refinement of the main result in [Ba3]: there, the fact that the orbital equivalence preserves also the weak stable and unstable foliations was established only outside the periodic orbits tangent to the boundary. Also, as we explained before, in the case of $\mathbb{R}$-covered Anosov flows there are no blow ups and the analysis is much simpler.

The general strategy of the proof here is quite different. The use of almost k-convergence groups is completely new. The hyperbolic blow ups of actions is also completely new. The alternative model of the geodesic flow is also new. We expect that these objects introduced here will be useful in other contexts.

The future

The results of this article can be used to understand/classify pseudo-Anosov flows. For example suppose that Φ is a pseudo-Anosov flow in M that is a *graph manifold*. This means that all pieces of the JSJ decomposition are Seifert fibered. This is a large and very important class of 3-manifolds. The results of [Ba-Fe1] and of this article show that one can understand the flow Φ up to orbital equivalence in carefully defined neighborhoods of each Seifert piece P of M. There are infinitely many different gluing maps, but the next goal is to show that up to orbit equivalence there are only finitely many pseudo-Anosov flows in a fixed graph manifold. This would be a substantial result.

2. Background

***p*-prong**

Let $\mathbb{R}^2_{1/2}$ be the quotient of $\mathbb{R}^2$ by $-id$, and let $\beta : \mathbb{R}^2_p \to \mathbb{R}^2_{1/2}$ be the finite p-covering over $\mathbb{R}^2_{1/2}$ branched over the origin 0 : in complex coordinates, one can simply define β as the map $z \mapsto z^{p/2}$. Let $\tau : \mathbb{R}^2_p \to \mathbb{R}^2_p$ be the rotation by $2\pi/p$: it is a generator of the Galois group of β. We denote by $\mathfrak{P}^s_0$, $\mathfrak{P}^u_0$ the preimage in $\mathbb{R}^2_p$ of the "vertical line" and the "horizontal line" through 0, respectively. Let λ be a real number (non necessarily positive) of modulus > 1. Let $f_\lambda : \mathbb{R}^2_p \to \mathbb{R}^2_p$ be the only lift of the linear map $(x, y) \mapsto (\lambda x, \lambda^{-1} y)$ such that:
– (if $\lambda > 1$) f_λ preserves every component of $\mathfrak{P}^s_0 - 0$,
– (if $\lambda < -1$) f_λ maps every component of $\mathfrak{P}^s_0 - 0$ onto its image by the rotation by π/p.

Definition 2.1. *(local model near a p-prong periodic orbit) For any integer $0 \leq k < p$, the composition $\tau^k \circ f_\lambda$ is the model p-prong map of index k and parameter λ. We denote it by $f_{\lambda,k}$. The suspension of $f_{\lambda,k}$, i.e. the quotient $M_{\lambda,k}$ of $\mathbb{R}^2_p \times \mathbb{R}$ by the transformation $(z, t) \mapsto (f_{\lambda,k}(z), t - 1)$ equipped with the projection of the horizontal vector field $\frac{\partial}{\partial t}$, is the model p-prong vector field of index k. The periodic orbit, projection of the line $\{z = 0\}$, is the model p-prong periodic orbit of index k. The projections of $\mathfrak{P}^s_0 \times \mathbb{R}$, $\mathfrak{P}^u_0 \times \mathbb{R}$ are respectively denoted by Λ^s_0, Λ^u_0, and called the stable (unstable) leaf of the model periodic orbit.*

Observe that, up to topological conjugacy, $f_{\lambda,k}$ does not depend on λ, just on its sign. Similarly, a model p-prong only depends, up to orbital equivalence, on p, the index k, and the sign of λ.

Pseudo-Anosov flows – definitions

Definition 2.2. *(pseudo-Anosov flow) Let Φ be a flow on a closed 3-manifold M. We say that Φ is a pseudo-Anosov flow if the following conditions are satisfied:*
- For each $x \in M$, the flow line $t \to \Phi(x, t)$ is C^1, it is not a single point, and the tangent vector bundle $D_t\Phi$ is C^0 in M.
- There are two (possibly) singular transverse foliations Λ^s, Λ^u which are two dimensional, with leaves saturated by the flow and so that Λ^s, Λ^u intersect exactly along the flow lines of Φ.
- There is a finite number (possibly zero) of periodic orbits, called singular orbits such that in the neighborhood of each of them the flow is locally orbit equivalent to a model p-prong flow as defined in definition 2.1, with $p \geq 3$.
- In a stable leaf all orbits are forward asymptotic, in an unstable leaf all orbits are backwards asymptotic.

Basic references for pseudo-Anosov flows are [Mo1, Mo2] and [An] for Anosov flows. A fundamental Remark is that the ambient manifold supporting a pseudo-Anosov flow is necessarily irreducible - the universal covering is homeomorphic to $\mathbb{R}^3$ ([Fe-Mo]). We stress that in our definition one prongs are not allowed. Even if they will not appear in the present paper, we mention that there are however "tranversely hyperbolic" flows with one prongs:

Definition 2.3. *(one prong pseudo-Anosov flows) A flow Φ is a one prong pseudo-Anosov flow in M^3 if it satisfies all the conditions of the definition of pseudo-Anosov flows except that the p-prong singularities can also be 1-prong ($p = 1$).*

Torus decomposition

Let M be an irreducible closed 3–manifold. If M is orientable, it has a unique (up to isotopy) minimal collection of disjointly embedded incompressible tori such that each component of M obtained by cutting along the tori is either atoroidal or Seifert-fibered [Ja, Ja-Sh] and the pieces are isotopically maximal with this property. If M is not orientable, a similar conclusion holds; the decomposition has to be performed along tori, but also along some incompressible embedded Klein bottles.

Hence the notion of maximal Seifert pieces in M is well-defined up to isotopy. If M admits a pseudo-Anosov flow, we say that a Seifert piece P is *periodic* if there is a Seifert fibration on P for which, up to finite powers, a regular fiber is freely homotopic to a periodic orbit of Φ. If not, the piece is called *free.*

Remark. In a few circumstances, the Seifert fibration is not unique: it happens for example when P is homeomorphic to a twisted line bundle over the Klein bottle or P is $T^2 \times I$. We stress out that our convention is to say that the Seifert piece is free if no Seifert fibration in P has fibers homotopic to a periodic orbit.

Birkhoff annuli

Definition 2.4. *(Birkhoff annulus) Let A be an immersed annulus in M. We say that A is a Birkhoff annulus if the boundary ∂A is a union of closed orbits of Φ and the interior of A is transverse to Φ.*

Observe that Birkhoff annuli are essentially topological objects: we may require differentiability at the (transverse) interior, but not at the tangent periodic orbits. Since the interior of A is transverse to Φ, it is transverse to the stable and unstable foliations, which then induce one dimensional foliations in A - each boundary component of A is a leaf of each of these foliations. We say that the Birkhoff annulus A is *elementary* if each of these foliations does not have a closed leaf in the interior of A. We stress that A need not be embedded in M. The easiest non embedded example occurs for geodesic flows in the unit tangent bundle $M = T^1 S$ where S is a hyperbolic surface. If γ is a non embedded closed geodesic in S, consider the annulus A obtained by turning the angles along γ from 0 to π. Since γ is not embedded, this generates a Birkhoff annulus that is not homotopic rel boundary to an embedded one.

Orbit space and leaf spaces of pseudo-Anosov flows

Notation/definition: We denote by $\pi : \widetilde{M} \to M$ the universal covering of M, and by $\pi_1(M)$ the fundamental group of M, considered as the group of deck transformations on $\widetilde{M}$. The singular foliations lifted to $\widetilde{M}$ are denoted by $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$. If $x \in M$ let $W^s(x)$ denote the leaf of Λ^s containing x. Similarly one defines $W^u(x)$ and in the universal cover $\widetilde{W}^s(x), \widetilde{W}^u(x)$. Similarly if $\tilde{\theta}$ is an orbit of Φ define $W^s(\tilde{\theta})$, etc... Let also $\widetilde{\Phi}$ be the lifted flow to $\widetilde{M}$.

We review the results about the topology of $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ that we will need. We refer to [Fe2, Fe3] for detailed definitions, explanations and proofs. The orbit space of $\widetilde{\Phi}$ in $\widetilde{M}$ is homeomorphic to the plane $\mathbb{R}^2$ [Fe-Mo] and is denoted by $\mathcal{O} \cong \widetilde{M}/\widetilde{\Phi}$. There is an induced action of $\pi_1(M)$ on $\mathcal{O}$. Let

$$\Theta : \ \widetilde{M} \ \rightarrow \ \mathcal{O} \ \cong \ \mathbb{R}^2$$

be the projection map: it is naturally $\pi_1(M)$-equivariant. If L is a leaf of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$, then $\Theta(L) \subset \mathcal{O}$ is a tree which is either homeomorphic to $\mathbb{R}$ if L is regular, or is a union of p-rays all with the same starting point if L has a singular p-prong orbit. The foliations $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ induce $\pi_1(M)$-invariant singular 1-dimensional foliations $\mathcal{O}^s, \mathcal{O}^u$ in $\mathcal{O}$. Its leaves are $\Theta(L)$ as above. If L is a leaf of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$, then a *sector* is a component of $\widetilde{M} - L$. Similarly for $\mathcal{O}^s, \mathcal{O}^u$. If B is any subset of $\mathcal{O}$, we denote by $B \times \mathbb{R}$ the set $\Theta^{-1}(B)$. The same notation $B \times \mathbb{R}$ will be used for any subset B of $\widetilde{M}$: it will just be the union of all flow lines through points of B. We stress that for pseudo-Anosov flows there are at least 3-prongs in any singular orbit ($p \geq 3$). For example, the fact that the orbit space in $\widetilde{M}$ is a 2-manifold is not true in general if one allows 1-prongs.

Definition 2.5. *Let L be a leaf of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$. A slice of L is $l \times \mathbb{R}$ where l is a properly embedded copy of the reals in $\Theta(L)$. For instance if L is regular then L is its only slice. If a slice is the boundary of a sector of L then it is called a line leaf of L. If a is a ray in $\Theta(L)$ then $A = a \times \mathbb{R}$ is called a half leaf of L. If ζ is an open segment in $\Theta(L)$ it defines a* flow band *L_1 of L by $L_1 = \zeta \times \mathbb{R}$. We use the same terminology of slices and line leaves for the foliations $\mathcal{O}^s, \mathcal{O}^u$ of $\mathcal{O}$.*

If $F \in \widetilde{\Lambda}^s$ and $G \in \widetilde{\Lambda}^u$ then F and G intersect in at most one orbit.

We abuse convention and call a leaf L of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$ *periodic* if there is a non-trivial covering translation γ of $\widetilde{M}$ with $\gamma(L) = L$. This is equivalent to $\pi(L)$ containing a periodic orbit of Φ. In the same way an orbit $\tilde{\theta}$ of $\widetilde{\Phi}$ is *periodic* if $\pi(\tilde{\theta})$ is a periodic orbit of Φ. Observe that in general, the stabilizer of an element $\tilde{\theta}$ of $\mathcal{O}$ is either trivial, or a cyclic subgroup of $\pi_1(M)$.

Leaf spaces of $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$

Let $\mathcal{H}^s, \mathcal{H}^u$ be the leaf spaces of $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ respectively, with the respective quotient topology. These are the same as the leaf spaces of $\mathcal{O}^s, \mathcal{O}^u$ respectively. The spaces $\mathcal{H}^s, \mathcal{H}^u$ are inherently one dimensional and they are simply connected. It is essential that there are no 1-prongs. For simplicity consider $\mathcal{H}^s$. Through each point passes either a germ of an interval, if the stable leaf is non singular; or a p-prong, if the stable leaf has a p-prong singular orbit. In this way the space $\mathcal{H}^s$ is "treelike". In addition it may not be Hausdorff, and this is extremely common, even for Anosov flows [Fe2, Ba-Fe1]. These spaces are what is called a non Hausdorff tree. This was defined in

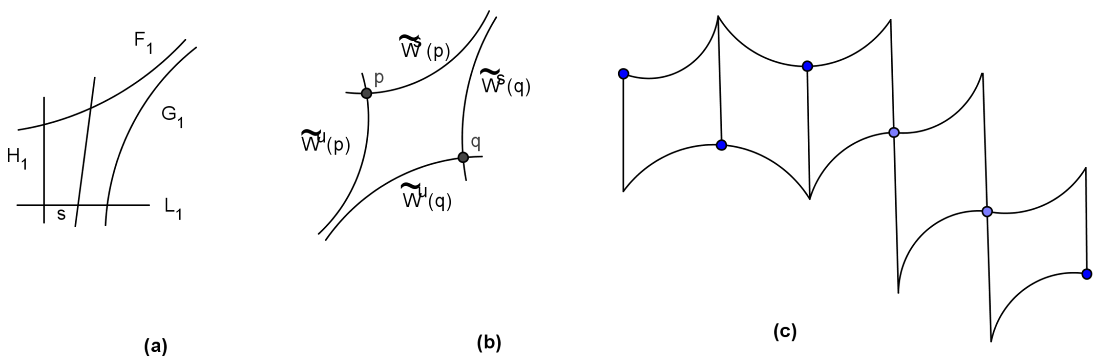

Figure 1: *a. Perfect fits in $\widetilde{M}$, b. A lozenge, c. A chain of lozenges.*

[Fe5, Ro-St]. In those articles, group actions on non Hausdorff trees are carefully studied, in particular the case of free actions. Even more general than non Hausdorff trees is the concept of "order trees" introduced by Gabai and Kazez [Ga-Ka2], where the local model is any space with a total order.

Product regions

Suppose that a leaf $F \in \widetilde{\Lambda}^s$ intersects two leaves $G, H \in \widetilde{\Lambda}^u$ and so does $L \in \widetilde{\Lambda}^s$. Then F, L, G, H form a *rectangle* in $\widetilde{M}$, ie. every stable leaf between F and L intersects every unstable leaf between G and H. In particular, there is no singularity in the interior of the rectangle [Fe3].

Definition 2.6. *Suppose A is a flow band in a leaf of $\widetilde{\Lambda}^s$. Suppose that for each orbit $\tilde{\theta}$ of $\widetilde{\Phi}$ in A there is a half leaf $B_{\tilde{\theta}}$ of $\widetilde{W}^u(\tilde{\theta})$ defined by $\tilde{\theta}$ so that: for any two orbits $\tilde{\theta}', \tilde{\theta}''$ in A then a stable leaf intersects $B_{\tilde{\theta}'}$ if and only if it intersects $B_{\tilde{\theta}''}$. This defines a stable product region which is the union of the $B_{\tilde{\theta}}$. Similarly define unstable product regions.*

The main property of product regions is the following: for any product region P, and for any $F \in \widetilde{\Lambda}^s$, $G \in \widetilde{\Lambda}^u$ so that (i) $F \cap P \neq \emptyset$ and (ii) $G \cap P \neq \emptyset$, then $F \cap G \neq \emptyset$. There are no singular orbits of $\widetilde{\Phi}$ in P.

Theorem 2.7. *(*[Fe3]*) Let Φ be a pseudo-Anosov flow. Suppose that there is a stable or unstable product region. Then Φ is topologically equivalent to a suspension Anosov flow. In particular Φ is non singular.*

We will occasionally use *product pseudo-Anosov flow* as an abbreviation for *pseudo-Anosov flow topologically equivalent to a suspension.*

Perfect fits, lozenges and scalloped chains

Recall that a foliation $\mathcal{F}$ in M is $\mathbb{R}$-covered if the leaf space of $\widetilde{\mathcal{F}}$ in $\widetilde{M}$ is homeomorphic to the real line $\mathbb{R}$ [Fe1].

Definition 2.8. *(*[Fe2, Fe3]*) Perfect fits - Two leaves $F \in \widetilde{\Lambda}^s$ and $G \in \widetilde{\Lambda}^u$, form a perfect fit if $F \cap G = \emptyset$ and there are half leaves F_1 of F and G_1 of G and also flow bands $L_1 \subset L \in \widetilde{\Lambda}^s$ and $H_1 \subset H \in \widetilde{\Lambda}^u$, so that the set*

$$\overline{F}_1 \cup \overline{H}_1 \cup \overline{L}_1 \cup \overline{G}_1$$

separates M and forms an a rectangle R with a corner removed: The joint structure of $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ in R is that of a rectangle with a corner orbit removed. The removed corner corresponds to the perfect of F and G which do not intersect.

We refer to fig. 1, a for perfect fits. There is a product structure in the interior of R: there are two stable boundary sides and two unstable boundary sides in R. An unstable leaf intersects one stable boundary side (not in the corner) if and only if it intersects the other stable boundary side (not in the corner). We also say that the leaves F, G are *asymptotic*.

Definition 2.9. *(*[Fe2, Fe3]*) Lozenges - A lozenge R is a region of $\widetilde{M}$ whose closure is homeomorphic to a rectangle with two corners removed. More specifically two points p, q define the corners of a lozenge if there are half leaves A, B of $\widetilde{W}^s(p), \widetilde{W}^u(p)$ defined by p and C, D half leaves of $\widetilde{W}^s(q), \widetilde{W}^u(q)$ defined by p, q, so that A and D form a perfect fit and so do B and C. The region bounded by the lozenge R does not have any singularities. The sides of R are A, B, C, D. The sides are not contained in the lozenge, but are in the boundary of the lozenge. There may be singularities in the boundary of the lozenge. See fig. 1, b.*

There are no singularities in the lozenges, which implies that R is an open region in $\widetilde{M}$.

Two lozenges are *adjacent* if they share a corner and there is a stable or unstable leaf intersecting both of them, see fig. 1, c. Therefore they share a side. A *chain of lozenges* is a collection $\mathcal{C} = \{C_i\}, i \in I$, where I is an interval (finite or not) in $\mathbb{Z}$; so that if $i, i+1 \in I$, then C_i and C_{i+1} share a corner, see fig. 1, c. Consecutive lozenges may be adjacent or not. The chain is finite if I is finite.

Definition 2.10. *(scalloped chain) Let $\mathcal{C}$ be a chain of lozenges. If any two successive lozenges in the chain are adjacent along one of their unstable sides (respectively stable sides), then the chain is called* s-scalloped *(respectively* u-scalloped*) (see fig. 2 for an example of a s-scalloped chain). Observe that a chain is s-scalloped if and only if there is a stable leaf intersecting all the lozenges in the chain. Similarly, a chain is u-scalloped if and only if there is an unstable leaf intersecting all the lozenges in the chain. The chains may be infinite. A scalloped chain is a chain that is either s-scalloped or u-scalloped.*

For simplicity when considering scalloped chains we also include any half leaf which is a boundary side of two of the lozenges in the chain. The union of these is called a *scalloped region* which is then a connected set.

We say that two orbits $\tilde{\theta}_1, \tilde{\theta}_2$ of $\widetilde{\Phi}$ (or the leaves $\widetilde{W}^s(\tilde{\theta}_1), \widetilde{W}^s(\tilde{\theta}_2)$) are connected by a chain of lozenges $\{C_i\}, 1 \leq i \leq n$, if $\tilde{\theta}_1$ is a corner of C_1 and $\tilde{\theta}_2$ is a corner of C_n.

Theorem 2.11. *(*[Fe2, Fe3]*) Let Φ be a pseudo-Anosov flow in M^3 closed and let $F_0 \neq F_1 \in \widetilde{\Lambda}^s$. Suppose that there is a non-trivial covering translation γ with $\gamma(F_i) = F_i, i = 0, 1$. Let $\tilde{\theta}_i, i = 0, 1$ be the periodic orbits of $\widetilde{\Phi}$ in F_i so that $\gamma(\tilde{\theta}_i) = \tilde{\theta}_i$. Then $\tilde{\theta}_0$ and $\tilde{\theta}_1$ are connected by a finite chain of lozenges $\{C_i\}, 1 \leq i \leq n$, and γ leaves invariant each lozenge C_i as well as their corners.*

In addition we always assume without mention that the chain is minimal: this means that there is no back-tracking and no three lozenges share a corner. That could happen if C_i, C_{i+1} share (say) a stable side (they are adjacent) and C_{i+1}, C_{i+2} share an unstable side (also adjacent), with all three sharing a corner. In this case we eliminate C_{i+1} from the chain. We could have a more complicated behavior if the corner is a singular orbit with many lozenges abutting it.

The main result concerning non Hausdorff behavior in the leaf spaces of $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ is the following:

Theorem 2.12. [Fe2, Fe3] *Let Φ be a pseudo-Anosov flow in M^3. Suppose that $F \neq L$ are not separated in the leaf space of $\widetilde{\Lambda}^s$. Then F is periodic and so is L. More precisely, there is a non-trivial element γ of $\pi_1(M)$ such that $\gamma(F) = F$ and $\gamma(L) = L$. Moreover, let $\tilde{\theta}_1$, $\tilde{\theta}_2$ be the unique γ-fixed points in F, L, respectively. Then, the chain of lozenges connecting $\tilde{\theta}_1$ to $\tilde{\theta}_2$ is s-scalloped.*

Remark 2.13. A key fact, first observed in [Ba3], and extensively used in [Ba-Fe1], is the following: the lifts in $\widetilde{M}$ of elementary Birkhoff annuli are precisely lozenges invariant by some cyclic subgroup of $\pi_1(M)$ (see [Ba3, Proposition 5.1] for the case of embedded Birkhoff annuli). It will also play a crucial role in the sequel. More precisely: let A be an elementary Birkhoff annulus. We say that A lifts to the lozenge C in $\widetilde{M}$ if the interior of A has a lift which intersects orbits only in C. It follows that this lift intersects every orbit in C exactly once and also that the two boundary closed orbits of A lift to the full corner orbits of C. This uses the fact that a lozenge cannot be properly contained in another lozenge.

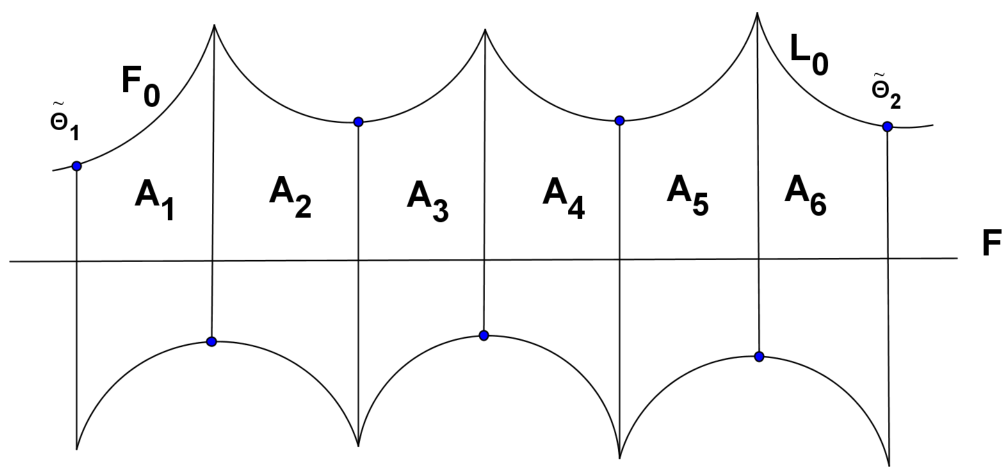

Figure 2: *A partial view of a scalloped region. Here F, F_0, L_0 are stable leaves, so this is a s-scalloped region.*

In particular the following important property also follows: if θ_1 and θ_2 are the periodic orbits in ∂A (traversed in the flow forward direction), then there are positive integers n, m so that θ_1^n is freely homotopic to $(\theta_2^m)^{-1}$. We emphasize the free homotopy between inverses.

Remark 2.14. According to Remark 2.13, chains of lozenges correspond to sequences of Birkhoff annuli, every Birkhoff annulus sharing a common periodic orbit with the previous element of the sequence, and also a periodic orbit with the next element in the sequence. When the sequence closes up, it provides an immersion $f : T^2$ (or K, in this article K denotes the Klein bottle) $\to M$, which is called a *Birkhoff torus* (if the cyclic sequence contains an even number of Birkhoff annuli), or a *Birkhoff Klein bottle* (in the other case).

Remark 2.15. – Birhoff annuli associated to the same lozenges. Let A_1, A_1' be two elementary Birkhoff annuli whose lifts in $\widetilde{M}$ project to the same lozenge. Then, every orbit intersecting the interior of A_1 intersects A_1' : there is a continuous map t from the interior of A_1 into $\mathbb{R}$ such that the application f_1 mapping every x in the interior of A_1 into $\Phi^{t(x)}(x)$ realizes a homeomorphism between A_1 and A_1'.

However, in general, the map t does not extend continuously on the boundary orbits of A_1. In particular it may be that the continuous function $t(x)$ is unbounded and accumulates to $\pm\infty$ as x approaches the boundary of the annulus along certain paths or sequences of points. This can be explicitly and easily constructed for any Birkhoff annulus: more specifically for any elementary Birkhoff annulus A there is a Birkhoff annulus A' so that A, A' have lifts to the same lozenge in $\widetilde{M}$ and so that the map t from the interior of A to the interior of A' does not extend continuously to the boundary, meaning for example that $t(x)$ converges to plus infinity as one approaches the boundary of A.

Furthermore, even if f_1 is defined on ∂A_1, and if A_2 and A_2' are other Birkhoff annuli, one isotopic to the other along the flow, and adjacent to A_1, A_1', there is no reason for the map $f_2 : A_2 \to A_2'$ to coincide with f_1 on the common boundary orbit. The point is that we construct Birkhoff annuli from lozenges in $\widetilde{M}$ that are invariant by some non trivial g in $\pi_1(M)$. In particular these Birkhoff annuli are inherently only topological objects, non differentiable, and can have a very wild behavior near the tangent periodic orbit. Consequently Birkhoff tori as defined in the previous remark may be homotopic along the flow only on the complement of the tangent periodic orbits.

Nevertheless, the following fact is true for embedded Birkhoff tori (the similar statement also holds for embedded Birkhoff Klein bottles and is left to the reader). Let (M, Φ), (N, Ψ) be two pseudo-Anosov flows, and T, T' embedded Birkhoff tori in respectively M, N. Let $\mathcal{C}$, $\mathcal{C}'$ be the associated chain of lozenges: they are preserved by subgroups H, H' of $\pi_1(M)$, $\pi_1(N)$, both isomorphic to $\mathbb{Z}^2$, and corresponding to the fundamental groups of T, T'. Assume that there is an equivariant (with respect to H and H') map $f : \mathcal{C} \to \mathcal{C}'$. Then, we can replace f by another map (still denoted by f) with the same properties but furthermore preserving the foliations on the lozenges induced by the stable/unstable foliations. It induces a continuous map F between T_{tr} and T'_{tr}, where T_{tr} and T'_{tr} are the complements in the tori of the tangent periodic orbits. Furthermore, F can be chosen so that it maps the foliations induced on T_{tr} by the stable/unstable foliations $\Lambda^{s,u}(\Phi)$ onto the foliations induced on T'_{tr} by the stable/unstable foliations $\Lambda^{s,u}(\Psi)$. This map F may not extend to a continuous map on the entire T, but it induces a bijection φ between the tangent periodic orbits of T and the tangent periodic orbits of T'. Assume moreover that every periodic orbit θ tangent to T has the same number of prongs and the same index than $\varphi(\theta)$. Then, there are neighborhoods U_θ, U'_θ of respectively θ, $\varphi(\theta)$ and an orbital equivalence $f_\theta : U_\theta \to U'_\theta$ between the restriction of Φ to U_θ and the restriction of Ψ to U'_θ such that this orbital equivalence maps the restrictions of $\Lambda^{s,u}(\Phi)$ to U_θ onto the restrictions of $\Lambda^{s,u}(\Phi)$ to U'_θ.

Claim: There are tubular neighborhoods W and W' of respectively T, T' such that the restriction of Φ to W is orbitally equivalent to the restriction of Ψ to W' (but this orbital equivalence does not necessarily maps T onto T'). Moreover, this orbital equivalence maps the restriction to W of the stable/unstable foliations $\Lambda^{s,u}(\Phi)$ onto the restriction to W' of $\Lambda^{s,u}(\Psi)$.

The proof of the claim is as follows: for every Birkhoff annulus A of T, let A' be an open relatively compact sub-annulus of A, and let U_A, U'_A be the open neighborhoods of A', $F(A')$ made of points of the form $\Phi^t(x)$ (respectively $\Psi^t(x)$) for t small and x in A' or $F(A')$. Then there is an orbital equivalence $f_A : U_A \to U'_A$ between the restrictions of Φ and Ψ to U_A and U'_A, which preserves the restrictions of the stable/unstable foliations. We can adjust U_θ so that:

– the union of all U_A and U_θ covers T, and the union of all U'_A and U'_θ covers T',

– for different periodic orbits θ_1 and θ_2 the neighborhoods U_{θ_1} and U_{θ_2} are disjoint,

– U_θ intersects U_A if and only if θ is a boundary component of A,

– the intersection with U_θ of any orbit ν of the restriction of Φ to U_A is either empty, or a connected relatively compact subset of ν.

The last condition means that for any point x in $U_A \cap U_\theta$, the orbit of x under Φ escapes in the past and in the future from U_θ still staying in U_A.

Then, the union of the domains U_θ and U_A, and the union of U'_A and U'_θ, are open neighborhoods W, W' of respectively T, T'. Introduce a partition of unity subordinate to the covering formed by U_θ and U_A: these are functions $\mu_A : W \to [0,1]$ and $\mu_\theta : W \to [0,1]$ such that:

– μ_A vanishes outside U_A,

– μ_θ vanishes outside U_θ,

– for every x in W, the sum of all $\mu_A(x)$ and $\mu_\theta(x)$ is 1 (observe that all the terms of this sum vanish except maybe at most two of them).

Finally, we equip W and W' with Riemannian metrics, that we use to reparametrize the orbits of Φ and Ψ by unit lentgh, so that these local flows are now defined over the entire $\mathbb{R}$.

Then, for any x in W, we define $g(x)$ as the average along the Ψ-orbit of $f_\theta(x)$ with weight $\mu_\theta(x)$ and $f_A(x)$ with weight $\mu_A(x)$ (observe that if, for example, x does not lie in $U_A(x)$, the affected weight $\mu_A(x)$ vanishes. Hence

– $g(x)$ is either $f_\theta(x)$ if x lies in U_θ but $\mu_A(x) = 0$ for any Birkhoff annulus A,

– $g(x)$ is $f_A(x)$ is x lies in U_A but no U_θ, and

– $g(x)$ is an average as defined if x belongs to an intersection $U_A \cap U_\theta$).

This defines a map $g : W \to W'$ which has all the required properties of the orbital equivalence we seek for, except that it may fail to be injective along orbits of Φ. But since we required that orbits of points in $U_A \cap U_\theta$ escape from U_θ by staying in U_A, it is easy to see that there is $t_0 > 0$ such that for any x in W we have $g(\Phi^{t_0}(x)) \neq g(x)$. It then follows that, by a classical procedure of averaging along orbits (explained for example in the proof of Proposition 3.25 in [Ba3]), we can modify f so that it is an orbital equivalence between the restrictions of Φ and Ψ (but W' may have changed). This proves the claim.

We emphasize that in general we cannot choose g that takes T to T' with the properties above.

3. **Actions on the circle**

In all this section, $\bar{\Gamma}$ will be a finitely generated group. We consider representations $\bar{\rho} : \bar{\Gamma} \to \mathrm{Homeo}(\mathbb{S}^1)$, that we will assume most of the time to be non-elementary (i.e. with no finite orbit). Then, there is a unique minimal invariant closed subset μ [He-Hi]. We will also assume that the action of $\bar{\Gamma}$ restricted to μ *almost commutes* with a homeomorphism $\tau : \mu \to \mu$, of finite order k, which preserves the cyclic order induced by the cyclic order on the circle. More precisely, we will require that:

– for every $\bar{\gamma}$ in $\bar{\Gamma}$, we have, on μ, $\bar{\rho}(\bar{\gamma}) \circ \tau = \tau \circ \bar{\rho}(\bar{\gamma})$ if $\bar{\rho}(\bar{\gamma})$ preserves the orientation, and $\bar{\rho}(\bar{\gamma}) \circ \tau = \tau^{-1} \circ \bar{\rho}(\bar{\gamma})$ if $\bar{\rho}(\bar{\gamma})$ reverses the orientation;

– for every x in μ, the (small) arcs $[x, \tau(x)]$ have the following properties:

- $\mathbb{S}^1 = [x, \tau(x)] \cup [\tau(x), \tau^2(x)] \cup ... \cup [\tau^{k-1}(x), \tau^k(x) = x]$
- the open arcs $]\tau^i(x), \tau^{i+1}(x)[$ for $i = 0, 1, ..., k-1$ are pairwise disjoint.

In other words, τ looks like the rotation by $1/k$, but is only defined on μ.

The small arcs $[x, \tau(x)]$ are uniquely defined unless $k = 2$. If $k = 1$, then μ is the identity and $]x, \tau(x)[= \mathbb{S}^1 - \{x\}$. When $k = 2$, choose an orientation in $\mathbb{S}^1$ and choose $[x, \tau(x)]$ to be the arc from x to $\tau(x)$ in this orientation. In this case $\mu = \mu^{-1}$. For all other k the small arc $[x, \tau(x)]$ is uniquely defined by the second condition on disjointness of the open arcs.

By $x < y < \tau(x)$ we mean that y is in $[x, \tau(x)]$ and is not one of the extremities, that is, y is in $]x, \tau(x)[$.

We will also assume that $\bar{\rho}$ is *μ-faithful*, meaning that the restriction of $\bar{\Gamma}$ to μ is faithful.

A *gap* is a connected component of $\mathbb{S}^1 - \mu$. For every gap I and for every element γ of $\bar{\Gamma}$, we have the dichotomy:

- $\gamma I \cap I = \emptyset$
- $\gamma I = I$

If I is disjoint from all its iterates γI for $\gamma \neq 1$, it is called a *wandering gap*. If not, I is called a *periodic gap*. The group $\bar{\Gamma}$ acts by permutations on the set $\mathcal{I}$ of gaps of μ. We will denote by $\sigma(\bar{\gamma})$ the permutation on $\mathcal{I}$ induced by $\bar{\rho}(\bar{\gamma})$. It commutes with the action induced by τ if $\bar{\rho}(\bar{\gamma})$ is orientation preserving. Observe that $\bar{\rho}$ is μ-faithful if and only if the morphism $\sigma : \bar{\Gamma} \to S(\mathcal{I})$ is injective, where $S(\mathcal{I})$ denotes the permutation group of gaps of μ.

Definition 3.1. *Let μ be a closed perfect subset of $\mathbb{S}^1$, $\tau : \mu \to \mu$ a fixed point free homeomorphism of order k that preserves cyclic order, and $\sigma : \bar{\Gamma} \to S(\mathcal{I})$ a morphism, where $\mathcal{I}$ is the set of gaps of μ. A (μ, τ, σ)-representation is a representation $\bar{\rho} : \bar{\Gamma} \to Homeo(\mathbb{S}^1)$ such that:*

– μ is the unique minimal invariant set in $\mathbb{S}^1$,

– τ almost commutes with the restriction of the action to μ,

– the induced action on the set $\mathcal{I}$ of gaps is σ.

3.1. **Orbifold groups.**

In this paper, we mainly focus on the case of orbifold groups. More precisely, let Γ denote the fundamental group of the Seifert fibered space P with boundary. The boundary components of P are tori. When the base orbifold B is orientable, Γ is generated by elements:

$$h, a_1, b_1, ..., a_g, b_g, d_1, ..., d_p, c_1, ..., c_q$$

satisfying the relations:

$$a_i h = h^\epsilon a_i, b_i h = h^\epsilon b_i, d_i h = h d_i, c_i h = h c_i, d_j^{\alpha_j} h^{\beta_j} = 1, h^e c_1 ... c_q = [a_1, b_1] ... [a_g, b_g] d_1 ... d_p$$

The integer g is the genus of B, the number ϵ is ± 1, according to the fiber-orientability of the Seifert bundle along the appropriate curve in the base orbifold B. Every d_j corresponds to a singularity of B (of type (α_j, β_j)) and every c_i corresponds to a boundary component. Since P has boundary it follows that $q \geq 1$.

When B is non-orientable, Γ is generated by elements

$$h, a_1, ..., a_g, d_1, ..., d_p, c_1, ..., c_q$$

satisfying the relations:

$$a_i h = h^{\epsilon_i} a_i, d_i h = h d_i, c_i h = h c_i, d_j^{\alpha_j} h^{\beta_j} = 1, h^e c_1 ... c_q = a_1^2 ... a_g^2 d_1 ... d_p$$

Here g is the number of crosscaps needed to generate B.

In either case, it follows that the fundamental group $\bar{\Gamma}$ of the base orbifold B, quotient of Γ by the cyclic subgroup H generated by h, has the following presentation:

$$\langle a_1, b_1, ..., a_g, b_g, d_1, ..., d_p, c_1, ..., c_q \ | \ d_j^{\alpha_j} = 1, \ c_1 ... c_q = [a_1, b_1] ... [a_g, b_g] d_1 ... d_p \rangle \text{ (when } B \text{ is orientable)} \quad (1)$$

$$\langle a_1, ..., a_g, d_1, ..., d_p, c_1, ..., c_q \ | \ d_j^{\alpha_j} = 1, \ c_1 ... c_q = a_1^2 ... a_g^2 d_1 ... d_p \rangle \text{ (when } B \text{ is not orientable)} \quad (2)$$

We call $\bar{\Gamma}$ an *orbifold group.*

We also need the following definition:

Definition 3.2. *Let $\tilde{\mu}$ be a closed, perfect subset of $\mathbb{R}$, $\tilde{\tau}:\tilde{\mu}\to\tilde{\mu}$ a fixed point free homeomorphism order preserving, and $\tilde{\sigma}:G\to S(\widetilde{\mathcal{I}})$ a morphism, where $\widetilde{\mathcal{I}}$ is the set of gaps of $\tilde{\mu}$. A $(\tilde{\mu},\tilde{\tau},\tilde{\sigma})$-representation is a representation $\rho:G\to Homeo(\mathbb{R})$ such that:*
– $\tilde{\mu}$ is the unique minimal invariant set in $\mathbb{R}$,
– $\tilde{\tau}$ almost commutes with the restriction of the action to $\tilde{\mu}$,
– the induced action on the set $\widetilde{\mathcal{I}}$ of gaps is $\tilde{\sigma}$.

3.2. **Modifying the action in a periodic gap.** From now on, $\bar{\Gamma}$ will be assumed to be an orbifold group. In this section, we explain how it is possible to modify a (μ,τ,σ)-representation of $\bar{\Gamma}$ to another (μ,τ,σ)-representation which essentially only differs on a periodic gap, and with any new prescribed action on this periodic gap. More precisely:

Proposition 3.3. *Let $\bar{\Gamma}$ be an orbifold group. Let $\bar{\rho}:\bar{\Gamma}\to Homeo(\mathbb{S}^1)$ be a (μ,τ,σ)-representation, and let I_0 be a periodic gap of $\bar{\rho}(\bar{\Gamma})$. Suppose that the stabilizer $Stab(I_0)$ of I_0 is generated by one of the generators c_i of $\bar{\Gamma}$ as described in the previous section. Denote by $\mathfrak{J}$ the union of all the iterates of I_0 by $\sigma(\bar{\Gamma})$.*

Let f_0 be any homeomorphism of I_0, coinciding with c_i on ∂I_0. Then, there is a new (μ,τ,σ)-representation $\bar{\rho}':\bar{\Gamma}\to Homeo(\mathbb{S}^1)$ such that:
– the action on the complement $\mathbb{S}^1-\mathfrak{J}$ is not modified i.e. coincides with the action induced by $\bar{\rho}$,
– the restriction of $\bar{\rho}'(c_i)$ on I_0 coincides with f_0.

We call such a representation ***a modification of $\bar{\rho}$ on the gap I_0 by f_0.***

Furthermore, if $\bar{\rho}$ and f_0 are C^k, and f_0 coincides with $\bar{\rho}(c_i)$ near ∂I_0, then the new representation $\bar{\rho}'$ is also C^k, and for every γ in $\bar{\Gamma}$ and every $r\leq k$, the r-derivatives of $\bar{\rho}(\gamma)$ and $\bar{\rho}'(\gamma)$ coincide on μ.

Remark 3.4. Suppose that g_0 is a homeomorphism of I_0 that is topologically conjugate to f_0 by some homeomorphism $\varphi_0:I_0\to I_0$ preserving the orientation. Let $\bar{\rho}'_1$ (respectively $\bar{\rho}'_2$) be a modification of $\bar{\rho}$ on the gap I_0 by f_0 (respectively by g_0). Then, one can extend φ_0 to a homeomorphism $\varphi:\bar{\rho}(\bar{\Gamma})(I_0)\to\bar{\rho}(\bar{\Gamma})(I_0)$ by $\bar{\rho}(\bar{\Gamma})$-equivariance, and thereafter to the entire circle simply by requiring to be the identity map on $\mathbb{S}^1-\mathfrak{J}$. This provides a topological conjugacy between $\bar{\rho}'_1$ and $\bar{\rho}'_2$.

In other words, modifications on I_0 are well defined up to topological conjugacy by the choice of I_0 and the topological conjugacy class of f_0.

Remark 3.5. More generally, let g_0 be a homeomorphism of I_0 which is topologically semi-conjugate to f_0. This means that there is a continuous, surjective map $\varphi_0:I_0\to I_0$ that is weakly monotone and preserves the orientation, and such that on I_0, we have:

$$g_0\circ\varphi_0=\varphi_0\circ f_0$$

Then, one can show as above that, given any pair $\bar{\rho}'_1$ and $\bar{\rho}'_2$ of modifications of $\bar{\rho}$ on the gap I_0 given respectively by f_0 and g_0, there is an extension of φ_0 on the entire circle defining a semiconjugacy between $\bar{\rho}'_1$ and $\bar{\rho}'_2$.

Proof of Proposition 3.3. The final statement in Proposition 3.3, i.e. the fact that if $\bar{\rho}$ and f_0 are C^k, and that if f_0 coincides with $\bar{\rho}(c_i)$ near ∂I_0, then the new action is still C^k, follows easily in every case considered in the proof below.

We can assume without loss of generality that c_i is c_1. Let us first consider the case $q\geq 2$: the orbifold group is then a free product of the cyclic subgroup generated by c_1 and a subgroup $\bar{\Gamma}_1$, where $\bar{\Gamma}_1$, in the orientable case, is the subgroup generated by a_1, b_1, ... , a_g, b_g, d_1, ... , d_p, c_3, ... , c_q, and in the non-orientable case, the subgroup generated by a_1, ... , a_g, d_1, ... , d_p, c_3, ... , c_q (observe that c_1 and c_2 have been removed). In other words we remove c_2 from the generators of $\bar{\Gamma}$ and c_2 can be recovered from the last relation in either (1) or (2). When that is done, the only relations that need to be satisfied for $\bar{\rho}(\bar{\Gamma})$ to be a representation are $(\bar{\rho}(d_j))^{\alpha_j}=1, 1\leq j\leq p$ (notice that p may be 0 in which case there are no relations at all).

The modification $\bar{\rho}'$ is then simply defined as the unique representation such that:
– $\bar{\rho}'$ and $\bar{\rho}$ coincide on $\bar{\Gamma}_1$,
– $\bar{\rho}'(c_1)$ is the map coinciding with f_0 on I_0, and equal to $\bar{\rho}(c_1)$ everywhere else.

Then the action of $\bar{\rho}(\bar{\Gamma})$ on the complement of $\mathfrak{J}$ is clearly not modified. In addition the action of the subgroup $\bar{\Gamma}_1$ is also not modified, so the defining relations are still satisfied. Therefore $\bar{\rho}'_1$ is still a representation and it is easy to see that the other properties of the modification are satisfied. The proposition follows in this case.

From now on we assume $q=1$, i.e. the orbifold B has exactly one boundary component.

Consider the following generating set $\mathfrak{G}$ for $\bar{\Gamma}$:

– when B is orientable we put $\mathfrak{S} = \{a_1^{\pm 1}, b_1^{\pm 1}, ..., a_g^{\pm 1}, b_g^{\pm 1}, d_1^{\pm 1}, ..., d_p^{\pm 1}\}$. Then:

$$c_1 = [a_1, b_1]...[a_g, b_g]d_1...d_p \quad (*)$$

– when B is not orientable, we define $\mathfrak{S} = \{a_1^{\pm 1}, ..., a_g^{\pm 1}, d_1^{\pm 1}, ..., d_p^{\pm 1}\}$. We have:

$$c_1 = a_1^2...a_g^2 d_1...d_p \quad (**)$$

In both cases, the only defining relations for $(\bar{\Gamma}, \mathfrak{S})$ are $d_i^{\alpha_i} = 1$. In other words $\bar{\Gamma}$ is a free product of two groups. One group is a free group generated by either $a_1, b_1, ..., a_g, ..., b_g$ in the orientable case or $a_1, ..., a_g$ in the non orientable case. The other group is freely generated by the torsion elements $d_1, ..., d_p$.

In order to treat simultaneously the orientable and non-orientable cases, let us write $w_0 = s_\ell ... s_1$ for the word $[a_1, b_1]...[a_g, b_g]d_1...d_p$ in the orientable case, and $a_1^2...a_g^2 d_1...d_p$ in the non-orientable case. The crucial observation is that w_0 is the unique word with letters in $\mathfrak{S}$ representing c_1 and of minimal length. This is because $\bar{\Gamma}$ is a free product of two groups as above and in either case c_1 is represented by the formulas $(*)$ or $(**)$ above. Furthermore, since w_0 is cyclically reduced, it can also be easily checked that for every integer r, any word with letters in $\mathfrak{S}$ representing a conjugate of c_1^r has word-length $\geq r\ell$.

We define by induction, for every integer $i \geq 1$:

$$I_i = \sigma(s_i) I_{i-1}$$

Claim: for $0 \leq i < j \leq \ell - 1$, we have $I_i \neq I_j$.

Indeed, if not, we would have an element $s_j ... s_{i+1}$ of length $< \ell$ whose image by $\bar{\rho}$ maps I_i onto itself. Therefore $s_j ... s_{i+1}$ is a power of a conjugate of c_1. As we have just observed, this is possible only if $s_j ... s_{i+1}$ represents the trivial element, but then we could write c_1 as a product of $\ell - (j - i)$ generators; contradiction.

Let us first consider the case $g > 0$: the last letter s_ℓ is then a_1. In this case, our new action $\bar{\rho}'$ is obtained by applying essentially one and only one modification to the generators: we only modify the restriction of $\bar{\rho}(s_\ell)$ to the arc $I_{\ell-1}$ (and hence $\bar{\rho}(a_1^{-1})$ on I_0). More precisely, for every element s of $\mathfrak{S}$ except for $a_1 = s_\ell$ and a_1^{-1}, we define $\bar{\rho}'(s) = \bar{\rho}(s)$. We define $\bar{\rho}'(a_1)$ as follows: outside $I_{\ell-1}$ we put $\bar{\rho}'(a_1)(x) = \bar{\rho}(a_1)(x)$, and on $I_{\ell-1}$ we put $\bar{\rho}'(a_1)(x) = \bar{\rho}'(s_\ell)(x) = f_0 \circ \bar{\rho}(s_{\ell-1}...s_1)^{-1}(x)$. By the claim no $I_i = I_{l-1}$ or I_0 if $i < l - 1$, therefore $\bar{\rho}'(a_1)$ is well defined. Finally we define $\bar{\rho}'(a_1^{-1})$ to be the inverse of $\bar{\rho}'(a_1)$.

Since the only relations in $\bar{\Gamma}$ are $d_i^{\alpha_i} = 1$, and we have not changed the representation on d_i, these prescriptions of $\bar{\rho}'$ on $\mathfrak{S}$ define a representation $\bar{\rho}' : \bar{\Gamma} \to \mathrm{Homeo}(\mathbb{S}^1)$. It follows directly from our construction that $\bar{\rho}$ and $\bar{\rho}'$ coincide outside $\mathfrak{J}$. Let us check the last statement to be proved, i.e. that the restriction of $\bar{\rho}'(c_1)$ to I_0 is f_0. In the orientable case we have:

$$\bar{\rho}'(c_1) = \bar{\rho}'(a_1)\bar{\rho}'(b_1)\bar{\rho}'(a_1)^{-1}...\bar{\rho}'(d_p)$$

Let x be in I_0. The point $\bar{\rho}'(b_1)^{-1}...\bar{\rho}'(d_p)(x)$ is equal to $\bar{\rho}(b_1)^{-1}...\bar{\rho}(d_p)(x)$ since $\bar{\rho}$ and $\bar{\rho}'$ coincide on elements of $\mathfrak{S} - \{a_1, a_1^{-1}\}$. This point belongs to $I_{\ell-3}$. According to the Claim above, $I_{\ell-3}$ is different from I_0, hence by our construction, $\bar{\rho}(a_1^{-1})$ and $\bar{\rho}'(a_1^{-1})$ coincide on $I_{\ell-3}$: we have $\bar{\rho}'(a_1^{-1})...\bar{\rho}'(d_p)(x) = \bar{\rho}(a_1^{-1})...\bar{\rho}(d_p)(x)$. The equality $\bar{\rho}'(b_1)\bar{\rho}'(a_1^{-1})...\bar{\rho}'(d_p)(x) = \bar{\rho}(b_1)\bar{\rho}(a_1^{-1})...\bar{\rho}(d_p)(x)$ follows. Now $\bar{\rho}'(c_1)(x)$ is the image under $\bar{\rho}'(a_1) = \bar{\rho}'(s_\ell)$ of $\bar{\rho}(b_1)\bar{\rho}(a_1)^{-1}...\bar{\rho}(d_p)(x)$. In addition this is equal to $\bar{\rho}(s_{l-1}...s_1)(x)$. But we have defined the restriction of $\bar{\rho}'(s_\ell)(x)$ to be $f_0 \circ \bar{\rho}(s_{\ell-1}...s_1)^{-1}$. The equality $\bar{\rho}'(c_1)(x) = f_0(x)$ follows.

The non-orientable case is treated in a similar way. We have:

$$\bar{\rho}'(c_1) = \bar{\rho}'(a_1)^2\bar{\rho}'(a_2)^2...\bar{\rho}'(d_p)$$

because $\bar{\rho}'$ is a representation. For x in I_0, we still have the equality $\bar{\rho}'(a_2)^2...\bar{\rho}'(d_p)(x) = \bar{\rho}(a_2)^2...\bar{\rho}(d_p)(x)$, one checks that this point lies in the region where $\bar{\rho}'(a_1)$ and $\bar{\rho}(a_1)$ coincide because I_{l-2} is not I_{l-1}. The next occurence of $\bar{\rho}'(a_1)$ has been designed so that it leads to the desired equality $\bar{\rho}'(c_1)(x) = f_0(x)$.

The last case to consider is the case $g = 0$, $q = 1$, ie. the case where the orbifold B is a disk with a finite number ≥ 2 of singular points. This is because if B is non orientable then $g \geq 1$ as at least one crosscap is needed to produce B. Then $\mathfrak{S}$ is the collection $\{d_1^{\pm 1}, ..., d_p^{\pm 1}\}$ of finite order elements. In this case we have $w_0 = d_1...d_p$. We essentially do the same procedure as in the previous case: the first idea is to define $\bar{\rho}'(d_i^{\pm 1}) = \bar{\rho}(d_i^{\pm 1})$ for every $i \geq 2$. This implies that $\bar{\rho}'(d_i^{\pm 1}) = (\bar{\rho}(d_i))^{\pm 1}$ and in addition $(\bar{\rho}'(d_i))^{\alpha_i} = 1$, for all $i \geq 2$. Then we first define $\bar{\rho}'(d_1)(x) = \bar{\rho}(d_1)(x)$ everywhere except on the interval $I_{\ell-1} = d_1^{-1}(I_0)$ where we put $\bar{\rho}'(d_1)(x) = f_0(\bar{\rho}(d_p)^{-1}...\bar{\rho}(d_2)^{-1}(x))$. Notice it does **not** matter if we put $\bar{\rho}$ or $\bar{\rho}'$ here as they are equal on $d_i, i \geq 2$. In this way the equality $\bar{\rho}'(c_1)(x) = f_0(x)$ automatically holds for $x \in I_0$. The problem is that after this change, the relation $\bar{\rho}'(d_1)^{\alpha_1} = 1$ is not satisfied. This can be fixed by changing $\bar{\rho}(d_1)$ in another interval as well in the following way: Notice that $\alpha_1 \geq 2$. In addition $I_{l-1} = \bar{\rho}(d_1^{\alpha_1 - 1})(I_0)$ as $d_1^{\alpha_1} = 1$. We want to define $\bar{\rho}'(d_1)$ on $(\bar{\rho}(d_1))^{-1}(I_{l-1})$ so that $(\bar{\rho}'(d_1))^{\alpha_1} = 1$. The only modification done so far has been on $\bar{\rho}'(d_1)$ in I_{l-1}. The concern is that $\bar{\rho}(d_1^m(I_0)) = I_{l-1}$ for some $m < \alpha_1 - 1$. But if that happens, d_1^{m+1} is in $Stab(I_0)$ and is not trivial in $\bar{\Gamma}$ because $m+1 < \alpha_1$. By hypothesis on the proposition this is not possible, because $Stab(I_0)$ is generated by c_1 and no nontrivial element in the torsion subgroup of d_1 is in the subgroup generated by c_1. Therefore we can now define $\bar{\rho}'(d_1)$ on $\bar{\rho}(d_1^{-1})(I_{l-1})$ (which would be equal to I_0 if $\alpha_1 = 2$) so that $\bar{\rho}'(d_1^{\alpha_1}) = 1$. This finishes the proof of Proposition 3.3.

□

The representations $\bar{\rho} : \bar{\Gamma} \to \mathrm{Homeo}(\mathbb{S}^1)$ we will consider are always coming from a representation $\rho : \Gamma \to \mathrm{Homeo}(\mathbb{R})$ such that $\rho(h)$ is the translation by $+1$, i.e. the Galois covering for the covering map $\mathbb{R} \to \mathbb{S}^1$. The homeomorphism τ is then the projection of a homeomorphism $\tilde{\tau} : \tilde{\mu} \to \tilde{\mu}$, where $\tilde{\mu}$ is the unique minimal closed invariant subset of the action of $\rho(\Gamma)$, and we can assume wlog that $\tilde{\tau}$ is the restriction of the translation by $1/k$.

Observe that the construction in Proposition 3.3 does not affect this property: this construction lifts to a modification $\rho^* : \Gamma \to \mathrm{Homeo}(\mathbb{R})$, and the lifting is uniquely characterized by the property that the restrictions of $\rho(\Gamma)$ and $\rho^*(\Gamma)$ to $\tilde{\mu}$ coincide.

In the same way as in Proposition 3.3 we define modifications of $(\tilde{\mu}, \tilde{\tau}, \tilde{\sigma})$:

Definition 3.6. *Let Γ be the fundamental group of a Seifert manifold. Let $\rho : \Gamma \to Homeo(\mathbb{R})$ be a $(\tilde{\mu}, \tilde{\tau}, \tilde{\sigma})$-representation, and let I_0 be a periodic gap of $\tilde{\mu}$, with stabilizer generated by some element c_i of Γ. Denote by $\mathfrak{I}$ the union of all the iterates of I_0 by $\sigma(\Gamma)$.*

Let f_0 be any homeomorphism of I_0, coinciding with c_i on ∂I_0. Then, there is a new $(\tilde{\mu}, \tilde{\tau}, \tilde{\sigma})$-representation $\rho' : \Gamma \to Homeo(\mathbb{R})$ such that:

– the action on the complement $\mathbb{R} - \mathfrak{I}$ is not modified i.e. coincides with the action induced by ρ,

– the restriction of $\rho'(c_i)$ on I_0 coincides with f_0.

*We call such a representation **a modification of ρ on the gap I_0 by f_0**.*

Furthermore, if ρ and f_0 are C^k, and f_0 coincides with $\rho(c_i)$ near ∂I_0, then the new representation ρ' is also C^k, and for every γ in Γ and every $r \leq k$, the r-derivatives of $\rho(\gamma)$ and $\rho'(\gamma)$ coincide on μ.

3.3. **Groups of almost-(k)-convergence.**

Definition 3.7. *A (μ, τ, σ)-representation $\bar{\rho} : \bar{\Gamma} \to Homeo(\mathbb{S}^1)$ has the **(discrete) (k)-convergence property** if, for every sequence $(\bar{\gamma}_n)_{n\in\mathbb{N}}$, up to a subsequence the following dichotomy holds:*

– either the sequence $(\bar{\rho}(\bar{\gamma}_n))_{n\in\mathbb{N}}$ is stationary,

– or there exist two τ-orbits $\{x_0^- = \tau(x_{k-1}^-), x_1^- = \tau(x_0^-), \ldots, x_{k-1}^- = \tau(x_{k-2}^-)\}$ and $\{x_0^+ = \tau(x_{k-1}^+), x_1^+ = \tau(x_0^+), \ldots, x_{k-1}^+ = \tau(x_{k-2}^+)\}$ such that, for any compact subset K of $]x_i^-, x_{i+1}^-[$ (with $0 \leq i < k$) the restriction of $\bar{\rho}(\bar{\gamma}_n)$ to K converges uniformly to x_i^+.

We also say that $(\bar{\gamma}_n)_{n\in\mathbb{N}}$ or $(\bar{\rho}(\bar{\gamma}_n))_{n\in\mathbb{N}}$ satisfies the (k)-convergence property.

Observe that, in particular, if $\bar{\rho} : \bar{\Gamma} \to \mathrm{Homeo}(\mathbb{S}^1)$ has the (k)-convergence property, then the fixed point set of every non-trivial $\bar{\rho}(\gamma)$ is a union of at most 2 orbits by τ, hence contain at most $2k$ elements. In the case $k = 1$ one recovers the usual notion of convergence group.

Typical examples of discrete (k)-convergence groups are Fuchsian groups. More precisely: for every integer $k \geq 1$, let $\mathrm{PGL}_k(2, \mathbb{R})$ denote the groups of projective transformations (orientation preserving or not) of the cyclic k-cover $\mathbb{RP}_k^1$ over the real projective line. It is also the quotient of the universal covering $\widetilde{\mathrm{PGL}}(2, \mathbb{R})$ of $\mathrm{PGL}(2, \mathbb{R})$ by the subgroup of index k of the center of $\mathrm{PGL}(2, \mathbb{R})$. Then, any discrete subgroup of $\mathrm{PGL}_k(2, \mathbb{R})$, as group of transformation of $\mathbb{RP}_k^1 \approx \mathbb{S}^1$, is a discrete (k)-convergence group.

In particular, the definition of $\mathrm{PGL}_k(2, \mathbb{R})$ immediately implies that there is a natural projection $\pi_k : \mathrm{PGL}_k(2, \mathbb{R}) \to \mathrm{PGL}(2, \mathbb{R})$ that is a k-fold covering map and a homomorphism.

Theorem 3.8. *Every (μ, τ, σ)-representation $\bar{\rho} : \bar{\Gamma} \to Homeo(\mathbb{S}^1)$ satisfying the (k)-convergence property is topologically conjugate to a Fuchsian action, i.e. there exists a homeomorphism $f : \mathbb{S}^1 \to \mathbb{RP}_k^1$ and a representation $\bar{\rho}_0 : \bar{\Gamma} \to PGL_k(2, \mathbb{R})$ such that:*

$$f \circ \bar{\rho} = \bar{\rho}_0 \circ f$$

Proof. The case $k = 1$ is simply a reformulation of the *convergence group Theorem* proved by Gabai and Casson-Jungreis ([Ga, Ca-Ju]) culminating a series of works by many others. Actually, the results in [Ga, Ca-Ju] are stated for actions preserving the orientation of $\mathbb{S}^1$, and here we have to take care of the general case allowing orientation reversing elements. The reference [Tuk] actually dealt with this general situation, and proved the conjecture except in the case of orientation preserving actions of a triangular group $< a, b, c \mid a^p = b^q = c^r = 1 >$. The triangular group case was solved thereafter independently by Gabai and Casson-Jungreis. This most difficult case will not be used in this paper, since we will apply this Theorem to an orbifold group of an orbifold admitting at least one boundary component, hence not a triangular group.

Now we give an outline of the way to reduce the general case $k > 1$ to the case $k = 1$ (compare with Lemma 3.6.2 in [Mon]). The idea is to extend τ to a homeomorphism $\tau : \mathbb{S}^1 \to \mathbb{S}^1$, almost commuting with the action, and to apply the convergence group Theorem to the induced action on the quotient circle $\mathbb{S}^1/\tau$.

Let I be a gap of μ. If I is wandering, one can choose arbitrarily any orientation preserving extension of τ inside $\tau^i(I)$ for $0 \leq i \leq k-2$, and then define the restriction of τ inside $\tau^{k-1}(I)$ so that the restriction of τ^k to I is the identity map. Then define the restriction of τ to every $\bar\rho(\bar\gamma)(\tau^i(I))$ as $\bar\rho(\bar\gamma)\tau_{|\tau^i(I)}\bar\rho(\bar\gamma)^{-1}$. Since I is wandering, there is no relation to obey, and we define in this way an extension of τ to the entire circle, except on periodic gaps.

We consider now the case where I is periodic. Let $\bar\Gamma_I$ be the stabilizer of I. Let us denote by a, b the two extremities of I, so that $I =]a, b[$. If some element $\bar\gamma$ of $\bar\Gamma_I$ admits a fixed point x inside I, then it would admit at least $2k+1$ fixed points: the orbits of a, b by τ and x, and it would contradict the (k)-convergence property applied to the sequence $(\bar\gamma^n)_{n\in\mathbb{N}}$. Therefore, the action of $\bar\Gamma_I$ on I is free, and therefore, according to Hölder's Theorem ([Ho]), topologically conjugate to an action by translations (once I is identified with the real line). If $\bar\rho(\bar\Gamma_I)$ is not cyclic, then one would once more obtain a contradiction by considering a sequence $(\bar\gamma_n)_{n\in\mathbb{N}}$ made of distinct elements mapping a point x in I to elements $\bar\rho(\bar\gamma_n)x$ converging to x.

We conclude that $\bar\rho(\bar\Gamma_I)$ is cyclic, generated by an element $\bar\rho(\bar\gamma_0)$, and that its action on I preserves the orientation. Therefore, there is a topological conjugacy between the action of $\bar\rho(\bar\Gamma_I)$ inside I and its action on $\tau(I)$. More precisely, we select a point x_i in every $\tau^i(I)$, and take any orientation preserving homeomorphism τ_i between $[x_i, \bar\rho(\bar\gamma_0)x_i]$ and $[x_{i+1}, \bar\rho(\bar\gamma_0)x_{i+1}]$, only adjusting so that the composition $\tau_{k-1} \circ ... \circ \tau_0$ is trivial on $[x_0, \bar\rho(\bar\gamma_0)x_0]$. We then extend every τ_i on every $\tau^i(I)$ by $\bar\rho(\bar\gamma_0)$-equivariance, and then on every $\bar\rho(\bar\gamma)(\tau^i(I))$ by $\bar\rho(\bar\Gamma)$-equivariance.

All these extensions are compatible with one another, and define an extension of τ to the entire circle, which almost commutes with the action of $\bar\Gamma$. One considers then the action on the circle $\mathbb{S}^1/\tau$, which has the (1)-convergence property, and therefore is topologically conjugate to a projective action. The lift of this topological conjugacy is the required topological conjugacy between $\bar\rho$ and a Fuchsian representation $\bar\rho_0 : \bar\Gamma \to \mathrm{PGL}_k(2, \mathbb{R})$. □

Of course, when one modifies the representation on a periodic gap as in Proposition 3.3, the new representation does not have anymore the (k)-convergence property, since one can increase arbitrarly the number of fixed points for a given element. However, we will see that a weak form of (k)-convergence property still holds:

Definition 3.9. *A (μ, τ, σ)-representation $\bar\rho : \bar\Gamma \to Homeo(\mathbb{S}^1)$ has the* ***almost (k)-convergence property*** *if, for every sequence $(\bar\gamma_n)_{n\in\mathbb{N}}$, up to a subsequence, the following trichotomy holds:*

– either the sequence $(\bar\rho(\bar\gamma_n))_{n\in\mathbb{N}}$ is stationary,

– or there are elements a, $\bar\gamma$ of $\bar\Gamma$ and a sequence $(p_n)_{n\in\mathbb{N}}$ of integers such that $\bar\gamma$ preserves a gap of μ and:

$$\forall n \in \mathbb{N} \;\; \bar\rho(\bar\gamma_n) = \bar\rho(\bar\gamma)^{p_n}\bar\rho(a)$$

– or there exist two τ-orbits $\{x_0^-, x_1^- = \tau(x_0^-), ..., x_{k-1}^- = \tau(x_{k-2}^-)\}$ and $\{x_0^+, x_1^+ = \tau(x_0^+), ..., x_{k-1}^+ = \tau(x_{k-2}^+)\}$ such that, for any compact subset K of $]x_i^-, x_{i+1}^-[$ (with $0 \leq i < k$) the restriction of $\bar\rho(\bar\gamma_n)$ to K converges uniformly to x_i^+.

Theorem 3.10. *Let $\bar\rho : \bar\Gamma \to Homeo(\mathbb{S}^1)$ be a μ-faithful non elementary (μ, τ, σ)-representation of an orbifold group $\bar\Gamma$, and let $\bar\rho' : \bar\Gamma \to Homeo(\mathbb{S}^1)$ be a modification of $\bar\rho$ on a periodic gap. Then, $\bar\rho'$ is a group of almost (k)-convergence if and only if the same is true for $\bar\rho$.*

Proof. Let $(\bar\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $\bar\Gamma$. Up to a subsequence, we can assume that $(\bar\rho(\bar\gamma_n))_{n\in\mathbb{N}}$ is in one the cases prescribed by almost (k)-convergence:

Case 1: the sequence $(\bar\rho(\bar\gamma_n))_{n\in\mathbb{N}}$ is stationary. Then, since $\bar\rho$ is μ-faithful, it follows that the sequence $(\bar\gamma_n)_{n\in\mathbb{N}}$ is stationary, and therefore, that $(\bar\rho'(\bar\gamma_n))_{n\in\mathbb{N}}$ is stationary.

Case 2: there are elements a, $\bar\gamma$ of $\bar\Gamma$ and a sequence $(p_n)_{n\in\mathbb{N}}$ of integers such that $\bar\gamma$ preserves a gap of μ and:

$$\forall n \in \mathbb{N} \;\; \bar\rho(\bar\gamma_n) = \bar\rho(\bar\gamma)^{p_n}\bar\rho(a)$$

Then, as in Case 1, since $\bar\rho$ is μ-faithful, it follows that $\bar\gamma_n = \bar\gamma^{p_n}a$. Applying $\bar\rho'$ shows the same property holds for $\bar\rho'$.

Case 3: there exist two τ-orbits $\{x_0^-, x_1^- = \tau(x_0^-), ..., x_{k-1}^- = \tau(x_{k-2}^-)\}$ and $\{x_0^+, x_1^+ = \tau(x_0^+), ..., x_{k-1}^+ = \tau(x_{k-2}^+)\}$ such that, for any compact subset K of $]x_i^-, x_{i+1}^-[$ (with $0 \leq i < k$) the restriction of $\bar\rho(\bar\gamma_n)$ to K converges uniformly to x_i^+.

In particular this implies that x_i^+ is in μ. In addition, considering the sequence $(\bar\rho(\bar\gamma_n)i)^{-1}$, one sees that x_i^-, x_{i+1}^- are also in μ. In this case, we will show that either the sequence $(\bar\rho'(\bar\gamma_n))_{n\in\mathbb{N}}$ satisfies the same property, or satisfies the property described in case 2.

Assume that for some compact arc $K = [a, b] \subset]x_i^-, x_{i+1}^-[$, the iterates $\bar{\rho}'(\bar{\gamma}_n)K$ do not shrink to the point x_i^+. We can assume wlog that $i = 0$. Let α be the unique element of $\mu \cap [x_0^-, a]$ such that $]\alpha, a]$ is disjoint from μ (if a lies in μ, then we have $\alpha = a$). Consider similarly the unique element β of $\mu \cap [b, x_1^-]$ for which $\mu \cap [b, \beta[= \emptyset$. If $\alpha \neq x_0^-$ and $\beta \neq x_1^-$, then the interval $[\alpha, \beta]$ shrinks under the action of $\bar{\rho}(\bar{\gamma}_n)$ to the point x_0^+. But since α and β are in μ, and since $\bar{\rho}'$ differs from $\bar{\rho}$ only by its action inside gaps, the same property holds for the sequence $\bar{\rho}'(\bar{\gamma}_n)[\alpha, \beta]$. It follows that $K \subset [\alpha, \beta]$ also shrinks to x_0^+ under the action of $\bar{\rho}'(\bar{\gamma}_n)$, contradiction.

Hence, we must have $\alpha = x_0^-$ (or $\beta = x_1^-$, but the treatment of this case is similar, and will not be considered here). In other words, a lies in a gap $]x_0^-, \alpha'[$. If we had also $\beta = x_1^-$, we would also conclude that there is a gap $]\beta', x_1^-[$, but then x_1^- would be at the boundary of two different gaps: $]\beta', x_1^-[$ and also $\tau(]x_0^-, \alpha'[) =]x_1^-, \tau(\alpha')[$. It is impossible since μ is perfect, therefore we have $\beta < x_1^-$. We conclude that the segment $[\alpha', \beta]$ shrinks to x_0^+ under the action of $\bar{\rho}(\bar{\gamma}_n)$, and under the action of $\bar{\rho}'(\bar{\gamma}_n)$ as well since α' and β both lie in μ.

It follows that the iterates $\bar{\rho}'(\bar{\gamma}_n)[a, \alpha']$ do not shrink to a point, but to a segment $[x^+, x_0^+]$ with $x^+ < x_0^+$ (up to a subsequence). Hence the iterates under $\bar{\rho}'(\bar{\gamma}_n)$ of the gap $]x_0^-, \alpha'[$ also do not converge to a point, but to a non-trivial segment. This limit segment I_∞ must be a gap; and since, for every ϵ, there is only a finite number of gaps of length $\geq \epsilon$, it follows that, up to a subsequence, the gaps $\bar{\rho}'(\bar{\gamma}_n)(]x_0^-, \alpha'[) = \sigma(\bar{\gamma}_n)(]x_0^-, \alpha'[) = \bar{\rho}(\bar{\gamma}_n)(]x_0^-, \alpha'[)$ are all equal to I_∞. Hence for every n, $\bar{\gamma}_n\bar{\gamma}_1^{-1}$ is in the stabilizer of I_∞. Since we are in case 3 for $\bar{\rho}(\bar{\gamma}_n)$, we know that the iterates $\bar{\rho}(\bar{\gamma}_n)[a, \alpha']$ shrink to x_0^+, hence the segment $[a, \alpha']$ lies in the domain where we have modified the action. It follows that I_∞ is in the $\sigma(\bar{\Gamma})$-orbit of the periodic gap I_0 where the action has been modified, and therefore, according to the hypothesis of Proposition 3.3, the stabilizer of I_∞ is cyclic, generated by some element $\bar{\gamma}$.

In summary, and denoting $\bar{\gamma}_1$ by a, we have proved that every $\bar{\gamma}_n$ is of the form $\bar{\gamma}^{p_n}a$. We are in case 2, and the Theorem is proved. □

Proposition 3.11. *Let $\bar{\rho} : \bar{\Gamma} \to Homeo(\mathbb{S}^1)$ be a μ-faithful non elementary (μ, τ, σ)-representation satisfying the almost (k)-convergence property. Then, $\bar{\rho}$ has the (k)-convergence property if and only if for every periodic gap I we have:*

- *the action of the stabilizer $\bar{\Gamma}_I$ of I on I is free,*
- *for every non-trivial element $\bar{\gamma}$ of $\bar{\Gamma}_I$ the points in $\mathbb{S}^1$ fixed by $\bar{\gamma}$ are exactly the iterates under τ of the extremities ∂I, and they are all hyperbolic fixed points.*

Proof. One implication is clear: if $\bar{\rho}$ has the (k)-convergence property, then it is topologically conjugate to a Fuchsian action, and the two conditions are necessarily satisfied.

Let now $\bar{\rho} : \bar{\Gamma} \to \text{Homeo}(\mathbb{S}^1)$ be a μ-faithful non elementary (μ, τ, σ)-representation satisfying the almost (k)-convergence property and the two conditions stated in the proposition. Notice that the fixed points of non-trivial $\bar{\rho}(\bar{\gamma})$ are in μ. By the first hypothesis, the action $\bar{\rho}(\bar{\Gamma})$ on $\mathbb{S}^1 - \mu$ is free. Let $(\bar{\gamma}_n)_{n\in\mathbb{N}}$ be a sequence in $\bar{\Gamma}$. Up to a subsequence, $(\bar{\rho}(\bar{\gamma}_n))_{n\in\mathbb{N}}$ is in one the three cases imposed by almost (k)-convergence. It satisfies the condition obeyed by sequences under the (k)-convergence property, except maybe if we are in the case where $\bar{\rho}(\bar{\gamma}_n) = \bar{\rho}(\bar{\gamma})^{p_n}\bar{\rho}(a)$ for some sequence $(p_n)_{n\in\mathbb{N}}$ of integers and for two elements a, $\bar{\gamma}$ of $\bar{\Gamma}$, where $\bar{\gamma}$ preserves a gap I of μ. Then since $\bar{\gamma} \in \bar{\Gamma}_I$, by hypothesis, it admits exactly $2k$ fixed points: the points in the τ-orbit of ∂I; and all these fixed points are hyperbolic. It follows that $\bar{\rho}(\bar{\gamma})$ is topologically conjugate to a projective transformation, hence that $(\bar{\rho}(\bar{\gamma})^{p_n})_{n\in\mathbb{N}}$ satisfies the (k)-convergence property, and therefore the same is true for $(\bar{\gamma}_n)_{n\in\mathbb{N}}$. □

Corollary 3.12. *Any modification of a (k)-convergence group on a periodic gap I by a homeomorphism $f : I \to I$ without fixed points and such that f is hyperbolic near ∂I is a (k)-convergence group.* □

3.4. **A dynamical characterization of (k)-convergence groups.**

Theorem 3.13. *Let $\bar{\rho} : \bar{\Gamma} \to Homeo(\mathbb{S}^1)$ be a (μ, τ, σ)-representation. Let $\bar{\Gamma}_0$ be the index 2 subgroup made of elements preserving the orientation. Assume that $\bar{\rho}$ satisfies the following properties:*

1. *every gap of μ is periodic,*
2. *for every x in $\mathbb{S}^1$, the stabilizer of x is trivial or cyclic,*
3. *for every non-trivial element $\bar{\gamma}$ of $\bar{\Gamma}_0$ the fixed point set of $\bar{\rho}(\bar{\gamma})$ is either trivial, or one orbit of τ, or the union of two orbits by τ, one made of attractive fixed points and the other made of repellent fixed points,*
4. *if (x_0, y_0) is a pair of fixed points of some element of $\bar{\Gamma}$ with $x_0 < y_0 < \tau(x_0)$, then the $\bar{\rho}(\bar{\Gamma}_0)$-orbit by the diagonal action of (x_0, y_0) in the space $U = \{(x, y) \in \mu \times \mu \mid x < y < \tau(x)\}$ is closed and discrete.*

Then, $\bar{\rho}$ has the (k)-convergence property, and thus is topologically conjugate to a Fuchsian representation.

Remark 3.14. Item (4) is coherent: for every non-trivial element $\bar{\gamma}_0$, the fixed points of $\bar{\rho}(\bar{\gamma}_0)$ all lie in μ. Indeed, if not, $\bar{\rho}(\bar{\gamma}_0)$ would admit a fixed point x in a gap $]a, b[$. Then, $\bar{\rho}(\bar{\gamma}_0^2)$ would admit at least $1 + 2k$ fixed points: x, and the orbits of a and b by τ. It contradicts item (3).

Therefore, if (x_0, y_0) is a pair of fixed points of $\bar{\gamma}_0$ with $x_0 < y_0 < \tau(x_0)$, then (x_0, y_0) lies indeed in $U = \{(x, y) \in \mu \times \mu \mid x < y < \tau(x)\}$. *Recall the definition of $x < y < \tau(x)$ in the beginning of this section.*

Sketch of proof. We skip the elementary case where $\bar{\Gamma}$ is a cyclic group. Let $\sim$ be the equivalence relation identifying all points in $\bar{I}$ for every gap I of μ. Then, the quotient space $\mathbb{S} = \mathbb{S}^1/\sim$ is homeomorphic to the circle, and $\bar{\rho}$ induces an action of $\bar{\Gamma}$ on $\mathbb{S}$ by homeomorphisms. After the collapsing the action is minimal. Moreover, τ induces a homeomorphism $\breve{\tau}$ on $\mathbb{S}$ of order k, and the quotient $\breve{\mathbb{S}} = \mathbb{S}/\breve{\tau}$ is a circle too, on which $\bar{\Gamma}$ acts naturally. We focus on the induced action of $\bar{\Gamma}_0$. This action is minimal, and satisfies the same properties with $k = 1$: 1) there is no gap, hence no wandering gap; 2) the stabilizer of any point is trivial or cyclic; 3) every element admits at most 2 fixed points, and if it admits 2, it is of hyperbolic type. Finally, if $(\breve{x}_0, \breve{y}_0)$ with $\breve{x}_0 \neq \breve{y}_0$ is fixed by some element, then its $\bar{\Gamma}_0$-orbit is closed and discrete in $\breve{\mathbb{S}} \times \breve{\mathbb{S}} - \Delta$, where Δ is the diagonal (for details, see the proof of Theorem 2.6 in [Ba3]).

Then, Theorem 2.6 in [Ba3] implies that this quotient action has the convergence property, hence is Fuchsian. Since $\bar{\Gamma}_0$ is finitely generated and the limit set is the entire circle, the quotient hyperbolic surface $\bar{\Gamma}_0\backslash\mathbb{H}^2$ has finite volume. Let now $\bar{\gamma}_0$ be any element of $\bar{\Gamma} - \bar{\Gamma}_0$. Then $\bar{\gamma}_0$ induces a involution on $\bar{\Gamma}_0\backslash\mathbb{H}^2$. It is a easy case for the Nielsen realization problem [Ke]: there exist a hyperbolic metric on the surface so that the involution is an isometry. It follows that the whole action of $\bar{\Gamma}$ on $\breve{\mathbb{S}}$ is Fuchsian.

The initial action $\bar{\rho}$ is obtained by opening some cusps in the associated hyperbolic orbifold, replacing them by funnels, and taking a finite covering. The Theorem follows (cf. in particular Corollary 3.12). □

Remark 3.15. Some of the results in this section (in particular Theorem 3.8, Proposition 3.11 and Theorem 3.13) are related to the results in Mann's article [Mann], and possibly could be implied by Mann's results or techniques if interpreted correctly. Mann's powerful techniques involve carefully analysing rotation numbers of homeomorphisms of the circle in the corresponding group representations.

4. Blowing up pieces of geodesic flows

In this section, we usually let Γ be the fundamental group of a Seifert fibered space P (or P_0) with boundary, and $\bar{\Gamma}$ is the orbifold fundamental group of the base space B of P. In addition $\rho : \Gamma \to \mathrm{Homeo}(\mathbb{R})$ is a representation, such that the image $\rho(h)$ of the element corresponding to regular fibers is the translation by $+1$. It will always be a lift of a (μ, τ, σ)-representation $\bar{\rho} : \bar{\Gamma} \to \mathrm{Homeo}(\mathbb{S}^1)$, satisfying the almost (k)-convergence property. Since $\bar{\Gamma}$ is the orbifold fundamental group of the base space B of P, then $\bar{\Gamma} = \Gamma/ < h >$. We denote by $\tilde{\mu}$ the lift of μ (it is therefore the minimal invariant closed subset for $\rho(\Gamma)$) and by $\tilde{\tau} : \tilde{\mu} \to \tilde{\mu}$ the lift of τ: it is a homeomorphism, almost commuting with the restriction of $\rho(\Gamma)$, and satisfying: $\tilde{\tau}^k = \rho(h)|_{\tilde{\mu}}$.

We will denote by $\tilde{\tau}_0$ any homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, coinciding with $\tilde{\tau}$ on $\tilde{\mu}$ and such that $\tilde{\tau}_0^k = \rho(h)$ (but not necessarily almost commuting with $\rho(\Gamma)$ outside $\tilde{\mu}$). Up to a conjugation in $\mathrm{Homeo}(\mathbb{S}^1)$ one may assume that $\tilde{\tau}_0$ is the translation by $+1/k$.

We will modify the representation along periodic gaps (we have already observed in section 3.1 that all the modifications of $\bar{\rho}$ lift to representations of the same group Γ). The goal is to construct two foliations on the Seifert fibered space P, transverse to each other, that can be considered as blow ups of the stable and unstable foliations of the geodesic flow for some hyperbolic metric on the base orbifold B.

For our blow up construction it will be helpful to present the geodesic flow of hyperbolic surfaces in the following manner.

4.1. **An alternative construction of geodesic flows.** We start with a convex cocompact subgroup $\bar{\Gamma}$ of $\mathrm{PGL}_k(2, \mathbb{R})$. Since we will have to consider the non-orientable case, we carefully define this notion which involves some subtleties. First: The group $\mathrm{PGL}(2, \mathbb{R})$ is the isometry group of the hyperbolic plane. This is not the usual approach, so let us point out for the reader's convenience that the Möbius transformation of the semi-plane model of the hyperbolic plane defined by an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with negative determinant is $z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d}$. It is a symmetry through a geodesic and it reverses orientation in $\mathbb{H}^2$.

Definition 4.1. *A subgroup $\bar{\Gamma}$ of $PGL_k(2, \mathbb{R})$ is convex cocompact if its projection $\breve{\Gamma}$ in $PGL(2, \mathbb{R})$ is discrete, finitely generated, admits no parabolic element, and such that elements of $\breve{\Gamma}$ reversing the orientation have infinite order.*

Remark 4.2. The last condition may look unnecessary, even a bit unusual. The reason for this condition is that we want the action of $\check{\Gamma}_0$ (the subgroup of index two of orientation preserving elements) on $\mathrm{T}^1\mathbb{H}^2$ to be free, so that the unit tangent bundle $\bar{\Gamma}\backslash T^1\mathbb{H}^2$ is a manifold. An elementary example of a discrete subgroup of $\mathrm{PGL}(2,\mathbb{R})$ satisfying all the hypotheses of the convex cocompact definition except the last one is the quotient of a hyperbolic genus 2 surface Σ by an involution on a "middle" simple closed geodesic of Σ.

Let Γ be the preimage of $\bar{\Gamma}$ in $\widetilde{\mathrm{PGL}}(2,\mathbb{R})$. Then, the inclusion $\rho_0 : \Gamma \hookrightarrow \widetilde{\mathrm{PGL}}(2,\mathbb{R}) \subset \mathrm{Homeo}(\mathbb{R})$ is a $(\tilde{\mu}, \tilde{\tau}_0, \tilde{\sigma})$-representation. We will also consider the preimage $\ddot{\Gamma}$ in $\widetilde{\mathrm{PGL}}(2,\mathbb{R})$ of $\check{\Gamma}$: Γ is a finite index subgroup of $\ddot{\Gamma}$.

Here, $\tilde{\tau}_0$ is a generator of the pseudo center of $\widetilde{\mathrm{PGL}}(2,\mathbb{R})$, in the sense of almost commutation defined in section 2. Therefore $\tilde{\tau}$ is defined on the entire universal covering $\widetilde{\mathbb{RP}}^1$.

There are natural projections $\widetilde{\mathrm{PGL}}(2,\mathbb{R}) \to \mathrm{PGL}_k(2,\mathbb{R}) \to \mathrm{PGL}(2,\mathbb{R})$. The first is infinite to one and the second is finite to one. We denote by $\tilde{\pi}_k$ the first projection and by $\tilde{\pi}$ the composition of the two projections. For convenience of the reader in future referencing we recall

$$\bar{\Gamma} \ \leq \ \mathrm{PGL}_k(2,\mathbb{R}), \qquad \check{\Gamma} = \pi_k(\bar{\Gamma}) \ \leq \ \mathrm{PGL}(2,\mathbb{R}),$$

$$\Gamma = \tilde{\pi}^{-1}(\bar{\Gamma}) \ \leq \ \widetilde{\mathrm{PGL}}(2,\mathbb{R}), \qquad \ddot{\Gamma} = \tilde{\pi}_k^{-1}(\check{\Gamma}) \ \leq \ \widetilde{\mathrm{PGL}}(2,\mathbb{R}).$$

Here $\leq$ means being a subgroup.

The data of an oriented geodesic in $\mathbb{H}^2$ is equivalent to the data of a pair of distinct points (x,y) in $\partial\mathbb{H}^2$. Hence the space of oriented geodesics is $\mathbb{RP}^1 \times \mathbb{RP}^1 - \Delta$, where Δ is the diagonal. The action of $\mathrm{PGL}(2,\mathbb{R})$ on $\mathbb{RP}^1 \times \mathbb{RP}^1 - \Delta$ corresponding to the action on geodesics is simply the diagonal action.

Let M_0 be the unit tangent bundle of the hyperbolic orbifold ${}_{\check{\Gamma}}\backslash\mathbb{H}^2$. Denote by Ψ_0^t the geodesic flow on M_0. Geodesics of $\mathbb{H}^2$ are simply projections of orbits of the lift of Ψ_0^t to $T^1\mathbb{H}^2$. Let Ψ_0 be the one-dimensional foliation of M_0 induced by Ψ_0^t.

The orbit space of the lift $\tilde{\Psi}_0$ of the geodesic flow in the universal covering $\widetilde{M}_0$ of M_0 is the universal covering of $\mathbb{RP}^1 \times \mathbb{RP}^1 - \Delta$. This is identified with the open domain Ω_0 in $\widetilde{\mathbb{RP}}^1 \times \widetilde{\mathbb{RP}}^1$ between the graphs of the identity map id and of the map $\tilde{\tau}_0$, that is,

$$\Omega_0 \ = \ \{(x,y) \in \widetilde{\mathbb{RP}}^1 \times \widetilde{\mathbb{RP}}^1, \ \ x < y < \tilde{\tau}_0(x)\}$$

The delicate point is to understand how the action of $\mathrm{PGL}(2,\mathbb{R})$ on $\mathbb{RP}^1 \times \mathbb{RP}^1 - \Delta$ lifts to an action of $\widetilde{\mathrm{PGL}}(2,\mathbb{R})$ on Ω_0: it is <u>not</u> the restriction of the diagonal action since this action does not preserve Ω_0: the diagonal action of elements of $\widetilde{\mathrm{PGL}}(2,\mathbb{R}) - \widetilde{\mathrm{PSL}}(2,\mathbb{R})$ permutes Ω_0 and the domain $\{(x,y) \in \widetilde{\mathbb{RP}}^1 \times \widetilde{\mathbb{RP}}^1 \mid \tilde{\tau}_0^{-1}(x) < y < x\}$. The lifted action is actually the following:

– if γ is an element of $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$, define $\gamma.(x,y) = (\gamma x, \gamma y)$;

– if $\gamma \in \widetilde{\mathrm{PGL}}(2,\mathbb{R})$ reverses the orientation, define $\gamma.(x,y) = (\gamma x, \tilde{\tau}_0(\gamma y))$.

One easily checks that this is an action, and a lift of the diagonal action on $\mathbb{RP}^1 \times \mathbb{RP}^1 - \Delta$. Moreover this action preserves the domain Ω_0.

A key fact for us is that one can reconstruct the geodesic flow Ψ_0^t on M_0 from the data of the action of $\ddot{\Gamma}$ on Ω_0: Let $\mathrm{PT}\Omega_0$ be the projectivized tangent bundle of Ω_0. The action of $\widetilde{\mathrm{PGL}}(2,\mathbb{R})$ on $\mathbb{R}$ is differentiable and hence so is the action on Ω_0. Therefore $\widetilde{\mathrm{PGL}}(2,\mathbb{R})$ acts on $\mathrm{PT}\Omega_0$. Let ∂_x and ∂_y be the horizontal and vertical vector fields on Ω_0. The action of $\widetilde{\mathrm{PGL}}(2,\mathbb{R})$ preserves the lines defined by these vector fields, hence restricts to a natural action on $\mathrm{PT}\Omega_0$ with the vertical and horizontal directions removed. We focus on this last action. It admits two orbits: one orbit is the open domain

$$\widetilde{M}(\Omega_0) \ = \ \{(x,y,\xi), \ \ \xi = a\partial_x + b\partial_y, \text{ with } ab > 0\}$$

The other orbit is the collection of (x,y,ξ) for which ab is negative. The action preserves each of these because the action on Ω_0 preserves orientation. Below we show that these are actually orbits, that is, the respective actions are transitive. Observe that $\widetilde{M}(\Omega_0)$ can also be naturally parametrized by (x,y,m), $(x,y) \in \Omega_0$, and where m is the real number $\log(b/a)$, where as above the point is given by $(x,y,a\partial_x + b\partial_y)$.

The action of $\widetilde{\mathrm{PGL}}(2,\mathbb{R})$ on $\widetilde{M}(\Omega_0)$ is not free: as an isometry of $\mathbb{H}^2$, a reflection R along a geodesic with extremities (the projections in $\mathbb{S}^1$ of) x and y fixes every element of the form (x,y,ξ): For simplicity we can think of R acting on $\mathbb{R}$ as $z \to -z$ and $x = y = 0$. Hence when acting on $\mathrm{T}\Omega_0$, R will preserve the lines in the tangent bundle $\mathrm{T}\Omega_0$ at the point (x, y). However the stabilizer of this action at any point of $\widetilde{M}(\Omega_0)$ over (x, y) is precisely the group of order 2 generated by R. Therefore, $\widetilde{M}(\Omega_0)$ is naturally identified with the right quotient $\widetilde{\mathrm{PGL}}(2, \mathbb{R})/R$, where the action of $\widetilde{\mathrm{PGL}}(2, \mathbb{R})$ on $\widetilde{M}(\Omega_0)$ corresponds to the (left) action of $\widetilde{\mathrm{PGL}}(2, \mathbb{R})$ on $\widetilde{\mathrm{PGL}}(2, \mathbb{R})/R$.

On the other hand, the action of $\widetilde{\mathrm{PGL}}(2, \mathbb{R})$ on the universal covering $\widetilde{T^1\mathbb{H}}^2$ is transitive, and the stabilizer of any unit vector tangent to the geodesic (x, y) is the group of order 2 generated by R: it follows that the quotient $\widetilde{\mathrm{PGL}}(2, \mathbb{R})/R$ can also be identified with $\widetilde{T^1\mathbb{H}}^2$.

Therefore, $\widetilde{M}(\Omega_0)$ is naturally identified with $\widetilde{T^1\mathbb{H}}^2$; orbits of the geodesic flow in $\widetilde{T^1\mathbb{H}}^2$ correspond to the fibers of the projection of $\widetilde{M}(\Omega_0) \subset \mathrm{PT}\Omega_0$ over Ω_0, and this identification is equivariant with respect to the actions of $\widetilde{\mathrm{PGL}}(2, \mathbb{R})$. In particular, the induced action of $\ddot{\Gamma}$ on $\widetilde{M}(\Omega_0)$ is properly discontinuous. Furthermore, by our requirement in the definition of convex cocompact subgroups (see Definition 4.1), this action is free.

The quotient $M_{\ddot{\Gamma}}(\Omega_0) = \ddot{\Gamma}\backslash\widetilde{M}(\Omega_0)$ is a 3-manifold equipped with an one-dimensional foliation (by an abuse of notation still denoted by) Ψ_0. This foliation is the projection by $\ddot{\Gamma}$ of the restriction to $\widetilde{M}(\Omega_0)$ of the foliation of $\mathrm{PT}\Omega_0$ induced by the fibration over Ω_0. Furthermore, this foliation is oriented: the orientation in the leaf defined by $(x, y) \in \Omega_0$ is the one for which the logarithmic slope m is increasing.

In summary, the oriented foliations (M_0, Ψ_0) and $(M_{\ddot{\Gamma}}(\Omega_0), \Psi_0)$ have the same associated orbit space Ω_0 in their universal covers, with the same group action (on Ω_0) of the fundamental group $\ddot{\Gamma}$. The identifications between $\widetilde{M}(\Omega_0)$, $\widetilde{T^1\mathbb{H}}^2$ and $\widetilde{\mathrm{PGL}}(2, \mathbb{R})/R$ produce a topological conjugacy between (M_0, Ψ_0) and $(M_{\ddot{\Gamma}}(\Omega_0), \Psi_0)$.

Moreover, $M_{\ddot{\Gamma}}(\Omega_0)$ has two invariant foliations: one provided by the vertical foliation $dx = 0$ of Ω_0, and denoted by $\Lambda^s(\Psi_0)$; the other by the horizontal foliation $dy = 0$ and denoted by $\Lambda^u(\Psi_0)$. It is easy to check that, as suggested by the notations, the first one is the weak stable foliation of the geodesic flow (M_0, Ψ_0), the other is the unstable foliation of this flow.

In a more general way, finite coverings of the geodesic flow of hyperbolic surfaces are all obtained as quotients $M_\Gamma(\Omega_0) := \Gamma\backslash\widetilde{M}(\Omega_0)$ where Γ is any convex cocompact subgroup of $\widetilde{\mathrm{PGL}}(2, \mathbb{R})$ (we mean not necessarily the entire preimage $\ddot{\Gamma}$ in $\widetilde{\mathrm{PGL}}(2, \mathbb{R})$ of its projection in $\mathrm{PGL}(2, \mathbb{R})$).

Remark 4.3. There is another well known model of $T^1\mathbb{H}^2$ as the triples (x, y, z) with the counterclockwise order. The identification is via the orientation preserving isometries of $\mathbb{H}^2$. In our situation we will have to consider isometries that are orientation reversing $-$ for example that would come from an orientation reversing geodesic in a non orientable hyperbolic surface. For this reason we were not able to use this model in our work and decided to use the projectivized model.

4.2. **Cutting a compact Seifert piece in the model geodesic flow.** In the previous section, we have shown how to identify finite coverings (M_0, Ψ_0) of the geodesic flow of a convex cocompact orbifold $O := \check{\Gamma}\backslash\mathbb{H}^2$ with one of our models $M_\Gamma(\Omega_0)$. The orbifold O has a compact convex core K delimited in O by a finite number of closed simple geodesics c_1, ... , c_k. More precisely, O is the union of K and a finite number of flaring annuli, one for each c_s. Observe that singularities of the orbifold O are all contained in K, their preimage in $M_\Gamma(\Omega_0)$ are precisely the singular fibers of the Seifert fibered structure. For each c_s, unit tangent vectors based at points of c_s form an embedded torus, which lifts to an embedded torus T_s in $(M_0, \Psi_0) \approx (M_\Gamma(\Omega_0), \Psi_0)$. These tori are quasi-tranverse: outside a finite number of periodic orbits, they are transverse to the flow Ψ_0 (or to Ψ_0 in $M_\Gamma(\Omega_0)$). More precisely, the region in the torus between two successive periodic orbits is an elementary Birkhoff annulus.

The submanifold P_0**:** The union of the tori above is the boundary of a compact submanifold P_0 of M_0 (equivalently $M_\Gamma(\Omega_0)$) that is a finite covering of the unit tangent bundle of the convex core K. In particular P_0 is a Seifert fibered space.

Definition 4.4. *The manifold P_0 with boundary, equipped with the restriction of the oriented foliation Ψ_0 and the restrictions $\hat{\Lambda}^s_0$, $\hat{\Lambda}^u_0$ of the stable, unstable foliations, is called a* ***piece of geodesic flow.***

Every torus T_s as above lifts in the universal covering $\widetilde{M}_0$ to infinitely many properly embedded planes. We will denote the collection of all such lifts for the union of the T_s as $\{\widetilde{T}_i\}$. Each $\widetilde{T}_i$ is invariant by a maximal free abelian subgroup H_i of rank 2 of Γ. The union of these planes bounds a unique region $\widetilde{P}_0$ in $\widetilde{M}(\Omega_0)$. For each $\widetilde{T}_i$, the complement $\widetilde{M}(\Omega_0) - \widetilde{T}_i$ has two connected components. One of these connected components contains the interior of $\widetilde{P}_0$ and all the others lifts of quasi-transverse tori: we denote it by $\widetilde{T}_i^+$. We denote by $\widetilde{T}_i^-$ the other connected component of $\widetilde{M}(\Omega_0) - \widetilde{T}_i$. Observe that the interior of $\widetilde{P}_0$ is the intersection of all $\widetilde{T}_i^+$.

One can describe precisely what is the projection in the orbit space Ω_0 of each of these regions (cf. [Ba3, Sect. 3.1]): the projection of $\widetilde{T}_i$ is a H_i-invariant string of lozenges $(R^i_j)_{j\in\mathbb{Z}}$ and their corners, where each R^i_j has two

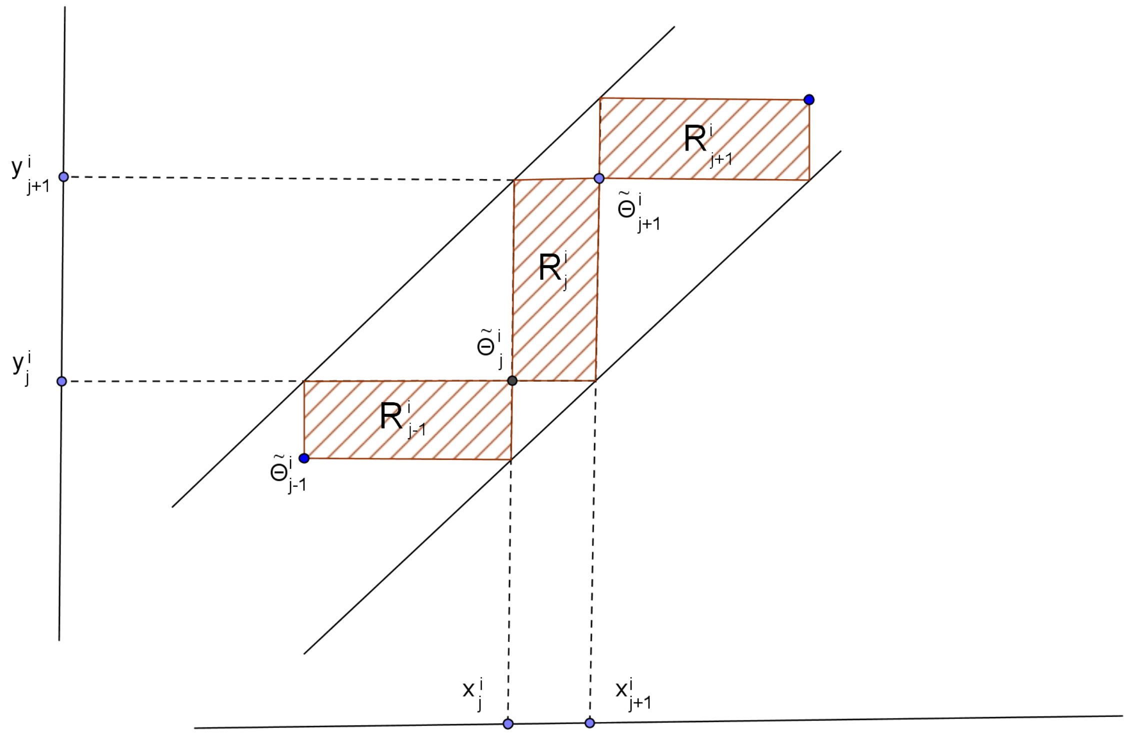

Figure 3: *Chain of lozenges for the geodesic flow.*

corners θ^i_j, θ^i_{j+1}. For each i the $\widetilde{T}_i$ produces $\mathbb{Z}$ many corners θ^i_j. These corners make up all orbits tangent to $\widetilde{T}_i$, each preserved by a cyclic subgroup of H_i. See figure 3.

More precisely, if x^i_j, y^i_j are the coordinates of θ^i_j in $\Omega_0 \subset \widetilde{\mathbb{RP}}^1 \times \widetilde{\mathbb{RP}}^1$ we have:

$$R^i_j := \{(x,y) \in \widetilde{\mathbb{RP}}^1 \times \widetilde{\mathbb{RP}}^1 \ \mid \ x^i_j < x < x^i_{j+1},\ y^i_j < y < y^i_{j+1}\}$$

so R^i_j is a lozenge. In addition each lozenge R^i_j is the projection of the lift of an elementary Birkhoff annulus A^i_j in some torus $T \subset \partial P_0$. The interior of the Birkhoff annulus is transverse to the flow. There are two cases: A^i_j is either an entrance region into P_0 or an exit region for Ψ_0. Equivalently the vector field generating $\widetilde{\Psi}_0$ might be pointing in the direction of $\widetilde{T}^+_i$ or in the direction of $\widetilde{T}^-_i$. On the other hand, one and only one of the two sides $]x^i_j, x^i_{j+1}[$ and $]y^i_j, y^i_{j+1}[$ of the rectangle R^i_j is a gap of the minimal set $\tilde{\mu}$. The following characterization will be crucial:

- If the gap of $\tilde{\mu}$ is the horizontal side $]x^i_j, x^i_{j+1}[$, then the corresponding Birkhoff annulus A^i_j is an exit annulus.
- If the gap is $]y^i_j, y^i_{j+1}[$, the annulus A^i_j is an entrance annulus.

Here μ is the minimal set of the associated action of $\check{\Gamma}$ on $\mathbb{S}^1 \cong \mathbb{RP}^1$. In the case of pieces of geodesic flows this characterization is easy to see. For an explicit proof that applies to a more general setting, see [Ba3, Corollaire 3.15].

We can say more: assume that we are in the case of entering annulus, i.e. the case where $]y^i_j, y^i_{j+1}[$ is a gap of $\tilde{\mu}$. Then, the following "triangle"

$$\Delta(\theta^i_j) := \{(x,y) \in \widetilde{\mathbb{RP}}^1 \times \widetilde{\mathbb{RP}}^1 \ \mid \ \tilde{\tau}_0^{-1}(x^i_{j+1}) < x < x^i_j,\ \ y^i_j < y < y^i_{j+1}\}$$

has the following geometric interpretation: it is the projection of the orbits in $\widetilde{T}^-_i$ trapped between the part of the stable and unstable leaves of θ^i_j contained in $\widetilde{T}^-_i$. In the case where A^i_j is an exit annulus, the triangle in Ω_0 satisfying this property is:

$$\Delta(\theta^i_j) := \{(x,y) \in \widetilde{\mathbb{RP}}^1 \times \widetilde{\mathbb{RP}}^1 \ \mid \ x^i_j < x < x^i_{j+1},\ \ \tilde{\tau}_0^{-1}(y^i_{j+1}) < y < y^i_j\}$$

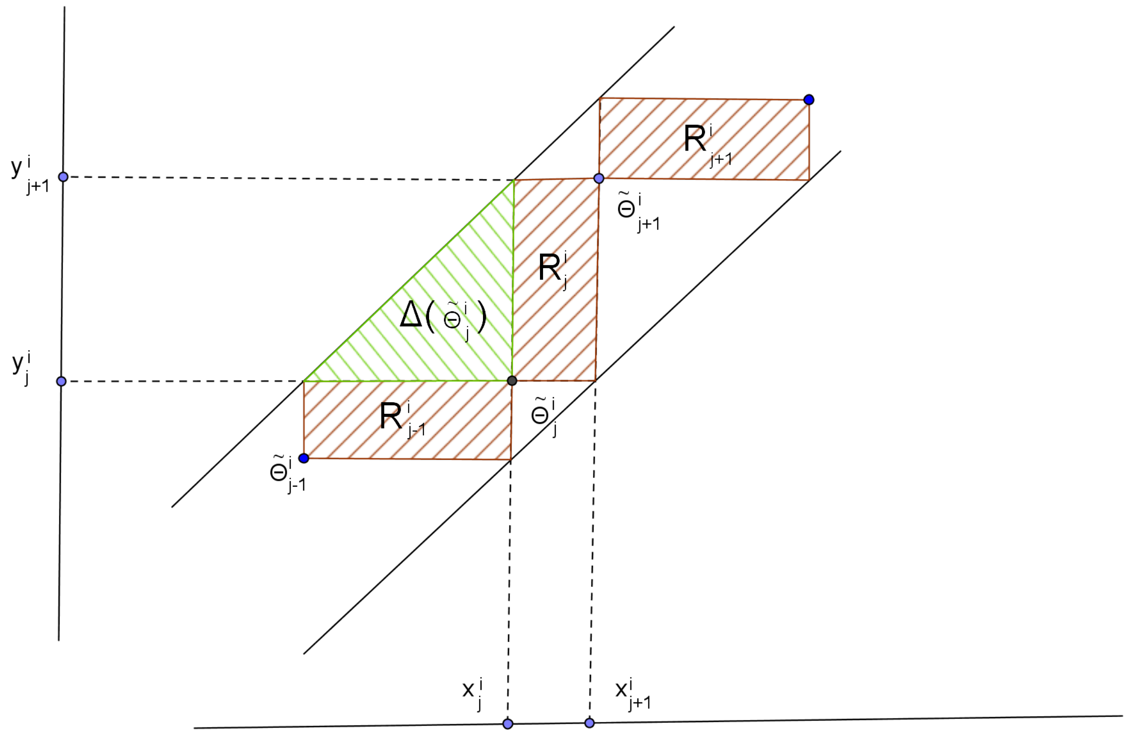

Figure 4: *Triangle for a tangent periodic orbit in the case where* $]y^i_j, y^i_{j+1}[$ *is a gap of* $\tilde{\mu}$*, hence in the case of entering annulus.*

These formulas define $\Delta(\theta^i_j)$ in the case of entering and exiting annuli.

The non-wandering set $\mathcal{M}_0$ of Ψ_0 is precisely the projection in M_0 of the union $\widetilde{\mathcal{M}}_0$ of the orbits whose projection in Ω_0 lies in $\tilde{\mu} \times \tilde{\mu}$. It is also the closure of the union of the periodic orbits of Ψ_0. Observe that the lift $\widetilde{\mathcal{M}}_0$ to the universal cover contains the tangent periodic orbits θ^i_j for every i, j. In addition this lift is contained in $\widetilde{P}_0$.

Finally, the projection of $\widetilde{P}_0$ (i.e. the set of orbits intersecting $\widetilde{P}_0$) is the complement in Ω_0 of the union of the closure of the triangles $\Delta(\theta^i_j)$. Here θ^i_j runs over all the lifted tori $\widetilde{T}_i$ and all lifts of periodic orbits tangent to these $\widetilde{T}_i$.

4.3. Construction of flows on 3-manifolds via group actions on the line.

4.3.1. *Hyperbolic blow up.* Recall that $\Gamma = \pi_1(P_0)$ where P_0 is a Seifert fibered space and $\bar{\Gamma} = \Gamma / < h >$ where h represents a regular fiber of the Seifert fibration of P_0.

Definition 4.5. *A modification of a* (σ, μ, τ)*-representation* $\rho_0 : \Gamma \to$ *Homeo*$(\mathbb{S}^1)$ *on a periodic gap* I_0 *is* ***hyperbolic*** *if the modified restriction* $f_0 : I_0 \to I_0$ *has only a finite number of fixed points in* I_0*, and that all these fixed points (including the extremities* ∂I_0 *as maps of* $\mathbb{S}^1$*) are hyperbolic.*

A hyperbolic blow up of ρ_0 *is a* (σ, μ, τ)*-representation obtained from the Fuchsian representation* ρ_0 *by a finite number of hyperbolic modifications on periodic gaps.*

A corollary of Remark 3.4 is that the topological conjugacy class of a hyperbolic modification is uniquely determined by the (even) number of fixed points introduced in the periodic gap. It also follows from Remark 3.5 that hyperbolic blow ups are semiconjugate to the initial Fuchsian representations they are constructed from.

The chronological order of the hyperbolic modifications has no incidence on the conjugacy class of the resulting hyperbolic blow up.

According to Theorem 3.10, the representation $\bar{\rho} : \bar{\Gamma} \to \mathrm{Homeo}(\mathbb{S}^1)$ associated to a hyperbolic blow up has the almost (k)-convergence property. It follows that every element $\bar{\gamma}$ for which $\bar{\rho}(\bar{\gamma})$ is not of finite order has only a finite number of fixed points, which are all of hyperbolic type, but the number of fixed points is not always $2k$: some elements may have more than $2k$ fixed points. Observe also that every gap is periodic, since it is already true for Fuchsian actions.

4.3.2. *Constructing the orbit space.* Let us now consider a Fuchsian representation $\rho_0 : \Gamma \to \widetilde{\mathrm{PGL}}(2,\mathbb{R})$ as in subsection 4.1 (the inclusion map of a convex cocompact subgroup). It is a $(\tilde{\sigma}, \tilde{\tau}_0, \tilde{\mu})$-representation. Let $\rho_0^* : \Gamma \to \widetilde{\mathrm{PGL}}(2,\mathbb{R})$ be the *twisted* representation: for every γ in Γ, if γ preserves the orientation we have $\rho_0^*(\gamma) = \rho_0(\gamma)$, and if γ reverses the orientation we have $\rho_0^*(\gamma) = \tilde{\tau}_0 \circ \rho_0(\gamma)$. We have already observed that ρ_0^* is indeed a representation, and that the action $(x,y) \mapsto (\rho_0(\gamma)x, \rho_0^*(\gamma)y)$ preserves the open domain $\Omega_0 = \{(x,y) \mid x < y < \tilde{\tau}_0(x)\}$. It is a $(\tilde{\sigma}^*, \tilde{\tau}_0, \tilde{\mu})$-representation, where $\tilde{\sigma}^*(\gamma)$ coincides with $\tilde{\sigma}(\gamma)$ when γ is orientation preserving, and $\tilde{\sigma}^*(\gamma)$ coincides with $\tilde{\tau}_0 \circ \sigma(\gamma)$ if not.

Orientation reversing elements do not stabilize any gap. The reason is the following. Suppose that an orientation reversing element preserves a gap. Then the unperturbed representation also satisfies this. This representation is a lift of a Fuchsian representation, so it comes from a representation into $\mathrm{PGL}(2,\mathbb{R})$. Then there would be an orientation reversing element preserving a gap. But gaps associated with Fuchsian representations correspond to boundary elements in the associated orbifold and these are orientation preserving, so this cannot happen and establishes this fact.

The representations ρ_1 and ρ_2 – Let now ρ_1, ρ_2 be hyperbolic blow ups of respectively ρ_0, ρ_0^*.

We do not require the hyperbolic blow ups to be performed on the same gaps for ρ_1 and ρ_2, they are performed in an independent way. We consider the action of Γ on $\mathbb{R} \times \mathbb{R}$ defined by:

$$\gamma.(x,y) := (\rho_1(\gamma)x, \rho_2(\gamma)y)$$

Let Γ^* be the subgroup of index at most 2 comprising orientation preserving elements of Γ.

We now very carefully construct the orbit space of our eventual "flow". Its orbit space will be a Γ-invariant open domain of $\Omega_0 \subset \mathbb{R} \times \mathbb{R}$, with properties similar to that of Ω_0. In particular, it will be the region between the graphs of two monotone non-decreasing maps from $\mathbb{R}$ into $\mathbb{R}$.

Let $I =]a,b[$ be a gap of $\tilde{\mu}$. It is periodic. Let Γ_I the stabilizer of I. It is generated by an element γ_I, that we can select so that a is a repelling fixed point for $\rho_i(\gamma_I)$. We compare the actions of $\rho_1(\gamma_I)$ and $\rho_2(\gamma_I)$ on I.

Construction of the map α_1

Let

$$x_1^1 = a < x_2^1 < \ldots < x_{2p}^1 = b \ \text{ be the fixed points of } \ \rho_1(\gamma_I) \ \text{ in } \ \bar{I}$$

and

$$x_1^2 = a < x_2^2 < \ldots < x_{2q}^2 = b \ \text{ be the fixed points of } \ \rho_2(\gamma_I) \ \text{ in } \ \bar{I}$$

Every x_k^j is a repelling fixed point of $\rho_j(\gamma_I)$ if k is odd, and an attracting fixed point if k is even. We select any Γ_I-equivariant increasing homeomorphism between $[a, x_2^1]$ and $[x_{2q-1}^2, b]$ realizing a topological conjugacy between $\rho_1(\gamma_I)$ and $\rho_2(\gamma_I)$ on these intervals. If $x_2^1 < b$, we then extend this map on $[x_2^1, b]$ as the constant map taking the value b. This defines a map $\alpha_1 : I \to I$ which is a semi-conjugacy between the restrictions to I of $\rho_1(\gamma_I)$ and $\rho_2(\gamma_I)$. We actually make the choices so that on I we have always $x < \alpha_1(x)$ if x is not an endpoint of I. Therefore

$$\alpha_1([x_2^1, b]) = b, \quad \alpha_1([a, x_2^1]) = [x_{2q-1}^2, b]$$

Construction of the map β_1

Similarly, let

$$y_1^2 = \tilde{\tau}_0(a) < y_2^2 < \ldots < y_{2r}^2 = \tilde{\tau}_0(b) \ \text{ be the fixed points of } \ \rho_2(\gamma_I) \ \text{ in } \ \tilde{\tau}_0(I)$$

The map β_1 is chosen to be the constant map on $[a, x_{2p-1}^1]$, taking the value $\tilde{\tau}_0(a)$, and to be an increasing Γ_I equivariant, topological conjugacy between the restrictions of $\rho_1(\gamma_I)$ and $\rho_2(\gamma_I)$ on respectively $[x_{2p-1}^1, b]$ and $[\tilde{\tau}_0(a), y_2^2]$. We actually adjust so that $\beta_1(x) < \tilde{\tau}_0(x)$ for x in I. Therefore

$$\beta_1([a, x_{2p-1}^1]) = \tilde{\tau}_0(a), \quad \beta_1([x_{2p-1}^1, b]) = [\tilde{\tau}_0(a), y_2^2]$$

We then extend α_1 and β_1 to $\bigcup_{\gamma \in \Gamma^*} \tilde{\sigma}(\gamma)I$ by Γ^*-equivariance: the restriction of α_1 (respectively β_1) to $\rho_1(\gamma)I$ is defined as the conjugate $\rho_2(\gamma) \circ \alpha_1 \circ \rho_1(\gamma)^{-1}$ (respectively $\rho_2(\gamma) \circ \beta_1 \circ \rho_1(\gamma)^{-1}$). Everywhere else we define α_1 to be the identity and β_1 to coincide with $\tilde{\tau}_0$.

For orientation reversing element γ, we define the restriction of α_1 to $\rho_1(\gamma)I$ as the conjugate $\rho_2(\gamma) \circ \beta_1 \circ \rho_1(\gamma)^{-1}$, and the restriction of β_1 on the same interval as $\rho_2(\gamma) \circ \alpha_1 \circ \rho_1(\gamma)^{-1}$.

This is done for one $\sigma(\Gamma)$-orbit of a gap. We then apply the same procedure for other gaps I in other $\sigma(\Gamma)$-orbits of gaps. The result are non-decreasing maps $\alpha_1, \beta_1 : \mathbb{R} \to \mathbb{R}$ such that, for every element γ of Γ, we have

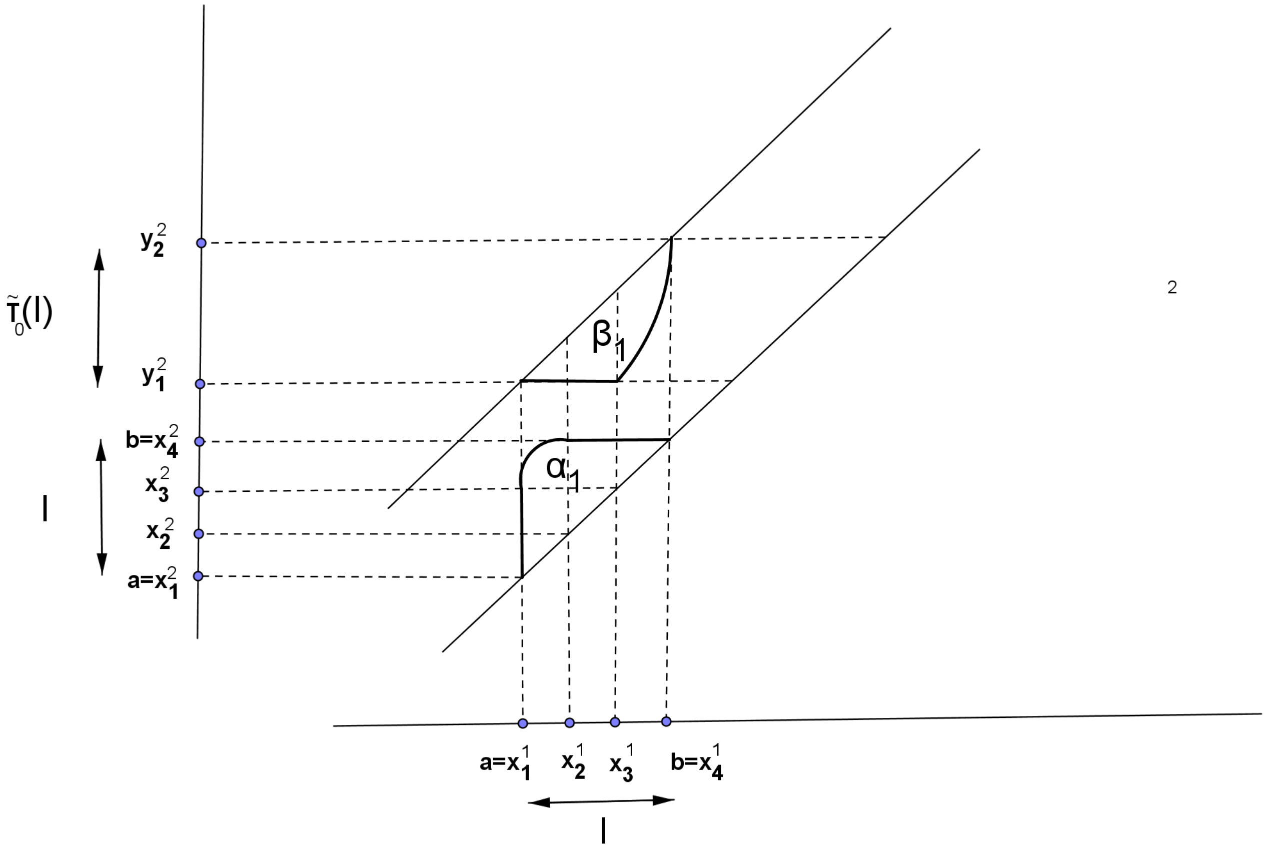

Figure 5: *The construction of α_1 and β_1 on a gap I. Here, $\rho_1(\gamma_I)$ has two fixed points in I, and $\rho_2(\gamma_I)$ has two fixed points in I and no fixed points in $\tilde{\tau}(I)$.*

$\rho_2(\gamma) \circ \alpha_1 = \alpha_1 \circ \rho_1(\gamma)$ and $\rho_2(\gamma) \circ \beta_1 = \beta_1 \circ \rho_1(\gamma)$ if γ preserves the orientation, and $\rho_2(\gamma) \circ \alpha_1 = \beta_1 \circ \rho_1(\gamma)$ and $\rho_2(\gamma) \circ \beta_1 = \alpha_1 \circ \rho_1(\gamma)$ if not. We furthermore have $x \leq \alpha_1(x)$ and $\beta_1(x) \leq \tilde{\tau}_0(x)$ for every x in $\mathbb{R}$.

It is easy to see that by construction the map α_1 is continuous on the right, but not on the left, and that β_1 is continuous on the left. In the above setup, for every gap $[a, b]$, the subinterval $]a, x^2_{2q-1}[$ (which may be empty if $x^2_{2q-1} = a$, that is, no blow up in that interval) is not in the image of α_1. We add every such vertical segment $\{a\} \times [a, x^2_{2q-1}[$ to the graph of α_1: the result is a closed embedded line L_- in $\mathbb{R}^2$. Similarly, we add vertical segments to the graph of β_1, obtaining a closed embedded line L_+ in the plane. The union $L_+ \cup L_-$ is the boundary of an open domain denoted by Ω, which is invariant by the action of Γ^*. The domain Ω can also be simply defined as follows:

$$\Omega := \{(x, y) \in \mathbb{R}^2 \mid \alpha_1(x) < y < \beta_1(x)\}$$

Observe that since $x \leq \alpha_1(x) < \beta_1(x) \leq \tilde{\tau}_0(x)$, the region Ω is naturally included in Ω_0: we call this (non-equivariant) inclusion the *canonical inclusion.*

We now introduce two maps $\alpha_1^-, \beta_1^+ : \mathbb{R} \to \mathbb{R}$ defined as follows: For every x in $\mathbb{R}$, the intersection between L_- and the vertical line $\{x\} \times \mathbb{R}$ is a segment $\{x\} \times [\alpha_1^-(x), \alpha_1(x)]$, whose maximal element is indeed $\alpha_1(x)$. Similarly, the intersection between L^+ and $\{x\} \times \mathbb{R}$ is a segment $[\beta_1(x), \beta_1^+(x)]$. The maps α_1^- and β_1^+ are non decreasing and Γ^*-equivariant, for instance because $\alpha_1^- = \alpha_1$ in the gaps of $\tilde{\mu}$. Moreover, α_1^- and β_1^+ coincide on $\tilde{\mu}$ with respectively the identity map and $\tilde{\tau}_0$.

It is easy to see that α_1^- is continuous on the left (whereas α_1 is continuous on the right), and that β_1^+ is continuous on the right. Actually, $\alpha_1^-(x)$ can be defined as the limit of $\alpha_1(x')$ for x' converging to x at the left.

Finally, we observe that L_- and L_+ can also be considered as generalized "graphs" of maps α_2, β_2 in another way: the intersection between L_+ (respectively L_-) and every horizontal $\{y\} \times \mathbb{R}$ is a segment (maybe reduced to a point) $[\alpha_2^-(y), \alpha_2(y)]$ (respectively $[\beta_2(y), \beta_2^+(y)]$. These maps are non decreasing and Γ^*-equivariant. The open domain Ω can also be defined as:

$$\Omega := \{(x, y) \in \mathbb{R}^2 \mid \alpha_2(y) < x < \beta_2(y)\}$$

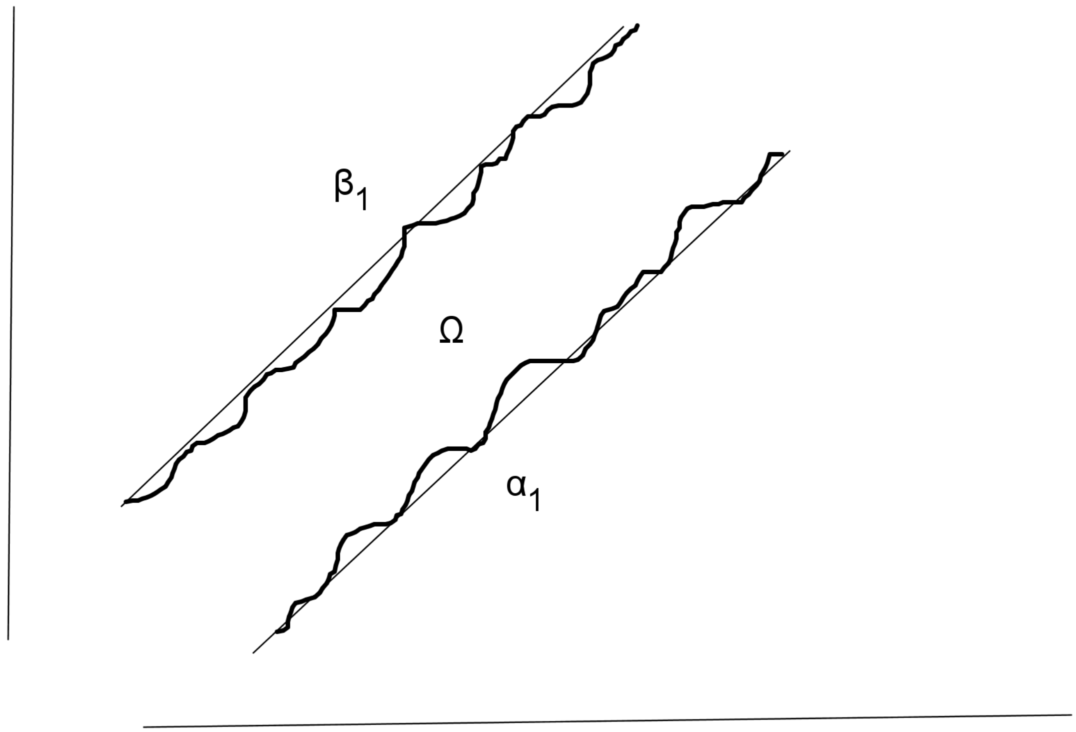

Figure 6: *The maps α_1, β_1 and the domain Ω.*

The action of Γ on $\mathbb{R}^2$ preserves Ω (but orientation reversing elements permute the connected components $L^{\pm}$ of the boundary).

4.3.3. *Constructing a model hyperbolic blow up flow.* According to Remark 3.5 there exist topological semiconjugacies $\varphi_1 : \mathbb{R} \to \mathbb{R}$ and $\varphi_2 : \mathbb{R} \to \mathbb{R}$ such that:

$$\forall \gamma \in \Gamma \quad \rho_0(\gamma) \circ \varphi_1 = \varphi_1 \circ \rho_1(\gamma)$$
$$\forall \gamma \in \Gamma \quad \rho_0^*(\gamma) \circ \varphi_2 = \varphi_2 \circ \rho_2(\gamma)$$

More precisely, we construct these semiconjugacies in the following way, using the notations introduced in the previous subsection 4.3.2: on $\tilde{\mu}$ the maps φ_1 and φ_2 coincide with the identity map. Using the notation introduced in the previous subsection, then on every (periodic) gap $I =]a, b[$, the map φ_1 takes the constant value b on $[x_2^1, b]$, and on the interval $]a, x_2^1[$, φ_1 is any conjugacy between the restriction of $\rho_1(\gamma_I)$ and the restriction of $\rho_0(\gamma_I)$ to I. In addition we want that φ_1 is the identity in any gap J that has not been modified.

We then define φ_2 on I as the unique map taking the constant value a on $]a, x_{2q-1}^2[$ and such that on $[x_{2q-1}^2, b]$ is satisfies:

$$\varphi_2 \circ \alpha_1 = \varphi_1$$

We then extend φ_1 and φ_2 on the entire $\mathbb{R}$ so that they are Γ-equivariant.

Then we define the map

$$\chi : \mathbb{R}^2 \to \mathbb{R}^2, \quad \text{by} \quad \chi(x, y) \ := \ (\varphi_1(x), \varphi_2(y))$$

This map is Γ-equivariant. Furthermore, it follows from our choices that χ maps L_- onto the graph of the identity map. Moreover, for every gap I we have the following dichotomy concerning the image by χ of the "triangle" $T_I := \{(x, y) \mid x \in \bar{I},\ \tilde{\tau}_0(a) \leq y \leq \beta_1(x)\}$:

- either ρ_0 has not been modified in I and ρ_0^* has not been modified in $\tilde{\tau}_0(I)$: in this case, χ maps the triangle T_I (that in this case is $\{(x, y) \mid x \in \bar{I},\ \tilde{\tau}_0(a) \leq y \leq \tilde{\tau}_0(x)\}$) onto itself;
- or one (maybe both) of the actions on I or $\tilde{\tau}_0(I)$ has been modified: in this case, χ maps T_I into the union of the two sides $\{(x, y) \mid x \in \bar{I},\ y = \tilde{\tau}_0(a)\}$ – the horizontal side of T_I, and $\{(x, y) \mid x = b,\ y \in \tilde{\tau}_0(\bar{I})\}$ – the vertical side of T_I.

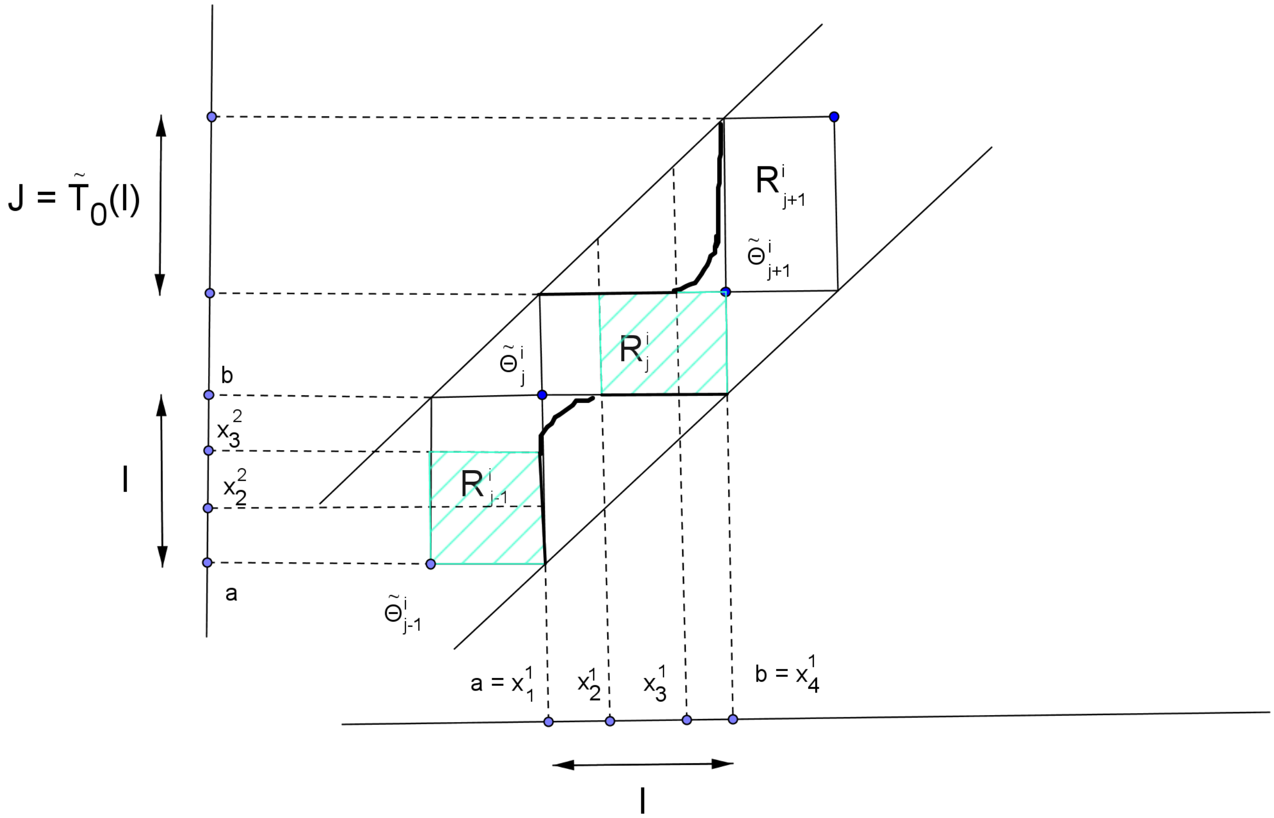

Figure 7: *The map χ collapses the stripped rectangles, and the triangle contained in $I \times J$.*

We explain this dichotomy. We use the notation of the definition of α_1 and β_1 in subsection 4.3.1, with the fixed points x's and y's. Let $J = \tilde{\tau}_0(I)$.

- In the first case there is no change of the representations, therefore $p = 1$ and $r = 1$ − notation from subsection 4.3.1, hence on I we have $\alpha_1 = id$ and $\beta_1 = \tilde{\tau}_0$. Therefore the triangle T_I is an actual triangle, and in this case $\varphi_1 = id$ in I and $\varphi_2 = id$ in J.
- In the second case, suppose first that there is some modification of ρ_0 in I. Therefore $p \geq 2$ and in particular $x^1_2 < x^1_{2p-1}$. Here α_1 sends $[x^1_2, b]$ to b and β_1 is constant $= \tilde{\tau}_0(a)$ on $[a, x^1_{2p-1}]$. If $x \leq x^1_{2p-1}$, then $T_I \cap \{x\} \times \mathbb{R} = (x, \tilde{\tau}_0(a))$, so $\beta_1(x) = \tilde{\tau}_0(a)$. Then χ sends the point $(x, \tilde{\tau}_0(a))$ to a point in the horizontal side of T_I. Recall that φ_1, φ_2 are equal to id in $\tilde{\mu}$. If on the other hand $x > x^1_{2p-1}$ then by definition $\varphi_1(x) = b$ and the image under χ of that part of T_I is in the vertical side of T_I.
- Finally suppose that ρ^*_0 has been modified in $J = \tilde{\tau}_0(I)$. Points in T_I have the form (x, y) with $y \in [\tilde{\tau}_0(a), \beta_1(x)]$. Since ρ^*_0 has been modified in $\tilde{\tau}_0(I)$, the definition of β_1 implies that $y \leq y^2_2$ in I. We now consider the image of this under φ_2. The subtle point is that we have to apply the definition of φ_2 to the interval $J = \tilde{\tau}_0(I)$ and not to I. In particular a of that definition corresponds to $\tilde{\tau}_0(a)$ in J and x^2_{2q-1} of that definition corresponds to y^2_{2q-1}. Since $q \geq 2$, $2q - 1 > 2$ and so φ_2 sends $[\tilde{\tau}_0(a), \beta_1(y)]$ to $\tilde{\tau}_0(a)$, so the image is in the horizontal side of T_I. This finishes the proof of the dichotomy.

It follows that χ maps L_+ into the closure of Ω_0, and therefore that the image of Ω by χ is contained in Ω_0 (see figure 7).

We now consider the oriented line bundle $\widetilde{M}(\Omega)$ over Ω which is the pull-back by χ of the line bundle $\widetilde{M}(\Omega_0) \to \Omega_0$ defined in section 4.1. In other words:

$$\widetilde{M}(\Omega) := \{(x, y, m) \in \mathbb{R}^3 \mid \alpha_1(x) < y < \beta_1(x)\}$$

equipped with the following action of Γ:

$$\forall \gamma \in \Gamma \quad \gamma.(x, y, m) \;=\; (\rho_1(\gamma)x, \rho_2(\gamma)y,\ m - \log|(\rho_0(\gamma))'(\varphi_1(x))| + \log|(\rho^*_0(\gamma))'(\varphi_2(y))|)$$

Therefore, χ induces a Γ-equivariant map from $\widetilde{M}(\Omega)$ into $\widetilde{M}(\Omega_0)$, simply defined by:

$$(x, y, m) \to (\chi(x, y), m)$$

Since the action on $\widetilde{M}(\Omega_0)$ is free and proper, the action of Γ on $\widetilde{M}(\Omega)$ is free and proper too. Let the quotient

$$M_\Gamma(\Omega) \ := \ \Gamma\backslash\widetilde{M}(\Omega)$$

This is a 3-manifold equipped with an oriented one dimensional foliation Ψ, which is the projection of the foliation induced by the fibers over Ω. More specifically the projection of $(x, y) \times \mathbf{R}$ for (x, y) in Ω. The manifold $M_\Gamma(\Omega)$ is also equipped with two foliations of codimension one $\Lambda^s(\Psi)$ and $\Lambda^u(\Psi)$ induced by the horizontal and vertical foliations of Ω. Since the modification is a hyperbolic blow up, it follows that these foliations have transversely hyperbolic behavior, that is, the flow Ψ is "essentially" an Anosov flow. The map χ induces a semi-conjugacy $\chi_\Gamma : M_\Gamma(\Omega) \to M_\Gamma(\Omega_0)$, mapping the oriented orbits of Ψ onto the oriented orbits of Ψ_0, and mapping the foliations $\Lambda^s(\Psi)$, $\Lambda^u(\Psi)$ onto the stable/unstable foliations $\Lambda^s(\Psi_0)$, $\Lambda^u(\Psi_0)$.

4.3.4. *Cutting a Seifert piece in the blow-up flow.* In this section, we show that there is in $M_\Gamma(\Omega)$ a compact manifold $P_\Gamma(\Omega)$ which is a Seifert bundle homeomorphic to the Seifert piece P_0 (see Section 4.2) and whose boundary is a union of embedded Birkhoff tori for the flow Ψ. Moreover, the submanifold $P_\Gamma(\Omega)$ is unique up to topological conjugacies preserving the restrictions of the stable and unstable foliations.

Recall that in subsection 4.2 we did the following: $M_\Gamma(\Omega_0) = \Gamma\backslash\widetilde{M}(\Omega_0)$ and P_0 is a compact submanifold bounded by Birkhoff tori. In this section we consider $M_\Gamma(\Omega_0) = \Gamma\backslash\widetilde{M}(\Omega)$. Notice that we abuse notation using the same Γ for the original action on $\widetilde{M}(\Omega_0)$ and the new action from the hyperbolic blow up on $\widetilde{M}(\Omega)$.

Let T_i be one peripheral torus of $M_0 \cong M_\Gamma(\Omega_0)$, and let $H_i \approx \mathbb{Z}^2 \subset \Gamma$ be its fundamental group. As recalled in Section 4.2, H_i preserves a string of lozenges $(R^i_j)_{j\in\mathbb{Z}}$ in Ω_0, and the stabilizer of the corners $\tilde{\theta}^i_j$ is a cyclic subgroup D_i. Recall that the blow up action is obtained by blowing up gaps of μ. These are associated with entering or exiting annuli for Ψ_0, which generated the lozenges R^i_j. These lozenges R^i_j are still contained in Ω, i.e. in the image of the canonical inclusion $\Omega \subset \Omega_0$, and are in some sense still preserved by H_i, for the new action through (ρ_1, ρ_2) (instead of (ρ_0, ρ_0^*)). The stabilizer of the corners is still the cyclic subgroup D_i. However, some R^i_j may have been decomposed into finitely many D_i-invariant sub-lozenges because there may be more fixed points under the (new) action of D_i (under (ρ_1, ρ_2)). But still there are invariant lozenges exactly because the blow up is hyperbolic. More specifically, suppose that the associated gap of $\tilde{\mu}$ is $]a, b[$ and the gap is exiting, that is, the gap is in the x direction. Then the lozenge R^i_j corresponds to the rectangle $]a, b[\times]b, \tilde{\tau}_0(a)[$. Here b is in $\tilde{\mu}$ and is isolated on the $]a, b[$ side, therefore b is not isolated in $\tilde{\mu}$ in the $]b, \tilde{\tau}_0(a)[$ side. In particular $]b, \tilde{\tau}_0(a)[$ is not a gap of $\tilde{\mu}$. It follows that there is no blow up in the interval $]b, \tilde{\tau}_0(a)[$ abutting b and the action of D_i on this interval is hyperbolic without fixed points. So if $a = x^1_1, x^1_2, ..., x^1_{2p} = b$ are the fixed points of the blown up action in $]a, b[$ it follows that the lozenge R^i_j splits into $2p - 1$ lozenges, forming a chain of adjacent lozenges, all intersecting a common unstable leaf.

It follows that one can easily find as in [Ba3] a D_i-invariant section of the bundle $\widetilde{M}(\Omega)$ over each new lozenge S^i_j, so that their images are D_i-invariant Birkhoff bands transverse to the fibers and with boundary the fibers above the (new) corners $\tilde{\theta}^i_j$ and $\tilde{\theta}^i_{j+1}$. Altogether all these Birkhoff bands form a closed embedded plane $\widetilde{T}'_i$ that one can select to be H_i-invariant (simply by taking, for every γ in H_i, the section over γS^i_j the image under γ of the section over S^i_j). As in [Ba3] or [Ba-Fe1], one shows, by cut and paste techniques, that the collection of these embedded planes, when we consider all the boundary tori T_i of M_0, can be chosen so that their projections in $M_\Gamma(\Omega)$ are embedded Birkhoff tori T'_i that are two by two disjoint.

Conclusion: There are finitely many embedded tori $\{T'_i\}$ in $M_\Gamma(\Omega)$ with union a compact subset of $M_\Gamma(\Omega)$.

Let $\widetilde{\mathcal{M}}$ be the preimage in $\widetilde{M}(\Omega)$ of $\tilde{\mu} \times \tilde{\mu}$, and let $\mathcal{M}$ be its projection in $M_\Gamma(\Omega)$. Observe that the restriction of χ_Γ to $(\tilde{\mu} \times \tilde{\mu}) \cap \Omega$ is a homeomorphism onto $(\tilde{\mu} \times \tilde{\mu}) \cap \Omega_0 = (\tilde{\mu} \times \tilde{\mu}) \cap \Omega$ (it is the identity map!). It follows that $\mathcal{M}$ is naturally identified with the non-wandering set for the geodesic flow $(M_\Gamma(\Omega_0), \Psi_0)$, and therefore the very important consequence that $\mathcal{M}$ is compact. Observe also that $\mathcal{M}$ intersects the tori T'_i only at (possibly a subcollection of) their tangent periodic orbits. In particular, there is no orbit in $\mathcal{M}$ crossing one T'_i. This is because an orbit crossing T'_i lifts to an orbit in a lozenge as above. Then either its stable or unstable leaf is in a gap of $\tilde{\mu}$.

Consider one of these tori, and denote it by T'_i. Consider one of its lifts $\widetilde{T}'_i$. Since $\widetilde{T}'_i$ is a closed embedded plane, it disconnects $\widetilde{M}(\Omega)$ in two connected components $(\widetilde{T}'_i)^+$, $(\widetilde{T}'_i)^-$. We show that one of these components does not intersect $\widetilde{\mathcal{M}}$. Assume by way of contradiction that $(\widetilde{T}'_i)^+$ and $(\widetilde{T}'_i)^-$ each contain one element $\tilde{\theta}^+$, $\tilde{\theta}^-$ respectively of $\widetilde{\mathcal{M}}$. Periodic orbits are dense in $\mathcal{M}$, hence we can assume that $\tilde{\theta}^+, \tilde{\theta}^-$ are both lifts of periodic orbits. Moreover, there is a sequence $\tilde{\theta}_0, \ldots, \tilde{\theta}_{2n}$ of elements in $\widetilde{\mathcal{M}}$ such that:

– $\tilde{\theta}_0 = \tilde{\theta}^+$ and $\tilde{\theta}_{2n} = \tilde{\theta}^-$,
– every $\tilde{\theta}_{2k}$ is the lift of a periodic orbit (for $0 \leq k \leq n$),
– for every k between 0 and $n-1$, $\tilde{\theta}_{2k+1}$ is the intersection between the stable leaf of $\tilde{\theta}_{2k}$ and the unstable leaf of $\tilde{\theta}_{2k+2}$, or the intersection between the unstable leaf of $\tilde{\theta}_{2k}$ and the stable leaf of $\tilde{\theta}_{2k+2}$ (this sequence corresponds to a polygonal line in Ω made of vertical and horizontal segments joining $\tilde{\theta}^+$ to $\tilde{\theta}^-$). This is certainly true for the geodesic flow and hence it follows for Ψ also. We can furthermore assume that no $\tilde{\theta}_k$ is one orbit tangent to $\widetilde{T}'_i$.

Then, as explained above, none of the $\tilde{\theta}_k$ crosses $\widetilde{T}'_i$. Since $\tilde{\theta}^+$ and $\tilde{\theta}^-$ lie in different connected components of the complement, there is some integer k such that $\tilde{\theta}_{2k}$ lies in $(\widetilde{T}'_i)^+$ and $\tilde{\theta}_{2k+2}$ lies in $(\widetilde{T}'_i)^-$. Then, $\tilde{\theta}_{2k+1}$ is an orbit that in the past gets closer and closer to, say $\tilde{\theta}_{2k}$, and in the future gets closer and closer to $\tilde{\theta}_{2k+2}$. Since $\tilde{\theta}_{2k}$ and $\tilde{\theta}_{2k+2}$ project to periodic orbits, they stay a minimum distance from $\widetilde{T}'_i$. Hence, $\tilde{\theta}_{2k+1}$ must cross $\widetilde{T}'_i$: contradiction.

Therefore, one of the connected components $(\widetilde{T}'_i)^+$, $(\widetilde{T}'_i)^-$ is disjoint from $\widetilde{\mathcal{M}}$: we fix the notation so that this connected component is $(\widetilde{T}'_i)^-$. Consider the intersection of all connected components $(\widetilde{T}'_i)^+$ through all the possible embedded Birkhoff planes $\widetilde{T}'_i$. This intersection is the interior of a manifold with boundary $\widetilde{P}(\Omega)$, which contains $\widetilde{\mathcal{M}}$ and also every Birkhoff plane $\widetilde{T}'_i$ (indeed, $\widetilde{T}'_i$ contains elements of $\widetilde{\mathcal{M}}$: lifts of some tangent periodic orbits, hence $\widetilde{T}'_i$ cannot be on the $(\widetilde{T}'_j)^-$ side of some other Birkhoff plane $\widetilde{T}'_j$). Moreover, $\widetilde{P}(\Omega)$ is Γ-invariant, hence projects in $M_\Gamma(\Omega)$ to a manifold with boundary $P_\Gamma(\Omega)$, whose boundary is the union of all the Birkhoff tori T'_i.

Since $\widetilde{T}_i$ is a Birkhoff plane, it follows as in [Ba3] that every orbit of $\widetilde{\Psi}$ can intersect any $\widetilde{T}'_i$ at most once. In addition $(\widetilde{T}'_i)^-$ is disjoint from the other Birkhoff planes $\widetilde{T}'_j$. It follows that if an orbit of $\widetilde{\Psi}$ crosses T'_i transversely and enters $(\widetilde{T}'_i)^-$, it cannot intersect $\widetilde{T}'_i$ after that, and hence stays trapped in $(\widetilde{T}'_i)^-$. Moreover an element of Γ preserving $(\widetilde{T}'_i)^-$ also preserves its boundary $(\widetilde{T}'_i)^-$, because $\widetilde{P}(\Omega)$ is Γ invariant. Hence this element of Γ must be an element of H_i. It follows that the projection of $(\widetilde{T}'_i)^-$ in $M_\Gamma(\Omega)$ is a domain $(T'_i)^-$ disjoint from $\mathcal{M}$, with boundary T'_i, homotopic to $T'_i \times]0, +\infty|$, and such that every orbit of Ψ crossing T'_i and entering $(T'_i)^-$ remains trapped in $(T'_i)^-$.

The next step is to identify what are the entrance/exit Birkhoff annuli in the boundary $\partial P_\Gamma(\Omega)$. As before let $S^i_j =]x^i_j, x^i_{j+1}[\times]y^i_j, y^i_{j+1}[$ be a lozenge in Ω, projection of a Birkhoff band of a Birkhoff plane $\widetilde{T}'_i$. It corresponds to a transverse annulus $(A')^i_j$ of T'_i. One (and only one) of the sides $]x^i_j, x^i_{j+1}[$, $]y^i_j, y^i_{j+1}[$ is a gap of the minimal set $\tilde{\mu}$. If the gap is the horizontal side $]x^i_j, x^i_{j+1}[$, we define:

$$\Delta'(\tilde{\theta}^i_j) := \{(x,y) \in \Omega \ | \ x^i_j < x < x^i_{j+1}, \ \tilde{\tau}_0^{-1}(y^i_{j+1}) < y < y^i_j\}$$

If the gap is $]y^i_j, y^i_{j+1}[$, we define:

$$\Delta'(\tilde{\theta}^i_j) := \{(x,y) \in \Omega \ | \ \tilde{\tau}_0^{-1}(x^i_{j+1}) < x < x^i_j, \ y^i_j < y < y^i_{j+1}\}$$

Then, as in the case of the geodesic flow (Section 4.2) the first case is the case where A'_i is an exit annulus, whereas in the second case, A'_i is an entrance annulus. Indeed, for example in the second case, the lozenge S^i_j cannot be crossed by a horizontal-unstable leaf containing an orbit of $\widetilde{\mathcal{M}}$, hence S^i_j is crossed by vertical-stable leaves containing orbits of $\widetilde{\mathcal{M}}$. Therefore, the projection of an orbit in such a stable leaf must accumulate in the future in an element of $\mathcal{M}$, meaning than it cannot enter in $(T'_i)^-$ since it would be a "non-return in $P_\Gamma(\Omega)$" option (compare with [Ba3, Corollaire 3.15]).

Furthermore, similarly to the situation in [Ba3], the triangles $\Delta'(\tilde{\theta}^i_j)$ are the projections in Ω of the orbits in $(\widetilde{T}'_i)^-$ trapped between the part of the stable and unstable leaves of $\tilde{\theta}^i_j$. The reason is that $\Delta'(\tilde{\theta}^i_j)$ is one of the four connected components of Ω with the stable and unstable leaves of $\tilde{\theta}^i_j$ removed. This is true even if one blows up the vertical interval $](\tilde{\tau}_0)^{-1}(y^i_{j+1}, y^i_j[)$ because we only consider points in Ω. This is different from what happened in [Ba3]. This component is not one of the components whose projection contains a lozenge adjacent to $\tilde{\theta}^i_j$, hence orbits in $\Delta'(\tilde{\theta}^i_j)$ do not cross $\widetilde{T}'_i$. Furthermore, since one of the sides of $\Delta'(\tilde{\theta}^i_j)$ is a gap of $\tilde{\mu}$, $\Delta'(\tilde{\theta}^i_j)$ contains no element of $\widetilde{\mathcal{M}}$, hence cannot be in the quadrant of $\tilde{\theta}^i_j$ containing elements of $\widetilde{\mathcal{M}}$ accumulating non-trivially in $\tilde{\theta}^i_j$, i.e. the $(\widetilde{T}'_i)^+$ side. The claim follows.

Now we prove a very important property:

Lemma 4.6. *The set $P_\Gamma(\Omega)$ is compact.*

Proof. Let U be a relatively compact open neighborhood in $M_\Gamma(\Omega)$ of the (compact) union of all the tori T'_i and the compact invariant set $\mathcal{M}$. Let $\theta = (x,y)$ be an element of Ω. Here we think of $\theta = (x,y)$ (or another element of Ω) as both a point in the plane and as an orbit in $\widetilde{M}(\Omega)$.

We have three possibilities:

– either x and y both lie in $\tilde{\mu}$; then $\theta \in \widetilde{\mathcal{M}}$,

– or one of them lies in $\tilde{\mu}$, and the other lies in a gap. Suppose that x lies in a gap $]a,b[$. There is (z,y) in $\widetilde{\mathcal{M}}$. Then the orbit of the flow associated with (x,y) is backards asymptotic to an orbit (z,y) of $\widetilde{\mathcal{M}}$ and since x is in a gap, the forward orbit leaves $\widetilde{P}(\Omega)$ so the orbit (x,y) crosses some $\widetilde{T}'_i$ in a point w. So this orbit is in $\widetilde{P}(\Omega)$ flow backwards from w and is in $(\widetilde{T}'_i)^-$ flow forwards from w,

– The final possibility is that x and y are both outside the minimal set $\tilde{\mu}$. In this case there are two possibilities. One possibility is that θ lies in a triangle $\Delta'(\tilde{\theta}^i_j)$. In this case the orbit does not intersect any $\widetilde{T}'_i$ and hence this orbit projects to an orbit in $M_\Gamma(\Omega)$ outside $P_\Gamma(\Omega)$. The other possibility is that the orbit enters $\widetilde{P}(\Omega)$. Since x is in a gap of $\tilde{\mu}$ then this orbit has to exit $\widetilde{P}(\Omega)$ through some $\widetilde{T}'_i$. Similarly since y is in a gap then the orbit has to enter $\widetilde{P}(\Omega)$. In other words θ lies in two different lozenges associated with chains of lozenges of tori, and it crosses two different Birkhoff planes.

It follows that for every point p in $P_\Gamma(\Omega)$, the future orbit of p either intersects one Birkhoff torus T'_i and afterwards enters in $(T'_i)^-$, or the orbit is forward asymptotic to an orbit in $\mathcal{M}$. In both cases, there is a non negative time t such $\Psi^t(p)$ lies in U. The same t will apply to a neighborhood of p as U is open. In the same way there is a non positive t so that $\Psi_t(p)$ is also in U.

Now since the boundary of U is compact, we claim that there is a uniform positive upper bound T, meaning that for every p in $P_\Gamma(\Omega)$ there is a time $0 < t < T$ such that $\Psi^t(p)$ lies in U. We explain further: for any p let t_p be the infimum of $t > 0$ so that $\Psi_t(p)$ is in U. If p is in U then this is zero and $\Psi_{t_p}(p)$ is in U. Otherwise $\Psi_{t_p}(p)$ is in ∂U and not in U. If the claim is not true there are p_i in $P_\Gamma(\Omega)$ so that (t_{p_i}) converges to infinity. We cannot get a convergent subsequence of the (p_i) as we do not know yet that $P_\Gamma(\Omega)$ is compact. But up to subsequence asssume that $q_i = \Psi_{t_{p_i}}(p_i)$ converges to q in ∂U as this is compact. The Ψ orbit segments from p_i to q_i intersect $U \cup \partial U$ only in q_i. These orbit segments have length converging to infinity at $t_i \to \infty$ and they are entirely contained in $P_\Gamma(\Omega)$. It follows that the backward orbit of q is contained in $P_\Gamma(\Omega)$ and does not intersect U. But this contradicts the fact that there is $t' \leq 0$ so that $\Psi_{t'}(q)$ is in U. This proves the claim.

Similarly, there is a time T' such that for every q in $P_\Gamma(\Omega)$ there is a time t with $-T' < t < 0$ such that $\Psi^t(p)$ lies in U. It follows that every p in $P_\Gamma(\Omega)U$ lies in a segment of orbit of time-length $< T + T'$ joining two points in ∂U. Since the closure of U is compact, it now follows that $P_\Gamma(\Omega)$ is compact.

This proves the lemma. □

Finally, $P_\Gamma(\Omega)$ is irreducible (since its universal covering $\widetilde{P}(\Omega)$ is contractible) and its fundamental group Γ is the fundamental group of the Seifert manifold P_0: it follows that $P_\Gamma(\Omega)$ is homeomorphic to P_0. This follows from Scott's result that there are no fake Seifert fibered spaces [Sco2]. The boundary components of $P_\Gamma(\Omega)$ are embedded Birkhoff tori, whose associated chain of lozenges in the orbit space Ω is prescribed from the beginning.

Definition 4.7. *The manifold with boundary $P_\Gamma(\Omega)$, equipped with the restriction of the oriented foliation Ψ and the restrictions $\hat{\Lambda}^s_\Gamma$, $\hat{\Lambda}^u_\Gamma$ of the stable, unstable foliations $\Lambda^s(\Psi)$, $\Lambda^u(\Psi)$, is a* ***hyperbolic blow up piece of a geodesic flow.***

Remark 4.8. During the construction, we have performed several choices: the maps α_1, β_1, χ, the boundary tori T'_i. But the orbits in $M_\Gamma(\Omega)$ intersecting $P_\Gamma(\Omega)$ are the projections of the elements of Ω which are not in the triangles $\Delta'(\tilde{\theta}^i_j)$. Consider a modification of ρ_0 in a gap I of $\tilde{\mu}$. In other words α_1 and β_1 are not id and $\tilde{\tau}_0$ in that gap. But the corresponding graphs of α_1 and β_1 are both in the excluded triangles: the graph of α_1 is in $\Delta'(\tilde{\theta}^i_j)$ for some j and the graph of β_1 is in $\Delta'(\tilde{\theta}^i_{j+1})$. Hence the region in Ω corresponding to orbits intersecting $P_\Gamma(\Omega)$ does not depend on these choices. In other words, the choices of α_1, β_1, χ only contribute to the definition of the flow in the regions $(T'_i)^-$, i.e. outside $P_\Gamma(\Omega)$.

Hence the only choice that matters is the selection of the embedded Birkhoff tori T'_i. But, as we will see in the proof of Theorem 7.1, it follows from Remark 2.15 that the hyperbolic blow up $(P_\Gamma(\Omega), \hat{\Lambda}^s_\Gamma, \hat{\Lambda}^u_\Gamma)$ does not depend on these choices, and therefore, is uniquely defined, up to orbital equivalence, by the initial piece of geodesic flow and the number of tangent periodic orbits introduced in every boundary torus.

5. Leaf spaces and orbit spaces associated to a free Seifert piece

From now on, we consider a free Seifert piece P of an arbitrary pseudo-Anosov flow (M, Φ). We denote by h an element of $\pi_1(P)$ represented by a regular fiber of the Seifert fibration. Therefore, h is a generator of the pseudo-center of $\pi_1(P)$. Recall that $\mathcal{H}^s$ (respectively $\mathcal{H}^u$) is the leaf space of $\widetilde{\Lambda}^s$ (resp. $\widetilde{\Lambda}^u$).

5.1. **Existence of h-invariant axis in the leaf spaces.**

Proposition 5.1. *The action of h on $\mathcal{H}^s$ (respectively $\mathcal{H}^u$) admits an axis $\mathcal{A}^s$ (respectively $\mathcal{A}^u$) which is a properly embedded real line. In other words these embedded lines are h-invariant and the action of h on each of them is free, i.e. an action by translation.*

Proof. In the case that P is all of M this was proved in Theorem 4.1 of [Ba-Fe1]. The proof in the case P is not all of M is similar. We refer to [Ba-Fe1] whenever details are the same as in the case $P = M$. Consider the action of h on $\mathcal{H}^s$. Since P is a free Seifert piece, h acts freely on $\mathcal{H}^s$. In [Fe5] it is shown that there is a unique axis $\mathcal{A}^s$ for h. There are two options: either $\mathcal{A}^s$ is a real line or $\mathcal{A}^s$ is an infinite union of closed intervals. First we show that the second case cannot happen. Suppose that

$$\mathcal{A}^s \ = \ \bigcup_{i\in\mathbb{Z}}[x_i, y_i] \ = \ \bigcup_{i\in\mathbb{Z}} B_i,$$

where $B_i = [x_i, y_i]$ are closed segments in $\mathcal{H}^s$ and y_i is not separated from x_{i+1}. Since the axis $\mathcal{A}^s$ is unique and h is in the pseudo-center of $\pi_1(P)$, then every element of $\pi_1(P)$ permutes the collection $\{B_i\}$. In fact $\pi_1(P)$ acts on the indexing set of this collection which is $\mathbb{Z}$. The action preserves elements being neighbors. Some elements can reverse the order in $\mathbb{Z}$. As in Case 2 of Theorem 4.1 of [Ba-Fe1], $\pi_1(P)$ acts on $\mathbb{Z}$ and has a subgroup of index ≤ 4 which is $\mathbb{Z}^2$. The difference from [Ba-Fe1] is that when $P = M$ this quickly implies a contradiction, which is not the case here. In any case since M is assumed orientable, then Lemma 5.3 of [Ba-Fe1] implies that P is either $T^2 \times [0, 1]$, where T^2 is the torus; or P is a twisted I bundle over the Klein bottle.

In the first case recall that the torus decomposition into Seifert fibered pieces is minimal, the only possibility is that P is the only piece of the JSJ decomposition and M would be obtained by gluing one to the other the boundary components of M. Then M would be a torus bundle over the circle, hence, by [Ba-Fe1], the pseudo-Anosov flow would be the suspension of a linear diffeormorphism: it is in contradiction with the hypothesis $\mathcal{A}^s \neq \mathbb{R}$.

In the second case $\pi_1(P) = \pi_1(K) =< a, b|aba^{-1} = b^{-1} >$, where K is the Klein bottle. In this situation the only possible regular fibers for a Seifert fibration of P are represented either by a^2 or b. In our situation $\pi_1(P)$ acts on $\mathbb{Z}$ and h acts freely on $\mathbb{Z}$. Otherwise for some j, $h(B_j) = B_j$ and either $h(x_j) = x_j$ or if h reversed the orientation in $[x_j, y_j]$ then h would fix an interior point of $[x_j, y_j]$, both options contradict the free action of h on $\mathcal{H}^s$. Therefore h acts as a translation on $\mathbb{Z}$.

Suppose first that h is represented by b. Then $aha^{-1} = h^{-1}$ $(*)$. If a acts freely on $\mathbb{Z}$, it is a translation and the equation is impossible. If a fixes an element in $\mathbb{Z}$, then so does a^2 and so P has a Seifert fibration where the fiber does not act freely and P would be a periodic Seifert piece, contradiction to assumption. Now suppose that h is represented by a^2. Hence a^2 acts freely on $\mathcal{H}^s$ and as proved in [Fe5], a also acts freely on $\mathcal{H}^s$. If b acts freely, this contradicts equation $(*)$ above. If b does not act freely then P has a Seifert fibration where the fiber is periodic, contradiction. We conclude that this case for $\mathcal{A}^s$ cannot happen.

Conclusion $-$ The axis of h is a real line $\mathcal{A}^s$.

Now we need to check whether $\mathcal{A}^s$ is properly embedded or not. Examples where an axis of an element acting freely on $\mathcal{H}^s$ is not properly embedded are very common and occur for instance in the Bonatti-Langevin example [Bo-La]. Parametrize the leaves of $\widetilde{\Lambda}^s$ in $\mathcal{A}^s$ as $\{s(t), t \in \mathbb{R}\}$ where $s(t)$ separates $s(t')$ from $s(t")$ if and only if t is between t' and $t"$ in $\mathbb{R}$. Without loss of generality suppose that $\mathcal{A}^s$ is not properly embedded for $t \to \infty$, so

$$\lim_{t\to\infty} s(t) \ = \ \bigcup_{i\in I} C_i$$

where C_i is a collection of leaves non separated from each other in the side the collection $\{s(t)\}$ is limiting on. The set I is an interval in $\mathbb{Z}$ which can be either finite or $\mathbb{Z}$ [Fe2, Fe3]. Since $< h >$ is normal in $\pi_1(P)$ and $\mathcal{A}^s$ is the unique axis for the action of h on $\mathcal{H}^s$, it follows that $\pi_1(P)$ preserves $\mathcal{A}^s$. In addition a subgroup G of index at most 2 in $\pi_1(P)$ preserves orientation in $\mathcal{A}^s$, so it preserves the collection $\{C_i, i \in I\}$. This already implies that $I = \mathbb{Z}$ and that $\pi_1(P)$ has a finite index subgroup isomorphic to $\mathbb{Z}^2$. In the first part of this proof we showed that then the pseudo Anosov flow is a suspension, and therefore $\mathcal{H}^s = \mathcal{A}^s$ is an embedded real line as required.

We conclude that both $\mathcal{A}^s$ and $\mathcal{A}^u$ are homeomorphic to $\mathbb{R}$ and each is properly embedded in $\mathcal{H}^s$ or $\mathcal{H}^u$ respectively. ☐

Remark 5.2. One checks easily that element of $\mathcal{A}^s$ are characterized by the following property: an element s of $\mathcal{H}^s$ lies in $\mathcal{A}^s$ if and only if there is a component U of the complement of s in $\mathcal{H}^s$ such that $h(U) \subset U$. Indeed, if s lies in $\mathcal{A}^s$, then we can take as U the component containing $h(s)$. If s does not lie in $\mathcal{A}^s$, then for any connected component U of $\mathcal{H}^s - \{s\}$ we have:

– either U contains $\mathcal{A}^s$: then s lies in $h(U)$ which therefore cannot be contained in U;

– or U is disjoint from $\mathcal{A}^s$: in this case, U and $h(U)$ are disjoint.

Similarly, an element u of $\mathcal{H}^u$ lies in $\mathcal{A}^u$ if and only if there is a component U of the complement of s in $\mathcal{H}^u$ such that $h(U) \subset U$.

Remark 5.3. From now on, we fix the orientation of $\mathcal{A}^s$ and $\mathcal{A}^u$ (hence of $\mathcal{H}^s$ and $\mathcal{H}^u$) so that, for the induced total order on $\mathcal{A}^s$ and $\mathcal{A}^u$ we have $h(x) > x$ for every leaf x.

5.2. **Identifying Ω' in the orbit space.** In this section, we construct a $\pi_1(P)$-invariant domain Ω_P in the orbit space $\mathcal{O}$ of our fixed flow Φ. This domain is naturally identified to a domain Ω' in $\mathcal{A}^s \times \mathcal{A}^u$, enjoying properties similar to the properties satisfied by the domain Ω described in Section 4.3.2.

Given our fixed pseudo-Anosov flow Φ, let

$$\Omega' \; = \; \{(s,u) \in \mathcal{A}^s \times \mathcal{A}^u \mid s \cap u \; \neq \; \emptyset\}$$

Then, the map $\flat : \Omega' \to \mathcal{O}$ mapping (s,u) into the unique intersection point between s and u is continuous. This intersection point is an orbit of $\widetilde{\Phi}$. We denote by Ω_P the image of $\flat$.

The first step is to prove that this set is non-empty.

The properties of the stable and unstable foliations are similar, up to inversion of the flow. In the following lemma, properties are stated for both foliations, but the proof is written for only one of them.

Lemma 5.4. *The set Ω_P intersects every leaf in $\mathcal{A}^s$ and every leaf in $\mathcal{A}^u$.*

Proof. Let s be an element of $\mathcal{A}^s$. We abuse notation and think of s both as a leaf in $\mathcal{A}^s$ and as a subset of the orbit space $\mathcal{O}$. Assume by contradiction that s does not intersect any leaf in $\mathcal{A}^u$. In particular since $\mathcal{A}^u$ is h-invariant then $h^n(s)$ also does not intersect any leaf in $\mathcal{A}^u$. Define C_0 as the unique connected component of $\mathcal{O} - s$ which contains $h^{-1}(s)$ (despite the notation, C_0 is not a lozenge). The set C_0 It is an open subset, saturated by $\mathcal{O}^s$, with boundary contained in s. For every integer n, let $C_n = h^n(C_0)$. Then every C_n is contained in C_{n+1}, more precisely, the closure of C_n is contained in C_{n+1}.

The union C_∞ of all the C_n is a h-invariant open subset of $\mathcal{O}$, saturated by $\mathcal{O}^s$. We claim that C_∞ is the entire $\mathcal{O}$. If not, since $\mathcal{O}$ is connected, C_∞ is not closed: there is an element p of $\mathcal{O} - C_\infty$ and a sequence of points p_n converging to p, such that every p_n lies in some $C_{k(n)}$. Consider a small neighborhood U of p: for n sufficiently big, every p_n lies in U. If U intersects only finitely many different iterates $h^k(s)$, then every p_n eventually belongs to the same C_k, hence p lies in the closure of C_k, i.e. in $h^k(s)$. But then $p \in C_{k+1}$, contradiction. Therefore, U intersects infinitely many iterates $h^n(s)$. It means that the leaf $\mathcal{O}^s(p)$ is a limit of iterates of the leaf s. Hence $h(\mathcal{O}^s(p))$ is non separated from $\mathcal{O}^s(p)$, a contradiction to the axis of h being properly embedded in $\mathcal{H}^s$.

We have proved that C_∞ is the entire orbit space. Let u be a leaf in $\mathcal{A}^u$, and q an element of u. Let n be the smaller integer such that q belongs to C_n. Since u is disjoint from $h^n(s)$, we have $u \subset C_n - C_{n-1}$. In fact, the union of unstable leaves belonging to $\mathcal{A}^u$ is connected because $\mathcal{A}^u$ is a properly embedded line in $\mathcal{H}^u$. It follows that all the elements of $\mathcal{A}^u$ are leaves contained in $C_n - C_{n-1}$. But this is a contradiction since $\mathcal{A}^u$ is h-invariant and that $C_n - C_{n-1}$ is disjoint from its h-iterates. □

Let $z \in \mathcal{O}$. A stable prong of z is a component of $\mathcal{O}^s(z) - \{z\}$. Sometimes we include z itself in the prong. The point z is singular if and only if there are more than two prongs at z. We also say these are the prongs of $\mathcal{O}^s(z)$ at z.

Lemma 5.5. *Let p be a point in Ω_P. Then there are at most two stable (respectively unstable) prongs at p intersecting Ω_P.*

Proof. Let s be the stable leaf of p. First assume that there are 3 points p_1, p_2, p_3 in Ω_P lying in different prongs of s at p. Let u_1, u_2, u_3 be the unstable leaves of p_1, p_2, p_3. The leaves u_1, u_2, u_3 all lie in the axis $\mathcal{A}^u$, which is a line. Hence one them, say u_1, must disconnect the other two. On the other hand, one can join u_2 to u_3 by a segment in s avoiding the prong containing p_1. This stable segment does not intersect u_1, hence u_1 does not disconnect u_2 from u_3. Contradiction. □

Lemma 5.6. *The intersection between Ω_P and a stable (or unstable) leaf is a segment.*

Proof. According to Lemma 5.5 we just have to prove that the intersection between a stable leaf s and Ω_P is connected. Let p_1, p_2 be two elements of $s \cap \Omega_P$, and p any element of the segment $[p_1, p_2]$ in s. Then the unstable leaf $\mathcal{O}^u(p)$ disconnects $\mathcal{O}^u(p_1)$ from $\mathcal{O}^u(p_2)$. It follows that it must be an element of $\mathcal{A}^u$. Therefore, $p = s \cap \mathcal{O}^u(p)$ lies in Ω_P. □

Lemma 5.7. *For any element z of Ω_P, there is a point p in $\mathcal{O}^s(z)$ so that p is in the interior of a segment I contained in $\mathcal{O}^u(p) \cap \Omega_P$. In particular if q is in $I \subset \mathcal{O}^u(p)$, then $\mathcal{O}^s(q)$ is in the stable axis $\mathcal{A}^s$.*

Proof. Let s be the stable leaf through z, and let u be the unstable leaf through z. We have $s \in \mathcal{A}^s$, $u \in \mathcal{A}^u$.

Case 1 - The stable leaf s is not singular. In this case, we just take $p = z$. Consider the projection $\mathcal{O} \to \mathcal{H}^s$ taking a point to the stable leaf containing it. The projection in $\mathcal{H}^s$ of the points in u near p lie in a neighborhood of s in $\mathcal{H}^s$. Since s is not a branch point of $\mathcal{H}^s$, then these projections are in $\mathcal{A}^s$ if the points in u are sufficiently close to z. Therefore, a small segment in u is the required interval I.

Case 2 - s is singular

Case 2.1 - z is the singular point. Let (s_i) be a sequence in $\mathcal{A}^s$, with $s_i > s$ and (s_i) converging to s in $\mathcal{H}^s$. Since z is the singular point in s then for big enough i the leaf u intersects s_i, consequently a prong of u at z intersects s_i. It is crucial here that z is the singular point, for otherwise this may not be true. The same reasoning applies if $s_i < s$ in $\mathcal{A}^s$. Hence there are exactly two prongs of u at z which project, near z, into the axis $\mathcal{A}^s$. Choose small segments of u in these prongs. The union (including z) is the segment I as desired.

Case 2.2 - z is not the singular point. Let q be the singular point in s. Once more, since a neighborhood of s in $\mathcal{H}^s$ is described by the projections of all the unstable prongs at q, there are exactly two prongs I_1, I_2 of $\mathcal{O}^u(q)$ at q, so that for any $w \in I_1 \cup I_2$ then $\mathcal{O}^s(w)$ is in $\mathcal{A}^s$.

Let s_1 be the prong of s at q containing z. Suppose first that there is no prong of $\mathcal{O}^s(q) = s$ at q between s_1 and I_1 and also no prong of $\mathcal{O}^s(q) = s$ at q between s_1 and I_2. Then, the projection of $I_1 \cup \{q\} \cup I_2$ in $\mathcal{H}^s$ near q coincides with the projection of $u = \mathcal{O}^u(z)$ near z. In this case let $p = z$ and I a small unstable segment in u containing p in the interior and we are done.

Hence we are left with the case that say there is a prong s' of s at q between s_1 and I_1. Let s^* be a leaf in $\mathcal{A}^s$ intersecting I_1 in w^*. Then, according to Lemma 5.4, s^* contains a point x_1 in Ω_P, and hence $\mathcal{O}^u(x_1)$ is in $\mathcal{A}^u$. Suppose first that x_1 is not w^*. Then, $\mathcal{O}^u(q)$ disconnects $\mathcal{O}^u(x_1)$ from u. Since u and $\mathcal{O}^u(x_1)$ both lie in $\mathcal{A}^u$, it follows that $\mathcal{O}^u(q)$ lies in $\mathcal{A}^u$. On the other hand if $x_1 = w^*$ then obviously $\mathcal{O}^u(x_1) = \mathcal{O}^u(w^*) = \mathcal{O}^u(q)$ (because w^* is in $I_1 \subset \mathcal{O}^u(q)$) is also in $\mathcal{A}^u$.

Hence $\mathcal{O}^u(q)$ is in $\mathcal{A}^u$ and $\mathcal{O}^s(q) = s$ is in $\mathcal{A}^s$. Therefore, q lies in Ω_P. We are back to the situation of Case 2.1: let $p = q$, and let I be a small segment contained in $I_1 \cup \{q\} \cup I_2$, with p in the interior.. □

Let $p_s : \Omega_P \to \mathcal{H}^s$ be the projection map.

Lemma 5.8. *Ω_P is pathwise connected, in fact any two points p, q in Ω_P are connected by a piecewise path made of stable and unstable segments.*

Proof. By Lemma 5.7, for any s in $\mathcal{A}^s$ there is an open segment O in $\mathcal{A}^s$ containing s so that any point q in $\Omega_P \cap (p_s)^{-1}(O)$ can be reached from any point r in $s \cap \Omega_P$ by a desired path. That is, from a point z in $s \cap \Omega_P$ go to the point p as in Lemma 5.7 then along the segment I along an unstable leaf as in Lemma 5.7 and then along the stable leaf to the point r. We are also using Lemma 5.6 to obtain this. Any closed interval J in $\mathcal{A}^s$ can be covered by these open intervals and hence has a finite subcovering. Concatenating the paths above proves the Lemma. □

By Lemmas 5.4 and 5.6, for every element s of $\mathcal{A}^s$ the intersection $s \cap \Omega_P$ is a non-empty segment (possibly a single point). The projection $I^u(s)$ of $s \cap \Omega_P$ in $\mathcal{H}^u$ is then a segment in $\mathcal{A}^u \approx \mathbb{R}$. Observe that this interval can be open or closed at each of its extremities. In fact, the interval may be closed at an extremity whenever there are singular orbits at the "boundary" of the Seifert piece P. Moreover, a priori the interval can be the entire axis $\mathcal{A}^u$ or a ray in it. Recall that we have a total order $<$ on $\mathcal{A}^u \approx \mathbb{R}$ such that for every u in $\mathcal{A}^u$ we have $h(u) > u$, and similarly for $\mathcal{A}^s$. We can then define functions α_s, $\beta_s : \mathcal{A}^s \to \mathcal{A}^u \cup \{\pm\infty\}$ with the following conventions:

– If $I^u(s)$ is not bounded from above in $\mathcal{A}^u$, then let $\beta_s(s) = +\infty$, otherwise let

$$\beta_s(s) \;=\; \mathrm{Sup}\{\, v \mid v \in I^u(s)\} \;\in\; \mathcal{A}^u$$

– If $I^u(s)$ is not bounded from below in $\mathcal{A}^u$, then let $\alpha_s(s) = -\infty$, otherwise let

$$\alpha_s(s) \;=\; \mathrm{Inf}\{\, v \mid v \in I^u(s)\} \;\in\; \mathcal{A}^u$$

We can define in a similar way two functions α_u, $\beta_u : \mathcal{A}^u \to \mathcal{A}^s \cup \{\pm\infty\}$. Observe that all these maps are clearly h-equivariant.

We insist on the fact that $\alpha_s(s)$ and $\beta_s(s)$ even if finite, may or may not belong to $I^u(s)$. Actually:

Lemma 5.9. *If $\alpha_s(s)$ is the projection of a point z in $s \cap \Omega_P$, that is, $\alpha_s(s)$ is in $I^u(s)$, then the leaf $\alpha_s(s) = \mathcal{O}^u(z)$ is singular. Similarly for $\beta_s(s)$, and the stables leaves $\alpha_u(u)$, $\beta_u(u)$.*

Proof. Suppose that $\mathcal{O}^u(z)$ is not singular. Then we are in Case 1 of Lemma 5.7, switching stable and unstable, and we can take $p = z$. The conclusion of that Lemma is that z is in the interior of a segment I contained in $\mathcal{O}^s(z) \cap \Omega_P$, and again since $\mathcal{O}^u(z)$ is non singular this segment projects in $\mathcal{H}^s$ to a neighborhood of $\mathcal{O}^u(z)$ in $\mathcal{H}^u$. In particular the segment also projects to a neighborhood of $\mathcal{O}^u(z)$ in $\mathcal{A}^u$. Therefore $\mathcal{O}^u(z)$ cannot be $\beta_s(s)$, contradiction. □

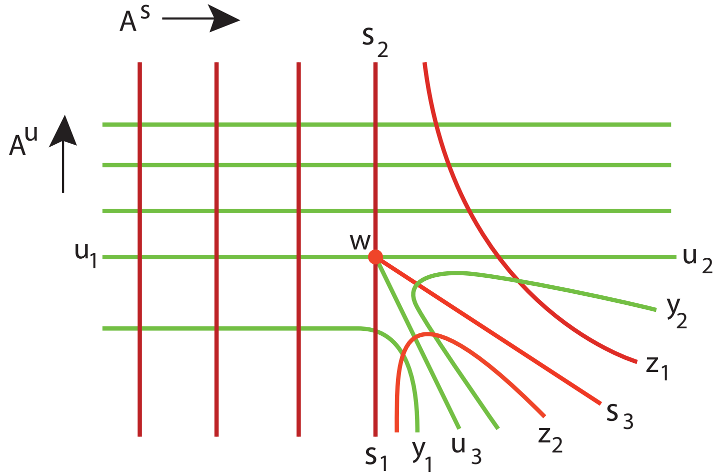

Figure 8: *Boundary points in Ω_P. The singular orbit w is in Ω_P. The figure depicts the situation of a 3-prong orbit. The unstable prongs u_1, u_2 of w are contained in Ω_P and the unstable prong u_3 is not in Ω_P. Similarly the stable prongs s_1, s_2 are contained in Ω_P and s_3 is not. We also depict additional leaves: y_1, y_2 unstable leaves, y_1 is in $\mathcal{A}^u$, y_2 is not in $\mathcal{A}^u$; and z_1, z_2 stable leaves, where z_1 is in $\mathcal{A}^s$ and z_2 is not in $\mathcal{A}^s$. It follows that the unstable prong u_2 is contained in the boundary of Ω_P and so is the stable prong s_1. Locally the boundary of Ω_P near w is made up of $s_1 \cup \{w\} \cup u_2$.*

Remark 5.10. The situation that $\alpha_s(s)$ is the projection of a point z in $s \cap \Omega_P$ is quite possible, and occurs if there is a p-prong singularity in the "boundary" of P, in other words, the singularity is in the union of the boundary Birkhoff annuli. An example of this is the following: consider a branched cover of the geodesic flow in the unit tangent bundle M^* of a hyperbolic surface S with a simple geodesic β of symmetry, so that $S - \{\beta\}$ is the union of two surfaces S_1, S_2. Then T_1S_1 survives in the branched cover M, and it generates a free piece P of the branched pseudo-Anosov flow that has a singularity in the boundary of P. Lift this to $\widetilde{M}$. We choose w a lift of the corresponding singular orbit in the boundary of P, so that $\mathcal{O}^s(w)$ and $\mathcal{O}^u(w)$ are in $\mathcal{A}^s, \mathcal{A}^u$ respectively. Since w is singular, Case 1 of Lemma 5.7 implies that there are exactly two prongs s_1, s_2 of $\mathcal{O}^s(w)$ at w so that they are contained in Ω_P. As w is singular there are other prongs of $\mathcal{O}^s(w)$ at p that are not contained in Ω_P. In the same way there are two unstable prongs u_1, u_2 at p contained in Ω_P. The four prongs s_1, s_2, u_1, u_2 have to be consecutive around w, that is, after rearranging we may assume that they go in the following order: s_1, u_1, s_2, u_2. In other words s_1, u_1 bound a "quadrant" of the stable and unstable foliations at w. A *quadrant* is a component of the complement of the union of $\mathcal{O}^s(w)$ and $\mathcal{O}^u(w)$. The quadrant in question is contained in Ω_P. Similarly for u_1, s_2 and s_2, u_2. There are exactly 3 quadrants at w contained in Ω_P. If w is a p-prong singular orbit, then in total there are $2p$ quadrants at w, and $2p-3$ open quadrants disjoint from Ω_P. It now follows that u_1 and s_2 are in the interior of Ω_P and s_1, u_2 are in the boundary of Ω_P. If s_3 is another stable prong at w and y_2 is an unstable leaf intersecting s_3 very near w, then y_2 is <u>not</u> in $\mathcal{A}^u$ and hence the intersection $u \cap s_3$ is not in Ω_P. We refer to figure 8.

Lemma 5.11. *The functions α_s, β_s (respectively α_u, β_u) take values in $\mathcal{A}^u$ (respectively $\mathcal{A}^s$).*

Proof. The meaning is that these functions never attain the values $\pm\infty$. We only prove the statement for α_s, the proof for the other functions is similar. Let s_0 be an element of $\mathcal{A}^s$ such that $\alpha_s(s_0) = -\infty$. Then for every integer n we have $\alpha_s(h^n(s_0)) = -\infty$, because h preserves the whole set $\mathcal{A}^u$. Assume that there is an element s in the interval $(s_0, h(s_0))$ such that $\alpha_s(s)$ is an element u of $\mathcal{A}^u$. Then, there is an element u' of $\mathcal{A}^u$ strictly smaller than all of u, $\beta_s(s_0)$, and $\beta_s(h(s_0))$, since $\alpha_s(s_0), \alpha_s(h(s_0))$ are both equal to $-\infty$. According to Lemma 5.8 there is a stable/unstable path in Ω_P connecting (s_0, u') to $(h(s_0), u')$. Since this path is contained in Ω_P, it must cross the set $I^u(s)$. Hence the path must contain a point (s, u'') with $u'' \geq u > u'$. Since the path starts at (s_0, u') and finishes at $(h(s_0), u')$, the path must cross, for every element v of the interval (u', u) in $\mathcal{A}^u$, the v-level (that is, intersects the unstable leaf v) twice. This means that the path contains an element of the form (s_-, v) with $s_0 \leq s_- < s$, and another element (s_+, v) with $s < s_+ \leq h(s_0)$. Hence, intersection between the unstable leaf v and Ω_P admits at least two connected components: one containing $s_- \cap v$, and the other containing $s_+ \cap v$. Observe that the intersection cannot contain $s \cap v$ since $v < u = \alpha_s(s)$. This contradicts Lemma 5.6.

Therefore, α_s attains the value $-\infty$ on the entire segment $[s_0, h(s_0)]$.

There is a stable/unstable path from (s_0, t_1) in Ω_P to $h(s_0, t_2)$ in Ω_P for some $t_1, t_2 \in \mathcal{A}^u$. By compactness this path attains a mininum $t \in \mathcal{A}^u$ therefore t intersects every leaf $s \in [s_0, h(s_0)]$ because $\alpha_s \equiv -\infty$ in this interval. Now recall that $\mathcal{A}^u$ is properly embedded in $\mathcal{H}^u$. Therefore for every stable leaf $s \in [s_0, h(s_0)]$, s does not intersect any other unstable leaf in the negative direction. This implies that the unstable segment in t from $(s_0 \cap t)$ to $(h(s_0) \cap t)$ is the base segment of a stable product region. By Theorem 2.7 it follows that Φ is topologically conjugate to a suspension, contrary to hypothesis that $\pi_1(P)$ is not elementary in our study of free Seifert pieces P. This finishes the proof of the Lemma. □

Lemma 5.12. *The functions α_s, $\beta_s : \mathcal{A}^s \to \mathcal{A}^u$ are both weakly monotone increasing.*

Proof. Suppose by way of contradiction that there is $s_2 > s_1$ in $\mathcal{A}^s$ with $\alpha_s(s_2) < \alpha_s(s_1)$. Let k be the unique non negative integer such that $s_1 < h^{-k}(s_2) \leq h(s_1)$. Then

$$\alpha_s(h^{-k}(s_2)) = h^{-k}\alpha_s(s_2) < \alpha_s(s_2) < \alpha_s(s_1),$$

hence we can assume without loss of generality that $k = 0$, ie. that $s_1 < s_2 \leq h(s_1)$. Then, if we put $s_3 = h^{-1}(s_2)$, we have:

$$\alpha_s(s_3) = h^{-1}\alpha_s(s_2) < \alpha_s(s_2) < \alpha_s(s_1)$$

The contradiction is then obtained as in the proof of Lemma 5.11. There is a unstable leaf v in $\mathcal{A}^u$ such that $\alpha_s(s_3) < \alpha_s(s_2) < v < \alpha_s(s_1)$. The leaf v intersects two leaves s_-, s_+ with $s_3 \leq s_- < s_1$ and $s_2 \geq s_+ > s_1$, but v does not intersect s_1. Again this contradicts Lemma 5.6. □

Lemma 5.13. *For any s in $\mathcal{A}^s$, we have $\alpha_s(s) < \beta_s(s)$.*

Proof. Assume by contradiction that the equality $\alpha_s(s) = \beta_s(s)$ holds for some s in $\mathcal{A}^s$. This means that $s \cap \Omega_P$ is a single point z. Let u be the unstable leaf through z. Then $u = \alpha_s(s)$. By Lemma 5.9 u is singular. Let p be the singular point in u. If $p = z$, then by Case 2.1 of Lemma 5.7, with stable and unstable switched, the following happens: there are two stable prongs J_1, J_2 of $s = \mathcal{O}^s(z) = \mathcal{O}^s(p)$ at p so that J_1 and J_2 project to $\mathcal{A}^u$. In other words for any t in $J_1 \cup J_2$ then $\mathcal{O}^u(t) \in \mathcal{A}^u$. Then $J_1 \cup \{p\} \cup J_2$ is an in s entirely contained in Ω_P, contradiction.

Now we consider the case that p is not z. Since $\mathcal{O}^u(p) \in \mathcal{A}^u$ and p is singular, then as in the proof of Lemma 5.7, case 2.2, there are exactly two <u>stable</u> prongs I_1, I_2 at p which locally project into $\mathcal{A}^u$. Let v_1, v_2 in these prongs very near p, so $U_1 = \mathcal{O}^u(v_1), U_2 = \mathcal{O}^u(v_2)$ are in $\mathcal{A}^u$. Notice that s does not intersect either U_1 or U_2 and s does not disconnect U_1 from U_2. Since both U_1 and U_2 are in the same complementary component of s, then s separates either $h^{-1}(s)$ or $h(s)$ from both U_1 and U_2. Assume wlog the second option that is s separates $h(s)$ from both U_1 and U_2. Let W be the component of $\mathcal{O} - (U_1 \cup U_2)$ containing s. Replacing U_1 with U_2 if necessary we may assume that $U_1 < U_2$ in $\mathcal{A}^u$. This implies that U_2 separates $h(W)$ from W. In particular since $s \subset W$ then $h(s)$ is separated from s by U_2. But we proved above that s separates $h(s)$ from U_2, contradiction. This finishes the proof of the Lemma. □

In summary, we have proved that Ω' is a region very similar to the region Ω studied in section 4.3.2: whereas Ω is the region between two maps α_1, β_1, Ω' is the region in $\mathcal{A}^s \times \mathcal{A}^u$ limited by the graphs of two functions α_s, β_s which are weakly monotone, and which do not coincide anywhere. Ω' may contain elements of the graph of α_s or β_s. Notice that it is not necessarily true that if $\alpha_s(s) < u < \beta_s(s)$ in $\mathcal{A}^u$ then the point (s, u) is in the interior of Ω_P. For example it could be that (s, u) is in the interior of a nondegenerate segment $\{s\} \times [\alpha_s(s), u_0]$ that is contained in the boundary of Ω'.

With these properties one can prove that there are infinitely many periodic orbits “contained” in the piece P and uncountably many full orbits “contained” in P, so the dynamics of the flow in P is extremely complicated. We do not prove this separately as it follows immediately from the Main theorem.

5.3. **The minimal set and fixed points.** The group $\pi_1(P)$ acts on $\mathcal{A}^s$. The following result is standard:

Lemma 5.14. *There is a unique minimal set $\tilde{\mu}_s \subset \mathcal{A}^s$ which is non empty and $\pi_1(P)$ invariant. Since $\pi_1(P)$ is not virtually cyclic, the set $\tilde{\mu}_s$ can either be the entire $\mathcal{A}^s$ or a Cantor set. In particular $\tilde{\mu}_s$ is a perfect set. Similarly, there exist a unique minimal non-empty and $\pi_1(P)$ invariant set $\tilde{\mu}_u \subset \mathcal{A}^u$.*

Lemma 5.15. *The restrictions to $\tilde{\mu}_s$ of α_s and β_s are almost injective. More precisely: if s, s' are two elements of $\tilde{\mu}_s$ such that $\alpha_s(s) = \alpha_s(s')$ or $\beta_s(s) = \beta_s(s')$, then $s = s'$ or $]s, s'[$ is a connected component of $\mathcal{A}^s - \tilde{\mu}_s$. Similarly, the same property holds for the restrictions of α_u and β_u to $\tilde{\mu}_u$.*

Proof. Let U be the open subset of $\mathcal{A}^s$ comprising points where α_s (or β_s) is locally constant. Then $\mathcal{A}^s - U$ is a closed invariant subset, hence contains $\tilde{\mu}_s$. The Lemma follows. □

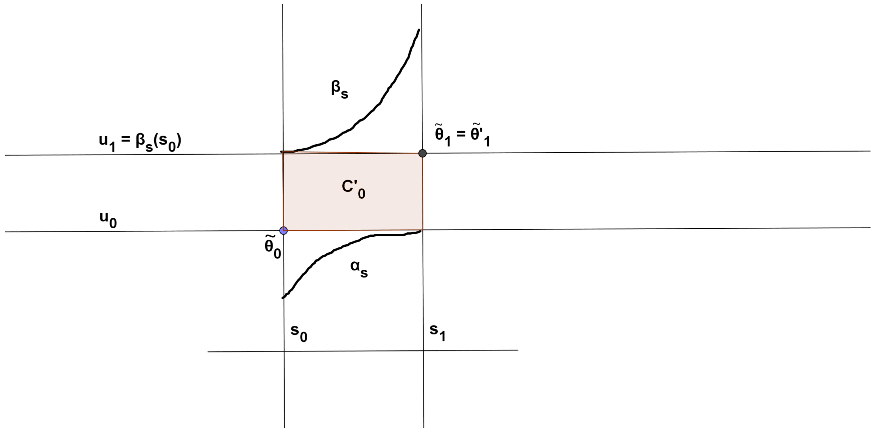

Figure 9: *A lozenge entirely in Ω_P.*

In the next section, we will improve Lemma 5.15 (see Corollary 6.5).

Let γ_0 be a non-trivial element of $\pi_1(P)$ preserving an element s_0 of $\mathcal{A}^s$. Replacing γ_0 by its square if necessary, we can assume that it commutes with h. Let $\tilde{\theta}_0$ be the unique fixed point of γ_0 in s_0, and let u_0 be the unstable leaf $\mathcal{O}^u(\tilde{\theta}_0)$. As we will see (s_0, u_0) may not belong to Ω', i.e. $\tilde{\theta}_0$ may be outside Ω_P. However, since s_0 lies in $\mathcal{A}^s$, and $u_0 = \mathcal{O}^u(\tilde{\theta}_0)$ with $\tilde{\theta}_0$ periodic, it follows that the leaf u_0 intersects a maximal segment $]s_{-1}, s_1[$ of stable leaves in $\mathcal{A}^s$ which has the property of being γ_0-invariant, and containing only one fixed point: the leaf s_0. Let $\tilde{\theta}_{-1}$ and $\tilde{\theta}_1$ be the γ_0-fixed points in respectively s_{-1} and s_1.

Notice that $\gamma_0 h(\tilde{\theta}_0) = h\gamma_0(\tilde{\theta}_0) = h(\tilde{\theta}_0)$. According to Theorem 2.11 there is a chain of lozenges C_1, C_2, ... C_k such that $\tilde{\theta}_0$ is corner of C_1 and $h(\tilde{\theta}_0)$ a corner of C_k. We extend it to a bi-infinite chain of lozenges $\{C_i\}_{(i\in\mathbb{Z})}$ which is H-invariant where H is the group generated by γ_0 and h: for every integer i we have $\gamma_0(C_i) = C_i$ and $h(C_i) = C_{i+k}$. But it is not clear that these lozenges, or their interiors, are contained in Ω_P. We will prove this fact, and this will be done by reconstructing the chain of lozenges in Ω_P.

Proposition 5.16. *The γ_0 fixed point $\tilde{\theta}_0$ is the corner of a lozenge C'_0 which is contained in Ω_P, except maybe the corners. The preimage by $\flat$ of C'_0 with the corners removed is the rectangle $]s_0, s_1[\, \times\,]u_0, \beta_s(s_0)[\, \subset \Omega'$.*

Proof. We distinguish two cases:

Case 1 - $\tilde{\theta}_0$ is in Ω_P

In this case $u_0 = \mathcal{O}^u(\tilde{\theta}_0)$ is in $\mathcal{A}^u$, and consequently , the segment $]s_{-1}, s_1[\times\{u_0\}$ lies in Ω'. Therefore, we have $\alpha_s(s) \leq u_0$ for every s in $[s_0, s_1[$. Moreover, since β_s is non-decreasing, we also have $\beta_s(s) \geq \beta_s(s_0)$ for every s in $[s_0, s_1[$. It follows that the rectangle $]s_0, s_1[\times]u_0, \beta_s(s_0)[$ is contained in Ω'. Moreover, we have $\alpha_s(s_1) \geq u_0$ (if not, $(\alpha_s(s_1), u_0)$ would correspond to a second fixed point in the unstable leaf u_0). In addition if $u_1 = \beta_s(s_0)$ then $\gamma_0(u_1) = u_1$ as γ_0 commutes with β_s and $\gamma_0(s_0) = s_0$.

We then have two subcases:

– *either $\beta_s(s) > \beta_s(s_0)$ for some s in $[s_0, s_1[$:* then it is true for every s in $[s_0, s_1[$ since β_s is non-decreasing and commutes with γ_0. It means that the leaf $u_1 = \beta_s(s_0)$ intersects every s in $]s_0, s_1[$. Let $\tilde{\theta}'_1$ be the unique γ_0 fixed point in u_1. The union of the intersections $u_1 \cap s$ for $s \in\,]s_0, s_1[$ is a component of $u_1 - \{\tilde{\theta}'_1\}$ which makes a perfect fit with s_0. Since $\tilde{\theta}'_1$ is a periodic orbit, every unstable leaf close to u_1 intersects $\mathcal{O}^s(\tilde{\theta}'_1)$. In particular, the intersections between $\mathcal{O}^s(\tilde{\theta}'_1)$ and leaves u in $]u_0, u_1[$ describes a half leaf in $\mathcal{O}^s(\tilde{\theta}'_1)$ which makes a perfect fit with s_0. It follows that $\tilde{\theta}_0$ and $\tilde{\theta}'_1$ are the corners of a lozenge C'_0, such that C'_0 is the rectangle $]s_0, s_1[\times]u_0, \beta_s(s_0)[$. Moreover, in this case we have $\tilde{\theta}'_1 = \tilde{\theta}_1 \in \Omega_P$: we have shown that $\tilde{\theta}_1$ is the corner of a lozenge entirely contained in Ω_P, corners included and also the side $]s_0, s_1[\times u_1$ included (see figure 9, this figure does not illustrate all subcases: as explained in Lemma 5.9, it could happen that one corner, if singular, lies on the graph of α_s or β_s).

– *or for every s in $[s_0, s_1[$ we have $\beta_s(s) = \beta_s(s_0)$:* Since $\tilde{\theta}_0$ is the periodic orbit in s_0, then for $s \in]s_0, s_1[$ near s_0 it follows that s intersects u_0 and hence $\alpha_s(s) \leq u_0$. Again by equivariance under γ_0 this is true for all s in the interval. Similarly $\beta_s \geq u_1$ throughout the interval. As before $\beta_s(s_0)$ makes a perfect with s_0. Similarly u_0 makes a perfect fit with s_1. In other words the γ_0 invariant lozenge $]s_0, s_1[\times]u_0, \beta_s(s_0)[$ is entirely contained in Ω_P. The difference is that in this case $]s_0, s_1[\times u_1$ is not in the interior of Ω_P, but rather it is in the boundary of Ω_P. It may be contained in Ω_P or not, and similarly for $\tilde{\theta}'_1$, see figure 10. In this case again $\tilde{\theta}'_1 = \tilde{\theta}_1$.

Case 2 - $\tilde{\theta}_0$ is not in Ω_P

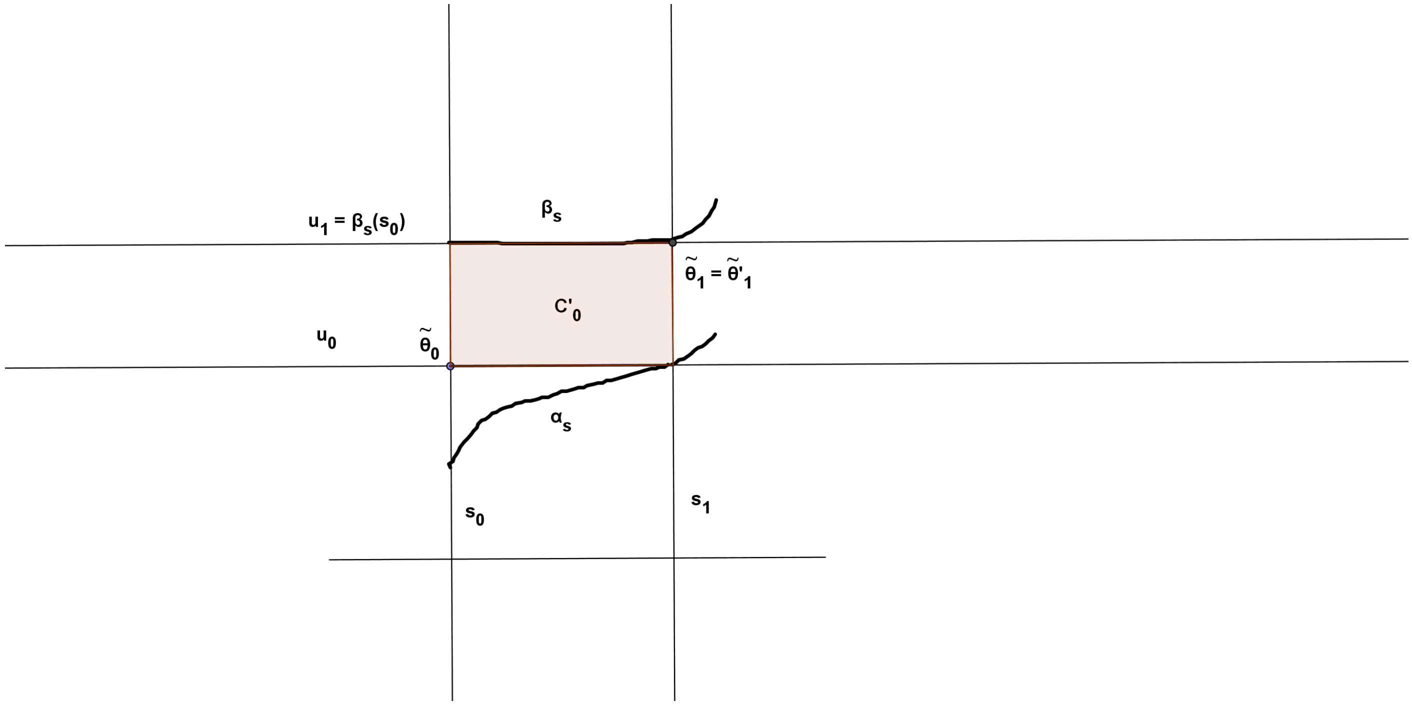

Figure 10: *A lozenge with a side not in the interior of* Ω_P*.*

In this case, u_0 is not in $\mathcal{A}^u$. Since s intersects Ω_P in an open interval and since it is γ_0 invariant then $s \cap \Omega_P$ is a component D of $s - \tilde{\theta}_0$. The projection of this component in the unstable leaf space is contained in $\mathcal{A}^u$ and it is exactly the segment $]\alpha_s(s_0), \beta_s(s_0)[$. In other words The segment $]\alpha_s(s_0), \beta_s(s_0)[$ corresponds to the unique component of $\mathcal{O}^s(\tilde{\theta}_0) - \{\tilde{\theta}_0\}$ intersecting $\mathcal{A}^u$. One of the extremities, $\alpha_s(s_0)$ or $\beta_s(s_0)$, is a leaf which is not separated from u_0. Let us assume that it is $\beta_s(s_0)$, the other case can be treated in a similar way. Then, again because $\tilde{\theta}_0$ is periodic, leaves in $\mathcal{A}^s$ close to s_0 all intersect u_0, which is not in $\mathcal{A}^u$. It follows that these leaves are elements s of $\mathcal{A}^s$ satisfying $\beta_s(s) = \beta_s(s_0)$. But this set of leaves is γ_0-invariant, hence it is the entire $]s_{-1}, s_1[$.

Recall that $\tilde{\theta}_1$ is the periodic orbit in s_1. The leaves u_0 and s_1 makes a perfect fit: it follows that there is a component b_1 of $u'_1 - \{\tilde{\theta}_1\}$ (where $u'_1 = \mathcal{O}^u(\tilde{\theta}_1)$) such that every leaf in $]s_0, s_1[$ intersects b_1 (see figure 11) - observe that u'_1 may still be outside $\mathcal{A}^u$.

Again u'_1 does not intersect s_0 and b_1 makes a perfect with s_0. It follows that we have $u'_1 = \alpha_s(s_0)$. The lozenge C'_0 with corners $\tilde{\theta}_0$ and $\tilde{\theta}_1$ is then a lozenge with interior contained in Ω_P, corresponding to the rectangle $]s_0, s_1[\times]\alpha_s(s_0), \beta_s(s_1)[$. □

Observe that it is still not totally clear from this proof that C'_0 coincides with the first lozenge C_0 of the chain connecting $\tilde{\theta}_0$ to $h(\tilde{\theta}_0)$. But it is easy to infer this statement: indeed, apply Proposition 5.16 to s_1: we get another lozenge at the right of s_1, that is, intersecting s with $s > s_1$ in $\mathcal{A}^s$. If we iterate this procedure, since there are only finitely many γ_0-fixed points in $\mathcal{A}^s$ between s_0 and $h(s_0)$, we finally reach the stable leaf $h(s_0)$. It follows that the sequence of lozenges obtained in this way must be the unique chain of lozenges connecting $\tilde{\theta}_0$ to $h(\tilde{\theta}_0)$. In figure 12, we have drawn what a possible example of this sequence of lozenges in $\mathcal{A}^s \times \mathcal{A}^u$. We have drawn in red the corners of the rectangles that cannot be corners of the lozenges (they are attracting or repelling fixed points for the action of γ_0 on $\mathcal{A}^s \times \mathcal{A}^u$), and in blue points that *may* be corners. In other words, the blue corners correspond to actual orbits of the flow in Ω'. The action of γ_0 is attracting on one of $\mathcal{A}^s, \mathcal{A}^u$ and repelling on the other one. The red corners correspond to perfect fits between stable/unstable leaves and they do not correspond to any orbit of $\widetilde{\Phi}$. In $\mathcal{A}^s$, $\mathcal{A}^u$ the actions at the corresponding points are either both attracting or both repelling.

As we can see in this picture, this chain of lozenges is a subdivision of a chain of rectangles, where each rectangle is a maximal union of s-adjacent or u-adjacent lozenges. Moreover, it follows from the analysis above that (blue) corners between two successive maximal rectangles (for example, the corner common to C_2 and C_3, or the one common to C_3 and C_4) cannot be as in case 2 of the proof of Proposition 5.16: they are points in Ω' corresponding to the common corner of the lozenges in question. We call them *true blue corners*. The other blue corners (for example, the one common to C_0 and C_1, are *fake blue corners*: they are precisely the ones appearing in case 2 of the proof of Proposition 5.16. They are never points in Ω'. More precisely:

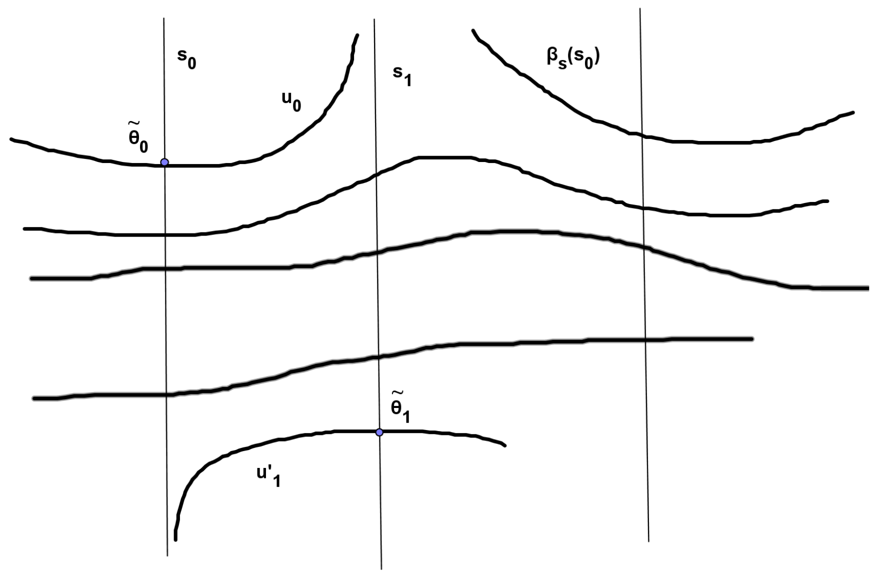

Figure 11: *The picture in $\mathcal{O}$ for case* 2.

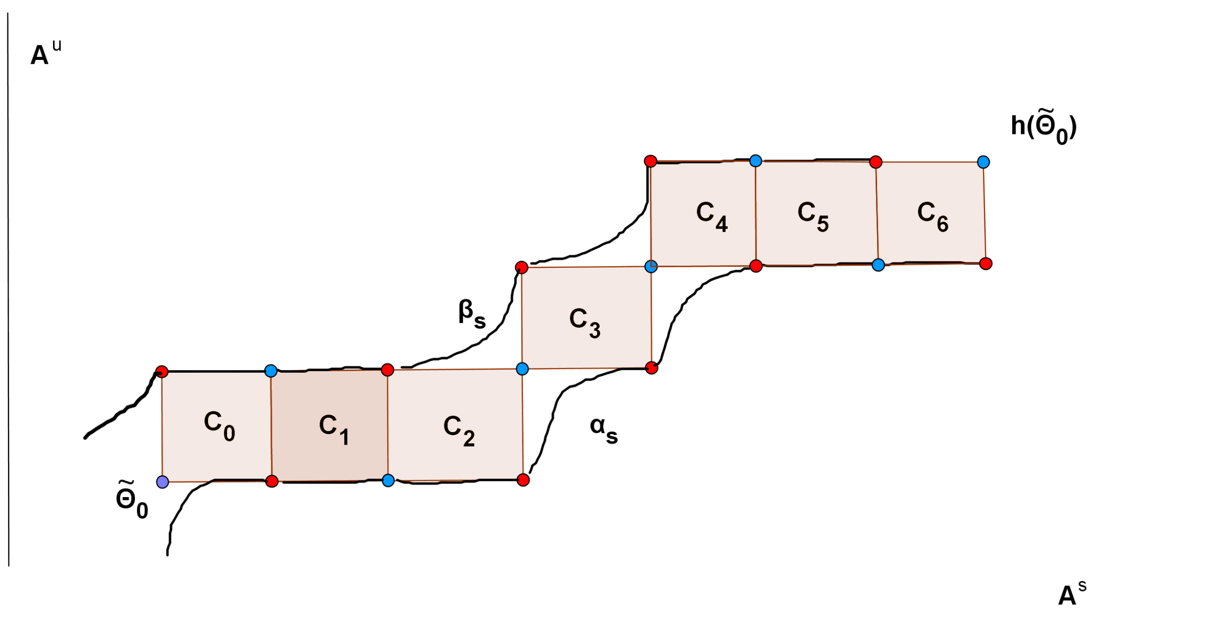

Figure 12: *A chain of lozenges in Ω' between the graphs of α_s and β_s.*

Type S and type N for fixed points in $\mathcal{A}^s$. There are two types of fixed points of γ_0 in $\mathcal{A}^s$: the stable leaves s_i containing their blue point (which is then a true blue corner), and the others, for which $s_i \cap \Omega'$ contains no γ_0-fixed point. We call the first type of type S, and the second type of type N - this terminology comes from the following observation left to the reader: let x_i be the fixed point of γ_0 in s_i. If s_i is of type S, then $\mathcal{O}^u(x_i)$ Separates x_{i_1} from x_{i+1} in $\mathcal{O}$. in s_{i+1} from the the γ_0-fixed point in s_{i-1}; and If s_i is of type N, then $\mathcal{O}^u(x_i)$ does not separate x_{i-1} from x_{i+1} in $\mathcal{O}$.

Lemma 5.17. *If s is of type N then s is not in $\tilde{\mu}_s$.*

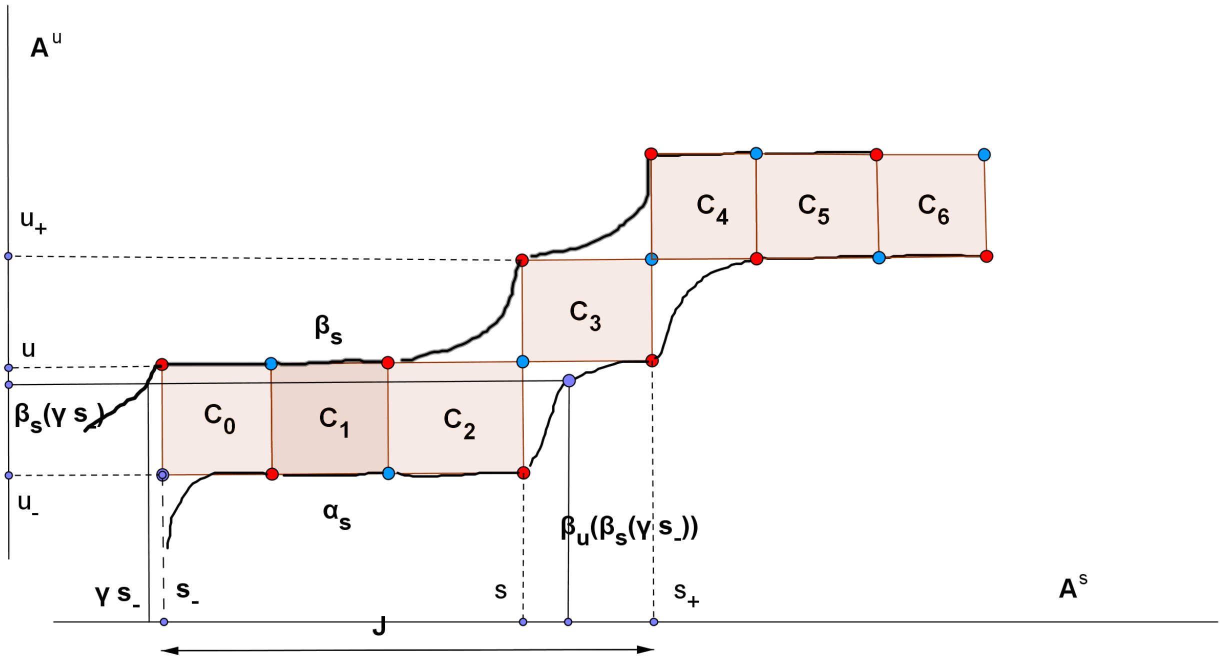

Figure 13: *The gap J of $\tilde{\mu}_s$ contains a γ_0'-fixed point.*

Proof. Observe that if s is of type N, then α_s or β_s is constant in a neighborhood of s. Now the lemma follows from Lemma 5.15. □

Conversely:

Lemma 5.18. *If s is of type S then s is in $\tilde{\mu}_s$.*

Proof. The proof is by contradiction. Suppose that s is not in $\tilde{\mu}_s$. Let J be the complementary interval of $\tilde{\mu}_s$ in $\mathcal{A}^s$ containing s. Observe that J is γ_0-invariant. Its endpoints s_-, s_+ with $s_- < s < s_+$ are also γ_0-fixed points. By the previous lemma s_- and s_+ are of type S since they are in $\tilde{\mu}_s$. Each of s_-, s_+ contains a γ_0-fixed point $x\pm = (s_\pm, u_\pm)$ in Ω which is a true blue corner, and there is also a fixed point (s, u) as well. Since $\tilde{\mu}_s$ is perfect and since the $\pi_1(P)$-orbit of s_- is dense in $\tilde{\mu}_s$ there is an iterate $\gamma(s_-)$ that intersects the unstable leaf $u_- = \mathcal{O}^u(x_-)$. The leaf u_- is in $\mathcal{A}^u$, because $\mathcal{O}^u(u_-)$ separates two stable leaves in $\mathcal{A}^s$ from each other since s_- is of type S and is associated with a true blue corner. Again, because x_- is the periodic point in s_- it follows that $\gamma(s_-)$ intersects elements in $\mathcal{A}^u$ that are bigger than u_-. In other words, $\beta_s(\gamma(s_-)) > u_-$. But since β_s is non decreasing, we have

$$u_- \ < \ \beta_s(\gamma(s_-)) \ \leq \ \beta_s(s_-) \ \leq \ u.$$

Observe that one can chose γ so that in addition γs_- is not fixed by γ_0: it is a fixed point of an element $\gamma_0' = \gamma\gamma_0\gamma^{-1}$ which has no common fixed points with γ_0 (indeed, observe that if two elements of $\pi_1(P)$ as above admit a common fixed point in $\mathcal{A}^s$, then either they are both orientation preserving or orientation reversing on $\mathcal{A}^s$ and it follows that they have exactly the same set of fixed points in $\mathcal{A}^s$).

The unstable leaf $\beta_s(\gamma(s_-))$ is fixed by γ_0', hence its image by β_u is a γ_0' fixed point too. But since β_u is non-decreasing, we have

$$\beta_u(u_-) \ \leq \ \beta_u(\beta_s(\gamma s_-)) \ \leq \ \beta_u(u). \quad \text{But also} \quad s_- \ \leq \ \beta_u(u_-), \ \text{ and } \ \beta_u(u) \ \leq s_+$$

(see figure 13). As a consequence we obtain that $[s_-, s_+]$ contains the γ_0'-fixed point $\beta_u(\beta_s(\gamma(s_-)))$.

But $[s_-, s_+]$ is the closure of the gap J of $\tilde{\mu}_s$: it follows that J is preserved by γ_0'. Its endpoints $s_\pm$ are fixed by γ_0 and γ_0': contradiction. This proves the Lemma. □

We stress the following important property. The arguments above show that $\tilde{\theta}$ is a true blue corner if and only if the following happens: the two consecutive lozenges in the chain that share $\tilde{\theta}$ as a corner are <u>not adjacent</u>. Otherwise these lozenges either intersect common stable or unstable leaves.

Corollary 5.19. *Let $\tilde{\theta} \in \mathcal{O}$ be a fixed point of a non-trivial element γ_0 of $\pi_1(P)$. Then, $\tilde{\theta}$ lies in Ω_P if and only if $\mathcal{O}^s(\tilde{\theta}) \in \tilde{\mu}_s$ or $\mathcal{O}^u(\tilde{\theta}) \in \tilde{\mu}_u$. Moreover, if it happens, then the preimage $\flat^{-1}(\tilde{\theta}) = (\mathcal{O}^s(\tilde{\theta}), \mathcal{O}^u(\tilde{\theta}))$ lies in $\tilde{\mu}_s \times \tilde{\mu}_u$. In particular, $\tilde{\mu}_s$ is the closure of the subset of $\mathcal{A}^s \subset \mathcal{H}^s$ that consists of $\mathcal{O}^s(\tilde{\theta}) \in \mathcal{A}^s$, where $\tilde{\theta}$ in Ω_P is fixed by γ in $\pi_1(P) - id$ and $\mathcal{O}^s(\tilde{\theta})$ is of type S.*

Proof. Let $\tilde{\theta} \in \mathcal{O}$ with $\gamma(\tilde{\theta}) = \tilde{\theta}$ and γ in $\pi_1(P) - id$. By the previous Lemmas $\mathcal{O}^s(\tilde{\theta})$ is in $\tilde{\mu}_s$ if and only if $\mathcal{O}^s(\tilde{\theta})$ is of type S. Hence $\tilde{\theta}$ is a true blue corner and as shown in the proof of the previous Lemma, $\mathcal{O}^u(\tilde{\theta})$ is in $\mathcal{A}^u$. This implies that $\tilde{\theta}$ is in Ω_P. In addition $\mathcal{O}^u(\tilde{\theta})$ is invariant under γ and it corresponds to a true blue corner when looking at the action on $\mathcal{A}^u$, so $\mathcal{O}^u(\tilde{\theta}) \in \tilde{\mu}_u$.

Conversely if $\tilde{\theta}$ lies in Ω_P then $\mathcal{O}^s(\tilde{\theta})$ and $\mathcal{O}^u(\tilde{\theta})$ are in $\mathcal{A}^s$ and $\mathcal{A}^u$ respectively. Since $\mathcal{O}^s(\tilde{\theta})$ separates $\mathcal{O}^s(h(\tilde{\theta}))$ from $\mathcal{O}^s(h^{-1}(\tilde{\theta}))$ and similarly for $\mathcal{O}^u(\tilde{\theta})$. It follows that $\mathcal{O}^s(\tilde{\theta})$ is in $\tilde{\mu}_s$ and $\mathcal{O}^u(\tilde{\theta})$ is in $\tilde{\mu}_u$. This proves the first two statements of the Corollary.

To prove the last statement all we need to do is to show that there is $\mathcal{O}^s(\tilde{\theta})$ in $\tilde{\mu}_s$ fixed by γ in $\pi_1(P) - id$. To prove that, recall that $\pi_1(P)$ is not abelian, and therefore there is $\gamma \in \pi_1(P) - id$ with a fixed point $\mathcal{O}^s(\tilde{\theta})$ in $\mathcal{A}^s$. If $\mathcal{O}^s(\tilde{\theta})$ is in $\tilde{\mu}_s$ then we are done. Otherwise $\mathcal{O}^s(\tilde{\theta})$ is in a complementary component J of $\tilde{\mu}_s$. In that case γ^2 fixes the endpoints of J that are in $\tilde{\mu}_s$.

This shows that the set B that is the closure of the fixed points of $\gamma \in \pi_1(P) - id$ of type S is a non empty set. It is closed and $\pi_1(P)$ invariant. Since any fixed point of type S is in $\tilde{\mu}_s$ then B is a subset of $\tilde{\mu}_s$. The minimality of $\tilde{\mu}_s$ shows that $B = \tilde{\mu}_s$. This finishes the proof of the Corollary. □

In the sequel, we will need to consider the maps α_s^-, β_s^+, α_u^-, β_u^+, analogous to the maps α_i^-, β_i^+ with $i = 1, 2$ introduced in section 4.3.2. They are defined as follows:

$$
\begin{aligned}
\alpha_s^-(s) &= \lim_{\epsilon \to 0} \alpha_s(s - |\epsilon|) \\
\alpha_u^-(u) &= \lim_{\epsilon \to 0} \alpha_u(u - |\epsilon|) \\
\beta_s^+(s) &= \lim_{\epsilon \to 0} \beta_s(s + |\epsilon|) \\
\beta_u^+(u) &= \lim_{\epsilon \to 0} \beta_u(u + |\epsilon|)
\end{aligned}
$$

The maps $\alpha_{s,u}^-$ are continuous on the left, and $\beta_{s,u}^+$ are continuous on the right. For every s in $\mathcal{A}^s$ we clearly have, $\alpha_s^-(s) \leq \alpha_s(s)$ and $\beta_s(s) \leq \beta_s^+(s)$. Similarly, for every u in $\mathcal{A}^u$ we have $\alpha_u^-(u) \leq \alpha_u(u)$ and $\beta_u(s) \leq \beta_u^+(u)$.

The proof of the following two lemmas is easy, with the help of figure 13):

Lemma 5.20. *Let (s, u) be a fixed point in $\tilde{\mu}_s \times \tilde{\mu}_u$ of an element γ_0 of $\pi_1(P) - id$. Then, (s, u) is the corner of an open rectangle R in $\mathcal{O}$, whose other corner is $(\beta_u^+(u), \beta_s^+(s))$. Observe that $\beta_u^+(u)$ is the lowest element s' of $\tilde{\mu}_s \cap Fix(\gamma_0)$ satisfying $s < s'$.*

Proof. It follows that (s, u) is in Ω_P and it is a true blue corner, and so is $(\beta_u^+(u), \beta_s^+(s))$. As in figure 13 these two corners as connected by a chain of adjacent lozenges. We let R to be this union of lozenges, plus the sides common to adjacent lozenges. By the definition of β_u^+, β_-^+, the rectangle R is the <u>maximal</u> open rectangle made of adjacent lozenges (all intersecting a common stable or unstable leaf) containing the initial lozenge with corner (s, u) and with stable leaves $> s$ in $\mathcal{A}^s$. Suppose that these are the lozenges $C_1, ..., C_i$ in the chain. Notice that the next lozenge in the chain, C_{i+1} cannot be adjacent with C_i. If $i = 1$ this is not possible for then $C_1 \cup C_2$ would be adjacent and C_1 would not be maximal. If $i > 1$ suppose that $C_1, ..., C_i$ all intersect common unstable leaves and C_i, C_{i+1} intersect common stable leaves. Then C_{i-1}, C_i, C_{i+1} all share a common corner and C_i could be eliminated from this chain of lozenges. This shows that both $\beta_u^+(u)$ and $\beta_s^+(s)$ both separate consecutive lozenges in the chain, and therefore by the above remark, both are in $\tilde{\mu}_s$ and $\tilde{\mu}_u$ respectively.

In addition by this description anyother stable leaf s' fixed by γ_0 and between s and $\beta_u^+(u)$ is a stable leaf in between two <u>adjacent</u> lozenges in the chain. It follows that s' is not in $\tilde{\mu}_s$. □

Lemma 5.21. *For every s in $\tilde{\mu}_s$ and every u in $\tilde{\mu}_u$ so that: s is fixed by $\gamma_0 - id$ and is of type S and u is also fixed by γ_0 :*

$$
\begin{aligned}
\alpha_s^-(\beta_u^+(u)) &= u \\
\alpha_u^-(\beta_s^+(s)) &= s
\end{aligned}
$$

Proof. For every (s,u) in $\tilde{\mu}_s \times \tilde{\mu}_u$ then $(v,w) = (\beta^+_u(u), \beta^+_s(s))$ is the other corner of the maximal rectangle R as in the previous Lemma. Starting from the corner (v,w) in $\tilde{\mu}_s \times \tilde{\mu}_u$ and going in the negative direction there is also a maximal rectangle (union of adjacent lozenges) and that is exactly the same rectangle R. The other corners are obtained by the maps α^-_u and α^-_+ and they recover the point (s,u). □

We define the maps $\tilde{\tau}_s := \beta^+_u \circ \beta^+_s$ from $\tilde{\mu}_s$ to $\tilde{\mu}_s$ and $\tilde{\tau}_u := \beta^+_s \circ \beta^+_u$ from $\tilde{\mu}_u$ to $\tilde{\mu}_u$. From the discussion above, when s is a fixed point of γ_0 in $\tilde{\mu}_s$, then $\tilde{\tau}_s(s)$ is the second γ_0-fixed point in $\tilde{\mu}_s$ after s. Clearly β^+_u sends a fixed point of type S of $\gamma \in \pi_1(P) - id$ to a fixed point of γ_0 acting on $\mathcal{A}^u$ of the analogous type S property. Hence restricted to the fixed points over all $\gamma \in \pi_1(P) - id$ of type S the map β^+_u is injective and clearly order preserving. We later show that all functions $\alpha^-_s, \alpha^-_u, \beta^+_s, \beta^+_u$ are strictly increasing when restricted to the appropriate $\tilde{\mu}_s$ or $\tilde{\mu}_u$. In particular they are order preserving homeomorphisms.

6. The lifted flow in the intermediate cover

In this section, we produce a good model for the free Seifert piece by lifting to a appropriate cover, and show that the stable and unstable foliations restrict to non singular foliations in this piece.

6.1. **Putting the boundary tori in good position in the intermediate cover.** Let M_P be the cover of M associated with $\pi_1(P)$. Notice that M_P is not compact. Let $\hat{\Phi}$ be the lift of Φ to M_P. Choose an embedded representative for P in M bounded by finitely many incompressible tori $\mathcal{T} = \{T'_1, ..., T'_{i_0}\}$. Each torus T'_i is homotopic to an a priori only immersed Birkhoff torus B_i.

Proposition 6.1. *Each torus T'_i is freely homotopic to a weakly embedded Birkhoff torus T^*_i so that the collection of such tori satisfies:*

– *T^*_i lifts to an embedded Birkhoff torus T_i in M_P.*

– *The collection $\{T_1, ..., T_{i_0}\}$ can be chosen to be pairwise disjoint and they bound a compact submanifold $\hat{P} \subset M_P$.*

– *$\pi_1(\hat{P})$ is isomorphic to $\pi_1(M_P) = \pi_1(P)$. The closure of each component of $M_P - \hat{P}$ is homeomorphic to $T^2 \times [0, \infty)$.*

Proof. In [Ba-Fe1] we proved that each T'_i is freely homotopic to a weakly embedded Birkhoff torus T^*_i. We will adjust T^*_i as needed. In [Ba-Fe1], Theorem 6.10 we proved that we can choose T^*_i to be embedded unless one of the following happens:

- 1) T'_i is isotopic to the boundary of a regular neighborhood V of an embedded Birkhoff-Klein bottle K. The neighborhood V is contained in a free Seifert piece P_1.
- 2) T'_i is homotopic to a weakly embedded Birkhoff torus contained in a periodic Seifert fibered piece P_1.

Consider first possibility 1). Here $\partial V = T'_i$ and V is contained in a Seifert piece P_1 and T'_i is a torus in the JSJ decomposition of M. Since T'_i is also in the boundary of a piece of the JSJ decomposition, it follows that $V = P_1$. Then $\pi_1(P_1) = \pi_1(K)$. As seen in the proof of Proposition 5.1 this implies that P_1 is not free. This contradiction shows that this case cannot happen.

Consider now possibility 2). In this case, it is quite possible that T^*_i is not embedded in M. What we will show is that we can choose T^*_i so that T_i is embedded in M_P. Here the Seifert piece P_1 is periodic and hence P, P_1 are distinct Seifert pieces and hence T'_i is a torus in the boundary of both P_1 and P. As described in [Ba-Fe1], T^*_i is weakly embedded, so if T^*_i is not embedded, it is because of the periodic orbits in T^*_i: either a periodic orbit in T^*_i is traversed multiple times as a loop in T^*_i or two periodic orbits collapse. In the first case there is an element $g \in \pi_1(M)$ associated with the periodic orbit so that g is not in $\pi_1(T'_i)$, but for some $n > 1$, g^n is in $\pi_1(T'_i)$. Here g is in $\pi_1(P_1)$ because in P_1 the orbit is represented by a closed orbit. The other option is that two periodic orbits collapse. Here there is an element g in $\pi_1(P_1)$ that is not in $\pi_1(T'_i)$ and which corresponds to the identification of orbits in the Birkhoff torus.

Suppose that this problem persists when lifting T^*_i to M_P. By Scott's core theorem [Sco1, He], there is a compact core in M_P that carries all the homotopy of M_P. The submanifold P lifts homeomorphically to M_P. The submanifold P_1 lifts to a non compact submanifold P'_1 of M_P, but the component of the intersection of P_1 and P corresponding to T'_i lifts homeomorphically to M_P. Hence we can apply the same analysis of the last paragraph to M_P. Then there is a g as above, but now this g is in $\pi_1(M_P) = \pi_1(P)$. It follows that this element g is in both $\pi_1(P)$ and $\pi_1(P'_1)$. But this intersection is only $\pi_1(T'_i)$. This is a contradiction and it shows that we can choose T_i to be embedded in M_P.

The next step is to analyse whether the collection $\{T_i\}$ can be chosen pairwise disjoint. Suppose that $T_i \cap T_j \neq \emptyset$ for some $i \neq j$. Suppose first that a closed orbit in T_i intersects a closed orbit in T_j. Then they are the same closed orbit of $\hat{\Phi}$. This produces a closed curve in T'_i which is freely homotopic to a closed curve in T'_j. This produces an essential annulus in P, and this annulus is isotopic to a vertical annulus. It follows that the periodic orbit in T'_i is freely homotopic to a regular fiber in P up to powers. But then P would be a periodic Seifert piece, contradiction to the assumption that P is free.

Suppose now that a closed orbit γ in T_i intersects the interior of a Birkhoff annulus in T_j in M_P. Lift this to the cover M^* of M_P associated to $\pi_1(T_j)$. Notice that in this proof we use several covers $\widetilde{M} \to M^* \to M_P \to M$. The torus T_j lifts homeomorphically to an embedded torus in M^*, which is still denoted by T_j for simplicity. We use the fact that if B is a lift to $\widetilde{M}$ of a Birkhoff torus, then an orbit $\tilde{\theta}$ of $\widetilde{\Phi}$ which intersects B transversely, intersects B only once and the components of $\tilde{\theta} - B$ are in different components of $\widetilde{M} - B$. This follows from the description of lifts of Birkhoff annuli and associated lozenges in $\widetilde{M}$. The closed orbit γ in M_P lifts to a curve γ_0 in M^* intersecting T_j transversely. If γ_0 is closed then it represents a power of γ in $\pi_1(M)$, hence this power of γ is freely homotopic into T_j in M_P. This was disallowed in the previous paragraph. Hence γ_0 is not compact. Suppose that $\gamma_)$ intersects T_j more than once in M^*. Then since $\pi_1(M^*) = \pi_1(T_j)$, the corresponding lift of γ_0 to $\widetilde{M}$ is an orbit intersecting the universal cover of T_j more than once, contradiction. Hence there are two rays of $\gamma_0 - T_j$ and they are in different components of $M^* - T_j$. Let δ be one such ray. If δ accumulates in a point of M^*, then because γ is compact this would imply that δ is actually a closed curve, as $M^* \to M$ is a cover. This is a contradiction. Therefore $d_{M^*}(p, T_j)$ goes to infinity as p escapes in the ray δ.

On the other hand, projecting to M we see that $\pi(\gamma_0)$ is homotopic to a curve disjoint from $\pi(T_j)$, since $\pi(\gamma_0)$ is homotopic into T'_i and $\pi(T_j)$ is homotopic to T'_j. Lifting this free homotopy to M^* we see that γ_0 is homotoped by a homotopy moving points a bounded distance to be disjoint from T_j. Hence there is a ray of $\gamma_0 - T_j$ which is a bounded distance from T_j. As seen above, this is a contradiction and it shows this situation cannot happen.

This shows that any closed orbit in T_i cannot intersect T_j. Finally we check what happens if the interior of a Birkhoff annulus in T_i intersects the interior of a Birkhoff annulus in T_j. Put T_i, T_j in general position. The analysis now follows standard arguments. First eliminate null homotopic intersections. Using innermost arguments one gets a disk in T_i and another in T_j which jointly produce an embedded sphere bounding a ball in M_P - as M_P is irreducible. This intersection can be eliminated by sliding T_j across the ball. The remaining intersections are freely homotopic to closed orbits of the flow (up to powers). By the first part of this analysis, this cannot happen.

We conclude that we can choose the collection $\{T_1, ..., T_{i_0}\}$ to be pairwise disjoint. Jointly they bound a compact submanifold, which is denoted by $\hat{P}$ in M_P. The submanifold $\hat{P}$ carries all the homotopy of M_P. Since $\widetilde{M} \cong \mathbb{R}^3$ and M is Haken one can use the general theory of Haken manifolds and compact cores [Sco1, He], and it follows that the closure of the components of $M_P - \hat{P}$ are homeomorphic to $T^2 \times [0, \infty)$. This finishes the proof of Proposition 6.1. □

6.2. **Analysing the free Seifert piece in the intermediate cover.**

Notation – In the cover M_P we have the compact submanifold $\hat{P}$ with boundary a union of pairwise disjoint Birkhoff tori $\{T_1, ..., T_{i_0}\}$. Let Λ^s_P, Λ^u_P be the lifts of Λ^s, Λ^u to M_P. Let Φ_P be the lift of Φ to M_P.

Preparatory step – We will adjust the boundary of $\hat{P}$ slightly near some of the boundary periodic orbits in order to have Λ^s_P, Λ^u_P to be non singular foliations when restricted to $\hat{P}$. Consider a Birkhoff torus T in the boundary of $\hat{P}$. Consider a periodic orbit θ in T and lifts $\widetilde{T}$ and $\widetilde{\theta}$ to $\widetilde{M}$. Let $\mathcal{C}$ be the chain of lozenges invariant under $\pi_1(T)$ associated with the Birkhoff representative T. If the two consecutive lozenges of $\mathcal{C}$ at $\widetilde{\theta}$ are not adjacent we do not make any adjustment to T near θ. Suppose now that these lozenges C_1, C_2 are adjacent along $\widetilde{\Lambda}^s(\widetilde{\theta})$. We adjust T as follows. Let L be the half leaf of $\widetilde{\Lambda}^s(\widetilde{\theta})$ in the boundary of both C_1 and C_2. Move $\widetilde{T}$ away from $\widetilde{\theta}$ and slightly into L. We can make this adjustment in M. Now T is not tangent to Φ near θ, but is actually transverse to Φ near θ. That is, we eliminated one tangency of Φ and T. See details of this in [Ba-Fe2, Lemma 3.3]. This can only be done if and only if C_1 and C_2 are adjacent. We assume that T is this slightly adjusted torus.

After this modification each boundary component of $\hat{P}$ is transverse to both Λ^s_P and Λ^u_P. Since T is transverse to $\hat{P}$ outside of the periodic orbits of $\hat{\Phi}$, then we only have to check what happens at a tangent orbit $\theta \subset T$. Lift to $\widetilde{M}$ producing lifts $\tilde{\theta}$ and $\widetilde{T}$ and lozenges D_1, D_2 abutting $\tilde{\theta}$. Since the lozenges D_1, D_2 are not adjacent at $\tilde{\theta}$, then both $\widetilde{\Lambda}^s(\tilde{\theta})$ and $\widetilde{\Lambda}^u(\tilde{\theta})$ separate D_1 from D_2. It follows that Λ^s_P, Λ^u_P are transverse to T at θ. If D_1, D_2 were adjacent along $\widetilde{\Lambda}^s(\theta)$ then Λ^u_p would be tangent to T at θ and not transverse. After these modifications we prove the following:

Proposition 6.2. *The foliations Λ^s_P, Λ^u_P induce non singular foliations in $\hat{P}$, which are transverse to $\partial\hat{P}$. They are denoted by $\hat{\Lambda}^s, \hat{\Lambda}^u$. These foliations are $\mathbb{R}$-covered. Each leaf of $\hat{\Lambda}^s$ intersects every component of $\hat{P}$ in a single component.*

Proof. We do the proof for $\hat{\Lambda}^s$. By the preliminary step Λ^s_P induces a foliations in $\hat{P}$ which is possibly singular with p-prong singularities. The preliminary step shows that there is no local leaf of Λ^s_P restricted to $\hat{P}$ which intersects $\hat{P}$ only in a tangent periodic orbit of $\hat{P}$. If that were the case then in the universal cover no stable prongs of the lifted periodic orbit would separate the lozenges abutting that orbit. Hence these lozenges would be adjacent along an unstable leaf. This was dealt with in the preliminary step. It follows that the restriction of Λ^s_P to $\hat{P}$ is an actual foliation in $\hat{P}$ with possible p-prong singularities.

Now double $\hat{P}$ along the boundary and double the foliations as well. This produces a manifold $2\hat{P}$ with a foliation $2\hat{\Lambda}^s$. We have the following properties:

- $\hat{P}$ is Seifert fibered and hence $2\hat{P}$ is also Seifert fibered.
- $2\hat{\Lambda}^s$ is a (possibly singular) foliation.
- Since every tangent orbit to $\partial\hat{P}$ has at least one prong of Λ^s_P entering $\hat{P}$, then in $2\hat{P}$ there are at least two prongs of $2\hat{\Lambda}^s$ for each such orbit.
- Therefore the singularities of $2\hat{\Lambda}^s$ are p-prong singular orbits with $p \geq 2$.

Now we use the theory of essential laminations [Ga-Oe] to deal with this situation. If there are singular leaves of $2\hat{\Lambda}^s$ blow them up to produce a lamination $\mathcal{L}$ in $2\hat{P}$. We will prove that $\mathcal{L}$ is an essential lamination.

Suppose first there is a compact leaf C of $\mathcal{L}$. First we have to rule out sphere leaves of $\mathcal{L}$ and tori leaves bounding solid tori. A lot of the arguments are standard in 3-dimensional topology and some details are left to the reader. We think of $\hat{P}$ as contained in $2\hat{P} = \hat{P} \cup P^*$. Suppose that C is a sphere. If C is contained in $\hat{P}$ or in P^* that produces a compact leaf of $\widetilde{\Lambda}^s$ in $\widetilde{M}$, contradiction. It follows that C intersects $\partial\hat{P}$. Now using the components of $C - \partial\hat{P}$ and innermost arguments there is a component of $C - \partial\hat{P}$ which is a disk D. Without loss of generality assume that $D \subset \hat{P}$. Consider the flow $\hat{\Phi}$ in $\hat{P}$. If ∂D is not a tangent orbit of the flow then the flow $\hat{\Phi}$ is transverse to T in ∂D. Hence it is transverse to ∂D. Suppose it is incoming in ∂D. This is impossible as it would generate a center singularity of the flow in D. If ∂D is tangent to the flow a similar argument ensues. It follows that C cannot be a sphere.

Suppose now that C is a torus. Consider a component E of $C - \partial\hat{P}$. Assume that E is contained in $\hat{P}$. Since C is compact then C is the double of E. So the problem can only happen if E is an annulus. The case that C bounds a solid torus in $2\hat{P}$ can only happen if E is an annulus that is boundary parallel in $\hat{P}$. So E together with an annulus in $\partial\hat{P}$ bound a solid torus V in $\hat{P}$. In particular both boundary components of E intersect the same component T of $\partial\hat{P}$. Lifting to the universal cover we obtain a lift $\widetilde{E}$ of E which has two boundary components in the same component $\widetilde{T}$ of $\partial\widetilde{P}$ because V is a solid torus. So this produces a leaf of $\widetilde{\Lambda}^s$ which intersects $\widetilde{T}$ in at least two components. This is a contradiction as T is a quasitransverse Birkhoff torus. Hence the leaves of $\mathcal{L}$ cannot be tori bonding solid tori.

Let W be the closure of a complementary component of $\mathcal{L}$ and Y a component of ∂W. We need to prove that Y is incompressible in W. Suppose this is not the case. Then there is a simple closed curve α in Y which is not null homotopic in Y but bounds a disk D in W. We look at the intersection of D and $\partial\hat{P}$ $-$ the surfaces are supposed to be in general position with respect to each other. In D look at an innermost arc intersection δ with $\partial\hat{P}$. This curve δ bounds a subdisk D_1 of D (that is δ and an arc in ∂D) with no arc intersections with $\partial\hat{P}$. If there is a circle intersection in D_1, it is null homotopic in D and hence bounds a disk in $\partial\hat{P}$ as well. These two disks can be used to create a sphere which bounds a ball and such intersections can be isotoped away. It follows that we may assume that the interior of D_1 does not intersect $\partial\hat{P}$. The complementary regions of $\mathcal{L}$ restricted to $\partial\hat{P}$ (that is, $(2\hat{P} - \mathcal{L}) \cap \partial\hat{P}$, are either annuli $-$ when one blows up a closed curve in $\hat{P}$; or infinite strips, if one blows up an infinite curve in $\partial\hat{P}$. Since the two endpoints of δ are in D which is a disk and has connected boundary; it follows that the endpoints of δ are in Y. It also follows that δ union an arc ϵ in Y bounds a disk $D_2 \subset \partial\hat{P}$. Furthermore δ and an arc in ∂D bounds a subdisk $D_1 \subset \hat{P}$, where D_1 is a subdisk of D. Then $D_1 \cup D_2$ is a disk D' which has boundary in a leaf of $\mathcal{L}$. Now D' is a disk enitrely contained in $\hat{P}$. And in $\hat{P}$ it is easy to see that this component of $Y \cap \hat{P}$ is incompressible in W. Hence $\partial D'$ also bounds a disk in Y. Since $\partial D'$ is null homotopic in Y, then the curve α cut up by D_2 and union with the arc ϵ produces another simple closed curve in Y which is not null homotopic. Proceed with one less intersection with $\partial\hat{P}$. By induction we arrive finally at a contradiction and therefore Y is incompressible in W.

In a similar way, using that there are no one prongs in Λ^s, then the boundary leaves of K are also end incompressible. One of the main results of [Ga-Oe] implies that $\mathcal{L}$ is an essential lamination in $2\hat{P}$. Brittenham's theorem [Br] implies that $\mathcal{L}$ has a minimal sublamination $\mathcal{L}_1$ which is vertical or horizontal.

Suppose that $\mathcal{L}_1$ is vertical. Let F be a leaf of $\mathcal{L}_1$. Then F has a closed curve α contained in say $\hat{P}$ so that α is not null homotopic in $\hat{P}$. Since it is vertical it is freely homotopic to a regular fiber in $\hat{P}$ and projects to a curve which is not null homotopic in M. So in M the stable leaf which contains it is not a plane and contains a periodic orbit β. But then the fiber in P is freely homotopic to a periodic orbit up to powers. This would imply that P is a periodic piece, contrary to assumption in this case.

We conclude that $\mathcal{L}_1$ is a horizontal sublamination. Since $\mathcal{L}_1$ is horizontal it follows that $\mathcal{L}$ is horizontal also, because one can do the isotopies preserving the tori in $\partial\hat{P}$. In particular this also implies that there are no singularities of $2\hat{\Lambda}^s$. We conclude the following:

- $\hat{\Lambda}^s$ is a non singular horizontal foliation in $\hat{P}$.
- Recall that each component of $\partial\hat{P}$ is a Birkhoff torus. It now follows that every compact component $\tilde{\theta}$ of $\hat{\Lambda}^s \cap \partial\hat{P}$ is a closed orbit of Φ_p and locally the stable leaf of $\tilde{\theta}$ has only one prong entering $\hat{P}$. Notice that there may be singularities of Φ_P contained in $\partial\hat{P}$, but only one stable prong of such an orbit enters $\hat{P}$.
- Given these facts it is immediate that in the universal cover of $\hat{P}$ every leaf of the lifted foliation $\widetilde{\hat{\Lambda}^s}$ intersects every component of $\partial\widetilde{\hat{P}}$. In particular $\hat{\Lambda}^s$ is an $\mathbb{R}$-covered foliation in $\hat{P}$.

The fact about $\tilde{\theta}$ in the second claim is true because $\tilde{\theta}$ is a closed curve in a Birkhoff torus and it is contained in a single stable leaf. By the structure of the induced stable foliation in Birkhoff annuli, it follows that $\tilde{\theta}$ has to be a boundary component of one of the annuli in the Birkhoff torus, and hence it is a closed orbit of the flow.

We now prove the last statement of the proposition. First we note the following fact: consider an orbit θ of Φ_P that intersects a component T^- of $M_P - \hat{P}$ and let p be a point of θ in this component. Suppose that the forward orbit of p is always in this component, that is it never enters $\hat{P}$. Then if the forward orbit of p only limits in points of the torus boundary T of T^- it follows that p is in the stable leaf of a boundary tangent orbit contained in T. This follows from the fact that T is a Birkhoff torus and the hyperbolic dynamics near a periodic orbit.

Let now L be a leaf of Λ^s_P. We show that L intersects $\hat{P}$ in a single component (could be empty). Let then p, q be points in $L \cap \hat{P}$. Let θ, β be the Φ_P orbits through p, q respectively. Suppose first that the forward orbit of p gets out of $\hat{P}$. Then it first exits $\hat{P}$ through a boundary Birkhoff torus T. As explained before θ can intersect T at most once. Hence once θ leaves $\hat{P}$ through T it has to stay in the component T^- of $M_P - \hat{P}$ bounded by T. We claim that θ cannot be in the stable leaf of a boundary periodic orbit contained in T. This is a key fact here. Suppose not. Then L is the stable leaf of this periodic orbit θ_1. But L also intersects T transversely in the orbit through p. Consider a segment in θ from the intersection with T and then to a point very near the boundary orbit θ_1. Add a small arc at the end to connect it to θ_1 and produce segment S. Then S is a segment in the closure of T^- intersecting T only in the endpoints. Since $\pi_1(M_P) = \pi_1(P)$, this segment is homotopic rel endpoints to an arc in T. Hence when we lift to $\widetilde{M}$, producing $\widetilde{T}, \widetilde{\theta}, \widetilde{L}$ we obtain that $\widetilde{\theta}$ is transverse to $\widetilde{T}$ $-$ in the lift of the interior of a Birkhoff annulus, hence in a lozenge. In addition $\widetilde{\theta}$ is also in the stable leaf of one of the lifts of a periodic orbit, that is in a corner of one of the lozenges associated with $\widetilde{T}$. This is a contradiction. This shows that θ cannot forward accumulate only in T.

Since q is also in L it is forward asymptotic with the p orbit, and so has to enter the component T^- as well (as θ gets sufficiently far from T). So we have the following: Points p_0, q_0 in the forward orbits of p, q that are in T. From p to p_0 the orbit θ is contained in $\hat{P}$ and after p_0 the orbit θ is entirely contained in T^-. Similarly for q and the orbit β. Go forward enough on both orbits so they are close. Then we produce an arc in L as follows: start in p_0, move forward on θ until very close to β, move to β along L and then backwards to q_0. As above call this arc S. As above S is homotopic into T. As above lift to $\widetilde{M}$ to produce $\widetilde{T}, \widetilde{L}, \widetilde{p}_0, \widetilde{q}_0$, etc.. By the homotopic property above, the two points $\widetilde{p}_0, \widetilde{q}_0$ are in $\widetilde{T}$ and also in $\widetilde{L}$. But the intersection of $\widetilde{L}$ with $\widetilde{T}$ is connected $-$ just consider the intersection of $\widetilde{L}$ with the chain of lozenges associated with $\widetilde{T}$. Therefore $\widetilde{p}_0, \widetilde{q}_0$ can be connected along $\widetilde{T}$ by an arc in $\widetilde{L}$. Therefore in $\hat{P}$, p and q are connected by an arc from p to p_0 in $\hat{P}$, then an arc along $L \cap T$ to q_0 then back to q. It follows that p, q are in the same component of $L \cap \hat{P}$.

This deals with the case that the forward orbit of p exits $\hat{P}$. This was the harder case. Let us now deal with the other case. Suppose first that θ forward accumulates in a point in the interior of $\hat{P}$. Then β also does and since the intersection of β with $\hat{P}$ is connected the forward orbit of q is contained in $\hat{P}$. Then p, q are connected in $L \cap \hat{P}$. If the forward orbit of p only accumulates in a component T of $\partial\hat{P}$, then it is in the stable leaf of a boundary periodic orbit θ_1. Similarly for q. By the explanation in the beginning, q cannot be in an orbit that intersects T transversely, therefore the forward orbit of q has to be entirely contained in $\hat{P}$. So again p, q are connected in $L \cap \hat{P}$.

This finishes the proof of Proposition 6.2. □

6.3. **Gaps are periodic and consequences.** In the previous section, we have shown that in the intermediate cover M_P there is a submanifold $\hat{P}$ whose boundary is an union of tori transverse to the foliations Λ^s_P, Λ^u_P, which are regular inside $\hat{P}$.

Let $\widetilde{P}$ be the lift of $\hat{P}$ to $\widetilde{M}$ which is $\pi_1(P)$-invariant. The induced foliations $\hat{\Lambda}^s$ and $\hat{\Lambda}^u$ on $\hat{P}$ are $\mathbb{R}$-covered, the last statement of Proposition 6.2 and the arguments in the proof of Proposition 6.2 imply that the projection of these $\mathbb{R}$'s into the leaf spaces of $\widetilde{\Lambda}^s$ and $\widetilde{\Lambda}^u$ respectively is injective. In addition these projections are obviously $\pi_1(P)$ invariant. From this it follows that the leaf space of $\widetilde{\hat{\Lambda}^s}$ is $\mathcal{A}^s$, and the leaf space of $\widetilde{\hat{\Lambda}^u}$ is $\mathcal{A}^u$, and that the projection of $\widetilde{P}$ in the orbit space is contained in the domain Ω' defined in Section 5.2. Recall that $\Gamma = \pi_1(P)$.

Actually, for every lift $\widetilde{T}$ of a torus boundary T of $\hat{P}$, the projection of $\widetilde{T}$ in $\mathcal{O}$ (or Ω') is a chain of lozenges. More precisely let $\widetilde{A}_j$ be a Birkhoff band, connected component of $\widetilde{T}$ with the tangent orbits removed. It projects in Ω in a region of the form:

$$R_j = \{(x,y) \in \Omega' \ \mid \ x_j < x < x_{j+1}, \ y_j < y < y_{j+1}\}$$

Due to our adjustment procedure (preparatory step in section 6.2), the set R_j is not necessarily a lozenge; it could be a s-scalloped or u-scalloped chain of lozenges, where we also include the sides between adjacent lozenges. We call it a *generalized lozenge*. In section 5.3, we have proved that the corners (x_i, y_i) of these rectangles all lie in Ω'. Let $\gamma_0 \in \Gamma$ be an element generating the group of orientation preserving elements of $\pi_1(\Gamma)$ fixing every x_i and every y_i: according to Corollary 5.19, the fixed points of γ_0 that are in $\tilde{\mu}_s$ are precisely the x_i's, and the y_i's are the fixed points of γ_0 that are in $\tilde{\mu}_u$. The other fixed points of γ_0 in $\mathcal{A}^s$ or in $\mathcal{A}^u$ correspond to subdivisions of R_j in lozenges as in section 5.

Lemma 6.3. *$\widetilde{A}_j$ is an entrance (respectively exit) transverse band if and only if $]y_j, y_{j+1}[$ is a gap of $\tilde{\mu}_u$ (respectively $]x_j, x_{j+1}[$ is a gap of $\tilde{\mu}_s$).*

Proof. This lemma was proved in [Ba3, Section 3.1] in the case of $\mathbb{R}$-covered Anosov flows. Here we give a simpler proof, and that applies to our much more general situation.

First of all irrespective of entering or exit annulus, we claim the following:

Claim: If γ is in $\Gamma = \pi_1(P)$ and $\gamma(R_j)$ intersects R_j, then they are equal and γ is a power of γ_0.

Suppose that $\gamma(R_j)$ non trivially intersects R_j. Since R_j is a rectangle, it follows that either $\gamma(]x_j, x_{j+1}[) \subset]x_j, x_{j+1}[$ or $\gamma(]y_j, y_{j+1}[) \subset]y_j, y_{j+1}[$ (but not both). In either case there is w in R_j with $\gamma(w) = w$. Projecting to M produces a closed orbit of Φ intersecting one of the Birkhoff tori transversely and so that it represents an element of $\pi_1(P)$. As we explained previously this is impossible. This proves the claim.

Let us consider the case where $\widetilde{A}_j$ is an exit transverse band. Then, an orbit starting from a point in $\widetilde{A}_j$ goes outside $\widetilde{P}$ and never go back since it crosses $\widetilde{T}$ at most once. Assume that $\tilde{\mu}_s$ intersects $]x_j, x_{j+1}[$. Then there is an iterate γx_j in $]x_j, x_{j+1}[$, because x_j is in $\tilde{\mu}_s$ and $\tilde{\mu}_s$ is the minimal set of the action. Then $\gamma(R_j)$ intersects $]x_j, x_{j+1}[\times \mathcal{A}^u$. But since $\gamma(R_j) \cap R_j = \emptyset$, it follows that this intersection is either "below" R_j (that is contains points (x,y) with $y < y_j$) or "above" R_j. Since $\beta_s(\gamma x_j) \geq y_{j+1}$ we have to have the second option. The band $\widetilde{A}_j$ does not intersect the unstable leaf y_{j+1} but intersects every unstable leaf y with $y < y_{j+1}$ in $\mathcal{A}^u$ near y_{j+1}. So this band escapes <u>down</u> as it nears the unstable leaf y_{j+1}. This is because in a stable leaf, say γx_j, as it nears the intersection with y_{j+1} one has to escape flow backwards for it not to intersect y_{j+1}, as flow forwards all orbits are asymptotic. It follows that γR_j is in the component of $\widetilde{M} - \widetilde{T}$ that is "flow forwards" of $\widetilde{A}_j$. But this is a contradiction to the property above.

This contradiction shows that $]x_j, x_{j+1}[$ is disjoint from $\tilde{\mu}_s$, i.e. is a gap.

In a similar way, one proves that if $\widetilde{A}_j$ is an entrance transverse band, then $]y_j, y_{j+1}[$ is a gap.

In order to conclude, we just have to prove that when $\widetilde{A}_j$ is an exit transverse band, then $]y_j, y_{j+1}[$ is **not** a gap (the proof that $]x_j, x_{j+1}[$ is not a gap when $\widetilde{A}_j$ is an entrance transverse band is similar).

Assume by a way of contradiction that $]x_j, x_{j+1}[$ and $]y_j, y_{j+1}[$ are both gaps. Let $p_n = \gamma_n x_j$ a sequence of iterates accumulating non-trivially to x_j. We can assume that no p_n is fixed by γ_0, i.e. that no $\gamma_n\gamma_0\gamma_n^{-1}$ is a power of γ_0. Since $]x_j, x_{j+1}[$ is a gap, we have $p_n < x_j$. On the other hand, for n sufficiently big, we have $y_j < \beta_s(p_n) \leq \beta_s(x_j)$. According to Lemma 5.15, the last inequality is strict, since $]p_n, x_j[$ cannot be a gap (x_j cannot be the extremity of two different gaps). Hence, since $y_{j+1} \geq \beta_s(x_j)$, the gap $]y_j, y_{j+1}[$ contains all the points $\beta_s(p_n)$, that are fixed points of $\gamma_n\gamma_0\gamma_n^{-1}$. It follows that y_j and y_{j+1} are fixed by $\gamma_n\gamma_0\gamma_n^{-1}$. Therefore, $\gamma_n\gamma_0\gamma_n^{-1}$ are all powers of γ_0. Contradiction.

The Lemma is proved. □

When $]x_j, x_{j+1}[$ is a gap, we define:

$$\Delta(x_j, y_j) := \{(x,y) \in \Omega' \ \mid \ x_j < x < x_{j+1}, \ \alpha_s(x) < y < y_j\}$$

When the gap is $]y_j, y_{j+1}[$, we define:

$$\Delta(x_j, y_j) := \{(x,y) \in \Omega' \ \mid \ \alpha_u(y) < x < x_j, \ y_j < y < y_{j+1}\}$$

The following is one of the most important facts in our analysis:

Proposition 6.4. *Every gap of $\tilde{\mu}_s$ (respectively $\tilde{\mu}_u$) is $\pi_1(P)$-periodic. More precisely, all gaps of $\tilde{\mu}_s$ and $\tilde{\mu}_u$ are sides of one generalized lozenges R corresponding to a Birkhoff annulus in $\partial\hat{P}$.*

Proof. Let I be a gap of $\tilde{\mu}_s$. Assume it is not a side of some generalized lozenge corresponding to a Birkhoff annulus in $\partial\hat{P}$. Let $\widetilde{\Lambda}^s(I)$ be the preimage of I in $\widetilde{P}$, and let $\hat{\Lambda}^s(I)$ be the projection of $\widetilde{\Lambda}^s(I)$ in $\hat{P}$. By hypothesis, I cannot be one side of a generalized lozenge. Hence $\hat{\Lambda}^s(I)$ contains no point in an exit annulus. It follows that no orbit in $\hat{\Lambda}^s(I)$ can escape from $\hat{P}$.

But since I is an open segment in $\mathcal{A}^s$ there is a segment J in an unstable leaf, transverse to the flow and so that J is contained in $\hat{\Lambda}^s(I)$. By the above, the forward orbit of J is entirely contained in $\hat{P}$, and similarly for any point in its closure. Since orbits in J are expanding away from each other we obtain unstable leaves entirely contained in $\hat{P}$. This contradicts the final statement of Proposition 6.2.

This finishes the proof of the Proposition. □

Recall that at the end of section 5.3 we have defined maps $\alpha^-_{s,u}$, $\beta^+_{s,u}$, and $\tilde{\tau}_s = \beta^+_u \circ \beta^+_s$.

Corollary 6.5. *The restrictions to $\tilde{\mu}_s$ of the maps α^-_s, β^+_s, and the restrictions of α^-_u, β^+_u to $\tilde{\mu}_u$ are all injective.*

Proof. We first prove that the restriction of α^-_s to $\tilde{\mu}_s$ is injective. Assume not: there are two element a, b of $\tilde{\mu}_s$ such that $\alpha^-_s(a) = \alpha^-_s(b)$. As in Lemma 5.15 one proves that $]a, b[$ is a gap of $\tilde{\mu}_s$. According to Proposition 6.4, a and b are preserved by a non-trivial element γ_0 of $\pi_1(P)$, and $]a, b[$ is the s-side of a γ_0-invariant generalized lozenge R corresponding to an exit Birkhoff band $\widetilde{A}$. Let (a, u_1) be the "lower left" corner of R and (b, u_2) the "upper right" corner of R in Ω'. It follows that they are both true blue corners. Then as in Lemma 5.21, we have that $\alpha^-_s(b) = u_1$. But since we assumed that $\alpha^-_s(a) = \alpha^-_s(b)$, we obtain $\alpha^-_s(a) = u_1$, which is impossible.

The proof of the injectivity of β^+_s on $\tilde{\mu}_s$, and of α^-_u, β^+_u on $\tilde{\mu}_u$ are similar. □

Proposition 6.6. *The image of $\tilde{\mu}_s$ by α^-_s is $\tilde{\mu}_u$. Similarly, $\beta^+_s(\tilde{\mu}_s) = \tilde{\mu}_u$ and $\alpha^-_u(\tilde{\mu}_u) = \beta^+_u(\tilde{\mu}_u) = \tilde{\mu}_s$. In particular, $\tilde{\tau}_s$ induces an order preserving homeomorphism from $\tilde{\mu}_s$ onto itself, and $\tilde{\tau}_u$ induces an order preserving homeomorphism from $\tilde{\mu}_u$ onto itself.*

Proof. We just deal with β^+_s, the other cases being similar. Let s be an element of $\tilde{\mu}_s$ fixed by a non-trivial element γ_0 of $\pi_1(P)$. According to Proposition 6.4 and Corollary 5.19 there is an element u of $\tilde{\mu}_u$ such that (s, u) is an element of Ω' fixed by a non-trivial element γ_0 of $\pi_1(P)$. More precisely, (s, u) is the lower left corner of a γ_0-invariant generalized lozenge whose other (upper right) corner is $(\beta^+_u(u), \beta^+_s(s))$ (see Lemma 5.20). In particular, $\beta^+_s(s)$ lies in $\tilde{\mu}_u$. Since elements with non-trivial $\pi_1(P)$-stabilizers are dense in $\tilde{\mu}_s$, we obtain $\beta^+_s(\tilde{\mu}_s) \subset \tilde{\mu}_u$. Moreover, by the same arguments, $\beta^+_s(\tilde{\mu}_s)$ contains all the elements of $\tilde{\mu}_u$ with non-trivial stabilizer. In order to conclude, we just have to prove $\tilde{\mu}_u \subset \beta^+_s(\tilde{\mu}_s)$.

Assume not. Let u be an element u of $\tilde{\mu}_u - \beta^+_s(\tilde{\mu}_s)$. Let y be the bigger element of $\tilde{\mu}_s$ such that $\beta^+_s(s) < u$ for every $s < y$. Then, for every $s > y$ we have $\beta^+_s(s) > u$. By hypothesis, $\beta^+_s(y)$ is different from u. There are two cases:

– either $\beta^+_s(y) < u$: in this case, for every s in $\tilde{\mu}_s$ we have $\beta^+_s(s) \leq \beta^+_s(y)$ if $s \leq y$ (because β^+_s is non decreasing) and $\beta^+_s(s) > u$ if $s > y$.

– or $\beta^+_s(y) > u$: in this case, for every s in $\tilde{\mu}_s$ we have $\beta^+_s(s) < u$ if $s < y$ and $\beta^+_s(s) > u$ if $s \geq y$.

In both cases, $\beta^+_s(\tilde{\mu}_s)$ is disjoint from the open interval I with extremities $\beta^+_s(y)$ and u. Since $\beta^+_s(\tilde{\mu}_s)$ is dense in $\tilde{\mu}_u$, it follows that I is a gap of $\tilde{\mu}_u$. By the previous Proposition it follows that u has non-trivial $\pi_1(P)$-stabilizer. It then follows by Lemma 5.21 that u is in $\beta^+_s(\tilde{\mu}_s)$. Contradiction.

The statements about $\tilde{\tau}_s$ and $\tilde{\tau}_u$ follow immediately. □

Once we proved that $\tilde{\tau}_s$ and $\tilde{\tau}_u$ are order preserving homeomorphisms, the following happens. For every element γ of $\pi_1(P)$:

– $\tilde{\tau}_s \circ \gamma = \gamma \circ \tilde{\tau}_s$ if γ preserves the orientation,

– $\tilde{\tau}_s \circ \gamma = \gamma \circ \tilde{\tau}_s^{-1}$ if γ reverses the orientation.

Corollary 6.7. *$\alpha^-_u \circ \beta^+_s(u) = u$ for all u in $\tilde{\mu}_u$ and $\alpha^-_s \circ \beta^+_u(s) = s$ for all s in $\tilde{\mu}_s$.*

Proof. The equalities are true in dense subsets of $\tilde{\mu}_u$, $\tilde{\mu}_s$ respectively by Lemma 5.21. The previous proposition shows that the maps $\alpha_s^-, \alpha_u^-, \beta_s^+, \beta_u^+$ are all homeomorphisms, so the result follows. □

Observe that $\alpha_s(\tilde{\mu}_s)$ in general is not necessarily contained in $\tilde{\mu}_u$, hence Proposition 6.6 is false if we replace α_s^- by α_s. But in the sequel we will need to understand where α_s and α_s^- may differ.

Lemma 6.8. *Let s be an element of $\tilde{\mu}_s$. We always have $\alpha_s^-(s) \leq \alpha_s(s)$, and if the strict inequality $\alpha_s^-(s) < \alpha_s(s)$ holds, then s is an element of $\tilde{\mu}_s$ with non-trivial $\pi_1(P)$-stabilizer. Similarly, we have $\beta_s(s) = \beta_s^+(s)$ unless s has a non-trivial stabilizer; and for every u in $\tilde{\mu}_u$, we have $\alpha_u^-(u) = \alpha_u(u)$ and $\beta_u(u) = \beta_u^+(u)$ unless u has a non-trivial stabilizer.*

Proof. The inequality $\alpha_s^-(s) \leq \alpha_s(s)$ is obvious. Assume $\alpha_s^-(s) < \alpha_s(s)$. Notice that $s \cap \Omega' \neq \emptyset$. By Lemma 5.7 there is p in s and an open interval I in $\mathcal{O}^u(p) \cap \Omega_P$ containing p in the interior. Let $u = \mathcal{O}^u(p)$. It is an element of $]\alpha_s(s), \beta_s(s)[$, and by the above there is $s_0 < s$ such that for for every element s' of $]s_0, s[$ the point (s', u) also lies in Ω'.

Let now u' be an element of $\mathcal{A}^u$ in the interval $]\alpha_s^-(s), \alpha_s(s)[$. For $s' < s$, we have $\alpha_s(s') \leq \alpha_s^-(s) < u'$. For s' in $]s_0, s[$ we obtain $\alpha_s(s') < u' < \beta_s(s')$, the last inequality holding since (s', u) lies in Ω'. Therefore, for every s' in $]s_0, s[$ and every u' in $]\alpha_s^-(s), \alpha_s(s)[$, the point (s', u') lies in Ω'. But (s, u') is not in Ω' (since $\alpha_s(s) > u'$). It follows that $\beta_u(u') = s$ for every u' in $]\alpha_s^-(s), \alpha_s(s)[$. The map β_u is therefore constant on $]\alpha_s^-(s), \alpha_s(s)[$. As in Lemma 5.15, we obtain that $]\alpha_s^-(s), \alpha_s(s)[$ is contained in a gap of $\tilde{\mu}_u$. By Proposition 6.4 this gap is periodic, and left invariant by some γ_0 in $\pi_1(P) - id$, hence s, which is the image of $]\alpha_s^-(s), \alpha_s(s)[$ by β_u, also is invariant by γ_0, and has a non-trivial stabilizer. □

Lemma 6.9. *There is an integer $k > 0$ such that $\tilde{\tau}_s^k$ and h coincide on $\tilde{\mu}_s$ and such that $\tilde{\tau}_u^k$ and h coincide on $\tilde{\mu}_u$.*

Proof. Recall that $\tilde{\tau}_s = \beta_u^+ \circ \beta_s^+$ and $\tilde{\tau}_u = \beta_s^+ \circ \beta_u^+$. Let (s, u) be an element of $(\tilde{\mu}_s \times \tilde{\mu}_u) \cap \Omega'$ fixed by a non-trivial element γ_0 of $\pi_1(P)$. Then $(h(s), h(u))$ is also a fixed point of γ_0 in $(\tilde{\mu}_s \times \tilde{\mu}_u) \cap \Omega$. There is a sequence of generalized lozenges connecting (s, u) to $(h(s), h(u))$. Moreover, if s is an attracting fixed point, $h(s)$ is attracting too. It follows that there is an integer $k(s)$ such that $h(s) = \tilde{\tau}_s^{k(s)}(s)$ and $h(u) = \tilde{\tau}_u^{k(s)}(u)$. Moreover, since $h^{-1} \circ \tilde{\tau}_s^\ell$ is continuous on $\tilde{\mu}_s$ for every integer ℓ, the map $s \mapsto k(s)$ is locally constant on $s \in \tilde{\mu}_s$, hence constant since $\pi_1(P)$-invariant. □

Here are a couple of remarks:

1) Suppose that γ_0 in $\pi_1(P)$ is associated with a boundary periodic in a Birkhoff torus. Consider the collection $\{x_i\}$, $i \in \mathbb{Z}$ of type S fixed points of γ_0 in $\mathcal{A}^s$. Notice that this set is invariant by h, hence bi-infinite. The intervals $]x_i, x_{i+1}[$ are associated with the lifts of the Birkhoff annuli and can be entering or exiting, alternatively. Hence, by Lemma 6.3, up to reindexing, the intervals $]x_{2i}, x_{2i+1}[$ are all complementary components of $\tilde{\mu}_s$ and the intervals $]x_{2i-1}, x_{2i}[$ all intersect $\tilde{\mu}_s$.

2) If $\mathcal{C}$ is the chain of lozenges associated with γ_0 and $\mathcal{D}$ is any other chain of lozenges invariant by some $\mathbb{Z}^2 < \pi_1(P)$, then $\mathcal{D}$ and $\mathcal{C}$ intersect each other a lot: for every lozenge C_i in $\mathcal{C}$, then it intersects one at least one lozenge in $\mathcal{D}$ and vice versa.

Proposition 6.10. *Let $\top$ be the map from $\widetilde{U} = \{(x, y) \in \tilde{\mu}_s \times \tilde{\mu}_s \mid x < y < \tilde{\tau}_s(x)\}$ into $\mathcal{A}^s \times \mathcal{A}^u$ defined by:*

$$\top(x, y) := (x, \alpha_s^-(y))$$

Then, $\top$ is a homeomorphism, with image $(\tilde{\mu}_s \times \tilde{\mu}_u) \cap \Omega'$.

Proof. In order to prove the Proposition, we just have to prove that the image of $\top(\widetilde{U})$ is precisely $(\tilde{\mu}_s \times \tilde{\mu}_u) \cap \Omega'$. This is because all maps are homeomorphisms.

Let (x, y) be an element of $\widetilde{U}$. The image $\top(x, y) = (x, \alpha_s^-(y))$ lies in $\tilde{\mu}_s \times \tilde{\mu}_u$. We have to show that this image is in Ω'.

Case 1: Consider first the case where x is fixed by a non-trivial element γ_0 of $\pi_1(P)$.

Then, there is a unique element u of $\tilde{\mu}_u$ fixed by γ_0 and such that (s, u) lies in Ω'. The point (x, u) is the lower left corner of some generalized lozenge R. The upper right corner of R is $(\beta_u^+(u), \beta_s^+(x))$. There is another generalized lozenge R' with corners

$$(\beta_u^+(u), \beta_s^+(x)) \quad \text{and} \quad (\beta_u^+(\beta_s^+(x)), \beta_s^+(\beta_u^+(u))) \;=\; (\tilde{\tau}_s(x), \tilde{\tau}_u(u)).$$

Finally there is a generalized lozenge W with corners $(\alpha_u^-(u), \alpha_s^-(x))$ and (x, u). We refer to figure 14 in this proof. Let $\mathcal{C}$ be the chain of generalized lozenges (each a union of adjacent lozenges) through (x, u). It contains R, R' and W. The only γ_0 fixed points of type S in $\mathcal{A}^s$ in $[x, \tilde{\tau}_s(x)]$ are the endpoints and $\beta_u^+(u)$. Suppose first that y

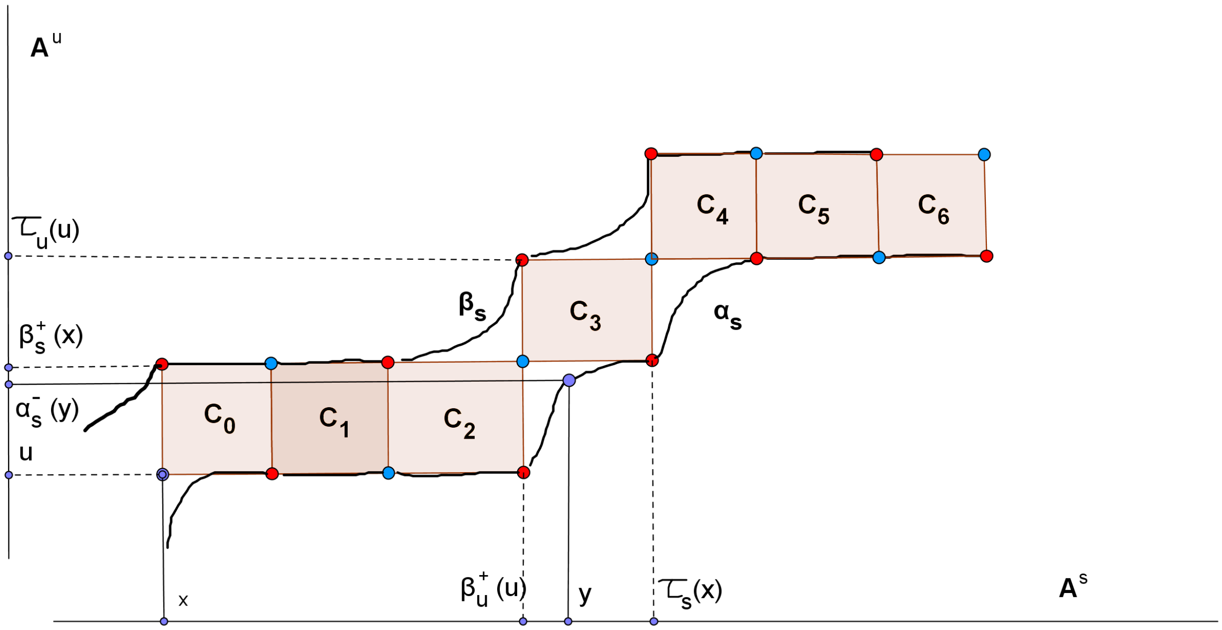

Figure 14: *The case where x has a non-trivial stabilizer.*

is in $]x, \beta_u^+(u)]$. If $y = \beta_u^+(u)$ then $\alpha_s^-(y) = u$ by Lemma 5.21, and hence $(x, \alpha_s^-(y))$ is in Ω'. Otherwise y is in $]x, \beta_u^+(u)[$. Clearly

$$\alpha_s(x) \leq \alpha_s^-(y) \leq \alpha_s^-(\beta_u^+(u)) = u.$$

So to have $\alpha_s^-(y)$ intersect x we only need to rule out $\alpha_s^-(y) = \alpha_s(x)$ and not intersecting x. Since y is in $]x, \beta_u^+(u)[$, $\tilde{\mu}_s$ intersects this interval. In addition $]x, y[$ cannot be a gap of $\tilde{\mu}_s$, as y is not a γ_0 fixed point of type S. In particular there is a translate γx in $]x, y[$, where we can assume γ preserves orientation in $\mathcal{A}^s, \mathcal{A}^u$. Hence the generalized lozenge $\gamma(W)$ in $\gamma(\mathcal{C})$ intersects the generalized lozenge R crossing it vertically: the interior of $\gamma(W)$ intersects parts of the unstable sides of R and the closure of $\gamma(W)$ does not intersect the stable sides of R. The lower left corner of $\gamma(W)$ is $(\gamma(\alpha_u^-(u), \alpha_s^-(x))$. The next generalized lozenge in $\gamma(\mathcal{C})$ in the negative direction has to intersect W horizontally and hence intersects s. This implies that

$$\alpha_s^-(y) \geq \alpha_s^-(\gamma(\alpha_u^-(u))) \ > \alpha_s(s)$$

so $\alpha_s^-(y)$ intersects s, which is what we wanted to prove in this subcase.

Suppose now that y is in $]\beta_u^+(u), \tilde{\tau}_s(x)[$. Here $\alpha_s^-(y) \geq \alpha_s^-(\beta_u^+(u)) = u$, so now we need to show that $\alpha_s^-(y) < \beta_s(s)$. By arguments similar to the subcase we just finished, there is γx in the interval $]y, \tilde{\tau}_s(x)[$. Then $\gamma(W)$ (W as above) intersects S horizontally, so $\alpha_s^-(\gamma(x)) < \beta_s(x)$. Since $\alpha_s^-(y) < \alpha_s^-(\gamma(x))$, it follows that $\top(x, y) = (x, \alpha_s^-(y))$ lies in Ω' This finishes the analysis in Case 1.

Case 2: From now on, we assume that x has a trivial $\pi_1(P)$-stabilizer.

Since $x < y$, we have $\alpha_s(x) \leq \alpha_s^-(y)$. Moreover, if we have equality $\alpha_s(x) = \alpha_s^-(y)$, then α_s is constant on $]x, y[$. It follows (Lemma 5.15) that $]x, y[$ is a gap of $\tilde{\mu}_s$, because both endpoints are in $\tilde{\mu}_s$. According to the important Proposition 6.4, x has then a non-trivial $\pi_1(P)$-stabilizer, that we have excluded. Therefore:

$$\alpha_s(x) < \alpha_s^-(y)$$

By hypothesis, we have $y < \tilde{\tau}_s(x) = \beta_u^+(\beta_s^+(x))$. It implies $\alpha_s^-(y) \leq \alpha_s^-(\beta_u^+(\beta_s^+(x)))$ By Lemma 5.21 we obtain $\alpha_s^-(y) \leq \beta_s^+(x)$.

We claim that this inequality is strict. Indeed, assume by a way of contradiction that the equality $\alpha_s^-(y) = \beta_s^+(x)$ holds. Then $\alpha_s^-(z) = \beta_s^+(x) = \alpha_s^-(y)$ for any z in $]y, \beta_u^+(\beta_s^+(x))[$. As we have already observed several times, since α_s^- is then constant on $]y, \beta_u^+(\beta_s^+(x))[$, this segment is contained in a periodic gap of $\tilde{\mu}_s$, preserved by a non-trivial element γ_0 of $\pi_1(P)$. But y and $\beta_u^+(\beta_s^+(x))$ are both elements of $\tilde{\mu}_s$, hence they are both fixed points of γ_0. We conclude that $(y, \beta_s^+(x))$ is a corner of a generalized lozenge R. But then we obtain $\alpha_s^-(y) < \beta_s^+(x)$: contradiction.

Therefore, as claimed, we have $\alpha_s^-(y) < \beta_s^+(x)$. According to Lemma 6.8, and since the $\pi_1(P)$-stabilizer of x is trivial, $\beta_s(x) = \beta_s^+(x)$: hence $\alpha_s(x) < \alpha_s^-(y) < \beta_s(x)$. These inequalities mean that $\top(x, y) = (x, \alpha_s^-(y))$ lies in Ω': we have proved $\top(\widetilde{U}) \subset (\tilde{\mu}_s \times \tilde{\mu}_u) \cap \Omega'$.

Conversely, let (x, z) be an element of $(\tilde{\mu}_s \times \tilde{\mu}_u) \cap \Omega'$. According to Proposition 6.6, $z = \alpha_s^-(y)$ for some element y of $\tilde{\mu}_s$. Clearly, since $(x, \alpha_s^-(y))$ lies in Ω', we have $y > x$. Assume that $y \geq \tilde{\tau}_s(x)$: then by Corollary 6.7

$$\alpha_s^-(y) \ \geq \ \alpha_s^-(\beta_u^+ \circ \beta_s^+(x)) \ \geq \ \beta_s^+(x).$$

But this is impossible since $(x, \alpha_s^-(y)) = (x, z)$ is in Ω'. Therefore, $y < \tilde{\tau}_s(x)$: (x, y) is an element of $\widetilde{U}$ such that $\top(x, y) = (x, z)$. This finishes the proof of the proposition. □

7. Proof of the Main theorem

We have now all the ingredients needed in the proof of the main theorem, which we state in more precise terms:

Theorem 7.1. *Let (M, Φ) be a pseudo-Anosov flow in a closed 3-manifold. Let P be a free Seifert piece in M. Assume that P is not elementary, i.e. that $\pi_1(P)$ does not contain a free abelian group of finite index. Then, in the intermediate cover M_P associated to $\pi_1(P)$ there is a compact submanifold $\hat{P}$ bounded by embedded Birkhoff tori, such that the restriction of the lifted flow $\hat{\Phi}$ to $\hat{P}$ is orbitally equivalent to a hyperbolic blow up $(P_\Gamma(\Phi), \Phi)$ of a geodesic flow associated to a convex cocompact subgroup $\Gamma \subset \widetilde{PGL}(2, \mathbb{R})$ isomorphic to $\pi_1(P)$. More precisely, this orbital equivalence maps the restricted foliations $\hat{\Lambda}^s$ and $\hat{\Lambda}^u$ to the restricted foliations $\hat{\Lambda}^s_\Gamma$ and $\hat{\Lambda}^u_\Gamma$ (see Definition 4.7). Moreover, $\hat{P}$ is almost unique up to isotopy along the lifted flow $\hat{\Phi}$: if $\hat{P}'$ is another compact submanifold bounded by embedded Birkhoff tori, and if $\hat{P}_*$, $\hat{P}'_*$ are the complements in $\hat{P}$, $\hat{P}'$ of the (finitely many) periodic orbits contained in $\partial\hat{P}$, there is a map $t : \hat{P}_* \to \mathbb{R}$ such that the map from $\hat{P}_*$ into M_P mapping x on $\hat{\Phi}^{t(x)}(x)$ is a homeomorphism, with image $\hat{P}'_*$.*

The manifold $\hat{P}$ has boundary and the flow $\hat{\Phi}$ is neither tangent nor transverse to the boundary everywhere, so the situation is way more subtle than a semi-flow in $\hat{P}$ transverse to the boundary. The semiconjugacy sends orbits to orbits, where an orbit may be defined only in an interval of the parameter that includes any possible boundary points. One added subtlety or difficulty in the proof involves the tangent orbits in $\partial\hat{P}$.

Proof. The proof of this theorem will take all of this section and it will use the previous constructions in the article.

We summarize what has been done in the previous sections: the action ρ_s of $\pi_1(P)$ on the stable leaf space $\mathcal{H}^s$ has an invariant axis $\mathcal{A}^s$, that is homeomorphic to the reals and is properly embedded in $\mathcal{H}^s$. We identify $\mathcal{A}^s$ with the reals $\mathbb{R}$. Let $\Gamma = \pi_1(P)$. Let h represent the regular fiber in $\pi_1(P)$. Then h acts freely on $\mathcal{A}^s \cong \mathbb{R}$. Since $< h >$ is a normal subgroup of $\pi_1(P)$ this induces an action of $\bar{\rho}_s$ of $\bar{\Gamma} = \pi_1(P)/ < h >$ on

$$\mathbb{S}^1 \ = \ \mathbb{R}/h \ \cong \ \mathcal{A}^s/h$$

The goal is to show that the induced representation $\bar{\rho}_s$ is a hyperbolic blow up of a Fuchsian representation and then use the results and constructions of section 4.

The unique Γ-invariant minimal set $\tilde{\mu}_s$ projects to an invariant set μ_s in $\mathbb{S}^1$, which is the unique $\bar{\rho}_s(\bar{\Gamma})$-invariant minimal invariant set.

We also have a homeomorphism $\tilde{\tau}_s : \tilde{\mu}_s \to \tilde{\mu}_s$ almost commuting with $\rho_s(\Gamma)$ and an integer $k > 0$ such that $\tilde{\tau}_s^k = h|_{\tilde{\mu}_s}$ (Proposition 6.6, Lemma 6.9).

Then, exactly as is done in Proposition 3.18 of [Ba3], one can extend $\tilde{\tau}_s$ to $\mathcal{A}^s$ so that it commutes with h and satisfies $\tilde{\tau}_s^k = h$. Since it commutes with h, it follows that $\tilde{\tau}_s$ induces a homeomorphism τ_s of $\mathbb{S}^1$. Let $\mathfrak{I}$ be the set of gaps of μ_s in $\mathbb{S}^1$. Let $\sigma_s : \bar{\Gamma} \to S(\mathfrak{I})$ be the representation describing the action of $\bar{\Gamma}$ on $\mathfrak{I}$. It is very easy to see that $\bar{\rho}_s$ is a $(\mu_s, \tau_s, \sigma_s)$-representation on $\mathbb{S}^1$, that lifts to the $(\tilde{\mu}_s, \tilde{\tau}_s, \tilde{\sigma}_s)$-representation ρ_s on $\mathcal{A}^s$.

We want to show that $\bar{\rho}_s$ is a hyperbolic blow up of a Fuchsian representation. We know that all gaps of μ_s are periodic (Proposition 6.4). First we modify the actions on gaps by "hyperbolic" modifications. Instead of adding new fixed points as we did previously, we replace the action of the stabilizer on the periodic gap under consideration by a map f_0 which has no fixed points in the gap, and only fixes the extremities. Hence it is more accurate to call this process a hyperbolic *blow down.* This produces another representation $\bar{\rho}'_s : \bar{\Gamma} \to \mathrm{Homeo}(\mathbb{S}^1)$. Let $\pi_s : \mathcal{A}^s \to \mathbb{S}^1$ be the projection map.

Lemma 7.2. *The representation $\bar{\rho}'_s$ is topologically conjugate to a Fuchsian representation.*

Proof. We need to verify the conditions of Theorem 3.13 that we reproduce here for the readers's convenience (we recall that $\bar{\Gamma}_0$ is the index 2 subgroup made of elements preserving the orientation):

(1) every gap of μ_s is periodic,
(2) for every x in $\mathbb{S}^1$, the stabilizer of x is trivial or cyclic,
(3) for every non-trivial element $\bar{\gamma}$ of $\bar{\Gamma}_0$ the fixed point set of $\bar{\rho}'_s(\bar{\gamma})$ is either trivial, or one orbit of τ_s, or the union of two orbits by τ_s, one made of attractive fixed points and the other made of repellent fixed points,
(4) if (x_0, y_0) is a pair of fixed points of some element of $\bar{\Gamma}$ with $x_0 < y_0 < \tau_s(x_0)$, then the $\bar{\rho}'_s(\bar{\Gamma}_0)$-orbit by the diagonal action of (x_0, y_0) in the space $U = \{(x,y) \in \mu_s \times \mu_s \mid x < y < \tau_s(x)\}$ is closed and discrete.

Condition (1) is the content of Proposition 6.4.

Condition (2) is also easy to deduce. The stabilizer of a point by ρ_s is at most cyclic, as Φ is a pseudo-Anosov flow. Hence condition (2) was already satisfied by the representation $\bar{\rho}_s$. During the hyperbolic blow down we have eliminated fixed points, so condition (2) follows.

Now consider condition (3). Let $\bar{\rho}'_s(\bar{\gamma})$ be an element of the representation. Look back at $\bar{\rho}_s(\bar{\gamma})$ before the blow down. Either it acts freely or has fixed points. If it has fixed points, then it has an even number of points, that are consecutively attracting and repelling. This is because when we pull back $\bar{\gamma}$ to an element γ of Γ so that it has fixed points, then the fixed points of γ are discrete and are alternatively attracting and repelling. So the same holds for $\bar{\rho}_s(\bar{\gamma})$. When we perform the hyperbolic blow down, the fixed points that are not in μ_s are removed. Hence the only remaining the fixed points are those corresponding to elements in $\tilde{\mu}_s$. This proves (3).

Finally consider condition (4). In Proposition 6.10 we proved that the map $\top$ from

$$\widetilde{U} \;=\; \{(x,y) \in \tilde{\mu}_s \times \tilde{\mu}_s \mid x < y < \tilde{\tau}_s(x)\}$$

into $\mathcal{A}^s \times A^u$ defined by $\top(x,y) = (x, \alpha_s^-(y))$ is a homeomorphism between $\widetilde{U}$ and $(\tilde{\mu}_s \times \tilde{\mu}_u) \cap \Omega$. Let (x_0, y_0) be an element of $\widetilde{U}$ fixed by some element γ_0: its image by $\top$ is an element of $\Omega' \approx \Omega_P \subset \mathcal{O}$ corresponding to a periodic orbit of $\widetilde{\Phi}$. Since every periodic orbit in M is compact, and since every periodic orbit of Φ admits a neighborhood in M in which there is no other periodic orbit freely homotopic to it, it follows that the $\bar{\rho}_s(\bar{\Gamma}_0)$-orbit of $(\pi_s(x_0), \pi_s(y_0))$ in $U = \{(x,y) \in \mu_s \times \mu_s \mid x < y < \tau_s(x)\}$ is closed and discrete (for more details, see for example Lemma 3.20 in [Ba3]). Now the $\bar{\rho}'_s(\bar{\Gamma}_0)$-orbit of $(\pi_s(x_0), \pi_s(y_0))$ coincides with its $\bar{\rho}_s(\bar{\Gamma}_0)$-orbit since the hyperbolic blow down does not modify nothing in $\mu_s \times \mu_s$. Condition (4) is proved.

By Theorem 3.13 it follows that $\bar{\rho}'_s$ has the (k)-convergence property, and is topologically conjugate to a Fuchsian group. □

The same happens with the representation $\bar{\rho}_u$ coming from the unstable axis $\mathcal{A}^u$: it is a hyperbolic blow up of a Fuchsian representation. Observe that α_s^+ induces a conjugacy between $\bar{\rho}_s$ and $\bar{\rho}_u$, at least on the minimal sets μ_s and μ_u. It follows that the Fuchsian representations conjugated to $\bar{\rho}'_s$ and $\bar{\rho}'_u$ are the same. This is because a Fuchsian action is determined by its action on the minimal set. It follows that $\bar{\rho}_s$ and $\bar{\rho}_u$ are hyperbolic blow ups of the <u>same</u> Fuchsian representation $\bar{\rho}_0 : \bar{\Gamma} \to \mathrm{PGL}(2, \mathbb{R})$.

Since every fixed point in $\mathcal{A}^s$ or $\mathcal{A}^u$ is attracting or repelling, the Fuchsian representation has no parabolic element and no non-trivial element of finite order: it is a convex cocompact Fuchsian representation as defined in Definition 4.1.

Therefore, we are precisely in the situation described in section 4: There is a hyperbolic blow up of a piece of geodesic flow $(P_\Gamma(\Omega), \Psi, \hat{\Lambda}^s_\Gamma, \hat{\Lambda}^u_\Gamma)$ associated to the hyperbolic blow ups $\bar{\rho}_s$ and $\bar{\rho}_u$ of the convex cocompact representation $\bar{\rho}_0$. As mentioned in Remark 4.8, this piece does not depend on the choice of the maps α_1 and β_1 involved in the construction. The orbit space $D(\Psi)$ of $(P_\Gamma(\Omega), \Psi)$ is identified with the domain D in $\mathcal{A}^s \times \mathcal{A}^u \approx \mathbb{R} \times \mathbb{R}$ obtained by removing from Ω' the triangles $\Delta(\tilde{\theta})$ associated to orbits $\tilde{\theta}$ tangent to lifted Birkhoff tori, and this identification is Γ-equivariant. In other words, the Seifert piece $\hat{P}$, equipped with the restriction of $\hat{\Phi}$, and the foliations $\hat{\Lambda}^s$, $\hat{\Lambda}^u$ has exactly the same transverse structure as $(P_\Gamma(\Omega), \Psi, \Lambda^s(\Psi), \Lambda^u(\Psi))$, meaning that there is a Γ-equivariant homeomorphism

$$\Upsilon : \; D(\Psi) \;\to\; D$$

between their orbit spaces. The usual key idea is that Υ should lift to an orbital equivalence between $(\hat{P}, \hat{\Phi})$ and $(P_\Gamma(\Omega), \Psi)$, which moreover maps $\hat{\Lambda}^{s,u}$ on $\hat{\Lambda}^{s,u}_\Gamma(\Psi)$ since Υ respects the horizontal/vertical foliations.

Actually this is not exactly true, since here we consider restrictions to Seifert pieces admitting toroidal components with <u>tangent</u> periodic orbits: $(P_\Gamma(\Omega), \Psi)$ depends on the choice on the boundary Birkhoff tori.

Moreover, there is an additional difficulty: the periodic orbits contained in the boundary Birkhoff tori of $\hat{P}$ might be singular periodic orbits, with p-prongs ($p \geq 3$) hence cannot be orbitally equivalent in their neighborhood to the corresponding periodic orbit in a Birkhoff torus $T \subset \partial P_\Gamma(\Omega)$. Such a periodic orbit lifts in $\widetilde{M}$ to an orbit $\tilde{\theta}$ contained in Ω_P but not in the interior of Ω_P: it is the situation described at the end of section 5.2. The associated triangle $\Delta(\tilde{\theta})$ is then empty.

We solve this difficulty as follows: let $\hat{\theta}$ be a periodic orbit of $\hat{\Phi}$, contained in a Birkhoff torus $T' \subset \partial\hat{P}$. Since the orbit space of the restriction of $\hat{\Phi}$ to $\hat{P}$ is $D \subset \Omega'$, there is only one component F^s of $\Lambda^s_P(\hat{\theta}) - \hat{\theta}$ intersecting $\hat{P}$, the other components are contained in $M_P - \hat{P}$. None of them can be in the interior of Ω_P (the triangle $\Delta(\tilde{\theta})$ is empty). What we do is to artificially modify the flow $\hat{\Phi}$ in the other side of T' in M_P in the neighborhood of $\hat{\theta}$ so that $\hat{\theta}$ is still a periodic orbit of the modified flow $\hat{\Phi}'$, but now regular.

More precisely, we consider pairwise disjoint small tubular open neighborhoods $U(T')$ in M_P of each Birkhoff torus $T' \subset \partial\hat{P}$, and the union $\mathcal{U}$ of $\hat{P}$ with all these open domains $U(T')$. We replace $\hat{\Phi}$ in $\mathcal{U}$ by a semi flow $\hat{\Phi}'$ such that:

– $\hat{\Phi}$ and $\hat{\Phi}'$ coincide in $\hat{P}$,

– every periodic orbit contained in $\partial\hat{P}$ is a 2-prong periodic orbit for $\hat{\Phi}'$,

– the restriction of $\hat{\Phi}'$ to every $U(T')$ is orbitally equivalent to an Anosov flow (not pseudo-Anosov) in the neighborhood of an embedded Birkhoff torus. By an abuse of notation we still denote by $\hat{\Lambda}^{s,u}$ the stable and unstable foliations in $U(T')$.

Recall that we proved in Remark 2.15 that embedded elementary Birkhoff annuli are unique up to orbital equivalence in their neighborhoods, meaning that any two of them admit tubular neighborhoods in which the restrictions of the flow are orbitally equivalent. This is important because we will now show that the arguments used in the proof in Remark 2.15 lead to the fact that $(\mathcal{U}, \hat{\Phi}')$ is orbitally equivalent to the restriction of Ψ to a neighborhood of $P_\Gamma(\Omega)$ in $M_\Gamma(\Omega)$.

Indeed: first, select pairwise disjoint tubular neighborhoods $U(T)$ for every T in $\partial P_\Gamma(\Omega)$, so that there are orbital equivalences f_T between the restriction of Ψ to $U(T)$ and the restriction of $\hat{\Phi}'$ to $U(T')$ (the torus T' corresponding to T being the one lifting to a Birkhoff plane having the same projection in $\Omega' \approx \Omega$ as a lift of T). Futhermore, f_T sends the restrictions of $\Lambda^{s,u}_\Gamma$ to $U(T)$ to the restrictions of $\hat{\Lambda}^{s,u}$ to $U(T')$ - more precisely, the lift of f_T induces on the orbit space the restriction of Υ to the chain of lozenges in Ω associated with (a lift of) T and the chain of lozenges in Ω' associated to (a lift of) T'. Observe that in general, $f_T(T)$ is not T', but a small deformation of it, which is isotopic to T' along the flow, outside the tangent periodic orbits. Since this deformation is along the flow, the foliation $\hat{\Phi}'$ still coincides with the foliation $\hat{\Phi}$ on one side of $f_T(T)$. We modify $\hat{P}$ so that we have $T = f_{T'}(T')$ for every T'.

Let $\mathcal{U}^*$ be the union in $M_\Gamma(\Omega)$ of $P_\Gamma(\Omega)$ and the $U(T')$'s, and equip $\mathcal{U}$, $\mathcal{U}^*$ with complete metrics, so that the orbits of the restrictions of Ψ to $\mathcal{U}^*$ and of $\hat{\Phi}'$ to $\mathcal{U}$, parametrized by unit length, are complete. Notice that $\mathcal{U}^*$,$\mathcal{U}$ are open. From now on let Ψ and $\hat{\Phi}'$ denote these restrictions, which after these reparametrizations are complete flows, not merely semi-flows.

By construction, the collection of maps f_T for T boundary components of $\partial P_\Gamma(\Omega)$, defines an orbital equivalence between Ψ and $\hat{\Phi}'$ in the neighborhoods of $\partial P_\Gamma(\Omega)$ and $\partial\hat{P}$ that we want to extend in the interior of $P_\Gamma(\Omega)$. Select a collection $(Z_i)_{i\in I}$ of open 2-dimensional disks in $P_\Gamma(\Omega)$ transverse to Ψ such that for some $\epsilon > 0$, the flow boxes U_i obtained by pushing Z_i along Ψ a time of absolute value $< \epsilon$, together with all the $U(T)$'s, define a locally finite covering of $\mathcal{U}^*$. We moreover can require that the flow boxes U_i for $i \in I$ are all contained in the interior of $P_\Gamma(\Omega)$.

For every $i \in I$, the projections in $\Omega \approx \Omega'$ of every lift of Z_i are open disks; take the image of each of them in Ω' by the map Υ, lift it to $\widetilde{M}$. We can choose this last lift so that it projects to a disk Z'_i contained in $\hat{P}$, and transverse to $\hat{\Phi}' = \hat{\Phi}$ (in $\hat{P}$). We then have a naturally defined homeomorphism $f_i : Z_i \to Z'_i$.

We incorporate the collection of indices I and the collection of boundary Birkhoff tori into a collection J. For every $j \in J$ is either an element of I, or a Birkhoff torus T. In order to simplify the redaction, in the last case, we define Z_j and Z'_j as T, $T' = f_T(T)$ respectively, and U_j as the tubular neighborhood $U(T)$.

Now we do as in Remark 2.15: select a partition of unity $(\mu_j)_{i\in J}$ for the covering $(U_j)_{j\in J}$, and for every x in $\mathcal{U}^*$ define $f(x)$ as the barycenter of the various ${\hat{\Phi}'}^{t_j}(f_j(x_j))$, balanced with the weight $\mu_j(x)$, where x_j is an element of Z_j and t_j a real number in absolute value $< \epsilon$ such that $\Psi^{t_j}(x_j) = x$.

In this way, we obtain a continuous homotopy equivalence $f : \mathcal{U}^* \to \mathcal{U}$, mapping orbits of Ψ into orbits of $\hat{\Phi}'$. Observe that since the flow boxes U_i for $i \in I$ are contained in the interior of $P_\Gamma(\Omega)$, if $x \in \mathcal{U}^*$ is outside $P_\Gamma(\Omega)$, or even in $\partial P_\Gamma(\Omega)$, then $f(x) = f_T(x)$ for some torus T: f restricted to the complement of $P_\Gamma(\Omega)$ is a homeomorphism, with image the complement of $\hat{P}$ in $\mathcal{U}$.

Moreover, f lifts to a map $\tilde{f}$ between the universal coverings that, restricted to $P_\Gamma(\Omega)$ induces the map $\Psi : \Omega \to \Omega'$ between the orbit spaces. In particular, it follows that f maps leaves of $\Lambda^{s,u}(\Psi)$ into leaves of $\hat{\Lambda}^{s,u}$, and it may fail to be injective only along orbits of Ψ: if $f(x) = f(x')$, then there is some real number t such that $x' = \Psi^t(x)$. There is a continuous map $(x,t) \mapsto u(x,t)$ such that for every x in $\mathcal{U}^*$ we have $f(\Psi^t(x)) = (\hat{\Phi}')^{u(x,t)}(f(x))$ (this continuous map is well-defined and unique despite of periodic orbits, it is due to the fact that the orbits lifted in the universal covering are non-periodic).

As mentioned in Remark 2.15, if we can find a real number $t_0 > 0$ such that $u(x,t) \neq 0$ for every $t > t_0$ for every x in $\mathcal{U}^*$, then one can average f along Ψ-orbits and obtain a map $f_0 : \mathcal{U}^* \to \mathcal{U}$ with the same properties than f, but which is moreover a homeomorphism (once more, for details, we refer to Proposition 3.25 of [Ba3]).

The existence of such a t_0 will follow, by compactness of $P_\Gamma(\Omega)$, if we are able to prove that for every x in $\mathcal{U}^*$, there is an upper bound on the set of real numbers t such that $u(x,t) = 0$. Assume by a way of contradiction that there is a sequence of elements x_n of $\mathcal{U}^*$ and an increasing sequence $0 < t_1 < t_2 < ...$ of positive real numbers with no upper bound such that $u(x_n, t_n) = 0$ for every integer $n > 0$.

If $\Psi^t(x)$ lies outside $P_\Gamma(\Omega)$ for some t, then, since an orbit can exit $P_\Gamma(\Omega)$ at most once, we have $\Psi^s(x) \notin P_\Gamma(\Omega)$ for all $s > t$ and therefore $f(\Psi^s(x)) \neq f(x)$. Since $f(\Psi^{t_n}(x_n)) = (\hat{\Phi}')^{u(x_n,t_n)}(f(x_n)) = f(x_n) \in P_\Gamma(\Omega)$, it follows that for every t in $[0, t_n]$ the iterate $\Psi^t(x)$ lies in $P_\Gamma(\Omega)$. Since $P_\Gamma(\Omega)$ is compact, up to a subsequence, we can assume that x_n and $y_n = \Psi^{t_n}(x_n)$ have limits x_∞ and y_∞. Moreover, $f(y_n) = f(\Psi^{t_n}(x_n)) = f(x_n)$, therefore x_∞ and y_∞ have the same image by f, and since f is transversely injective, there is some real number t_1 such that $y_\infty = \Psi^{t_1}(x_\infty)$.

For every integer n, consider the loop γ_n in $P_\Gamma(\Omega)$ starting from x_∞, going to x_n, then following the Ψ-orbit during the time t_n until reaching y_n, when going to $y_\infty = \Psi^{t_1}(x_\infty)$, and then going back to x_∞ along the Ψ-orbit backward. The bigger n, the smaller the intermediate steps outside the Ψ-orbits, and the bigger the time t_n of travel along the first orbit. Due to the pseudo-Anosov character of Ψ, the action of γ_n on the orbit space $D(\Psi)$ near the orbit of a fixed lift of x_∞ is contracting on the vertical, and expanding on the horizontal, and the bigger n is, the bigger are these expansions and contractions.

On the other hand, the image of γ_n by f is a loop homotopic to the loop γ'_n in $\hat{P}'$, going from $f(x_\infty)$ to $f(x_n)$, then going quickly to $f(y_\infty) = f(x_\infty)$. Indeed, we can replace the portion on the $\hat{\Phi}'$-orbits by constant maps, since $u(x_n, t_n) = 0 = u(x_\infty, t_1)$. Therefore, this loop is homotopically trivial and acts trivially on the orbit space D. It is a contradiction, since Υ is a conjugacy between the action of γ_n on $D(\Psi)$ and the action of γ'_n on D.

Therefore, we have shown that $(\mathcal{U}^*, \Psi)$ and $(\mathcal{U}, \hat{\Phi}')$ are orbitaly equivalent, in a way preserving the stable and unstable foliations It follows that the same is true for the restrictions of Ψ and $\hat{\Phi}$ to respectively $P_\Gamma(\Omega)$ and $\hat{P}$, since $\hat{\Phi}$ and $\hat{\Phi}'$ coincide on $\hat{P}$.

This finishes the proof of Theorem 7.1, except that we still have to show that $\hat{P} \subset M_P$ is unique up to isotopy along the flow outside the periodic orbits contained in the boundary. This essentially follows from the "uniqueness of Birkhoff tori up to flow isotopy", established in [Ba-Fe1]. However, in [Ba-Fe1] this statement was imprecise, since as pointed out in Remark 2.15, the flow isotopy does not in general extend to the tangent periodic orbits. We clarify the situation here.

The key fact is that for any other compact core $\hat{P}'$ of M_P, admitting as boundary components embedded Birkhoff tori, every component T' of $\partial\hat{P}'$ is homotopic to a boundary torus of $\hat{P}$, hence with the same fundamental group (as subgroups of $\pi_1(M)$). Therefore, the chains of lozenges in Ω_P preserved by $\pi_1(T)$ and $\pi_1(T')$ are the same. Therefore, we can push along the flow every Birkhoff torus in $\hat{P}'$ onto the corresponding torus in $\partial\hat{P}$ (except the tangent periodic orbits that they already have in common). Moreover, the order in which entrance/exit tori in the boundary are crossed is encoded in the orbit space D: we can therefore push every boundary torus one-by-one, without perturbing at any moment what has be done previously. It then follows that this procedure lead to an isotopy along the flow between the convex cores $\hat{P}'$ and $\hat{P}$ outside their boundary components.

Theorem 7.1 is proved. □

8. **Examples of pseudo-Anosov flows and tori decompositions**

Here we produce a variety of examples showing how two adjoining pieces P, P' of the JSJ decomposition can behave with respect to a pseudo-Anosov flow. The examples we mention here were constructed in one way or another previously in other references. Therefore we only describe them briefly with an emphasis on the properties we want to analyze and refer to the appropriate references. First a preliminary lemma:

Lemma 8.1. *Let Φ be a pseudo-Anosov flow in M^3 and P, P' adjoining pieces along a torus T. Suppose that P is a free Seifert fibered piece and Φ is not orbitally equivalent to a suspension Anosov flow. Then T is not homotopic to a torus transverse to Φ, that is, it can only be homotopic to a quasi-transverse torus.*

Proof. Let h represent the fiber in P. According to Prop. 5.1 the axes $\mathcal{A}^s, \mathcal{A}^u$ are properly embedded lines in the respective leaf spaces. Since Φ is not orbitally equivalent to a suspension Anosov flow, then $\pi_1(T)$ leaves invariant a chain of lozenges $\mathcal{C}$ [Fe5]. The lozenges intersect the stable and unstable leaves in $\mathcal{A}^s, \mathcal{A}^u$. If two lozenges C_1, C_2 are adjacent, then they either intersect the same stable leaves or the same unstable leaves. Since h acts monotonically on both $\mathcal{A}^s, \mathcal{A}^u$ there are consecutive lozenges that are not adjacent. According to [Ba-Fe2] this implies that T cannot be homotopic to a torus transverse to Φ. $\square$

Note that there may be many consecutive lozenges in $\mathcal{C}$ that are adjacent. The proof shows that *not all* such pairs can be adjacent. Now we begin the examples.

Free adjoining free

Examples of these are the Handel-Thurston flows [Ha-Th]. For example suppose that Ψ_0 is the geodesic flow in a hyperbolic surface S. Cut the manifold along the torus above a (say) separating, simple geodesic c of S and reglue with an appropriate Dehn twist. Handel and Thurston showed that the resulting flow is Anosov. If S_1, S_2 are the closures of the complementary components of c and M_1, M_2 are the $\mathbb{S}^1$ bundles over these, then they survive in the surgered manifold M. In addition they form the JSJ decomposition of M. Finally both pieces are free.

Free adjoining periodic

Start with say the geodesic flow Φ_0 in a hyperbolic surface S of high enough genus. Let c be (say) a separating, simple geodesic of S. Let A_0 be a closed annulus neighborhood of c in S, and let S_-, S_+ be the closures of the complementary components of A_0 in S. Let V_0, V_-, V_+ be the unit tangent bundles of A_0, S_-, S_+ in T^1S. The unit tangent bundle T^1S is the union of V_0, V_- and V_+.

Now do a blow up of one periodic flow line θ_+ of Φ_0 over c. In doing so, we create 3 periodic orbits, $\theta_1, \theta_2, \theta_3$, two of which are hyperbolic and the last one (say) θ_3 is an expanding orbit. Remove a solid torus neighborhood of θ_3 to get a manifold M_1 with a semi-flow transverse to the boundary. In the terminology of [BBY], M_1 is a *hyperbolic attracting plug*: the maximal invariant subset is hyperbolic, and there is a unique entrance torus. The entrance lamination (see [BBY]) is then a foliation. It is easy to check that the stable leaves of θ_1, θ_2 intersects the entrance torus along closed leaves. Moreover, since θ_1 and θ_2 are freely homotopic, the closed leaves in the entrance torus have the same contracting direction (once more, in the terminology of [BBY]). It follows that the entrance foliation admits at least one Reeb annulus - actually, a more careful analysis as in [Fr-Wi] would show that it is the union of two Reeb annuli.

Now consider a copy M_2 of M_1 but with the reversed flow. In [BBY], Béguin, Bonatti and Yu showed that if we glue M_1 to M_2 by a map h sending the exit foliation of ∂M_2 to a foliation of ∂M_1 transverse to the entrance lamination, then the resulting flow Φ is Anosov. Since the entrance and exit foliations have a Reeb annulus, such a gluing map must send the isotopy class of closed exit leaves to the isotopy class of closed entrance leaves. This is because the only possible way for h to map the boundary of a Reeb annulus transversely to the entrance foliation (which has a Reeb annulus) is for h to map this curve to the isotopy class of closed curves in the entrance foliation.

Each submanifold V_-, V_+ survives in M_1 and hence in M. The same is true for their copies V'_-, V'_+ in M_2. By construction these are Seifert fibered manifolds and they are free. The remaining part of M is the gluing of $W_0 \subset M_1$ with a copy of $W'_0 \subset M_2$, where W_0 is V_0 with a solid torus removed. Observe that W_0 and W'_0 are both diffeomorphic to the product of a punctured annulus with the circle. In addition θ_1 is a representative of the Seifert fiber in M_1 and so is any closed curve in the entrance foliation of ∂M_1. Similarly for the exiting foliation in ∂M_2. Therefore condition on the gluing maps we pointed out in the previous paragraph implies that the gluing map h sends the Seifert fibers of W_0 to the Seifert fibers of W'_0. Hence the union $W = W_0 \cup W'_0$ is also a Seifert piece, diffeomorphic to the product of the circle with the sphere minus fours disks. The fiber is freely homotopic to the periodic orbits θ_1, θ_2, hence *not* freely homotopic in M with the fiber of M_1. This shows that the pieces of the JSJ decomposition of M are V_-, V_+, V'_-, V'_+ and W. These pieces, except W, are free pieces for the Anosov flow in M. The piece W, adjoining all the others, is periodic.

This produces examples of Anosov flows with free pieces adjoing periodic pieces.

Other examples were constructed in [Ba-Fe1]: start with the geodesic flow Φ_0 in a closed hyperbolic surface S with a symmetry over a (say) simple geodesic c. Let S_1, S_2 be the closures of the complementary components of c in S and let M_1, M_2 be the unit tangent bundles of S_1, S_2 respectively. Do branched covers of S along c that generate branched covers of the unit tangent bundle. In [Ba-Fe1] we explain that the resulting branched flow Φ is a pseudo-Anosov flow in a manifold M. Unlike in the previous examples, this cannot be an Anosov flow. The pieces M_i lift to free Seifert pieces of the final flow Φ. The torus over the geodesic c lifts to an $\mathbb{S}^1$ bundle over a graph E that has singularities. This bundle is Seifert fibered. As in the previous example one can easily show that this is a piece of the JSJ decomposition of M and it is a periodic piece. It is adjoining to the free pieces. One can also do this construction with a disjoint union of geodesics in S.

Hyperbolic blow up

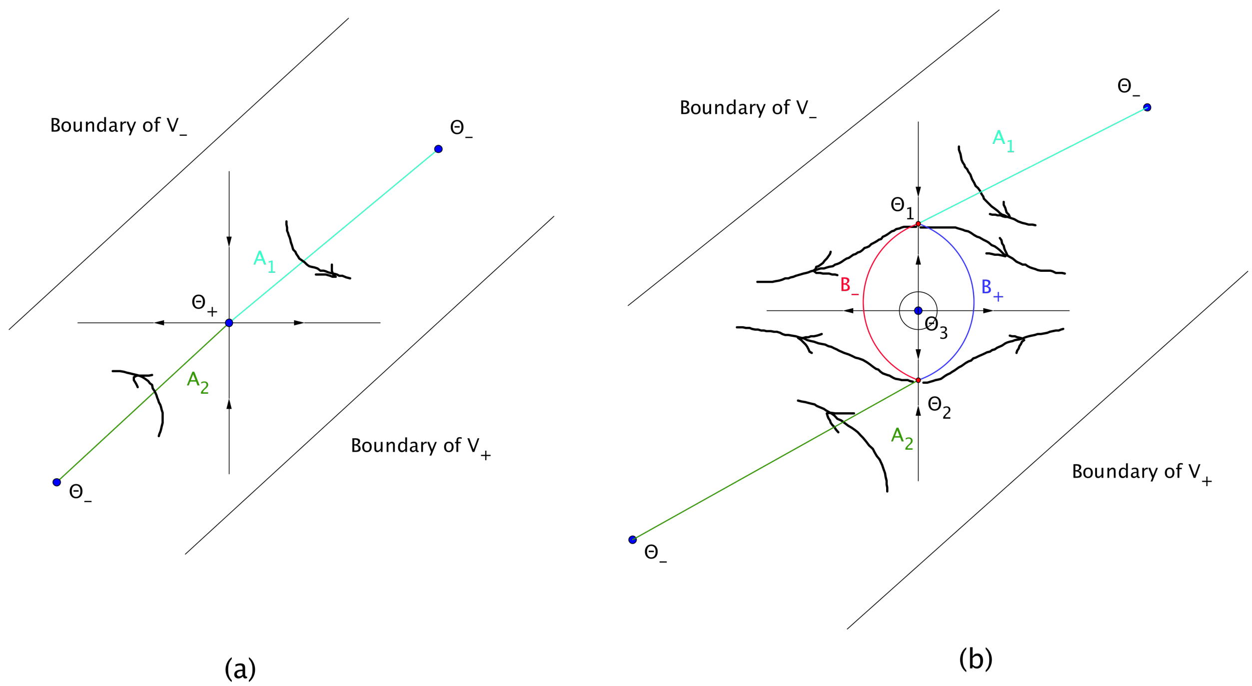

Figure 15: *Birkhoff annuli: (a) Before the DA blow up fo orbit θ_+, (b) After the DA blow up (the periodic orbit θ_3 is the repelling one created during the DA operation, it is removed during the contruction.*

We further explain the first class of examples in the previous subsection Free adjoining periodic. See Figure 15. Recall that $W = W_0 \cup W_0'$ is a (periodic) Seifert piece. The original flow Φ_0 had a Birkhoff torus associated with the geodesic c. This torus was made up from two Birkhoff annuli between the orbit θ_+ and another orbit θ_- corresponding to the geodesic traversed in the opposite direction. After the DA blow up of the orbit θ_+ creating periodic orbits θ_1, θ_2 and θ_3, these two Birkhoff annuli are modified to two Birkhoff annuli A_1, A_2, with A_1 from θ_1 to θ_- and A_2 from θ_2 to θ_-. These annuli are in the submanifold $W_0 \subset M_1$. There is an embedded annulus B_- from θ_1 to θ_2 contained in W_0 and so that the torus $A_1 \cup A_2 \cup B_-$ is isotopic to ∂V_- in $V_- \cup W_0$. The annulus B_- can be chosen transverse to the blow up flow in $T_1 S$, and so is a Birkhoff annulus for this flow, even though this flow is <u>not</u> Anosov or pseudo-Anosov. There is also a Birkhoff annulus B_+ in W_0 so that $A_1 \cup A_2 \cup B_+$ is isotopic to ∂V_+ in $W_0 \cup V_+$.

The union $A_1 \cup B_- \cup A_2$ is a Birkhoff torus for Φ in M as well (Φ is Anosov), again isotopic to ∂V_- in M.

Note that the Birkhoff annuli B_- and B_+ cannot be elementary for the Anosov flow Φ. This is because these Birkhoff annuli realize an oriented isotopy between the orbits θ_1 and θ_2 oriented by the direction of the flow. Therefore each of B_-, B_+ must be isotopic to a union of an even number elementary annuli (cf. [Ba-Fe1]). We analize this situation in much more detail. There is a symetric picture in W_0', with three periodic orbits $\theta_1', \theta_2', \theta_-'$ in W_0' that are freely homotopic to Seifert fibers and 4 Birkhoff annuli B_-', B_+', A_1', A_2'. Here A_1', A_2' are elementary and B_-', B_+' are not elementary. All the periodic orbits in the boundary of these Birkhoff annuli are fibers, but taking into account their orientation given by the direction of the flow, we see that θ_1, θ_2 are freely homotopic to θ_-', and to the <u>inverse</u> of θ_1', θ_2', θ_-.

Consider an elementary Birkhoff annulus containing θ_1 in its boundary. There are at most four such elementary Birkhoff annuli, because the flow is Anosov and therefore a given point in $\mathcal{O}$ is the corner of at most four lozenges. Recall that a quadrant is a connected component of a small tubular neighborhood of θ_1 with the stable/unstable local leaves removed. It is clear from the picture that one of the four quadrants of θ_1 cannot contain an elementary

Birkhoff annulus: the quadrant between the local stable and unstable half-leaves crossing the boundary of V_-. The three others quadrants contain A_1, B_- and B_+. We conclude that each of them contains one elementary Birkhoff annulus:, and we already know one of them, which is A_1. The other boundary component of any one of the two other Birkhoff annuli must be an oriented periodic orbit freely homotopic to the inverse of θ_1: it only can be θ'_1 or θ'_2. We explain this in more detail. Any closed orbit α of Φ what is freely homotopic to θ_1 cannot intersect ∂M_1 because ∂M_1 is transverse to the flow and separates M. In addition if $\alpha \subset M_1$ then since M_1, M_2 are union of Seifert spaces it follows that one can do cut and paste and produce a free homotopy from α to θ_1 entirely contained in M_1. But the only periodic orbit of Φ in M_1 that is freely homotopic to the inverse of θ_1 in M_1 is θ_-, because of the structure of the geodesic flow in T^1S. The same reasoning applies to M_2, hence the other orbit in the boundary of these Birkhoff annuli must be either θ'_1 or θ'_2.

It follows that there is an elementary Birkhoff annulus C_1 between θ_1 and θ'_1, and another elementary Birkhoff annulus D_1 between θ_1 and θ'_2. In a symmetric way, there is an elementary Birkhoff annulus C_2 between θ_2 and θ'_1, and another elementary Birkhoff annulus D_2 between θ_2 and θ'_2. The union

$$T_1 = C_1 \cup D_1 \cup C_2 \cup D_2$$

can be chosen to be embedded and it is isotopic to $\partial M_1 = \partial M_2$. Notice that T_1 has two periodic orbits θ_1 and θ_2 in M_1 and two other periodic orbits θ'_1, θ'_2 in M_2. The four Birkhoff annuli above are transverse to ∂M_1 in their interiors. Modulo changing the indices we can assume that θ'_1 is the orbit in the intersection of C_1 and C_2, and θ'_2 is the orbit in the intersection of D_1 and D_2.

Now we can describe the free Seifert pieces of M with respect to the flow. We can choose a representative of the free Seifert piece V_- in M bounded by the embedded Birkhoff torus $A_1 \cup A_2 \cup C_2 \cup C_1$. Similarly V_+ has a representative with boundary $A_1 \cup A_2 \cup D_2 \cup D_1$, also $\underline{V'_-}$ has a representative with boundary $A'_1 \cup A'_2 \cup D_1 \cup C_1$, and finally V'_+ has a representative with boundary $A'_1 \cup A'_2 \cup D_2 \cup C_2$.

Finally we discuss the periodic Seifert piece W. The periodic Seifert piece W has a two dimensional spine that can be chosen to be

$$Z \ = \ A_1 \cup A_2 \cup A'_1 \cup A'_2 \cup C_1 \cup C_2 \cup D_1 \cup D_2.$$

The associated fat graph to this Seifert piece (see explanation of this in [Ba-Fe1]) has 8 edges corresponding exactly to the 8 Birkhoff annuli in Z. In addition it has 6 vertices corresponding to $\theta_1, \theta_2, \theta'_1, \theta'_2, \theta_-, \theta_+$. The first 4 vertices have valence 3 in the fat graph and the last 2 have valence 2. For example θ_1 is a boundary of orbit of A_1, C_1 and D_1 only and θ_- is a boundary orbit of A_1 and A_2 only.

In terms of actions on the circle, the flow Φ in a Seifert fibered piece of M can be achieved by blowing up the action on one of the intervals associated with the action in the original Birkhoff annuli in T_1c to a homeomorphism with two new hyperbolic fixed points in the interior. This splits a lozenge into three adjacent lozenges.

Remark 8.2. Let us denote by Φ_1 the blow up flow of Φ_0 in $N = T^1S$. The two Birkhoff annuli A_1, A_2 of Φ_1 and the annulus B_+ bound a submanifold Y in N. The annulus B_+ is transverse to Φ_1 in the interior. The flow Φ_1 restricted to Z is already exactly in the format prescribed by the Main theorem (notice that Z is not a Seifert piece of the JSJ decomposition of N as N is Seifert fibered). What we mean is that it is orbitally equivalent to one obtained by the Hyperbolic blow up of geodesic flow operation in section 4. However the flow Φ_1 in N is not an Anosov flow, since it has a repelling orbit. That is why one does the operation of removing a solid torus neighborhood of the repelling orbit and gluing a time reversal flow to obtain an Anosov flow in the final manifold M. But the final flow in M contains a copy of the flow $\Phi_1|_Y$ in it and Y is associated with a Seifert fibered piece of the JSJ decomposition of M.

Free adjoining atoroidal

We give some examples of Anosov flows. Start with a geodesic flow Φ_0 in a closed hyperbolic surface S and let c be a *non simple* geodesic that fills a subsurface S_1 so that the complement is also not a union of annuli. For simplicity assume that the complement S_2 is connected. Let M_i be the bundles over S_i. Now do a high enough Dehn surgery on an orbit of Φ_0 over c. In [Fe6] the second author proves that the resulting manifold M has two pieces in the JSJ decomposition: one corresponds to M_1 that survives the surgery intact. The other, call it M', is obtained from the surgery in M_2. Here M_1 is Seifert fibered and free and M' is atoroidal [Fe6]. This produces the examples.

Atoroidal adjoining atoroidal

One can get examples very similar to the ones in the previous case. Use the same notation as in the previous setup, but now also do Dehn surgery on an orbit over a geodesic c' that fills S_2. Then the resulting manifold has two atoroidal pieces in its JSJ decomposition.

Periodic adjoining atoroidal

Examples of Anosov flows were constructed by Franks and Williams [Fr-Wi] in their seminal paper. These are not the original set of examples in that paper − the famous intransitive ones, but are slightly more complicated. Start with a suspension Φ_0 do a blow up of an orbit and remove a solid torus neighborhood to produce an atoroidal manifold M_1 with a semi-flow. Now glue to a more complicated manifold: for example it could be an $\mathbb{S}^1$ bundle P over a twice punctured sphere. The flow is the suspension of a simple diffeomorphism with Morse-Smale singularities. This produces a Seifert piece P in the resulting manifold and the piece is periodic.

Examples of pseudo-Anosov flows that have singularities can be obtained by the branched construction as explained in the free adjoining free examples as follows. Before doing the branching, do Dehn surgery to have the original manifold with two symmetric atoroidal pieces M_1, M_2 glued along a torus. Then proceed as before. The atoroidal pieces lift to atoroidal pieces in the resulting manifold. The torus lifts to a Seifert piece P that is periodic.

Periodic adjoining periodic

A very large class of examples were constructed in [Ba-Fe1]. The construction is not immediate and we refer the reader to [Ba-Fe1]. Then if P, P' are adjoining periodic pieces, we showed in [Ba-Fe2] that the adjoining torus T is isotopic to a torus *transverse* to the pseudo-Anosov flow. We showed that there are many examples that are Anosov.

Adjoining tori that are transverse

The original Franks-Williams examples have two atoroidal pieces glued along a transverse torus. As for atoroidal adjoining a periodic piece, we explained above that Franks-Williams produced examples where the adjoining torus is again transverse to the flow.

So unless the piece is free, there are examples where the adjoining torus is transverse to the flow.

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Thierry Barbot, Université d'Avignon et des pays de Vaucluse, LANLG, Faculté des Sciences, 33 rue Louis Pasteur, 84000 Avignon, France.

E-mail address: `thierry.barbot@univ-avignon.fr`

Sérgio Fenley, Florida State University, Tallahassee, FL 32306-4510, USA and Princeton University, Princeton, NJ 08544-1000, USA

E-mail address: `fenley@math.princeton.edu`