REGULAR UNIMODULAR TRIANGULATIONS OF REFLEXIVE IDP 2-SUPPORTED WEIGHTED PROJECTIVE SPACE SIMPLICES

BENJAMIN BRAUN AND DEREK HANELY

ABSTRACT. For each integer partition $\mathbf{q}$ with d parts, we denote by $\Delta_{(1,\mathbf{q})}$ the lattice simplex obtained as the convex hull in $\mathbb{R}^d$ of the standard basis vectors along with the vector $-\mathbf{q}$. For $\mathbf{q}$ with two distinct parts such that $\Delta_{(1,\mathbf{q})}$ is reflexive and has the integer decomposition property, we establish a characterization of the lattice points contained in $\Delta_{(1,\mathbf{q})}$. We then construct a Gröbner basis with a squarefree initial ideal of the toric ideal defined by these simplices. This establishes the existence of a regular unimodular triangulation for reflexive 2-supported $\Delta_{(1,\mathbf{q})}$ having the integer decomposition property. As a corollary, we obtain a new proof that these simplices have unimodal Ehrhart h^*-vectors.

1. Introduction & Background

Consider an integer partition $\mathbf{q} \in \mathbb{Z}^d_{\geq 1}$ where $q_1 \leq \cdots \leq q_d$.

Definition 1.1. The lattice simplex associated with $\mathbf{q}$ is

$$\Delta_{(1,\mathbf{q})} := \operatorname{conv}\left\{\mathbf{e}_1, \ldots, \mathbf{e}_d, -\sum_{i=1}^{d} q_i \mathbf{e}_i\right\} \subset \mathbb{R}^d,$$

where $\mathbf{e}_i$ denotes the i-th standard basis vector in $\mathbb{R}^d$. Set $N(\mathbf{q}) := 1 + \sum_i q_i$.

It is straightforward to prove [13, Proposition 4.4] that $N(\mathbf{q})$ is the normalized volume of $\Delta_{(1,\mathbf{q})}$. Let $\mathcal{Q}$ denote the set of all lattice simplices of the form $\Delta_{(1,\mathbf{q})}$. The simplices in $\mathcal{Q}$ correspond to a subset of the simplices defining weighted projective spaces [9]; the vector $(1, \mathbf{q})$ gives the weights of the associated weighted projective space. Simplices in $\mathcal{Q}$ have been the subject of active recent study [1, 5, 6, 7, 12, 14]. Given a vector of distinct positive integers $\mathbf{r} = (r_1, \ldots, r_t)$, we write

$$(r_1^{x_1}, r_2^{x_2}, \ldots, r_t^{x_t}) := (\underbrace{r_1, r_1, \ldots, r_1}_{x_1 \text{ times}}, \underbrace{r_2, r_2, \ldots, r_2}_{x_2 \text{ times}}, \ldots, \underbrace{r_t, r_t, \ldots, r_t}_{x_t \text{ times}}).$$

There is a natural stratification of $\mathcal{Q}$ based on the distinct entries in the vector $\mathbf{q}$, leading to the following definition.

Definition 1.2. If $\mathbf{q} = (q_1, \ldots, q_d) = (r_1^{x_1}, r_2^{x_2}, \ldots, r_t^{x_t})$, we say that both $\mathbf{q}$ and $\Delta_{(1,\mathbf{q})}$ are *supported* by the vector $\mathbf{r} = (r_1, \ldots, r_t)$ with *multiplicity* $\mathbf{x} = (x_1, \ldots, x_t)$. We write $\mathbf{q} = (\mathbf{r}, \mathbf{x})$ in this case, and say that $\mathbf{q}$ is *t-supported*.

Because $\Delta_{(1,\mathbf{q})}$ contains the origin, the geometric dual $\Delta^*_{(1,\mathbf{q})}$ is a rational polytope, where

$$\Delta^*_{(1,\mathbf{q})} := \{\mathbf{y} \in \mathbb{R}^d \mid \langle \mathbf{a}, \mathbf{y} \rangle \leq 1 \text{ for all } \mathbf{a} \in \Delta_{(1,\mathbf{q})}\}.$$

Date: 24 May 2021.

2020 *Mathematics Subject Classification.* Primary: 52B20.

BB was partially supported by NSF award DMS-1953785. DH was partially supported by NSF award DUE-1356253.

A lattice polytope P is *reflexive* if the dual of P is also a lattice polytope. It is known [9] that $\Delta_{(1,\mathbf{q})}$ is reflexive if and only if

$$q_i \text{ divides } 1+\sum_{j=1}^{d} q_j \quad \text{for all } 1\le i\le d\,. \tag{1}$$

Thus, when (1) holds, we refer to both $\mathbf{q}$ and $\Delta_{(1,\mathbf{q})}$ as reflexive.

We say a lattice polytope P has the *integer decomposition property*, or *is IDP*, if for every $M\in\mathbb{Z}_{\ge 1}$ and $p\in(M\cdot P)\cap\mathbb{Z}^d$, there exist $p_1,\ldots,p_M\in P\cap\mathbb{Z}^d$ such that $p=p_1+\cdots+p_M$. A detailed study of reflexive IDP $\Delta_{(1,\mathbf{q})}$ was initiated in [6], motivated by several open problems regarding unimodality in Ehrhart theory (see [4, Conjectures 1 and 2]). Braun, Davis, and Solus [6, Theorem 4.1] classified the 2-supported reflexive IDP $\Delta_{(1,\mathbf{q})}$, proving that every such $\mathbf{q}$ is of the form $(r_1^{x_1},r_2^{x_2})$ where either

$$r_1>1 \text{ with } r_2=1+r_1x_1 \text{ and } x_2=r_1-1, \text{ or}$$
$$r_1=1 \text{ with } r_2=1+x_1 \text{ and } x_2 \text{ arbitrary.}$$

They further established that every $\Delta_{(1,\mathbf{q})}$ has a unimodal Ehrhart h^*-vector.

It is well known that if a lattice polytope P admits a unimodular triangulation, then P is IDP. Further, Bruns and Römer proved the following theorem.

Theorem 1.3 (Bruns and Römer [8])**.** *If P is reflexive and admits a regular unimodular triangulation, then P has a unimodal Ehrhart h^*-vector.*

Thus, it is of interest to determine whether or not reflexive IDP lattice polytopes admit regular unimodular triangulations. In particular, while we know that reflexive IDP 2-supported $\Delta_{(1,\mathbf{q})}$'s are h^*-unimodal, producing regular unimodular triangulations of these simplices demonstrates that this unimodality is a consequence of the more general Theorem 1.3.

It has been shown [6] that each 2-supported reflexive IDP $\Delta_{(1,\mathbf{q})}$ with $\mathbf{q}=(1^{x_1},(1+x_1)^{x_2})$ arises as an affine free sum of $\Delta_{(1,1^{x_1})}$ and $\Delta_{(1,1^{x_2})}$. Thus, every $\Delta_{(1,\mathbf{q})}$ of this form admits a regular unimodular triangulation, for example the triangulation arising as the join of the boundary of $\Delta_{(1,1^{x_1})}\times(0^{x_2})$ with the unique unimodular triangulation of $(0^{x_1})\times\left(\Delta_{(1,1^{x_2})}-\mathbf{e}_{x_1+1}\right)$ in $\mathbb{R}^{x_1+x_2}$. (Note that this latter simplex has two triangulations, one being the entire simplex and the other being the cone of the interior point with the boundary complex, and only one of these is unimodular.)

In this paper, we study regular unimodular triangulations for the other 2-supported case. Thus, for the remainder of this paper, we assume that $\mathbf{q}=(r_1^{x_1},(1+r_1x_1)^{r_1-1})$ with $r_1>1$. Observe that $\dim\Delta_{(1,\mathbf{q})}=d=x_1+x_2=r_1+x_1-1$. Define $\mathcal{A}'(\mathbf{q}):=\{\mathbf{a}'_1,\ldots,\mathbf{a}'_{r_1+3},\mathbf{b}'_1,\ldots,\mathbf{b}'_d\}\subset\mathbb{Z}^d$, where:

$$\begin{aligned}
\mathbf{a}'_{r_1+1}&=((-1)^{x_1},(-x_1)^{r_1-1})\\
\mathbf{a}'_{r_1+2}&=(0^{x_1},(-1)^{r_1-1})\\
\mathbf{a}'_{r_1+3}&=(0^{x_1},0^{r_1-1})\\
\mathbf{a}'_i&=(r_1-i+1)\mathbf{a}'_{r_1+1}+\mathbf{a}'_{r_1+2} \text{ for } 1\le i\le r_1\\
\mathbf{b}'_j&=\mathbf{e}_{d-j+1} \text{ for } 1\le j\le d
\end{aligned}$$

Observe that $\mathbf{a}'_1=-\mathbf{q}$, so all vertices of $\Delta_{(1,\mathbf{q})}$ are contained in $\mathcal{A}'(\mathbf{q})$. Note that later in this work we will use the notation $\mathcal{A}(\mathbf{q})$ to denote the set of these vectors where each vector has a 1 appended. Thus, we use $\mathcal{A}'(\mathbf{q})$ for the vectors defined above.

Example 1.4. Let $r_1=6$ and $x_1=4$, so $\mathbf{q}=(6^4,25^5)\in\mathbb{Z}^9$. The elements of $\mathcal{A}'(\mathbf{q})$ are given by the columns of the matrix in Figure 1.

In this paper, we prove the following theorems.

$$
\begin{array}{cccccccccccccccccc}
\mathbf{a}'_1 & \mathbf{a}'_2 & \mathbf{a}'_3 & \mathbf{a}'_4 & \mathbf{a}'_5 & \mathbf{a}'_6 & \mathbf{a}'_7 & \mathbf{a}'_8 & \mathbf{a}'_9 & \mathbf{b}'_1 & \mathbf{b}'_2 & \mathbf{b}'_3 & \mathbf{b}'_4 & \mathbf{b}'_5 & \mathbf{b}'_6 & \mathbf{b}'_7 & \mathbf{b}'_8 & \mathbf{b}'_9
\end{array}
$$

$$
\begin{pmatrix}
-6 & -5 & -4 & -3 & -2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-6 & -5 & -4 & -3 & -2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-6 & -5 & -4 & -3 & -2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-6 & -5 & -4 & -3 & -2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-25 & -21 & -17 & -13 & -9 & -5 & -4 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-25 & -21 & -17 & -13 & -9 & -5 & -4 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-25 & -21 & -17 & -13 & -9 & -5 & -4 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-25 & -21 & -17 & -13 & -9 & -5 & -4 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-25 & -21 & -17 & -13 & -9 & -5 & -4 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

FIGURE 1. $\mathcal{A}'(\mathbf{q})$ for $\mathbf{q} = (6^4, 25^5)$.

Theorem 1.5. *For $\mathbf{q} = (r_1^{x_1}, (1 + r_1x_1)^{r_1-1})$ with $r_1 > 1$, the lattice points of the IDP simplex $\Delta_{(1,\mathbf{q})}$ are given by $\mathcal{A}'(\mathbf{q})$.*

Theorem 1.6. *For $\mathbf{q} = (r_1^{x_1}, (1 + r_1x_1)^{r_1-1})$ with $r_1 > 1$, there exists a lexicographic squarefree initial ideal of the toric ideal associated with $\Delta_{(1,\mathbf{q})}$.*

Corollary 1.7. *Every 2-supported IDP reflexive simplex $\Delta_{(1,\mathbf{q})}$ admits a regular unimodular triangulation. When $\mathbf{q} = (r_1^{x_1}, (1+r_1x_1)^{r_1-1})$ with $r_1 > 1$, this triangulation is induced by a lexicographic term order $<_{lex}$.*

The remainder of this paper is structured as follows. In Section 2 we prove Theorem 1.5. In Section 3 we introduce needed algebraic machinery and prove Theorem 1.6. In Section 4 we describe the facets of the resulting triangulation and discuss connections to the Ehrhart h^*-vector of $\Delta_{(1,\mathbf{q})}$.

2. Proof of Theorem 1.5

Our strategy is to determine the number of lattice points in $\Delta_{(1,\mathbf{q})}$, show that this value equals the number of columns of $\mathcal{A}'(\mathbf{q})$, and then show that all of the columns of $\mathcal{A}'(\mathbf{q})$ are contained in $\Delta_{(1,\mathbf{q})}$.

Proposition 2.1. *For $\mathbf{q}$ as given in Theorem 1.5, we have $\left|\Delta_{(1,\mathbf{q})} \cap \mathbb{Z}^d\right| = r_1 + d + 3$.*

Proof. Using [6, Theorem 2.2], we know the *Ehrhart h^*-polynomial* of $\Delta_{(1,\mathbf{q})}$, denoted $h^*(\Delta_{(1,\mathbf{q})}; z) := h_0^* + h_1^*z + \cdots + h_d^*z^d$, is given by

$$h^*(\Delta_{(1,\mathbf{q})}; z) = \sum_{b=0}^{r_1(x_1r_1+1)-1} z^{w(\mathbf{q},b)},$$

where

$$w(\mathbf{q}, b) := b - x_1 \left\lfloor \frac{b}{1 + x_1r_1} \right\rfloor - (r_1 - 1) \left\lfloor \frac{b}{r_1} \right\rfloor .$$

It is well known (see, e.g., [2]) that the coefficient h_1^* is given by the formula

$$h_1^* = \left|\Delta_{(1,\mathbf{q})} \cap \mathbb{Z}^d\right| - (\dim \Delta_{(1,\mathbf{q})} + 1) . \tag{2}$$

To compute h_1^*, we must determine all b for which $w(\mathbf{q}, b) = 1$. Since $0 \le b \le r_1(x_1r_1+1)-1$, the division algorithm allows us to write $b = \alpha(1 + x_1r_1) + \beta$, where $0 \le \alpha < r_1$ and $0 \le \beta < 1 + x_1r_1$.

Hence,

$$\begin{aligned} w(\mathbf{q},b) &= w(\mathbf{q},\alpha(1+x_1r_1)+\beta) \\ &= \alpha(1+x_1r_1)+\beta - x_1\left\lfloor \frac{\alpha(1+x_1r_1)+\beta}{1+x_1r_1}\right\rfloor - (r_1-1)\left\lfloor \frac{\alpha(1+x_1r_1)+\beta}{r_1}\right\rfloor \\ &= \alpha(1+x_1r_1)+\beta-\alpha x_1-(r_1-1)\left(\alpha x_1+\left\lfloor\frac{\alpha+\beta}{r_1}\right\rfloor\right) \\ &= \alpha+\beta-(r_1-1)\left\lfloor\frac{\alpha+\beta}{r_1}\right\rfloor. \end{aligned}$$

Therefore, the equation $w(\mathbf{q},b)=1$ becomes

$$\alpha+\beta-(r_1-1)\left\lfloor\frac{\alpha+\beta}{r_1}\right\rfloor = 1 \quad\Longleftrightarrow\quad \alpha+\beta = 1+(r_1-1)\left\lfloor\frac{\alpha+\beta}{r_1}\right\rfloor.$$

Now, let $\ell = \left\lfloor\frac{\alpha+\beta}{r_1}\right\rfloor$. By the previous equation, $\alpha+\beta = 1+(r_1-1)\ell$. Substituting this equivalent expression for $\alpha+\beta$ into both sides of the previous equation, it follows that solving $w(\mathbf{q},b)=1$ is equivalent to finding all pairs (α,β) such that

$$\begin{aligned} 1+(r_1-1)\ell &= 1+(r_1-1)\left\lfloor\frac{1+(r_1-1)\ell}{r_1}\right\rfloor \\ &= 1+(r_1-1)\left(\ell+\left\lfloor\frac{1-\ell}{r_1}\right\rfloor\right). \end{aligned}$$

Rearranging this equation yields

$$(r_1-1)\left\lfloor\frac{1-\ell}{r_1}\right\rfloor = 0.$$

Therefore, since $r_1>1$, this implies

$$\left\lfloor\frac{1-\ell}{r_1}\right\rfloor = 0 \quad\Longrightarrow\quad \ell = \begin{cases} 0 \\ 1 \end{cases} \quad\Longrightarrow\quad \alpha+\beta = \begin{cases} 1 \\ r_1 \end{cases}.$$

If $\alpha+\beta=1$, then $(\alpha,\beta)=(1,0)$ or $(\alpha,\beta)=(0,1)$. Otherwise, in the case that $\alpha+\beta=r_1$, there are r_1 possible pairs (α,β) where $\alpha\in\{0,\ldots,r_1-1\}$ and $\beta=r_1-\alpha$. Thus,

$$h_1^* = |\{b : w(\mathbf{q},b)=1\}| = r_1+2.$$

Consequently, (2) implies

$$\left|\Delta_{(1,\mathbf{q})}\cap\mathbb{Z}^d\right| = r_1+d+3,$$

as desired. ☐

Proposition 2.2. *For* $\mathbf{t}=(t_1,\ldots,t_d)\in\mathbb{R}^d$, *define*

$$\lambda_k(\mathbf{t}) := \begin{cases} \displaystyle\sum_{\substack{j=1\\ j\neq k}}^{d} t_j - x_1r_1t_k, & \text{if } 1\le k\le x_1 \\ \displaystyle\sum_{\substack{j=1\\ j\neq k}}^{d} t_j-(r_1-1)t_k, & \text{if } x_1+1\le k\le d \\ \displaystyle\sum_{j=1}^{d} t_j, & \text{if } k=d+1\,. \end{cases}$$

An irredundant $\mathcal{H}$-description of $\Delta_{(1,\mathbf{q})}$ is given by $\lambda_k(\mathbf{t}) \leq 1$ for all $1 \leq k \leq d+1$.

Proof. Observe that for all $1 \leq j \leq d$, $\mathbf{e}_j$ satisfies all of the given inequalities tightly except when $k = j$ (i.e., $\lambda_k(\mathbf{e}_j) = 1$ for all $k \neq j$ and $\lambda_j(\mathbf{e}_j) < 1$). Moreover, $-\mathbf{q}$ satisfies the first d inequalities tightly (i.e., $\lambda_k(-\mathbf{q}) = 1$ for all $1 \leq k \leq d$), but not $\sum_j t_j \leq 1$. Thus, as each vertex of the simplex $\Delta_{(1,\mathbf{q})}$ satisfies exactly d of the given inequalities with equality, the inequalities necessarily constitute an $\mathcal{H}$-description of $\Delta_{(1,\mathbf{q})}$. □

Proof of Theorem 1.5. To begin, observe that $|\mathcal{A}'(\mathbf{q})| = r_1 + d + 3 = \left|\Delta_{(1,\mathbf{q})} \cap \mathbb{Z}^d\right|$. Therefore, as each element in $\mathcal{A}'(\mathbf{q})$ is an integer vector, it suffices to show that each point satisfies the inequalities in Proposition 2.2. To this end, let λ_k be defined as in Proposition 2.2; we evaluate each vector in $\mathcal{A}'(\mathbf{q})$ on λ_k. For each $1 \leq i \leq r_1$, note that

$$\mathbf{a}'_i = (r_1 - i + 1)\mathbf{a}'_{r_1+1} + \mathbf{a}'_{r_1+2} = \left((-(r_1 - i + 1))^{x_1}, ((-(1 + (r_1 - i + 1)x_1))^{r_1 - 1}\right).$$

Therefore, we have that

$$\lambda_k(\mathbf{a}'_1) = 1 \text{ if } 1 \leq k \leq d \quad \text{and} \quad \lambda_{d+1}(\mathbf{a}'_1) < 1,$$

and for each $i \in \{2, \ldots, r_1\} \cup \{r_1 + 2\}$,

$$\lambda_k(\mathbf{a}'_i) = 1 \text{ if } x_1 + 1 \leq k \leq d \quad \text{and} \quad \lambda_k(\mathbf{a}'_i) < 1 \text{ otherwise.}$$

Also,

$$\lambda_k(\mathbf{a}'_{r_1+1}) = 1 \text{ if } 1 \leq k \leq x_1 \quad \text{and} \quad \lambda_k(\mathbf{a}'_{r_1+1}) < 1 \text{ otherwise,}$$

and

$$\lambda_k(\mathbf{a}'_{r_1+3}) < 1 \text{ for all } 1 \leq k \leq d+1.$$

Lastly, for all $1 \leq j \leq d$,

$$\lambda_k(\mathbf{b}'_j) = 1 \text{ if } k \neq d - j + 1 \quad \text{and} \quad \lambda_k(\mathbf{b}'_j) < 1 \text{ if } k = d - j + 1.$$

Thus, $\mathcal{A}'(\mathbf{q}) \subseteq \Delta_{(1,\mathbf{q})} \cap \mathbb{Z}^d$, and the result follows. □

3. Proof of Theorem 1.6

We next seek to prove the existence of a regular unimodular triangulation of the convex hull of these points. Given a field K, there are natural parallels between properties of lattice polytopes and algebraic objects such as semigroup algebras, toric varieties, and monomial ideals. The following one-to-one correspondence between lattice points and Laurent monomials plays a central role:

$$\mathbf{a}' = (a_1, \ldots, a_d) \in \mathbb{Z}^d \quad \longleftrightarrow \quad \mathbf{t}^{\mathbf{a}'} := t_1^{a_1} \cdots t_d^{a_d} \in K[t_1^{\pm 1}, \ldots, t_d^{\pm 1}].$$

For details regarding the significance of this correspondence, see [16, Chapter 8]. Furthermore, for all notation related to combinatorial commutative algebra, we refer the reader to [10].

Let K be a field, and define $\mathcal{A}(\mathbf{q}) = (\mathbf{a}_1, \ldots, \mathbf{a}_{r_1+3}, \mathbf{b}_1, \ldots, \mathbf{b}_d) \subset \mathbb{Z}^{(d+1)\times(r_1+d+3)}$ to be the homogenization of $\mathcal{A}'(\mathbf{q})$ where $\mathbf{a}_i = (\mathbf{a}'_i, 1)$ and $\mathbf{b}_j = (\mathbf{b}'_j, 1)$; that is, $\mathcal{A}(\mathbf{q})$ is the matrix associated with the point configuration consisting of all vectors in $\mathcal{A}'(\mathbf{q})$ lifted to height 1. (Note that we can view the columns of $\mathcal{A}(\mathbf{q})$ as the intersection of $\mathbb{Z}^{d+1}$ with the degree 1 slice of the polyhedral cone over $\Delta_{(1,\mathbf{q})}$.) Let $K[\mathcal{A}(\mathbf{q})] := K[z_1, \ldots, z_{r_1+3}, y_1, \ldots, y_d]$ be the polynomial ring associated with the columns of $\mathcal{A}(\mathbf{q})$ in $r_1 + d + 3$ variables over K. Moreover, let $\mathcal{M}(K[\mathcal{A}(\mathbf{q})])$ denote the set of monomials contained in $K[\mathcal{A}(\mathbf{q})]$, and let $\mathcal{R}_K[\mathcal{A}(\mathbf{q})]$ be the K-subalgebra of the Laurent polynomial ring $K[\mathbf{t}^{\pm 1}] := K[t_1^{\pm 1}, \ldots, t_{d+1}^{\pm 1}]$ generated by all monomials $\mathbf{t}^{\mathbf{a}}$ with $\mathbf{a} \in \mathcal{A}(\mathbf{q})$, where

$\mathbf{t^a} = t_1^{a_1} \cdots t_{d+1}^{a_{d+1}}$ if $\mathbf{a} = (a_1, \ldots, a_{d+1})$. The toric ideal $I_{\mathcal{A}(\mathbf{q})}$ is the kernel of the surjective ring homomorphism $\pi : K[\mathcal{A}(\mathbf{q})] \to \mathcal{R}_K[\mathcal{A}(\mathbf{q})]$ defined by

$$\pi(z_i) = \mathbf{t}^{\mathbf{a}_i}, \text{ for } 1 \le i \le r_1 + 3$$
$$\pi(y_j) = \mathbf{t}^{\mathbf{b}_j}, \text{ for } 1 \le j \le d.$$

A generating set for $I_{\mathcal{A}(\mathbf{q})}$ is given by the set of all homogeneous binomials $f - g$ with $\pi(f) = \pi(g)$ and $f, g \in \mathcal{M}(K[\mathcal{A}(\mathbf{q})])$, see [16, Lemma 4.1]. We fix the lexicographic term order $<_{lex}$ on $K[\mathcal{A}(\mathbf{q})]$ induced by the ordering of the variables

$$z_1 > z_2 > \cdots > z_{r_1+3} > y_1 > y_2 > \cdots > y_d.$$

Moreover, for $f = z_1^{\gamma_1} \cdots z_{r_1+3}^{\gamma_{r_1+3}} y_1^{\delta_1} \cdots y_d^{\delta_d} \in \mathcal{M}(K[\mathcal{A}(\mathbf{q})])$, we introduce the notation

$$\operatorname{supp}_{\mathbf{z}}(f) := \{i \in \{1, \ldots, r_1 + 3\} : \gamma_i > 0\}.$$

Given this setup, we restate Theorem 1.6. Note that Corollary 1.7 follows immediately from Theorem 3.1 as it indicates the existence of a squarefree initial ideal of the toric ideal $I_{\mathcal{A}(\mathbf{q})}$ [16, Corollary 8.9].

Theorem 3.1 (Restatement of Theorem 1.6)**.** *Let B be the set of all $(i, j) \in \mathbb{N}^2$ satisfying the following conditions:*

(i) $j - i \ge 2$
(ii) $1 \le i \le r_1$
(iii) $j \le r_1 + 3$
(iv) $j \ne r_1 + 1$
(v) $(i, j) \ne (r_1, r_1 + 2)$

Given $(i, j) \in B$, define (k, ℓ) as follows:

$$\begin{aligned}
&k = \left\lfloor \frac{i+j}{2} \right\rfloor, \ell = \left\lceil \frac{i+j}{2} \right\rceil && \text{if } j < r_1 + 1\\
&k = \left\lfloor \frac{i+j-1}{2} \right\rfloor, \ell = \left\lceil \frac{i+j-1}{2} \right\rceil && \text{if } j = r_1 + 2\\
&k = i + 1,\, \ell = r_1 + 1 && \text{if } j = r_1 + 3, i \ne r_1\\
&k = r_1 + 1,\, \ell = r_1 + 2 && \text{if } j = r_1 + 3, i = r_1.
\end{aligned}$$

If $x_1 \ge r_1 - 2$, then the set of binomials $\mathcal{G}$ given by

$$z_i z_j - z_k z_\ell, \qquad (i, j) \in B \tag{3}$$

$$z_{k+1} \prod_{\ell=1}^{r_1-1} y_\ell - z_{r_1+1}^{r_1-k} z_{r_1+3}^{k}, \qquad 0 \le k \le r_1 - 1 \tag{4}$$

$$z_{r_1-k} \prod_{\ell=r_1}^{d} y_\ell - z_{r_1}^{k} z_{r_1+2}^{x_1+1-k}, \qquad 0 \le k \le r_1 - 1 \tag{5}$$

$$z_{r_1+2} \prod_{\ell=1}^{r_1-1} y_\ell - z_{r_1+3}^{r_1}, \tag{6}$$

$$z_{r_1+1} \prod_{\ell=r_1}^{d} y_\ell - z_{r_1+2}^{x_1} z_{r_1+3} \tag{7}$$

is a Gröbner basis of $I_{\mathcal{A}(\mathbf{q})}$ with respect to the lexicographic term order $<_{lex}$ as specified above. In the case that $x_1 < r_1 - 2$, replace (5) above with

$$\begin{cases} z_{r_1-k}\prod_{\ell=r_1}^{d} y_\ell - z_{r_1}^{k} z_{r_1+2}^{x_1+1-k}, & 0 \leq k \leq x_1+1 \\ z_{r_1-k}\prod_{\ell=r_1}^{d} y_\ell - z_{r_1-1}^{k-x_1-1} z_{r_1}^{2x_1+2-k}, & x_1+2 \leq k \leq r_1-1. \end{cases} \tag{5*}$$

Note that regardless of case (either $x_1 \geq r_1 - 2$ or $x_1 < r_1 - 2$), the initial terms of the k-th binomial in (5) and (5*) are identical. Therefore, whenever we are considering only leading terms of these polynomials, we can ignore any relationship between x_1 and $r_1 - 2$.

Remark 3.2. The intuition for most of these binomial relations is that they are encoding the additive structure on the columns of $\mathcal{A}(\mathbf{q})$. Specifically, in the definition of $\mathcal{A}'(\mathbf{q})$, we see that $\mathbf{a}'_i = (r_1 - i + 1)\mathbf{a}'_{r_1+1} + \mathbf{a}'_{r_1+2}$ for $1 \leq i \leq r_1$, and there are natural syzygies that result from this structure. We require the replacement of (5) with (5*) in the case that $x_1 < r_1 - 2$ since otherwise, the exponent of z_{r_1+2}, namely $x_1 + 1 - k$, would fail to be positive when $x_1 + 2 \leq k \leq r_1 - 1$.

To prove Theorem 3.1, we employ the following well-known lemma, e.g. [11, (0.1)], for proving a finite subset of the toric ideal $I_{\mathcal{A}(\mathbf{q})}$ is a Gröbner basis of $I_{\mathcal{A}(\mathbf{q})}$. For a finite set of polynomials $\mathcal{G}$ in a polynomial ring with a term order $<$, let $in_<(\mathcal{G})$ denote the ideal generated by the set of initial terms of elements of $\mathcal{G}$.

Lemma 3.3 ([11])**.** *A finite set $\mathcal{G}$ of $I_{\mathcal{A}(\mathbf{q})}$ is a Gröbner basis of $I_{\mathcal{A}(\mathbf{q})}$ with respect to the term order $<$ if and only if $\{\pi(f) : f \in \mathcal{M}(K[\mathcal{A}(\mathbf{q})]), f \notin in_<(\mathcal{G})\}$ is linearly independent over K; i.e., if and only if $\pi(f) \neq \pi(g)$ for all $f \notin in_<(\mathcal{G})$ and $g \notin in_<(\mathcal{G})$ with $f \neq g$.*

We will also require the following fact which provides an upper bound on the supported z-variables for any monomial outside the initial ideal generated by the binomials in Theorem 3.1 with respect to $<_{lex}$.

Lemma 3.4. *Let $\mathcal{G}$ be the set of binomials given in Theorem 3.1. Suppose*

$$f = z_1^{\gamma_1} \cdots z_{r_1+3}^{\gamma_{r_1+3}} y_1^{\delta_1} \cdots y_d^{\delta_d} \in \mathcal{M}(K[\mathcal{A}(\mathbf{q})])$$

with $f \notin in_{<_{lex}}(\mathcal{G})$ and $|\mathrm{supp}_{\mathbf{z}}(f)| \geq 1$. Let m denote the minimal index such that z_m divides f. Then, $|\mathrm{supp}_{\mathbf{z}}(f)| \leq 3$ and we are restricted to the following possibilities:

(1) if $1 \leq m \leq r_1 - 1$, then $\gamma_{m+1}, \gamma_{r_1+1} \geq 0$ and $\gamma_i = 0$ for all $i \in \{1, \ldots, r_1+3\} \setminus \{m, m+1, r_1+1\}$.
(2) if $m = r_1$, then $\gamma_{r_1+1}, \gamma_{r_1+2} \geq 0$ and $\gamma_i = 0$ for all $i \in \{1, \ldots, r_1 - 1\} \cup \{r_1 + 3\}$.
(3) if $m \in \{r_1+1, r_1+2, r_1+3\}$, then $\gamma_i = 0$ for all $i < m$ and $\gamma_i \geq 0$ for all $i > m$.

Proof. Suppose $1 \leq m \leq r_1 - 1$. Since $z_m z_{m+1} \notin in_{<_{lex}}(\mathcal{G})$ and $z_m z_{r_1+1} \notin in_{<_{lex}}(\mathcal{G})$, z_{m+1} and z_{r_1+1} possibly divide f. However, given the structure of B as defined in Theorem 3.1, it follows that $z_m z_{r_1+2}, z_m z_{r_1+3}, z_m z_n \in in_{<_{lex}}(\mathcal{G})$ for all n with $n > m+1, n \neq r_1+1$. Therefore, since m is minimal, $|\mathrm{supp}_{\mathbf{z}}(f)| \leq 3$ and we precisely satisfy the conditions of Lemma 3.4(1).

Now, suppose $m = r_1$. By the minimality of m, we need only consider indices greater than r_1. Observe that $z_{r_1} z_{r_1+1} \notin in_{<_{lex}}(\mathcal{G})$, $z_{r_1} z_{r_1+2} \notin in_{<_{lex}}(\mathcal{G})$, and $z_{r_1} z_{r_1+3} \in in_{<_{lex}}(\mathcal{G})$. Thus, we have that $|\mathrm{supp}_{\mathbf{z}}(f)| \leq 3$ and we end up in Lemma 3.4(2).

Finally, for $m \in \{r_1+1, r_1+2, r_1+3\}$, minimality of m immediately implies $|\mathrm{supp}_{\mathbf{z}}(f)| \leq 3$. To see that this case yields Lemma 3.4(3), observe that $z_m z_n \notin in_{<_{lex}}(\mathcal{G})$ for $m, n \in \{r_1+1, r_1+2, r_1+3\}$ with $m \neq n$. □

Proof of Theorem 3.1. One easily checks that each binomial $h = m_1 - m_2 \in \mathcal{G}$ is contained in $I_{\mathcal{A}(\mathbf{q})}$ by showing $\pi(m_1) = \pi(m_2)$. To show $\mathcal{G}$ is a Gröbner basis of $I_{\mathcal{A}(\mathbf{q})}$, we employ Lemma 3.3. Suppose $f, g \in \mathcal{M}(K[\mathcal{A}(\mathbf{q})])$ with $f \neq g$, $f \notin in_{<_{lex}}(\mathcal{G})$, and $g \notin in_{<_{lex}}(\mathcal{G})$. Write

$$f = z_1^{\alpha_1} \cdots z_{r_1+3}^{\alpha_{r_1+3}} y_1^{\beta_1} \cdots y_d^{\beta_d} \quad \text{and} \quad g = z_1^{\alpha'_1} \cdots z_{r_1+3}^{\alpha'_{r_1+3}} y_1^{\beta'_1} \cdots y_d^{\beta'_d},$$

where $\alpha_i, \alpha'_i, \beta_j, \beta'_j \geq 0$. We may assume f and g are relatively prime (since otherwise, we could simply factor out the common variables and consider the images of the reduced monomials). Further assume to the contrary that $\pi(f) = \pi(g)$, and without loss of generality, assume $|\mathrm{supp}_{\mathbf{z}}(f)| \geq |\mathrm{supp}_{\mathbf{z}}(g)|$. For convenience, let $\mathbf{f}^\pi, \mathbf{g}^\pi \in \mathbb{Z}^{d+1}$ denote the exponent vectors associated with $\pi(f)$ and $\pi(g)$, respectively, and let $\mathbf{f}^\pi[k]$ (resp. $\mathbf{g}^\pi[k]$) denote the k-th entry of $\mathbf{f}^\pi$ (resp. $\mathbf{g}^\pi$). With this notation, observe that $\pi(f) = \pi(g)$ if and only if $\mathbf{f}^\pi[k] = \mathbf{g}^\pi[k]$ for all $1 \leq k \leq d+1$.

The general structure for the rest of the proof is to consider cases based on the size of the $\mathbf{z}$-support for monomials g and f. Throughout, we identify the minimal indices of the $\mathbf{z}$-variables dividing both g and f, and we repeatedly apply Lemma 3.4 to deduce a contradiction in each of the resulting cases.

Case 1: $|\mathrm{supp}_{\mathbf{z}}(g)| = 0$. By definition, it follows that $\alpha'_i = 0$ for all $1 \leq i \leq r_1 + 3$. Therefore, we know that

$$\mathbf{g}^\pi = \left(\beta'_d, \ldots, \beta'_1, \textstyle\sum_j \beta'_j\right).$$

Subcase 1.1: $|\mathrm{supp}_{\mathbf{z}}(f)| = 0$. Thus,

$$\mathbf{f}^\pi = \left(\beta_d, \ldots, \beta_1, \textstyle\sum_j \beta_j\right).$$

Since $\pi(f) = \pi(g)$, this implies $\beta_j = \beta'_j$ for all $1 \leq j \leq d$, and consequently, $f = g$, a contradiction.

Subcase 1.2: $|\mathrm{supp}_{\mathbf{z}}(f)| \geq 1$. Let m denote the minimal index such that z_m divides f (i.e., $\alpha_m > 0$ and $\alpha_i = 0$ for all $i < m$).

(a) Suppose $1 \leq m \leq r_1 + 1$. Since $z_m y_{r_1} \cdots y_d \in in_{<_{lex}}(\mathcal{G})$ (by (5) and (7)) and $f \notin in_{<_{lex}}(\mathcal{G})$, there exists an index $\ell \in \{r_1, \ldots, d\}$ such that $\beta_\ell = 0$. Hence,

$$\mathbf{f}^\pi[d-\ell+1] = \underbrace{\sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-\ell+1,i}}_{<0} + \underbrace{\sum_{j=1}^{d} \beta_j \mathcal{A}(\mathbf{q})_{d-\ell+1,r_1+3+j}}_{=0} < 0.$$

However, $\mathbf{g}^\pi[d-\ell+1] = \beta'_\ell \geq 0$, a contradiction.

(b) Suppose $m = r_1 + 2$. Since $z_{r_1+2} y_1 \cdots y_{r_1-1} \in in_{<_{lex}}(\mathcal{G})$ (by (6)) and $f \notin in_{<_{lex}}(\mathcal{G})$, there exists an index $k \in \{1, \ldots, r_1 - 1\}$ such that $\beta_k = 0$. Since $k < r_1$, it follows that $d - k + 1 > x_1$. Therefore, $\mathcal{A}(\mathbf{q})_{d-k+1,r_1+2} = -1$. Hence,

$$\mathbf{f}^\pi[d-k+1] = \underbrace{\sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-k+1,i}}_{=-\alpha_{r_1+2}<0} + \underbrace{\sum_{j=1}^{d} \beta_j \mathcal{A}(\mathbf{q})_{d-k+1,r_1+3+j}}_{=0} < 0.$$

However, $\mathbf{g}^\pi[d-k+1] = \beta'_k \geq 0$, a contradiction.

(c) Suppose $m = r_1 + 3$. Since m is minimal, we know $\alpha_i = 0$ for all $1 \leq i \leq r_1 + 2$. Since $\mathcal{A}(\mathbf{q})$ is homogenized, we also know $\sum_i \alpha_i + \sum_j \beta_j = \sum_i \alpha'_i + \sum_j \beta'_j$ (this can be seen directly from $\mathbf{f}^\pi[d+1] = \mathbf{g}^\pi[d+1]$). Hence, in this case, the equation simplifies to $\alpha_{r_1+3} + \sum_j \beta_j = \sum_j \beta'_j$, and moreover,

$$\mathbf{f}^\pi = (\beta_d, \ldots, \beta_1, \alpha_{r_1+3} + \textstyle\sum_j \beta_j).$$

Since $\pi(f) = \pi(g)$, $\beta_j = \beta'_j$ for all $1 \le j \le d$. Therefore, substituting into the above equation,

$$\alpha_{r_1+3} + \sum_j \beta_j = \sum_j \beta'_j = \sum_j \beta_j,$$

but $\alpha_{r_1+3} > 0$, a contradiction.

Case 2: $|\text{supp}_{\mathbf{z}}(g)| \ge 1$. Let n denote the minimal index such that z_n divides g (i.e., $\alpha'_n > 0$ and $\alpha'_i = 0$ for all $i < n$). Since $|\text{supp}_{\mathbf{z}}(f)| \ge |\text{supp}_{\mathbf{z}}(g)|$ and $|\text{supp}_{\mathbf{z}}(g)| \ge 1$, we know $\text{supp}_{\mathbf{z}}(f) \neq \emptyset$. Hence, let m denote the minimal index such that z_m divides f. Via Lemma 3.4, this case naturally lends itself to the following subcases of consideration.

Subcase 2.1: $n \in \{1, \dots, r_1 - 1\}$. By Lemma 3.4, we know $\alpha'_n > 0$, $\alpha'_{n+1}, \alpha'_{r_1+1} \ge 0$, and $\alpha'_i = 0$ for all $i \in \{1, \dots, r_1+3\} \setminus \{n, n+1, r_1+1\}$. Since $z_n y_1 \cdots y_{r_1-1} \in in_{<_{lex}}(\mathcal{G})$ (by (4)), $z_n y_{r_1} \cdots y_d \in in_{<_{lex}}(\mathcal{G})$ (by (5)), and $g \notin in_{<_{lex}}(\mathcal{G})$, there exist indices $k_1 \in \{1, \dots, r_1 - 1\}$ and $\ell_1 \in \{r_1, \dots, d\}$ such that $\beta'_{k_1} = \beta'_{\ell_1} = 0$. Then,

$$\mathbf{f}^\pi[d - k_1 + 1] = \sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-k_1+1,i} + \beta_{k_1} \tag{8}$$

$$\mathbf{g}^\pi[d - k_1 + 1] = -(1 + (r_1 - n + 1)x_1)\alpha'_n - (1 + (r_1 - n)x_1)\alpha'_{n+1} - x_1\alpha'_{r_1+1} \tag{9}$$

$$\mathbf{f}^\pi[d - \ell_1 + 1] = \sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-\ell_1+1,i} + \beta_{\ell_1} \tag{10}$$

$$\mathbf{g}^\pi[d - \ell_1 + 1] = -(r_1 - n + 1)\alpha'_n - (r_1 - n)\alpha'_{n+1} - \alpha'_{r_1+1}. \tag{11}$$

Note that $\pi(f) = \pi(g)$ implies (8) = (9) and (10) = (11). Now, we claim $m \in \{1, \dots, r_1 + 1\}$. Indeed, assume otherwise, that is, $\text{supp}_{\mathbf{z}}(f) \subseteq \{r_1 + 2, r_1 + 3\}$. Then, $\mathbf{f}^\pi[d - \ell + 1] = \beta_\ell \ge 0$ for all $\ell \in \{r_1, \dots, d\}$, but from (11), $\mathbf{g}^\pi[d - \ell_1 + 1] < 0$ since $\alpha'_n > 0$ and $\alpha'_{n+1}, \alpha'_{r_1+1} \ge 0$. This contradicts $\pi(f) = \pi(g)$. Hence, given the structure of Lemma 3.4, we consider the following subsubcases.

(a) $m \in \{1, \dots, r_1 - 1\}$. Since $z_m y_1 \cdots y_{r_1-1}$ (by (4)), $z_m y_{r_1} \cdots y_d \in in_{<_{lex}}(\mathcal{G})$ (by (5)), and $f \notin in_{<_{lex}}(\mathcal{G})$, there exist indices $k_2 \in \{1, \dots, r_1 - 1\}$ and $\ell_2 \in \{r_1, \dots, d\}$ such that $\beta_{k_2} = \beta_{\ell_2} = 0$. Then, we have that

$$\mathbf{f}^\pi[d - k_2 + 1] = \sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-k_2+1,i} \tag{12}$$

$$\begin{aligned}\mathbf{g}^\pi[d - k_2 + 1] = &-(1 + (r_1 - n + 1)x_1)\alpha'_n - (1 + (r_1 - n)x_1)\alpha'_{n+1} \\ &- x_1\alpha'_{r_1+1} + \beta'_{k_2}\end{aligned} \tag{13}$$

$$\mathbf{f}^\pi[d - \ell_2 + 1] = \sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-\ell_2+1,i} \tag{14}$$

$$\mathbf{g}^\pi[d - \ell_2 + 1] = -(r_1 - n + 1)\alpha'_n - (r_1 - n)\alpha'_{n+1} - \alpha'_{r_1+1} + \beta'_{\ell_2}, \tag{15}$$

where (12) = (13) and (14) = (15) as $\pi(f) = \pi(g)$. Since $1 \le k_i \le r_1 - 1$ for $i \in \{1, 2\}$, subtracting the equation (12) = (13) from (8) = (9) implies $\beta_{k_1} = -\beta'_{k_2}$. Similarly, since $r_1 \le \ell_i \le d$ for $i \in \{1, 2\}$, subtracting equation (14) = (15) from (10) = (11) implies $\beta_{\ell_1} = -\beta'_{\ell_2}$. Since $\beta_j, \beta'_j \ge 0$ for all $1 \le j \le d$, this implies $\beta_{k_1} = \beta'_{k_2} = \beta_{\ell_1} = \beta'_{\ell_2} = 0$. Also, by Lemma 3.4, we know $\alpha_m > 0$, $\alpha_{m+1}, \alpha_{r_1+1} \ge 0$, and $\alpha_i = 0$ for all $i \in \{1, \dots, r_1 + 3\} \setminus \{m, m+1, r_1+1\}$. Consequently, equations (8) and (10) simplify to

$$\mathbf{f}^\pi[d - k_1 + 1] = -(1 + (r_1 - m + 1)x_1)\alpha_m - (1 + (r_1 - m)x_1)\alpha_{m+1} - x_1\alpha_{r_1+1} \tag{16}$$

$$\mathbf{f}^\pi[d - \ell_1 + 1] = -(r_1 - m + 1)\alpha_m - (r_1 - m)\alpha_{m+1} - \alpha_{r_1+1}. \tag{17}$$

Since $\pi(f) = \pi(g)$, (16) = (9) and (17) = (11), thereby implying $x_1(11) - (9) = x_1(17) - (16)$. Observe that $x_1(11) - (9) = x_1(17) - (16)$ is the following

$$\alpha_m + \alpha_{m+1} = \alpha'_n + \alpha'_{n+1}. \tag{18}$$

Now, consider the equation (17) = (11):

$$-(r_1 - m + 1)\alpha_m - (r_1 - m)\alpha_{m+1} - \alpha_{r_1+1} = -(r_1 - n + 1)\alpha'_n - (r_1 - n)\alpha'_{n+1} - \alpha'_{r_1+1}.$$

Adding (18) to this equation r_1 times yields

$$(m - 1)\alpha_m + m\alpha_{m+1} - \alpha_{r_1+1} = (n - 1)\alpha'_n + n\alpha'_{n+1} - \alpha'_{r_1+1}.$$

Either $m < n$ or $m > n$ (note that $m \neq n$ since f and g are relatively prime). First, suppose $m < n$. Subtracting (18) from our previous equation $m - 1$ times gives

$$\alpha_{m+1} - \alpha_{r_1+1} = (n - m)\alpha'_n + (n - m + 1)\alpha'_{n+1} - \alpha'_{r_1+1} \tag{19}$$

As $m < n$, we have that

$$\begin{aligned}
\alpha_{m+1} - \alpha_{r_1+1} &= \underbrace{(n-m)}_{>0}\underbrace{\alpha'_n}_{>0} + \underbrace{(n-m+1)}_{>0}\underbrace{\alpha'_{n+1}}_{\geq 0} - \alpha'_{r_1+1} \\
&> \alpha'_n + \alpha'_{n+1} - \alpha'_{r_1+1} \\
&\overset{(18)}{=} \alpha_m + \alpha_{m+1} - \alpha'_{r_1+1},
\end{aligned}$$

which implies

$$\alpha'_{r_1+1} > \underbrace{\alpha_m}_{>0} + \alpha_{r_1+1} \quad \Longrightarrow \quad \alpha'_{r_1+1} > 0.$$

Since f and g are relatively prime, this forces $\alpha_{r_1+1} = 0$. Thus, $\mathrm{supp}_{\mathbf{z}}(f) \subseteq \{m, m+1\}$. Moreover, $\alpha'_{n+1} = 0$ since $|\mathrm{supp}_{\mathbf{z}}(f)| \geq |\mathrm{supp}_{\mathbf{z}}(g)|$ and we have found $\alpha'_n, \alpha'_{r_1+1} > 0$. Consequently, (18) reduces to $\alpha'_n = \alpha_m + \alpha_{m+1}$ and (19) reduces to

$$\alpha'_{r_1+1} = (n - m)\alpha_m + (n - m - 1)\alpha_{m+1}. \tag{20}$$

Now, $\mathbf{f}^\pi[d + 1] = \mathbf{g}^\pi[d + 1]$ gives that

$$\alpha_m + \alpha_{m+1} + \sum_j \beta_j = \alpha'_n + \alpha'_{r_1+1} + \sum_j \beta'_j.$$

Since $\alpha'_n = \alpha_m + \alpha_{m+1}$ and $\alpha'_{r_1+1} > 0$, this implies $\sum_j \beta_j > \sum_j \beta'_j$. For each $r_1 \leq j \leq d$, $-\mathbf{f}^\pi[d - j + 1] = -\mathbf{g}^\pi[d - j + 1]$ is given by

$$(r_1 - m + 1)\alpha_m + (r_1 - m)\alpha_{m+1} - \beta_j = (r_1 - n + 1)\alpha'_n + \alpha'_{r_1+1} - \beta'_j.$$

Solving for α'_{r_1+1} and substituting $\alpha'_n = \alpha_m + \alpha_{m+1}$ yields

$$\alpha'_{r_1+1} = (n - m)\alpha_m + (n - m - 1)\alpha_{m+1} + \beta'_j - \beta_j.$$

Adding these equations for each $r_1 \leq j \leq d$ gives

$$\begin{aligned}
(d - r_1 + 1)\alpha'_{r_1+1} = (d - r_1 + 1)&\left[(n - m)\alpha_m + (n - m - 1)\alpha_{m+1}\right] \\
&+ \sum_{r_1 \leq j \leq d} (\beta'_j - \beta_j).
\end{aligned} \tag{21}$$

Similarly, for each $1 \leq j \leq r_1 - 1$, $-\mathbf{f}^\pi[d - j + 1] = -\mathbf{g}^\pi[d - j + 1]$ is given by

$$(1 + (r_1 - m + 1)x_1)\alpha_m + (1 + (r_1 - m)x_1)\alpha_{m+1} - \beta_j = (1 + (r_1 - n + 1)x_1)\alpha'_n + x_1\alpha'_{r_1+1} - \beta'_j.$$

Solving for $x_1\alpha'_{r_1+1}$ and making the appropriate substitutions yields

$$x_1\alpha'_{r_1+1} = (n - m)x_1\alpha_m + (n - m - 1)x_1\alpha_{m+1} + \beta'_j - \beta_j.$$

Adding these equations for each $1 \le j \le r_1 - 1$ gives

$$\begin{aligned}(r_1-1)x_1\alpha'_{r_1+1} = (r_1-1)\left[(n-m)x_1\alpha_m + (n-m-1)x_1\alpha_{m+1}\right] \\ + \sum_{1\le j\le r_1-1}(\beta'_j - \beta_j).\end{aligned} \tag{22}$$

Combining (21) and (22) gives

$$r_1x_1\alpha'_{r_1+1} = r_1x_1\underbrace{[(n-m)\alpha_m + (n-m-1)\alpha_{m+1}]}_{=\alpha'_{r_1+1}\text{ by (20)}} + \underbrace{\sum_{j=1}^{d}(\beta'_j-\beta_j)}_{<0},$$

a contradiction. Now, suppose $m > n$. In this case, rather than subtracting $m-1$ copies of (18), we instead subtract $n-1$ copies of (18) yielding

$$(m-n)\alpha_m + (m-n+1)\alpha_{m+1} - \alpha_{r_1+1} = \alpha'_{n+1} - \alpha'_{r_1+1}.$$

Then, since $m-n>0$, the same argument from the $m<n$ case will follow through by appropriately replacing each occurrence of α'_n with α_m, α_m with α'_n, α'_{n+1} with α_{m+1}, α_{m+1} with α'_{n+1}, α'_{r_1+1} with α_{r_1+1}, and α_{r_1+1} with α'_{r_1+1}.

(b) $m = r_1$. Since $z_{r_1}y_1\cdots y_{r_1-1}$ (by (4)), $z_{r_1}y_{r_1}\cdots y_d \in in_{<_{lex}}(\mathcal{G})$ (by (5)), and $f \notin in_{<_{lex}}(\mathcal{G})$, there exist indices $k_2 \in \{1,\dots,r_1-1\}$ and $\ell_2 \in \{r_1,\dots,d\}$ such that $\beta_{k_2} = \beta_{\ell_2} = 0$. Then, we have that

$$\mathbf{f}^\pi[d-k_2+1] = \sum_{i=1}^{r_1+3}\alpha_i\mathcal{A}(\mathbf{q})_{d-k_2+1,i} \tag{23}$$

$$\begin{aligned}\mathbf{g}^\pi[d-k_2+1] = -(1+(r_1-n+1)x_1)\alpha'_n - (1+(r_1-n)x_1)\alpha'_{n+1} \\ - x_1\alpha'_{r_1+1} + \beta'_{k_2}\end{aligned} \tag{24}$$

$$\mathbf{f}^\pi[d-\ell_2+1] = \sum_{i=1}^{r_1+3}\alpha_i\mathcal{A}(\mathbf{q})_{d-\ell_2+1,i} \tag{25}$$

$$\mathbf{g}^\pi[d-\ell_2+1] = -(r_1-n+1)\alpha'_n - (r_1-n)\alpha'_{n+1} - \alpha'_{r_1+1} + \beta'_{\ell_2}, \tag{26}$$

where (23) = (24) and (25) = (26) as $\pi(f) = \pi(g)$. Subtracting the equation (23) = (24) from (8) = (9) implies $\beta_{k_1} = -\beta'_{k_2}$. Similarly, subtracting equation (25) = (26) from (10) = (11) implies $\beta_{\ell_1} = -\beta'_{\ell_2}$. Since $\beta_j, \beta'_j \ge 0$ for all $1 \le j \le d$, this implies $\beta_{k_1} = \beta'_{k_2} = \beta_{\ell_1} = \beta'_{\ell_2} = 0$. Also, by Lemma 3.4, we know $\alpha_{r_1} > 0$, $\alpha_{r_1+1}, \alpha_{r_1+2} \ge 0$, and $\alpha_i = 0$ for all $i \in \{1,\dots,r_1-1\} \cup \{r_1+3\}$. Consequently, equations (8) and (10) simplify to

$$\mathbf{f}^\pi[d-k_1+1] = -(1+x_1)\alpha_{r_1} - x_1\alpha_{r_1+1} - \alpha_{r_1+2} \tag{27}$$

$$\mathbf{f}^\pi[d-\ell_1+1] = -\alpha_{r_1} - \alpha_{r_1+1}. \tag{28}$$

Since $\pi(f) = \pi(g)$, (27) = (9) and (28) = (11), thereby implying $x_1(11) - (9) = x_1(28) - (27)$. Observe that $x_1(11) - (9) = x_1(28) - (27)$ is the following

$$\alpha_{r_1} + \alpha_{r_1+2} = \alpha'_n + \alpha'_{n+1}. \tag{29}$$

Now, consider the equation $-(28) = -(11)$:

$$\alpha_{r_1} + \alpha_{r_1+1} = (r_1-n+1)\alpha'_n + (r_1-n)\alpha'_{n+1} + \alpha'_{r_1+1}.$$

Substituting (29) into this equation yields

$$\alpha'_n + \alpha'_{n+1} - \alpha_{r_1+2} + \alpha_{r_1+1} = (r_1-n+1)\alpha'_n + (r_1-n)\alpha'_{n+1} + \alpha'_{r_1+1}.$$

Rearranging by subtracting $\alpha'_n + \alpha'_{n+1}$ on both sides yields

$$\alpha_{r_1+1} - \alpha_{r_1+2} = \underbrace{(r_1 - n)\alpha'_n}_{>0} + \underbrace{(r_1 - n - 1)\alpha'_{n+1}}_{\geq 0} + \alpha'_{r_1+1}. \tag{30}$$

Observe that (30) implies $\alpha_{r_1+1} > 0$, so since f and g are relatively prime, this forces $\alpha'_{r_1+1} = 0$. Therefore, subtracting $r_1 - n$ copies of (29) from (30) gives

$$\alpha_{r_1+1} - (r_1 - n)\alpha_{r_1} - (r_1 - n + 1)\alpha_{r_1+2} = -\alpha'_{n+1},$$

which implies

$$\alpha'_{n+1} = (r_1 - n)\alpha_{r_1} + (r_1 - n + 1)\alpha_{r_1+2} - \alpha_{r_1+1}. \tag{31}$$

Now, $\mathbf{f}^\pi[d+1] = \mathbf{g}^\pi[d+1]$ gives that

$$\alpha_{r_1} + \alpha_{r_1+1} + \alpha_{r_1+2} + \sum_j \beta_j = \alpha'_n + \alpha'_{n+1} + \sum_j \beta'_j.$$

Since $\alpha'_n = \alpha_{r_1} + \alpha_{r_1+2} - \alpha'_{n+1}$ by (29) and $\alpha_{r_1+1} > 0$, this implies $\sum_j \beta_j < \sum_j \beta'_j$. For each $r_1 \leq j \leq d$, $-\mathbf{f}^\pi[d-j+1] = -\mathbf{g}^\pi[d-j+1]$ is given by

$$\alpha_{r_1} + \alpha_{r_1+1} - \beta_j = (r_1 - n + 1)\alpha'_n + (r_1 - n)\alpha'_{n+1} - \beta'_j.$$

Solving for α'_{n+1} and substituting $\alpha'_n = \alpha_{r_1} + \alpha_{r_1+2} - \alpha'_{n+1}$ yields

$$\alpha'_{n+1} = (r_1 - n)\alpha_{r_1} + (r_1 - n + 1)\alpha_{r_1+2} - \alpha_{r_1+1} + \beta_j - \beta'_j.$$

Adding these equations for each $r_1 \leq j \leq d$ gives

$$\begin{aligned}(d - r_1 + 1)\alpha'_{n+1} = (d - r_1 + 1)\left[(r_1 - n)\alpha_{r_1} + (r_1 - n + 1)\alpha_{r_1+2} - \alpha_{r_1+1}\right] \\ + \sum_{r_1 \leq j \leq d} (\beta_j - \beta'_j).\end{aligned} \tag{32}$$

Similarly, for each $1 \leq j \leq r_1 - 1$, $-\mathbf{f}^\pi[d-j+1] = -\mathbf{g}^\pi[d-j+1]$ is given by

$$(1 + x_1)\alpha_{r_1} + x_1\alpha_{r_1+1} + \alpha_{r_1+2} - \beta_j = (1 + (r_1 - n + 1)x_1)\alpha'_n + (1 + (r_1 - n)x_1)\alpha'_{n+1} - \beta'_j.$$

Solving for $x_1\alpha'_{n+1}$ and making the appropriate substitutions yields

$$x_1\alpha'_{n+1} = (r_1 - n)x_1\alpha_{r_1} + (r_1 - n + 1)x_1\alpha_{r_1+2} - x_1\alpha_{r_1+1} + \beta_j - \beta'_j.$$

Adding these equations for each $1 \leq j \leq r_1 - 1$ gives

$$\begin{aligned}(r_1 - 1)x_1\alpha'_{n+1} = (r_1 - 1)\left[(r_1 - n)x_1\alpha_{r_1} + (r_1 - n + 1)x_1\alpha_{r_1+2} - x_1\alpha_{r_1+1}\right] \\ + \sum_{1 \leq j \leq r_1 - 1} (\beta_j - \beta'_j).\end{aligned} \tag{33}$$

Combining (32) and (33) gives

$$r_1 x_1 \alpha'_{n+1} = r_1 x_1 \underbrace{\left[(r_1 - n)\alpha_{r_1} + (r_1 - n + 1)\alpha_{r_1+2} - \alpha_{r_1+1}\right]}_{=\alpha'_{n+1} \text{ by } (31)} + \underbrace{\sum_{j=1}^{d} (\beta_j - \beta'_j)}_{<0},$$

a contradiction.

(c) $m = r_1 + 1$. Since $z_{r_1+1}y_{r_1}\cdots y_d \in in_{<_{lex}}(\mathcal{G})$ (by (7)) and $f \notin in_{<_{lex}}(\mathcal{G})$, there exists an index $\ell_2 \in \{r_1, \ldots, d\}$ such that $\beta_{\ell_2} = 0$. Then, we have that

$$\mathbf{f}^\pi[d - \ell_2 + 1] = \sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-\ell_2+1,i} \tag{34}$$

$$\mathbf{g}^\pi[d - \ell_2 + 1] = -(r_1 - n + 1)\alpha'_n - (r_1 - n)\alpha'_{n+1} - \alpha'_{r_1+1} + \beta'_{\ell_2}, \tag{35}$$

where (34) = (35) as $\pi(f) = \pi(g)$. Subtracting the equation (34) = (35) from (10) = (11) implies $\beta_{\ell_1} = -\beta'_{\ell_2}$. Since $\beta_j, \beta'_j \geq 0$ for all $1 \leq j \leq d$, this implies $\beta_{\ell_1} = \beta'_{\ell_2} = 0$. Also, by Lemma 3.4, we know $\alpha_{r_1+1} > 0$, $\alpha_{r_1+2}, \alpha_{r_1+3} \geq 0$, and $\alpha_i = 0$ for all $i \in \{1, \ldots, r_1\}$. Consequently, since $\beta_{\ell_1} = 0$, equations (8) and (10) simplify to

$$\mathbf{f}^\pi[d - k_1 + 1] = -x_1\alpha_{r_1+1} - \alpha_{r_1+2} + \beta_{k_1} \tag{36}$$

$$\mathbf{f}^\pi[d - \ell_1 + 1] = -\alpha_{r_1+1}. \tag{37}$$

Furthermore, since f and g are relatively prime, $\alpha_{r_1+1} > 0$ implies $\alpha'_{r_1+1} = 0$, so equations (9) and (11) simplify to

$$\mathbf{g}^\pi[d - k_1 + 1] = -(1 + (r_1 - n + 1)x_1)\alpha'_n - (1 + (r_1 - n)x_1)\alpha'_{n+1} \tag{38}$$

$$\mathbf{g}^\pi[d - \ell_1 + 1] = -(r_1 - n + 1)\alpha'_n - (r_1 - n)\alpha'_{n+1}. \tag{39}$$

Since $\pi(f) = \pi(g)$, (36) = (38) and (37) = (39). Therefore, we have that −(36) = −(38) and −(37) = −(39), that is,

$$x_1\alpha_{r_1+1} + \alpha_{r_1+2} - \beta_{k_1} = (1 + (r_1 - n + 1)x_1)\alpha'_n + (1 + (r_1 - n)x_1)\alpha'_{n+1} \tag{40}$$

and

$$\alpha_{r_1+1} = (r_1 - n + 1)\alpha'_n + (r_1 - n)\alpha'_{n+1}. \tag{41}$$

Now, $\mathbf{f}^\pi[d + 1] = \mathbf{g}^\pi[d + 1]$ gives that

$$\alpha_{r_1+1} + \alpha_{r_1+2} + \alpha_{r_1+3} + \sum_j \beta_j = \alpha'_n + \alpha'_{n+1} + \sum_j \beta'_j.$$

Substituting (41) and since $(r_1 - n)\alpha'_n > 0$, we obtain

$$\sum_j \beta_j < \sum_j \beta'_j\,. \tag{42}$$

For each $r_1 \leq j \leq d$, $-\mathbf{f}^\pi[d - j + 1] = -\mathbf{g}^\pi[d - j + 1]$ is given by

$$\alpha_{r_1+1} - \beta_j = (r_1 - n + 1)\alpha'_n + (r_1 - n)\alpha'_{n+1} - \beta'_j,$$

which readily implies

$$\alpha_{r_1+1} = (r_1 - n + 1)\alpha'_n + (r_1 - n)\alpha'_{n+1} + \beta_j - \beta'_j.$$

Adding these equations for each $r_1 \leq j \leq d$ gives

$$\begin{aligned}(d - r_1 + 1)\alpha_{r_1+1} = (d - r_1 + 1)\left[(r_1 - n + 1)\alpha'_n + (r_1 - n)\alpha'_{n+1}\right] \\ + \sum_{r_1 \leq j \leq d} (\beta_j - \beta'_j).\end{aligned}$$

Using (41), this simplifies to

$$0 = \sum_{r_1 \leq j \leq d} (\beta_j - \beta'_j)\,. \tag{43}$$

Similarly, for each $1 \leq j \leq r_1 - 1$, $-\mathbf{f}^\pi[d - j + 1] = -\mathbf{g}^\pi[d - j + 1]$ is given by

$$x_1\alpha_{r_1+1} + \alpha_{r_1+2} - \beta_j = (1 + (r_1 - n + 1)x_1)\alpha'_n + (1 + (r_1 - n)x_1)\alpha'_{n+1} - \beta'_j,$$

which implies

$$x_1\alpha_{r_1+1} = (1 + (r_1 - n + 1)x_1)\alpha'_n + (1 + (r_1 - n)x_1)\alpha'_{n+1} - \alpha_{r_1+2} + \beta_j - \beta'_j.$$

Adding these equations for each $1 \leq j \leq r_1 - 1$ gives

$$(r_1 - 1)x_1\alpha_{r_1+1} = (r_1 - 1)\big[(1 + (r_1 - n + 1)x_1)\alpha'_n + (1 + (r_1 - n)x_1)\alpha'_{n+1} - \alpha_{r_1+2}\big] + \sum_{1 \leq j \leq r_1-1} (\beta_j - \beta'_j).$$

Using (40), this simplifies to

$$0 = -(r_1 - 1)\beta_{k_1} + \sum_{1 \leq j \leq r_1-1} (\beta_j - \beta'_j). \tag{44}$$

Combining (43) and (44), and observing (42), gives

$$0 = -(r_1 - 1)\beta_{k_1} + \underbrace{\sum_{j=1}^{d}(\beta_j - \beta'_j)}_{<0},$$

which implies $(r_1 - 1)\beta_{k_1} < 0$, a contradiction.

Subcase 2.2: $n = r_1$. By Lemma 3.4, we know $\alpha'_{r_1} > 0$, $\alpha'_{r_1+1}, \alpha'_{r_1+2} \geq 0$, and $\alpha'_i = 0$ for all $i \in \{1, \ldots, r_1 - 1\} \cup \{r_1 + 3\}$. Since $z_{r_1}y_1 \cdots y_{r_1-1} \in in_{<_{lex}}(\mathcal{G})$ (by (4)), $z_{r_1}y_{r_1} \cdots y_d \in in_{<_{lex}}(\mathcal{G})$ (by (5)), and $g \notin in_{<_{lex}}(\mathcal{G})$, there exist indices $k_1 \in \{1, \ldots, r_1 - 1\}$ and $\ell_1 \in \{r_1, \ldots, d\}$ such that $\beta'_{k_1} = \beta'_{\ell_1} = 0$. Then,

$$\mathbf{f}^\pi[d - k_1 + 1] = \sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-k_1+1,i} + \beta_{k_1} \tag{45}$$

$$\mathbf{g}^\pi[d - k_1 + 1] = -(1 + x_1)\alpha'_{r_1} - x_1\alpha'_{r_1+1} - \alpha'_{r_1+2} \tag{46}$$

$$\mathbf{f}^\pi[d - \ell_1 + 1] = \sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-\ell_1+1,i} + \beta_{\ell_1} \tag{47}$$

$$\mathbf{g}^\pi[d - \ell_1 + 1] = -\alpha'_{r_1} - \alpha'_{r_1+1}. \tag{48}$$

Note that $\pi(f) = \pi(g)$ implies (45) = (46) and (47) = (48). Now, we claim $m \in \{1, \ldots, r_1 - 1\} \cup \{r_1 + 1\}$ (we need not consider $m = r_1$ since f and g are relatively prime and $n = r_1$ in this case). Indeed, assume otherwise, that is, $\mathrm{supp}_{\mathbf{z}}(f) \subseteq \{r_1 + 2, r_1 + 3\}$. Then, $\mathbf{f}^\pi[d - \ell + 1] = \beta_\ell \geq 0$ for all $\ell \in \{r_1, \ldots, d\}$, but from (48), $\mathbf{g}^\pi[d - \ell_1 + 1] < 0$ since $\alpha'_{r_1} > 0$ and $\alpha'_{r_1+1} \geq 0$. This contradicts $\pi(f) = \pi(g)$. Hence, given the structure of Lemma 3.4 and since m cannot be r_1, we consider the following subsubcases.

(a) $m \in \{1, \ldots, r_1 - 1\}$. Since $z_m y_1 \cdots y_{r_1-1}$ (by (4)), $z_m y_{r_1} \cdots y_d \in in_{<_{lex}}(\mathcal{G})$ (by (5)), and $f \notin in_{<_{lex}}(\mathcal{G})$, there exist indices $k_2 \in \{1, \ldots, r_1 - 1\}$ and $\ell_2 \in \{r_1, \ldots, d\}$ such that $\beta_{k_2} = \beta_{\ell_2} = 0$. Then, we have that

$$\mathbf{f}^\pi[d - k_2 + 1] = \sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-k_2+1,i} \tag{49}$$

$$\mathbf{g}^\pi[d - k_2 + 1] = -(1 + x_1)\alpha'_{r_1} - x_1\alpha'_{r_1+1} - \alpha'_{r_1+2} + \beta'_{k_2} \tag{50}$$

$$\mathbf{f}^\pi[d - \ell_2 + 1] = \sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-\ell_2+1,i} \tag{51}$$

$$\mathbf{g}^\pi[d - \ell_2 + 1] = -\alpha'_{r_1} - \alpha'_{r_1+1} + \beta'_{\ell_2}, \tag{52}$$

where (49) = (50) and (51) = (52) as $\pi(f) = \pi(g)$. Subtracting the equation (49) = (50) from (45) = (46) implies $\beta_{k_1} = -\beta'_{k_2}$. Similarly, subtracting equation (51) = (52) from (47) = (48) implies $\beta_{\ell_1} = -\beta'_{\ell_2}$. Since $\beta_j, \beta'_j \geq 0$ for all $1 \leq j \leq d$, this implies $\beta_{k_1} = \beta'_{k_2} =$

$\beta_{\ell_1} = \beta'_{\ell_2} = 0$. Also, by Lemma 3.4, we know $\alpha_m > 0$, $\alpha_{m+1}, \alpha_{r_1+1} \geq 0$, and $\alpha_i = 0$ for all $i \in \{1, \ldots, r_1+3\} \setminus \{m, m+1, r_1+1\}$. Consequently, equations (45) and (47) simplify to

$$\mathbf{f}^\pi[d - k_1 + 1] = -(1 + (r_1 - m + 1)x_1)\alpha_m - (1 + (r_1 - m)x_1)\alpha_{m+1} - x_1\alpha_{r_1+1} \tag{53}$$

$$\mathbf{f}^\pi[d - \ell_1 + 1] = -(r_1 - m + 1)\alpha_m - (r_1 - m)\alpha_{m+1} - \alpha_{r_1+1}. \tag{54}$$

Since $\pi(f) = \pi(g)$, (53) = (46) and (54) = (48), thereby implying $x_1(48) - (46) = x_1(54) - (53)$. Observe that $x_1(48) - (46) = x_1(54) - (53)$ is the following

$$\alpha_m + \alpha_{m+1} = \alpha'_{r_1} + \alpha'_{r_1+2}. \tag{55}$$

Now, consider the equation $-(54) = -(48)$:

$$(r_1 - m + 1)\alpha_m + (r_1 - m)\alpha_{m+1} + \alpha_{r_1+1} = \alpha'_{r_1} + \alpha'_{r_1+1}.$$

Substituting (55) into this equation and solving for α'_{r_1+1} yields

$$\alpha'_{r_1+1} = (r_1 - m)\alpha_m + (r_1 - m - 1)\alpha_{m+1} + \alpha_{r_1+1} + \alpha'_{r_1+2}. \tag{56}$$

Observe that (56) implies $\alpha'_{r_1+1} > 0$, so since f and g are relatively prime, this forces $\alpha_{r_1+1} = 0$. Thus, $\mathrm{supp}_{\mathbf{z}}(f) \subseteq \{m, m+1\}$. Moreover, since $|\mathrm{supp}_{\mathbf{z}}(f)| \geq |\mathrm{supp}_{\mathbf{z}}(g)|$, $\alpha_{r_1+1} = 0$, and we have $\alpha'_{r_1}, \alpha'_{r_1+1} > 0$, it follows that $\alpha_{m+1} > 0$ and $\alpha'_{r_1+2} = 0$. Consequently, (55) reduces to $\alpha'_{r_1} = \alpha_m + \alpha_{m+1}$ and (56) reduces to

$$\alpha'_{r_1+1} = (r_1 - m)\alpha_m + (r_1 - m - 1)\alpha_{m+1}.$$

Summing these reduced equations yields

$$\alpha'_{r_1} + \alpha'_{r_1+1} = (r_1 - m + 1)\alpha_m + (r_1 - m)\alpha_{m+1}. \tag{57}$$

Now, $\mathbf{f}^\pi[d+1] = \mathbf{g}^\pi[d+1]$ gives that

$$\alpha_m + \alpha_{m+1} + \sum_j \beta_j = \alpha'_{r_1} + \alpha'_{r_1+1} + \sum_j \beta'_j.$$

Since $\alpha'_{r_1} = \alpha_m + \alpha_{m+1}$ and $\alpha'_{r_1+1} > 0$, this implies

$$\sum_j \beta_j > \sum_j \beta'_j. \tag{58}$$

For each $r_1 \leq j \leq d$, $-\mathbf{f}^\pi[d - j + 1] = -\mathbf{g}^\pi[d - j + 1]$ is given by

$$(r_1 - m + 1)\alpha_m + (r_1 - m)\alpha_{m+1} - \beta_j = \alpha'_{r_1} + \alpha'_{r_1+1} - \beta'_j,$$

which, via (57), implies $\beta_j = \beta'_j$. Similarly, for each $1 \leq j \leq r_1 - 1$, $-\mathbf{f}^\pi[d - j + 1] = -\mathbf{g}^\pi[d - j + 1]$ is given by

$$(1 + (r_1 - m + 1)x_1)\alpha_m + (1 + (r_1 - m)x_1)\alpha_{m+1} - \beta_j = (1 + x_1)\alpha'_{r_1} + x_1\alpha'_{r_1+1} - \beta'_j,$$

which, via (57), implies $\beta_j = \beta'_j$. Thus, we have that $\beta_j = \beta'_j$ for all $1 \leq j \leq d$, but we had in (58) that $\sum_j \beta_j > \sum_j \beta'_j$, a contradiction.

(b) $m = r_1 + 1$. Since $z_{r_1+1} y_{r_1} \cdots y_d \in in_{<_{lex}}(\mathcal{G})$ (by (7)) and $f \notin in_{<_{lex}}(\mathcal{G})$, there exists an index $\ell_2 \in \{r_1, \ldots, d\}$ such that $\beta_{\ell_2} = 0$. Then, we have that

$$\mathbf{f}^\pi[d - \ell_2 + 1] = \sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-\ell_2+1,i} \tag{59}$$

$$\mathbf{g}^\pi[d - \ell_2 + 1] = -\alpha'_{r_1} - \alpha'_{r_1+1} + \beta'_{\ell_2}, \tag{60}$$

where (59) = (60) as $\pi(f) = \pi(g)$. Subtracting the equation (59) = (60) from (47) = (48) implies $\beta_{\ell_1} = -\beta'_{\ell_2}$. Since $\beta_j, \beta'_j \geq 0$ for all $1 \leq j \leq d$, this implies $\beta_{\ell_1} = \beta'_{\ell_2} = 0$. Also, by Lemma 3.4, we know $\alpha_{r_1+1} > 0$, $\alpha_{r_1+2}, \alpha_{r_1+3} \geq 0$, and $\alpha_i = 0$ for all $i \in \{1, \ldots, r_1\}$. Consequently, since $\beta_{\ell_1} = 0$, equations (45) and (47) simplify to

$$\mathbf{f}^\pi[d - k_1 + 1] = -x_1\alpha_{r_1+1} - \alpha_{r_1+2} + \beta_{k_1} \tag{61}$$

$$\mathbf{f}^\pi[d - \ell_1 + 1] = -\alpha_{r_1+1}. \tag{62}$$

Furthermore, since f and g are relatively prime, $\alpha_{r_1+1} > 0$ implies $\alpha'_{r_1+1} = 0$, so equations (46) and (48) simplify to

$$\mathbf{g}^\pi[d - k_1 + 1] = -(1 + x_1)\alpha'_{r_1} - \alpha'_{r_1+2} \tag{63}$$

$$\mathbf{g}^\pi[d - \ell_1 + 1] = -\alpha'_{r_1}. \tag{64}$$

Since $\pi(f) = \pi(g)$, (61) = (63) and (62) = (64). Therefore, we have that $-(61) = -(63)$ and $-(62) = -(64)$, that is,

$$(1 + x_1)\alpha'_{r_1} + \alpha'_{r_1+2} = x_1\alpha_{r_1+1} + \alpha_{r_1+2} - \beta_{k_1} \tag{65}$$

and

$$\alpha'_{r_1} = \alpha_{r_1+1}. \tag{66}$$

Plugging (66) into (65) and solving for β_{k_1} gives

$$\beta_{k_1} = \alpha_{r_1+2} - \alpha_{r_1+1} - \alpha'_{r_1+2}. \tag{67}$$

Note that if $\alpha'_{r_1+2} > 0$, the relatively prime condition would force $\alpha_{r_1+2} = 0$, thereby implying $\beta_{k_1} < 0$, a contradiction. Hence, we may assume $\alpha'_{r_1+2} = 0$, and since $\beta_{k_1} \geq 0$, it must be that $\alpha_{r_1+2} > 0$. Now, $\mathbf{f}^\pi[d + 1] = \mathbf{g}^\pi[d + 1]$ gives that

$$\alpha_{r_1+1} + \alpha_{r_1+2} + \alpha_{r_1+3} + \sum_j \beta_j = \alpha'_{r_1} + \sum_j \beta'_j.$$

Substituting (66) and since $\alpha_{r_1+2} > 0$, this implies

$$\sum_j \beta_j < \sum_j \beta'_j\,. \tag{68}$$

For each $r_1 \leq j \leq d$, $-\mathbf{f}^\pi[d - j + 1] = -\mathbf{g}^\pi[d - j + 1]$ is given by

$$\alpha_{r_1+1} - \beta_j = \alpha'_{r_1} - \beta'_j,$$

which, via (66), implies $\beta_j = \beta'_j$. Similarly, for each $1 \leq j \leq r_1 - 1$, $-\mathbf{f}^\pi[d - j + 1] = -\mathbf{g}^\pi[d - j + 1]$ is given by

$$x_1\alpha_{r_1+1} + \alpha_{r_1+2} - \beta_j = (1 + x_1)\alpha'_{r_1} - \beta'_j,$$

which, via (65), implies $\beta_{k_1} = \beta_j - \beta'_j$. Therefore,

$$0 < \sum_{j=1}^{d}(\beta'_j - \beta_j) = \sum_{j=1}^{r_1-1}(\beta'_j - \beta_j) + \sum_{j=r_1}^{d}(\beta'_j - \beta_j) = -(r_1 - 1)\beta_{k_1} \leq 0\,,$$

a contradiction.

<u>Subcase 2.3</u>: $n = r_1 + 1$. By Lemma 3.4, we know $\alpha'_{r_1+1} > 0$, $\alpha'_{r_1+2}, \alpha'_{r_1+3} \geq 0$, and $\alpha'_i = 0$ for all $i \in \{1, \ldots, r_1\}$. Since $z_{r_1+1}y_{r_1}\cdots y_d \in in_{<_{lex}}(\mathcal{G})$ (by (7)) and $g \notin in_{<_{lex}}(\mathcal{G})$, there exists an index $\ell_1 \in \{r_1, \ldots, d\}$ such that $\beta'_{\ell_1} = 0$. Then,

$$\mathbf{f}^\pi[d - \ell_1 + 1] = \sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-\ell_1+1,i} + \beta_{\ell_1} \tag{69}$$

$$\mathbf{g}^\pi[d - \ell_1 + 1] = -\alpha'_{r_1+1}, \tag{70}$$

where (69) = (70) as $\pi(f) = \pi(g)$. Now, we claim $m \in \{1, \ldots, r_1\}$ (we need not consider $m = r_1 + 1$ since f and g are relatively prime and $n = r_1 + 1$ in this case). Indeed, assume otherwise, that is, $\mathrm{supp}_{\mathbf{z}}(f) \subseteq \{r_1+2, r_1+3\}$. Then, $\mathbf{f}^\pi[d-\ell+1] = \beta_\ell \geq 0$ for all $\ell \in \{r_1, \ldots, d\}$, but from (70), $\mathbf{g}^\pi[d - \ell_1 + 1] < 0$ since $\alpha'_{r_1+1} > 0$. This contradicts $\pi(f) = \pi(g)$. Hence, given the structure of Lemma 3.4 and since m cannot be r_1+1, we consider the following subsubcases.

(a) $m \in \{1, \ldots, r_1 - 1\}$. Since $z_m y_1 \cdots y_{r_1-1} \in in_{<_{lex}}(\mathcal{G})$ (by (4)), $z_m y_{r_1} \cdots y_d \in in_{<_{lex}}(\mathcal{G})$ (by (5)), and $f \notin in_{<_{lex}}(\mathcal{G})$, there exist indices $k_2 \in \{1, \ldots, r_1 - 1\}$ and $\ell_2 \in \{r_1, \ldots, d\}$ such that $\beta_{k_2} = \beta_{\ell_2} = 0$. Then, we have that

$$\mathbf{f}^\pi[d - k_2 + 1] = \sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-k_2+1,i} \tag{71}$$

$$\mathbf{g}^\pi[d - k_2 + 1] = -x_1\alpha'_{r_1+1} - \alpha'_{r_1+2} + \beta'_{k_2} \tag{72}$$

$$\mathbf{f}^\pi[d - \ell_2 + 1] = \sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-\ell_2+1,i} \tag{73}$$

$$\mathbf{g}^\pi[d - \ell_2 + 1] = -\alpha'_{r_1+1} + \beta'_{\ell_2}, \tag{74}$$

where (71) = (72) and (73) = (74) since $\pi(f) = \pi(g)$. Subtracting the equation (73) = (74) from (69) = (70) implies $\beta_{\ell_1} = -\beta'_{\ell_2}$. Since $\beta_j, \beta'_j \geq 0$ for all $1 \leq j \leq d$, this implies $\beta_{\ell_1} = \beta'_{\ell_2} = 0$. Also, by Lemma 3.4 and since $n = r_1 + 1$, we know $\alpha_m > 0$, $\alpha_{m+1} \geq 0$, $\alpha_{r_1+1} = 0$, and $\alpha_i = 0$ for all $i \in \{1, \ldots, r_1 + 3\} \setminus \{m, m+1\}$. Consequently, since $\beta_{\ell_1} = 0$, the equation (69) = (70) simplifies to

$$-(r_1 - m + 1)\alpha_m - (r_1 - m)\alpha_{m+1} = -\alpha'_{r_1+1},$$

which implies

$$\alpha'_{r_1+1} = (r_1 - m + 1)\alpha_m + (r_1 - m)\alpha_{m+1}. \tag{75}$$

Furthermore, the equation $-(71) = -(72)$ simplifies to

$$(1 + (r_1 - m + 1)x_1)\alpha_m + (1 + (r_1 - m)x_1)\alpha_{m+1} = x_1\alpha'_{r_1+1} + \alpha'_{r_1+2} - \beta'_{k_2}.$$

Via (75), this equation is equivalent to

$$\alpha_m + \alpha_{m+1} + x_1\alpha'_{r_1+1} = x_1\alpha'_{r_1+1} + \alpha'_{r_1+2} - \beta'_{k_2},$$

which implies

$$\beta'_{k_2} = \alpha'_{r_1+2} - \alpha_m - \alpha_{m+1}. \tag{76}$$

Note that if $\alpha'_{r_1+2} = 0$, $\beta'_{k_2} < 0$ by (76), a contradiction. Hence, we may assume $\alpha'_{r_1+2} > 0$. Also, since $|\mathrm{supp}_{\mathbf{z}}(f)| \geq |\mathrm{supp}_{\mathbf{z}}(g)|$ and $\alpha_{r_1+1} = 0$, it follows that $\alpha_{m+1} > 0$ and $\alpha'_{r_1+3} = 0$. Now, $\mathbf{f}^\pi[d+1] = \mathbf{g}^\pi[d+1]$ gives that

$$\alpha_m + \alpha_{m+1} + \sum_j \beta_j = \alpha'_{r_1+1} + \alpha'_{r_1+2} + \sum_j \beta'_j.$$

Substituting (75), this implies

$$\sum_j \beta_j > \sum_j \beta'_j\,. \tag{77}$$

For each $r_1 \leq j \leq d$, $-\mathbf{f}^\pi[d - j + 1] = -\mathbf{g}^\pi[d - j + 1]$ is given by

$$(r_1 - m + 1)\alpha_m + (r_1 - m)\alpha_{m+1} - \beta_j = \alpha'_{r_1+1} - \beta'_j,$$

which, via (75), implies $\beta_j = \beta'_j$. Similarly, for each $1 \leq j \leq r_1 - 1$, $-\mathbf{f}^\pi[d-j+1] = -\mathbf{g}^\pi[d-j+1]$ is given by

$$(1+(r_1-m+1)x_1)\alpha_m + (1+(r_1-m)x_1)\alpha_{m+1} - \beta_j = x_1\alpha'_{r_1+1} + \alpha'_{r_1+2} - \beta'_j,$$

which, via (75) and (76), implies

$$\beta'_{k_2} = \beta'_j - \beta_j. \tag{78}$$

Therefore, by (77) and (78),

$$0 < \sum_{j=1}^{d}(\beta_j - \beta'_j) = \sum_{j=1}^{r_1-1}(\beta_j - \beta'_j) + \sum_{j=r_1}^{d}(\beta_j - \beta'_j) = \sum_{j=1}^{r_1-1}(\beta_j - \beta'_j) = -(r_1-1)\beta'_{k_2} \leq 0,$$

a contradiction.

(b) $m = r_1$. Since $z_{r_1}y_1\cdots y_{r_1-1} \in in_{<_{lex}}(\mathcal{G})$ (by (4)), $z_{r_1}y_{r_1}\cdots y_d \in in_{<_{lex}}(\mathcal{G})$ (by (5)), and $f \notin in_{<_{lex}}(\mathcal{G})$, there exist indices $k_2 \in \{1,\ldots,r_1-1\}$ and $\ell_2 \in \{r_1,\ldots,d\}$ such that $\beta_{k_2} = \beta_{\ell_2} = 0$. Then, we have that

$$\mathbf{f}^\pi[d-k_2+1] = \sum_{i=1}^{r_1+3}\alpha_i\mathcal{A}(\mathbf{q})_{d-k_2+1,i} \tag{79}$$

$$\mathbf{g}^\pi[d-k_2+1] = -x_1\alpha'_{r_1+1} - \alpha'_{r_1+2} + \beta'_{k_2} \tag{80}$$

$$\mathbf{f}^\pi[d-\ell_2+1] = \sum_{i=1}^{r_1+3}\alpha_i\mathcal{A}(\mathbf{q})_{d-\ell_2+1,i} \tag{81}$$

$$\mathbf{g}^\pi[d-\ell_2+1] = -\alpha'_{r_1+1} + \beta'_{\ell_2}, \tag{82}$$

where (79) = (80) and (81) = (82) since $\pi(f) = \pi(g)$. Subtracting the equation (81) = (82) from (69) = (70) implies $\beta_{\ell_1} = -\beta'_{\ell_2}$. Since $\beta_j, \beta'_j \geq 0$ for all $1 \leq j \leq d$, this implies $\beta_{\ell_1} = \beta'_{\ell_2} = 0$. We know $\alpha_{r_1+1} = 0$ since $\alpha'_{r_1+1} > 0$. Also, by Lemma 3.4, we know $\alpha_{r_1} > 0$ and $\alpha_{r_1+2} \geq 0$, so it follows that $\alpha_i = 0$ for all $i \in \{1,\ldots,r_1+3\} \setminus \{r_1, r_1+2\}$. Consequently, since $\beta_{\ell_1} = 0$, the equation (69) = (70) simplifies to

$$\alpha_{r_1} = \alpha'_{r_1+1}. \tag{83}$$

Furthermore, the equation $-(79) = -(80)$ simplifies to

$$(1+x_1)\alpha_{r_1} + \alpha_{r_1+2} = x_1\alpha'_{r_1+1} + \alpha'_{r_1+2} - \beta'_{k_2}.$$

Via (83), this equation is equivalent to

$$(1+x_1)\alpha'_{r_1+1} + \alpha_{r_1+2} = x_1\alpha'_{r_1+1} + \alpha'_{r_1+2} - \beta'_{k_2},$$

which implies

$$\beta'_{k_2} = \alpha'_{r_1+2} - \alpha'_{r_1+1} - \alpha_{r_1+2}. \tag{84}$$

Note that if $\alpha'_{r_1+2} = 0$, $\beta'_{k_2} < 0$ by (84), a contradiction. Hence, it must be that $\alpha'_{r_1+2} > 0$. However, by the relatively prime condition, this implies $\alpha_{r_1+2} = 0$. As a consequence, since $\alpha_{r_1+1} = \alpha_{r_1+2} = 0$ and $\alpha'_{r_1+1}, \alpha'_{r_1+2} > 0$, we have that

$$|\mathrm{supp}_{\mathbf{z}}(f)| = 1 < 2 \leq |\mathrm{supp}_{\mathbf{z}}(g)|,$$

contradicting our assumption that $|\mathrm{supp}_{\mathbf{z}}(f)| \geq |\mathrm{supp}_{\mathbf{z}}(g)|$.

<u>**Subcase 2.4:**</u> $n \in \{r_1+2, r_1+3\}$. In this case, $\mathrm{supp}_{\mathbf{z}}(g) \subseteq \{r_1+2, r_1+3\}$. Consequently, for $1 \leq j \leq d$, we have that

$$\mathbf{g}^\pi[d-j+1] = \begin{cases} -\alpha'_{r_1+2} + \beta'_j, & \text{for } 1 \leq j \leq r_1 - 1 \\ \beta'_j, & \text{for } r_1 \leq j \leq d. \end{cases} \tag{85}$$

Now, we consider the possibilities for m.

(a) $m \in \{1, \dots, r_1+1\}$. Since $z_m y_{r_1} \cdots y_d \in in_{<_{lex}}(\mathcal{G})$ (by (5) or (7)) and $f \notin in_{<_{lex}}(\mathcal{G})$, there exists an index $\ell_1 \in \{r_1, \dots, d\}$ such that $\beta_{\ell_1} = 0$. Therefore, since $\alpha_m > 0$, we have that

$$\mathbf{f}^{\pi}[d-\ell_1+1] = \underbrace{\sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-\ell_1+1,i}}_{<0} + \underbrace{\sum_{j=1}^{d} \beta_j \mathcal{A}(\mathbf{q})_{d-\ell_1+1,r_1+3+j}}_{=0} < 0,$$

but this contradicts $\pi(f) = \pi(g)$ since $\mathbf{g}^{\pi}[d-\ell_1+1] = \beta'_{\ell_1} \geq 0$ from (85).

(b) $m \in \{r_1+2, r_1+3\}$. Note that since the relatively prime condition implies $m \neq n$, it follows that $|\mathrm{supp}_{\mathbf{z}}(f)| = |\mathrm{supp}_{\mathbf{z}}(g)| = 1$ in this case. Therefore, we may assume without loss of generality that $m = r_1+2$ and $n = r_1+3$. Since $z_{r_1+2} y_1 \dots y_{r_1-1} \in in_{<_{lex}}(\mathcal{G})$ (by (6)) and $f \notin in_{<_{lex}}(\mathcal{G})$, there exists an index $k_1 \in \{1, \dots, r_1-1\}$ such that $\beta_{k_1} = 0$. Therefore, since $\alpha_{r_1+2} > 0$, we have that $\alpha'_{r_1+2} = 0$ and

$$\mathbf{f}^{\pi}[d-k_1+1] = \underbrace{\sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-k_1+1,i}}_{<0} + \underbrace{\sum_{j=1}^{d} \beta_j \mathcal{A}(\mathbf{q})_{d-k_1+1,r_1+3+j}}_{=0} < 0.$$

However, this contradicts $\pi(f) = \pi(g)$ since $\mathbf{g}^{\pi}[d-k_1+1] = -\underbrace{\alpha'_{r_1+2}}_{=0} + \beta'_{k_1} \geq 0$ from (85).

Since each of the above cases (which together cover all possible pairs (m, n)) yields a contradiction, Lemma 3.3 implies that $\mathcal{G}$ forms a Gröbner basis of $I_{\mathcal{A}(\mathbf{q})}$ with respect to $<_{lex}$, as required. ☐

In sum, since we have demonstrated that $\mathcal{G}$ is a Gröbner basis of $I_{\mathcal{A}(\mathbf{q})}$ with respect to $<_{lex}$, we know $in_{<_{lex}}(\mathcal{G}) = in_{<_{lex}}(I_{\mathcal{A}(\mathbf{q})})$. Therefore, since we can clearly see $in_{<_{lex}}(\mathcal{G})$ is squarefree, Theorem 1.6 holds and [16, Corollary 8.9] proves Corollary 1.7. As such, there exists a regular unimodular triangulation of the points in $\mathcal{A}'(\mathbf{q})$, as desired.

4. Facets of the Triangulation

For $\mathbf{q} = (r_1^{x_1}, (1+r_1x_1)^{r_1-1})$ with $r_1 > 1$, let $\mathcal{T}(\mathbf{q})$ denote the regular unimodular triangulation induced by the lexicographic term order $<_{lex}$ used in the previous section. This triangulation is identical to the placing triangulation obtained by placing the columns of $\mathcal{A}(\mathbf{q})$ from left to right in the order as given in Figure 1. Throughout this section, we will abuse notation in that the variable in $K[\mathcal{A}(\mathbf{q})]$ associated with each vertex of the triangulation $\mathcal{T}(\mathbf{q})$ will represent that vertex. The Gröbner basis $\mathcal{G}$ for $I_{\mathcal{A}(\mathbf{q})}$ in Theorem 3.1 indicates which elements of $\mathcal{M}(K([\mathcal{A}(\mathbf{q})]))$ generate the minimal non-faces (i.e., minimal subsets of vertices that are not faces) of $\mathcal{T}(\mathbf{q})$. From this, we can deduce the facets of $\mathcal{T}(\mathbf{q})$ as outlined in the following corollary. More specifically, the facets correspond to the squarefree monomials of degree $d+1$ in $K[\mathcal{A}(\mathbf{q})]$ that are not contained in $in_{<_{lex}}(\mathcal{G})$.

Corollary 4.1. *Let $f \in \mathcal{M}(K[\mathcal{A}(\mathbf{q})])$ be squarefree with $f \notin in_{<_{lex}}(\mathcal{G})$. Let m denote the minimal index such that z_m divides f. Then, f defines a facet of $\mathcal{T}(\mathbf{q})$ when it is one of the following possible forms (the notation $\widehat{y_k}$ indicates the variable y_k is omitted):*

(i) if $1 \leq m \leq r_1-1$, then $f = z_m z_{m+1} z_{r_1+1} y_1 \cdots \widehat{y_i} \cdots y_{r_1-1} y_{r_1} \cdots \widehat{y_j} \cdots y_d$ for any $1 \leq i \leq r_1-1$ and $r_1 \leq j \leq d$;

(ii) if $m = r_1$, then $f = z_{r_1} z_{r_1+1} z_{r_1+2} y_1 \cdots \widehat{y_i} \cdots y_{r_1-1} y_{r_1} \cdots \widehat{y_j} \cdots y_d$ for any $1 \leq i \leq r_1-1$ and $r_1 \leq j \leq d$;

(iii) if $m = r_1+1$, then $f = z_{r_1+1} z_{r_1+2} z_{r_1+3} y_1 \cdots \widehat{y_i} \cdots y_{r_1-1} y_{r_1} \cdots \widehat{y_j} \cdots y_d$ for any $1 \leq i \leq r_1-1$ and $r_1 \leq j \leq d$ or $f = z_{r_1+1} z_{r_1+3} y_1 \cdots y_{r_1-1} y_{r_1} \cdots \widehat{y_j} \cdots y_d$ for any $r_1 \leq j \leq d$;

(iv) *if* $m = r_1 + 2$*, then* $f = z_{r_1+2}z_{r_1+3}y_1 \cdots \widehat{y_i} \cdots y_{r_1-1}y_{r_1} \cdots y_d$ *for any* $1 \leq i \leq r_1 - 1$*;*
(v) *if* $m = r_1 + 3$*, then* $f = z_{r_1+3}y_1 \cdots y_d$*.*

Proof. The normalized volume of $\Delta_{(1,\mathbf{q})}$, denoted $N(\mathbf{q})$, is given by $N(\mathbf{q}) = 1 + \sum_{i=1}^{d} q_i = 1 + x_1r_1 + (r_1 - 1)(1 + r_1x_1) = r_1(1 + r_1x_1)$. Since $\mathcal{T}(\mathbf{q})$ is unimodular, we know the number of facets of $\mathcal{T}(\mathbf{q})$ should equal $N(\mathbf{q})$. Indeed, since $d = r_1 + x_1 - 1$, it is straightforward to verify that there are precisely $r_1(1 + r_1x_1)$ squarefree monomials given by the forms (i)-(v) above. Moreover, note that any facet of $\mathcal{T}(\mathbf{q})$ will require the inclusion of at least one z-variable since facets must consist of $d + 1$ points and there are a total of d y-variables.

Now, suppose $1 \leq m \leq r_1 - 1$. By Lemma 3.4, we know $\mathrm{supp}_{\mathbf{z}}(f) \subseteq \{m, m + 1, r_1 + 1\}$. Since $z_my_1 \cdots y_{r_1-1} \in in_{<_{lex}}(\mathcal{G})$ by (4) and $z_my_{r_1} \cdots y_d \in in_{<_{lex}}(\mathcal{G})$ by (5), there exist indices $1 \leq i \leq r_1 - 1$ and $r_1 \leq j \leq d$ such that $y_i \nmid f$ and $y_j \nmid f$. As facets of $\mathcal{T}(\mathbf{q})$ must contain exactly $d + 1$ points, this forces the inclusion of all other y-variables, z_{m+1}, and z_{r_1+1}. With no further restriction on i and j, we obtain form (i).

Now suppose $m = r_1$. By Lemma 3.4, we know $\mathrm{supp}_{\mathbf{z}}(f) \subseteq \{r_1, r_1 + 1, r_1 + 2\}$. Again, (4) and (5) indicate that $z_{r_1}y_1 \cdots y_{r_1-1} \in in_{<_{lex}}(\mathcal{G})$ and $z_{r_1}y_{r_1} \cdots y_d \in in_{<_{lex}}(\mathcal{G})$, so there exist indices $1 \leq i \leq r_1 - 1$ and $r_1 \leq j \leq d$ such that $y_i \nmid f$ and $y_j \nmid f$. Thus, to have a collection of $d+1$ points, it must be that f is of the form $f = z_{r_1}z_{r_1+1}z_{r_1+2}y_1 \cdots \widehat{y_i} \cdots y_{r_1-1}y_{r_1} \cdots \widehat{y_j} \cdots y_d$, giving form (ii).

Next, suppose $m = r_1 + 1$. Lemma 3.4 gives that $\mathrm{supp}_{\mathbf{z}}(f) \subseteq \{r_1 + 1, r_1 + 2, r_1 + 3\}$, and we have that $z_{r_1+1}y_{r_1} \cdots y_d \in in_{<_{lex}}(\mathcal{G})$ by (7). Therefore, there exists some index $r_1 \leq j \leq d$ such that $y_j \nmid f$. Now, suppose $z_{r_1+2} \mid f$. Since $z_{r_1+2}y_1 \cdots y_{r_1-1} \in in_{<_{lex}}(\mathcal{G})$ by (6), there exists some index $1 \leq i \leq r_1 - 1$ such that $y_i \nmid f$. The exclusion of y_i and y_j necessarily requires the inclusion of all other y-variables and z_{r_1+3} to have a total of $d + 1$ points. As such, f is of the form $f = z_{r_1+1}z_{r_1+2}z_{r_1+3}y_1 \cdots \widehat{y_i} \cdots y_{r_1-1}y_{r_1} \cdots \widehat{y_j} \cdots y_d$. Otherwise, if $z_{r_1+2} \nmid f$, then the exclusion of y_j forces the inclusion of all other y-variables and z_{r_1+3} to have a total of $d + 1$ points. Thus, f is of the form $f = z_{r_1+1}z_{r_1+3}y_1 \cdots y_{r_1-1}y_{r_1} \cdots \widehat{y_j} \cdots y_d$. Combining these two possibilities gives form (iii).

Next, suppose $m = r_1 + 2$. Since $z_{r_1+2}y_1 \cdots y_{r_1-1} \in in_{<_{lex}}(\mathcal{G})$ by (6), there exists some index $1 \leq i \leq r_1 - 1$ such that $y_i \nmid f$. To have a total of $d + 1$ points, this forces the inclusion of all other y-variables and z_{r_1+3}. Therefore, f is of the form $f = z_{r_1+2}z_{r_1+3}y_1 \cdots \widehat{y_i} \cdots y_{r_1-1}y_{r_1} \cdots y_d$, giving (iv).

Finally, suppose $m = z_{r_1+3}$. Then, for f to be supported on $d + 1$ points, we must necessarily include all y-variables, yielding the form $f = z_{r_1+3}y_1 \cdots y_d$. Note that $f \notin in_{<_{lex}}(\mathcal{G})$, so we obtain form (v). □

Given that we know an explicit description of the facets of the unimodular triangulation $\mathcal{T}(\mathbf{q})$, a natural problem is to find a shelling of the facets from which we can recover the Ehrhart h^*-polynomial using standard techniques [3, 15]. This would provide another proof of Ehrhart h^*-unimodality, and give an explicit combinatorial interpretation to the coefficients of the h^*-polynomial. It is not clear how to construct a shelling in which both the shelling and the restriction sets admit a reasonable description. For example, one natural way to list the facets is to list them in lexicographic order; however, while this works for some small values of r_1 and x_1, computations with SageMath [17] show that this is not a shelling order when x_1 is sufficiently large compared to r_1.

It would be of interest to describe the regular unimodular triangulations of the 2-supported reflexive IDP $\Delta_{(1,\mathbf{q})}$, and to connect these shellings explicitly to the Ehrhart theory of these simplices. However, the most important aspect of the existence of the regular unimodular triangulation given in this work is to establish that the h^*-unimodality of these simplices falls within the framework of Theorem 1.3.

References

[1] Gabriele Balletti, Takayuki Hibi, Marie Meyer, and Akiyoshi Tsuchiya. Laplacian simplices associated to digraphs. *Arkiv för matematik*, 56, 12 2018.

[2] M. Beck and S. Robins. *Computing the Continuous Discretely: Integer-point Enumeration in Polyhedra*. Undergraduate Texts in Mathematics. Springer New York, 2007.

[3] Matthias Beck and Sinai Robins. *Computing the continuous discretely*. Undergraduate Texts in Mathematics. Springer, New York, second edition, 2015. Integer-point enumeration in polyhedra, With illustrations by David Austin.

[4] Benjamin Braun. Unimodality problems in Ehrhart theory. In *Recent trends in combinatorics*, volume 159 of *IMA Vol. Math. Appl.*, pages 687–711. Springer, [Cham], 2016.

[5] Benjamin Braun and Robert Davis. Ehrhart series, unimodality, and integrally closed reflexive polytopes. *Ann. Comb.*, 20(4):705–717, 2016.

[6] Benjamin Braun, Robert Davis, and Liam Solus. Detecting the integer decomposition property and Ehrhart unimodality in reflexive simplices. *Advances in Applied Mathematics*, 100, 08 2018.

[7] Benjamin Braun and Fu Liu. h^*-polynomials with roots on the unit circle. *Experimental Mathematics*, pages 1–17, 2019.

[8] Winfried Bruns and Tim Römer. h-vectors of Gorenstein polytopes. *J. Combin. Theory Ser. A*, 114(1):65–76, 2007.

[9] Heinke Conrads. Weighted projective spaces and reflexive simplices. *Manuscripta Math.*, 107(2):215–227, 2002.

[10] David A. Cox, John Little, and Donal O'Shea. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra, 3/e (Undergraduate Texts in Mathematics)*. Springer-Verlag, Berlin, Heidelberg, 2007.

[11] Takayuki Hibi and Hidefumi Ohsugi. Quadratic initial ideals of root systems. *Proc. Amer. Math. Soc.*, 130(7):1913–1922, 2001.

[12] Fu Liu and Liam Solus. On the relationship between Ehrhart unimodality and Ehrhart positivity. *Ann. Comb.*, 23(2):347–365, 2019.

[13] Benjamin Nill. Volume and lattice points of reflexive simplices. *Discrete Comput. Geom.*, 37(2):301–320, 2007.

[14] Liam Solus. Simplices for numeral systems. *Trans. Amer. Math. Soc.*, 371(3):2089–2107, 2019.

[15] Richard P. Stanley. *Combinatorics and commutative algebra*, volume 41 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, second edition, 1996.

[16] Bernd Sturmfels. *Gröbner Bases and Convex Polytopes*, volume 8 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1996.

[17] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 8.9)*, 2019. `https://www.sagemath.org`.

Department of Mathematics, University of Kentucky, Lexington, KY 40506–0027
Email address: `benjamin.braun@uky.edu`

Department of Mathematics, University of Kentucky, Lexington, KY 40506–0027
Email address: `derek.hanely@uky.edu`