MULTIDIMENSIONAL TRANSONIC SHOCKS AND FREE BOUNDARY PROBLEMS FOR THE EULER EQUATIONS AND RELATED NONLINEAR EQUATIONS

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Abstract. We are concerned with free boundary problems originating from the analysis of multidimensional transonic shocks for the Euler equations in compressible fluid dynamics. We survey some recent developments in the analysis of multidimensional transonic shocks and corresponding free boundaries for the Euler equations and related nonlinear partial differential equations (PDEs). The nonlinear PDEs under our analysis include the steady Euler equations for potential flow, the steady full Euler equations, the unsteady Euler equations for potential flow, and related nonlinear PDEs of mixed elliptic-hyperbolic type. The transonic shock problems especially include the problem of steady transonic flow past solid wedges, von Neumann's problem for shock reflection-diffraction, and the Prandtl-Meyer problem for unsteady supersonic flow onto solid wedges. We first show how these longstanding multidimensional transonic problems can be formulated as free boundary problems for the Euler equations and related nonlinear PDEs of mixed type. Then we present an effective nonlinear method and related techniques for solving these free boundary problems, which should also be useful to analyze other longstanding or newly emerging free boundary problems for nonlinear PDEs.

Contents

1. Introduction

The purpose of this expository paper is to survey some recent developments in the analysis of free boundary problems originating from the analysis of multidimensional (M-D) transonic shock waves for the Euler equations in compressible fluid dynamics and related nonlinear nonlinear partial differential equations (PDEs). We show how several multidimensional transonic problems can be formulated as free boundary problems for the Euler equations and then present an efficient nonlinear method and related techniques for solving these free boundary problems.

Shock waves are steep wavefronts that are fundamental in nature, especially in high-speed fluid flows governed by the compressible Euler equations in fluid mechanics. The time-dependent compressible Euler equations are a nonlinear second-order nonlinear waves equations for potential flow or a first-order system of hyperbolic conservation laws for full Euler flow. One of the main features of such nonlinear PDEs is that, no mater how smooth the initial data starts with, the solution develops singularity in a finite time to form shock waves generically, so that the notion of solutions has to be extended to the notion of entropy solutions in order to accommodate the discontinuity waves such as the shock waves, that is, the weak solutions satisfying the entropy condition that is consistent with the second law of thermodynamics. The general entropy solutions involving shock waves (shocks for short) for this system have extremely complicated and rich structures. On the other hand, many fundamental problems in physics and engineering involve steady solutions (*i.e.*, time-independent solutions) or self-similar solutions (*i.e.*, the solutions depend only on the self-similar variables with form $\frac{\mathbf{x}}{t}$ for the space variables $\mathbf{x}$ and time-variable t). Such solutions are governed by the steady or self-similar compressible Euler equations for potential flow or, more generally, the full Euler flow. These governing PDEs in the new forms are time-independent and often are of mixed elliptic-hyperbolic type.

Generally speaking, *multidimensional transonic shocks* are codimension-one discontinuity fronts in the solutions of the steady or self-similar Euler equations and related nonlinear PDEs of mixed elliptic-hyperbolic type, which separate two phases: one of them is supersonic phase (*i.e.*, the fluid speed is larger than the sonic speed) which is hyperbolic, while the other is subsonic phase (*i.e.*, the fluid speed is smaller than the sonic speed) which is often elliptic (for potential flow) or elliptic-hyperbolic composite (for full Euler flow; i.e., elliptic equations composited with some hyperbolic transport equations). They are formed in many physical situations, for example, by smooth supersonic flows or supersonic shock waves impinging onto solid wedges/cones or passing through a de Laval nozzle, around supersonic or near-sonic flying bodies, or other natural processes. Mathematical analysis of shock waves (shocks, for short) can date back Stokes [112], Riemann [101], starting for the one-dimensional case. Mathematical understanding of multidimensional transonic shocks has been one of the most challenging and long-standing scientific research directions, since the solutions involved are discontinuous across the shocks. Such transonic shocks can be formulated as *free boundary problems* (FBPs) in the mathematical theory of nonlinear PDEs involving mixed elliptic-hyperbolic type.

General speaking, a *free boundary problem* is a boundary value problem for a PDE or a system of PDEs which is defined in a domain, a part of whose boundary is *a priori unknown*; this part is accordingly named as a *free boundary*. The mathematical problem is then to determine both the location of the free boundary and the solution of the PDE or the system of PDEs in the resulting domain, which requires to combine Analysis and Geometry in sophisticated ways, together with mathematical modelling based on Physics and Engineering. FBPs are one of the most important research directions in the analysis of PDEs, with wide applications across the sciences and real world problems; on the other hand, it is widely regarded as a truly challenging field of Mathematics.

Transonic shocks problems for steady or self-similar solutions are typically formulated as boundary-value problems for a nonlinear PDE or system of mixed elliptic-hyperbolic type, where the type of PDE at a point is determined by the solution (as well as its gradient for some models). For a system, the type is more complicated and may be either hyperbolic, or mixed-composite elliptic-hyperbolic (simply also called "mixed", for short when no confusion arises). General solutions of such nonlinear PDEs can be nonsmooth and of complicated structure, and their uniqueness is not known in many cases. However, in many problems, especially the those motivated by physical phenomena, the expected structure of solutions is known from the physics. The solutions are expected to be piecewise smooth and to have some hyperbolic and elliptic regions separated by shocks, or sonic surfaces (or curves in the 2-D case) with continuous type-transition (*i.e.*, the type of the PDEs changes without discontinuities in the quantities corresponding to the physical velocities). In this paper, we present the problems in which the hyperbolic part of the solution is known *apriori*, or can be determined separately from the elliptic part, in some larger region. Then the problem is reduced to determining the region in which the solution is elliptic, with the transonic shock as a part of its boundary is the transonic shock. In other words, we need to solve a free boundary problem with the transonic shock as a free boundary for the elliptic phase of the solution. Since the type of the PDE or system depends on the solution, the ellipticity in the region to be determined is a part of the results to be proved. We note that, in some other problems involving shocks, a FBP needs to be solved in order to find the hyperbolic part of the solution as well. We will not discuss such problems in this paper.

For several problems which we discuss below, the PDEs are a single second-order nonlinear equation of second order, whose type (elliptic or hyperbolic) depend on the gradient of solution. That is, the quasilinear elliptic PDE of second order, whose coefficients (and thus the type of the PDE) depend on the gradient of solution. In the other problems, the PDEs are a first-order nonlinear system, whose type is either hyperbolic or composite elliptic-hyperbolic, and is determined by the solution only. In these problems as FBPs, the key is to determine the expected elliptic region in which the solution is solved, while the hyperbolic part of the solution is *apriori* known. However, since the equation or system is of mixed type, the ellipticity in the region depends on the solution and thus is not determined *apriori*, and needs to be controlled in the process of solving the problem.

In all the problems discussed in the paper, the equation (or a part of the system) is elliptic for our solution in the region determined by the free boundary problem. That is, we solve an (expected) elliptic free boundary problem. However, the methods of elliptic FBPs, starting from the variational methods of Alt-Caffarelli [1] and Alt-Caffarelli-Friedman [2–4], and the Harnack inequality approach of Caffarelli [14–16] to other methods of many further works, do not directly apply to our problems. One of the reasons is that the type of equation needs to be controlled in order to apply these methods, which requires some strong estimates already. Another technical reason is that the mixed elliptic-hyperbolic problems do not directly fit into a standard variational framework, because the Euler-Lagrange equation is elliptic for convex functionals. On the other hand, the equations and boundary conditions with elliptic truncation have a complicated structure, which does not fit in the framework of [1–4] and other works on variational free boundary problems. The reasons of why methods of [14–16] do not apply directly include that a boundary comparison principle for positive solutions of nonlinear elliptic PDEs in Lipschitz domains is unavailable yet, in the case that nonlinear PDEs are not homogeneous with respect to the unknown function and its derivatives, which is the case for the problem. To overcome these difficulties, we use the global structure of our problem. It allows to derive certain properties of the solution (such as monotonicities), which allow to control the type of the PDEs and the geometry of the problem. With this, we solve the free boundary problem by the iteration procedure.

Because of the nonlinearity and mixed type of the equation/system, it is not clear if general weak solutions of the problem we consider is unique. On the other hand, we are interested in the solutions of a specific structure, motivated by physical applications. Thus, we construct the solution in a carefully defined class of solutions, which we call admissible solutions. This class needs to be defined with two somewhat opposite features: the conditions determining this class need not only to be flexible enough so that this class contains all possible solutions of the problem which are of the desired structure, but also to be rigid enough so that the conditions in the definition of admissible solutions force the desired structure of the solution and give the sufficient analytic and geometric control such that one can derive the estimates for these solutions and eventually construct a solution in this class by the iteration procedure. In order to define such class, we start with the solutions near some background solutions

(i) to make sure that the solutions obtained are still in the same phase (elliptic) via careful estimates;
(ii) to gain the insight and motivation for the solution structures and properties to form an admissible class of solutions on which the apriori estimates and fixed point argument are based.

In several problems below, we consider only the solutions near the background solution, as in (i) above. In the other problems, see Section 4, we carry out both steps described above and construct admissible solutions which are not close to any known background solution.

Furthermore, The elliptic and hyperbolic regions may be separated not only by shocks, which are discontinuity fronts for velocities, but also by sonic surfaces (or curves in the 2-D case) where the type of the equation changes without discontinuities in the quantities corresponding to the physical velocities, as pointed out earlier. This means that the ellipticity and hyperbolicity degenerate near the sonic surfaces. This introduces additional difficulties in the analysis of such solutions. Also, the sonic surfaces (or curves) may intersect the transonic shocks (see *e.g.* Figure 4.1, point P_1) so that, near such points, the analysis of solutions is even more involved.

The organization of this expository paper is as follows: In §2, we start with our presentation of multidimensional transonic shocks and free boundary problems for the Euler equations in a setup as simple as possible, and show how a transonic shock problem can be formulated as a free boundary problem for the corresponding nonlinear PDEs of mixed elliptic-hyperbolic type. Then we describe a nonlinear method and related ideas/techniques, first developed in Chen-Feldman [31], with focus on the key points for solving such free boundary problems through this simplest setup. In §3, we describe how the nonlinear method and related techniques presented in §2 can be applied to solve the existence, stability, and asymptotic behavior of two-dimensional steady transonic flows with transonic shocks past curved wedges for the full Euler equations, by reformulating the problems as free boundary problems via two different approaches. In §4, we describe how transonic shocks and free boundary problems for self-similar shock reflection/diffraction for the Euler equations for potential flow. In §5, we discuss some recent developments in the analysis of geometric properties of transonic shocks as free boundaries in two-dimensional self-similar coordinates for compressible fluid flows with focus on convexity properties of the self-similar transonic shocks in §4. Finally, in §6, we give several concluding remarks including some open problems for further developments.

2. Multidimensional Transonic Shocks and Free Boundary Problems for the Steady Euler Equations for Potential Flow

For clarity, we start with our presentation of multidimensional transonic shocks and free boundary problems for the Euler equations in a setup as simple as possible, and show how a transonic shock problem can be formulated as a free boundary problem for the corresponding nonlinear PDEs of mixed elliptic-hyperbolic type. Then we describe a nonlinear method and related ideas/techniques, first developed in Chen-Feldman [31], with focus on the key points for solving such free boundary problems through this simplest setup.

The steady Euler equations for potential flow, consisting of the conservation law of mass and the Bernoulli law for the velocity, can be written into the following second-order, nonlinear PDE of mixed elliptic-hyperbolic type for the velocity potential $\varphi : \mathbb{R}^d \to \mathbb{R}$ by scaling:

$$\operatorname{div}\,(\rho(|D\varphi|^2)D\varphi) = 0, \tag{2.1}$$

where the density function $\rho(q^2)$ is

$$\rho(q^2) = \left(1 - \theta q^2\right)^{\frac{1}{2\theta}} \tag{2.2}$$

with $\theta = \frac{\gamma-1}{2} > 0$ for the adiabatic exponent $\gamma > 1$, and $D := (\partial_{x_1}, \dots, \partial_{x_d})$, *i.e.*, the gradient with respect to $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$.

Equation (2.1) can also be written in the non-divergence form:

$$\sum_{i,j=1}^{d} \left(\rho(|D\varphi|^2)\delta_{ij} + 2\rho'(|D\varphi|^2)\varphi_{x_i}\varphi_{x_j}\right)\varphi_{x_ix_j} = 0, \tag{2.3}$$

where the coefficients of the second-order PDE (2.3) depend on $D\varphi$, the gradient of the unknown function φ.

The second-order nonlinear PDE (2.1), or equivalently (2.3) for smooth solutions, is strictly elliptic at $D\varphi$ with $|D\varphi| = q$ if

$$\rho(q^2) + 2q^2\rho'(q^2) > 0; \tag{2.4}$$

and is strictly hyperbolic if

$$\rho(q^2) + 2q^2\rho'(q^2) < 0. \tag{2.5}$$

In fluid mechanics, the elliptic regions of equation (2.1) correspond to the *subsonic flow*, the hyperbolic regions of (2.1) to the *supersonic flow*, and the regions with $\rho(q^2) + 2q^2\rho'(q^2) = 0$ for $q = |D\varphi|$ to the *sonic flow*.

2.1. **Multidimensional Transonic Shocks and Free Boundary Problems.** Let $\Omega \subset \mathbb{R}^d$ be a domain. A function $\varphi \in W^{1,\infty}(\Omega)$ is a *weak solution* of (2.1) in Ω if

(i) $|D\varphi(\mathbf{x})| \leqslant 1/\sqrt{\theta}$ *a.e.* $\mathbf{x} \in \Omega$ (*i.e.*, physical region after the scaling);
(ii) for any test function $\zeta \in C_0^\infty(\Omega)$,

$$\int_\Omega \rho(|D\varphi|^2)D\varphi \cdot D\zeta \,\mathrm{d}\mathbf{x} = 0. \tag{2.6}$$

We are interested in the weak solutions with shocks (surfaces of jump discontinuity of the solutions with codimension-one), motivated from Continuum Physics. More precisely, let Ω^+ and Ω^- be open nonempty subsets of Ω such that

$$\Omega^+ \cap \Omega^- = \varnothing, \qquad \overline{\Omega^+} \cup \overline{\Omega^-} = \overline{\Omega},$$

and $\mathcal{S} := \partial\Omega^+ \backslash \partial\Omega$. Let $\varphi \in W^{1,\infty}(\Omega)$ be a weak solution of (2.1) so that $\varphi \in C^2(\Omega^\pm) \cap C^1(\overline{\Omega^\pm})$ and $D\varphi$ has a jump across $\mathcal{S}$.

We now derive the necessary conditions on $\mathcal{S}$ that is a C^1 surface of codimension-one. First, the requirement that φ is in $W^{1,\infty}(\Omega)$ yields $\operatorname{curl}(D\varphi) = 0$ in the sense of distributions, which implies

$$\varphi_{\boldsymbol{\tau}}^+ = \varphi_{\boldsymbol{\tau}}^- \qquad \text{on } \mathcal{S}, \tag{2.7}$$

where

$$\varphi_{\boldsymbol{\tau}}^\pm := D\varphi^\pm - (D\varphi^\pm \cdot \boldsymbol{\nu})\boldsymbol{\nu}$$

are the trace values of the tangential gradients of φ on $\mathcal{S}$ in the tangential space with $(d-1)$-dimension on the $\Omega^\pm$ sides, respectively, and $\boldsymbol{\nu}$ is the unit normal to $\mathcal{S}$ from Ω^- to Ω^+. Then we simply write $\varphi_{\boldsymbol{\tau}} := \varphi_{\boldsymbol{\tau}}^\pm$ on $\mathcal{S}$ and choose

$$\varphi^+ = \varphi^- \qquad \text{on } \mathcal{S}. \tag{2.8}$$

Now, for $\zeta \in C_0^\infty(\Omega)$, we use (2.6) to compute

$$\begin{aligned} 0 &= \Big(\int_{\Omega^+} + \int_{\Omega^-}\Big) \rho(|D\varphi|^2) D\varphi \cdot D\zeta \,\mathrm{d}\mathbf{x} \\ &= -\int_{\partial\Omega^+} \rho(|D\varphi|^2) D\varphi \cdot \boldsymbol{\nu}\,\zeta \,\mathrm{d}\mathcal{H}^{d-1} + \int_{\partial\Omega^-} \rho(|D\varphi|^2) D\varphi \cdot \boldsymbol{\nu}\,\zeta \,\mathrm{d}\mathcal{H}^{d-1} \\ &= \int_{\mathcal{S}} \big(-\rho(|D\varphi^+|^2) D\varphi^+ \cdot \boldsymbol{\nu} + \rho(|D\varphi^-|^2) D\varphi^- \cdot \boldsymbol{\nu}\big)\,\zeta \,\mathrm{d}\mathcal{H}^{d-1}, \end{aligned}$$

where $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure, *i.e.*, the surface area measure. Thus, the other condition on $\mathcal{S}$, which measures the trace jump of the normal derivative of φ across $\mathcal{S}$, is

$$\rho(|D\varphi^+|^2)\varphi^+_{\boldsymbol{\nu}} = \rho(|D\varphi^-|^2)\varphi^-_{\boldsymbol{\nu}} \qquad \text{on } \mathcal{S}, \tag{2.9}$$

where $\varphi^\pm_{\boldsymbol{\nu}} = D\varphi^\pm \cdot \boldsymbol{\nu}$ are the trace values of the normal derivative of φ along $\mathcal{S}$ on the $\Omega^\pm$ sides, and

$$\rho(|D\varphi^\pm|^2) = \big(1 - \theta|\varphi^\pm_{\boldsymbol{\tau}}|^2 - \theta|\varphi^\pm_{\boldsymbol{\nu}}|^2\big)^{\frac{1}{2\theta}},$$

respectively.

Conditions (2.8)–(2.9) are called the Rankine-Hugoniot conditions for potential flow in fluid mechanics. On the other hand, it can also be shown that any $\varphi \in C^2(\Omega^\pm) \cap C^1(\overline{\Omega^\pm})$ such that $D\varphi$ has a jump across $\mathcal{S}$ satisfying the Rankine-Hugoniot conditions (2.8)–(2.9), must be a weak solution of (2.1).

Therefore, the necessary and sufficient conditions for $\varphi \in C^2(\Omega^\pm) \cap C^1(\overline{\Omega^\pm})$ to be a weak solution of (2.1) are the Rankine-Hugoniot conditions (2.8)–(2.9).

For given $K > 0$, consider the function:

$$\Phi_K(p) := \big(K - \theta p^2\big)^{\frac{1}{2\theta}}\, p \qquad \text{for } p \in [0, \sqrt{K/\theta}]. \tag{2.10}$$

This function satisfies

$$\Phi_K(p) > 0 \text{ for } p \in (0, \sqrt{K/\theta}), \qquad \lim_{p\to 0} \Phi_K(p) = \lim_{p\to\sqrt{K/\theta}} \Phi_K(p) = 0, \tag{2.11}$$

$$0 < \Phi_K'(p) \leqslant K^{\frac{1}{2\theta}} \text{ for } p \in (0, p^K_{\mathrm{sonic}}), \qquad \Phi_K'(p) < 0 \text{ for } p \in (p^K_{\mathrm{sonic}}, \sqrt{K/\theta}), \tag{2.12}$$

$$\Phi_K''(p) < 0 \text{ for } p \in (0, p^K_{\mathrm{sonic}}], \tag{2.13}$$

where

$$p^K_{\mathrm{sonic}} := \sqrt{K/(\theta+1)}. \tag{2.14}$$

By direct calculation, condition (2.4) is equivalent to $\Phi_1'(q) > 0$, and condition (2.5) is equivalent to $\Phi_1'(q) < 0$. Thus, using (2.12), we obtain that PDE (2.1) is strictly elliptic at $D\varphi$ if $|D\varphi| < p^1_{\mathrm{sonic}}$ and is strictly hyperbolic if $|D\varphi| > p^1_{\mathrm{sonic}}$, where we use the notation (2.14).

Suppose that $\varphi(x)$ is a solution satisfying

$$|D\varphi| < p^1_{\mathrm{sonic}} = 1/\sqrt{\theta+1} \quad \text{in } \Omega^+, \qquad |D\varphi| > p^1_{\mathrm{sonic}} \quad \text{in } \Omega^-, \tag{2.15}$$

and

$$D\varphi^\pm \cdot \boldsymbol{\nu} > 0 \quad \text{on } \mathcal{S}, \tag{2.16}$$

besides (2.8) and (2.9). Then $\varphi(x)$ is a *transonic shock solution* with *transonic shock* $\mathcal{S}$ that divides the *subsonic region* Ω^+ from the *supersonic region* Ω^-. In addition, $\varphi(\mathbf{x})$ satisfies the physical entropy condition (see Courant-Friedrichs [56]; also see [57, 76]):

$$\rho(|D\varphi^-|^2) < \rho(|D\varphi^+|^2) \tag{2.17}$$

which implies, by (2.16), that the density ρ increases in the flow direction. Note that equation (2.1) is elliptic in the subsonic region Ω^+ and hyperbolic in the supersonic region Ω^-.

For clarity of presentation of the nonlinear method, first developed in Chen-Feldman [31], we focus first on the free boundary problem in the simplest setup, while the method and related ideas and techniques have been applied to more general free boundary problems involving transonic shocks, which will be discussed later.

Let $(\mathbf{x}', x_d)$ be the coordinates of $\mathbb{R}^d$ with $x_d \in \mathbb{R}$ and $\mathbf{x}' = (x_1, \cdots, x_{d-1}) \in \mathbb{R}^{d-1}$. From now on, in this section, we focus on $\Omega := (0,1)^{d-1} \times (-1,1)$ for simplicity without loss of our main objectives.

Let $q^- \in (p^1_{\rm sonic}, 1/\sqrt{\theta})$ and $\varphi_0^-(x) := q^- x_d$. Then φ_0^- is a supersonic solution in Ω. From (2.11)–(2.13), there exists a unique $q^+ \in (0, p^1_{\rm sonic})$ such that

$$\big(1-\theta(q^+)^2\big)^{\frac{1}{2\theta}}\, q^+ = \big(1-\theta(q^-)^2\big)^{\frac{1}{2\theta}}\, q^-. \tag{2.18}$$

In particular, $q^+ < q^-$. Define $\varphi_0^+(\mathbf{x}) := q^+ x_d$ in Ω. Then the function

$$\varphi_0(\mathbf{x}) = \min(\varphi_0^+(\mathbf{x}), \varphi_0^-(\mathbf{x})) \tag{2.19}$$

is a transonic shock solution in Ω, in which $\Omega_0^\pm = \{x_d \gtrless 0\} \cap \Omega$ are the subsonic and supersonic regions of $\varphi_0(\mathbf{x})$, respectively. Also note that, on $\partial(0,1)^{d-1} \times [-1,1]$, the boundary condition $(\varphi_0)_{\boldsymbol{\nu}} = 0$ holds.

We start with perturbations of the background solution $\varphi_0(\mathbf{x})$ defined in (2.19). We use the following Hölder norms: For $\alpha \in (0,1)$ and any non-negative integer k,

$$[u]_{k,0,\Omega} = \sum_{|\boldsymbol{\beta}|=k} \sup_{x\in\Omega}(|D^{\boldsymbol{\beta}} u(\mathbf{x})|, \qquad [u]_{k,\alpha,\Omega} = \sum_{|\boldsymbol{\beta}|=k} \sup_{\mathbf{x},\mathbf{y}\in\Omega,\mathbf{x}\neq\mathbf{y}} \frac{|D^{\boldsymbol{\beta}} u(\mathbf{x}) - D^{\boldsymbol{\beta}} u(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\alpha}, \tag{2.20}$$

$$\|u\|_{k,0,\Omega} = \sum_{j=0}^{k} [u]_{j,0,\Omega}, \qquad \|u\|_{k,\alpha,\Omega} = \|u\|_{k,0,\Omega} + [u]_{k,\alpha,\Omega},$$

where $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_d), \beta_l \geqslant 0$ integers, $D^{\boldsymbol{\beta}} = \partial_{x_1}^{\beta_1} \cdots \partial_{x_d}^{\beta_d}$, and $|\boldsymbol{\beta}| = \beta_1 + \cdots + \beta_d$.

Then the transonic shock problem can be formulated as:

Problem 2.1. *Given a supersonic solution φ^- of (2.1) in Ω, which is a $C^{2,\alpha}$ perturbation of φ_0^-:*

$$\|\varphi^- - \varphi_0^-\|_{2,\alpha,\Omega} \leqslant \sigma \tag{2.21}$$

for some $\alpha \in (0,1)$ with small $\sigma > 0$ and satisfies

$$\varphi^-_{\boldsymbol{\nu}} = 0 \qquad \text{on } \ \partial(0,1)^{d-1} \times [-1,1], \tag{2.22}$$

find a transonic shock solution φ in Ω such that

$$\varphi = \varphi^- \qquad \text{in } \ \Omega^- := \Omega \backslash \overline{\Omega^+},$$

where $\Omega^+ := \{\mathbf{x} \in \Omega \ : \ |D\varphi(\mathbf{x})| < p^1_{\rm sonic}\}$ is the subsonic region of φ, which is the complementary set of the supersonic region of φ in Ω, and

$$\begin{cases} \varphi = \varphi^- & \text{on } \ (0,1)^{d-1} \times \{-1\}, \\ \varphi = \varphi_0^+ & \text{on } \ (0,1)^{d-1} \times \{1\}, \\ \varphi_{\boldsymbol{\nu}} = 0 & \text{on } \ \partial(0,1)^{d-1} \times [-1,1]. \end{cases} \tag{2.23}$$

Since $\varphi = \varphi^-$ in Ω^-, $|D\varphi| < p^1_{\rm sonic} < |D\varphi^-|$ in Ω^+, $|D\varphi^-| \sim \partial_{x_d}\varphi^- > p^1_{\rm sonic}$ in Ω, and it is expected that $\Omega^+ = \{x_d > f(\mathbf{x}')\} \cap \Omega$ and $|D\varphi| \sim \partial_{x_d}\varphi < p^1_{\rm sonic}$ in Ω^+ with (2.8) across the transonic shock $\mathcal{S} = \{x_d = f(\mathbf{x}')\} \cap \Omega$, then φ should satisfy

$$\varphi(\mathbf{x}) \leqslant \varphi^-(\mathbf{x}), \qquad \text{for } \mathbf{x} \in \Omega. \tag{2.24}$$

This motivates the following reformulation of Problem 2.1 into a more general free boundary problem for the subsonic (elliptic) part of the solution:

Problem 2.2 (Free Boundary Problem)**.** *Find $\varphi \in C(\overline{\Omega})$ such that*

(i) *φ satisfies (2.24) in Ω and (2.23) on $\partial\Omega$;*
(ii) *$\varphi \in C^{2,\alpha}(\overline{\Omega^+})$ is a solution of (2.1) in $\Omega^+ = \{\mathbf{x} \in \Omega \, : \, \varphi(\mathbf{x}) < \varphi^-(\mathbf{x})\}$, the non-coincidence set;*
(iii) *the free boundary $\mathcal{S} = \partial\Omega^+ \cap \Omega$ is given by $x_d = f(\mathbf{x}')$ for $\mathbf{x}' \in (0,1)^{d-1}$ so that $\Omega^+ = \{x_d > f(\mathbf{x}') \mid x' \in (0,1)^{d-1}\}$ with $f \in C^{2,\alpha}([0,a]^{d-1})$;*
(iv) *the free boundary condition (2.9) holds on $\mathcal{S}$.*

In the free boundary problem (Problem 2.2) above, phase φ^- is not required to be a solution of (2.1) and φ is not necessary to be subsonic in Ω^+, although we require the subsonicity in Problem 2.1 so that the free boundary is a transonic shock.

It is proved in Chen-Feldman [31] that, if the perturbation $\varphi^- - \varphi_0^-$ is small enough in $C^{2,\alpha}$, then the free boundary problem (Problem 2.2) has a solution, and this implies that Problem 2.1 has a transonic shock solution. Furthermore, the transonic shock is stable under any small $C^{2,\alpha}$ perturbation of φ^-.

Theorem 2.1 (Chen-Feldman [31])**.** *Let $q^+ \in (0, p^1_{\rm sonic})$ and $q^- \in (p^1_{\rm sonic}, 1/\sqrt{\theta})$ satisfy (2.18). Then there exist positive constants σ_0, C_1, and C_2 depending only on q^+, d, γ, and Ω such that, for every $\sigma \leqslant \sigma_0$ and any function φ^- satisfying (2.21) and (2.22), there exists a unique solution φ of the free boundary problem, Problem 2.2, satisfying*

$$\|\varphi - \varphi_0^+\|_{2,\alpha,\Omega^+} \leqslant C_1\sigma.$$

In addition, $\Omega^+ = \{x_d > f(\mathbf{x}')\} \cap \Omega$ with $f : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying

$$\|f\|_{2,\alpha,\mathbb{R}^{d-1}} \leqslant C_2\sigma, \qquad D_{\mathbf{x}'}f(\mathbf{x}') = 0 \;\; \text{on } \partial(0,1)^{d-1},$$

that is, the free boundary $\mathcal{S} = \{(\mathbf{x}', x_d) \, : \, x_d = f(\mathbf{x}'), \mathbf{x}' \in \mathbb{R}^{d-1}\} \cap \Omega$ is in $C^{2,\alpha}$ and orthogonal to $\partial\Omega$ at their intersection points.

In particular, we obtain

Corollary 2.3. *Let $q^{\pm}$ be as in Theorem 2.1, and let σ_0 be the constant defined in Theorem 2.1. If φ^- satisfies conditions of Theorem 2.1 with $\sigma \leqslant \sigma_0$, and if $\varphi^-(\mathbf{x})$ is a supersonic solution of (2.1) satisfying the conditions stated in Problem 2.1, then there exists a transonic shock solution φ of Problem 2.1 with the shock $\mathcal{S} = \{(\mathbf{x}', x_d) \, : \, x_d = f(\mathbf{x}'), \mathbf{x}' \in \mathbb{R}^{d-1}\} \cap \Omega$. Functions φ and f have the properties stated in Theorem 2.1.*

Indeed, under conditions of Corollary 2.3, solution φ of Problem 2.2 obtained in Theorem 2.1, along with the free boundary $\mathcal{S} = \{(\mathbf{x}', x_d) \, : \, x_d = f(\mathbf{x}'), \mathbf{x}' \in \mathbb{R}^{d-1}\} \cap \Omega$, forms a transonic shock solution of Problem 2.1.

The following features of equation (2.1) and the free boundary condition (2.9) are employed in the proof of Theorem 2.1.

(i) The nonlinear equation (2.1) is uniformly elliptic only if $|D\varphi| < p^1_{\rm sonic} - \varepsilon$ in Ω^+ for some $\varepsilon > 0$;
(ii) $|D\varphi^+| = \left(|\varphi^+_{\boldsymbol{\nu}}|^2 + |\varphi_{\boldsymbol{\tau}}|^2\right)^{1/2}$ on $\mathcal{S}$ is subsonic only if $\varphi_{\boldsymbol{\tau}}$ is sufficiently small;

(iii) The free boundary condition (2.9) is uniformly non-degenerate (*i.e.*, $\varphi^-_{\boldsymbol{\nu}} - \varphi^+_{\boldsymbol{\nu}}$ is bounded from below by a positive constant on $\mathcal{S}$) only if $\varphi^-_{\boldsymbol{\nu}} > p^K_{\rm sonic} - \varepsilon$ on $\mathcal{S}$ for some $\epsilon > 0$ with $K = 1 - \theta|\varphi_{\boldsymbol{\tau}}|^2$.

By (2.21), these conditions hold if, for any $\mathbf{x} \in \mathcal{S}$, the unit normal $\boldsymbol{\nu}(\mathbf{x})$ to $\mathcal{S}$ is sufficiently close to being orthogonal to $\{x_d = 0\}$.

2.2. **A Nonlinear Method for Solving the Free Boundary Problems for Nonlinear PDEs of Mixed Type.** We now describe an iterative method and related techniques, developed first in Chen-Feldman [31], for the construction of solutions of free boundary problems for equations of mixed elliptic hyperbolic type, through Problem 2.2 for the simplest setup. In Theorem 2.1 we construct a solution which is close to a given background solution, and we describe now the version of the method which is restricted to this setup. The key ingredient is an iteration scheme, based on the non-degeneracy of the free boundary condition: the jump of the normal derivative of solutions across the free boundary has a strict lower bound. Since the equation is of mixed type, we make a cutoff (truncation) of the nonlinearity near the value related to the background solution in order to fix the type of equation (make it elliptic) and, at the fixed point of the iteration, we remove the cutoff by an estimate. The iteration set consists of functions close to the background solution, in the present case in $C^{2,\alpha}$ norm. Then, for each function from the iteration set, the nondegeneracy allows to use one of the Rankine-Hugoniot conditions, equality (2.8), to define the iteration free boundary, which is a smooth graph. In the domain Ω^+ determined by the iteration free boundary, we solve a boundary-value problem, with the truncated PDE, and with the condition on the shock derived from another Rankine-Hogoniot condition, (2.9), by a truncation similar to the truncation of the equation, to achieve the uniform obliqueness, and with other appropriate modifications, in some cases with the use of condition (2.8). On the rest of the boundary of the iteration domain the conditions for the iteration problem are same as in the original problem. The solution of this iteration problem defines the value of the iteration map. We use estimates for the iteration problem to prove existence of a fixed point of the iteration map. Then we show that a fixed point is a solution of the original problem.

In some further problems, we look for solutions which are not close to a known background solution. Some of these problems, and the corresponding version of the nonlinear method described above, are discussed in Section 4.

2.2.1. Subsonic Truncations – Shiffmanization. In order to solve the free boundary problem, we first reformulate Problem 2.2 as a truncated one-phase free boundary problem, motivated by the argument in Shiffman [109], so called the *shiffmanization* (*cf.* Lax [77]); also see [4, pp. 87–90]. This is achieved by modifying both the nonlinear equation (2.1) and the free boundary condition (2.9), to make the equation uniformly elliptic away from the elliptic region and the free boundary condition non-degenerate. Then we solve the truncated one-phase free boundary problem with the modified equation in the downstream region, the modified free boundary condition, and the given hyperbolic phase in the upstream region. By a careful gradient estimate later on, we prove that the solution in fact solves the original problem. We note that for steady potential flow equation (2.1), the coefficients of it non-divergent form (2.3)) depend on $D\varphi$, so the type of equation depends on $D\varphi$. Because of this the iteration procedure has no additional compactness effect, which is different from that in [18].

We first recall that the ellipticity condition for (2.1) at $|D\varphi| = q$ is (2.4), which is equivalent to

$$\Phi_1'(q) > 0, \tag{2.25}$$

where $\Phi_K(p)$ is the function defined in (2.10). By (2.12), inequality (2.25) holds for $q \in (0, p^1_{\rm sonic})$.

The shiffmanization is done by modifying $\Phi_1(q)$ so that the new function $\tilde{\Phi}_1(q)$ satisfies (2.25) uniformly for all $q > 0$ and, around q^+, $\tilde{\Phi}_1(q) = \Phi_1(q)$. More precisely, the procedure is in the following steps:

1. Denote $\varepsilon := \frac{p^1_{\rm sonic}-q^+}{2}$. Let $y = c_0q + c_1$ be the tangent line of the graph of $y = \Phi_1(q)$ at $q = p^1_{\rm sonic} - \varepsilon$. Then, using (2.12), we obtain $c_0 = \Phi_1'(p^1_{\rm sonic} - \varepsilon) > 0$.

Define $\tilde{\Phi}_1 : [0,\infty) \to \mathbb{R}$ as

$$\tilde{\Phi}_1(q) = \begin{cases} \Phi_1(q) & \text{if } 0 \leqslant q < p^1_{\rm sonic} - \varepsilon, \\ c_0q + c_1 & \text{if } q > p^1_{\rm sonic} - \varepsilon, \end{cases} \tag{2.26}$$

which satisfies $\tilde{\Phi}_1 \in C^{1,1}([0,\infty))$.

2. Define

$$\tilde{\rho}(s) = \frac{\tilde{\Phi}_1(\sqrt{s})}{\sqrt{s}} \qquad \text{for } s \in [0,\infty). \tag{2.27}$$

Then $\tilde{\rho} \in C^{1,1}([0,\infty))$ and

$$\tilde{\rho}(q^2) = \rho(q^2) \qquad \text{if } \ 0 \leqslant q < p^1_{\rm sonic} - \varepsilon. \tag{2.28}$$

By (2.12)–(2.13) and the definition of $\tilde{\Phi}_1$ in (2.26),

$$0 < c_0 = \Phi_1'(p^1_{\rm sonic} - \varepsilon) \leqslant \tilde{\Phi}_1'(q) = \tilde{\rho}(q^2) + 2q^2\tilde{\rho}'(q^2) \leqslant C \qquad \text{for } \ q \in (0,\infty)$$

for some constant $C > 0$. Then the equation

$$\tilde{\mathcal{L}}\varphi := \operatorname{div}(\tilde{\rho}(|D\varphi|^2)D\varphi) = 0 \tag{2.29}$$

is uniformly elliptic, with ellipticity constants depending only on q^+ and γ.

3. We also do the corresponding truncation of the free boundary condition (2.9):

$$\tilde{\rho}(|D\varphi|^2)\varphi_{\boldsymbol{\nu}} = \rho(|D\varphi^-|^2)D\varphi^- \cdot \boldsymbol{\nu} \qquad \text{on } \mathcal{S}. \tag{2.30}$$

On the right-hand side of (2.30), we use the non-truncated function ρ since $\rho \neq \tilde{\rho}$ on the range of $|D\varphi^-|^2$. Note that (2.30), with the right-hand side considered as a known function, is the conormal boundary condition for the uniformly elliptic equation (2.29).

4. Introduce the function

$$u := \varphi^- - \varphi.$$

Then, by (2.24), the problem is to find $u \in C(\overline{\Omega})$ with $u \geqslant 0$ such that

(i) $u \in C^{2,\alpha}(\overline{\Omega^+})$ is a solution of

$$\operatorname{div} A(Du, \mathbf{x}) = f(\mathbf{x}) \qquad \text{in } \Omega^+ := \{u > 0\} \cap \Omega \text{ (the non-coincidence set)}, \tag{2.31}$$

$$A(Du, \mathbf{x}) \cdot \boldsymbol{\nu} = G(\boldsymbol{\nu}, \mathbf{x}) \qquad \text{on } \mathcal{S} := \partial\Omega^+ \backslash \partial\Omega, \tag{2.32}$$

and the boundary condition on $\partial\Omega$ determined by (2.23) and $\varphi^-(\mathbf{x})$:

$$\begin{cases} u = 0 & \text{on } \ (0,1)^{d-1} \times \{-1\}, \\ u = \varphi^- - \varphi_0^+ & \text{on } \ (0,1)^{d-1} \times \{1\}, \\ u_{\boldsymbol{\nu}} = 0; & \text{on } \ \partial(0,1)^{d-1} \times [-1,1], \end{cases} \tag{2.33}$$

where $\boldsymbol{\nu}$ is the unit normal to $\mathcal{S}$ towards the unknown phase and

$$A(P,\mathbf{x}) = \tilde{\rho}(|D\varphi^-(\mathbf{x}) - P|^2)(D\varphi^-(\mathbf{x}) - P) - \tilde{\rho}(|D\varphi^-(\mathbf{x})|^2)D\varphi^-(\mathbf{x}), \qquad P \in \mathbb{R}^d,$$
$$f(\mathbf{x}) = -\operatorname{div}(\tilde{\rho}(|D\varphi^-(\mathbf{x})|^2)D\varphi^-(\mathbf{x})),$$
$$G(\boldsymbol{\nu},\mathbf{x}) = \big(\rho(|D\varphi^-(\mathbf{x})|^2) - \tilde{\rho}(|D\varphi^-(\mathbf{x})|^2)\big)D\varphi^-(\mathbf{x}) \cdot \boldsymbol{\nu}.$$

Note that we used (2.22) to determine the condition on third line of (2.33).

(ii) the free boundary $\mathcal{S} := \partial\Omega^+ \cap \Omega = \{x_d = f(\mathbf{x}') \ : \ \mathbf{x}' \in (0,1)^{d-1}\}$, so that $\Omega^+ = \{x_d > f(\mathbf{x}')\} \cap \Omega$ with $f \in C^{2,\alpha}([0,a]^{d-1})$ and $D_{\mathbf{x}'}f = 0$ on $\partial((0,1)^{d-1} \times [-1,1])$;

(iii) the free boundary condition (2.32) holds on $\mathcal{S}$.

2.2.2. Domain Extension. We then extend domain Ω of the truncated free boundary problem in §2.2.1 above to domain Ω_e, so that the whole free boundary lies in the interior of the extended domain. This is possible because we consider the simple geometry of the domain in this section.

Notice that, for a function $\phi \in C^{2,\alpha}(\overline{\Omega})$ with $\Omega := (0,1)^{d-1} \times (-1,1)$ and

$$\phi_{\boldsymbol{\nu}} = 0 \qquad \text{on } \partial(0,1)^{d-1} \times [-1,1], \tag{2.34}$$

we can extend ϕ to $\mathbb{R}^{d-1} \times [-1,1]$ so that the extension (still denoted) ϕ satisfies

$$\phi \in C^{2,\alpha}(\mathbb{R}^{d-1} \times [-1,1]),$$

and, for every $m = 1, \cdots, n-1$, and $k = 0, \pm 1, \pm 2, \cdots$,

$$\phi(x_1, \cdots, x_{m-1}, k - z, x_{m+1}, \cdots, x_d) = \phi(x_1, \cdots, x_{m-1}, k + z, x_{m+1}, \cdots, x_d), \tag{2.35}$$

that is, ϕ is symmetric with respect to every hyperplane $\{x_m = k\}$. Indeed, for $\mathbf{k} = (k_1, \cdots, k_{d-1}, 0)$ with $k_1, \cdots, k_{d-1}$ integers, we define

$$\phi(\mathbf{x} + \mathbf{k}) = \phi(\eta(x_1, k_1), \cdots, \eta(x_{d-1}, k_{d-1}), x_d) \qquad \text{for } \mathbf{x} \in (0,1)^{d-1} \times [-1,1],$$

with

$$\eta(t,k) = \begin{cases} t & \text{if } k \text{ is even,} \\ 1-t & \text{if } k \text{ is odd.} \end{cases}$$

It follows from (2.35) that $\phi(\mathbf{x}', x_d)$ is $2a$-periodic in each variable of $(x_1, \cdots, x_{d-1})$:

$$\phi(\mathbf{x} + 2\mathbf{e}_m) = \phi(\mathbf{x}), \qquad \text{for } \mathbf{x} \in \mathbb{R}^{d-1} \times [-1,1],\ m = 1, \cdots, d-1,$$

where $\mathbf{e}_m$ is the unit vector in the direction of x_m.

Thus, with respect to this 2-periodicity, we can consider ϕ as a function on $\Omega_{\mathrm{e}} := \mathbb{T}^{d-1} \times [-1,1]$, where $\mathbb{T}^{d-1}$ is an $(d-1)$-dimensional flat torus with its coordinates given by cube $(0,2)^{d-1}$. Note that (2.35) represents an extra symmetry condition, in addition to $\phi \in C^{2,\alpha}(\mathbb{T}^{d-1} \times [-1,1])$, and (2.35) implies (2.34).

Then, by (2.22), we can extend φ^- in the same way, that is, $\varphi^- \in C^{2,\alpha}(\Omega_{\mathrm{e}})$ satisfies (2.35). Also, $\varphi_0^\pm$ can be also considered as the functions in Ω_{e} satisfying (2.35), since $\varphi_0^\pm(\mathbf{x}) = q^\pm x_d$ in $\mathbb{R}^{d-1} \times [-1,1]$, which is independent of $\mathbf{x}'$.

Therefore, we have reduced the transonic shock problem, Problem 2.2, into the following free boundary problem:

Problem 2.4. *Find $u \in C(\overline{\Omega_{\mathrm{e}}})$ with $u \geqslant 0$ such that*

(i) $u \in C^{2,\alpha}(\overline{\Omega_{\mathrm{e}}{}^+})$ *is a solution of* (2.31) *in* $\Omega_{\mathrm{e}}{}^+ := \{u(\mathbf{x}) > 0\} \cap \Omega_{\mathrm{e}}$*, the non-coincidence set;*

(ii) *First two conditions in* (2.33) *hold on* $\partial\Omega_{\mathrm{e}}$*, i.e.* $u = 0$ *on* $\partial\Omega_{\mathrm{e}} \cap \{x_n = -1\}$ *and*

$$u = \varphi^- - \varphi_0^+ \quad \text{on } \partial\Omega_{\mathrm{e}} \cap \{x_n = 1\}; \tag{2.36}$$

(iii) *the free boundary* $\mathcal{S} = \partial\Omega^+ \cap \Omega_{\mathrm{e}}$ *is given by the equation:* $x_d = f(\mathbf{x}')$ *for* $\mathbf{x}' \in \mathbb{T}^{d-1}$*, so that* $\Omega^+ = \{x_d > f(\mathbf{x}') \mid \mathbf{x}' \in \mathbb{T}^{d-1}\}$ *with* $f \in C^{2,\alpha}(\mathbb{T}^{d-1})$ *and* $D_{\mathbf{x}'} f = 0$ *on* $\partial(\mathbb{T}^{d-1} \times [-1,1])$*;*

(iv) *the free boundary condition* (2.32) *holds on* $\mathcal{S}$*.*

As indicated in §2.1, similarly, one of the main difficulties for solving the modified free boundary problem, Problem 2.4, is that the methods of previous works on elliptic free boundary problems do not directly apply to it. Indeed, equation (2.31) is quasilinear, uniformly elliptic, but does not have a clear variational structure, while the function $G(\boldsymbol{\nu}, \mathbf{x})$ in (2.32) depends on $\boldsymbol{\nu}$. Because of these features, the variational methods in [1, 3] do not directly apply to Problem 2.4. Moreover, the nonlinearity in our problem makes it difficult to apply the Harnack inequality approach of Caffarelli [14–16]. In particular, a boundary comparison principle for positive solutions of elliptic equations in Lipschitz domains is unavailable yet in our case that nonlinear equations are not homogeneous with respect to (D^2u, Du, u) here. Therefore, a different nonlinear method is required to overcome these difficulties for solving Problem 2.4.

2.2.3. Iteration Scheme for Solving Free Boundary Problems. The iteration scheme, developed in Chen-Feldman [31], is based on the non-degeneracy of the free boundary condition: the jump of the normal derivative of a solution across the free boundary has a strictly positive lower bound.

Denote $u_0 := \varphi^- - \varphi_0^+$. Note that u_0 satisfies the nondegeneracy condition: $\partial_{x_d} u_0 = q^- - q^+ > 0$ in $\Omega_{\rm e}$. Assume that (2.21) holds with $\sigma \leqslant \frac{q^- - q^+}{10}$. Let function v on $\Omega_{\rm e}$ be given such that $\|v - (\varphi^- - \varphi_0^+)\|_{C^{2,\alpha}(\overline{\Omega_{\rm e}})} \leqslant \frac{q^- - q^+}{10}$, then v satisfies the nondegeneracy condition: $\partial_{x_d} v \geqslant \frac{q^- - q^+}{2} > 0$ in $\Omega_{\rm e}$. Define domain $\Omega^+(v) = \{v > 0\} \subset \Omega_{\rm e}$. Then $\Omega^+(v) = \{x_d > f(\mathbf{x}') \mid \mathbf{x}' \in \mathbb{T}^{d-1}\}$ and $\mathcal{S}(v) := \partial\Omega^+(v) \backslash \partial\Omega_{\rm e} = \{x_d = f(\mathbf{x}') \mid \mathbf{x}' \in \mathbb{T}^{d-1}\}$ with $f \in C^{2,\alpha}(\mathbb{T}^{d-1})$. We solve the oblique derivative problem (2.31)–(2.32) and (2.36) in $\Omega^+(v)$ to obtain the solution $u \in C^{2,\alpha}(\overline{\Omega^+(v)})$. However, u is not identically zero on $\mathcal{S}(v)$ in general, and then u is not a solution of the free boundary problem. Next, estimates for the problem (2.31)–(2.32) and (2.36) in $\Omega^+(v)$ show that $\|u - (\varphi^- - \varphi_0^+)\|_{C^{2,\alpha}(\overline{\Omega^+(v)})}$ is small. Then we extend u to the whole domain $\Omega_{\rm e}$ so that $\|u - (\varphi^- - \varphi_0^+)\|_{C^{2,\alpha}(\overline{\Omega_{\rm e}})}$ is small. This defines iteration $v \mapsto u$. The fixed point $u = v$ of this process determines a solution of the free boundary problem, since u is a solution of (2.31)–(2.32) and (2.36) in $\Omega^+(u)$, and u satisfies $u = v > 0$ on $\Omega^+(u) = \Omega^+(v) := \{v > 0\}$, and $u = v = 0$ on $\mathcal{S} := \partial\Omega^+(v) \backslash \partial\Omega_{\rm e}$. Then we need to show existence of a fixed point. Because of the dependence on $\boldsymbol{\nu}$ on the right-hand side of the free boundary condition (2.32), the elliptic estimates alone are not sufficient for that. However, the structure of our problem allows to obtain better estimates for the iteration and to prove the existence of a fixed point. More precisely, the nonlinear method can be described in the following five steps:

1. Iteration set. Let $M \geqslant 1$. Set

$$\mathcal{K}_M := \big\{w \in C^{2,\alpha}(\overline{\Omega_{\rm e}}) \;\;:\;\; \|w - (\varphi^- - \varphi_0^+)\|_{2,\alpha,\Omega_{\rm e}} \leqslant M\sigma, \quad w \text{ satisfies } (2.35)\big\}, \tag{2.37}$$

where $\varphi_0^+(\mathbf{x}) = q^+ x_d$. According to the definition, $\mathcal{K}_M$ is convex and compact in $C^{2,\beta}(\Omega_{\rm e}), 0 < \beta < \alpha$.

Let $w \in \mathcal{K}_M$. Since $q^- > q^+$, it follows that, if

$$\sigma \leqslant \frac{q^- - q^+}{10(M+1)}, \tag{2.38}$$

then (2.37) and (2.21) imply

$$w_{x_d}(\mathbf{x}) \geqslant \frac{q^- - q^+}{2} > 0. \tag{2.39}$$

Then, by the implicit function theorem, $\Omega^+(w) := \{w(\mathbf{x}) > 0\} \cap \Omega_{\rm e}$ has the form:

$$\Omega^+(w) = \{x_d = f(\mathbf{x}') \mid \mathbf{x}' \in \mathbb{T}^{d-1}\}, \qquad \|f\|_{2,\alpha,\mathbb{T}^{d-1}} \leqslant CM\sigma < 1, \tag{2.40}$$

with C depending upon $q^- - q^+$, and the last inequality is obtained by choosing small σ. The corresponding unit normal

$$\boldsymbol{\nu}(\mathbf{x}') = \frac{(-D_{\mathbf{x}'} f(\mathbf{x}'), 1)}{\sqrt{1 + |D_{\mathbf{x}'} f(\mathbf{x}')|^2}} \in C^{1,\alpha}(\mathbb{T}^{d-1}; \mathbb{S}^{d-1}),$$

and

$$\|\boldsymbol{\nu} - \boldsymbol{\nu}_0\|_{1,\alpha,\mathbb{R}^{d-1}} \leqslant CM\sigma, \tag{2.41}$$

where $\boldsymbol{\nu}_0$ is defined by

$$\boldsymbol{\nu}_0 := \frac{D(\varphi_0^- - \varphi_0^+)}{|D(\varphi_0^- - \varphi_0^+)|} = (0, \cdots, 0, 1)^\top. \tag{2.42}$$

Also, $\boldsymbol{\nu}(\cdot)$ can be considered as a function on $\mathcal{S}_w := \{x_d = f(\mathbf{x}')\}$. From the definition of $f(\mathbf{x}')$ in (2.40), it follows that, for $\mathbf{x} \in \mathcal{S}_w$,

$$\boldsymbol{\nu}(\mathbf{x}) = \frac{Dw(\mathbf{x})}{|Dw(\mathbf{x})|}. \tag{2.43}$$

By the definition of $\mathcal{K}_M$ and (2.38) with (2.21), $\boldsymbol{\nu}(\mathbf{x})$ can be extended to Ω_{e} via formula (2.43) and

$$\|\boldsymbol{\nu} - \boldsymbol{\nu}_0\|_{1,\alpha,\Omega_{\mathrm{e}}} \leqslant CM\sigma \tag{2.44}$$

with $C = C(q^+, q^-)$. Motivated by the free boundary condition (2.30), we define the function, G_w, on Ω_{e}:

$$G_w(\mathbf{x}) := \big(\rho(|D\varphi^-(\mathbf{x})|^2) - \tilde{\rho}(|D\varphi^-(\mathbf{x})|^2)\big) D\varphi^-(\mathbf{x}) \cdot \boldsymbol{\nu}(\mathbf{x}), \tag{2.45}$$

where $\boldsymbol{\nu}(\cdot)$ is defined by (2.43).

We now solve the following fixed-boundary value problem for u in domain $\Omega^+(w)$:

$$\operatorname{div} A(Du, \mathbf{x}) = f(\mathbf{x}) \qquad \text{in } \Omega^+ := \{w > 0\}, \tag{2.46}$$

$$A(Du, \mathbf{x}) \cdot \boldsymbol{\nu} = G(\boldsymbol{\nu}, \mathbf{x}) \qquad \text{on } \mathcal{S} := \partial\Omega^+ \backslash \partial\Omega_{\mathrm{e}}, \tag{2.47}$$

$$u = \varphi^- - q^+ \qquad \text{on } \{x_d = 1\} = \partial\Omega^+(w) \backslash \mathcal{S}_w, \tag{2.48}$$

and show that its unique solution u can be extended to the whole domain Ω_{e} so that $u \in \mathcal{K}_M$.

2. Existence and uniqueness of the solution for the fixed boundary value problem (2.46)–(2.48). We establish the existence and uniqueness of solution u for problem (2.46)–(2.48) and show that u is close in $C^{2,\alpha}(\overline{\Omega^+(w)})$ to the unperturbed subsonic solution $\varphi^- - \varphi_0^+$: Let $M \geqslant 1$. There exists $\sigma_0 > 0$, depending only on M, q^+, Ω, d, and γ such that, if $\sigma \in (0, \sigma_0)$ so that φ^- satisfies (2.21) and $w \in \mathcal{K}_M$, there exists a unique solution $u \in C^{2,\alpha}(\overline{\Omega^+(w)})$ of problem (2.46)–(2.48) that satisfies (2.35) and

$$\|u - (\varphi^- - \varphi_0^+)\|_{2,\alpha,\Omega^+(w)} \leqslant C\sigma, \tag{2.49}$$

where C depends only on (q^+, Ω, d, γ), and is independent of M, $w \in \mathcal{K}_M$, and $\sigma \in (0, \sigma_0)$.

To achieve this, it requires to combine the existence arguments with careful Schauder estimates for nonlinear oblique boundary value problems for nonlinear elliptic equations, based on the results in Gilbarg-Trudinger [65], Lieberman [84], Lieberman-Trudinger [85], and the references cited therein.

3. Construction and continuity of the iteration map. Then we construct the iteration map by an extension of the unique solution of (2.46)–(2.48), which satisfies (2.49), and show the continuity of the iteration map: Let $u(\mathbf{x})$ be a solution of problem (2.46)–(2.48) in domain $\Omega^+(w)$ established in Step 2 above. Then $u(\mathbf{x})$ can be extended to the whole domain Ω_{e} in such way that this extension, denoted as $\mathcal{P}_w u(\mathbf{x})$, satisfies the following two properties:

(i) There exists $C_0 > 0$, which depends only on (q^+, Ω, d, γ) and is independent of (M, σ) and $w(\mathbf{x})$, such that

$$\|\mathcal{P}_w u - (\varphi^- - \varphi_0^+)\|_{2,\alpha,\Omega_{\mathrm{e}}} \leqslant C_0\sigma. \tag{2.50}$$

(ii) Let $\beta \in (0, \alpha)$. Let a sequence $w_j \in \mathcal{K}_M$ converge in $C^{2,\beta}(\overline{\Omega_{\mathrm{e}}})$ to $w \in \mathcal{K}_M$. Let $u_j \in C^{2,\alpha}(\overline{\Omega^+(w_j)})$ and $u \in C^{2,\alpha}(\overline{\Omega^+(w)})$ be the solutions of problems (2.46)–(2.48) for $w_j(x)$ and $w(x)$, respectively. Then $\mathcal{P}_{w_j} u_j \to \mathcal{P}_w u$ in $C^{2,\beta}(\overline{\Omega_{\mathrm{e}}})$.

Define the iteration map $J : \mathcal{K}_M \to C^{2,\alpha}(\overline{\Omega_{\mathrm{e}}})$ by

$$Jw := \mathcal{P}_w u, \tag{2.51}$$

where $u(\mathbf{x})$ is the unique solution of problem (2.46)–(2.48) for $w(\mathbf{x})$. By (2.50), J is continuous in the $C^{2,\beta}(\overline{\Omega_{\mathrm{e}}})$-norm for any positive $\beta < \alpha$.

Now we denote by $u(\mathbf{x})$ both the function $u(\mathbf{x})$ in $\Omega^+(w)$ and its extension $\mathcal{P}_w u(\mathbf{x})$.

Choose M to be the constant C_0 from (2.50). Then, for $u \in \mathcal{K}_M$, we see that $u := Jw \in \mathcal{K}_M$ if $\sigma > 0$ is sufficiently small, depending only on q^+, Ω, d, and γ, since M is now fixed. Thus, (2.51) defines the iteration map $J : \mathcal{K}_M \to \mathcal{K}_M$ and, from (2.50), J is continuous on $\mathcal{K}_M$ in the $C^{2,\beta}(\overline{\Omega_e})$-norm for any positive $\beta < \alpha$.

4. Existence of a fixed point of the iteration map. We then prove the existence of solutions of the free boundary problem, Problem 2.2.

First, in order to solve Problem 2.4, we seek a fixed point of map J. We use the Schauder fixed point theorem (cf. [65, Theorem 11.1]) in the following setting:

Let $\sigma > 0$ satisfy the conditions in Step 3. Let $\beta \in (0, \alpha)$. Since Ω_e is a compact manifold with boundary and $\mathcal{K}_M$ is a bounded convex subset of $C^{2,\alpha}(\overline{\Omega_e})$, it follows that $\mathcal{K}_M$ is a compact convex subset of $C^{2,\beta}(\overline{\Omega_e})$. We have shown that $J(\mathcal{K}_M) \subset \mathcal{K}_M$, and J is continuous in the $C^{2,\beta}(\overline{\Omega_e})$-norm. Then, by the Schauder fixed point theorem, J has a fixed point $\varphi \in \mathcal{K}_M$.

If $u(\mathbf{x})$ is such a fixed point, then

$$\tilde{u}(\mathbf{x}) := \max(0, u(\mathbf{x}))$$

is a classical solution of Problem 2.4, and $\mathcal{S}(u)$ is its free boundary.

It follows that $\varphi := \varphi^- - \tilde{u}$ is a solution of Problem 2.2, provided that σ is small enough so that (2.49) implies that $|D\varphi| = |D(\varphi^- - u)(\mathbf{x})| < p^1_{\text{sonic}} - \varepsilon$ on $\Omega^+(u)$, where $\varepsilon = \frac{p_{\text{sonic}} - q^+}{2}$ defined in §2.2.1. Indeed, then (2.28) implies that $\varphi(\mathbf{x})$ lies in the non-truncated region for equation (2.29). Note also that boundary condition $\varphi_{\boldsymbol{\nu}} = 0$ on $\partial(0,1)^{d-1} \times [-1, 1]$ is satisfied because u and φ^- satisfy (2.35) on $\mathbb{T}^{d-1} \times [-1, 1]$.

For such values of σ, if $\varphi^-(\mathbf{x})$ is a supersonic solution of (2.1) satisfying the conditions stated in Problem 2.1, the function $\varphi(x)$ is a solution of Problem 2.1. Indeed, $|D\varphi| = |D(\varphi^- - \tilde{u})| < p^1_{\text{sonic}} - \varepsilon$ on $\Omega^+(\varphi) := \{\varphi < \varphi^-\} = \{\tilde{u}(\mathbf{x}) > 0\}$ since $\tilde{u} = u$ on $\Omega^+(\tilde{u})$, and $|D\varphi| = |D(\varphi^-| > p^1_{\text{sonic}}$ on $\Omega \backslash \Omega^+(\varphi)$, and equation (2.1) is satisfied in $\Omega^+(\varphi)$ and in $\Omega \backslash \Omega^+(\varphi)$, and Rankine-Hugoniot conditions (2.8)–(2.9) are satisfied on $\mathcal{S} = \partial\Omega^+(\varphi) \backslash \partial\Omega$.

This completes the construction of the global solutions. The uniqueness and stability of solutions of the free boundary problem are obtained by using the regularity and nondegeneracy of solutions.

Remark 2.5. *For clarity, in this section, we focus on the simplest setup of the domain as* $\Omega = (0,1)^{d-1} \times (-1, 1)$, *which can be extended directly to* $\Omega_R = \Pi_{j=1}^{d-1}(0, a_j) \times (-1, R)$ *for any* $R > 0$, *then to* $\Omega_R = \Pi_{j=1}^{d-1} \times (-1, \infty)$ *by analysing the asymptotic behavior of the solution when* $R \to \infty$, *as well as to* $\Omega = \mathbb{R}^{d-1} \times (-1, \infty)$; *see Chen-Feldman* [31–33]. *See also* Chen [47] *for the extension to the isentropic Euler case.*

If the hyperbolic phase is C^∞, *then the solution and the corresponding free boundary in Theorem* 2.1 *are also* C^∞. *Furthermore, our results can be extended to the problem with a steady* $C^{1,\alpha}, \alpha \in (0, 1)$, *perturbation of the upstream supersonic flow and/or general Dirichlet data* $h(\mathbf{x}'), \mathbf{x}' \in \mathbb{R}^{d-1}$, *at* $x_d = 1$ *satisfying*

$$\|h - \varphi_0^+\|_{1,\alpha,\mathbb{R}^{d-1}} \leqslant C\sigma.$$

Also, the Dirichlet data in Problem 2.2 *may be replaced by the corresponding Neumann data satisfying the global solvability condition.*

The global uniqueness of the piecewise constant transonic shocks in straight ducts modulo translations was analyzed in [42, 63].

Remark 2.6. *The setup domains have also been extended to multidimensional infinite nozzles of arbitrary cross-section in Chen-Feldman* [34]; *also see Xin-Yin* [123] *and Yuan* [125] *for the two-dimensional case with the downstream pressure exit.*

For the analysis of geometric effects of the nozzles on the uniqueness and stability of steady transonic shocks, see Bae-Feldman [5]*, Chen-Yuan* [43]*, Liu-Yuan* [89]*,Liu-Xu-Yuan* [90]*, Li-Xin-Yin* [79]*,and the references cited therein.*

Remark 2.7. *The iteration scheme can also be reformulated such that the free boundary normal $\boldsymbol{\nu}$ as unknown in the iteration by replacing in* (2.43) *the known function w by the unknown u, i.e. defining*

$$\boldsymbol{\nu}(\mathbf{x}) = \frac{Du(\mathbf{x})}{|Du(\mathbf{x})|}. \tag{2.52}$$

Note that, at the fixed point, when $u = w$, (2.52) *coincides with* (2.43)*, i.e. defines the normal to $\mathcal{S}$. By using expression* (2.52) *for $\boldsymbol{\nu}$ in the iteration boundary condition, we improve the regularity and structure of the boundary condition, in particular make it independent of the regularity and constants in the iteration set. This is useful in many cases, see e.g.* [35]*. Moreover, this allows to obtain the compactness of the iteration map, which was used in* [37]*.*

This nonlinear method and related techniques described above for free boundary problems has played a key role in many recent developments in the analysis of multidimensional transonic shock problems, as shown in §3–§5.

3. Two-Dimensional Transonic Shocks and Free Boundary Problems for the Steady Full Euler Equations

We now describe how the nonlinear method and related techniques presented in §2 can be applied to prove the existence, stability, and asymptotic behavior of two-dimensional steady transonic flows with transonic shocks past curved wedges for the full Euler equations, by reformulating the problems as free boundary problems via two different approaches.

The two-dimensional steady Euler equations for polytropic gases are of the form (*cf.* [37, 56]):

$$\begin{cases} \operatorname{div}(\rho\mathbf{u}) = 0, \\ \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p = 0, \\ \operatorname{div}\big(\rho\mathbf{u}(E + \frac{p}{\rho})\big) = 0, \end{cases} \tag{3.1}$$

where $\mathbf{u} = (u_1, u_2)$ is the velocity, ρ the density, p the pressure, and $E = \frac{1}{2}|\mathbf{u}|^2 + e$ the total energy with internal energy e.

Choose density ρ and entropy S as the independent thermodynamical variables. Then the constitutive relations can be written as $(p, e, T) = (p(\rho, S), e(\rho, S), T(\rho, S))$ governed by

$$TdS = de - \frac{p}{\rho^2}d\rho,$$

where T represents the temperature. For a polytropic gas,

$$p = p(\rho, S) = \kappa\rho^\gamma e^{S/c_v}, \quad e = e(\rho, S) = \frac{\kappa}{\gamma - 1}\rho^{\gamma-1}e^{S/c_v}, \quad T = T(\rho, S) = \frac{\kappa}{(\gamma-1)c_v}\rho^{\gamma-1}e^{S/c_v}, \tag{3.2}$$

where $\gamma > 1$ is the adiabatic exponent, $c_v > 0$ is the specific heat at constant volume, and $\kappa > 0$ is any constant under scaling.

System (3.1) can be written as a first-order system of conservation laws:

$$\partial_{x_1}F(U) + \partial_{x_2}G(U) = 0, \qquad U = (\mathbf{u}, p, \rho) \in \mathbb{R}^4. \tag{3.3}$$

Solving $\det(\lambda\nabla_U F(U) - \nabla_U G(U)) = 0$ for λ, we obtain four eigenvalues:

$$\lambda_1 = \lambda_2 = \frac{u_2}{u_1}, \qquad \lambda_j = \frac{u_1u_2 + (-1)^j c\sqrt{|\mathbf{u}|^2 - c^2}}{u_1^2 - c^2} \text{ for } j = 3, 4,$$

where

$$c = \sqrt{\frac{\gamma p}{\rho}} \tag{3.4}$$

is the sonic speed of the flow for a polytropic gas.

The repeated eigenvalues λ_1 and λ_2 are real and correspond to the two linear degenerate characteristic families which generate vortex sheets and entropy waves, respectively. The eigenvalues λ_3 and λ_4 are real when the flow is supersonic (*i.e.*, $|\mathbf{u}| > c$), and complex when the flow is subsonic (*i.e.*, $|\mathbf{u}| < c$) in which case the elliptic equations are involved,

For a transonic flow, in which both the supersonic and subsonic phases occur in the flow, system (3.1) is of mixed-composite hyperbolic-elliptic type, which consists of two equations of mixed elliptic-hyperbolic type and two equations of hyperbolic type (*i.e.*, two transport-type equations).

In the regimes with $\rho|\mathbf{u}| > 0$, from the first equation in (3.1), considered in simply-connected domain containing the origin, there exists a unique stream function ψ such that

$$D\psi = (-\rho u_2, \rho u_1) \qquad \text{with } \psi(\mathbf{0}) = 0. \tag{3.5}$$

We use the following coordinate transformation to the Lagrangian coordinates:

$$(x_1, x_2) \to (y_1, y_2) = (x_1, \psi(x_1, x_2)), \tag{3.6}$$

under which the original curved streamlines become straight. In the new coordinates $\mathbf{y} = (y_1, y_2)$, we still denote the unknown variables $U(\mathbf{x}(\mathbf{y}))$ by $U(\mathbf{y})$ for simplicity of notation. Then the original Euler equations in (3.1) become the following equations in divergence form:

$$\big(\frac{1}{\rho u_1}\big)_{y_1} - \big(\frac{u_2}{u_1}\big)_{y_2} = 0, \tag{3.7}$$

$$\big(u_1 + \frac{p}{\rho u_1}\big)_{y_1} - \big(\frac{p u_2}{u_1}\big)_{y_2} = 0, \tag{3.8}$$

$$(u_2)_{y_1} + p_{y_2} = 0, \tag{3.9}$$

$$\big(\frac{1}{2}|\mathbf{u}|^2 + \frac{\gamma p}{(\gamma-1)\rho}\big)_{y_1} = 0. \tag{3.10}$$

One of the advantages of the Lagrangian coordinates is to straighten the streamlines so that the streamline may be employed as one of the coordinates to simplify the formulations, since the Bernoulli variable and entropy are conserved along the streamlines.

3.1. **Supersonic Flow onto Solid Wedges and Free Boundary Problems.** For an upstream steady uniform supersonic flow past a symmetric straight-sided wedge (see Fig. 3.1):

$$W := \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \,:\, |x_2| < x_1 \tan\theta_{\mathrm{w}}, x_1 > 0\} \tag{3.11}$$

whose angle θ_{w} is less than the detachment angle $\theta_{\mathrm{w}}^{\mathrm{d}}$, there exists an oblique shock emanating from the wedge vertex. Since the upper and lower subsonic regions do not interact with each other, it suffices to study the upper part. More precisely, if the upstream steady flow is a uniform supersonic state, we can find the corresponding constant downstream flow along the straight-sided upper wedge boundary, together with a straight shock separating the two states. The downstream flow is determined by the shock polar whose states in the phase space are governed by the Rankine-Hugoniot conditions and the entropy condition; see Fig. 3.1. Indeed, Prandtl in [100] first employed the shock polar analysis to show that there are two possible steady oblique shock configurations when the wedge angle θ_{w} is less than the detachment angle $\theta_{\mathrm{w}}^{\mathrm{d}}$ — The steady weak shock with supersonic or subsonic downstream flow (determined by the wedge angle that is less or larger than the sonic angle $\theta_{\mathrm{w}}^{\mathrm{s}}$) and the steady strong shock with subsonic downstream flow, both of which satisfy the entropy condition, provided that no additional conditions are assigned at downstream. See also [13, 56, 97] and the references cited therein.

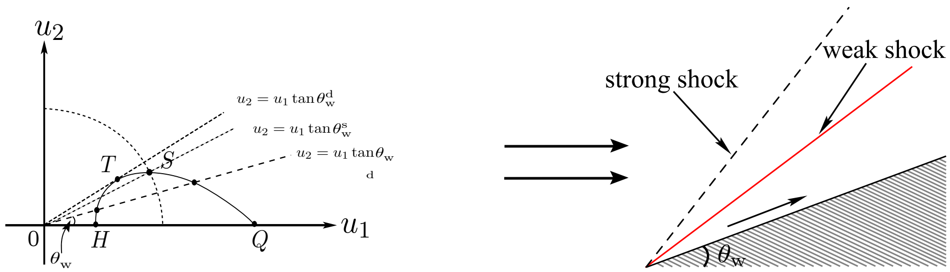

FIGURE 3.1. The shock polar in the $\mathbf{u}$-plane and uniform steady (weak/strong) shock flows (see [24])

The fundamental issue – whether one or both of the steady weak and strong shocks are physically admissible – has been vigorously debated over the past eight decades (*cf.* [56, 57, 91, 106, 120]). Experimental and numerical evidence has strongly indicated that the steady weak shock solution would be physically admissible, as Prandtl conjectured in [100]. One natural approach to single out the physically admissible steady shock solutions is via the stability analysis: The stable ones are physical. See Courant-Friedrichs [56] and von Neumann [120]; see also [91, 106].

A piecewise smooth solution $U = (\mathbf{u}, p, \rho)$ separated by a front $\mathcal{S} := \{\mathbf{x} : x_2 = \sigma(x_1), x_1 \geqslant 0\}$ becomes a weak solution of the Euler equations (3.1) as in §2.1 if and only in the Rankine-Hugoniot conditions are satisfied along $\mathcal{S}$:

$$\begin{cases} \sigma'(x_1)[\,\rho u_1\,] = [\,\rho u_2\,], \\ \sigma'(x_1)[\,\rho u_1^2 + p\,] = [\,\rho u_1 u_2\,], \\ \sigma'(x_1)[\,\rho u_1 u_2\,] = [\,\rho u_2^2 + p\,], \\ \sigma'(x_1)[\,\rho u_1(E + \frac{p}{\rho})\,] = [\,\rho u_2(E + \frac{p}{\rho})\,], \end{cases} \tag{3.12}$$

where $[\,\cdot\,]$ denotes the jump between the quantity of two states across front $\mathcal{S}$ as before.

Such a front $\mathcal{S}$ is called a shock if the entropy condition holds along $\mathcal{S}$: *The density increases in the fluid direction across $\mathcal{S}$.*

For given state U^-, where we use notation (3.3), all states U that can be connected with U^- through the relations in (3.12) form a curve in the state space $\mathbb{R}^4$; the part of the curve whose states satisfy the entropy condition is called the *shock polar*. The projection of the shock polar onto the $\mathbf{u}$–plane is shown in Fig. 3.1. In particular, for an upstream uniform horizontal flow $U_0^- = (u_{10}^-, 0, p_0^-, \rho_0^-)$ past the upper part of a straight-sided wedge whose angle is θ_w, the downstream constant flow can be determined by the shock polar (see Fig. 3.1). Note that downstream flow must be parallel to the wedge and the upstream flow is parallel to the axis of wedge, so the angle between the upstream and downstream flow is equal to the wedge (half)-angle. According to the shock polar, the two flow angles (or, equivalently, wedge angles) are important:

One is the detachment angle θ_w^d, such that line $u_2 = u_1 \tan\theta_w^d$ is tangential to the shock polar at point T and there is no intersection between line $u_2 = u_1 \tan\theta_w$ and the shock polar when $\theta_w > \theta_w^d$. For wedge angles $\theta_w \in (0, \theta_w^d)$ there are two intersection points of the line $u_2 = u_1 \tan\theta_w$ and the shock polar, one intersection point is on the arc $\widehat{TH}$ and it determines velocity (u_1, u_2) of the downstream flow corresponding to the strong shock, and another intersection point is on the arc $\widehat{TQ}$ and it corresponds to the weak shock. Thus for wedge angles $\theta_w \in (0, \theta_w^d)$, shock polar ensures the existence of two attached shocks at the wedge: strong and weak.

Other important angle is the sonic angle $\theta_w^s < \theta_w^d$ such that line $u_2 = u_1 \tan\theta_w^s$ intersects with the shock polar at point S on the circle of radius c_0, for which the downstream fluid velocity is at the sonic speed. Point S divides arc $\widehat{HS}$, which corresponds to the weak shocks, into the two open arcs $\widehat{TS}$ and $\widehat{TH}$; see Fig. 3.1. The nature of these two cases, as well as the case for arc $\widehat{SQ}$, is very different. When the wedge angle $\theta_{\rm w}$ is between $\theta_{\rm w}^{\rm s}$ and $\theta_{\rm w}^{\rm d}$, there are two subsonic solutions (corresponding to the strong and weak shocks); while for the wedge angle $\theta_{\rm w}$ is smaller than $\theta_{\rm w}^{\rm s}$, there one subsonic solution (for the strong shock) and one supersonic solution (for the weak shock). Such an oblique shock $\mathcal{S}_0$ is straight, described by $x_2 = s_0 x_1$. The question is whether the steady oblique shock solution is stable under a perturbation of both the upstream supersonic flow and the wedge boundary.

Since we are interested in determining the downstream flow, we can restrict the domain to the first quadrant, see Fig. 3.2.

Fix a constant upstream flow U^-, a wedge angle $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm d})$, and a constant downstream state U_0^+ which is one of downstream states (weak or strong) determined by the shock polar. States U_0^- and U_0^+ determine the oblique shock $x_2 = s_0 x_1$, and the transonic shock solution U_0 in $\{\mathbf{x} \mid x_1 > 0,\, x_2 > 0\}\backslash W$ such that $U = U_0^-$ in $\Omega_0^- = \{\mathbf{x} \in \mathbb{R}^2 : x_2 > s_0 x_1, x_1 > 0\}$ and $U = U_+^-$ in $\Omega_0^+ = \{\mathbf{x} \in \mathbb{R}^2 : s_0 x_1 > x_2 > x_1 \tan\theta_{\rm w}, x_1 > 0\}$, see Fig. 3.1. We will refer to this solution as constant transonic solution (U_0^-, U_0^+).

Assume that the perturbed upstream flow U_I^- is close to U_0^-, thus U_I^- is supersonic and almost horizontal, and that the wedge is close to a straight-sided wedge. Then, for any suitable wedge angle (smaller than a detachment angle), it is expected that there should be a shock attached to the wedge vertex, see Fig. 3.2. We now use a function $b(x_1) \geqslant 0$ to describe the upper perturbed wedge boundary:

$$\partial W = \{\mathbf{x} \in \mathbb{R}^2 : x_2 = b(x_1),\ x_1 > 0\}, \quad \text{where } b(0) = 0. \tag{3.13}$$

Then the wedge problem can be formulated as the following problem:

Problem 3.1 (Initial-Boundary Value Problem). *Find a global solution of system* (3.1) *in* $\Omega := \{x_2 > b(x_1), x_1 > 0\}$ *such that the following conditions hold*:

(i) *Cauchy condition at* $x_1 = 0$:

$$U|_{x_1=0} = U_I^-(x_2); \tag{3.14}$$

(ii) *Boundary condition on* ∂W *as the slip boundary*:

$$\mathbf{u} \cdot \boldsymbol{\nu}_{\rm w}|_{\partial W} = 0, \tag{3.15}$$

where $\boldsymbol{\nu}_{\rm w}$ *is the outer unit normal vector to* ∂W.

Assume that the background shock is the straight line given by $x_2 = \sigma_0(x_1)$ for $\sigma_0(x_1) := s_0 x_1$. When the upstream steady supersonic perturbation $U_I^-(x_2)$ at $x_1 = 0$ is suitably regular and small, the upstream steady supersonic smooth solution $U^-(\mathbf{x})$ exists in region $\Omega^- = \{\mathbf{x} : x_2 > \frac{s_0}{2} x_1, x_1 \geqslant 0\}$, beyond the background shock, but is still close to U_0^-.

Assume that the shock front (the free boundary) $\mathcal{S}$ we seek is

$$\mathcal{S} = \{\mathbf{x} : x_2 = \sigma(x_1),\, x_1 \geqslant 0\}, \quad \text{where } \sigma(0) = 0,\ \sigma(x_1) > 0 \text{ for } x_1 > 0. \tag{3.16}$$

The domain for the downstream flow behind $\mathcal{S}$ is denoted by

$$\Omega = \{\mathbf{x} \in \mathbb{R}^2 : b(x_1) < x_2 < \sigma(x_1), x_1 > 0\}. \tag{3.17}$$

Then Problem 3.1 can be further reformulated into the following free boundary problem:

Problem 3.2 (Free Boundary Problem; see Fig. 3.2). *Let* (U_0^-, U_0^+) *be a constant transonic solution for wedge angle* $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm d})$, *with transonic shock* $\mathcal{S}_0 := \{x_2 = \sigma_0(x_1) : x_1 > 0\}$ *for* $\sigma_0(x_1) := s_0 x_1$. *For any upstream flow* U^- *for system* (3.1) *in domain* Ω^- *which is a small perturbation of* U_0^-, *and any wedge boundary function* $b(x_1)$, *which is a small perturbation of* $b_0(x_1) = x_1 \tan\theta_{\rm w}$, *find a shock* $\mathcal{S}$ *as a free boundary* $x_2 = \sigma(x_1)$ *and a solution* U *in* Ω, *which are small perturbations of* $\mathcal{S}_0$ *and* U_0^+, *respectively, such that*

(i) U *satisfies* (3.1) *in domain* Ω;

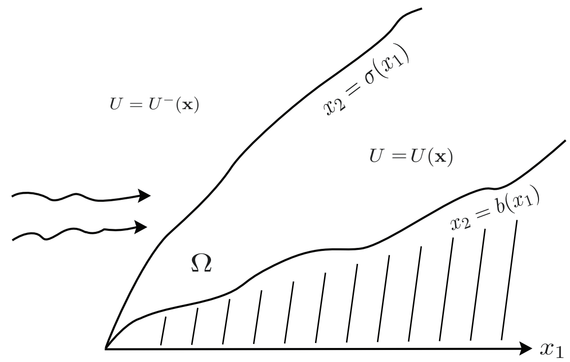

FIGURE 3.2. The leading steady shock $x_2 = \sigma(x_1)$ as a free boundary under the perturbation (see [24])

(ii) *The slip condition* (3.15) *holds along the wedge boundary* ∂W;
(iii) *The Rankine-Hugoniot conditions* (3.12) *as free boundary conditions hold along the transonic shock-front* $\mathcal{S}$.

U_0^+ *corresponding to a state on arc* $\widehat{SQ}$ *gives a weak supersonic shock* (*i.e., both the upstream and downstream states are supersonic*) (*see Fig.* 3.1)*, the problem is denoted by* Problem 3.2 (WS)*;* U_0^+ *corresponding to a subsonic state on arc* $\widehat{TS}$ *gives a weak transonic shock* (*i.e., the upstream state is supersonic and the downstream state is subsonic*) (*see Fig.* 3.1)*, the problem is denoted by* Problem 3.2 (WT)*; while the strong transonic shock problem corresponds to arc* $\widehat{TH}$*, denoted by* Problem 3.2 (ST).

In general, uniqueness in the initial-boundary value problem (Problem 3.1) is not known (as it is problem for a nonlinear system of a composite elliptic-hyperbolic type), so is possible that (Problem 3.1) has solutions which are not of steady oblique shock structure, i.e. which are not solutions of (Problem 3.2). On the other hand, the global solution of the free boundary problem (Problem 3.2) provides the global structural stability of the steady oblique shock, as well as more detailed structure of the solution.

Supersonic (*i.e.*, supersonic-supersonic) shocks correspond to arc $\widehat{SQ}$ (which is a stronger shock) (see Fig. 3.1). The local stability of such shocks was first established in [67, 81, 102]. The global stability of the supersonic shocks for potential flow past piecewise smooth perturbed curved wedges was established in Zhang [129]; also see [45, 52, 53] and the references therein. The global stability and uniqueness of the supersonic shocks for the full Euler equations, Problem 3.2 (WS), was solved for more general perturbations of both the initial data and wedge boundary even in BV in Chen-Zhang-Zhu [44] and Chen-Li [40].

For transonic (*i.e.*, supersonic-subsonic) shocks, the strong shock case corresponding to arc $\widehat{TH}$ was first studied in Chen-Fang [52] for the potential flow (see Fig. 3.1). In Fang [62], the full Euler equations were studied with a uniform Bernoulli constant for both weak and strong transonic shocks. Because the framework is a weighted Sobolev space, the asymptotic behavior of the shock slope or subsonic solution was not derived. In Yin-Zhou [124], the Hölder norms were used for the estimates of solutions of the full Euler equations with the assumption on the sharpness of the wedge angle, which means that the subsonic state is near point H in the shock polar, by Approach I, first introduced in [25] which will described §3.2. In Chen-Chen-Feldman [26], the weaker transonic shock, which corresponds to arc $\widehat{TS}$, was first investigated by Approach I. Then, in [27], the weak and strong transonic shocks, which correspond to arcs $\widehat{TS}$ and $\widehat{TH}$, respectively, were solved, by Approach II which will described in §3.3, so that the existence, uniqueness, stability, and asymptotic behavior of subsonic solutions of Problem 3.2 (WT) & (ST) in a weighted Hölder space were obtained.

We now describe two approaches for the wedge problem, based on the nonlinear method and related techniques presented in §2. First, we need to introduce the weighed Hölder norms in the subsonic domain Ω, where Ω is either a truncated triangular domain or an unbounded domain with the vertex at origin O and one side as the wedge boundary. There are two weights: One is the distance function to origin O and the other is to the wedge boundary $\partial\mathcal{W}$. For any $\mathbf{x},\mathbf{x}'\in\Omega$, define

$$\delta^{\rm o}_{\mathbf{x}}:=\min(|\mathbf{x}|,1),\quad \delta^{\rm o}_{\mathbf{x},\mathbf{x}'}:=\min(\delta^{\rm o}_{\mathbf{x}},\delta^{\rm o}_{\mathbf{x}'}),\quad \delta^{\rm w}_{\mathbf{x}}:=\min(\mathrm{dist}(\mathbf{x},\partial\mathcal{W}),1),\quad \delta^{\rm w}_{\mathbf{x},\mathbf{x}'}:=\min(\delta^{\rm w}_{\mathbf{x}},\delta^{\rm w}_{\mathbf{x}'}),$$

$$\Delta_{\mathbf{x}}:=|\mathbf{x}|+1,\quad \Delta_{\mathbf{x},\mathbf{x}'}:=\min(\Delta_{\mathbf{x}},\Delta_{\mathbf{x}'}),\quad \widetilde{\Delta}_{\mathbf{x}}:=\mathrm{dist}(\mathbf{x},\partial\mathcal{W})+1,\quad \widetilde{\Delta}_{\mathbf{x},\mathbf{x}'}:=\min(\widetilde{\Delta}_{\mathbf{x}},\widetilde{\Delta}_{\mathbf{x}'}).$$

Let $\alpha\in(0,1)$, and $l_1,l_2,\gamma_1,\gamma_2\in\mathbb{R}$ with $\gamma_1\geqslant\gamma_2$, and let $k\geqslant 0$ be an integer. Let $\mathbf{k}=(k_1,k_2)$ be an integer-valued vector, where $k_1,k_2\geqslant 0$, $|\mathbf{k}|=k_1+k_2$, and $D^{\mathbf{k}}=\partial_{x_1}^{k_1}\partial_{x_2}^{k_2}$. We define

$$[f]^{(\gamma_1;O)(\gamma_2;\partial\mathcal{W})}_{k,0;(l_1,l_2);\Omega}=\sup_{\substack{\mathbf{x}\in\Omega\\|\mathbf{k}|=k}}\big\{(\delta^{\rm o}_{\mathbf{x}})^{\hat{\gamma}_0}(\delta^{\rm w}_{\mathbf{x}})^{\max\{k+\gamma_2,0\}}\,\Delta_{\mathbf{x}}^{l_1}\widetilde{\Delta}_{\mathbf{x}}^{l_2+k}|D^{\mathbf{k}}f(\mathbf{x})|\big\},\tag{3.18}$$

$$[f]^{(\gamma_1;O)(\gamma_2;\partial\mathcal{W})}_{k,\alpha;(l_1,l_2);\Omega}=\sup_{\substack{\mathbf{x},\mathbf{x}'\in\Omega\\\mathbf{x}\neq\mathbf{x}',|\mathbf{k}|=k}}\Big\{(\delta^{\rm o}_{\mathbf{x},\mathbf{x}'})^{\hat{\gamma}_\alpha}(\delta^{\rm w}_{\mathbf{x},\mathbf{x}'})^{\max\{k+\alpha+\gamma_2,0\}}\Delta_{\mathbf{x},\mathbf{x}'}^{l_1}\widetilde{\Delta}_{\mathbf{x},\mathbf{x}'}^{l_2+k+\alpha}\frac{|D^{\mathbf{k}}f(\mathbf{x})-D^{\mathbf{k}}f(\mathbf{x}')|}{|\mathbf{x}-\mathbf{x}'|^\alpha}\Big\},\tag{3.19}$$

$$\|f\|^{(\gamma_1;O)(\gamma_2;\partial\mathcal{W})}_{k,\alpha;(l_1,l_2);\Omega}=\sum_{i=0}^{k}[f]^{(\gamma_1;O)(\gamma_2;\partial\mathcal{W})}_{i,0;(l_1,l_2);\Omega}+[f]^{(\gamma_1;O)(\gamma_2;\partial\mathcal{W})}_{k,\alpha;(l_1,l_2);\Omega},\tag{3.20}$$

where $\hat{\gamma}_\beta=\max\{\gamma_1+\min\{k+\beta,-\gamma_2\},0\}$ for $\beta\in[0,1)$. Similarly, the Hölder norms for a function of one variable on $(0,\infty)$ $\mathbb{R}$ with the weight near $\{0\}$ and decay at infinity are denoted by $\|f\|^{(\gamma_2;0)}_{k,\alpha;(l);(0,\infty)}$.

For a vector-valued function $\mathbf{f}=(f_1,f_2,\cdots,f_n)$, we define

$$\|\mathbf{f}\|^{(\gamma_1;O)(\gamma_2;\partial\mathcal{W})}_{k,\alpha;(l_1,l_2);\Omega}=\sum_{i=1}^{d}\|f_i\|^{(\gamma_1;O)(\gamma_2;\partial\mathcal{W})}_{k,\alpha;(l_1,l_2);\Omega}.$$

Let

$$C^{k,\alpha;(l_1,l_2)}_{(\gamma_1;O)(\gamma_2;\partial\mathcal{W})}(\Omega)=\Big\{\mathbf{f}\,:\,\|\mathbf{f}\|^{(\gamma_1;O)(\gamma_2;\partial\mathcal{W})}_{k,\alpha;(l_1,l_2);\Omega}<\infty\Big\}.\tag{3.21}$$

The requirement that $\gamma_1\geqslant\gamma_2$ in the definition above means that the regularity up to the wedge boundary is no worse than the regularity up to the wedge vertex. When $\gamma_1=\gamma_2$, the $\delta^{\rm o}$–terms disappear so that (γ_1,O) can be dropped in the superscript. If there is no weight $(\gamma_2,\partial\mathcal{W})$ in the superscript, the δ–terms for the weights should be understood as $(\delta^{\rm o}_{\mathbf{x}})^{\max\{k+\gamma_1,0\}}$ and $(\delta^{\rm o}_{\mathbf{x}})^{\max\{k+\alpha+\gamma_1,0\}}$ in (3.18) and (3.19), respectively. Moreover, when no weight appears in the superscripts of the seminorms in (3.18) and (3.19), it means that neither $\delta^{\rm o}$ nor $\delta^{\rm w}$ is present. For a function of one variable defined on $(0,\infty)$, the weighted norm $\|f\|^{(\gamma_1;0)}_{k,\alpha;(l);\mathbb{R}^+}$ is understood in the same as the definition above with the weight to $\{0\}$ and the decay at infinity.

In the study of Problem 3.2 for a transonic solution (U_0^-,U_0^+) with wedge angle $\theta_{\rm w}$, the variables in U are expected to have different levels of regularity, so we distinguish these variables by defining

$$\begin{aligned}&U_1=(\mathbf{u}\cdot\boldsymbol{\tau}^0_{\rm w},\rho)\text{ and }U_2=(w,p)\text{ for}\\&w=\frac{\mathbf{u}\cdot\boldsymbol{\nu}^0_{\rm w}}{\mathbf{u}\cdot\boldsymbol{\tau}^0_{\rm w}},\quad\text{where }\ \boldsymbol{\nu}^0_{\rm w}=(-\sin\theta_{\rm w},\cos\theta_{\rm w}),\ \ \boldsymbol{\tau}^0_{\rm w}=(\cos\theta_{\rm w},\sin\theta_{\rm w}).\end{aligned}\tag{3.22}$$

Note that $U^+_{10}=(|\mathbf{u}^+_0|,\rho^+_0)$ and $U^+_{20}=(0,p^+_0)$ are the corresponding quantities for the background subsonic state.

Note that $\boldsymbol{\nu}^0_{\rm w}$ is the interior (for Ω_0) unit normal to ∂W_0, and $\boldsymbol{\tau}^0_{\rm w}$ is the tangential to ∂W_0 unit vector, where ∂W_0 and Ω_0 are defined by (3.13) and (3.17) for the background solution (U_0^-,U_0^+), i.e. $\mathbf{u}\cdot\boldsymbol{\tau}^0_{\rm w}$

and $\mathbf{u}\cdot\boldsymbol{\nu}_{\rm w}^0$ are components u_1 and u_2 of $\mathbf{u}$ in the coordinates rotated clockwise by angle $\theta_{\rm w}$, so that the background downstream flow becomes horizontal.

Theorem 3.1 (Chen-Chen-Feldman [27])**.** *Let (U_0^-, U_0^+) be a constant transonic solution for wedge angle $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm d})$. There are positive constants α, β, C_0, and ε, depending only on the background states (U_0^-, U_0^+), such that:*

(i) *If (U_0^-, U_0^+) corresponds to the state on arc $\widehat{TS}$, and*

$$\|U^- - U_0^-\|_{2,\alpha;(1+\beta,0);\Omega^-} + \|b' - \tan\theta_{\rm w}\|^{(-\alpha;0)}_{1,\alpha;(1+\beta);\mathbb{R}^+} < \varepsilon, \tag{3.23}$$

then there exists a solution (U,σ) of Problem 3.2 (WT) *and a function $U^\infty = (u_1^\infty, 0, p_0^+, \rho^\infty) = Z^\infty(-x_1\sin\theta_{\rm w} + x_2\cos\theta_{\rm w})$ for an appropriate function $Z^\infty : [0,\infty) \to \mathbb{R}^4$ such that U_1 and U_2 defined by* (3.22) *satisfy*

$$\begin{aligned}&\|U_1 - U_1^\infty\|^{(-\alpha;\partial\mathcal{W})}_{2,\alpha;(\beta,1);\Omega} + \|U_2 - U_{20}^+\|^{(-\alpha;O)(-1-\alpha;\partial\mathcal{W})}_{2,\alpha;(1+\beta,0);\Omega} + \|\sigma' - s_0\|^{(-\alpha;0)}_{2,\alpha;(1+\beta);\mathbb{R}^+}\\ &+\|U_1^\infty - U_{10}^+\|^{(-\alpha;0)}_{2,\alpha;(1+\beta);[0,\infty)} \leqslant C_0\left(\|U^- - U_0^-\|_{2,\alpha;(1+\beta,0);\Omega^-} + \|b' - \tan\theta_{\rm w}\|^{(-\alpha;0)}_{1,\alpha;(1+\beta);\mathbb{R}^+}\right),\end{aligned} \tag{3.24}$$

where we have denoted $U_1^\infty := (u_\tau^\infty, \rho^\infty)$;

(ii) *If (U_0^-, U_0^+) corresponds to the state on arc $\widehat{TH}$, and*

$$\|U^- - U_0^-\|_{2,\alpha;(\beta,0);\Omega^-} + \|b' - \tan\theta_{\rm w}\|^{(-\alpha-1;0)}_{2,\alpha;(\beta);\mathbb{R}^+} < \varepsilon, \tag{3.25}$$

then there exists a solution (U,σ) of Problem 3.2 (ST)*, such that U_1 and U_2 defined by* (3.22) *satisfy*

$$\begin{aligned}&\|U_1 - U_{10}^+\|^{(-1-\alpha;\partial\mathcal{W})}_{2,\alpha;(0,\beta);\Omega} + \|U_2 - U_{20}^+\|^{(-1-\alpha;O)}_{2,\alpha;(\beta,0);\Omega} + \|\sigma' - s_0\|^{(-1-\alpha;0)}_{2,\alpha;(\beta);\mathbb{R}^+}\\ &\leqslant C_0\left(\|U^- - U_0^-\|_{2,\alpha;(\beta);\Omega^-} + \|b' - \tan\theta_{\rm w}\|^{(-1-\alpha;0)}_{2,\alpha;(\beta);\mathbb{R}^+}\right).\end{aligned} \tag{3.26}$$

The solution, (U,σ), is unique within the class of solutions such that the left-hand side of (3.24) *for* Problem 3.2 (WT) *or* (3.26) *for* Problem 3.2 (ST) *is less than $C_0\varepsilon$.*

The dependence of constants α, β, C_0, and ε in Theorem 3.1 is as follows: α and β depend on (U_0^-, U_0^+), but are independent of (C_0, ε); C_0 depends on $(U_0^-, U_0^+, \alpha, \beta)$, but is independent of ε; and ε depends on all $(U_0^-, U_0^+, \alpha, \beta, C_0)$.

The difference in the results of the two problems is that the solution of Problem 3.2 (WT) has less regularity at corner O and decays faster with respect to $|\mathbf{x}|$ (or the distance from the wedge boundary) than the solution of Problem 3.2 (ST).

Note that part (i) of Theorem 3.1 gives asymptotics os the solution U as $|\mathbf{x}| \to \infty$ within Ω, and U^∞ is an asymptotic profile. Moreover, convergence of U_2 to $U_2^\infty = U_{20}^+$ as $|\mathbf{x}| \to \infty$ is of polynomial rate $|\mathbf{x}|^{-(\beta+1)}$, which is faster than convergence rate of U_1, which is $|\mathbf{x}|^{-\beta}$. But as $x_2 \to +\infty$, both U_1 and U_2 decay to U_{10}^+ and U_{20}^+ resp. with the rate $x_2^{-(\beta+1)}$, which for U_1 can be seen by combining the estimates of the first and last terms in the right-hand side of (3.24). Part (ii) of Theorem 3.1 does not give asymptotic limit of U_1 as $|\mathbf{x}| \to \infty$, while U_2 converges to to U_{20}^+ with the rate $|\mathbf{x}|^{-\beta}$. Also, both both U_1 and U_2 decay to U_{10}^+ and U_{20}^+ resp. with the rate $x_2^{-\beta}$ in case (ii).

Furthermore, for both cases (i) and (ii) of Theorem 3.1, the asymptotic profile in Lagrangian coordinates is given in Theorem 3.3.

3.2. **Approach I for** Problem 3.2 (WT)**.** We now describe Approach I for solving Problem 3.2 (WT).

We work in Lagrangian coordinates introduced in (3.6). From the slip condition (3.15) on the wedge boundary ∂W it follow that ∂W is a streamline, and so in the Lagrangian coordinates, ∂W becomes the half-line $\mathcal{L}_1 = \{(y_1, y_2) \,:\, y_1 > 0, y_2 = 0\}$. Let $\mathcal{S} = \{y_2 = \hat{\sigma}(y_1)\}$ be a shock-front. Then, from equations (3.7)–(3.10), we can derive the Rankine-Hugoniot conditions along $\mathcal{S}$:

$$\hat{\sigma}'(y_1)\big[\frac{1}{\rho u_1}\big] = -\big[\frac{u_2}{u_1}\big], \tag{3.27}$$

$$\hat{\sigma}'(y_1)\big[u_1 + \frac{p}{\rho u_1}\big] = -\big[\frac{pu_2}{u_1}\big], \tag{3.28}$$

$$\hat{\sigma}'(y_1)[\,u_2\,] = [\,p\,], \tag{3.29}$$

$$\big[\frac{1}{2}|\mathbf{u}|^2 + \frac{\gamma p}{(\gamma-1)\rho}\big] = 0. \tag{3.30}$$

The background shock-front in the Lagrangian coordinates is $\mathcal{S}_0 = \{y_2 = s_1 y_1\}$ with $s_1 = \rho_0^+ u_{10}^+(s_0 - \tan\theta_0) > 0$.

Without loss of generality, we assume that, in the Lagrangian coordinates, the supersonic solution U^- exists in domain $\mathbb{D}^-$ defined by

$$\mathbb{D}^- = \Big\{\mathbf{y} \,:\, y_2 > \frac{s_1}{2} y_1,\, y_1 > 0\Big\}. \tag{3.31}$$

For a given shock function $\hat{\sigma}(y_1)$, let

$$\mathbb{D}^-_{\hat{\sigma}} = \big\{\mathbf{y} : y_2 > \hat{\sigma}(y_1),\, y_1 > 0\big\}, \tag{3.32}$$

$$\mathbb{D}_{\hat{\sigma}} = \big\{\mathbf{y} : 0 < y_2 < \hat{\sigma}(y_1),\, y_1 > 0\big\}. \tag{3.33}$$

Then Approach I consists of three steps:

1. Potential function $\phi(\mathbf{y})$. We first use a potential function to reduce the full Euler equations to a scalar nonlinear elliptic equation of second-order in the subsonic region. This method was first proposed in [25] in which the advantage of the conservation properties of the Euler system is taken for the reduction.

More precisely, since $\rho u_1 \neq 0$ in either the supersonic or subsonic region, using (3.7), there exists a potential function of the vector field $(\frac{u_2}{u_1}, \frac{1}{\rho u_1})$ such that

$$D\phi = (\frac{u_2}{u_1}, \frac{1}{\rho u_1}) \qquad \text{with } \phi(\mathbf{0}) = 0. \tag{3.34}$$

Equation (3.10) implies the Bernoulli law:

$$\frac{1}{2}q^2 + \frac{\gamma p}{(\gamma-1)\rho} = B(y_2), \tag{3.35}$$

where $B = B(y_2)$ is known, in fact it is completely determined by the incoming flow U^- at the initial position $\mathcal{I}$, because of the Rankine-Hugoniot condition (3.30), and $q = |\mathbf{u}| = \sqrt{u_1^2 + u_2^2}$.

From equations (3.7)–(3.10), we find

$$\Big(\frac{p}{\rho^\gamma}\Big)_{y_1} = 0, \tag{3.36}$$

which implies

$$p = A(y_2)\rho^\gamma \qquad \text{in the subsonic region } \mathbb{D}_{\hat{\sigma}}. \tag{3.37}$$

With equations (3.34) and (3.37), we can rewrite the Bernoulli law (3.35) as

$$\frac{\phi_{y_1}^2+1}{2\phi_{y_2}^2}+\frac{\gamma}{\gamma-1}A\rho^{\gamma+1}=B\rho^2. \tag{3.38}$$

In the subsonic region, $q=|\mathbf{u}|<c:=\sqrt{\frac{\gamma p}{\rho}}$. Therefore, the Bernoulli law (3.35) implies

$$\rho^{\gamma-1}>\frac{2(\gamma-1)B}{\gamma(\gamma+1)A}. \tag{3.39}$$

Condition (3.39) guarantees that ρ can be solved from (3.38) as a smooth function of $(D\phi, A, B)$.

Assume that $A=A(y_2)$ has been known. Then $(\mathbf{u}, p, \rho)$ can be expressed as functions of $D\phi$:

$$\rho=\rho(D\phi, A, B), \quad \mathbf{u}=(\frac{1}{\rho\phi_{y_2}}, \frac{\phi_{y_1}}{\rho\phi_{y_2}}), \quad p=A\rho^\gamma, \tag{3.40}$$

since $B=B(y_2)$ is given by the incoming flow.

Similarly, in the supersonic region $\mathbb{D}^-$, we employ the corresponding variables (ϕ^-, A^-, B) to replace U^-, where B is the same as in the subsonic region because of the Rankine-Hugoniot condition (3.30).

We now choose (3.9) to derive a *second-order nonlinear elliptic equation for* ϕ so that the full Euler system is reduced to this equation in the subsonic region:

$$(N^1(D\phi, A, B))_{y_1}+(N^2(D\phi, A, B))_{y_2}=0, \tag{3.41}$$

where $(N^1, N^2)(D\phi, A, B)=(u_2, p)(D\phi, A, B)$ are given by

$$N^1(D\phi, A, B)=\frac{\phi_{y_1}}{\phi_{y_2}\rho(D\phi, A(y_2), B(y_2))}, \quad N^2(D\phi, A, B)=A(y_2)\rho(D\phi, A(y_2), B(y_2))^\gamma. \tag{3.42}$$

Then a careful calculation shows that the discriminant

$$N^1_{\phi_{y_1}}N^2_{\phi_{y_2}}-N^1_{\phi_{y_2}}N^2_{\phi_{y_1}}=\frac{c^2\rho^2u_1^2}{c^2-q^2}>0 \tag{3.43}$$

in the subsonic region with $\rho u_1\neq 0$. Therefore, when ϕ is sufficiently close to ϕ_0^+ (determined by the subsonic background state U_0^+) in the C^1–norm, equation (3.41) is uniformly elliptic, and the Euler system (3.7)–(3.10) is reduced to the elliptic equation (3.41) in domain $\mathbb{D}_{\hat{\sigma}}$, where $\hat{\sigma}$ is the function for the free boundary (transonic shock).

The boundary condition for ϕ on the wedge boundary $\{y_2=0\}$ is derived from the fact that $\phi(y_1, y_2)=x_2(y_1, y_2)$ by (3.5), (3.6), (3.34). Then, recalling that $\partial W=\{\mathbf{x}: x_2=b(x_1), x_1>0\}$ in $\mathbf{x}$-coordinates, which is $\{\mathbf{y}: y_2=0, y_1>0\}$ in $\mathbf{y}$-coordinates, and $y_1=x_1$ by (3.6), we get

$$\phi(y_1, 0)=b(y_1). \tag{3.44}$$

The condition on $\mathcal{S}$ is derived from the Rankine-Hugoniot conditions (3.27)–(3.29). Condition (3.27) is equivalent to the continuity of ϕ across $\mathcal{S}$:

$$[\phi]|_{\mathcal{S}}=0, \tag{3.45}$$

which also gives

$$\hat{\sigma}'(y_1)=-\frac{[\phi_{y_1}]}{[\phi_{y_2}]}(y_1, \hat{\sigma}(y_1)). \tag{3.46}$$

Replacing $\hat{\sigma}'(y_1)$ in (3.28) and (3.29) with (3.46) gives rise to the conditions on $\mathcal{S}$:

$$G(D\phi, A, U^-)\equiv[\phi_{y_1}][\frac{1}{\rho\phi_{y_2}}+A\rho^\gamma\phi_{y_2}]-[\phi_{y_2}][A\rho^\gamma\phi_{y_1}]=0, \tag{3.47}$$

$$H(D\phi, A, U^-)\equiv[\phi_{y_1}][N^1]+[\phi_{y_2}][N^2]=0. \tag{3.48}$$

We now combine the above two conditions into the boundary condition for (3.41) by eliminating A. By calculation, we have

$$H_A = N_A^1[\phi_{y_1}] + N_A^2[\phi_{y_2}] = \frac{\gamma}{\gamma-1}\frac{\rho^{\gamma-1}u_2}{c^2-q^2}\left[\frac{u_2}{u_1}\right] - \frac{\rho^\gamma(q^2+\frac{c^2}{\gamma-1})}{c^2-q^2}\left[\frac{1}{\rho u_1}\right] > 0,$$

and

$$\begin{aligned} G_A &= [\phi_{y_1}]\Big(\frac{N_A^1}{\phi_{y_1}} + \phi_{y_2}N_A^2\Big) - [\phi_{y_2}]\phi_{y_1}N_A^2 \\ &= \frac{u_2\rho^\gamma(q^2+\frac{c^2}{\gamma-1})}{u_1(c^2-q^2)}\left[\frac{1}{\rho u_1}\right] - \frac{\rho^{\gamma-1}}{u_1(c^2-q^2)}\left({u_2}^2 + \frac{c^2-u_1^2}{\gamma-1}\right)\left[\frac{u_2}{u_1}\right] < 0, \end{aligned}$$

since $[\frac{1}{\rho u_1}] < 0$ and u_{2-} is close to 0. Therefore, both equations (3.47) and (3.48) can be solved for A to obtain $A = g_1(D\phi, U^-)$ and $A = g_2(D\phi, U^-)$, respectively. Then we obtain our desired condition on the free boundary (*i.e.*, the shock-front):

$$\bar{g}(D\phi, U^-) := (g_2 - g_1)(D\phi, U^-) = 0. \tag{3.49}$$

Then the original free boundary problem, Problem 3.2, is reduced to the following free boundary problem for the elliptic equation (3.41):

Problem 3.3 (Free Boundary Problem)**.** *Seek $(\hat\sigma, \phi, A)$ such that ϕ is a solution of the elliptic equation* (3.41) *in the region with the fixed boundary condition* (3.44)*, and the free boundary conditions* (3.45) *and* (3.49)*, and equalities* (3.47) *and* (3.48) *hold.*

2. Hodograph transformation and fixed boundary value problem. In order to solve the free boundary problem, we employ the hodograph transformation to make the shock-front a fixed boundary. This allows to find ϕ for each A from an appropriately chosen set. After that, we only need to perform iteration for the unknown function A, to satisfy (3.47) and (3.48).

Note that the solutions in Theorem 3.1 satisfy that $\|U - U_0^+\|_{L^\infty(\Omega)} \leqslant C_0\varepsilon$. Then, denoting by ϕ_0^+ the potential function (3.34) for the subsonic background state U_0^+, we obtain that ϕ is close to ϕ_0^+ in C^1 on the closure of the subsonic region. Then on the iteration, we will consider (and eventually obtain) solutions U for which the same property holds. Thus below we assume that ϕ is close to ϕ_0^+ in $C^1(\overline{\mathbb{D}_{\hat\sigma}})$, see (3.33).

We now extend the domain of ϕ^- from $\mathbb{D}^-$ to the first quadrant $\mathbb{D}^- \cup \mathbb{D}_{\hat\sigma}$. Let $\phi_0^- = \frac{1}{\rho_0^- u_{20}^-}y_2$, which is the the potential function (3.34) for the supersonic background state U_0^-. Then ϕ^- is close to ϕ_0^- in $C^1(\overline{\mathbb{D}^-})$ since U^- is close to U_0^- in L^∞ (and in stronger norm, see Theorem 3.1). We can extend ϕ^- into $\mathbb{D}^- \cup \mathbb{D}_{\hat\sigma}$ so that it remains close to ϕ_0^- in C^1 on the closure of $\mathbb{D}^- \cup \mathbb{D}_{\hat\sigma}$. We then use the following partial hodograph transformation:

$$(y_1, y_2) \to (z_1, z_2) = (\phi - \phi^-, y_2). \tag{3.50}$$

Note that, using (3.34), we have $\partial_{y_1}(\phi_0^+ - \phi_0^-) = \frac{u_{22}^0}{u_{21}^0} > 0$. Thus, since ϕ and ϕ^- are close in C^1 to ϕ_0^+ and ϕ_0^- resp., then the transformation (3.50) is invertible, that is y_1 is a function of $\mathbf{z} := (z_1, z_2)$: $y_1 = \varphi(\mathbf{z})$.

Let

$$M^1(D\phi, A, U^-) = N^1(D\phi, A, B) + N^2(D\phi, A, B)\frac{[\phi_{y_2}]}{[\phi_{y_1}]}, \quad M^2(D\phi, A, U^-) = \frac{N^2(D\phi, A, B)}{[\phi_{y_1}]},$$

and

$$\overline{M}^i(D\varphi,\varphi,A,\mathbf{z}) = -M^i(\partial_{y_1}\phi^-(\varphi,z_2)+\frac{1}{\varphi_{z_1}},\partial_{y_2}\phi^-(\varphi,z_2)-\frac{\varphi_{z_2}}{\varphi_{z_1}},A,U^-(\varphi,z_2)),\quad i=1,2.$$

Then equation (3.41) becomes

$$\big(\overline{M}^1(D\varphi,\varphi,A,\mathbf{z})\big)_{z_1}+\big(\overline{M}^2(D\varphi,\varphi,A,\mathbf{z})\big)_{z_2}=0. \tag{3.51}$$

Notice that

$$\overline{M}^1_{\varphi_{z_1}}\overline{M}^2_{\varphi_{z_2}}-\frac{1}{4}\big(\overline{M}^1_{\varphi_{z_2}}+\overline{M}^2_{\varphi_{z_1}}\big)^2=[\phi_{y_1}]^2\big(N^1_{\phi_{y_1}}N^2_{\phi_{y_2}}-(N^1_{\phi_{y_2}})^2\big)>0,$$

which implies that equation (3.51) is uniformly elliptic, for any solution φ that is close to φ_0^+ (determined by (3.50) with $\phi=\phi_0^+$) in the C^1–norm.

Under the transform (3.50), the unknown shock-front $\mathcal{S}$ becomes a fixed boundary, which is the z_2-axis (where we use that ϕ is close in C^1 to ϕ_0^+ in $\overline{\mathbb{D}_{\hat\sigma}}$ and to ϕ_0^- in $\overline{\mathbb{D}_{\hat\sigma}^-}$, to conclude from (3.34) that ϕ is Lipschitz across $\mathcal{S}$, and then that $\phi=\phi^-$ on $\mathcal{S}$ but $\phi\neq\phi^-$ in $\overline{\mathbb{D}_{\hat\sigma}}\backslash\mathcal{S}$). Along the z_2-axis, condition (3.49) is now

$$\begin{aligned}\tilde g(D\varphi,\varphi,\mathbf{z}) &:= \bar g(\partial_{y_1}\phi^-(\varphi,z_2)+\frac{1}{\varphi_{z_1}},\partial_{y_2}\phi^-(\varphi,z_2)-\frac{\varphi_{z_2}}{\varphi_{z_1}},U^-(\varphi,z_2))\\ &=0 \qquad \text{on } \{z_1=0,\ z_2>0\}.\end{aligned} \tag{3.52}$$

We also convert condition (3.48) into $\mathbf{z}$–coordinates:

$$\widetilde H(D\varphi,\varphi,A,\mathbf{z}) := H(\partial_{y_1}\phi^-(\varphi,z_2)+\frac{1}{\varphi_{z_1}},\partial_{y_2}\phi^-(\varphi,z_2)-\frac{\varphi_{z_2}}{\varphi_{z_1}},A,U^-(\varphi,z_2))=0 \tag{3.53}$$

along the z_2-axis.

The condition on the z_1-axis can be derived from (3.44) as follows: Restricted on $z_2=0$, the coordinate transformation (3.50) becomes

$$z_1=b(y_1)-\phi_-(y_1,0).$$

Then y_1 can be expressed in terms of z_1 as $y_1=\tilde b(z_1)$ so that $\varphi(z_1,0)=y_1$ satisfies

$$\varphi(z_1,0)=\tilde b(z_1) \quad \text{ on } \{z_2=0,\ z_1>0\}. \tag{3.54}$$

Therefore, the original wedge problem is now reduced to the following problem on the first quadrant $\mathbb{Q}$.

Problem 3.4 (Fixed Boundary Value Problem)**.** *Seek* (φ,A) *such that* φ *is a solution of the second-order nonlinear elliptic equation* (3.51) *in the unbounded domain* $\mathbb{Q}$ *with the boundary conditions* (3.52) *and* (3.54)*, and such that* (3.53) *holds.*

3. Solution to the fixed boundary value problem – Problem 3.4. Through the shock polar, we can determine the values of U at the origin, and hence $A(0)$ is fixed, depending on the values of $U^-(\mathbf{0})$ and $b'(0)$. Then we solve (3.53) to obtain a unique solution $\tilde A=h(\mathbf{z},\phi,D\phi)$ that defines the iteration map.

This is achieved by the following fixed point arguments. Consider a Banach space:

$$X=\{\mathcal{A}\,:\,\mathcal{A}(0)=0,\ \|\mathcal{A}\|^{(-\alpha);\{0\}}_{1,\alpha;(1+\beta);(0,\infty)}<\infty\}.$$

Then we define our iteration map $\mathcal{J}:X\longrightarrow X$ through the following:

First, we define a smooth cutoff function χ on $[0,\infty)$ such that

$$\chi(s)=\begin{cases}1, & 0\leqslant s<1,\\ 0, & s>2.\end{cases}$$

Set

$$A(0):=t(\omega(0),b'(0)) \qquad \text{for } \omega=U^- - U_0^-, \tag{3.55}$$

where t is a function determined by the Rankine-Hugoniot conditions (3.47)–(3.48). Then we define $w_t(z_2)$ as

$$w_t(z_2):=A_0^+ + \big(t(\omega(0),b'(0))-A_0^+\big)\chi(z_2). \tag{3.56}$$

Consider any $A=A(z_2)$ so that $A-w_t\in X$ satisfying

$$\|A-A_0^+\|^{(-\alpha);\{0\}}_{1,\alpha;(1+\beta);(0,\infty)}\leqslant C_0\varepsilon \tag{3.57}$$

for some fixed constant $C_0>0$, where $A_0^+=\frac{p_0^+}{(\rho_0^+)^\gamma}$.

With this A, we solve equation (3.51) for $\varphi=\varphi_A$ in the unbounded domain $\mathbb{Q}$ with the boundary conditions (3.52) and (3.54), and with the asymptotic condition $\varphi\to\varphi^\infty$ as $\mathbf{x}\to\infty$, where the limit is understood in the appropriate sense, where φ^∞ is the solution of

$$z_1=(\phi^\infty-\phi^-)(\varphi^\infty,z_2), \tag{3.58}$$

with $\phi^\infty=\frac{u_{20}^+}{u_{10}^+}y_1+l(y_2)$, where $l(y_2)$ is determined by the Bernoulli law (3.38), replacing ϕ and ρ with their asymptotic values ϕ^∞ and $\rho^\infty(y_2)=\Big(\frac{p_0^+}{A(y_2)}\Big)^{\frac{1}{\gamma}}$, and noting that $B=B(y_2)$ is determined by the upstream state U^-. Specifically, we show existence of a solution φ in the set:

$$\Sigma_\delta=\big\{\varphi:\ \|\varphi-\varphi^\infty\|^{(-1-\alpha);\partial\mathcal{W}}_{2,\alpha;(\beta,0);Q}\leqslant\delta\big\} \qquad \text{for sufficiently small } \delta>0,$$

which is compact and convex subset of the Banach space:

$$\mathcal{B}=\big\{\varphi\ :\ \|\varphi-\varphi^\infty\|^{(-1-\alpha');\partial\mathcal{W}}_{2,\alpha';(\beta',0);Q}<\infty\big\} \qquad \text{with } 0<\alpha'<\alpha,\ 0<\beta'<\beta.$$

Equation (3.51) is uniformly elliptic for $\varphi\in\Sigma_\delta$ if $\delta>0$ is small. This allows to solve the problem for $\varphi=\varphi_A\in\Sigma_\delta$ by the Schauder fixed point theorem if the perturbation is small, i.e. if ε is small in the conditions of Theorem 3.1 and in (3.57). Then, with this $\varphi=\varphi_A$, we solve (3.53) to obtain a unique $\tilde{A}$ that defines the iteration map J by $\mathcal{J}(A-w_t):=\tilde{A}-w_t$.

Finally, by the implicit function theorem, we prove that $\mathcal{J}$ has a fixed point $A-w_t$, for which A satisfies (3.57).

For more details for this approach, see Chen-Chen-Feldman [25, 26]. This approach can also be applied to Problem 3.2 (ST); see [124] for the case when the wedge angle is sufficiently small.

3.3. **Approach II for Problem 3.2 (ST) & (WT).** We now describe the second approach, Approach II. It allows to handle both cases in Theorem 3.1: case of Problem 3.2 (WT) and of Problem 3.2 (ST). Moreover, in case of Problem 3.2 (WT), this approach yields a better asymptotic decay rate, as stated in (3.24).

It will be convenient to rotate the $\mathbf{x}$-coordinates clockwise by angle $\theta_{\rm w}$, so that the background downstream flow becomes horizontal, as discussed in the paragraph before Theorem 3.1. We still use the same notations in the rotated coordinates, in particular we write $\mathbf{x} = (x_1, x_2)$ and $\mathbf{u} = (u_1, u_2)$ in the rotated basis. Then in the new coordinates:

$$\frac{u_{20}^-}{u_{10}^-} = -\tan\theta_{\rm w}, \quad U_0^- = (u_{10}^-,\, -u_{10}^- \tan\theta_{\rm w},\, p_0^-,\, \rho_0^-), \quad U_0^+ = (u_{10}^+,\, 0,\, p_0^+,\, \rho_0^+). \tag{3.59}$$

And since the new coordinates (x_1, x_2) are along the vectors $(\boldsymbol{\tau}_{\rm w}^0, \boldsymbol{\nu}_{\rm w}^0)$, i.e. $u_1 = \mathbf{u}\cdot\boldsymbol{\tau}_{\rm w}^0$, $u_2 = \mathbf{u}\cdot\boldsymbol{\nu}_{\rm w}^0$, we have by (3.22)

$$U_1 = (u_1, \rho) \text{ and } U_2 = (w, p) \text{ for } w = \frac{u_2}{u_1}. \tag{3.60}$$

Furthermore, we have from (3.13) and (3.23) or (3.25) with small ε, that in the rotated coordinates

$$\partial W = \{\mathbf{x} \in \mathbb{R}^2 \,:\, x_2 = b_{rot}(x_1),\ b_{rot}(0) = 0\}, \tag{3.61}$$

and the function b_{rot} satisfies the estimates in (3.63) or (3.65) below, resp., with $C\varepsilon$ instead of ε. For the background solution, $b_{rot,0} = 0$, i.e. ∂W_0 is the positive x_1-axis.

We will construct a solution with a shock front $\mathcal{S}$ expressed in the rotated coordinates as (3.16) with a function $\tilde{\sigma}(x_1)$. The background shock is now expressed as $\mathcal{S}_0 := \{x_2 = \tilde{\sigma}_0(x_1) \,:\, x_1 > 0\}$ for $\sigma_0(x_1) := \tilde{s}_0 x_1$, where $\tilde{s}_0 = \tan(\arctan s_0 - \theta_w)$. Then the subsonic region of the solution has the form:

$$\Omega = \{\mathbf{x} \in \mathbb{R}^2 \,:\, \tilde{b}(x_1) < x_2 < \tilde{\sigma}(x_1), x_1 > 0\}. \tag{3.62}$$

We can assume that the upstream steady supersonic smooth solution $U^-(\mathbf{x})$ exists in region $\Omega^- = \{\mathbf{x} \,:\, 2\tilde{s}_0 > x_2 > \frac{\tilde{s}_0}{2} x_1, x_1 \geqslant 0\}$, beyond the background shock, but is still close to U_0^-.

Moreover, in part (i) of Theorem 3.1, in the rotated coordinates U^∞ is independent of x_1, and $U^\infty = Z^\infty$.

Specifically, we will prove the following in the rotated coordinates:

Theorem 3.2 (Chen-Chen-Feldman [27]). *Let (U_0^-, U_0^+), given by (3.59), be a constant transonic solution for wedge angle $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm d})$. There are positive constants α, β, C_0, and ε, depending only on the background states (U_0^-, U_0^+), such that:*

(i) *If (U_0^-, U_0^+) corresponds to the state on arc $\widehat{TS}$, and*

$$\|U^- - U_0^-\|_{2,\alpha;(1+\beta,0);\Omega^-} + \|b'_{rot}\|^{(-\alpha;0)}_{1,\alpha;(1+\beta);\mathbb{R}^+} < \varepsilon, \tag{3.63}$$

then there exists a solution $(U, \tilde{\sigma})$ of Problem 3.2 (WT) *and a function $U^\infty(y_2) = (u_1^\infty(y_2), 0, p_0^+, \rho^\infty(y_2))$, and we denote $U_1^\infty = (u_\tau^\infty, \rho^\infty)$, such that U_1 and U_2 defined by (3.22) satisfy*

$$\begin{aligned} &\|U_1 - U_1^\infty\|^{(-\alpha;\partial\mathcal{W})}_{2,\alpha;(\beta,1);\Omega} + \|U_2 - U_{20}^+\|^{(-\alpha;O)(-1-\alpha;\partial\mathcal{W})}_{2,\alpha;(1+\beta,0);\Omega} + \|\tilde{\sigma}' - \tilde{s}_0\|^{(-\alpha;0)}_{2,\alpha;(1+\beta);\mathbb{R}^+} \\ &+ \|U_1^\infty - U_{10}^+\|^{(-\alpha;0)}_{2,\alpha;(1+\beta);[0,\infty)} \leqslant C_0 \left(\|U^- - U_0^-\|_{2,\alpha;(1+\beta,0);\Omega^-} + \|b'_{rot}\|^{(-\alpha;0)}_{1,\alpha;(1+\beta);\mathbb{R}^+} \right); \end{aligned} \tag{3.64}$$

(ii) *If (U_0^-, U_0^+) corresponds to the state on arc $\widehat{TH}$, and*

$$\|U^- - U_0^-\|_{2,\alpha;(\beta,0);\Omega^-} + \|b'_{rot}\|^{(-\alpha-1;0)}_{2,\alpha;(\beta);\mathbb{R}^+} < \varepsilon, \tag{3.65}$$

then there exists a solution $(U, \tilde{\sigma})$ of Problem 3.2 (ST)*, such that U_1 and U_2 defined by (3.22) satisfy*

$$\begin{aligned} &\|U_1 - U_{10}^+\|^{(-1-\alpha;\partial\mathcal{W})}_{2,\alpha;(0,\beta);\Omega} + \|U_2 - U_{20}^+\|^{(-1-\alpha;O)}_{2,\alpha;(\beta,0);\Omega} + \|\tilde{\sigma}' - \tilde{s}_0\|^{(-1-\alpha;0)}_{2,\alpha;(\beta);\mathbb{R}^+} \\ &\leqslant C_0 \left(\|U^- - U_0^-\|_{2,\alpha;(\beta);\Omega^-} + \|b'_{rot}\|^{(-1-\alpha;0)}_{2,\alpha;(\beta);\mathbb{R}^+} \right). \end{aligned} \tag{3.66}$$

The solution, $(U, \tilde{\sigma})$, is unique within the class of solutions such that the left-hand side of (3.24) for Problem 3.2 (WT) *or (3.26) for* Problem 3.2 (ST) *is less than $C_0\varepsilon$.*

Clearly, Theorem 3.1 follows from Theorem 3.2 if ε is small so that from the estimates of $\tilde{\sigma}$ in (3.64) or (3.66) shock remains graph $x_2 = \sigma(x_1)$ after rotating coordinates back.

To prove Theorem 3.2, we will work in the Lagrangian coordinates (3.6) defined for the rotated coordinates $\mathbf{x} = (x_1, x_2)$. Then, as in the previous case, using that, from the slip condition (3.15) on the wedge boundary, the curve ∂W is a streamline, we obtain that in the present Lagrangian coordinates, ∂W becomes the half-line

$$\mathcal{L}_1 = \{(y_1, y_2) \,:\, y_1 > 0, y_2 = 0\}.$$

We can assume that, in the Lagrangian coordinates, the supersonic solution U^- exists in domain $\mathbb{D}^-$ defined by (3.31). Shock $\mathcal{S}$ is given by $y_2 = \hat{\sigma}(y_1)$, $y_1 > 0$, where the function $\hat{\sigma}$ differs from the one in Approach 1 because Lagrangian coordinates are now defined differently. Supersonic region $\mathbb{D}^-_{\hat{\sigma}}$ and subsonic region $\mathbb{D}_{\hat{\sigma}}$ of the solution are given by (3.32) and (3.33) resp., with the present function $\hat{\sigma}$.

Background shock front $\mathcal{S}_0$ is now given by $y_2 = s_1 y_1$, $y_1 > 0$, where $s_1 = \rho_0^+ u_{10}^+ \tilde{s}_0$.

We prove first existence and estimates of solution in Lagrangian coordinates:

Theorem 3.3. *Let (U_0^-, U_0^+) be a constant transonic solution for wedge angle $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm d})$. There are positive constants α, β, C_0, and ε, depending only on the background states (U_0^-, U_0^+), such that if ∂W in (3.61) and U^- satisfy*

(i) (3.63) *for* Problem 3.2 (WT)
(ii) (3.65) *for* Problem 3.2 (ST)

then there exists a transonic shock $\mathcal{S}_L = \{y_2 = \hat{\sigma}(y_1),\, y_1 > 0\}$ and a subsonic solution $U = U(\mathbf{y})$ of (3.7)–(3.10) in $\mathbb{D}_{\hat{\sigma}}$, satisfying Rankine-Hugoniot conditions (3.27)–(3.30) along $\mathcal{S}_L$ with U^- expressed in Lagrangian coordinates in $\mathbb{D}^-_{\hat{\sigma}}$, and the slip condition $w_{|\mathcal{L}_1} = b'_{rot}$, and there exists a function $\mathcal{U}^\infty(y_2) = (u_1^\infty(y_2), 0, p_0^+, \rho^\infty(y_2))$, where we denote $\mathcal{U}_1^\infty(y_2) := (u_1^\infty(y_2), \rho^\infty(y_2))$, such that $U(\mathbf{y})$ satisfies the following estimates:

(i) *For* Problem 3.2 (WT)*:*

$$\begin{aligned}
&\|U_1 - \mathcal{U}_1^\infty\|^{(-\alpha;\mathcal{L}_1)}_{2,\alpha;(1+\beta,0);\mathbb{D}_{\hat{\sigma}}} + \|U_2 - U_{20}^+\|^{(-\alpha;O)(-1-\alpha;\mathcal{L}_1)}_{2,\alpha;(1+\beta,0);\mathbb{D}_{\hat{\sigma}}} + \|\hat{\sigma}' - s_1\|^{(-\alpha;0)}_{2,\alpha;(1+\beta);\mathbb{R}^+} \\
&+ \|\mathcal{U}_1^\infty - U_{10}^+\|^{(-\alpha;0)}_{2,\alpha;(1+\beta);\mathbb{R}^+} \leqslant C_0 \left(\|U^- - U_0^-\|_{2,\alpha;(1+\beta,0);\mathbb{D}^-_{\hat{\sigma}}} + \|b'_{rot}\|^{(-\alpha;0)}_{1,\alpha;(1+\beta);\mathbb{R}^+} \right);
\end{aligned} \tag{3.67}$$

(ii) *For* Problem 3.2 (ST)

$$\begin{aligned}
&\|U_1 - \mathcal{U}_1^\infty\|^{(-1-\alpha;\partial\mathcal{W})}_{2,\alpha;(\beta,0);\mathbb{D}_{\hat{\sigma}}} + \|U_2 - U_{20}^+\|^{(-1-\alpha;O)}_{2,\alpha;(\beta,0);\mathbb{D}_{\hat{\sigma}}} + \|\hat{\sigma}' - s_1\|^{(-1-\alpha;0)}_{2,\alpha;(\beta);\mathbb{R}^+} \\
&+ \|\mathcal{U}_1^\infty - U_{10}^+\|^{(-1-\alpha;0)}_{2,\alpha;(\beta);\mathbb{R}^+} \leqslant C_0 \left(\|U^- - U_0^-\|_{2,\alpha;(\beta);\mathbb{D}^-_{\hat{\sigma}}} + \|b'_{rot}\|^{(-1-\alpha;0)}_{2,\alpha;(\beta);\mathbb{R}^+} \right).
\end{aligned} \tag{3.68}$$

The function $\mathcal{U}^\infty(y_2)$ can be understood as asymptotic limit of $U(\mathbf{y})$ as $y_1 \to \infty$.

Now we describe the proof of Theorem 3.3, which is the main part of Approach 2.

Rewrite system (3.7)–(3.10) into the following nondivergence form for $U = (\mathbf{u}, p, \rho)^\top$:

$$A(U) U_{y_1} + B(U) U_{y_2} = 0, \tag{3.69}$$

where

$$A(U) = \begin{bmatrix} -\frac{1}{\rho u_1^2} & 0 & 0 & -\frac{1}{\rho^2 u_1} \\ 1 - \frac{p}{\rho u_1^2} & 0 & \frac{1}{\rho u_1} & -\frac{p}{\rho^2 u_1} \\ 0 & 1 & 0 & 0 \\ u_1 & u_2 & \frac{\gamma}{(\gamma-1)\rho} & -\frac{\gamma p}{(\gamma-1)\rho^2} \end{bmatrix}, \quad B(U) = \begin{bmatrix} \frac{u_2}{u_1^2} & -\frac{1}{u_1} & 0 & 0 \\ \frac{p u_2}{u_1^2} & -\frac{p}{u_1} & -\frac{u_2}{u_1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solving $\det(\lambda A - B) = 0$ for λ, we obtain four eigenvalues:

$$\lambda_1 = \lambda_2 = 0, \qquad \lambda_j = -\frac{c\rho}{c^2 - u_1^2}\big(cu_2 + (-1)^j u_1\sqrt{c^2 - q^2}i\big) \text{ for } j = 3, 4,$$

where $q = \sqrt{u_1^2 + u_2^2} < c$ in the subsonic region. The corresponding left eigenvectors are

$$\mathbf{l}_1 = (0, 0, 0, 1), \qquad \mathbf{l}_2 = (-pu_1, u_1, u_2, -1),$$

$$\mathbf{l}_{3,4} = \Big(\frac{p(\gamma p - \rho u_1^2)}{(\gamma - 1)\rho u_1}\lambda_{3,4} + \frac{\gamma p^2 u_2}{(\gamma - 1)u_1}, -(u_1 + \frac{\gamma p}{(\gamma - 1)\rho u_1})\lambda_{3,4} - \frac{\gamma p u_2}{(\gamma - 1)u_1}, \frac{\gamma p}{\gamma - 1} - u_2\lambda_{3,4}, \lambda_{3,4}\Big).$$

Then

(i) Multiplying equations (3.69) from the left by $\mathbf{l}_1$ leads to the same equation (3.10). This, together with the Rankine-Hugoniot condition (3.30), implies the Bernoulli law (3.35) in both supersonic and subsonic domains, and across the shock-front. Therefore, $B(y_2)$ can be computed from the upstream flow U^-. If u_1 is a small perturbation of u_{10}^+, then $u_1 > 0$. Therefore, we can solve (3.35) for u_1:

$$u_1 = \frac{\sqrt{2B - \frac{2\gamma p}{(\gamma-1)\rho}}}{\sqrt{1 + w^2}} \qquad \text{with } w = \tfrac{u_2}{u_1}. \tag{3.70}$$

(ii) Multiplying system (3.69) from the left by $\mathbf{l}_2$ also gives (3.36).

(iii) Multiplying equations (3.69) from the left by $\mathbf{l}_3$ and separating the real and imaginary parts of the equation lead to the elliptic system:

$$\begin{aligned} D_R w + e D_I p = 0,\\ D_I w - e D_R p = 0, \end{aligned} \tag{3.71}$$

where $D_R = \partial_{y_1} + \lambda_R \partial_{y_2}$, $D_I = \lambda_I \partial_{y_2}$, $\lambda_R = -\frac{c^2 \rho u_2}{c^2 - u_1^2}$, $\lambda_I = \frac{c\rho u_1\sqrt{c^2 - q^2}}{c^2 - u_1^2}$, and $e = \frac{\sqrt{c^2 - q^2}}{c\rho u_1^2}$.

Therefore, equations (3.7)–(3.10) are decomposed into (3.70)–(3.71).

We solve this problem by iterations. Given U^- which is close to U_0^- as defined in Theorem 3.3, working in Lagrangian coordinates, we solve for U. However, since $\mathcal{U}^\infty$ is not known, we cannot directly solve for U satisfying (3.67) for Problem 3.2 (WT) or (3.68) for Problem 3.2 (ST). Instead, we solve for U which is close to U_0^+ as in (3.26) for Problem 3.2 (ST) and similar norms with appropriate growth for Problem 3.2 (WT), but using these norms in Lagrangian coordinates (more precisely, in (z_1, z_2)-coordinates defined by (3.74)). Note that these norms are weaker than the ones in (3.67) or (3.68) resp., in particular they do not determine any limit for $U_1 = (u_1, \rho)$ as $|\mathbf{y}| \to \infty$ within the subsonic region. On the other hand, these norms determine that the quantities (w, p) have the limit $(0, p_0^+)$ at infinity within the subsonic region, and this asymptotic condition is sufficient to have the iteration problem well-defined (in fact, we use only the asymptotic decay of w, then we can prove the asymptotic decay of $p - p_0^+$), and obtain existence and uniqueness in the iteration problem. After we find by iteration (the unique) solution U of the problem stated in Theorem 3.3, we identify $\mathcal{U}_1^\infty = (\rho^\infty, u_1^\infty)$, and show the faster convergence of (ρ, u_1) to $(\rho^\infty, u_1^\infty)$, thus prove (3.67) or (3.68) resp. Note that in estimates discussed above, $U - U_0^+$ (rather than U itself) lies in weighted spaces (3.21). For this reason, it is convenient to perform iteration in terms of

$$\delta U_1 = U - U_{10}^+, \quad \delta U_2 = U - U_{20}^+ \quad \text{and } \delta\hat{\sigma} = \hat{\sigma} - \hat{\sigma}_0 = \hat{\sigma} - s_1 y_1, \tag{3.72}$$

where U_1, U_2 are defined by (3.60).

Then we follow the steps below to solve this problem:

1. Introduce a linear boundary value problem for iteration. For a given shock-front $\hat{\sigma}$, the subsonic domain $\mathbb{D}^{\hat{\sigma}}$ is fixed, and depends on $\hat{\sigma}$. We make the coordinate transformation to transform the domain from $\mathbb{D}^{\hat\sigma}$ to $\mathbb{D}$, where $\mathbb{D} = \mathbb{D}^{\hat\sigma_0}$ with $\hat\sigma_0(y_2) = s_1 y_1$ is the domain corresponding to the background solution:

$$\begin{aligned} &\mathbb{D} = \big\{\mathbf{y} : 0 < y_2 < s_1 y_1\big\}, \quad \text{with } \partial\mathbb{D} = \overline{\mathcal{L}_1} \cup \mathcal{L}_2 \quad \text{where} \\ &\mathcal{L}_1 = \{(y_1, y_2) \,:\, y_1 > 0, y_2 = 0\}, \quad \mathcal{L}_2 = \{(y_1, y_2) \,:\, y_1 > 0, y_2 = s_1 y_1\}. \end{aligned} \tag{3.73}$$

This transformation is:

$$(y_1, y_2) \to (z_1, z_2) := (y_1, y_2 - \delta\hat\sigma(y_1)), \tag{3.74}$$

where $\delta\hat\sigma(y_1) = \hat\sigma(y_1) - \hat\sigma_0(y_1)$. In the $\mathbf{z}$–coordinates, $\mathcal{L}_1$ corresponds to ∂W, and $\mathcal{L}_2$ corresponds to $\partial\mathcal{S}$. Also, $U(\mathbf{y})$ becomes $U_{\hat\sigma}(\mathbf{z})$, depending on $\hat\sigma$. Then the upstream flow U^- involves an unknown variable explicitly depending on $\hat\sigma$:

$$U^-_{\hat\sigma}(\mathbf{z}) = U^-(z_1, z_2 + \delta\hat\sigma(z_1)), \tag{3.75}$$

where U^- is the given upstream flow in the $\mathbf{y}$–coordinates. Equations (3.71) in $\mathbf{z}$-coordinates are:

$$\begin{aligned} &\widetilde{D}_R w + e\widetilde{D}_I p = 0, \\ &\widetilde{D}_I w - e\widetilde{D}_R p = 0, \end{aligned} \tag{3.76}$$

in $\mathbb{D}$. where $\widetilde{D}_R = \partial_{z_1} + (\lambda_R - \delta\hat\sigma')\partial_{z_2}$ and $\widetilde{D}_I = \lambda_I\partial_{z_2}$. Since U_0^+ is a constant vector and $w_0^+ = 0$, then the same system holds for δp, δw, where we use notation (3.72). Moreover, as we consider iteration $(\delta U, \delta w) \to (\delta\tilde U, \delta\tilde w)$, we use $U = U_0^+ + \delta U$ to determine the coefficients in (3.76), and $\delta\tilde p, \delta\tilde w$ for the unknown functions. Thus we have

$$\begin{aligned} &\widetilde{D}_R\delta\tilde w + e\widetilde{D}_I\delta\tilde p = 0, \\ &\widetilde{D}_I\delta\tilde w - e\widetilde{D}_R\delta\tilde p = 0, \end{aligned} \tag{3.77}$$

in $\mathbb{D}$. We use system (3.77) as a linear system for iterations.

In the $\mathbf{z}$-coordinates, the Rankine-Hugoniot conditions (3.27)–(3.30) keep the same form, except that $\hat\sigma'(y_1)$ is replaced by $\hat\sigma'(z_1)$ and U^- is replaced by $U^-_{\hat\sigma}$ along line $z_2 = s_1 z_1$. Among the four Rankine-Hugoniot conditions, (3.30) is used in the Bernoulli law. From condition (3.29), we have

$$\hat\sigma'(z_1) = \frac{[p]}{[u_1 w]}(z_1, s_1 z_1), \tag{3.78}$$

which will be used to update the shock-front later. Now, because of (3.70), we can use $\bar U = (w, p, \rho)$ as the unknown variables along $z_2 = s_1 z_1$. Using (3.78) to eliminate $\hat\sigma'$ in conditions (3.27)–(3.28) gives

$$G_1(U^-_{\hat\sigma}, \bar U) := [p]\Big[\frac{1}{\rho u_1}\Big] + [w][u_1 w] = 0, \tag{3.79}$$

$$G_2(U^-_{\hat\sigma}, \bar U) := [p]\Big[u_1 + \frac{p}{\rho u_1}\Big] + [pw][u_1 w] = 0, \tag{3.80}$$

on $\mathcal{L}_2$. We use conditions (3.79)–(3.80) to define the linear conditions for iteration $\bar U \to \tilde{\bar U}$, such that at a fixed point $\bar U = \tilde{\bar U}$ these iteration conditions imply that the original conditions (3.79)–(3.80) hold. Specifically, we define conditions

$$\nabla_{\bar U} G_i(U_0^-, \bar U_0^+) \cdot \delta\tilde{\bar U} = \nabla_{\bar U} G_i(U_0^-, \bar U_0^+) \cdot \delta\bar U - G_i(U^-_{\hat\sigma}, \bar U) \qquad \text{on } \mathcal{L}_2, \tag{3.81}$$

which can be written as:

$$b_{i1}\delta\tilde w + b_{i2}\delta\tilde p + b_{i3}\delta\tilde\rho = g_i(U^-_{\hat\sigma}, \bar U) \qquad \text{for } i = 1, 2 \qquad \text{on } \mathcal{L}_2, \tag{3.82}$$

where $(b_{i1}, b_{i2}, b_{i3}) := \nabla_{\bar U} G_i(U_0^-, \bar U_0^+)$ and $g_i(U^-_{\hat\sigma}, \bar U) := \nabla_{\bar U} G_i(U_0^-, \bar U_0^+) \cdot \delta\bar U - G_i(U^-_{\hat\sigma}, \bar U)$.

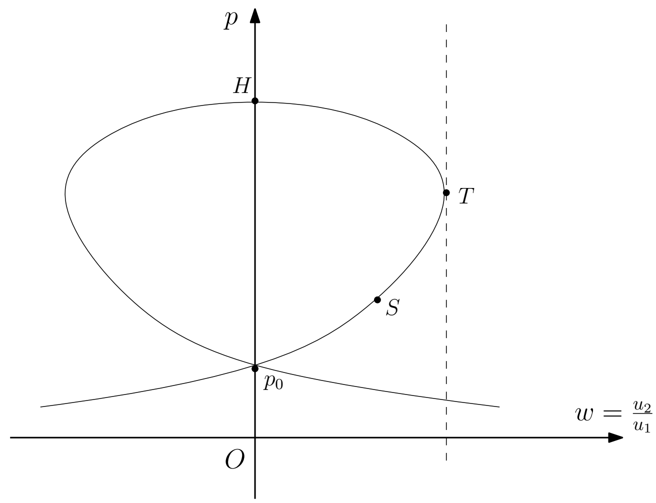

FIGURE 3.3. The shock polar in the (w, p)–variables

Since there are two conditions in (3.82), $i = 1, 2$, we can eliminate $\delta\tilde{\rho}$, thus obtain

$$\delta\tilde{w} + b_1\delta\tilde{p} = g_3 \qquad \text{on } \mathcal{L}_2, \tag{3.83}$$

where

$$b_1 = \frac{b_{12}b_{23} - b_{22}b_{13}}{b_{11}b_{23} - b_{21}b_{13}}, \qquad g_3 = \frac{b_{23}g_1 - b_{13}g_2}{b_{11}b_{23} - b_{21}b_{13}} \tag{3.84}$$

with

$$b_{11}b_{23} - b_{21}b_{13} = (-u_{20}^-)[p_0]\Big(\frac{\gamma p_0^+}{(\gamma-1)(\rho_0^+)^2 u_{10}^+} + \frac{p_0^-}{u_{10}^-}\big(\frac{1}{(\rho_0^+)^2} + \frac{\gamma p_0^+}{(\gamma-1)(\rho_0^+)^3(u_{10}^+)^2}\big)\Big) > 0.$$

Notice that the shock polar is a one-parameter curve determined by the Rankine-Hugoniot conditions. If p is used as the parameter, by equation (3.83), we obtain that $\delta w = -b_1\delta p + g_3(\delta p)$, which shows that $-b_1\delta p$ is the linear term and $g_3(\delta p)$ is the higher order term. From Fig. 3.3, we know that $w(p)$ is decreasing in p on arc $\widehat{TH}$ and increasing on $\widehat{TS}$. Therefore, it is easy to see that

$$b_1 > 0 \textit{ corresponds to the state on arc } \widehat{TH}, \ b_1 < 0 \textit{ to } \widehat{TS}, \textit{ and } b_1 = 0 \textit{ at the tangent point } T. \tag{3.85}$$

This difference in the sign of b_1 is the reason of different rates of decay at infinity and near the origin in cases (i) and (ii) of Theorems 3.1 and 3.3.

We compute

$$b_{13} = -[p_0]\Big(\frac{p_0^+}{(\rho_0^+)^2 u_{10}^+} + \frac{\gamma p_0^+}{(\gamma-1)(\rho_0^+)^3(u_{10}^+)^3}\Big) < 0.$$

Thus condition (3.82) for $i = 1$ can be rewritten as

$$\delta\tilde{\rho} = g_4 - b_2\delta\tilde{w} - b_3\delta\tilde{p} \qquad \text{on } \mathcal{L}_2, \tag{3.86}$$

where $g_4 = \frac{g_1}{b_{13}}, b_2 = \frac{b_{11}}{b_{13}}$, and $b_3 = \frac{b_{12}}{b_{13}}$.

We notice that conditions (3.83)–(3.86) are equivalent to conditions (3.82) for $i = 1, 2$.

Boundary condition on $\mathcal{L}_1$ comes from the slip condition (3.15) on ∂W. Specifically, using (3.61) and (3.15), we obtain $w = b'_{rot}$ on ∂W. Then, in $\mathbf{z}$-coordinates, this must hold on $\mathcal{L}_1$. Also, for the background solution, $b_{rot} = b_0 - b_0 = 0$ by (3.61). Then we prescribe

$$\delta\tilde{w} = b'_{rot} \qquad \text{on } \mathcal{L}_1. \tag{3.87}$$

2. Design the iteration map $\mathcal{Q}$ and existence of a fixed point for $\mathcal{Q}$. We perform iteration in terms of δU_k, $k = 1, 2$ and $\delta\hat{\sigma}$ as defined by (3.72), in z-coordinates defined in (3.74). In fact, for $\hat{\sigma}$, we only need $\hat{\sigma}'$ since $\hat{\sigma}(0) = 0$, i.e. shock is attached to the tip of wedge. Note also that $\delta\hat{\sigma}' = \hat{\sigma}' - s_1$. We thus denote $V = (U_1, U_2, \delta\hat{\sigma}')$, and perform the following iteration $\delta V \to \delta\tilde{V}$. For a given δV, we determine $V = \delta V + V_0^+$. Then we find $\tilde{V}$ by solving the linear system (3.77) in $\mathbb{D}$, with boundary conditions (3.83) and (3.87), to determine $(\tilde{w}, \tilde{p})$. Then determine u_1 from (3.70), and ρ from (3.36) which holds in $\mathbf{z}$-coordinates without change, and the boundary condition (3.86). Final step is to use solution $(\delta u_1, \delta\rho, \delta w, \delta p)$ and $U_{\hat{\sigma}}^-$ defined by (3.75) in the right-hand side of (3.78) to update the $\delta\hat{\sigma}'$. This defines of the iteration map $\mathcal{Q}$ from V to $\tilde{V}$, except we discuss below how we solve the boundary-value problem for (3.77) in $\mathbb{D}$, with boundary conditions (3.83) and (3.87).

As we discussed above, we perform iteration in the spaces from (3.68) for Problem 3.2 (ST) and similar norms with appropriate growth for Problem 3.2 (WT), expressed in z-coordinates (3.74). We discuss below the case of Problem 3.2 (WT), another case is similar. For $\tau > 0$, define:

$$
\begin{aligned}
\Sigma_1^\tau &= \{v \,:\, \|v\|_{2,\alpha;(0,1+\beta);\mathbb{D}}^{(-\alpha;\mathcal{L}_1)} + \|v_{z_1}\|_{2,\alpha;(1+\beta,1);\mathbb{D}}^{(1-\alpha;\mathcal{L}_1)} \leqslant \tau\}\\
\Sigma_2^\tau &= \{v \,:\, \|v\|_{2,\alpha;(1+\beta,0);\mathbb{D}}^{(-\alpha;O)(-1-\alpha;\mathcal{L}_1)} \leqslant \tau\}, \qquad \Sigma_3^\tau = \{v \,:\, \|v\|_{2,\alpha;(1+\beta);\mathbb{R}^+}^{(-\alpha;0)} \leqslant \tau\},\\
\Sigma^\tau &= \{(\delta U_1, \delta U_2, \delta\hat{\sigma}') \,:\, \delta U_1 \in \Sigma_1^\tau \times \Sigma_1^\tau,\;\; \delta U_2 \in \Sigma_2^\tau \times \Sigma_2^\tau,\;\; \delta\hat{\sigma}' \in \Sigma_3^\tau\}.
\end{aligned}
\tag{3.88}
$$

The condition on v_{z_1} in Σ_1^τ is added for technical reasons.

It remains to discuss how we find $(\delta\tilde{w}, \delta\tilde{p}) \in \Sigma_2^{C_0\varepsilon} \times \Sigma_2^{C_0\varepsilon}$ which solve (3.77) in $\mathbb{D}$, with boundary conditions (3.83) and (3.87). From system (3.77), we obtain

$$
(\delta\tilde{p})_{z_1} = \frac{(\lambda_R - \delta\hat{\sigma}')}{e\lambda_I}(\delta\tilde{w})_{z_1} + \frac{(\lambda_R - \delta\hat{\sigma}')^2 + \lambda_I^2}{e\lambda_I}(\delta\tilde{w})_{z_2}, \tag{3.89}
$$

$$
(\delta\tilde{p})_{z_2} = -\frac{1}{e\lambda_I}(\delta\tilde{w})_{z_1} - \frac{(\lambda_R - \delta\hat{\sigma}')}{e\lambda_I}(\delta\tilde{w})_{z_2}. \tag{3.90}
$$

Now, differentiating and subtracting the equations, we eliminate $\delta\tilde{p}$, and obtain a second order equation for $\delta\tilde{w}$ of the form

$$
\sum_{i,j=1}^{2} (a_{ij}(\delta\tilde{w})_{z_j})_{z_i} = 0, \tag{3.91}
$$

where the coefficients are computed explicitly from (3.89)–(3.90). Note that, at the subsonic background solution (3.59), we obtain $\lambda_{R0} = 0$, $\lambda_{I0} > 0$, $e_0 > 0$, where the left-hand sides are constants, and also $\delta\hat{\sigma}_0 = 0$. Then, computing the coefficients at the background solution, equation (3.91) becomes

$$
\frac{1}{\lambda_{I0}}(\delta\tilde{w})_{z_1z_1} + \lambda_{I0}(\delta\tilde{w})_{z_2z_2} = 0,
$$

i.e. the equation is uniformly elliptic. Then for the coefficients computed at $(U_{10}^+ + \delta U_1, U_{20}^+ + \delta U_2, \delta\hat{\sigma}')$ for $(\delta U_1, \delta U_2, \delta\hat{\sigma}') \in \Sigma^{C_0\varepsilon}$, the equation (3.91) is uniformly elliptic if ε is small. This allows to obtain the unique solution $\delta\tilde{w} \in \Sigma_2^{C_0\varepsilon}$ of (3.91) in $\mathbb{D}$ with boundary conditions (3.83) and (3.87). Note that the inclusion $\delta\tilde{w} \in \Sigma_2^{C_0\varepsilon}$ involves the asymptotic condition at infinity, and this makes the boundary-value problem well-defined and allows to prove uniqueness. After $\delta\tilde{w}$ is determined, we find $\delta\tilde{p}$ by z_2-integration from (3.90) with the initial condition (3.83), where it can be shown that $b_1 \neq 0$. Then we show that $\delta p \in \Sigma_2^{C_0\varepsilon}$. This completes the definition of the iteration map.

Iteration set for Problem 3.2 (WT) is $\Sigma^{C_0\varepsilon}$. We show that if ε is small, then $\mathcal{Q}(\Sigma^{C_0\varepsilon}) \subset \Sigma^{C_0\varepsilon}$, and obtain a fixed point by the Schauder fixed point theorem, by considering the set $\Sigma^{C_0\varepsilon}$ as a compact subset in the Banach space defined by the same norms as in the definition of Σ^τ, except that α is replaced by $\alpha' \in (0, \alpha)$ and showing that the map $\mathcal{Q}$ is continuous in this norm.

3. Fixed point: Asymptotic limit in y-coordinates Let $(\delta U_1, \delta U_2, \delta\hat{\sigma}') \in \Sigma^{C_0\varepsilon}$ be a fixed point of the iteration map, and let $(U_1, U_2, \hat{\sigma}') = (U_{10}^+ + \delta U_1, U_{20}^+ + \delta U_2, \delta\hat{\sigma}')$.

We change from $\mathbf{z}$ to $\mathbf{y}$ coordinates by inverting (3.74):

$$(z_1, z_2) \;\to\; (y_1, y_2) := (z_1, z_2 + \delta\hat{\sigma}(z_1)).$$

Note that since $\delta\hat{\sigma}' \in \Sigma_3^{C_0\varepsilon}$, then both (3.74) and its inverse are close to the identity map in $C^{2,\alpha}(\mathbb{D}_{\hat{\sigma}}; \mathbb{R}^2)$ and $C^{2,\alpha}(\mathbb{D}; \mathbb{R}^2)$ resp. Then it follows that, in $\mathbf{y}$-coordinates, $(\delta U_1, \delta U_2, \delta\hat{\sigma}') \in \tilde{\Sigma}^{2C_0\varepsilon}$ if ε is small, where

$$\begin{aligned} &\tilde{\Sigma}_1^\tau = \{v \,:\, \|v\|^{(-\alpha;\mathcal{L}_1)}_{2,\alpha;(0,1+\beta);\mathbb{D}_{\hat{\sigma}}} + \|v_{z_1}\|^{(1-\alpha;\mathcal{L}_1)}_{2,\alpha;(1+\beta,1);\mathbb{D}_{\hat{\sigma}}} \leqslant \tau\}, \quad \tilde{\Sigma}_2^\tau = \{v \,:\, \|v\|^{(-\alpha;O)(-1-\alpha;\mathcal{L}_1)}_{2,\alpha;(1+\beta,0);\mathbb{D}_{\hat{\sigma}}} \leqslant \tau\}, \\ &\tilde{\Sigma}^\tau = \{(\delta U_1, \delta U_2, \delta\hat{\sigma}') \,:\, \delta U_1 \in \tilde{\Sigma}_1^\tau \times \tilde{\Sigma}_1^\tau, \;\; \delta U_2 \in \tilde{\Sigma}_2^\tau \times \tilde{\Sigma}_2^\tau, \;\; \delta\hat{\sigma}' \in \Sigma_3^\tau\}. \end{aligned} \tag{3.92}$$

In particular, this proves the estimate of second and third terms in the left-hand side of (3.67).

Note that for $v \in \tilde{\Sigma}_2^\tau$, we have $v \to 0$ as $|\mathbf{y}| \to \infty$ in $\mathbb{D}_{\hat{\sigma}}$, with rate $|\mathbf{y}|^{-(\beta+1)}$. But for $v \in \tilde{\Sigma}_1^\tau$, no asymptotic limit as $|\mathbf{y}| \to \infty$ in $\mathbb{D}_{\hat{\sigma}}$ is defined.

Then, from (3.59)–(3.60) it follows that $U_2 = (w, p) \to (0, p_0^+)$ as $|\mathbf{y}| \to \infty$ in $\mathbb{D}$, but for $U_1 = (u_1, \rho)$ the limit is no determined by the spaces Σ_1^τ, and (u_1, ρ) does not converge to (u_{10}^+, ρ_0^+) in general, as we will see below. Then we determine the limiting profiles $(u_1^\infty(y_2), \rho^\infty(y_2))$.

To determine $\rho^\infty(y_2)$, we note that from (3.7)–(3.10) we obtain (3.36), and thus (3.37). Since function $\hat{\sigma}(y_1)$ is determined, the function $A(y_2)$ in (3.37) is determined by the upstream state $U^-(\mathbf{y})$ from the Rankine-Hugoniot conditions (3.27)–(3.30). Then, noting that $p \to p_0^\infty$, we obtain formally

$$\rho \to \rho^\infty(y_2) = \left(\frac{p_0^+}{A(y_2)}\right)^{\frac{1}{\gamma}} \quad \text{as} \;\; |\mathbf{y}| \to \infty \;\; \text{in} \;\; \mathbb{D}_{\hat{\sigma}}.$$

Similarly, we use (3.70) to obtain

$$u_1 \to u_1^\infty(y_2) = \sqrt{2B(y_2) - \frac{2\gamma p_0^+}{(\gamma-1)\rho^\infty(y_2)}} \quad \text{as} \;\; |\mathbf{y}| \to \infty \;\; \text{in} \;\; \mathbb{D}_{\hat{\sigma}}.$$

Then, defining $\mathcal{U}^\infty(y_2) = (u_1^\infty(y_2)$, we can show that estimate of the first and the last terms in the left-hand side of in (3.67) holds. This completes the argument for case (i) of Theorem 3.3.

Case (ii) is handled similarly. Note that the slower decay at infinity for case (ii), i.e. $|\mathbf{y}|^{-\beta}$, comes from elliptic estimates even if we require faster decay at infinity in (3.25). The reason for the difference in the rates in cases (i) and (ii) is (3.85).

4. Return to x-coordinates

We obtain Theorem 3.2 from Theorem 3.3 by changing coordinates. Recall that when we define the Lagrangian coordinates for Theorem 3.3, we use the rotated coordinates $\mathbf{x}$ in (3.6), see the discussion in the paragraph before Theorem 3.3.

From estimates in Theorem 3.3, it follows that in the Lagrangian coordinates, $|U - U_0^+| \leqslant C\varepsilon$ in $\mathbb{D}_{\hat{\sigma}}$, where C depends only on (U_0^-, U_0^+). Thus the same is true in $\mathbf{x}$-coordinates in Ω. Then it follows from (3.5), (3.6) and (3.59) where u_{10}^+, ρ_0^+ are positive, that the change of coordinates $\mathbf{x} \to \mathbf{y}$ given by (3.6) is bi-Lipschitz. Then (3.66) follows from (3.68) directly.

Similarly, estimates of the second and third terms in the left-hand side of (3.64) follow from (3.67) directly. In order to obtain the estimates of the remaining terms in the left-hand side of (3.64), we need to identify $U^\infty(x_2)$.

Note that on the shock $\mathcal{S}$, using (3.6) and the estimate of the third term in the left-hand side of (3.64), we have that for small ε,

$$\partial_{\boldsymbol{\tau}_{\mathcal{S}}}\psi = \rho\mathbf{u} \cdot \boldsymbol{\nu}_{\mathcal{S}} \geqslant \rho\mathbf{u}_0^+ \cdot \boldsymbol{\nu}_{\mathcal{S}_0} - C\varepsilon \geqslant \frac{1}{2}\rho\mathbf{u}_0^+ \cdot \boldsymbol{\nu}_{\mathcal{S}_0} > 0.$$

Recall also that $\psi(\mathbf{0}) = 0$ by (3.5). Then for each $y_2 > 0$ there exists a unique $\mathbf{x}^{in}(y_2) = (x_1^{in}(y_2), x_2^{in}(y_2)) \in \mathcal{S}$ such that $\psi(\mathbf{x}^{in}(y_2)) = y_2$, and it satisfies

$$\|\mathbf{x}^{in}\|_{C^{2,\alpha}([0,\infty))} \leqslant C \quad \text{and} \quad (\mathbf{x}^{in})' \geqslant \frac{1}{C} > 0 \;\; \text{on} \;\; [0,\infty).$$

From this and (3.5), it follows that for each $y_2 > 0$, we have

$$\Omega \cap \{\mathbf{x} \,:\, \psi(\mathbf{x}) = y_2\} = \{(x_1, x_2^*(x_1; y_2)) \;:\; x_1 > x_1^{in}(y_2)\},$$

where $x_2^*(\cdot; y_2)$ is the solution of the initial-value problem for ODE:

$$\begin{aligned} &\partial_{x_1} x_2^*(x_1; y_2) = w(x_1, x_2^*(x_1; y_2)), \\ &x_2^*(x_1^{in}(y_2); y_2) = x_2^{in}(y_2), \end{aligned} \tag{3.93}$$

where $w = \frac{u_2}{u_1}$ by (3.60). Since we have obtained the estimate of the second term in the left-hand side of (3.24), and using (3.59), we have

$$|D^k w(\mathbf{x})| \leqslant C_0 \varepsilon (1 + |\mathbf{x}|)^{-1-\beta} \quad \text{in } \Omega, \quad \text{for } \; k = 0, 1, 2. \tag{3.94}$$

In particular, for each $y_2 \geqslant 0$ and $k = 0, 1, 2$

$$\int_{x_1^{in}(y_2)}^{\infty} |D^k w(x_1, x_2^*(x_1; y_2))| dx_1 \leqslant C_0 \varepsilon \int_0^{\infty} (1 + x_1)^{-1-\beta} dx_1 \leqslant C\varepsilon. \tag{3.95}$$

Applying this with $k = 0$, we obtain that $\lim_{x_1 \to \infty} x_2^*(x_1; y_2)$ exists for each $y_2 \geqslant 0$; we denote it $x_2^{\infty}(y_2)$.

Differentiating (3.93) two times with respect to y_2 and using C^2 estimate of $\mathbf{x}^{in}$ and(3.95), we obtain $\|x_2^*(T; \cdot)\|_{C^2([0,\infty))} \leqslant C$, and from this

$$x_2^*(T; \cdot) \to x_2^{\infty}(\cdot) \text{ in } C^1 \text{ on compact subsets on } [0, \infty), \text{ and } \; \|x_2^{\infty}\|_{C^2([0,\infty))} \leqslant C. \tag{3.96}$$

$$\begin{aligned} &x_2^*(x_1; \cdot) \to x_2^{\infty}(\cdot) \text{ in } C^1 \text{ on compact subsets on } [0, \infty) \text{ as } x_1 \to \infty, \\ &\text{with } \|x_2^{\infty}\|_{C^2([0,\infty))} \leqslant C. \end{aligned} \tag{3.97}$$

Also, by

Also, by a similar argument, using $C^{2,\alpha}$ regularity of $\mathbf{x}^{in}$ and estimate w in the second term in (3.64), we get $x_2^{\infty} \in C^{2,\alpha}([0,\infty))$.

Furthermore, we note that for the background solution, using (3.59), the potentials ψ_0^- of U_0^-, ψ_0^+ of U_0^+, and ψ_0 of the transonic shock solution (U_0^-, U_0^+) in $\{x_1 > 0, x_2 > 0\}$ are:

$$\psi_0^-(\mathbf{x}) = \rho_0^- u_{10}^-(x_1 - x_2 \tan\theta_w), \quad \psi_0^+(\mathbf{x}) = \rho_0^+ u_{10}^+ x_1, \quad \psi_0(\mathbf{x}) = \begin{cases} \psi_0^-(\mathbf{x}), & \text{if } x_2 < \tilde{s} x_1 \\ \psi_0^+(\mathbf{x}), & \text{if } x_2 > \tilde{s} x_1, \end{cases}$$

where ψ_0 is Lipschitz. Then, estimating $\psi - \psi_0^-$ in Ω^- using (3.63), where the polynomial decay is of degree $-(1+\beta)$ and so we can use calculations similar to (3.95), and then using Rankine-Hugoniot condition on $\mathcal{S}$, we obtain

$$|(\mathbf{x}^{in})' - (\mathbf{x}_0^{in})'| \leqslant C\varepsilon \quad \text{ on } [0, \infty), \quad \text{where } \; \mathbf{x}_0^{in}(y_2) = \frac{y_2}{\rho_0^+ u_{10}^+}(1, \tilde{s}_0).$$

Here $\mathbf{x}_0^{in}$ is the function $\mathbf{x}^{in}$ of the background solution.

Denote by $x_{20}^*(x_1; y_2)$ the function $x_2^*(x_1; y_2)$ of the background solution. We have $x_{20}^*(x_1; y_2) = \frac{y_2}{\rho_0^+ u_{10}^+}$ on $x_1 > \frac{y_2}{\rho_0^+ u_{10}^+ \tilde{s}_0}$ for each $y_2 \geqslant 0$. Thus $x_{20}^*(x_1; y_2)$ does not depend on x_1, so $x_{20}^*(x_1; y_2) = x_{20}^*(y_2)$. Then, denoting

$$g(x_1; y_2) = x_2^*(x_1; y_2) - x_{20}^*(y_2),$$

we obtain that g satisfies

$$\begin{aligned}&\partial_{x_1} g(x_1; y_2) = w(x_1, x_2^*(x_1; y_2)),\\ &|g(x_1^{in}(y_2); y_2)| \leqslant C\varepsilon.\end{aligned} \tag{3.98}$$

Then, from (3.95) and (3.97), we get $|(x_2^\infty)' - (x_{20}^\infty)'| \leqslant C\varepsilon$, where $(x_{20}^\infty)'(y_2) = (x_{20}^*)'(y_2) = \frac{1}{\rho_0^+ u_{10}^+}$. Thus we obtain

$$(x_2^\infty)' \geqslant \frac{1}{2\rho_0^+ u_{10}^+} \qquad \text{on } [0,\infty)$$

if ε is small. In particular, noting that $x_2^\infty(0) = 0$ since ∂W is a streamline corresponding to $\psi = 0$ and $\lim_{x_1\to\infty} b_{rot}(x_1) = 0$ by (3.63), we obtain $x_2^\infty([0,\infty)) = [0,\infty)$. Then there exists the inverse to $x_2^\infty(\cdot)$ function $y_2^* : [0,\infty) \to [0,\infty)$ and $y_2^* \in C^{2,\alpha}([0,\infty)$ with $y_2^*(0) = 0$ and $(y_2^*)' \geqslant \frac{1}{C} > 0$.

Then we show that defining $U^\infty(x_2) = \mathcal{U}^\infty(y_2^*(x_2))$, we obtain (3.64) from (3.67).

For more details, see Chen-Chen-Feldman [27].

Remarks. ??

1. The nozzle problem (infinite nozzle, uniform nozzle) for the 2-D full Euler equations; [25] (2007)
2. Other results: Wedges, Nozzles,... Some further works on transonic shocks in nozzles include the study of shocks in de Laval nozzles [79], and uniqueness of transonic shocks [63].
3. Euler-Poisson: Bae, Park, ...

4. Two-Dimensional Transonic Shocks and Free Boundary Problems for the Self-Similar Euler Equations for Potential Flow

In §2–§3, we have discussed free boundary problems for steady transonic shock solutions of the compressible Euler equations. Now we discuss free boundary problems for time-dependent solutions.

General time-dependent solutions of the compressible Euler equations are of extremely complicated structure, so that very few results are currently available. On the other hand, many fundamental physical phenomena, including shock reflection/diffraction, are determined by time-dependent solutions of self-similar structure. In this section, we focus on this case. More precisely, we describe transonic shocks and free boundary problems for self-similar shock reflection/diffraction for the Euler equations for potential flow.

The compressible potential flow is governed by the conservation law of mass and the Bernoulli law for $\mathbb{R}^+ := (0,\infty)$ and $\mathbf{x} \in \mathbb{R}^2$:

$$\partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \nabla_{\mathbf{x}} \Phi) = 0, \tag{4.1}$$

$$\partial_t \Phi + \frac{1}{2}|\nabla_{\mathbf{x}}\Phi|^2 + h(\rho) = B \tag{4.2}$$

for density ρ and velocity potential Φ, where B is the Bernoulli constant, and $h(\rho)$ is given by

$$h(\rho) = \frac{\rho^{\gamma-1} - 1}{\gamma - 1} \qquad \text{for the adiabatic exponent } \gamma > 1. \tag{4.3}$$

By (4.2)–(4.3), ρ can be expressed as

$$\rho(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi) = h^{-1}(B - \partial_t\Phi - \frac{1}{2}|\nabla_{\mathbf{x}}\Phi|^2). \tag{4.4}$$

Then system (4.1)–(4.2) can be rewritten as the following second-order nonlinear wave equation:

$$\partial_t \rho(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi) + \nabla_{\mathbf{x}} \cdot \big(\rho(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi)\nabla_{\mathbf{x}}\Phi\big) = 0 \tag{4.5}$$

with $\rho(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi)$ determined by (4.4).

Note that system (4.1)–(4.2) is invariant under the self-similar scaling:

$$(t, \mathbf{x}) \to (\alpha t, \alpha\mathbf{x}), \quad (\rho, \Phi) \to (\rho, \frac{\Phi}{\alpha}) \qquad \text{for } \alpha \neq 0, \tag{4.6}$$

and thus it admits self-similar solutions in the form of

$$\rho(t, \mathbf{x}) = \rho(\boldsymbol{\xi}), \quad \Phi(t, \mathbf{x}) = t\phi(\boldsymbol{\xi}) \qquad \text{for } \boldsymbol{\xi} = \frac{\mathbf{x}}{t}. \tag{4.7}$$

Then the pseudo-potential function

$$\varphi(\boldsymbol{\xi}) = \phi(\boldsymbol{\xi}) - \frac{1}{2}|\boldsymbol{\xi}|^2$$

and the density function $\rho(\boldsymbol{\xi})$ satisfy the following Euler equations for self-similar solutions:

$$\operatorname{div}(\rho D\varphi) + 2\rho = 0, \qquad \frac{\rho^{\gamma-1} - 1}{\gamma - 1} + \big(\frac{1}{2}|D\varphi|^2 + \varphi\big) = B, \tag{4.8}$$

where the divergence div and gradient D are with respect to $\boldsymbol{\xi} \in \mathbb{R}^2$. From this, we obtain the following equation for the pseudo-potential function $\varphi(\boldsymbol{\xi})$:

$$\operatorname{div}(\rho(|D\varphi|^2, \varphi)D\varphi) + 2\rho(|D\varphi|^2, \varphi) = 0 \tag{4.9}$$

for

$$\rho(|D\varphi|^2, \varphi) = \big(B_0 - \theta(|D\varphi|^2 + 2\varphi)\big)^{\frac{1}{\gamma-1}}, \tag{4.10}$$

where $B_0 = (\gamma - 1)B + 1$ and $\theta = \frac{\gamma-1}{2}$. Equation (4.9) written in the non-divergence form is

$$(c^2 - \varphi_{\xi_1}^2)\varphi_{\xi_1\xi_1} - 2\varphi_{\xi_1}\varphi_{\xi_2}\varphi_{\xi_1\xi_2} + (c^2 - \varphi_{\xi_2}^2)\varphi_{\xi_2\xi_2} + 2c^2 - |D\varphi|^2 = 0, \tag{4.11}$$

where the sonic speed $c = c(|D\varphi|^2, \varphi)$ is determined by

$$c^2(|D\varphi|^2, \varphi) = \rho^{\gamma-1}(|D\varphi|^2, \varphi) = B_0 - (\gamma - 1)\big(\frac{1}{2}|D\varphi|^2 + \varphi\big). \tag{4.12}$$

Another form of (4.11), which uses both the potential ϕ and the pseudo-potential φ is:

$$(c^2 - \varphi_{\xi_1}^2)\phi_{\xi_1\xi_1} - 2\varphi_{\xi_1}\varphi_{\xi_2}\phi_{\xi_1\xi_2} + (c^2 - \varphi_{\xi_2}^2)\phi_{\xi_2\xi_2} = 0. \tag{4.13}$$

Equation (4.9) is a nonlinear PDE of mixed elliptic-hyperbolic type. It is elliptic at $\boldsymbol{\xi}$ if and only if

$$|D\varphi| < c(|D\varphi|^2, \varphi) \qquad \text{at } \boldsymbol{\xi}, \tag{4.14}$$

and is hyperbolic if the opposite inequality holds. This can be seen more clearly from the rotational invariance of (4.11), by fixing $\boldsymbol{\xi}$ and choosing coordinates (ξ_1, ξ_2) so that ξ_1 is along the direction of $D\varphi(\boldsymbol{\xi})$.

Moreover, from (4.11)–(4.12), equation (4.9) satisfies the Galilean invariance property: If $\varphi(\boldsymbol{\xi})$ is a solution, then its shift $\varphi(\boldsymbol{\xi} - \boldsymbol{\xi}_0)$ for any constant vector $\boldsymbol{\xi}_0$ is also a solution. Furthermore, $\varphi(\boldsymbol{\xi}) + const.$ is a solution of (4.9) with adjusted constant B correspondingly in (4.10), (4.12).

One class of solutions of (4.9) is that of *constant states* that are the solutions with constant velocities $\mathbf{v} = (u, v)$. This implies that the pseudo-potential of a constant state satisfies $D\varphi = \mathbf{v} - \boldsymbol{\xi}$ so that

$$\varphi(\boldsymbol{\xi}) = -\frac{1}{2}|\boldsymbol{\xi}|^2 + \mathbf{v} \cdot \boldsymbol{\xi} + C, \tag{4.15}$$

where C is a constant. For such φ, the expressions in (4.10), (4.12) imply that the density and sonic speed are positive constants ρ and c, *i.e.*, independent of $\boldsymbol{\xi}$. Then, from (2.4) and (4.15), the ellipticity condition for the constant state is

$$|\boldsymbol{\xi} - \mathbf{v}| < c.$$

Thus, for a constant state $\mathbf{v}$, equation (4.9) is elliptic inside the *sonic circle*, with center $\mathbf{v}$ and radius c, and hyperbolic outside this circle.

We also note that, if density ρ is a constant, then the solution is a constant state; that is, the corresponding pseudo-potential φ is of form (4.15).

Since the problem involves transonic shocks, we have to consider weak solutions of equation (4.9), which admits shocks. As in [35], it is defined in the distributional sense.

Definition 4.1. *A function $\varphi \in W^{1,1}_{\rm loc}(\Omega)$ is called a weak solution of* (4.9) *if*

(i) $\rho_0^{\gamma-1} - (\gamma-1)(\varphi + \frac{1}{2}|D\varphi|^2) \geqslant 0$ *a.e. in* Ω;

(ii) $(\rho(|D\varphi|^2, \varphi), \rho(|D\varphi|^2, \varphi)|D\varphi|) \in (L^1_{loc}(\Omega))^2$;

(iii) *For every* $\zeta \in C^\infty_{\rm c}(\Omega)$,

$$\int_\Omega \big(\rho(|D\varphi|^2, \varphi) D\varphi \cdot D\zeta - 2\rho(|D\varphi|^2, \varphi)\zeta\big) \mathrm{d}\boldsymbol{\xi} = 0. \tag{4.16}$$

A shock is a curve across which $D\varphi$ is discontinuous. If Ω^+ and $\Omega^- (:= \Omega \backslash \overline{\Omega^+})$ are two nonempty open subsets of a domain $\Omega \subset \mathbb{R}^2$, and $S := \partial\Omega^+ \cap \Omega$ is a C^1-curve where $D\varphi$ has a jump, then $\varphi \in W^{1,1}_{\rm loc} \cap C^1(\Omega^\pm \cup S) \cap C^2(\Omega^\pm)$ is a global weak solution of (4.9) in Ω if and only if φ is in $W^{1,\infty}_{\rm loc}(\Omega)$ and satisfies equation (4.9) and the Rankine-Hugoniot condition on $\mathcal{S}$:

$$\rho(|D\varphi|^2, \varphi) D\varphi \cdot \boldsymbol{\nu}|_{\Omega^+ \cap S} = \rho(|D\varphi|^2, \varphi) D\varphi \cdot \boldsymbol{\nu}|_{\Omega^- \cap S}. \tag{4.17}$$

Note that the condition, $\varphi \in W^{1,\infty}_{\rm loc}(\Omega)$, requires that

$$\varphi_{\Omega^+ \cap S} = \varphi_{\Omega^- \cap S}, \tag{4.18}$$

which is consistent with $\mathrm{curl}(\nabla\varphi) = 0$ in the distributional sense. *The front, $\mathcal{S}$, is called a shock if density ρ increases in the pseudo-flow direction across $\mathcal{S}$, i.e., in the direction of* $D\varphi|_{\Omega^+ \cap S}$. A piecewise smooth solution whose discontinuities are all shocks is called an *entropy solution.*

4.1. **von Neumann's Problem for Shock Reflection-Diffraction.** We now describe von Neumann's problem proposed for mathematical analysis first in [118–120]. When a vertical planar shock perpendicular to the flow direction x_1 and separating two uniform states (0) and (1), with constant velocities $(u_0, v_0) = (0, 0)$ and $(u_1, v_1) = (u_1, 0)$ and constant densities $\rho_1 > \rho_0$ (state (0) is ahead or to the right of the shock, and state (1) is behind the shock), hits a symmetric wedge:

$$W := \{(x_1, x_2) : |x_2| < x_1 \tan\theta_{\rm w}, x_1 > 0\}$$

head on at time $t = 0$, a reflection-diffraction process takes place when $t > 0$. Then a fundamental question is what types of wave patterns of reflection-diffraction configurations may be formed around the wedge. The complexity of reflection-diffraction configurations was first reported by Ernst Mach [93] in 1878, who first observed two patterns of reflection-diffraction configurations: Regular reflection (two-shock configuration; see Figs. 4.1–4.2) and Mach reflection (three-shock/one-vortex-sheet configuration); also see [9, 37, 56, 116]. The issues remained dormant until the 1940s when John von Neumann [118–120], as well as other mathematical/experimental scientists (*cf.* [9, 37, 56, 66, 116] and the references cited therein) began extensive research into all aspects of shock reflection-diffraction phenomena, due to its importance in applications. It has been found that the situations are much more complicated than what Mach originally observed: The Mach reflection can be further divided into more specific sub-patterns, and various other patterns of shock reflection-diffraction configurations may occur such as the double Mach reflection, the von Neumann reflection, and the Guderley reflection; see [9, 37, 56, 66, 116] and the references cited therein. Then the fundamental scientific issues include:

(i) Structure of the shock reflection-diffraction configurations;

(ii) Transition criteria between the different patterns of shock reflection-diffraction configurations;

(iii) Dependence of the patterns upon the physical parameters such as the wedge angle $\theta_{\rm w}$, the incident-shock-wave Mach number, and the adiabatic exponent $\gamma > 1$.

In particular, several transition criteria between the different patterns of shock reflection-diffraction configurations have been proposed, including the sonic conjecture and the detachment conjecture by von Neumann [118–120].

Careful asymptotic analysis has been made for various reflection-diffraction configurations in Lighthill [86, 87], Keller-Blank [73], Hunter-Keller [71], Harabetian [70], Morawetz [99], and the references cited therein; also see Glimm-Majda [66]. Large or small scale numerical simulations have been also made; *cf.* [9,66,122] and the references cited therein. However, most of the fundamental issues for shock reflection-diffraction phenomena have not been understood, especially the global structure and transition between the different patterns of shock reflection-diffraction configurations. This is partially because physical and numerical experiments are hampered by many difficulties and have not yielded clear transition criteria between the different patterns. In particular, numerical dissipation or physical viscosity smear the shocks and cause boundary layers that interact with the reflection-diffraction patterns and can cause spurious Mach steams; *cf.* [122]. Furthermore, some different patterns occur when the wedge angles are only fractions of a degree apart, a resolution even by sophisticated experiments has not been able to reach (*cf.* [9, 92]). For this reason, it is almost impossible to distinguish experimentally between the sonic and detachment criteria, as pointed out in [9]. In this regard, the necessary approach to understand fully the shock reflection-diffraction phenomena, especially the transition criteria, is via rigorous mathematical analysis. To achieve this, it is essential to formulate the shock reflection-diffraction problem as a free boundary problem and establish the global existence, regularity, and structural stability of its solution.

Mathematically, the shock reflection-diffraction problem is a two-dimensional lateral Riemann problem in domain $\mathbb{R}^2\backslash\overline{W}$.

Problem 4.2 (Two-Dimensional Lateral Riemann Problem)**.** *Piecewise constant initial data, consisting of state* (0) *on* $\{x_1 > 0\}\backslash\bar{W}$ *and state* (1) *on* $\{x_1 < 0\}$ *connected by a shock at* $x_1 = 0$*, are prescribed at* $t = 0$*. Seek a solution of the Euler system* (4.1)*–*(4.2) *for* $t \geqslant 0$ *subject to these initial data and the boundary condition* $\nabla\Phi\cdot\boldsymbol{\nu} = 0$ *on* ∂W*.*

In order to define the notion of weak solutions of Problem 4.2, it is noted that the boundary condition can be written as $\rho\nabla\Phi\cdot\boldsymbol{\nu} = 0$ on ∂W, which is spatial conormal to the equation (4.5). Then we have

Definition 4.3 (Weak Solutions of Problem 4.2)**.** *A function* $\Phi \in W^{1,1}_{\rm loc}(\mathbb{R}_+ \times (\mathbb{R}^2\backslash W))$ *is called a weak solution of* Problem 4.2 *if* Φ *satisfies the following properties*:

(i) $B_0 - \big(\partial_t\Phi + \frac{1}{2}|\nabla_{\mathbf{x}}\Phi|^2\big) \geqslant h(0+)$ *a.e. in* $\mathbb{R}_+ \times (\mathbb{R}^2\backslash W)$;

(ii) *For* $\rho(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi)$ *determined by* (4.4)*,*

$$(\rho(\partial_t\Phi, |\nabla_{\mathbf{x}}\Phi|^2), \rho(\partial_t\Phi, |\nabla_{\mathbf{x}}\Phi|^2)|\nabla_{\mathbf{x}}\Phi|) \in (L^1_{\rm loc}(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}^2\backslash W}))^2;$$

(iii) *For every* $\zeta \in C^\infty_c(\overline{\mathbb{R}_+} \times \mathbb{R}^2)$,

$$\int_0^\infty \int_{\mathbb{R}^2\backslash W} \Big(\rho(\partial_t\Phi, |\nabla_{\mathbf{x}}\Phi|^2)\partial_t\zeta + \rho(\partial_t\Phi, |\nabla_{\mathbf{x}}\Phi|^2)\nabla\Phi\cdot\nabla\zeta\Big)\,{\rm d}\mathbf{x}{\rm d}t + \int_{\mathbb{R}^2\backslash W}\rho(0,\mathbf{x})\zeta(0,\mathbf{x}){\rm d}\mathbf{x} = 0,$$

where

$$\rho|_{t=0} = \begin{cases} \rho_0 & \text{for } |x_2| > x_1\tan\theta_{\rm w},\ x_1 > 0, \\ \rho_1 & \text{for } x_1 < 0. \end{cases}$$

Remark 4.4. *Since* ζ *does not need to be zero on* $\partial\Lambda$*, the integral identity in Definition* 4.3 *is a weak form of equation* (4.5) *and the boundary condition* $\rho\nabla\Phi\cdot\boldsymbol{\nu} = 0$ *on* ∂W*.*

Notice that Problem 4.2 is invariant under scaling (4.6), so it admits self-similar solutions determined by equation (4.9) with (4.10), along with the appropriate boundary conditions, through (4.7). We now show how such solutions in self-similar coordinates $\boldsymbol{\xi} = (\xi_1, \xi_2) = \frac{\mathbf{x}}{t}$ can be constructed.

First, by the symmetry of the problem with respect to the ξ_1–axis, we consider only the upper half-plane $\{\xi_2 > 0\}$ and prescribe the boundary condition: $\varphi_{\boldsymbol{\nu}} = 0$ on the symmetry line $\{\xi_2 = 0\}$. Note that state (1) satisfies this condition. Then Problem 4.2 is reformulated as a boundary value problem in unbounded domain

$$\Lambda := \mathbb{R}^2_+ \backslash \{\boldsymbol{\xi} : |\xi_2| \leqslant \xi_1 \tan\theta_{\mathrm{w}}, \xi_1 > 0\}$$

in the self-similar coordinates $\boldsymbol{\xi} = (\xi_1, \xi_2)$, where $\mathbb{R}^2_+ = \mathbb{R}^2 \cap \{\xi_2 > 0\}$. The incident shock in the self-similar coordinates is the half-line $S_0 = \{\boldsymbol{\xi} = \boldsymbol{\xi}_1^0\} \cap \Lambda$, where

$$\xi_1^0 = \rho_1 \sqrt{\frac{2(c_1^2 - c_0^2)}{(\gamma - 1)(\rho_1^2 - \rho_0^2)}} = \frac{\rho_1 u_1}{\rho_1 - \rho_0}, \tag{4.19}$$

which is determined by the Rankine-Hugoniot conditions between states (0) and (1) on S_0. Now Problem 4.2 for self-similar solutions is:

Problem 4.5 (Boundary Value Problem)**.** *Seek a solution φ of equation* (4.9)–(4.10) *in the self-similar domain Λ with the slip boundary condition $D\varphi \cdot \boldsymbol{\nu}|_{\partial\Lambda} = 0$ on the wedge boundary $\partial\Lambda$ and the asymptotic boundary condition at infinity*:

$$\varphi \to \bar{\varphi} = \begin{cases} \varphi_0 & \text{for } \xi_1 > \xi_1^0, \xi_2 > \xi_1 \tan\theta_{\mathrm{w}}, \\ \varphi_1 & \text{for } \xi_1 < \xi_1^0,\ \xi_2 > 0, \end{cases} \qquad \text{when } |\boldsymbol{\xi}| \to \infty,$$

where $\varphi_0 = -\frac{1}{2}\boldsymbol{\xi}|^2$ *and* $\varphi_1 = -\frac{1}{2}|\boldsymbol{\xi}|^2 + u_1(\xi_1 - \xi_1^0)$.

A weak solution of Problem 4.5 is defined by taking $\Omega = \Lambda$ in Definition 4.1 and using $\zeta \in C_c^\infty(\mathbb{R}^2)$ in Definition 4.1(iii) to take into account the boundary condition, which can be written in the conormal form $\rho D\varphi \cdot \boldsymbol{\nu} = 0$ on $\partial\Lambda$; see Remark 4.4.

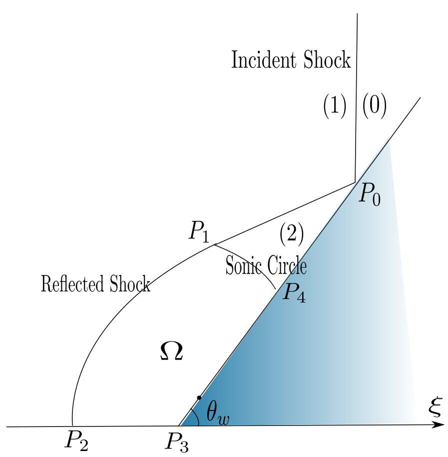

FIGURE 4.1. Supersonic regular shock reflection-diffraction configuration

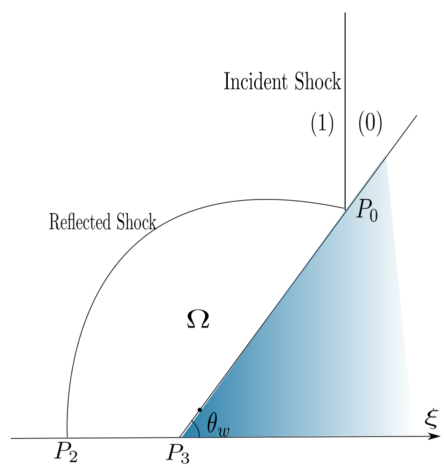

FIGURE 4.2. Subsonic regular shock reflection-diffraction configuration

If a solution has one of the regular shock reflection-diffraction configurations as shown in Figs. 4.1–4.2, and if its pseudo-potential φ is smooth in the subregion $\mathcal{D}$ between the wedge and the reflected shock, then it should satisfy both the slip boundary condition on the wedge and the Rankine-Hugoniot conditions with state (1) across the flat shock $S_1 = \{\varphi_1 = \varphi_2\}$, which passes through point P_0 where the incident shock meets the wedge boundary. We define the uniform state (2) with pseudo-potential $\varphi_2(\boldsymbol{\xi})$ such that

$$\varphi_2(P_0) = \varphi(P_0), \qquad D\varphi_2(P_0) = \lim_{P\to P_0,\, P\in\mathcal{D}} D\varphi(P).$$

Then the constant density ρ_2 of state (2) is equal to $\rho(|D\varphi|^2, \varphi)(P_0)$ defined by (4.9):

$$\rho_2 = \rho(|D\varphi|^2, \varphi)(P_0).$$

From the properties of φ discussed above, it follows that $D\varphi_2 \cdot \boldsymbol{\nu} = 0$ on the wedge boundary and the Rankine-Hugoniot conditions (4.17)–(4.18) hold on the flat shock $S_1 = \{\varphi_1 = \varphi_2\}$ between states (1) and (2), which passes through P_0. In particular, φ_2 satisfies the following three conditions at P_0:

$$\begin{aligned} &D\varphi_2 \cdot \boldsymbol{\nu}_{\rm w} = 0, \qquad \varphi_2 = \varphi_1, \qquad \rho(|D\varphi_2|^2, \varphi_2) D\varphi_2 \cdot \boldsymbol{\nu}_{S_1} = \rho_1 D\varphi_1 \cdot \boldsymbol{\nu}_{S_1}, \\ &\text{for } \ \boldsymbol{\nu}_{S_1} = \frac{D(\varphi_1 - \varphi_2)}{|D(\varphi_1 - \varphi_2)|}, \end{aligned} \tag{4.20}$$

where $\boldsymbol{\nu}_{\rm w}$ is the outward normal to the wedge boundary.

Solution φ, and correspondingly state (2), can be either supersonic or subsonic at P_0. This determines the supersonic or subsonic type of regular shock reflection-diffraction configurations. Regular reflection solution in the supersonic region is expected to consist of the constant states separated by straight shocks, in some cases this is proved, cf. [106, Theorem 4.1]. Then, when state (2) is supersonic at P_0, the constant state (2), extended up to arc P_1P_4 of the sonic circle of state (2) between the wall and the straight shock $P_0P_1 \subset S_1$ separating it from state (1), as shown in Fig. 4.1, is expected to be a part of the regular reflection configuration. The supersonic regular shock reflection-diffraction configuration on Fig. 4.1 consists of three uniform states (0), (1), (2), and a non-uniform state in domain $\Omega = P_1P_2P_3P_4$, where the equation (4.9) is elliptic. The reflected shock $P_0P_1P_2$ has a straight part P_0P_1. The elliptic domain Ω is separated from the hyperbolic region $P_0P_1P_4$ of state (2) by the sonic arc P_1P_4 which lies on the sonic circle of state (2), and the ellipticity in Ω degenerates on the sonic arc P_1P_4. The subsonic regular shock reflection-diffraction configuration as shown in Fig. 4.2 consists of two uniform states (0) and (1), and a non-uniform state in domain $\Omega = P_0P_2P_3$, where the equation is elliptic, and $\varphi_{|\Omega}(P_0) = \varphi_2(P_0)$ and $D(\varphi_{|\Omega})(P_0) = D\varphi_2(P_0)$.

For the supersonic regular shock reflection-diffraction configurations in Fig. 4.1, we use $\Gamma_{\rm sonic}$, $\Gamma_{\rm shock}$, $\Gamma_{\rm wedge}$, and $\Gamma_{\rm sym}$ for the sonic arc P_1P_4, curved part of the reflected shock P_1P_2, wedge boundary P_3P_4, and symmetry line segment P_2P_3, respectively.

For the subsonic regular shock reflection-diffraction configurations in Fig. 4.2, $\Gamma_{\rm shock}$, $\Gamma_{\rm wedge}$, and $\Gamma_{\rm sym}$ denote P_0P_2, P_0P_3, and P_2P_3, respectively. We unify the notations with the supersonic reflection case by introducing points P_1 and P_4 for the subsonic reflection case as

$$P_1 := P_0, \quad P_4 := P_0, \quad \overline{\Gamma_{\rm sonic}} := \{P_0\}. \tag{4.21}$$

The corresponding solution for $\theta_{\rm w} = \frac{\pi}{2}$ is called *normal reflection*. In this case, the incident shock normally reflects from the flat wall, see Fig. 4.3. The reflected shock is also a plane $\{\xi = \bar{\xi}_1\}$, where $\bar{\xi}_1 < 0$.

From the discussion above, it follows that a necessary condition for the existence of a regular reflection solution is the existence of the uniform state (2) with pseudo-potential φ_2 determined by the boundary condition $D\varphi_2 \cdot \boldsymbol{\nu} = 0$ on the wedge and the Rankine-Hugoniot conditions (4.17)–(4.18) across the flat shock $S_1 = \{\varphi_1 = \varphi_2\}$ separating it from state (1), and satisfying the entropy conditions $\rho_2 > \rho_1$. These conditions lead to the system of algebraic equations (4.20) for the constant velocity (u_2, v_2) and density ρ_2 of state (2). System (4.20) has solutions for some but not all of the wedge angles. More

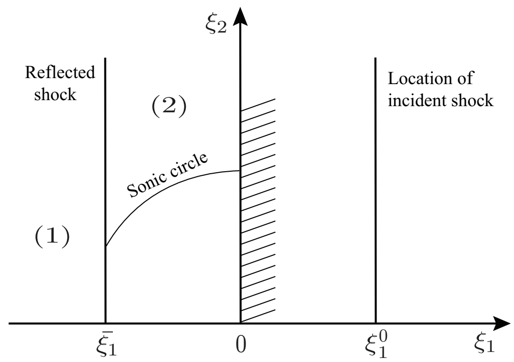

FIGURE 4.3. Normal reflection configuration

specifically, for any fixed densities $0 < \rho_0 < \rho_1$ of states (0) and (1), there exist a sonic angle $\theta_{\rm w}^{\rm s}$ and a detachment angle $\theta_{\rm w}^{\rm d}$ satisfying

$$0 < \theta_{\rm w}^{\rm d} < \theta_{\rm w}^{\rm s} < \frac{\pi}{2}$$

such that the algebraic system (4.20) has two solutions for each $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$, which become equal when $\theta_{\rm w} = \theta_{\rm w}^{\rm d}$. Thus, for each $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$, there exist two states (2), called weak and strong, with densities $\rho_2^{\rm weak} < \rho_2^{\rm strong}$. The weak state (2) is supersonic at the reflection point $P_0(\theta_{\rm w})$ for $\theta_{\rm w} \in (\theta_{\rm w}^{\rm s}, \frac{\pi}{2})$, sonic for $\theta_{\rm w} = \theta_{\rm w}^{\rm s}$, and subsonic for $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \hat{\theta}_{\rm w}^{\rm s})$ for some $\hat{\theta}_{\rm w}^{\rm s} \in (\theta_{\rm w}^{\rm d}, \theta_{\rm w}^{\rm s}]$. The strong state (2) is subsonic at $P_0(\theta_{\rm w})$ for all $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$.

There had been a long debate to determine which of the two states (2) for $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$, weak or strong, is physical for the local theory; see [9,37,56] and the references cited therein. It was conjectured that the strong shock reflection-diffraction configuration would be non-physical; indeed, it is shown as in Chen-Feldman [35, 37] that the weak shock reflection-diffraction configuration tends to the unique normal reflection in Fig. 4.3, but the strong reflection-diffraction configuration does not, when the wedge angle $\theta_{\rm w}$ tends to $\frac{\pi}{2}$. The entropy condition and the definition of weak and strong states (2) imply that $0 < \rho_1 < \rho_2^{\rm weak} < \rho_2^{\rm strong}$, which shows that the strength of the corresponding reflected shock near P_0 in the weak shock reflection-diffraction configuration is relatively weak, compared to the other shock given by the strong state (2).

If the weak state (2) is supersonic, the propagation speeds of the solution are finite, and state (2) is completely determined by the local information: state (1), state (0), and the location of point P_0. That is, any information from the reflection-diffraction region, especially the disturbance at corner P_3, cannot travel towards the reflection point P_0. However, if it is subsonic, the information can reach P_0 and interact with it, potentially altering a different reflection-diffraction configuration. This argument motivated the following conjecture by von Neumann in [118, 119]:

The Sonic Conjecture: *There exists a supersonic regular shock reflection-diffraction configuration when $\theta_{\rm w} \in (\theta_{\rm w}^{\rm s}, \frac{\pi}{2})$ for $\theta_{\rm w}^{\rm s} > \theta_{\rm w}^{\rm d}$. That is, the supersonicity of the weak state* (2) *implies the existence of a supersonic regular reflection solution, as shown in Fig.* 4.1.

Another conjecture is that global regular shock reflection-diffraction configuration is possible whenever the local regular reflection at the reflection point is possible:

The von Neumanm Detachment Conjecture: *There exists a regular shock reflection-diffraction configuration for any wedge angle $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$. That is, the existence of state* (2) *implies the existence of a regular reflection solution, as shown in Figs.* 4.1–4.2.

It is clear that the supersonic/subsonic regular shock reflection-diffraction configurations are not possible without a local two-shock configuration at the reflection point on the wedge, so the detachment conjecture is the weakest possible criterion for the existence of supersonic/subsonic regular shock reflection-diffraction configurations.

We now show how the solutions of regular shock reflection-diffraction configurations can be constructed to solve the von Neumann's conjectures. Note that the weak state (2) is obtained from the algebraic conditions described above, which determines point P_0, line S_1, and the sonic arc P_1P_4 in the case when the weak state (2) is supersonic at P_0. Thus, the unknowns are domain Ω (or equivalently, the curved part of the reflected shock $\Gamma_{\rm shock}$) and the pseudo-potential φ in Ω. Then, from (4.17)–(4.18), in order to construct a solution of Problem 4.5 of the supersonic or subsonic regular shock reflection-diffraction configuration, it suffices to solve the following problem:

Problem 4.6 (Free Boundary Problem). *For $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$, find a free boundary (curved reflected shock) $\Gamma_{\rm shock} \subset \Lambda \cap \{\xi_1 < \xi_{1P_1}\}$ ($\Gamma_{\rm shock} = P_1P_2$ on Fig. 4.1 and $\Gamma_{\rm shock} = P_0P_2$ on Fig. 4.2) and a function φ defined in region Ω as shown in Figs. 4.1–4.2 such that*

(i) *Equation (4.9) is satisfied in Ω, and the equation is strictly elliptic for φ in $\overline{\Omega}\backslash\overline{\Gamma_{\rm sonic}}$,*
(ii) *$\varphi = \varphi_1$ and $\rho D\varphi \cdot \boldsymbol{\nu}_s = D\varphi_1 \cdot \boldsymbol{\nu}_{\rm s}$ on the free boundary $\Gamma_{\rm shock}$,*
(iii) *$\varphi = \varphi_2$ and $D\varphi = D\varphi_2$ on P_1P_4 in the supersonic case as shown in Fig. 4.1 and at P_0 in the subsonic case as shown in Fig. 4.1,*
(iv) *$D\varphi \cdot \boldsymbol{\nu}_{\rm w} = 0$ on $\Gamma_{\rm wedge}$, and $D\varphi \cdot \boldsymbol{\nu}_{\rm sym} = 0$ on $\Gamma_{\rm sym}$,*

where $\boldsymbol{\nu}_{\rm s}$, $\boldsymbol{\nu}_{\rm w}$, and $\boldsymbol{\nu}_{\rm sym}$ are the interior unit normals to Ω on $\Gamma_{\rm shock}$, $\Gamma_{\rm wedge}$, and $\Gamma_{\rm sym}$, respectively.

Indeed, if φ is a solution of Problem 4.6, define its extension from Ω to Λ by setting:

$$\varphi = \begin{cases} \varphi_0 & \text{for } \xi_1 > \xi_1^0 \text{ and } \xi_2 > \xi_1 \tan\theta_{\rm w}, \\ \varphi_1 & \text{for } \xi_1 < \xi_1^0 \text{ and above curve } P_0P_1P_2, \\ \varphi_2 & \text{in region } P_0P_1P_4, \end{cases} \tag{4.22}$$

where we have used the notational convention (4.21) for the subsonic reflection case, in which region $P_0P_1P_4$ is one point and curve $P_0P_1P_2$ is P_0P_2; see Figs. 4.1 and 4.2. Also, ξ_1^0 used in (4.22) is the location of the incident shock (*cf.* (4.19)), and the extension by (4.22) is well-defined because of the requirement that $\Gamma_{\rm shock} \subset \Lambda \cap \{\xi_1 < \xi_{1P_1}\}$ in Problem 4.6.

Note that the conditions in Problem 4.6(ii) are the Rankine-Hugoniot conditions (4.17)–(4.18) on $\Gamma_{\rm shock}$ between $\varphi_{|\Omega}$ and φ_1. Since $\Gamma_{\rm shock}$ is a free boundary and equation (4.9) is strictly elliptic for φ in $\overline{\Omega}\backslash\overline{\Gamma_{\rm sonic}}$, then two conditions — the Dirichlet and oblique derivative conditions — on $\Gamma_{\rm shock}$ are consistent with one-phase free boundary problems for elliptic equations of second order (*cf.* [1, 3]).

In the supersonic case, the conditions in Problem 4.6(iii) are the Rankine-Hugoniot conditions on $\Gamma_{\rm sonic}$ between $\varphi_{|\Omega}$ and φ_2. Indeed, since state (2) is sonic on $\Gamma_{\rm sonic}$, then it follows from (4.17)–(4.18) that no gradient jump occurs on $\Gamma_{\rm sonic}$.

Then, if φ is a solution of Problem 4.6, its extension by (4.22) is a weak solution of Problem 4.5. From now on, we consider a solution of Problem 4.6 to be a function defined in Λ by extension via (4.22).

Since $\Gamma_{\rm sonic}$ is not a free boundary (its location is fixed), it is not possible in general to prescribe two conditions given in Problem 4.6(iii) on $\Gamma_{\rm sonic}$ for an elliptic equation of second order. In the iteration problem, we prescribe the condition: $\varphi = \varphi_2$ on $\Gamma_{\rm sonic}$, and then prove that $D\varphi = D\varphi_2$ on $\Gamma_{\rm sonic}$ by using the elliptic degeneracy on $\Gamma_{\rm sonic}$, as we describe below.

We observe that the key obstacle to the existence of regular shock reflection-diffraction configurations as conjectured by von Neumann [118, 119] is an additional possibility that, for some wedge angle

$\theta_{\rm w}^{\rm a} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$, shock P_0P_2 may attach to the wedge-vertex P_3, as observed by experimental results (*cf.* [116, Fig. 238]). To describe the conditions of such an attachment, we note that

$$\rho_1 > \rho_0, \qquad u_1 = (\rho_1 - \rho_0)\sqrt{\frac{2(\rho_1^{\gamma-1} - \rho_0^{\gamma-1})}{\rho_1^2 - \rho_0^2}}, \qquad c_1 = \rho_1^{\frac{\gamma-1}{2}}.$$

Then, for each ρ_0, there exists $\rho^{\rm c} > \rho_0$ such that

$$u_1 \leqslant c_1 \quad \text{if } \rho_1 \in (\rho_0, \rho^{\rm c}]; \qquad u_1 > c_1 \quad \text{if } \rho_1 \in (\rho^{\rm c}, \infty).$$

If $u_1 \leqslant c_1$, we can rule out the solution with a shock attached to the wedge vertex. This is based on the fact that, if $u_1 \leqslant c_1$, then the wedge vertex $P_3 = (0,0)$ lies within the sonic circle $\overline{B_{c_1}((u_1,0))}$ of state (1), and $\Gamma_{\rm shock}$ does not intersect $\overline{B_{c_1}((u_1,0))}$, as we show below.

If $u_1 > c_1$, there would be a possibility that the reflected shock could be attached to the wedge vertex as the experiments show (*e.g.*, [116, Fig. 238]).

Thus, in [35, 37], we have obtained the following results:

Theorem 4.1. *There are two cases:*

(i) *If ρ_0 and ρ_1 are such that $u_1 \leqslant c_1$, then the supersonic/subsonic regular reflection solution exists for each wedge angle $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$. That is, for each $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$, there exists a solution φ of* Problem 4.6 *such that*

$$\Phi(t, \mathbf{x}) = t\,\varphi(\frac{\mathbf{x}}{t}) + \frac{|\mathbf{x}|^{\mathbf{2}}}{2t} \qquad \text{for } \frac{\mathbf{x}}{t} \in \Lambda,\ t > 0$$

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$$\Phi(t, \mathbf{x}) = t\,\varphi(\frac{\mathbf{x}}{t}) + \frac{|\mathbf{x}|^2}{2t} \qquad \text{for } \frac{\mathbf{x}}{t} \in \Lambda,\ t > 0$$

with

$$\rho(t, \mathbf{x}) = \Big(\rho_0^{\gamma-1} - (\gamma-1)\big(\Phi_t + \frac{1}{2}|\nabla_{\mathbf{x}}\Phi|^2\big)\Big)^{\frac{1}{\gamma-1}}$$

is a global weak solution of Problem 4.2 *in the sense of Definition* 4.3 *satisfying the entropy condition; that is, $(\Phi, \rho)(t, \mathbf{x})$ is an entropy solution.*

(ii) *If ρ_0 and ρ_1 are so that $u_1 > c_1$, then there exists $\theta_{\rm w}^{\rm a} \in [\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$ such that the regular reflection solution exists for each wedge angle $\theta_{\rm w} \in (\theta_{\rm w}^{\rm a}, \frac{\pi}{2})$, and the solution is of self-similar structure described in* (i) *above. Moreover, if $\theta_{\rm w}^{\rm a} > \theta_{\rm w}^{\rm d}$, then, for the wedge angle $\theta_{\rm w} = \theta_{\rm w}^{\rm a}$, there exists an attached solution, i.e., φ is a solution of* Problem 4.6 *with $P_2 = P_3$.*

The type of regular shock reflection-diffraction configurations (supersonic as in Fig. 4.1 *or subsonic as in Fig.* 4.2*) is determined by the type of state* (2) *at P_0.*

(a) *For the supersonic and sonic reflection case, the reflected shock P_0P_2 is $C^{2,\alpha}$–smooth and its curved part P_1P_2 is C^∞ away from P_1. The solution satisfies $\varphi \in C^{1,\alpha}(\overline{\Omega}) \cap C^\infty(\Omega)$, and φ is $C^{1,1}$ across the sonic arc which is optimal; that is, φ is* not *C^2 across sonic arc.*

(b) *For the subsonic reflection case (Fig.* 4.2*), the reflected shock P_0P_2 and the solution in Ω is $C^{1,\alpha}$ near P_0 and P_3, and C^∞ away from $\{P_0, P_3\}$.*

Moreover, the regular reflection solution tends to the unique normal reflection (as in Fig. 4.3*) when the wedge angle $\theta_{\rm w}$ tends to $\frac{\pi}{2}$. In addition, for both supersonic and subsonic reflection cases,*

$$\varphi_1 > \varphi > \varphi_2 \qquad \text{in } \Omega. \tag{4.23}$$

Furthermore, φ is an admissible solution in the sense of Definition 4.10 *below, so that φ satisfies further properties listed in Definition* 4.10*.*

Theorem 4.1 is proved by solving Problem 4.6. The first results on the existence of global solutions of the free boundary problem (Problem 4.6) were obtained for the wedge angles sufficiently close to $\frac{\pi}{2}$ in Chen-Feldman [35]. Later, in Chen-Feldman [37], these results were extended up to the detachment angle as stated in Theorem 4.1. For this extension, the techniques developed in [35], notably the estimates near the sonic arc, were the starting point.

Case I: The Wedge Angles close to $\frac{\pi}{2}$. Let us first discuss the techniques in [35], where we employ the approach of Chen-Feldman [31] to develop an iteration scheme for constructing a global solution of Problem 4.6, when the wedge angle is close to $\frac{\pi}{2}$. For this case, the solutions are of the supersonic regular shock reflection-diffraction configuration as in Fig. 4.1. The general procedure is similar to the one described in §2.2, which can be presented in the following four steps:

1. Fix $\theta_{\rm w}$ sufficiently close to $\frac{\pi}{2}$ so that various constants in the argument can be controlled. The iteration set consists of functions defined on a region $\mathcal{D}$, where $\mathcal{D}$ contains all possible Ω for the fixed $\theta_{\rm w}$. Specifically, an important property of the regular shock reflection-diffraction configurations is (4.23), which implies that $\Omega \subset \{\varphi_2 < \varphi_1\}$; that is, Ω lies "below" line S_1 passing through P_0 and P_1 on Fig. 4.1. Note that, when $\theta_{\rm w}$ close to $\frac{\pi}{2}$, this line is close to the vertical reflected shock of normal reflection on Fig. 4.3. Then $\mathcal{D}$ is defined as a region bounded by S_1, $\Gamma_{\rm sonic} = P_1P_4$, $\Gamma_{\rm wedge} = P_3P_4$, and the symmetry line $\xi = 0$. The iteration set is a set of functions φ on $\mathcal{D}$, defined by $\varphi \geqslant \varphi_2$ on $\mathcal{D}$ and the bound of norm of $\varphi - \varphi_2$ on $\mathcal{D}$ in the scaled and weighted $C^{2,\alpha}$ space defined in (4.38) below. Such functions satisfy

$$\|\varphi - \varphi_2\|_{C^{1,\alpha}(\overline{\mathcal{D}})} \leqslant C(\frac{\pi}{2} - \theta_{\rm w}),$$

which is small when $\frac{\pi}{2} - \theta_w \ll 1$, and

$$\|\varphi - \varphi_2\|_{C^{1,1}(\overline{\mathcal{D}}\cap\mathcal{N}_\varepsilon(\Gamma_{\rm sonic}))} \leqslant C_1.$$

However, $\|\varphi - \varphi_2\|_{C^{1,1}(\overline{\mathcal{D}}\cap\mathcal{N}_\varepsilon(\Gamma_{\rm sonic}))}$ is not small even if $\frac{\pi}{2} - \theta_w$ is small; the reasons for that will be discussed below.

Given a function $\hat{\varphi}$ from the iteration set, we define domain $\Omega(\hat{\varphi}) := \{\hat{\varphi} < \varphi_1\}$ so that the iteration free boundary is $\Gamma_{\rm shock}(\hat{\varphi}) = \partial\Omega(\hat{\varphi}) \cap \mathcal{D}$. This is similar to (2.40), and the corresponding non-degeneracy similar to (2.39) in the present case is: $\partial_{\xi_1}(\varphi_1 - \varphi_2 - \phi) \geqslant u_1/2$ in $\mathcal{D}$ if $\|\phi\|_{C^1(\overline{\mathcal{D}})}$ and $\frac{\pi}{2} - \theta_w$ are small. Then we define the iteration equation by using form (4.13) of equation (4.9), by making an elliptic truncation (which is somewhat different from Step 1 in §2.2) and substituting $\hat{\varphi}$ in some terms of the coefficients of (4.13). The iteration boundary condition on $\Gamma_{\rm shock}(\hat{\varphi})$ is an oblique derivative condition obtained by combining two conditions in Problem 4.6(ii) and making some truncations. On $\Gamma_{\rm sonic}$, we prescribe $\varphi = \varphi_2$, *i.e.*, one of two conditions in Problem 4.6(iii). On $\Gamma_{\rm wedge}$ and $\Gamma_{\rm sym}(\hat{\varphi})$, we prescribe the conditions given in Problem 4.6(iv). The iteration map $\hat{\varphi} \to \varphi$ is defined by solving the iteration problem to obtain φ and then extending φ from $\Omega(\hat{\varphi})$ to $\mathcal{D}$.

The fundamental differences between the iteration procedure in the shock reflection-diffraction problem and the previous procedures on transonic shocks in the steady case in §2–§3 (such as [31,32,34,123] and further works) include:

(i) The procedures on steady transonic shocks in §2–§3 are for the perturbation case. In particular, the ellipticity of the iteration equation and the removal of the elliptic cutoff are achieved by making the iteration set sufficiently close to the background solution in C^1 or a stronger norm. For the regular reflection problem, this cannot be done because of elliptic degeneracy near the sonic arc.
(ii) Only one condition on $\Gamma_{\rm sonic}$ is prescribed; however, both $\varphi = \varphi_2$ and $D\varphi = D\varphi_2$ on $\Gamma_{\rm sonic}$ are required to be matched to obtain a global weak solution. This is resolved by using the elliptic degeneracy on $\Gamma_{\rm sonic}$.

2. In order to see the elliptic degeneracy on $\Gamma_{\rm sonic}$ more explicitly, we fix the wedge angle $\theta_{\rm w}$ and the corresponding pseudo-potential $\varphi_2 = \varphi_2^{(\theta_{\rm w})}$ of the weak state (2), and rewrite equation (4.11) in terms of the function:

$$\psi = \varphi - \varphi_2$$

in the following coordinates flattening $\Gamma_{\rm sonic}$:

$$x = c_2 - r, \qquad y = \theta - \theta_{\rm w},$$

where (r, θ) are the polar coordinates centered at center $O_2 = (u_2, v_2)$ of the sonic circle of state (2). Then

$$\Omega_\varepsilon := \Omega \cap \mathcal{N}_\varepsilon(\Gamma_{\rm sonic}) \subset \{x > 0\} \ \text{ for small } \varepsilon > 0, \qquad \Gamma_{\rm sonic} \subset \{x = 0\}.$$

Below we always assume that $\varphi \in C^{1,1}(\overline{\Omega}_\varepsilon)$ as in Theorem 4.1 for the supersonic case. Then, by the conditions in Problem 4.6(iii) and the definition of ψ,

$$\psi = 0 \qquad \text{on } \Gamma_{\rm sonic}, \tag{4.24}$$

$$D\psi = 0 \qquad \text{on } \Gamma_{\rm sonic}. \tag{4.25}$$

Moreover, we *apriori* assume that the solutions, φ, satisfy (4.23) in Ω to derive the required estimates of the solutions; with these estimates, we then construct such solutions. The heuristic motivation of (4.23) is the following: From Figs. 4.1–4.2, it appears that $\Gamma_{\rm shock}$ (and hence Ω) is located "below" line S_1, *i.e.*, in the half-plane $\{\varphi_1 > \varphi_2\}$. Thus, $\varphi = \varphi_1 > \varphi_2$ on $\Gamma_{\rm shock}$, and $\varphi_1 > \varphi_2 = \varphi$ on $\Gamma_{\rm sonic}$. Also, the potentials ϕ_1 and ϕ_2 of states (1) and (2) are linear functions, thus they satisfy equation (4.13) with coefficients determined by φ, considered as a linear equation for ϕ. Then, taking into account the inequalities on $\Gamma_{\rm shock}$ and $\Gamma_{\rm sonic}$ noted above, and the oblique boundary conditions on $\Gamma_{\rm wedge}$ and $\Gamma_{\rm sym}$, we obtain (4.23) by the maximum principle. Then, from (4.23), we have

$$\psi > 0 \qquad \text{in } \Omega. \tag{4.26}$$

The previous argument is heuristic, but the fact that it comes from the structure of the problem allows to include the condition that $\psi \geqslant 0$ in the definition of the iteration set and close the iteration argument for constructing the solutions within this set.

Equation (4.11) in $\Omega \cap \mathcal{N}_\varepsilon(\Gamma_{\rm sonic})$ for ψ in the (x, y)–coordinates is

$$\big(2x - (\gamma+1)\psi_x + O_1\big)\psi_{xx} + O_2\psi_{xy} + \big(\frac{1}{c_2} + O_3\big)\psi_{yy} - (1 + O_4)\psi_x + O_5\psi_y = 0, \tag{4.27}$$

where

$$\begin{aligned}
O_1(\nabla\psi, \psi, x) &= -\frac{x^2}{c_2} + \frac{\gamma+1}{2c_2}(2x - \psi_x)\psi_x - \frac{\gamma-1}{c_2}\big(\psi + \frac{1}{2(c_2-x)^2}\psi_y^2\big),\\
O_2(\nabla\psi, \psi, x) &= -\frac{2(\psi_x + c_2 - x)\psi_y}{c_2(c_2-x)^2}\\
O_3(\nabla\psi, \psi, x) &= \frac{1}{c_2(c_2-x)^2}\Big(x(2c_2 - x) - (\gamma-1)(\psi + (c_2-x)\psi_x + \frac{1}{2}\psi_x^2) - \frac{\gamma+1}{2(c_2-x)^2}\psi_y^2\Big),\\
O_4(\nabla\psi, \psi, x) &= \frac{1}{c_2 - x}\Big(x - \frac{\gamma-1}{c_2}\big(\psi + (c_2-x)\psi_x + \frac{1}{2}\psi_x^2 + \frac{(\gamma+1)\psi_y^2}{2(\gamma-1)(c_2-x)^2}\big)\Big),\\
O_5(\nabla\psi, \psi, x) &= -\frac{2(\psi_x + c_2 - x)\psi_y}{c_2(c_2-x)^3}.
\end{aligned} \tag{4.28}$$

From (4.24)–(4.25), and since $\psi \in C^{1,1}(\overline{\Omega}_\varepsilon)$, it follows that $|\psi(x,y)| \leqslant Cx^2$ and

$$|D\psi(x,y)| \leqslant Cx \qquad \text{in } \Omega_\varepsilon, \tag{4.29}$$

so that

$$|O_1(D\psi,\psi,x)| \leqslant N|x|^k, \quad |O_k(D\psi,\psi,x)| \leqslant N|x| \qquad \text{for } k=2,\dots,5, \tag{4.30}$$

which shows that $O_k(\nabla\psi,\psi,x)$ are small perturbations of the leading terms of equation (4.27) in $\Omega_\varepsilon = \Omega \cap \mathcal{N}_\varepsilon(\Gamma_{\rm sonic})$. Also, if (4.29) holds, equation (4.27) is strictly elliptic in $\overline{\Omega}_\varepsilon \backslash \overline{\Gamma_{\rm sonic}}$ if

$$\psi_x(x,y) \leqslant \frac{2\mu}{\gamma+1}x \tag{4.31}$$

for $\mu \in (0,1)$, when $\varepsilon = \varepsilon(\mu, N)$ is small. For $\theta_{\rm w}$ close to $\frac{\pi}{2}$, it can be shown that any solution of Problem 4.6 (with some natural regularity properties) satisfies that, for any small $\delta > 0$,

$$|\psi_x(x,y)| \leqslant \frac{1+\delta}{\gamma+1}x \qquad \text{in } \Omega_\varepsilon \text{ for small } \varepsilon = \varepsilon(\delta). \tag{4.32}$$

3. The iteration equation near $\Gamma_{\rm sonic}$ is defined based on the above facts. The iteration set $\mathcal{K}_M$ used in [35] is such that every $\hat\psi = \hat\varphi - \varphi_2 \in \mathcal{K}_M$ satisfies (4.24) and (4.29) for some $N, \varepsilon > 0$. Then the iteration equation for ψ is

$$\big(2x-(\gamma+1)x\eta(\frac{\psi_x}{x}) + O_1^{(\hat\psi)}\big)\psi_{xx} + O_2^{(\hat\psi)}\psi_{xy} + (\frac{1}{c_2} + O_3^{(\hat\psi)})\psi_{yy} - (1+O_4^{(\hat\psi)})\psi_x + O_5^{(\hat\psi)}\psi_y = 0, \tag{4.33}$$

where the cutoff function $\eta \in C^\infty(\mathbb{R})$ satisfies $|\eta| \leqslant \frac{5}{3(\gamma+1)}$, $\eta' \geqslant 0$, and $\eta(s) = s$ if $|s| \leqslant \frac{4}{3(\gamma+1)}$, and some other technical conditions. The terms, $O_k^{(\hat\psi)}$ for $k = 1,\dots,5$, are obtained from O_k by substituting $\hat\psi$ into certain terms in (4.28) and performing the ellipticity cutoff in the remaining terms, so that estimates (4.30) with $k = \frac{3}{2}$ hold. Then (4.33) is strictly elliptic in $\overline{\Omega}_\varepsilon \backslash \overline{\Gamma_{\rm sonic}}$ for small ε, and the ellipticity degenerates on $\Gamma_{\rm sonic}$. Since the solution of Problem 4.6 satisfies equation (4.27) and (4.32) with $\delta = \frac{1}{3}$ in Ω_ε for small ε, then it satisfies equation (4.33) in Ω_ε with $\hat\psi = \psi$. Indeed, we have the estimate: $|\psi_x| \leqslant \frac{4}{3(\gamma+1)}x$, so that $x\eta\left(\frac{\psi_x}{x}\right) = \psi_x$; and the cutoffs in the terms of $O_k^{(\hat\psi)}$ are removed similarly.

We also note that the degenerate ellipticity structure of equation (4.33) is the following: Writing (4.33) in the form

$$\sum_{i,j=1}^{2} A_{ij}(D\psi,\psi,x)D_{ij}\psi + \sum_{i=1}^{2} A_i(D\psi,\psi,x)D_i\psi = 0 \tag{4.34}$$

with $A_{12} = A_{21}$, we have

$$\lambda|\boldsymbol{\xi}|^2 \leqslant A_{11}(\mathbf{p},z,x)\frac{\xi_1^2}{x} + 2A_{12}(\mathbf{p},z,x)\frac{\xi_1\xi_2}{x^{1/2}} + A_{22}(\mathbf{p},z,x)\xi_2^2 \leqslant \frac{1}{\lambda}|\boldsymbol{\xi}|^2 \tag{4.35}$$

for all $(\mathbf{p},z) \in \mathbb{R}^2 \times \mathbb{R}$ and $\mathbf{x} \in (0,\varepsilon)$.

We consider the solutions of (4.33) in Ω_ε satisfying (4.24) and (4.26). Note that condition (4.25) can not be prescribed in the iteration problem as discussed above, so we have to obtain (4.25) from the estimates of the solutions by exploiting the elliptic degeneracy. The estimates for the positive solutions of (4.33) with (4.24) in Ω_ε are based on the fact that, for any $\delta > 0$, the function

$$w_\delta(x,y) = \frac{1+\delta}{2(\gamma+1)}x^2$$

is a supersolution of (4.33) in Ω_ε if $\varepsilon = \varepsilon(\delta)$ is small; that is, $\mathcal{N}(w_\delta) < 0$ in Ω_ε, where $\mathcal{N}(\cdot)$ denotes the operator in the left-hand side of (4.33). Using this, the boundary conditions on $\Gamma_{\rm shock}$ and $\Gamma_{\rm wedge}$, and (4.26), we obtain

$$0 \leqslant \psi \leqslant Cx^2 \qquad \text{in } \Omega_\varepsilon, \tag{4.36}$$

where ε and C are uniform for the wedge angles near $\frac{\pi}{2}$. Note that $-w_\delta$ is *not* a subsolution of (4.27) so that it cannot be used to bound ψ from below. Thus, (4.26), which is derived from the global structure of the solution, is crucially used in this argument. Then, in (4.36), the upper bound comes from the local estimates near $\Gamma_{\rm sonic}$, while the lower bound is from the global structure of the problem.

In particular, (4.36) with (4.24) implies that $D\psi = 0$ on $\Gamma_{\rm sonic}$, which resolves the issue described in (ii) above. Furthermore, from (4.36), using the non-isotropic "parabolic" rescaling corresponding to the elliptic degeneracy (4.35) of equation (4.33) near $x = 0$, we obtain the estimates in the appropriately weighted and scaled Hölder norm in Ω_ε, which in particular imply the uniform $C^{1,1}$ estimates:

$$|D^2\psi| \leqslant C \qquad \text{in } \Omega_\varepsilon. \tag{4.37}$$

More precisely, we denote this norm by $\|\psi\|^{(par)}_{2,\alpha,\Omega_\varepsilon}$, and define it as follows: Denote $z = (x, y)$ and $\tilde{z} = (\tilde{x}, \tilde{y})$ with $x, \tilde{x} \in (0, 2\varepsilon)$ and

$$\delta^{(par)}_\alpha(z, \tilde{z}) := \left(|x - \tilde{x}|^2 + \max(x, \tilde{x})|y - \tilde{y}|^2\right)^{\alpha/2}.$$

Then, for $\psi \in C^2(\Omega_\varepsilon) \cap C^{1,1}(\overline{\Omega_\varepsilon})$ written in the (x, y)–coordinates, we define

$$\begin{aligned}
\|\psi\|^{(par)}_{2,0,\Omega_\varepsilon} &:= \sum_{0\leqslant k+l\leqslant 2} \sup_{z\in\Omega_\varepsilon} \left(x^{k+l/2-2}|\partial_x^k\partial_y^l\psi(z)|\right),\\
[u]^{(par)}_{2,\alpha,\Omega_\varepsilon} &:= \sum_{k+l=2} \sup_{z,\tilde{z}\in\Omega_\varepsilon, z\neq\tilde{z}} \left(\min(x^{k+l/2-2}, \tilde{x}^{k+l/2-2})\frac{|\partial_x^k\partial_y^l\psi(z) - \partial_x^k\partial_y^l\psi(\tilde{z})|}{\delta^{(par)}_\alpha(z, \tilde{z})}\right),\\
\|\psi\|^{(par)}_{2,\alpha,\Omega_\varepsilon} &:= \|\psi\|^{(par)}_{2,0,\Omega_\varepsilon} + [\psi]^{(par)}_{2,\alpha,\Omega_\varepsilon}.
\end{aligned} \tag{4.38}$$

Now we obtain estimates in the norm (4.38), assuming that (4.36) holds in $\Omega_{2\varepsilon}$. For every $z_0 = (x_0, y_0) \in \overline{\Omega}_\varepsilon \backslash \Gamma_{\rm sonic}$ (thus $x_0 \in (0, \varepsilon]$), we define

$$R_{z_0} = \left\{(x, y) \;\; : \;\; |x - x_0| < \frac{x_0}{10},\; |y - y_0| < \frac{\sqrt{x_0}}{10}\right\} \cap \Omega. \tag{4.39}$$

Note that $\mathrm{dist}(R_{z_0}, \Gamma_{\rm sonic}) = \frac{3}{4}x_0 > 0$. We rescale the rectangle in (4.39) to the unit square $Q_1 = (-1, 1)^2$:

$$Q_1^{(z_0)} := \left\{(S, T) \in Q_1 \;:\; (x_0 + \frac{x_0}{10}S,\; y_0 + \frac{\sqrt{x_0}}{10}T) \in \Omega\right\}, \tag{4.40}$$

and define the scaled version of ψ in the (S, T) coordinates in $Q_1^{(z_0)}$:

$$\psi^{(z_0)}(S, T) := \frac{1}{x_0^2}\psi(x_0 + \frac{x_0}{10}S,\; y_0 + \frac{\sqrt{x_0}}{10}T) \qquad \text{for } (S, T) \in Q_1^{(z_0)}. \tag{4.41}$$

Note that this rescaling is non-isotropic with respect to x and y variables. By (4.36), we have

$$\|\psi^{(z_0)}\|_{L^\infty(\overline{Q_1^{(z_0)}})} \leqslant C \qquad \text{for any } z_0 = (x_0, y_0) \in \overline{\Omega}_\varepsilon \backslash \overline{\Gamma_{\rm sonic}}. \tag{4.42}$$

Rewriting equation (4.33) in terms of $\psi^{(z_0)}$ in the (S, T)–coordinates and noting the degenerate ellipticity structure (4.35), we find that $\psi^{(z_0)}$ satisfies a uniformly elliptic equation in $Q_1^{(z_0)}$ with the ellipticity constants and certain Hölder norms of coefficients independent of z_0. We also rescale the boundary conditions on $\Gamma_{\rm shock} \cap \partial\Omega_\varepsilon$ and $\Gamma_{\rm wedge} \cap \partial\Omega_\varepsilon$ in the similar way, when z_0 is on the corresponding part of the boundary. Then we apply the local elliptic $C^{2,\alpha}$ estimates for $\psi^{(z_0)}$ in $Q_1^{(z_0)}$ in the following cases:

(i) Interior rectangles R_{z_0}, *i.e.*, all z_0 such that $Q_1^{(z_0)} = Q_1$ holds;
(ii) Rectangles R_{z_0} centered on the shock: $z_0 \in \Gamma_{\rm shock} \cap \partial\Omega_\varepsilon$;
(iii) Rectangles R_{z_0} centered on the wedge: $z_0 \in \Gamma_{\rm wedge} \cap \partial\Omega_\varepsilon$,

where, in the last two cases, we have used the local estimates for the corresponding boundary-value problems. Using (4.42), we obtain

$$\|\psi^{(z_0)}\|_{C^{2,\alpha}(\overline{Q^{(z_0)}_{1/2}})} \leqslant C \qquad \text{with } C \text{ independent of } z_0,$$

where $Q^{(z_0)}_{1/2} = Q^{(z_0)}_1 \cap (-1/2,\ 1/2)^2$. Rewriting in terms of ψ in the (x, y)–coordinates and combining the estimates for all z_0 as above, we obtain the estimate $\|\psi\|^{(par)}_{2,\alpha,\Omega_\varepsilon} \leqslant C$ in the norm (4.38), which in particular implies the $C^{1,1}$ estimates (4.37).

Remark 4.7. *Note that $\psi^{(z_0)}_{SS}(S, T) = \frac{1}{100}\psi_{xx}(x_0 + \frac{x_0}{10}S,\ y_0 + \frac{\sqrt{x_0}}{10}T)$. It follows that $\|D^2\psi\|_{L^\infty}$ cannot be made small by choosing parameters, e.g. choosing ε small or $\theta_{\rm w}$ close to $\frac{\pi}{2}$.*

Remark 4.8. *The above argument, beginning from* (4.39)*, is also used for the apriori estimates of the positive solutions of* (4.27)*–*(4.28) *with condition* (4.24)*, satisfying* (4.29) *and the ellipticity condition* (4.31) *with some $\mu \in (0, 1)$. Note that* (4.24), (4.29), *and $\psi \geqslant 0$ imply* (4.36)*, which is used in the argument.*

Remark 4.9. *Remark* 4.8 *applies only to the positive solutions of* (4.27) *with condition* (4.24)*. For the negative solutions of* (4.27) *with condition* (4.24)*, the equation is* uniformly *elliptic up to $\{x = 0\}$ and, similar to Hopf's lemma, the negative solutions have* linear *growth: $|\psi(x, y)| \geqslant \frac{1}{C}x$, in a contrast with* (4.36)*. This feature is used in proving certain geometric properties of the free boundary for the wedge angles away from $\frac{\pi}{2}$, where we note that $\varphi - \varphi_1 < 0$ by* (4.23)*.*

4. In order to remove the elliptic cutoff in (4.33), *i.e.*, to show that the fixed point solution of (4.33) (*i.e.*, with $\psi = \hat{\psi}$) actually satisfies (4.27), we need to show that $|\psi_x| \leqslant \frac{4}{3(\gamma+1)}x$, as we discussed in the lines after (4.33). Combining (4.37) with $D\psi = 0$ on $\Gamma_{\rm sonic}$, we obtain that $|D\psi(x, y)| \leqslant Cx$ in Ω_ε, which does not remove the ellipticity cutoff, unless we show the explicit bound $C \leqslant \frac{4}{3(\gamma+1)}$. However, this bound does not follow from our estimates, cf. Remark 4.7.

Note that the only explicit solution we have known is the normal reflection for $\theta_{\rm w} = \frac{\pi}{2}$, for which $\varphi = \varphi_2^{(\frac{\pi}{2})}$, *i.e.*, $\psi = 0$ in Ω. Also, the analysis by Bae-Chen-Feldman [6] shows that the solutions of Problem 4.6 of supersonic reflection-diffraction structure satisfy, for small ε,

$$\psi_x \sim \frac{x}{\gamma+1} \qquad \text{in } \Omega_\varepsilon \cap \{(x, y)\ :\ \mathrm{dist}\,((x, y), \Gamma_{\rm shock}) > \sqrt{x}\},$$

but

$$D\psi = o(x) \qquad \text{in } \Omega_\varepsilon \cap \{(x, y)\ :\ \mathrm{dist}\,((x, y), \Gamma_{\rm shock}) < x^2\}.$$

This shows that the convergence of solutions $\varphi^{(\theta_{\rm w})}$ of Problem 4.6 to $\varphi^{(\frac{\pi}{2})}$ as $\theta_{\rm w} \to \frac{\pi}{2}^-$ does not hold in the norm sufficiently strong to capture the behaviour near $\Gamma_{\rm sonic}$ described above. In particular, this convergence does not hold in C^2 (but holds in $C^{1,\alpha}$) after mapping $\Omega^{(\theta_{\rm w})}$ to a fixed domain for all $\theta_{\rm w}$. Thus, there is no clear background solution such that the appropriate iteration set would lie in its small neighborhood in the norm sufficiently strong to remove the cutoff by the smallness of the norm. Then, in order to remove elliptic cutoff for the fixed point of the iteration, we derive an equation in Ω_ε and boundary conditions and estimates on $\Gamma_{\rm shock} \cap \{x < \varepsilon\}$ and $\Gamma_{\rm wedge} \cap \{x < \varepsilon\}$ for ψ_x in Ω_ε, and prove that

$$\psi_x \leqslant \frac{4}{3(\gamma+1)}x$$

from this boundary value problem. The estimate from below

$$\psi_x \geqslant -\frac{4}{3(\gamma+1)}x$$

is proved from the global setting of Problem 4.6. This use of the local and global structure is similar to that in the proof of (4.36).

Note that, in this argument, for the wedge angles near $\frac{\pi}{2}$, the non-perturbative nature of the problem is seen only in the estimates of the solution near Γ_{sonic}. The free boundary Γ_{shock} in this case is near $S_1(\theta_{\text{w}})$, and also close to the reflected shock of the normal reflection as in Fig. 4.3, which is the vertical line $S_1(\frac{\pi}{2})$. Also, $\|\varphi - \varphi_2^{(\theta_{\text{w}})}\|_{C^1(\Omega)} \leqslant C(\frac{\pi}{2} - \theta_{\text{w}})$, which is small. Thus, away from Γ_{sonic}, the argument is perturbative for the wedge angles near $\frac{\pi}{2}$. In the case of general wedge angles in Theorem 4.1, the free boundary Γ_{shock} is no longer close to a line, its structure is not known *apriori*, thus the study of geometric properties of the free boundary is a part of the argument.

Case II. General Wedge Angles up to the Detachment Angle. For this case and the proof of Theorem 4.1, we follow the approach introduced in Chen-Feldman [37]. Similar to the case of wedge angles near $\frac{\pi}{2}$ where we restricted consideration to the class of solutions satisfying $\psi \geqslant 0$ in Ω and showed the existence of such solutions, for the general case, we define a class of *admissible solutions*, make *apriori* estimates of such solutions, and prove the existence of solutions in this class. Motivation for the definition of admissible solutions comes from the following properties of supersonic regular reflection solutions φ for the wedge angles close to $\frac{\pi}{2}$, or more generally, for the supersonic regular reflection solutions satisfying that $\|\varphi - \varphi_2^{(\theta_{\text{w}})}\|_{C^1(\Omega)}$ is small: If (4.9) is strictly elliptic for φ in $\overline{\Omega}\backslash\overline{\Gamma_{\text{sonic}}}$, then it satisfies (4.23) and the monotonicity properties:

$$\partial_{\xi_2}(\varphi_1 - \varphi) \leqslant 0, \quad D(\varphi_1 - \varphi)\cdot \mathbf{e}_{S_1} \leqslant 0 \qquad \text{in } \Omega \tag{4.43}$$

where $\mathbf{e}_{S_1} = \frac{P_0P_1}{|P_0P_1|}$.

We present the outline of the proof of Theorem 4.1 in the following four steps:

1. Motivated by the discussion above, for the general wedge angles, we define the admissible solutions as the solutions of Problem 4.6 (thus the solutions with weak regular reflection-diffraction configuration of either supersonic or subsonic type) satisfying the following properties:

Definition 4.10. *Let $\theta_{\text{w}} \in (\theta_{\text{w}}^{\text{d}}, \frac{\pi}{2})$. A function $\varphi \in C^{0,1}(\overline{\Lambda})$ is an admissible solution of the regular reflection problem if φ is a solution of Problem* 4.6 *extended to Λ by* (4.22) (*where $P_0P_1P_4$ is a point in the subsonic and sonic cases*) *and satisfies the following properties*:

(i) *The structure of solutions*:

- *If $|D\varphi_2(P_0)| > c_2$, then φ is of the* supersonic *regular shock reflection-diffraction configuration shown on Fig.* 4.1 *and satisfies that the curved part of reflected-diffracted shock Γ_{shock} is C^2 in its relative interior*; *curves Γ_{shock}, Γ_{sonic}, Γ_{wedge}, and Γ_{sym} do not have common points except their endpoints*; $\varphi \in C^{0,1}(\Lambda)\cap C^1(\Lambda\backslash(S_0\cup\overline{P_0P_1P_2}))$ *and* $\varphi \in C^1(\overline{\Omega})\cap C^3(\overline{\Omega}\backslash(\overline{\Gamma_{\text{sonic}}}\cup\{P_2,P_3\}))$.
- *If $|D\varphi_2(P_0)| \leqslant c_2$, then φ is of the* subsonic *regular shock reflection-diffraction configuration shown on Fig.* 4.2 *and satisfies that the reflected-diffracted shock Γ_{shock} is C^2 in its relative interior*; *curves Γ_{shock}, Γ_{wedge}, and Γ_{sym} do not have common points except their endpoints*; $\varphi \in C^{0,1}(\Lambda) \cap C^1(\Lambda\backslash(S_0 \cup \overline{\Gamma_{\text{shock}}}))$ *and* $\varphi \in C^1(\overline{\Omega}) \cap C^3(\overline{\Omega}\backslash\{P_0, P_3\})$.

Moreover, in both the supersonic and subsonic cases, the curve $\Gamma_{\text{shock}}^{\text{ext}} := \Gamma_{\text{shock}} \cup \{P_0\} \cup \Gamma_{\text{shock}}^-$ is C^1 in its relative interior, where Γ_{shock}^- is the reflection of Γ_{shock} with respect to the ξ_1–axis.

(ii) *Equation* (4.9) *is strictly elliptic in $\overline{\Omega}\backslash\overline{\Gamma_{\text{sonic}}}$, i.e., $|D\varphi| < c(|D\varphi|^2, \varphi)$ in $\overline{\Omega}\backslash\overline{\Gamma_{\text{sonic}}}$.*

(iii) *$\partial_{\boldsymbol{\nu}}\varphi_1 > \partial_{\boldsymbol{\nu}}\varphi > 0$ on Γ_{shock}, where $\boldsymbol{\nu}$ is the normal to Γ_{shock}, pointing to the interior of Ω.*

(iv) *Inequalities hold:*

$$\varphi_1 \geqslant \varphi \geqslant \varphi_2 \quad \textit{in } \Omega, \tag{4.44}$$

(v) (4.43) *is satisfied, where the vector* $\mathbf{e}_{S_1}$ *is defined as the unit vector parallel to* S_1 *and pointing into* Λ *at* P_0 *for the general case.*

Note that (4.43) implies that

$$D(\varphi_1-\varphi)\cdot\mathbf{e}\leqslant 0 \qquad \text{in } \overline{\Omega} \quad \text{for all } \mathbf{e}\in\overline{Cone(\mathbf{e}_{\xi_2},\mathbf{e}_{S_1})}, \tag{4.45}$$

where $Cone(\mathbf{e}_{\xi_2},\mathbf{e}_{S_1})=\{a\,\mathbf{e}_{\xi_2}+b\,\mathbf{e}_{S_1}\ :\ a,b>0\}$ with $\mathbf{e}_{\xi_2}=(0,1)$. Notice that $\mathbf{e}_{\xi_2}$ and $\mathbf{e}_{S_1}$ are not parallel if $\theta_{\rm w}\neq\frac{\pi}{2}$.

2. To prove the existence of admissible solutions for each wedge angle in Theorem 4.1, we derive uniform *a priori* estimates for admissible solutions with any wedge angle $\theta_{\rm w}\in[\theta_{\rm w}^{\rm d}+\sigma,\frac{\pi}{2}]$ for each small $\sigma>0$, show compactness of this subset of admissible solutions in the appropriate norm, and then apply the degree theory to obtain the existence of admissible solutions for each $\theta_{\rm w}\in[\theta_{\rm w}^{\rm d}+\sigma,\frac{\pi}{2}]$, starting from the unique normal reflection solution for $\theta_{\rm w}=\frac{\pi}{2}$. To derive the *a priori* estimates for admissible solutions, we first obtain the required estimates related to the geometry of the shock and domain Ω, as well as the basic estimates of solution φ. We show:

(a) The inequality in (4.45) is strict for any $\mathbf{e}\in Cone(\mathbf{e}_{\xi_2},\mathbf{e}_{S_1})$. Combined with (4.44) and the fact that $\varphi=\varphi_1$ on $\Gamma_{\rm shock}$, this implies that $\Gamma_{\rm shock}$ is a Lipschitz graph with uniform Lipschitz estimate for all admissible solutions.
(b) The uniform bounds on $\mathrm{diam}(\Omega)$, $\|\varphi\|_{C^{0,1}(\Omega)}$, and the directional monotonicity of $\varphi-\varphi_2$ near the sonic arc for a cone of directions;
(c) The uniform positive lower bound for the distance between the shock and the wedge, and the uniform separation of the shock and the symmetry line (that is, $\Gamma_{\rm shock}$ is away from a uniform conical neighborhood of $\Gamma_{\rm sym}$ with vertex at their common endpoint P_2);
(d) The uniform positive lower bound for the distance between the shock and the sonic circle $B_{c_1}((u_1,0))$ of state (1), by using the properties described in Remark 4.9. This allows to estimate the ellipticity of (4.9) for φ in Ω (depending on the distance to the sonic arc P_1P_4 for the supersonic regular shock reflection-diffraction configuration and to P_0 for the subsonic regular shock reflection-diffraction configuration).
(e) Estimate (4.29) holds in the supersonic case, by using the monotonicity of $\psi=\varphi-\varphi_2$ near the sonic arc in a cone of directions shown in (b) and the conditions on $\Gamma_{\rm sonic}$ in Problem 4.6.

The results of (a)–(c) are obtained by the maximum principle, by considering equation (4.13) as a linear elliptic equation for ϕ and using the boundary conditions on $\Gamma_{\rm shock}$, $\Gamma_{\rm sonic}$, $\Gamma_{\rm wedge}$, and $\Gamma_{\rm sym}$ in Problem 4.6 and (4.44)–(4.45). The results of (c), combined with (a), show the structure of Ω which allows to perform the uniform local elliptic estimates in various parts of Ω: the interior, near a point P in a relative interior of $\Gamma_{\rm shock}$, $\Gamma_{\rm wedge}$, and $\Gamma_{\rm sym}$, and locally near the corners P_2 and P_3.

Based on estimates (a)–(d), we show the uniform regularity estimates for the solution and the free boundary in weighted/scaled $C^{2,\alpha}$ norms away from the sonic arc in the supersonic case and away from P_0 in the subsonic case, *i.e.*, in $\Omega\backslash\Omega_\varepsilon$, for any small $\varepsilon>0$. The equation is uniformly elliptic in this region, with ellipticity constant depending on ε. Thus, the estimates depend on ε.

3. Below we discuss the estimates near $\Gamma_{\rm sonic}$ (resp. near P_0 in the subsonic/sonic case), *i.e.*, the estimates in $\Omega_{2\varepsilon}$ for some ε independent of $\theta_{\rm w}\in[\theta_{\rm w}^{\rm d}+\sigma,\frac{\pi}{2}]$, which allows to complete uniform *apriori* estimates for admissible solutions with wedge angles $\theta_{\rm w}\in[\theta_{\rm w}^{\rm d}+\sigma,\frac{\pi}{2}]$. We obtain the estimates near $\Gamma_{\rm sonic}$ (or P_0 for the subsonic reflection), *i.e.*, in $\Omega_{2\varepsilon}$, in scaled and weighted $C^{2,\alpha}$ for φ and the free boundary $\Gamma_{\rm shock}\cap\partial\Omega_{2\varepsilon}$, by considering separately four cases depending on $\frac{|D\varphi_2|}{c_2}$ at P_0:

(i) Supersonic: $\frac{|D\varphi_2|}{c_2}\geqslant 1+\delta$;

(ii) Supersonic (almost sonic): $1 < \frac{|D\varphi_2|}{c_2} < 1 + \delta$;

(iii) Subsonic (almost sonic, including sonic): $1 - \delta \leqslant \frac{|D\varphi_2|}{c_2} \leqslant 1$;

(iv) Subsonic: $\frac{|D\varphi_2|}{c_2} \leqslant 1 - \delta$,

for a small $\delta > 0$ chosen so that the estimates can be obtained. The choice of δ determines ε.

For cases (i)–(ii), equation (4.9) is degenerate elliptic in Ω near P_1P_4 on Fig. 4.1. For case (iii), except the sonic case $\frac{|D\varphi_2(P_0)|}{c_2} = 1$, the equation is uniformly elliptic in $\overline{\Omega}$, but the ellipticity constant is small and tends to zero near P_0 on Fig. 4.2 as $\frac{|D\varphi_2^{(\theta_{\rm w})}(P_0)|}{c_2} \to 1^-$, i.e. for subsonic angles $\theta_{\rm w}$ which tend to the sonic angle. Thus, for cases (i)–(iii), we use the local elliptic degeneracy, which allows to find a comparison function in each case, to show the appropriately fast decay of $\varphi - \varphi_2$ near P_1P_4 for cases (i)–(ii) and near P_0 for case (iii); furthermore, combining with appropriate local non-isotropic rescaling to obtain the uniform ellipticity, we obtain the *a priori* estimates in the weighted and scaled $C^{2,\alpha}$–norms. In cases (i)–(ii), the norms are (4.38). For the case (iii), we use the different norms, and the estimates we obtain implysa the standard $C^{2,\alpha}$–estimates for case (iii). To obtain these estimates, in case (i) we use the argument developed in Chen-Feldman [35] and described above (see Remark 4.8), where we note that the ellipticity estimate (4.31) follows from the estimates described in (d) above, and (4.29) was obtained in (e). These estimates hold in Ω_ε with $\varepsilon \lesssim (\text{length}(\Gamma_{\rm sonic}))^2$ because the "rectangles" $R_{(x_0,y_0)}$ defined by (4.39) do not fit into Ω for larger x_0, which means, for example, that $R_{(x_0,y_0)} \cap \Gamma_{\rm wedge} \neq \varnothing$ for $(x_0, y_0) \in \Gamma_{\rm shock} \cap \partial\Omega_\varepsilon$ with $x_0 \geqslant C(\text{length}(\Gamma_{\rm sonic}))^2$ if C is large and length$(\Gamma_{\rm sonic})$ is small, because the length of y-side of $R_{(x_0,y_0)}$ is $\frac{\sqrt{x_0}}{10}$, and $\Gamma_{\rm shock}$ and $\Gamma_{\rm wedge}$ are smooth curves which intersect $\Gamma_{\rm sonic}$ transversally. However, length$(\Gamma_{\rm sonic})$) tends to zero, as $\dfrac{|D\varphi_2^{(\theta_{\rm w})}(P_0)|}{c_2} \to 1^+$, i.e. when the supersonic wedge angle tends to the sonic angle. Thus, a different argument, involving an appropriate scaling, is employed for case (ii) in order to keep ε uniform for all $\theta_{\rm w} \in [\theta_{\rm w}^{\rm d} + \delta, \frac{\pi}{2}]$. Another version of that argument (with a different scaling) is applied for case (iii). For both cases (ii)–(iii), we need to use smaller rectangles than those for case (i), but this requires stronger growth estimates than (4.36) to obtain a bound in $C^{1,1}$ from the corresponding weighted and scaled estimates. We obtain such growth estimates by using the conditions of cases (ii)–(iii) for sufficiently small δ. For case (iv), the equation is uniformly elliptic in Ω for the admissible solution, where the ellipticity constant is not small, and the estimates are more technically challenging than those for cases (i)–(iii). This can be seen as follows: For all cases (i)–(iv), the free boundary has a lower *apriori* regularity in the sense that only the Lipschitz estimate of $\Gamma_{\rm shock}$ is obtained in (a) above; however, for case (iv), the uniform ellipticity combined with oblique boundary conditions does not allow a comparison function that leads to the fast decay of $|\varphi - \varphi_2|$ near P_0. Thus, we prove the C^α–estimates of $D(\varphi - \varphi_2)$ near P_0, by deriving the equations and boundary conditions for two directional derivatives of $\varphi - \varphi_2$ near P_0, and performing hodograph transform to flatten the free boundary.

4. In order to prove the existence of solutions, we perform an iteration, which is an extension of the iteration process used in Chen-Feldman [35]. First, given an admissible solution φ for the wedge angle $\theta_{\rm w}$, we map its elliptic domain $\Omega(\varphi, \theta_{\rm w})$ to a unit square $Q = (0,1)^2$ so that, for the supersonic case, the boundary parts $\Gamma_{\rm shock}$, $\Gamma_{\rm sonic}$, $\Gamma_{\rm wedge}$, and $\Gamma_{\rm sym}$ are mapped to the respective sides of Q, and other properties of this map are satisfied. For the subsonic case, the map is discontinuous at $P_0 = \overline{\Gamma_{\rm sonic}}$ (mapping the triangular domain to a square). Moreover, we define a function u on Q by expressing $\varphi - \tilde{\varphi}_2^{(\theta_{\rm w})}$ in the coordinates on Q, where $\tilde{\varphi}_2^{(\theta_{\rm w})}$ is a function determined by $\theta_{\rm w}$ and equal to φ_2 near $\overline{\Gamma_{\rm sonic}}$; we skip the complete technical definition here. For appropriate functions u on Q and the wedge angle $\theta_{\rm w}$, this mapping can be inverted, *i.e.*, the elliptic domain $\Omega(u, \theta_{\rm w})$ and the iteration free boundary

$\Gamma_{\text{shock}}(u,\theta_{\rm w})$ can be determined, and a function $\varphi^{(u,\theta_{\rm w})}$ on $\Omega(u,\theta_{\rm w})$ is defined by expressing u in the coordinates on $\Omega(u,\theta_{\rm w})$ and adding $\tilde{\varphi}_2^{(\theta_{\rm w})}$ so that, if u is obtained from the admissible solution φ with elliptic domain Ω as described above, then $\Omega(u,\theta_{\rm w}) = \Omega$ and $\varphi^{(u,\theta_{\rm w})} = \varphi$ in Ω. Moreover, the map: $\Omega(u,\theta_{\rm w}) \to Q$ and its inverse satisfy certain continuity properties with respect to $(u,\theta_{\rm w})$. The iteration is performed in terms of functions defined on Q. The iteration set consists of pairs $(u,\theta_{\rm w})$, where u is in a weighted and scaled $C^{2,\alpha}$ space on Q, denoted as $C^{2,\alpha}_{**}$ (its definition is technical and we skip it here), and satisfy

(i) $\|u\|_{C^{2,\alpha}_{**}} \leqslant M(\theta_{\rm w})$, where $M(\theta_{\rm w})$ is defined explicitly, based on the *apriori* estimates discussed above;

(ii) $\Omega(u,\theta_{\rm w})$, $\Gamma_{\text{shock}}(u,\theta_{\rm w})$, and function $\varphi^{(u,\theta_{\rm w})}$ on $\Omega(u,\theta_{\rm w})$ satisfy some geometric and analytical properties.

The iteration map: $(\hat{u},\theta_{\rm w}) \to (u,\theta_{\rm w})$ is defined by solving the iteration problem in $\Omega(u,\theta_{\rm w})$ and then mapping the solution, φ, to a function u on Q. This mapping includes additional steps comparing to the one described above to modify the iteration free boundary by using solution φ of the iteration problem, and using this modified domain Ω in the mapping: $(\varphi,\theta_w) \to u$, so that the resulting function u on Q keeps the regularity gain obtained from solving the iteration problem. This yields the compactness of the iteration map. We show that, for a fixed point $(u,\theta_{\rm w})$ of the iteration map, $\varphi^{(u,\theta_{\rm w})}$ on $\Omega(u,\theta_{\rm w})$ is an admissible solution. We use the degree theory to show the existence of admissible solutions for each $\theta_{\rm w} \in [\theta_{\rm w}^{\rm d}+\delta, \frac{\pi}{2}]$, starting from the unique normal reflection solution for $\theta_{\rm w} = \frac{\pi}{2}$. The compactness of the iteration map described above is necessary for that. The *apriori* estimates of admissible solutions discussed above are used in the degree theory argument in order to define the iteration set such that a fixed point of the iteration map (*i.e.*, admissible solution) cannot occur on the boundary of the iteration set, since that would contradict the *apriori* estimates. With all of these arguments, we complete the proof of Theorem 4.1. This provides a solution to the von Neumann's conjectures.

More details can be found in Chen-Feldman [37]; also see [35].

4.2. **Prandtl-Meyer Problem for Shock Reflection.** As we discussed in §2–§3, steady shocks appear when a steady supersonic flow hits a straight wedge; see Figure 3.1. Since both weak and strong steady shock solutions are stable in the steady regime, the static stability analysis alone is not able to single out one of them in this sense, unless an additional condition is posed on the speed of the downstream flow at infinity. Then the dynamic stability analysis becomes more significant to understand the non-uniqueness issue of the steady oblique shock solutions. However, the problem for the dynamic stability of the steady shock solutions for supersonic flow past solid wedges involves several additional mathematical difficulties. The recent efforts have been focused on the construction of the global Prandtl-Meyer configurations in the self-similar coordinates for potential flow.

As we discussed earlier, if a supersonic flow with a constant density $\rho_0 > 0$ and a velocity $\mathbf{u}_0 = (u_{10}, 0)$, $u_{10} > c_0 := c(\rho_0)$, impinges toward wedge W in (3.11), and if $\theta_{\rm w}$ is less than the detachment angle $\theta_{\rm w}^{\rm d}$, then the well-known *shock polar analysis* shows that there are two different steady weak solutions: *the steady weak shock solution* $\bar{\Phi}$ and *the steady strong shock solution*, both of which satisfy the entropy condition and the slip boundary condition (see Fig. 3.1).

Then the dynamic stability of the weak transonic shock solution for potential flow can be formulated as the following problem:

Problem 4.11 (Initial-Boundary Value Problem). *Given $\gamma > 1$, fix (ρ_0, u_{10}) with $u_{10} > c_0$. For a fixed $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm d})$, let W be given by (3.11). Seek a global weak solution $\Phi \in W^{1,\infty}_{\rm loc}(\mathbb{R}_+ \times (\mathbb{R}^2\backslash W))$ of Eq. (4.5) with ρ determined by (4.4) and $B = \frac{u_{10}^2}{2} + h(\rho_0)$ so that Φ satisfies the initial condition at $t = 0$:*

$$(\rho, \Phi)|_{t=0} = (\rho_0, u_{10}x_1) \qquad \text{for } \mathbf{x} \in \mathbb{R}^2\backslash W, \tag{4.46}$$

and the slip boundary condition along the wedge boundary ∂W:

$$\nabla_{\mathbf{x}}\Phi \cdot \boldsymbol{\nu}_{\mathrm{w}}|_{\partial W} = 0, \tag{4.47}$$

where $\boldsymbol{\nu}_{\mathrm{w}}$ *is the exterior unit normal to* ∂W.

In particular, we seek a solution $\Phi \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}_+ \times (\mathbb{R}^2 \backslash W))$ *that converges to the steady weak oblique shock solution* $\bar{\Phi}$ *corresponding to the fixed parameters* $(\rho_0, u_{10}, \gamma, \theta_{\mathrm{w}})$ *with* $\bar{\rho} = h^{-1}(B - \frac{1}{2}|\nabla\bar{\Phi}|^2)$, *when* $t \to \infty$, *in the following sense: For any* $R > 0$, Φ *satisfies*

$$\lim_{t\to\infty} \|(\nabla_{\mathbf{x}}\Phi(t,\cdot) - \nabla_{\mathbf{x}}\bar{\Phi}, \rho(t,\cdot) - \bar{\rho})\|_{L^1(B_R(\mathbf{0})\backslash W)} = 0 \tag{4.48}$$

for $\rho(t,\mathbf{x})$ *given by* (4.4).

Since the initial data functions in (4.46) do not satisfy the boundary condition (4.47), a boundary layer is generated along the wedge boundary starting at $t = 0$, which forms the Prandtl-Meyer configurations, as proved in Bae-Chen-Feldman [7].

Notice that the initial-boundary value problem, Problem 4.11, is invariant under the scaling (4.6), so we may study the existence of self-similar solutions determined by equation (4.9) with (4.10) through (4.7).

As the upstream flow has the constant velocity $(u_{10}, 0)$, and noting the choice of B in Problem 4.11, the corresponding pseudo-potential φ_0 has the expression of

$$\varphi_0 = -\frac{1}{2}|\boldsymbol{\xi}|^2 + u_{10}\xi_1 \tag{4.49}$$

in self-similar coordinates $\boldsymbol{\xi} = \frac{\mathbf{x}}{t}$, as shown directly from (4.15). Notice also the symmetry of the domain and the upstream flow in Problem 4.11 with respect to the x_1–axis. Problem 4.11 can then be reformulated as the following boundary value problem in the domain:

$$\Lambda := \mathbb{R}^2_+ \backslash \{\boldsymbol{\xi} \,:\, \xi_2 \leqslant \xi_1 \tan\theta_{\mathrm{w}},\, \xi_1 \geqslant 0\}$$

in the self-similar coordinates $\boldsymbol{\xi}$, which corresponds to domain $\{(t,\mathbf{x}) \,:\, \mathbf{x} \in \mathbb{R}^2_+ \backslash W,\, t > 0\}$ in the $(t,\mathbf{x})$–coordinates, where $\mathbb{R}^2_+ = \{\boldsymbol{\xi} \,:\, \xi_2 > 0\}$.

Problem 4.12 (Boundary Value Problem)**.** *Seek a solution* φ *of equation* (4.9) *in the self-similar domain* Λ *with the slip boundary condition*:

$$D\varphi \cdot \boldsymbol{\nu}|_{\partial\Lambda} = 0 \tag{4.50}$$

and the asymptotic boundary condition at infinity:

$$\varphi - \varphi_0 \longrightarrow 0 \tag{4.51}$$

along each ray $R_\theta := \{\xi_1 = \xi_2 \cot\theta, \xi_2 > 0\}$ *with* $\theta \in (\theta_{\mathrm{w}}, \pi)$ *as* $\xi_2 \to \infty$ *in the sense that*

$$\lim_{r\to\infty} \|\varphi - \varphi_0\|_{C(R_\theta \backslash B_r(0))} = 0. \tag{4.52}$$

In particular, we seek a global entropy solution of Problem 4.12 with two types of Prandtl-Meyer configurations whose occurrence is determined by the wedge angle θ_{w} for the two different cases: One contains a straight weak oblique shock $\mathcal{S}_0$ attached to the wedge vertex O and connected to a normal shock $\mathcal{S}_1$ through a curved shock Γ_{shock} when $\theta_{\mathrm{w}} < \theta^{\mathrm{s}}_{\mathrm{w}}$, as shown in Fig. 4.4; the other contains a curved shock Γ_{shock} attached to the wedge vertex and connected to a normal shock $\mathcal{S}_1$ when $\theta^{\mathrm{s}}_{\mathrm{w}} \leqslant \theta_{\mathrm{w}} < \theta^{\mathrm{d}}_{\mathrm{w}}$, as shown in Fig. 4.5, in which the curved shock Γ_{shock} is tangential to the straight weak oblique shock $\mathcal{S}_0$ at the wedge vertex.

To seek a global entropy solution of Problem 4.12 with the structure of Fig. 4.4 or Fig. 4.5, one needs to compute the pseudo-potential function φ below $\mathcal{S}_0$.

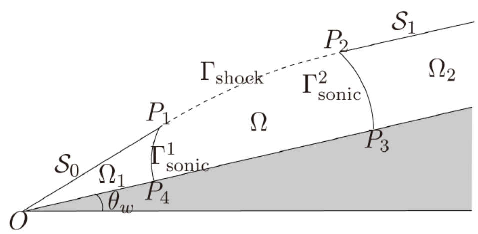

FIGURE 4.4. Self-similar solutions for $\theta_{\rm w} \in (0, \theta^{\rm s}_{\rm w})$ in the self-similar coordinates $\boldsymbol{\xi}$ (*cf.* [7])

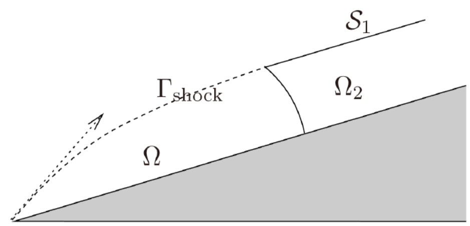

FIGURE 4.5. Self-similar solutions for $\theta_{\rm w} \in [\theta^{\rm s}_{\rm w}, \theta^{\rm d}_{\rm w})$ in the self-similar coordinates $\boldsymbol{\xi}$ (*cf.* [7])

Given $M_0 > 1$, ρ_1, and $\mathbf{u}_1$ are determined by using the shock polar in Fig. 3.1 for steady potential flow. For any wedge angle $\theta_{\rm w} \in (0, \theta^{\rm s}_{\rm w})$, line $u_2 = u_1 \tan\theta_{\rm w}$ and the shock polar intersect at a point $\mathbf{u}_1$ with $|\mathbf{u}_1| > c_1$ and $u_{11} < u_{10}$; while, for any wedge angle $\theta_{\rm w} \in [\theta^{\rm s}_{\rm w}, \theta^{\rm d}_{\rm w})$, they intersect at a point $\mathbf{u}_1$ with $u_{11} > u_{1\rm d}$ and $|\mathbf{u}_1| < c_1$. The intersection state $\mathbf{u}_1$ is the velocity for steady potential flow behind an oblique shock $\mathcal{S}_0$ attached to the wedge vertex with angle $\theta_{\rm w}$. The strength of shock $\mathcal{S}_0$ is relatively weak compared to the other shock given by the other intersection point on the shock polar, thus we call $\mathcal{S}_0$ *weak oblique shock*, and the corresponding state $\mathbf{u}_1$ is a *weak state*.

We also note that states $\mathbf{u}_1$ depend smoothly on u_{10} and $\theta_{\rm w}$, and such states are supersonic when $\theta_{\rm w} \in (0, \theta^{\rm s}_{\rm w})$ and subsonic when $\theta_{\rm w} \in [\theta^{s}_{\rm w}, \theta^{\rm d}_{\rm w})$.

Once $\mathbf{u}_1$ is determined, by (4.18) and (4.49), the pseudo-potentials φ_1 and φ_2 below the weak oblique shock $\mathcal{S}_0$ and the normal shock $\mathcal{S}_1$ are respectively in the form of

$$\varphi_1 = -\frac{1}{2}|\boldsymbol{\xi}|^2 + \mathbf{u}_1 \cdot \boldsymbol{\xi}, \qquad \varphi_2 = -\frac{1}{2}|\boldsymbol{\xi}|^2 + \mathbf{u}_2 \cdot \boldsymbol{\xi} + k_2 \tag{4.53}$$

for constant states $\mathbf{u}_1$ and $\mathbf{u}_2$, and constant k_2; see (4.15). Then it follows from (4.10) and (4.53) that the corresponding densities ρ_1 and ρ_2 are constants, respectively. In particular, we have

$$\rho_k^{\gamma-1} = \rho_0^{\gamma-1} + \frac{\gamma-1}{2}\big(u_{10}^2 - |\mathbf{u}_k|^2\big) \qquad \text{for } k = 1, 2. \tag{4.54}$$

Denote $\Gamma_{\rm wedge} := \partial W \cap \partial\Lambda$. Next we define the sonic arcs $\Gamma^1_{\rm sonic} = P_1P_4$ on Fig. 4.4 and $\Gamma^2_{\rm sonic} = P_2P_3$ on Figs. 4.4–4.5. The sonic circle $\partial B_{c_1}(\mathbf{u}_1)$ of the uniform state φ_1 intersects line $\mathcal{S}_0$, where $c_1 = \rho_1^{\frac{\gamma-1}{2}}$ by (4.12). For the supersonic case $\theta_{\rm w} \in (0, \theta^{\rm s}_{\rm w})$, there are two arcs of this sonic circle between $\mathcal{S}_0$ and $\Gamma_{\rm wedge}$ in Λ. We denote by $\Gamma^1_{\rm sonic}$ the *lower* arc (*i.e.*, located to the left from another arc) in the orientation on Fig. 4.4. Note that $\Gamma^1_{\rm sonic}$ tends to point O as $\theta_{\rm w} \nearrow \theta^{\rm s}_{\rm w}$, and is outside of Λ for the subsonic case $\theta_{\rm w} \in [\theta^{s}_{\rm w}, \theta^{\rm d}_{\rm w})$. Similarly, the sonic circle $\partial B_{c_2}(\mathbf{u}_2)$ of the uniform state φ_2 intersects line $\mathcal{S}_1$, where $c_2 = \rho_2^{\frac{\gamma-1}{2}}$. There are two arcs of this circle between $\mathcal{S}_1$ and the line containing $\Gamma_{\rm wedge}$. For all $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm d})$, the *upper* arc (*i.e.*, located to the right of the other arc) in the orientation on Figs. 4.4–4.5 is within Λ, which is denoted as $\Gamma_{\rm sonic}^2$.

Then Problem 4.12 can be reformulated into the following free boundary problem.

Problem 4.13 (Free Boundary Problem)**.** *For $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm d})$, find a free boundary* (*curved shock*) *$\Gamma_{\rm shock}$ and a function φ defined in domain Ω, as shown in Figs.* 4.4–4.5*, such that φ satisfies*

- (i) *Equation* (4.9) *in Ω;*
- (ii) *$\varphi = \varphi_0$ and $\rho D\varphi \cdot \boldsymbol{\nu}_{\rm s} = \rho_0 D\varphi_0 \cdot \boldsymbol{\nu}_{\rm s}$ on $\Gamma_{\rm shock}$;*
- (iii) *$\varphi = \hat{\varphi}$ and $D\varphi = D\hat{\varphi}$ on $\Gamma_{\rm sonic}^1 \cup \Gamma_{\rm sonic}^2$ when $\theta_{\rm w} \in (0, \theta_{\rm w}^{s})$ and on $\Gamma_{\rm sonic}^2 \cup \{O\}$ when $\theta_{\rm w} \in [\theta_{\rm w}^{\rm s}, \theta_{\rm w}^{\rm d})$ for $\hat{\varphi} := \max(\varphi_1, \varphi_2)$;*
- (iv) *$D\varphi \cdot \boldsymbol{\nu}_{\rm w} = 0$ on $\Gamma_{\rm wedge}$,*

where $\boldsymbol{\nu}_{\rm s}$ and $\boldsymbol{\nu}_{\rm w}$ are the interior with respect to Ω unit normals to $\Gamma_{\rm shock}$ and $\Gamma_{\rm wedge}$, respectively.

Remark 4.14. *Similar as in Problem* 4.6*, the conditions in Problem* 4.13(ii)–(iii) *are the Rankine-Hugoniot conditions* (4.17)–(4.18) *on $\Gamma_{\rm shock}$ and $\Gamma_{\rm sonic}^1 \cup \Gamma_{\rm sonic}^2$, respectively; see the discussions in the paragraphs after Problem* 4.6*.*

Let φ be a solution of Problem 4.13 such that $\Gamma_{\rm shock}$ is a C^1–curve up to its endpoints and $\varphi \in C^1(\overline{\Omega})$. To obtain a solution of Problem 4.12 from φ, we have two cases:

For the supersonic case $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm s})$, we divide region Λ into four separate regions; see Fig. 4.4. We denote by $\mathcal{S}_{0,\rm seg}$ the line segment $OP_1 \subset S_0$, and by $\mathcal{S}_{1,\rm seg}$ the portion (half-line) of S_1 with left endpoint P_2 so that $\mathcal{S}_{1,\rm seg} \subset \Lambda$. Let $\Omega_{\mathcal{S}}$ be the unbounded domain below curve $\overline{\mathcal{S}_{0,\rm seg} \cup \Gamma_{\rm shock} \cup \mathcal{S}_{1,\rm seg}}$ and above $\Gamma_{\rm wedge}$ (see Fig. 4.4). In $\Omega_{\mathcal{S}}$, let Ω_1 be the bounded domain enclosed by $\mathcal{S}_0, \Gamma_{\rm sonic}^1$, and $\Gamma_{\rm wedge}$. Set $\Omega_2 := \Omega_{\mathcal{S}} \backslash \overline{(\Omega_1 \cup \Omega)}$. Define a function φ_* in Λ by

$$\varphi_* = \begin{cases} \varphi_0 & \text{in}\, \Lambda \backslash \Omega_{\mathcal{S}}, \\ \varphi_1 & \text{in}\, \Omega_1, \\ \varphi & \text{in}\, \Gamma_{\rm sonic}^1 \cup \Omega \cup \Gamma_{\rm sonic}^2, \\ \varphi_2 & \text{in}\, \Omega_2. \end{cases} \tag{4.55}$$

By Problem 4.13(ii)–(iii), φ_* is continuous in $\Lambda \backslash \Omega_S$ and C^1 in $\overline{\Omega_{\mathcal{S}}}$. In particular, φ_* is C^1 across $\Gamma_{\rm sonic}^1 \cup \Gamma_{\rm sonic}^2$. Moreover, using Problem 4.13(i)–(iii), we obtain that φ_* is a global entropy solution of equation (4.9) in Λ.

For the subsonic case $\theta_{\rm w} \in [\theta_{\rm w}^{\rm s}, \theta_{\rm w}^{\rm d})$, region $\Omega_1 \cup \Gamma_{\rm sonic}^1$ in φ_* reduces to one point $\{O\}$; see Fig. 4.5. The corresponding φ_* is a global entropy solution of equation (4.9) in Λ.

The first rigorous unsteady analysis of the steady supersonic weak shock solution as the long-time behavior of an unsteady flow is due to Elling-Liu [61], in which they succeeded in establishing a stability theorem for an important class of physical parameters determined by certain assumptions for the wedge angle $\theta_{\rm w}$ less than the sonic angle $\theta_{\rm w}^{\rm s} \in (0, \theta_{\rm w}^{\rm d})$ for potential flow.

Recently, in Bae-Chen-Feldman [7], we have successfully removed the assumptions in Elling-Liu's theorem [61] and established the stability theorem for the steady (supersonic or transonic) weak shock solutions as the long-time asymptotics of the global Prandtl-Meyer configurations for unsteady potential flow for all the admissible physical parameters even up to the detachment angle $\theta_{\rm w}^{\rm d}$ (beyond the sonic angle $\theta_{\rm w}^{\rm s} < \theta_{\rm w}^{\rm d}$).

To achieve this, we solve the free boundary problem (Problem 4.13), involving transonic shocks, for all wedge angles $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm d})$ by employing the techniques developed in Chen-Feldman [37], described in §4.1 above. Similar to Definition 4.10, we define admissible solutions in the present case:

Definition 4.15. *Let $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$. A function $\varphi \in C^{0,1}(\overline{\Lambda})$ is an admissible solution of Problem 4.13 if φ is a solution of Problem 4.13 extended to Λ by (4.55) and satisfies the following properties:*

(i) *The structure of solutions is as follows:*
- *If $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm s})$, then φ has the configuration shown on Fig. 4.4 such that the reflected-diffracted shock $\Gamma_{\rm shock}$ is C^2 in its relative interior, $\varphi \in C^{0,1}(\Lambda) \cap C^1(\Lambda\backslash(\overline{\mathcal{S}_{0,\rm seg}} \cup \overline{\Gamma_{\rm shock}} \cup \overline{\mathcal{S}_{1,\rm seg}}))$, and $\varphi \in C^1(\overline{\Omega}) \cap C^2(\overline{\Omega}\backslash(\overline{\mathcal{S}_{0,\rm seg}} \cup \overline{\mathcal{S}_{1,\rm seg}})) \cap C^3(\Omega)$.*
- *If $\theta_{\rm w} \in [\theta_{\rm w}^{\rm s}, \theta_{\rm w}^{\rm d})$, then φ has the configuration shown on Fig. 4.5 such that the reflected-diffracted shock $\Gamma_{\rm shock}$ is C^2 in its relative interior, $\varphi \in C^{0,1}(\Lambda) \cap C^1(\Lambda\backslash(\Gamma_{\rm shock} \cup \overline{\mathcal{S}_{1,\rm seg}}))$, and $\varphi \in C^1(\overline{\Omega}) \cap C^2(\overline{\Omega}\backslash(\{O\} \cup \overline{\mathcal{S}_{1,\rm seg}})) \cap C^3(\Omega)$.*

(ii) *Equation (4.9) is strictly elliptic in $\overline{\Omega}\backslash\overline{\Gamma_{\rm sonic}}$, i.e., $|D\varphi| < c(|D\varphi|^2, \varphi)$ in $\overline{\Omega}\backslash\overline{\Gamma_{\rm sonic}}$.*

(iii) *$\partial_{\boldsymbol{\nu}}\varphi_0 > \partial_{\boldsymbol{\nu}}\varphi > 0$ on $\Gamma_{\rm shock}$, where $\boldsymbol{\nu}$ is the normal to $\Gamma_{\rm shock}$, pointing to the interior of Ω.*

(iv) *The inequalities hold:*
$$\max\{\varphi_1, \varphi_2\} \leqslant \varphi \leqslant \varphi_0 \qquad in\ \Omega, \tag{4.56}$$

(v) *The monotonicity properties hold:*
$$D(\varphi_0 - \varphi)\cdot \mathbf{e}_{\mathcal{S}_1} \geqslant 0, \quad D(\varphi_0 - \varphi)\cdot \mathbf{e}_{\mathcal{S}_0} \leqslant 0 \qquad in\ \Omega, \tag{4.57}$$
where $\mathbf{e}_{\mathcal{S}_0}$ and $\mathbf{e}_{\mathcal{S}_1}$ are the unit tangential directions to lines $\mathcal{S}_0$ and $\mathcal{S}_1$, respectively, pointing to the positive ξ_1–direction.

Similar to (4.45), we note that (4.57) implies that
$$D(\varphi_1 - \varphi)\cdot \mathbf{e} \leqslant 0 \qquad \text{in } \overline{\Omega} \ \text{ for all } \mathbf{e} \in \overline{Cone(-\mathbf{e}_{\mathcal{S}_1}, \mathbf{e}_{\mathcal{S}_0})}, \tag{4.58}$$
where $Cone(-\mathbf{e}_{\mathcal{S}_1}, \mathbf{e}_{\mathcal{S}_0}) = \{-a\,\mathbf{e}_{\mathcal{S}_1} + b\,\mathbf{e}_{\mathcal{S}_0} \ : \ a, b > 0\}$. We note that $\mathbf{e}_{\mathcal{S}_0}$ and $\mathbf{e}_{\mathcal{S}_1}$ are not parallel if $\theta_{\rm w} \neq 0$.

Then we establish the following theorem.

Theorem 4.2. *Let $\gamma > 1$ and $u_{10} > c_0$. For any $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm d})$, there exists a global entropy solution φ of Problem 4.13 such that the following regularity properties are satisfied:*

(i) *If $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm s})$, the reflected shock $\overline{\mathcal{S}_{0,\rm seg}} \cup \Gamma_{\rm shock} \cup \overline{\mathcal{S}_{1,\rm seg}}$ is $C^{2,\alpha}$–smooth, and $\varphi \in C^{1,\alpha}(\overline{\Omega}) \cap C^\infty(\overline{\Omega}\backslash(\overline{\Gamma^1_{\rm sonic}} \cup \overline{\Gamma^2_{\rm sonic}}))$,*

(ii) *If $\theta_{\rm w} \in [\theta_{\rm w}^{\rm s}, \theta_{\rm w}^{\rm d})$, the reflected shock $\overline{\Gamma_{\rm shock}} \cup \overline{\mathcal{S}_{1,\rm seg}}$ is $C^{1,\alpha}$ near O and $C^{2,\alpha}$ away from O, and $\varphi \in C^{1,\alpha}(\overline{\Omega}) \cap C^\infty(\overline{\Omega}\backslash(\{O\} \cup \overline{\Gamma^2_{\rm sonic}}))$.*

Moreover, in both cases, φ is $C^{1,1}$ across the sonic arcs, and $\Gamma_{\rm shock}$ is C^∞ in its relative interior.

Furthermore, φ is an admissible solution in the sense of Definition 4.15, so φ satisfies further properties listed in Definition 4.15.

We follow the argument described in §4.1 so that, for any small $\delta > 0$, we obtain the required uniform estimates of admissible solutions with wedge angles $\theta_{\rm w} \in [0, \theta_{\rm w}^{\rm d} - \delta]$. Using these estimates, we apply the Leray-Schauder degree theory to obtain the existence for each $\theta_{\rm w} \in [0, \theta_{\rm w}^{\rm d} - \delta]$ in the class of admissible solutions, starting from the unique normal solution for $\theta_{\rm w} = 0$. Since $\delta > 0$ is arbitrary, the existence of a weak solution for any $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm d})$ can be established. More details can be found in Bae-Chen-Feldman [7]; see also Chen-Feldman [37].

The existence results in Bae-Chen-Feldman [7] indicate that the steady weak supersonic/transonic shock solutions are the asymptotic limits of the dynamic self-similar solutions, the Prandtl-Meyer configurations, in the sense of (4.52) in Problem 5.1.

On the other hand, it is shown in Elling [60] and Bae-Chen-Feldman [7] that, for each $\gamma > 1$, there is no self-similar *strong* Prandtl-Meyer configuration for the unsteady potential flow in the class of admissible solutions (*cf.* [7]). This means that the situation for the dynamic stability of the strong steady oblique shocks is more sensitive.

5. Convexity of Self-Similar Transonic Shocks and Free Boundaries

We now discuss some recent developments in the analysis of geometric properties of transonic shocks as free boundaries in two-dimensional self-similar coordinates for compressible fluid flows. In Chen-Feldman-Xiang [38], we have developed a general framework for the analysis of the convexity of the transonic shocks as free boundaries. For both applications discussed above, the regular reflection problem in §4.1 and the Prandtl-Meyer reflection problem in §4.2, admissible solutions satisfy the conditions of this abstract framework, as shown in [38]. For simplicity, we present below the results on the convexity properties of transonic shocks for these two problems (without discussing the abstract framework).

For the regular shock reflection-diffraction configurations, we recall that, for admissible solutions in the sense of Definition 4.10, the inequality in (4.45) is shown to be strict for any $\mathbf{e} \in Cone(\mathbf{e}_{\xi_2}, \mathbf{e}_{S_1})$. From this, it is proved that, for admissible solutions, the shock is a graph in the coordinate system (S,T) with respect to basis $\{\mathbf{e}, \mathbf{e}^{\perp}\}$ for any unit vector $\mathbf{e} \in Cone(\mathbf{e}_{\xi_2}, \mathbf{e}_{S_1})$, where $\mathbf{e}^{\perp}$ is the unit vector orthogonal to $\mathbf{e}$ and oriented so that $T_{P_1} > T_{P_2}$, and we use notation (S_P, T_P) for the coordinates of point P. That is, there exists $f_{\mathbf{e}} \in C^{\infty}((T_{P_2}, T_{P_1})) \cap C^1([T_{P_2}, T_{P_1}])$ such that

$$\Gamma_{\text{shock}} = \{(S,T):\ S = f_{\mathbf{e}}(T),\ T_{P_2} < T < T_{P_1}\}, \qquad \Omega \cap \{T_{P_2} < T < T_{P_1}\} \subset \{S < f_{\mathbf{e}}(T)\}, \tag{5.1}$$

where we have used the notational convention (4.21) in the subsonic/sonic case.

Then we have

Theorem 5.1 (Convexity of transonic shocks for the regular shock reflection-diffraction configurations)**.** *If a solution of the regular reflection problem is admissible in the sense of Definition* 4.10*, then its shock curve* Γ_{shock} *is a strictly convex graph in the following sense*: *for any* $\mathbf{e} \in Cone(\mathbf{e}_{\xi_2}, \mathbf{e}_{S_1})$*, the function* $f_{\mathbf{e}}$ *in* (5.1) *satisfies*

$$f''_{\mathbf{e}} < 0 \qquad \text{on } (T_{P_2}, T_{P_1}).$$

That is, Γ_{shock} *is uniformly convex on any closed subset of its relative interior.*

Moreover, for the solution of Problem 4.6 *extended to* Λ *by* (4.22)*, with pseudo-potential* $\varphi \in C^{0,1}(\Lambda)$ *satisfying Definition* 4.10(i)–(iv)*, the shock is strictly convex if and only if Definition* 4.10(v) *holds.*

For the Prandtl-Meyer reflection problem, the results are similar. We first note that, based of (4.58) (which is strict for $\mathbf{e} \in Cone(-\mathbf{e}_{\mathcal{S}_1}, \mathbf{e}_{\mathcal{S}_0})$) and the maximum principle, it is proved that, for admissible solutions in the sense of Definition 4.15, the shock is a graph in the coordinate system (S,T) with respect to basis $\{\mathbf{e}, \mathbf{e}^{\perp}\}$ for any unit vector $\mathbf{e} \in Cone(-\mathbf{e}_{\mathcal{S}_1}, \mathbf{e}_{\mathcal{S}_0})$, *i.e.*, (5.1) holds, with $f_{\mathbf{e}} \in C^{\infty}((T_{P_2}, T_{P_1})) \cap C^1([T_{P_2}, T_{P_1}])$, where we have used the notational convention $P_1 = P_0$ in the subsonic/sonic case $\theta_{\text{w}} \in [\theta_{\text{w}}^{\text{s}}, \theta_{\text{w}}^{\text{d}})$.

Theorem 5.2 (Convexity of transonic shocks for the Prandtl-Meyer reflection configurations)**.** *If a solution of the Prandtl-Meyer reflection problem is admissible in the sense of Definition* 4.15*, then its shock curve* Γ_{shock} *is a strictly convex graph in the following sense*: *function* $f_{\mathbf{e}}$ *in* (5.1) *satisfies*

$$f''_{\mathbf{e}} < 0 \qquad \text{on } (T_{P_2}, T_{P_1}).$$

That is, Γ_{shock} *is uniformly convex on any closed subset of its relative interior.*

Moreover, for the solution of Problem 4.13 *extended to* Λ *by* (4.55) (*with the appropriate modification in the subsonic/sonic case*) *with pseudo-potential* $\varphi \in C^{0,1}(\Lambda)$ *satisfying Definition* 4.15(i)–(iv)*, the shock is strictly convex if and only if Definition* 4.15(v) *holds.*

Now we discuss the techniques developed in [38] by giving the main steps in the proofs of Theorems 5.1–5.2. While the argument in [38] is for a general domain Ω, we focus here on the specific cases of the regular shock reflection-diffraction and Prandtl reflection configurations; see [38] for the results in the more general cases and the detailed proofs.

For the regular reflection problem, define

$$\phi := \varphi - \varphi_1.$$

For the Prandtl-Meyer reflection problem, define

$$\phi := \varphi - \varphi_0.$$

Then, in both cases, $\phi = 0$ on Γ_{shock}. From this, using Definition 4.10(iii) for the regular reflection case, and Definition 4.15(iii) for the Prandtl reflection case, it follows that, in both problems, $\phi < 0$ in Ω near Γ_{shock}. Since Γ_{shock} is the zero level set of ϕ, the conclusion of Theorems 5.1–5.2 on the strict convexity of Γ_{shock} is equivalent to the following: $\phi_{\boldsymbol{\tau\tau}} > 0$ along Γ^0_{shock}, where Γ^0_{shock} is the relative interior of Γ_{shock}.

If the conclusion of Theorems 5.1–5.2 holds, then the curvature of Γ_{shock}:

$$\kappa = -\frac{f''_{\mathbf{e}}(T)}{\big(1 + (f'_{\mathbf{e}}(T))^2\big)^{3/2}}$$

has a positive lower bound on any closed subset of (T_{P_2}, T_{P_1}).

Now we briefly discuss the proofs of Theorems 5.1–5.2. Below ϕ denotes an admissible solution of either the regular reflection problem or the Prandtl-Meyer reflection problem. Also, Con denotes the cone from (4.45) for the regular reflection problem and the cone from (4.58) for the Prandtl reflection problem.

First, we establish the relation between the strict convexity/concavity of a portion of the shock and the possibility for $\partial_{\mathbf{e}}\phi$, with $\mathbf{e} \in Con$, to attain its local minimum or maximum with respect to $\overline{\Omega}$ on that portion of the shock. More precisely, on a portion of "wrong" convexity on which $f''_{\mathbf{e}} > 0$ (equivalently, $\phi_{\boldsymbol{\tau\tau}} < 0$), $\phi_{\mathbf{e}}$ may attain its local maximum, but not a local minimum. Then, assuming that a portion of the free boundary has a "wrong" convexity $f''_{\mathbf{e}} > 0$, we show that $\phi_{\mathbf{e}}$ for $\mathbf{e} \in Con$ attains its local minimum relative to Γ_{shock} on the closure of that portion. As we discussed above, it cannot be a local minimum with respect to $\overline{\Omega}$. Starting from that, through a nonlocal argument, with the use of the maximum principle for equation (4.13), considered as a linear elliptic equation for ϕ, in Ω, and boundary conditions on various parts of $\partial\Omega$, we reach a contradiction, thus showing that the shock is convex, possibly non-strictly, *i.e.*, $f''_{\mathbf{e}} \leqslant 0$ on (T_{P_2}, T_{P_1}), or equivalently, $\phi_{\boldsymbol{\tau\tau}} \geqslant 0$ on Γ_{shock}. Extending the previous argument with use of real analyticity of Γ^0_{shock}, we improve this to the locally uniform convexity as in Theorems 5.1–5.2.

Furthermore, with the convexity of reflected-diffracted transonic shocks, the uniqueness and stability of global regular shock reflection-diffraction configurations have also been established in the class of *admissible solutions*; see Chen-Feldman-Xiang [39] for the details.

The nonlinear method and related techniques/approaches we have presented above for solving multidimensional transonic shocks and free boundary problems should be useful to analyze other longstanding or newly emerging problems. Examples of such problems include the unsolved multidimensional steady transonic shock problems for the full Euler equations (including steady detached shock problems), the unsolved multidimensional self-similar transonic shock problems for potential flow (such as the two-dimensional Riemann problems and the conic body problems), as well as the longstanding open transonic shock problems for both the isentropic and the full Euler equations; also see Chen-Feldman [37]. Certainly, further new ideas, techniques, and methods are still required to be developed in order to solve these mathematically challenging and fundamental important problems.

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