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ON AN INVERSE PROBLEM OF NONLINEAR IMAGING WITH FRACTIONAL DAMPING

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Abstract. This paper considers the attenuated Westervelt equation in pressure formulation. The attenuation is by various models proposed in the literature and characterised by the inclusion of non-local operators that give power law damping as opposed to the exponential of classical models. The goal is the inverse problem of recovering a spatially dependent coefficient in the equation, the parameter of nonlinearity $\kappa(x)$, in what becomes a nonlinear hyperbolic equation with non-local terms. The overposed measured data is a time trace taken on a subset of the domain or its boundary. We shall show injectivity of the linearised map from κ to the overposed data and from this basis develop and analyse Newton-type schemes for its effective recovery.

1. Introduction

The problem of nonlinear B/A parameter imaging with ultrasound [2,3,5,15,33,36,37] in lossy media amounts to identification of the space dependent coefficient $\kappa(x)$ for the attenuated Westervelt equation in pressure formulation

$$\begin{aligned} &\left(v-\kappa(x)v^2\right)_{tt} - c_0^2\triangle v + Dv = r \quad \text{ in } \Omega\times(0,T) \\ &v=0 \text{ on } \partial\Omega\times(0,T)\,; \qquad v(0)=0, \quad v_t(0) \qquad\qquad =0 \ \text{ in } \ \Omega \end{aligned} \tag{1}$$

from observations Here $c_0 > 0$ is the wave speed (possibly space dependent as well), and Dv a damping term that will be specified below. For simplicity we impose homogeneous Dirichlet boundary conditions here, but the ideas in this paper extend to more realistic boundary conditions, such as absorbing boundary conditions for avoiding spurious reflections and/or inhomogeneous Neumann boundary conditions for modelling excitation via, e.g., some transducer array. Note that the excitation here is modelled by an interior source r, and we refer to a discussion on this in [21].

By letting our v equation satisfy boundary (or possibly interior) observations we obtain an inverse problem for the recovery of κ. These measurements will be taken to be

$$g(x,t) = v(x,t)\,,\ x\in\Sigma\,, \quad t\in(0,T) \tag{2}$$

either at single point $\Sigma=\{x_0\}$ or – in the spatially higher dimensional case – on some surface Σ contained in $\overline{\Omega}$.

The inverse problem represented by equations (1) and (2) is challenging on at least three counts. First, the underlying model equation is nonlinear and in fact

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the nonlinearity occurs in the highest order term. Second, the unknown coefficient $\kappa(x)$ is directly coupled to this term and third, is spatially varying whereas the data $g(t)$ is in the "orthogonal" time direction and this is well known to lead to severe ill-conditioning of the inversion of the map from data to unknown.

The use of ultrasound is a well-established protocol in the imaging of human tissue and, besides the classical sonography methodology, there exist several novel imaging principles, such as harmonic imaging or nonlinearity imaging. The latter [2, 3, 5, 15, 33, 36, 37] relies on tissue-dependence, hence spatial variation of the parameter of nonlinearity B/A that is contained in κ. It thus inherently needs a nonlinear acoustic model as an underlying PDE and we refer to e.g., the review [17] and the references therein for a brief derivation of the fundamental nonlinear acoustic equations. The quantity of interest from an imaging perspective is the coefficient κ and its recovery in the case when the damping term was $Du = b\triangle u_t$ was the subject of [21]. This is the classical formulation of damping being proportional to velocity but there are may alternative models that are prominent in the literature. We mention some of these in the next section but the main change is the incorporation of non-local terms involving either fractional derivatives in time or modifying the operator $(-\triangle)$ to have the Laplacian raised to a fractional power $(-\triangle)^\beta$. These have the effect of ameliorating the exponential decay of the solution, by a fractional exponent in the frequencies in the case of $(-\triangle)^\beta$ and by a power law decay in the case of a fractional time derivative. The use of such operators in inverse problems is now well documented in the literature (see [16] and in particular, for the wave equation in [22]).

In this paper we will provide analysis for the forward problem and in particular regularity and well-posedness for the coefficient-to-state map $G : \kappa \mapsto v$ where v solves (1). The forward map is defined by $F(\kappa) = \text{tr}_\Sigma v$, where $\text{tr}_\Sigma v$ denotes the time trace of the space–and–time dependent function $v : \Omega \times (0, T) \to \mathbb{R}$ at the observation surface Σ (which may also just be a single point $\Sigma = \{x_0\}$). Its linearization at $\kappa = 0$ is $F'(0)$ and we will prove an injectivity result in section 4 which will both show local uniqueness and pave the way for the use of Newton's method which we formulate and apply to obtain reconstructions of κ in section 6.

2. The imaging problem

As already mentioned in the introduction, the inverse problem under consideration is to recover the space dependent coefficient $\kappa(x)$ in the attenuated Westervelt equation which can also be written in the form

$$\begin{aligned} &u_{tt} + c^2\mathcal{A}u + Du = \kappa(x)(u^2)_{tt} + r \text{ in } \Omega \times (0, T) \\ &u(0) = 0, \quad u_t(0) = 0 \text{ in } \Omega \end{aligned} \tag{3}$$

from observations

$$g(x, t) = u(x, t), \quad x \in \Sigma,\ t \in (0, T), \tag{4}$$

where $\Sigma \subset \overline{\Omega}$ typically consists of a surface or a collection of discrete points or even just a single point. Some comments on the question about how rich it needs to be in order to allow for unique recovery of κ can be found in Remark 4.1. Note that we do not make any smoothness assumption on Σ. Here, $c > 0$ is the constant mean wave speed, and $\mathcal{A} = -(c_0(x)^2/c^2)\triangle$ contains the possibly spatially varying coefficient $c_0(x) > 0$ and is equipped – for simplicity – with homogeneous boundary conditions.

Moreover $r = r(x,t)$ is a known source term modelling excitation of the acoustic wave by a transducer array, see [22]. Throughout this paper we assume $\Omega \subseteq \mathbb{R}^d$, $d \in \{1,2,3\}$ to be a bounded domain with $C^{1,1}$ boundary and the coefficient $c_0(x)$ contained in $\mathcal{A}$ to be bounded away from zero and infinity.

The damping operator D appearing in (3) is a differential operator containing space and/or time derivatives. Classically, D will consist of integer derivatives, typical examples being $D = \mathcal{A}\partial_t$ or $D = \partial_t$ often referred to as strong and weak damping, respectively. We here list some of the (due to experimentally found power law frequency dependence) practically relevant fractional damping models, that we have already discussed in [22] in a different imaging context, namely for the inverse PAT/TAT problem:

Time fractional models.

- Caputo-Wismer model [6], [35, eq. (5)], called Kelvin wave equation in [4, eq. (19)]

(5) $$D = b\mathcal{A}\partial_t^\alpha,$$

where typically $\alpha \in [0,1]$.

- (Modified) Szabo model [30], [4, eq. (42)]

(6) $$D = b\partial_t^{\alpha+2},$$

where $\alpha \in [-1,1]$, $b \geq 0$.

- Fractional Zener (combined Caputo-Wismer-Szabo) model [14, 23], [4, eq. (30)]

(7) $$D = b_1\mathcal{A}\partial_t^{\alpha_1} + b_2\partial_t^{\alpha_2+2},$$

where $\alpha_1 \geq \alpha_2 \in [0,1]$, $b_1 \geq b_2 c^2$, cf. [14, Section III.B].

In these models ∂_t^α denotes the Djrbashian-Caputo fractional time derivative, which here, due to the homogeneous initial conditions, coincides with the Riemann-Liouville one.

Space fractional models.

- Chen-Holm model [7, eq. (21)]

(8) $$D = b\mathcal{A}^\beta$$

typically with $\beta \in [0,1]$, where Kelvin-Voigt damping is recovered when $\beta = 1$.

- Treeby-Cox model [32, eq. (28)]

(9) $$D = b_1\mathcal{A}^\beta\partial_t + b_2\mathcal{A}^{\beta+1/2}$$

typically $\beta \in [0,1]$, which is an extension of the former.

Here we use the spectral definition of the Laplacian which coincides with the Riesz version on $\mathbb{R}^d$; however, they differ in case of bounded Ω.

In this paper, we will focus on two damping models namely (a) a combination of (5) and (8), since we find it interesting to investigate the interplay of space- and time-fractional derivatives and its influence on the ill-posedness of the inverse problem; (b) (7) as it contains higher than second order time derivatives which are in case $\alpha_2 = 1$ known to make the equation have wave-like behavior (finite speed of propagation) in spite of the damping, which is expected to influence the degree of ill-posedness of the inverse problem as well. Thus we here focus on the two damping models

$D = b\mathcal{A}^\beta\partial_t^\alpha$ (combination of Caputo-Wismer-Kelvin and Chen-Holm model – CH)

$D = b_1\mathcal{A}\partial_t^{\alpha_1} + b_2\partial_t^{\alpha_2+2}$ (fractional Zener – FZ).

3. Analysis of the forward problem

The purpose of the analysis in this section is to provide the mathematical basis for writing the inverse problem (3), (4) as an operator equation on appropriate function spaces and for applying Newton's method as a reconstruction scheme for κ, see section 6.1.

To this end, we study well-posedness of the initial value problem for the parameter-to-state map $G : \kappa \mapsto u$ where u solves (3) and its linearisation $z = G'(\kappa)\underline{\delta\kappa}$

(10)
$$\begin{aligned}(1-2\kappa u)z_{tt} + c^2\mathcal{A}z + Dz - 4\kappa u_t\, z_t - 2\kappa u_{tt}\, z &= 2\underline{\delta\kappa}(u\,u_{tt} + u_t^2) \text{ in } \Omega\times(0,T)\\ z(0) = 0, \quad z_t(0) &= 0 \text{ in } \Omega\end{aligned}$$

for given κ and $\underline{\delta\kappa}$, respectively, in the context of the two damping models D defined according to the CH and the FZ case. In order to prove Fréchet differentiability, we will also have to consider the difference $v = G(\tilde\kappa) - G(\kappa) = \tilde u - u$, which solves

$$\begin{aligned}(1-2\kappa u)v_{tt} + c^2\mathcal{A}v + Dv - 2\kappa(\tilde u_t + u_t)\, v_t - 2\kappa\tilde u_{tt}\, v&\\ = 2(\tilde\kappa-\kappa)(\tilde u\tilde u_{tt} + \tilde u_t^2) \text{ in } \Omega\times(0,T)&\\ v(0) = 0, \quad v_t(0) = 0 \text{ in } \Omega&\end{aligned} \tag{11}$$

as well as the first order Taylor remainder $w = G(\tilde\kappa) - G(\kappa) - G'(\kappa)(\tilde\kappa-\kappa)$ which satisfies

(12)
$$\begin{aligned}(1-2\kappa u)w_{tt} + c^2\mathcal{A}w + Dw - 4\kappa u_t\, w_t - 2\kappa u_{tt}\, w&\\ = 2\underline{\delta\kappa}(v\tilde u_{tt} + uv_{tt} + (\tilde u_t + u_t)v_t) + 2\kappa(vv_{tt} + v_t^2) \text{ in } \Omega\times(0,T)&\\ w(0) = 0, \quad w_t(0) = 0 \text{ in } \Omega\,,&\end{aligned}$$

with $\underline{\delta\kappa} = \tilde\kappa - \kappa$.

Here we can allow for spatially varying sound speed $c_0(x)$ for which we only require

$$c_0 \in L^\infty(\Omega) \text{ and } c_0(x) \geq c > 0 \tag{13}$$

unless otherwise stated, by setting

$$\mathcal{A} = -\frac{c_0^2}{c^2}\triangle, \tag{14}$$

where $-\triangle$ is the Laplace operator equipped with homogeneous Dirichlet boundary conditions. We denote by (ϕ_j, λ_j) an eigensystem of the operator $\mathcal{A}$ with domain $\dot H^2(\Omega) := \mathcal{D}(\mathcal{A})$ which is selfadjoint and positive definite with respect to the weighted L^2 space $\dot L^2(\Omega) := L^2_{c^2/c_0^2}(\Omega)$. Note that by these assumptions the operator $\mathcal{A}^{-1} : \dot L^2(\Omega) \to \dot L^2(\Omega)$ is compact (based on the fact that Ω is bounded; for some comments on more general domain and boundary settings we point to [22]), so that the eigensystem exists and is complete with $\lambda_j \to \infty$ as $j \to \infty$. Moreover, this defines a scale of Hilbert spaces $\dot H^s(\Omega) := \mathcal{D}(\mathcal{A}^{s/2})$, $s \in \mathbb{R}$, whose norm can be defined via the eigensystem as $\|v\|_{\dot{H}^s(\Omega)} = \left(\sum_{j=1}^\infty \lambda_j^s |\langle v, \phi_j\rangle|^2\right)^{1/2}$ in case $s \geq 0$ and as the dual norm of $\dot{H}^{-s}(\Omega)$ in case $s < 0$. We will denote by $\langle\cdot,\cdot\rangle$ the $\dot{L}^2$ inner product (that is, the weighted one) on Ω whereas the use of the ordinary L^2 inner product will be indicated by a subscript $\langle\cdot,\cdot\rangle_{L^2}$. Moreover, we use the abbreviations $\|u\|_{L^p_t(L^q)} = \|u\|_{L^p(0,t;L^q(\Omega))}$, $\|u\|_{L^p(L^q)} = \|u\|_{L^p(0,T;L^q(\Omega))}$ for space-time norms.

Throughout this paper, we denote by ∂_t^α the (partial) Caputo-Djrbashian fractional time derivative of order $\alpha \in (n-1,n)$ with $n \in \mathbb{N}$ by $\partial_t^\alpha u = I_t^{n-\alpha}[\partial_t^n u]$, where ∂_t^n denotes the n-th integer order partial time derivative and for $\gamma \in (0,1)$, and I_t^γ is the Abel integral operator defined by

$$I_t^\gamma[v](t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{v(s)}{(t-s)^{1-\gamma}} ds\,.$$

For details on fractional differentiation and subdiffusion equations, we refer to, e.g., [8,9,24,28,29]. See also the tutorial paper on inverse problems for anomalous diffusion processes [16]. Whenever we use the Riemann-Liouville fractional derivative $\partial_t^n I_t^{n-\alpha}$, this will be denoted by ${}^{RL}\partial_t^\alpha$. These two versions of the fractional derivative coincide when applied to a function whose initial derivatives up to order $n-1$ vanish at $t=0$.

The crucial tool we need in obtaining the required estimates is the following consequence of Alikhanov's Lemma [1, Lemma 1]

$$\partial_t^\gamma[w](t)w(t) \geq \tfrac{1}{2}(\partial_t^\gamma w^2)(t) \tag{15}$$

for any absolutely continuous function w. We apply it to $w = \partial_t^\alpha v$ with $\gamma = 1-\alpha$, using the identities

$$\begin{aligned} &\partial_t^{1-\alpha} w = \partial_t^{1-\alpha}\partial_t^\alpha v = \partial_t^{1-\alpha} I_t^{1-\alpha} v_t = v_t\,,\\ &\int_0^t (\partial_s^{1-\alpha} w^2)(s)\,ds = \int_0^t \partial_s I_t^\alpha[w^2](s)\,ds = I_t^\alpha[w^2](t) \end{aligned}$$

that hold for $v_t \in L^\infty(0,T)$ and for $w^2 \in W^{1,1}(0,T)$ with $w(0) = 0$. Note that for $w = \partial_t^\alpha v$ we automatically have $w(0) = 0$ and $I_t^\alpha[w^2](t) = 0$. After integration with respect to time this implies the following result.

Lemma 3.1. *For $v \in W^{1,\infty}(0,T)$ with $(\partial_t^\alpha v)^2 \in W^{1,1}(0,T)$, and $t \in (0,T)$, the following estimate holds.*

$$\int_0^t \partial_s^\alpha[v](s) v_t(s)\,ds \geq I_t^\alpha\left[(\partial_t^\alpha v)^2\right] \geq \frac{1}{2\Gamma(\alpha)t^{1-\alpha}} \|\partial_t^\alpha v\|^2_{L^2(0,t)}\,. \tag{16}$$

A stronger version of this with respect to temporal regularity, however with a coefficient that vanishes as $\alpha \nearrow 1$, is the following coercivity estimate [11, Lemma 2.3], see also [34, Theorem 1]: For any $w \in H^{-(1-\alpha)/2}(0,t)$,

$$\int_0^t \langle I_t^{1-\alpha} w(s), w(s)\rangle\,ds \geq \cos(\tfrac{\pi(1-\alpha)}{2})\|w\|^2_{H^{-(1-\alpha)/2}(0,t)}. \tag{17}$$

In order to prove well-posedness of the nonlinear equation (3) (with initial conditions) needed for defining the forward operator, as well as the linear equations (10), (11) required for establishing Fréchet differentiability, we will proceed similarly in both damping model cases: First of all, we analyse a related linear equation with general coefficients that allows us to formulate the nonlinear equation as a fixed point problem and to apply Banach's Fixed Point Theorem for proving its well-posedness. Then we apply the same general coefficient linear result to the two linear problems (10), (11) in order to prove differentiability. It will turn out that in the CH case we need three different regularity levels in the solution spaces, whereas the analysis is somewhat simpler for FZ and allows us to work on the same solution space for all purposes pointed out above.

3.1. **Caputo-Wismer-Kelvin-Chen-Holm damping.** We start with the Caputo-Wismer-Kelvin-Chen-Holm model

$$D = b\mathcal{A}^\beta \partial_t^\alpha \quad \text{ with } \alpha \in [0,1],\ \beta \in [0,1],\ b \geq 0 \tag{18}$$

and first of all consider the initial boundary value problem for the general linear PDE

$$(1-\sigma)u_{tt} + c^2\mathcal{A}u + b\mathcal{A}^\beta\partial_t^\alpha u + \mu u_t + \rho u = h \tag{19}$$

$$u(0) = u_0\,, \quad u_t(0) = u_1 \tag{20}$$

with constants b, $c > 0$ and given space and time dependent functions σ, μ, ρ, h where σ satisfies the non-degeneracy condition

$$\sigma(x,t) \leq \overline{\sigma} < 1 \text{ for all } x \in \Omega \quad t \in (0,T)\,. \tag{21}$$

In order to prove existence and uniqueness of solutions to (19), (20), we apply the usual Faedo-Galerkin approach of discretisation in space with eigenfunctions of $\mathcal{A}$, $u(x,t) \approx u^n(x,t) = \sum_{i=1}^n u_i^n(t)\phi_i(x)$ and testing with ϕ_j, that is,

$$\langle (1-\sigma)u^n_{tt} + c^2\mathcal{A}u^n + b\mathcal{A}^\beta\partial_t^\alpha u^n + \mu u^n_t + \rho u^n - h, v\rangle = 0 \quad v \in \text{span}(\phi_1,\ldots,\phi_n)\,. \tag{22}$$

This leads to the ODE system

$$(I - S^n(t))\underline{u}^{n\prime\prime}(t) + b\,(\Lambda^n)^\beta(\partial_t^\alpha \underline{u}^n)(t) + M^n(t)\underline{u}^{n\prime}(t) + \Big(c^2\,\Lambda^n + R^n(t)\Big)\underline{u}^n(t) = \underline{h}^n \tag{23}$$

with matrices and vectors defined by

$$\begin{aligned}
&\underline{u}^n(t) = (u_i^n(t))_{i=1,\ldots n}\,, \quad \underline{h}^n(t) = (\langle h(t),\phi_i\rangle)_{i=1,\ldots n}\,, \quad \Lambda^n = \text{diag}(\lambda_1,\ldots,\lambda_n)\,,\\
&S^n(t) = (\langle\sigma(t)\phi_i,\phi_j\rangle)_{i,j=1,\ldots n}\,, \quad M^n(t) = (\langle\mu(t)\phi_i,\phi_j\rangle)_{i,j=1,\ldots n}\,,\\
&R^n(t) = (\langle\rho(t)\phi_i,\phi_j\rangle)_{i,j=1,\ldots n}\,.
\end{aligned} \tag{24}$$

Existence of a unique solution $\underline{u}^n \in C^2([0,T];\mathbb{R}^n)$ to (23) follows from standard ODE theory (Picard-Lindelöf Theorem and Gronwall's Inequality), as long as σ, μ and ρ are in $C([0,T];\dot{H}^s(\Omega))$ for some $s \in \mathbb{R}$ (noting that the eigenfunctions ϕ_j are contained in $\dot{H}^k(\Omega)$ for any $k \in \mathbb{N}$ and therefore the vector and matrix functions $\underline{h}^n$, S^n, M^n, R^n in (24) are well-defined and contained in $C([0,T];\mathbb{R}^n)$ and $C([0,T];\mathbb{R}^{n\times n})$, respectively). Moreover due to (21), the symmetric matrix $S^n(t)$ is positive definite with smallest eigenvalue bounded away from zero by $1-\overline{\sigma}$ cf (21).

We multiply (23) with $(\Lambda^n)^2\underline{u}^{n\prime}(t)$ and integrate with respect to time, using the identity

$$(S^n(t)\underline{u}^{n\prime\prime}(t))^T(\Lambda^n)^2\underline{u}^{n\prime}(t) = \sum_{i=1}^n\sum_{j=1}^n \langle\sigma(t)\phi_i,\phi_j\rangle\lambda_j^2 u_i^{n\prime\prime}(t)u_j^{n\prime}(t)$$

$$= \sum_{i=1}^n\sum_{j=1}^n \langle\sigma(t)\phi_i,\phi_j\rangle\lambda_i u_i^{n\prime\prime}(t)\lambda_j u_j^{n\prime}(t) + \sum_{i=1}^n\sum_{j=1}^n \langle\sigma(t)\phi_i,\phi_j\rangle(\lambda_j-\lambda_i)u_i^{n\prime\prime}(t)\lambda_j u_j^{n\prime}(t)$$

$$= \frac{1}{2}\frac{d}{dt}\sum_{i=1}^n\sum_{j=1}^n \langle\sigma(t)\phi_i,\phi_j\rangle\lambda_i u_i^{n\prime}(t)\lambda_j u_j^{n\prime}(t) - \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n \langle\sigma_t(t)\phi_i,\phi_j\rangle\lambda_i u_i^{n\prime}(t)\lambda_j u_j^{n\prime}(t)$$

$$+ \sum_{i=1}^n\sum_{j=1}^n \Big(\langle\sigma(t)\phi_i,\mathcal{A}\phi_j\rangle - \langle\sigma(t)\phi_j,\mathcal{A}\phi_i\rangle\Big)u_i^{n\prime\prime}(t)\lambda_j u_j^{n\prime}(t),$$

where

$$\langle\sigma(t)\phi_i,\mathcal{A}\phi_j\rangle - \langle\sigma(t)\phi_j,\mathcal{A}\phi_i\rangle = \langle\mathcal{A}[\sigma(t)\phi_i] - \sigma(t)\mathcal{A}\phi_i,\phi_j\rangle$$
$$= \langle-\triangle[\sigma(t)\phi_i] + \sigma(t)\triangle\phi_i,\phi_j\rangle_{L^2} = -\langle(c_0^2/c^2)(\triangle\sigma(t)\,\phi_i + \nabla\sigma(t)\cdot\nabla\phi_i),\phi_j\rangle$$

provided $\sigma(t)\phi_i \in \dot{H}^s(\Omega))$ for some $s\in\mathbb{R}$. (Note that the latter identity also holds true in case of spatially varying c_0 since we use the c^2/c_0^2 weighted L^2 inner product then.) Thus we have

$$\int_0^t ((I-S^n(s))\underline{u}^{n\prime\prime}(s))^T(\Lambda^n)^2\underline{u}^{n\prime}(s)\,ds$$
$$= \int_0^t \Big(\frac{1}{2}\frac{d}{dt}\|\sqrt{1-\sigma}\mathcal{A}u_t^n\|^2_{\dot{L}^2(\Omega)}(s) + \frac{1}{2}\langle\sigma_t(s)\mathcal{A}u_t^n(s),\mathcal{A}u_t^n(s)\rangle$$
$$+ \langle(c_0^2/c^2)(\triangle\sigma(s)\,u_{tt}^n(s) + \nabla\sigma(s)\cdot\nabla u_{tt}^n(s)),\mathcal{A}u_t^n(s)\rangle\Big)ds$$
$$= \frac{1}{2}\|\sqrt{1-\sigma(t)}\mathcal{A}u_t^n(t)\|^2_{\dot{L}^2(\Omega)} - \frac{1}{2}\|\sqrt{1-\sigma(0)}\mathcal{A}u_t^n(0)\|^2_{\dot{L}^2(\Omega)}$$
$$+ \int_0^t \langle\frac{1}{2}\sigma_t(s)\mathcal{A}u_t^n(s) + (c_0^2/c^2)(\triangle\sigma(s)\,u_{tt}^n(s) + \nabla\sigma(s)\cdot\nabla u_{tt}^n(s)),\mathcal{A}u_t^n(s)\rangle\,ds\,.$$

Similarly, we have

$$(M^n(t)\underline{u}^{n\prime}(t))^T(\Lambda^n)^2\underline{u}^{n\prime}(t)$$
$$= \langle\mu(t)\mathcal{A}u_t^n(t),\mathcal{A}u_t^n(t)\rangle - \langle(c_0^2/c^2)(\triangle\mu(t)\,u_t^n(t) + \nabla\mu(t)\cdot\nabla u_t^n(t)),\mathcal{A}u_t^n(t)\rangle$$

and

$$(R^n(t)\underline{u}^n(t))^T(\Lambda^n)^2\underline{u}^{n\prime}(t)$$
$$= \langle\rho(t)\mathcal{A}u^n(t),\mathcal{A}u_t^n(t)\rangle - \langle(c_0^2/c^2)(\triangle\rho(t)\,u^n(t) + \nabla\rho(t)\cdot\nabla u^n(t)),\mathcal{A}u_t^n(t)\rangle\,.$$

Finally,

$$\int_0^t ((\Lambda^n)^\beta(\partial_t^\alpha\underline{u}^n)(s))^T(\Lambda^n)^2\underline{u}^{n\prime}(s)ds = \int_0^t\sum_{j=1}^n \lambda_j^{2+\beta}(\partial_t^\alpha u_j^n)(s)u_j^{n\prime}(s)\,ds$$
$$\geq \frac{1}{2\Gamma(\alpha)t^{1-\alpha}}\sum_{j=1}^n\lambda_j^{2+\beta}\int_0^t\Big(\partial_t^\alpha u_j^n(s)\Big)^2\,ds = \frac{1}{2\Gamma(\alpha)t^{1-\alpha}}\|\mathcal{A}^{1+\beta/2}\partial_t^\alpha u^n\|^2_{L_t^2(\dot{L}^2)}\,,$$

and

$$\int_0^t (\Lambda^n \underline{u}^n(t))^T (\Lambda^n)^2 \underline{u}^{n\prime}(t) = \sum_{j=1}^n \lambda_j^3 \int_0^t u_j^n(s) u_j^{n\prime}(s)\, ds$$
$$= \frac{1}{2}\sum_{j=1}^n \lambda_j^3 \int_0^t \frac{d}{dt}\Big(u_j^n\Big)^2(s)\, ds = \frac{1}{2}\|\nabla \mathcal{A} u^n(t)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\nabla\mathcal{A}u^n(0)\|^2_{L^2(\Omega)}\,.$$

This together with Young's inequality yields the energy estimate

(25)

$$\begin{aligned}
&\frac{1}{2}\|\sqrt{1-\sigma(t)}\mathcal{A}u_t^n(t)\|^2_{\dot L^2(\Omega)} + \frac{b}{2\Gamma(\alpha)t^{1-\alpha}}\|\mathcal{A}^{1+\beta/2}\partial_t^\alpha u^n\|^2_{L^2_t(\dot L^2)} + \frac{c^2}{2}\|\nabla\mathcal{A}u^n(t)\|^2_{L^2(\Omega)}\\
&\le \frac{1}{2}\|\sqrt{1-\sigma(0)}\mathcal{A}u_t^n(0)\|^2_{\dot L^2(\Omega)} + \frac{c^2}{2}\|\nabla\mathcal{A}u^n(0)\|^2_{L^2(\Omega)} + \frac{1}{2\epsilon}\|\mathcal{A}u_t^n\|^2_{L^2_t(\dot L^2)}\\
&\quad + \frac{\epsilon}{2}\| -\frac{1}{2}\sigma_t\mathcal{A}u_t^n - \mu\mathcal{A}u_t^n - \rho\mathcal{A}u^n + \mathcal{A}h + (c_0^2/c^2)\\
&\qquad \cdot(-\triangle\sigma\, u_{tt}^n - \nabla\sigma\cdot\nabla u_{tt}^n + \triangle\mu\, u_t^n + \nabla\mu\cdot\nabla u_t^n + \triangle\rho\, u^n + \nabla\rho\cdot\nabla u^n)\|^2_{L^2_t(\dot L^2)}\\
&\le \frac{1}{2}\|\sqrt{1-\sigma(0)}\mathcal{A}u_t^n(0)\|^2_{\dot L^2(\Omega)} + \frac{c^2}{2}\|\nabla\mathcal{A}u^n(0)\|^2_{L^2(\Omega)} + \frac{1}{2\epsilon}\|\mathcal{A}u_t^n\|^2_{L^2_t(\dot L^2)}\\
&\quad + \frac{\epsilon}{2}\Big(\frac{1}{2}\|\sigma_t\|_{L^2_t(L^\infty)}\|\mathcal{A}u_t^n\|_{L^\infty_t(L^2)} + \|\mu\|_{L^2_t(L^\infty)}\|\mathcal{A}u_t^n\|_{L^\infty_t(L^2)}\\
&\qquad + \|\rho\|_{L^2_t(L^\infty)}\|\mathcal{A}u^n\|_{L^\infty_t(L^2)} + \|\mathcal{A}h\|_{L^2_t(L^2)}\\
&\qquad + \frac{\|c_0\|_{L^\infty(\Omega)}}{c}\big(\|\triangle\sigma\|_{L^\infty(L^4)}\|u_{tt}^n\|_{L^2(L^4)} + \|\nabla\sigma\|_{L^\infty_t(L^\infty)}\|\nabla u_{tt}^n\|_{L^2_t(L^2)}\\
&\qquad + \|\triangle\mu\|_{L^2_t(L^2)}\|u_t^n\|_{L^\infty_t(L^\infty)} + \|\nabla\mu\|_{L^2(L^4)}\|\nabla u_t^n\|_{L^\infty(L^4)}\\
&\qquad + \|\triangle\rho\|_{L^2_t(L^2)}\|u^n\|_{L^\infty_t(L^\infty)} + \|\nabla\rho\|_{L^2(L^4)}\|\nabla u^n\|_{L^\infty(L^4)}\big)\Big)^2.
\end{aligned}$$

Here we can make use of the fact that $\|u^n\|_{L^\infty(0,t;Z)} \le \|u^n(0)\|_Z + \sqrt{T}\|u_t^n\|_{L^2(0,t;Z)}$ and the embedding estimates

$$\text{(26)}\qquad \begin{aligned}
&\|v\|_{L^4(\Omega)} \le C_{H^1,L^4}\|\nabla v\|\,, \quad v\in H_0^1(\Omega)\,,\\
&\|v\|_{L^\infty(\Omega)} \le C_{H^2,L^\infty}\|\mathcal{A}v\|_{\dot L^2(\Omega)}\,, \quad v\in H_0^1(\Omega)\cap H^2(\Omega)\,,
\end{aligned}$$

(the latter resulting from elliptic regularity and continuity of the embedding $H^2(\Omega) \to L^\infty(\Omega)$) in order to further estimate

$$\begin{aligned}
\|u_{tt}^n\|_{L^2(L^4)} &\le C_{H^1,L^4}\|\nabla u_{tt}^n\|_{L^2_t(L^2)}\\
\|u_t^n\|_{L^\infty_t(L^\infty)} &\le C_{H^2,L^\infty}\|\mathcal{A}u_t^n\|_{L^\infty_t(\dot L^2)}\\
\|u^n\|_{L^\infty_t(L^\infty)} &\le C_{H^2,L^\infty}\|\mathcal{A}u^n\|_{L^\infty_t(\dot L^2)}\\
\|\nabla u^n\|_{L^\infty_t(L^\infty)} &\le C_{H^2,L^\infty}\|\nabla\mathcal{A}u^n\|_{L^\infty_t(L^2)}.
\end{aligned}$$

For the last inequality to hold, we need more smoothness than the globally assumed L^∞ boundedness on the variable wave speed c_0 contained in $\mathcal{A}$ and its reciprocal, namely, bearing in mind the identity $\nabla\mathcal{A}u^n = [-(c_0(x)^2/c^2)\triangle - \nabla[(c_0(x)^2/c^2)]\nabla\cdot]\nabla u^n$,

$$\text{(27)}\qquad c_0 \in W^{1,\infty}(\Omega).$$

Now we proceed with estimating $\|\nabla u_{tt}^n\|^2_{L^2}$ by multiplying the ODE (23) with $\Lambda^n\underline{u}^{n\prime\prime}$, that is testing (22) with $v = \mathcal{A}u_{tt}^n(t)$, and using integration by parts (note that all terms in $(1-\sigma)u_{tt}^n + c^2\mathcal{A}u^n + b\mathcal{A}^\beta\partial_t^\alpha u^n + \mu u_t^n + \rho u^n - h$ vanish on $\partial\Omega$) as well as Young's inequality

$$\begin{aligned}
0 &= \langle(1-\sigma)u_{tt}^n + c^2\mathcal{A}u^n + b\mathcal{A}^\beta\partial_t^\alpha u^n + \mu u_t^n + \rho u^n - h, \mathcal{A}u_{tt}^n\rangle \\
&= \|\sqrt{1-\sigma}\nabla u_{tt}^n\|_{L^2(\Omega)}^2 \\
&\quad + \langle -\nabla\sigma\, u_{tt}^n + \nabla\Big(c^2\mathcal{A}u^n + b\mathcal{A}^\beta\partial_t^\alpha u^n + \mu u_t^n + \rho u^n - h\Big), \nabla u_{tt}^n\rangle_{L^2(\Omega)} \\
&\geq \|\sqrt{1-\sigma}\nabla u_{tt}^n\|_{L^2(\Omega)}^2 - \frac{1}{2}(1-\overline{\sigma})\|\nabla u_{tt}^n\|_{L^2(\Omega)}^2 \\
&\quad - \frac{1}{2(1-\overline{\sigma})}\| - \nabla\sigma\, u_{tt}^n + c^2\nabla\mathcal{A}u^n + b\nabla\mathcal{A}^\beta\partial_t^\alpha u^n \\
&\quad + \nabla\mu\, u_t^n + \mu\nabla u_t^n + \nabla\rho\, u^n + \rho\nabla u^n - \nabla h\|_{L^2(\Omega)}^2
\end{aligned}$$

which yields

$$\begin{aligned}
\|\nabla u_{tt}^n\|_{L_t^2(L^2)} &\leq \frac{1}{1-\overline{\sigma}} \\
&\cdot \Big(\|\nabla\sigma\|_{L^\infty(L^4)}\|u_{tt}^n\|_{L^2(L^4)}^2 + c^2\|\nabla\mathcal{A}u^n\|_{L_t^2(L^2)} + b\|\mathcal{A}^{1/2+\beta}\partial_t^\alpha u^n\|_{L_t^2(\dot{L}^2)} \\
&\quad + \|\nabla\mu\|_{L^2(L^4)}\|u_t^n\|_{L^\infty(L^4)} + \|\mu\|_{L_t^2(L^\infty)}\|\nabla u_t^n\|_{L_t^\infty(L^2)} \\
&\quad + \|\nabla\rho\|_{L_t^2(L^2)}\|u^n\|_{L_t^\infty(L^\infty)} + \|\rho\|_{L_t^2(L^\infty)}\|\nabla u^n\|_{L_t^\infty(L^2)}\Big),
\end{aligned}$$

where we can again employ the embedding estimates (26) and assume

$$\|\nabla\sigma\|_{L^\infty(L^4)} < \frac{1-\overline{\sigma}}{C_{H^1\to L^4}} \tag{28}$$

in order to extract an estimate of the form

$$\begin{aligned}
&\|\nabla u_{tt}^n\|_{L_t^2(L^2)} \\
&\leq C\Big(\|\nabla\mathcal{A}u^n\|_{L_t^2(L^2)} + \|\mathcal{A}^{1/2+\beta}\partial_t^\alpha u^n\|_{L_t^2(\dot{L}^2)} + \|\nabla u_t^n\|_{L_t^\infty(L^2)} + \|\nabla u^n\|_{L_t^\infty(L^2)}\Big).
\end{aligned} \tag{29}$$

Adding a multiple (factor $\epsilon(\|\triangle\sigma\|_{L^\infty(L^4)}^2(C_{H^1,L^4})^2 + \|\nabla\sigma\|_{L_t^\infty(L^\infty)}^2)$) of the square of (29) to (25), making ϵ small enough (so that $C^2\epsilon < \frac{b}{2\Gamma(\alpha)t^{1-\alpha}}$) and all terms containing $L^\infty(0,T)$ norms of u^n on the right hand side of (25) can be dominated by left hand side terms) using the fact that $1/2+\beta \leq 1+\beta/2$ for $\beta\in[0,1]$ and Gronwall's inequality we end up with an estimate of the form

$$\begin{aligned}
&\|\nabla u_{tt}^n\|_{L^2(L^2)}^2 + \|\mathcal{A}u_t^n\|_{L_t^\infty(\dot{L}^2)}^2 + \|\mathcal{A}^{1+\beta/2}\partial_t^\alpha u^n\|_{L_t^2(\dot{L}^2)}^2 + \|\nabla\mathcal{A}u^n\|_{L_t^\infty(L^2)}^2 \\
&\leq C(T)\Big(\|\mathcal{A}u_t^n(0)\|_{\dot{L}^2(\Omega)}^2 + \|\nabla\mathcal{A}u^n(0)\|_{L^2(\Omega)}^2 + \|\mathcal{A}h\|_{L_t^2(\dot{L}^2)}^2\Big)
\end{aligned} \tag{30}$$

which via weak limits shows the existence of a solution to the homogeneous initial boundary value problem for (19) and transfers to u as

$$\begin{aligned}
\|u\|_U^2 &:= \|\nabla u_{tt}\|_{L^2(L^2)}^2 + \|\mathcal{A}u_t\|_{L_t^\infty(\dot{L}^2)}^2 + \|\mathcal{A}^{1+\beta/2}\partial_t^\alpha u\|_{L_t^2(\dot{L}^2)}^2 + \|\nabla\mathcal{A}u\|_{L_t^\infty(L^2)}^2 \\
&\leq C(T)\Big(\|\mathcal{A}u_t(0)\|_{\dot{L}^2(\Omega)}^2 + \|\nabla\mathcal{A}u(0)\|_{L^2(\Omega)}^2 + \|\mathcal{A}h\|_{L_t^2(\dot{L}^2)}^2\Big).
\end{aligned} \tag{31}$$

Uniqueness of a solution follows from an energy estimate obtained in a lower regularity regime (see Proposition 3.3)

The required regularity on σ, μ, ρ, h, u_0, u_1 is, besides (21) and (28)

$$\begin{aligned}&\sigma \in H^1(0,T;L^\infty(\Omega)) \cap L^\infty(0,T;W^{2,4}(\Omega)\cap W^{1,\infty}(\Omega))\\ &\mu \in L^2(0,T;H^2(\Omega))\,, \quad \rho \in L^2(0,T;H^2(\Omega))\\ &h \in L^2(0,T;\dot H^2(\Omega))\,, \quad u_0 \in \dot H^3(\Omega)\,, \quad u_1 \in \dot H^2(\Omega).\end{aligned} \tag{32}$$

Proposition 3.1. *Under conditions (21), (27), (28), (32), there exists a unique solution*

$$u \in U := H^2(0,T;\dot L^2(\Omega)) \cap W^{1,\infty}(0,T;\dot H^2(\Omega)) \cap L^\infty(0,T;\dot H^3(\Omega)) \tag{33}$$

to the initial boundary value problem (19), (20). This solution satisfies the estimate (31).

The energy estimate leading to this result has been obtained by basically "multiplying (19) with $\mathcal{A}^2 u_t$", that is, taking the $\dot L^2(\Omega)$ inner product of the PDE with $\mathcal{A}^2 u_t$ and using selfadjointness of $\mathcal{A}$ in $\dot L^2(\Omega)$.

Later on, we will also need less regular solutions along with estimates on them. Since the proofs are actually somewhat simpler then, we skip the details on Galerkin approximation and only provide the energy estimates.

Multiplying (19) with $\mathcal{A}u_t$ we obtain

(34)
$$\begin{aligned}&\frac12\|\sqrt{1-\sigma(t)}\nabla u_t(t)\|^2_{L^2(\Omega)} + \frac{c^2}{2}\|\mathcal{A}u(t)\|^2_{\dot L^2(\Omega)} + \frac{1}{2\Gamma(\alpha)t^{1-\alpha}}\|\mathcal{A}^{(1+\beta)/2}\partial_t^\alpha u\|^2_{L^2_t(\dot L^2)}\\
&\le \frac12\|\sqrt{1-\sigma(0)}\nabla u_t(0)\|^2_{L^2(\Omega)} + \frac{c^2}{2}\|\mathcal{A}u(0)\|^2_{\dot L^2(\Omega)}\\
&\quad - \int_0^t \langle \frac12\sigma_t(s)\nabla u_t(s) + \nabla\sigma(s)\, u_{tt}(s) + \nabla\mu\, u_t + \mu\nabla u_t\\
&\qquad + \nabla\rho\, u + \rho\nabla u - \nabla h, \nabla u_t(s)\rangle_{L^2(\Omega)}\, ds\\
&\le \frac12\|\sqrt{1-\sigma(0)}\nabla u_t(0)\|^2_{L^2(\Omega)} + \frac{c^2}{2}\|\mathcal{A}u(0)\|^2_{\dot L^2(\Omega)} + \frac{1}{2\epsilon}\|\nabla u_t\|^2_{L^2_t(L^2)}\\
&\quad + \frac{\epsilon}{2}\Big(\frac12\|\sigma_t\|_{L^2_t(L^\infty)}\|\nabla u_t\|_{L^\infty_t(L^2)} + \|\nabla\sigma\|_{L^\infty_t(L^\infty)}\|u_{tt}\|_{L^2_t(L^2)}\\
&\qquad + \|\nabla\mu\|_{L^2(L^4)}\|u_t\|_{L^\infty(L^4)} + \|\mu\|_{L^2_t(L^\infty)}\|\nabla u_t\|_{L^\infty_t(L^2)}\\
&\qquad + \|\nabla\rho\|_{L^2_t(L^2)}\|u\|_{L^\infty_t(L^\infty)} + \|\rho\|_{L^2(L^4)}\|\nabla u\|_{L^\infty(L^4)} + \|\nabla h\|_{L^2_t(L^2)}\Big)^2.\end{aligned}$$

The PDE provides us with

$$\begin{aligned}\|u_{tt}\|_{L^2_t(L^2)} &= \|-\frac{1}{1-\sigma}\Big(c^2\mathcal{A}u + b\mathcal{A}^\beta\partial_t^\alpha u + \mu u_t + \rho u - h\Big)\|_{L^2(L^2)}\\
&\le C(\|\mu\|_{L^2_t(L^\infty)} + \|\rho\|_{L^2(L^4)})\end{aligned}$$
$$\times \sup_{t\in(0,T)}\Big(\|\sqrt{1-\sigma(t)}\nabla u_t(t)\|^2_{L^2(\Omega)} + \frac{c^2}{2}\|\mathcal{A}u(t)\|^2_{\dot L^2(\Omega)} + \frac{1}{2\Gamma(\alpha)t^{1-\alpha}}\|\mathcal{A}^\beta\partial_t^\alpha u\|^2_{L^2_t(\dot L^2)}\Big)$$

which, due to the condition $\beta \le (1+\beta)/2$, allows us to dominate the ϵu_{tt} term on the right hand side of (34). Thus, using Gronwall's inequality, we get an estimate of the form

$$\begin{aligned}\|u\|^2_{U_{lo}} &:= \|u_{tt}\|^2_{L^2_t(\dot L^2)} + \|\nabla u_t\|^2_{L^\infty_t(\dot L^2)} + \|\mathcal{A}^{(1+\beta)/2}\partial_t^\alpha u\|^2_{L^2_t(\dot L^2)} + \|\mathcal{A}u\|^2_{L^\infty_t(\dot L^2)}\\
&\le C(T)\Big(\|\nabla u_t(0)\|^2_{L^2(\Omega)} + \|\mathcal{A}u(0)\|^2_{\dot L^2(\Omega)} + \|\nabla h\|^2_{L^2(L^2)}\Big)\end{aligned} \tag{35}$$

provided

$$\begin{aligned}
&\sigma \in H^1(0,T;L^\infty(\Omega)) \cap L^\infty(0,T;W^{1,\infty}(\Omega))\,,\\
&\mu \in L^2(0,T;L^\infty(\Omega)\cap W^{1,4}(\Omega))\,, \quad \rho \in L^2(0,T;H^1(\Omega))\\
&h \in L^2(0,T;H^1(\Omega))\,, \quad u_0 \in \dot H^2(\Omega)\,, \quad u_1 \in \dot H^1(\Omega).
\end{aligned} \tag{36}$$

Again, uniqueness of a solution follows from an energy estimate in a lower regularity regime (see Proposition 3.3).

Proposition 3.2. *Under the conditions* (21), (36), *there exists a unique solution* $u \in U_{lo} \subseteq H^2(0,T;\dot L^2(\Omega)) \cap W^{1,\infty}(0,T;\dot H^1(\Omega)) \cap L^\infty(0,T;\dot H^2(\Omega))$, *to the initial boundary value problem* (19), (20), *and this solution satisfies the estimate* (35).

Uniqueness and an even lower regularity estimate can be obtained by multiplication of (19) with u_t, (note that this is admissible for $u \in U_{lo}$, cf., (35), so we do not move to Galerkin discretizations here,) which yields

$$\begin{aligned}
&\frac12\|\sqrt{1-\sigma(t)}u_t(t)\|^2_{\dot L^2(\Omega)} + \frac{c^2}{2}\|\nabla u(t)\|^2_{L^2(\Omega)} + \frac{1}{2\Gamma(\alpha)t^{1-\alpha}}\|\mathcal{A}^{\beta/2}\partial_t^\alpha u\|^2_{L^2_t(\dot L^2)}\\
&\le \frac12\|\sqrt{1-\sigma(0)}u_t(0)\|^2_{\dot L^2(\Omega)} + \frac{c^2}{2}\|\nabla u(0)\|^2_{L^2(\Omega)}\\
&\qquad - \int_0^t \langle \frac12\sigma_t(s)u_t(s) + \mu\, u_t + \rho u - h, u_t(s)\rangle\, ds\\
&\le \frac12\|\sqrt{1-\sigma(0)}u_t(0)\|^2_{\dot L^2(\Omega)} + \frac{c^2}{2}\|\nabla u(0)\|^2_{L^2(\Omega)} + \frac{1}{2\epsilon}\|u_t\|^2_{L^2_t(\dot L^2)}\\
&\qquad + \frac{\epsilon}{2}\Big(\frac12\|\sigma_t\|_{L^2_t(L^\infty)}\|u_t\|_{L^\infty_t(\dot L^2)} + \|\mu\|_{L^2_t(L^\infty)}\|u_t\|_{L^\infty_t(\dot L^2)}\\
&\qquad\quad + \|\rho\|_{L^2(L^4)}\|u\|_{L^\infty(L^4)} + \|h\|_{L^2_t(\dot L^2)}\Big)^2,
\end{aligned} \tag{37}$$

(where we have used the fact that $\|\frac{c}{c_0}\|_{L^\infty(\Omega)} \le 1$), hence an estimate of the form

$$\begin{aligned}
\|u\|^2_{U_{vl}} :=\ &\|u_t\|^2_{L^\infty_t(\dot L^2)} + \|\mathcal{A}^{\beta/2}\partial_t^\alpha u\|^2_{L^2(L^2)} + \|\nabla u\|^2_{L^\infty_t(L^2)}\\
&\le C(T)\Big(\|u_t(0)\|^2_{\dot L^2(\Omega)} + \|\nabla u(0)\|^2_{L^2(\Omega)} + \|h\|^2_{L^2_t(\dot L^2)}\Big).
\end{aligned} \tag{38}$$

Here it obviously suffices to assume

$$\begin{aligned}
&\sigma \in H^1(0,T;L^\infty(\Omega))\,, \quad \mu \in L^2_t(L^\infty)\,, \quad \rho \in L^2(L^4)\\
&h \in L^2_t(\dot L^2)\,, \quad u_0 \in \dot H^1(\Omega)\,, \quad u_1 \in \dot L^2(\Omega)\,.
\end{aligned} \tag{39}$$

The estimate (38) also yields uniqueness of higher regularity solutions (see Propositions 3.1, 3.2), since in the linear setting we are considering here, for this purpose it suffices to prove that any solution with zero right hand side and initial data $h = 0$, $u_0 = 0$, $u_1 = 0$ needs to vanish.

Proposition 3.3. *Under conditions* (21) *and* (39), *any solution u lying in the space U_{lo} and satisfying the initial boundary value problem* (19), (20) *satisfies the estimate* (38).

Remark 3.1. Existence of a very low regularity solution $u \in U_{vl} = H^2(0,T;\dot H^{-1}(\Omega)) \cap W^{1,\infty}(0,T;\dot L^2(\Omega)) \cap L^\infty(0,T;\dot H^1(\Omega))$ can be shown by approximating σ, μ, ρ, h, u_0, u_1 by sequences $(\sigma_k)_{k\in\mathbb N} \subseteq H^1(0,T;L^\infty(\Omega)) \cap L^\infty(0,T;W^{1,\infty}(\Omega))$, $(\mu_k)_{k\in\mathbb N} \subseteq$

$L^2(0,T;L^\infty(\Omega)\cap W^{1,4}(\Omega))$, $(\rho_k)_{k\in\mathbb{N}}\subseteq L^2(0,T;H^1(\Omega))$, $(h_k)_{k\in\mathbb{N}}\subseteq L^2(0,T;H^1(\Omega))$, $(u_{0,k})_{k\in\mathbb{N}}\subseteq \dot{H}^2(\Omega)$, $(u_{1,k})_{k\in\mathbb{N}}\subseteq \dot{H}^1(\Omega)$, cf. (36), and applying Proposition 3.2 to obtain a sequence $(u_k)_{k\in\mathbb{N}}\subseteq U_{lo}$ of corresponding solutions. Due to the energy estimate (38), this sequence is bounded in U_{vl} and therefore has a weakly(*) convergent subsequence, whose limit (by testing with a smooth space and time dependent function) can be shown to satisfy (19), (20).

Uniqueness is inherently harder to obtain – first of all just conceptually, since it affects a larger space, secondly also technically, since testing with such low regularity solutions in (37) is not feasible. Still, one might possibly use arguments in the spirit of [31, §2.4] (see also the proof of [20, Proposition 4.1]) to establish uniqueness even in U_{vl}.

Note however, that for the purpose of proving Fréchet differentiability, the energy estimate stated in Proposition 3.3 is enough and neither existence nor uniqueness is really needed for that, see the comment preceding Theorem 3.2.

We proceed to proving well-posedness of the nonlinear problem (3) with $D = b\mathcal{A}^\beta\partial_t^\alpha$ by applying a fixed point argument to the operator $\mathcal{T}$ mapping v to the solution of

$$
\begin{aligned}
(1-2\kappa v)u_{tt} + c^2\mathcal{A}u + b\mathcal{A}^\beta\partial_t^\alpha u - 2\kappa v_t u_t = r \text{ in } \Omega\times(0,T)\\
u(0)=0, \quad u_t(0)=0 \text{ in } \Omega
\end{aligned}
\tag{40}
$$

that is, of (19), (20) with $\sigma(x,t)=-2\kappa(x)v(x,t)$, $\mu(x,t)=-2\kappa(x)v_t(x,t)$, $\rho(x,t)=0$, $h(x,t)=r(x,t)$. We first consider self-mapping of $\mathcal{T}$. Even in case of constant κ, the regularity requirements on σ, μ, ρ force us into the high regularity scenario of Proposition 3.1. For spatially variable κ, due to the estimates

$$
\begin{aligned}
\|\triangle\mu\|_{L^2(L^2)} &= 2\|\triangle(\kappa v_t)\|_{L^2(L^2)}\\
&\le 2\Big(\|\triangle\kappa\|_{L^2(\Omega)}\|v_t\|_{L^2(0,T;L^\infty(\Omega)} + 2\|\nabla\kappa\|_{L^4(\Omega)}\|\nabla v_t\|_{L^2(L^4)}\\
&\qquad + \|\kappa\|_{L^\infty(\Omega)}\|\triangle v_t\|_{L^2(L^2)}\Big)\\
\|\sigma_t\|_{L^2_t(L^\infty)} &\le 2C_{H^2,L^\infty}\|\triangle(\kappa v_t)\|_{L^2(L^2)}\\
\|\triangle\sigma\|_{L^\infty(L^4)} &= 2\|\triangle(\kappa v)\|_{L^\infty(L^4)}\\
&\le 2\Big(\|\triangle\kappa\|_{L^4(\Omega)}\|v\|_{L^\infty(L^\infty)}+2\|\nabla\kappa\|_{L^4(\Omega)}\|\nabla v\|_{L^\infty(L^\infty)}+\|\kappa\|_{L^\infty(\Omega)}\|\triangle v\|_{L^\infty(L^4)}\Big)\\
\|\nabla\sigma\|_{L^\infty_t(L^\infty)} &= 2\|\nabla(\kappa v)\|_{L^\infty_t(L^\infty)}\\
&\le 2\Big(\|\nabla\kappa\|_{L^\infty(\Omega)}\|v\|_{L^\infty(L^\infty)} + \|\kappa\|_{L^\infty(\Omega)}\|\nabla v\|_{L^\infty(L^\infty)}\Big)
\end{aligned}
$$

the regularity

$$
\kappa\in W^{2,4}(\Omega)\cap W^{1,\infty}(\Omega)
\tag{41}
$$

is sufficient for obtaining the regularity (32) for any for any $v\in U$. To achieve the nondegeneracy and smallness conditions (21), (28), we use the estimates

$$
\begin{aligned}
\|\sigma\|_{L^\infty_t(L^\infty)} &\le 2\|\kappa\|_{L^\infty(\Omega)}\|v\|_{L^\infty_t(L^\infty)}\\
\|\nabla\sigma\|_{L^\infty(L^4)} &\le 2\|\nabla\kappa\|_{L^4(\Omega)}\|v\|_{L^\infty(L^\infty)} + 2\|\kappa\|_{L^\infty(\Omega)}\|\nabla v\|_{L^\infty(L^4)}
\end{aligned}
\tag{42}
$$

and additionally to (41) require smallness of v.

Proposition 3.1 yields that $\mathcal{T}$ is a self-mapping on

$$
B_R = \{v\in U \ :\ \|v\|_U\le R\}
$$

provided the initial and right hand side data are sufficiently small so that

$$C(T)\Big(\|\mathcal{A}u_1\|^2_{\dot{L}^2(\Omega)} + \|\nabla\mathcal{A}u_0\|^2_{L^2(\Omega)} + \|\mathcal{A}r\|^2_{L^2_t(\dot{L}^2)}\Big) \le R^2 \tag{43}$$

with $C(T)$ as in Proposition 3.1. In view of (21), (28), (42), we choose R such that

$$\Big(C_{H^1\to L^4}\Big(\|\nabla\kappa\|_{L^4(\Omega)}C_{H^2,L^\infty} + \|\kappa\|_{L^\infty(\Omega)}C_{H^1,L^4}\Big) + \|\kappa\|_{L^\infty(\Omega)}C_{H^2,L^\infty}\Big)R < \frac{1}{2}\,. \tag{44}$$

Contractivity of $\mathcal{T}$ can be shown by taking $v^{(1)}$ and $v^{(2)}$ in B_R and considering $u^{(1)} = \mathcal{T}v^{(1)}$ and $u^{(2)} = \mathcal{T}v^{(2)}$, whose differences $\overline{u} = u^{(1)} - u^{(2)}$ and $\overline{v} = v^{(1)} - v^{(2)}$ solve

$$(1-2\kappa v^{(1)})\overline{u}_{tt} + c^2\mathcal{A}\overline{u} + b\mathcal{A}^\beta\partial_t^\alpha\overline{u}_t - 2\kappa v_t^{(1)}\overline{u}_t = 2\kappa\overline{v}_t u_t^{(2)} + 2\kappa\overline{v}u_{tt}^{(2)} \tag{45}$$

with homogeneous initial conditions. Similarly to the above, with $\sigma = -2\kappa v^{(1)}$, $\mu = -2\kappa v_t^{(1)}$, $\rho(x,t) = 0$, $h = 2\kappa\overline{v}_t u_t^{(2)} + 2\kappa\overline{v}u_{tt}^{(2)}$, since $v^{(1)}$, $u^{(2)} \in B_R$ (the latter due to the already shown self-mapping property of $\mathcal{T}$) we satisfy the conditions (21), (28), (32) on σ and μ. However h in general fails to be contained in $L^2(0,T;\dot{H}^2(\Omega))$ (in particular the term $2\kappa\overline{v}u_{tt}^{(2)}$), hence we move to the lower order regularity regime from Proposition 3.2. To this end, we estimate

$$\|\nabla h\|_{L^2(L^2)} \le 2\|\kappa\|_{L^4(\Omega)}\Big(\|\overline{v}_t\|_{L^2(L^4)}\|u_t^{(2)}\|_{L_t^\infty(L^\infty)} + \|\overline{v}\|_{L_t^\infty(L^\infty)}\|u_{tt}^{(2)}\|_{L^2(L^4)}\Big)\,.$$

Thus imposing the additional smallness condition

$$\theta := 2\|\kappa\|_{L^4(\Omega)}\sqrt{C}C_{H^1,L^4}C_{H^2,L^\infty}(\sqrt{T}+1)R < 1$$

on R and employing from Proposition 3.2, we obtain contractivity

$$\|\mathcal{T}v^{(1)} - \mathcal{T}v^{(2)}\|_{U_{lo}} = \|\overline{u}\|_{U_{lo}} \le \theta\|\overline{v}\|_{U_{lo}} = \theta\|v^{(1)} - v^{(2)}\|_{U_{lo}}.$$

Theorem 3.1. *Assume that (27) holds. For any $\alpha \in (0,1)$, $T > 0$, $\kappa \in W^{2,4}(\Omega) \cap W^{1,\infty}(\Omega)$ there exists $R_0 > 0$ such that for any data $u_0 \in \dot{H}^3(\Omega)$, $u_1 \in \dot{H}^2(\Omega)$, $r \in L^2(0,T;\dot{H}^2(\Omega))$ satisfying*

$$\|\mathcal{A}u_1\|^2_{\dot{L}^2(\Omega)} + \|\nabla\mathcal{A}u_0\|^2_{L^2(\Omega)} + \|\mathcal{A}r\|^2_{L^2(\dot{L}^2)} \le R_0^2 \tag{46}$$

there exists a unique solution $u \in U$ of

$$\begin{aligned}(1-2\kappa u)u_{tt} + c^2\mathcal{A}u + b\mathcal{A}^\beta\partial_t^\alpha u &= 2\kappa(u_t)^2 + r \ \textit{in}\ \Omega\times(0,T)\\ u(0) = u_0, \quad u_t(0) &= u_1 \ \textit{in}\ \Omega\,.\end{aligned} \tag{47}$$

Existence of the linearisation of G requires well-posedness of (10) with $D = b\mathcal{A}\partial_t^\alpha$, that is, (19), (20) with $\sigma = -2\kappa u$, $\mu = -4\kappa u_t$, $\rho = -2\kappa u_{tt}$, $h = 2\underline{\delta\kappa}(u\,u_{tt} + u_t^2)$. Due to the appearance of a u_{tt} term we are in a similar situation to the contractivity proof above and therefore the lower regularity Proposition 3.2 is the right framework for analysing the linearisation of the forward problem.

Proposition 3.4. *Under the assumptions of Theorem 3.1, for any $\underline{\delta\kappa} \in W^{1,\infty}(\Omega)$ there exists a unique solution $z \in U_{lo}$ of*

$$\begin{aligned}(1-2\kappa u)z_{tt} + c^2\mathcal{A}z + b\mathcal{A}^\beta\partial_t^\alpha z - 4\kappa u_t\, z_t - 2\kappa u_{tt}\, z &= 2\underline{\delta\kappa}(u\,u_{tt} + u_t^2) \ \textit{in}\ \Omega\times(0,T)\\ z(0) = 0, \quad z_t(0) &= 0 \ \textit{in}\ \Omega\,,\end{aligned} \tag{48}$$

where $u \in U$ solves (47).

In order to prove Fréchet differentiability we also need to bound the solution w of (12) with (18) that is, (19), (20) with $\sigma = -2\kappa u$, $\mu = -4\kappa u_t$, $\rho = -2\kappa u_{tt}$, $h = 2\underline{\delta\kappa}(v\tilde{u}_{tt}+uv_{tt}+(\tilde{u}_t+u_t)v_t)+2\kappa(vv_{tt}+v_t^2)$, where v can be bounded analogously to z by Proposition 3.2; in particular we can only expect to have $v_{tt} \in L^2(0,T;L^2(\Omega))$, so $h \in L^2(0,T;H^1(\Omega))$ is out of reach and we show Fréchet differentiability in the very low regularity regime of Proposition 3.3. Note that we actually only need the energy estimate from Proposition 3.3 in order to bound w. Existence is already guaranteed by existence of the three terms which compose $w = G(\tilde{\kappa}) - G(\kappa) - G'(\kappa)(\tilde{\kappa} - \kappa)$, namely $u = G(\kappa)$ (solving (47)), $\tilde{u} = G(\tilde{\kappa})$ (solving (47) with κ replaced by $\tilde{\kappa}$) and z (solving (48)). Clearly, uniqueness is not needed either.

Theorem 3.2. *For any $\alpha \in (0,1)$, $T > 0$, $\bar{R} > 0$ there exists $R_0 > 0$ such that for any data $u_0 \in \dot{H}^3(\Omega)$, $u_1 \in \dot{H}^2(\Omega)$, $r \in L^2(0,T;\dot{H}^2(\Omega))$ satisfying (46), the parameter-to-state map $G : B_{\bar{R}}(0) \to U$ is well-defined according to Theorem 3.1. Moreover, it is Fréchet differentiable as an operator $G : B_{\bar{R}}(0) \to U_{vl}$. Here $B_{\bar{R}}(0) = \{\kappa \in W^{2,4}(\Omega) \cap W^{1,\infty}(\Omega) : \|\kappa\|_{W^{2,4}(\Omega)\cap W^{1,\infty}(\Omega)} \leq \bar{R}\}$.*

3.2. **Fractional Zener damping.** Consider

$$D = b_1\mathcal{A}\partial_t^{\alpha_1} + b_2\partial_t^{\alpha_2+2} \quad \text{with } b_2 > 0,\ b_1 \geq b_2c^2,\ 1 \geq \alpha_1 \geq \alpha_2 > 0, \tag{49}$$

where based on the analysis in [22] we expect to get well-posedness of the nonlinear forward problem only in case $\alpha_1 = 1$, so we first of all focus on this case. Later on we will also prove a well-posedness result on the equation linearized at $\kappa = 0$ in the practically relevant case $\alpha_1 = \alpha_2 =: \alpha$. We refer to [25] for an analysis of the linear fractional Zener wave equation, even in the tensorial setting of viscoelasticity. See also [20] for an analysis of several different linear and nonlinear fractional acoustic wave equations as well as a derivation of these models and justification of their limits as $\alpha_1 = \alpha_2 = \alpha \nearrow 1$.

As in the previous section, we first of all consider the initial boundary value problem for the general linear PDE

$$(1-\sigma)u_{tt} + c^2\mathcal{A}u + b_1\mathcal{A}u_t + b_2\partial_t^{\alpha_2+2}u + \mu u_t + \rho u = h \tag{50}$$

$$u(0) = u_0\,, \quad u_t(0) = u_1\,, \quad (u_{tt}(0) = u_2 \text{ in case } \alpha_2 > \frac{1}{2}) \tag{51}$$

with given space and time dependent functions σ, μ, ρ, h.

Again we skip the details about the Faedo-Galerkin approach and the discretisation index n and only provide the crucial energy estimate. We multiply (50) with $\mathcal{A}u_{tt}$ and integrate with respect to time, using the inequalities and identities

$$\int_0^t \langle \mathcal{A}u(s), \mathcal{A}u_{tt}(s)\rangle\, ds = \langle \mathcal{A}u(t), \mathcal{A}u_t(t)\rangle - \langle \mathcal{A}u(0), \mathcal{A}u_t(0)\rangle - \int_0^t \|\mathcal{A}u_t(s)\|^2_{\dot{L}^2(\Omega)}$$

and

$$\begin{aligned}\int_0^t \langle \partial_t^{\alpha_2+2}[u](s), \mathcal{A}u_{tt}(s)\rangle\, ds &= \int_0^t \langle \partial_t^{\alpha_2}[\nabla u_{tt}](s), \nabla u_{tt}(s)\rangle\, ds \\ &\geq \frac{1}{2}\int_0^t \partial_t^{\alpha_2}\left[\|\nabla u_{tt}\|^2_{\dot{L}^2(\Omega)}\right](s)\, ds \\ &= \frac{1}{2} I_t^{1-\alpha}\left[\|\nabla u_{tt}\|^2_{L^2(\Omega)}\right](t) \geq \frac{1}{2\Gamma(1-\alpha)t^\alpha}\|\nabla u_{tt}\|^2_{L^2_t(L^2)}\,,\end{aligned}$$

where the latter equality holds provided $u_{tt}(0)=0$ and we have applied (15) (actually to the Fourier components of the Galerkin discretisation $w=\lambda_j^{1/2}u_j^{n\prime\prime})$ with $\gamma=\alpha_2$.

This yields the energy estimate

$$\begin{aligned}
&\Big(\frac{b_2}{2\Gamma(1-\alpha)t^\alpha}+1-\overline{\sigma}\Big)\|\nabla u_{tt}\|^2_{L^2_t(L^2)}+\frac{b_1}{2}\|\mathcal{A}u_t(t)\|^2_{\dot L^2(\Omega)}\\
&\le \frac{b_1}{2}\|\mathcal{A}u_t(0)\|^2_{L^2_t(\dot L^2)}+c^2\langle\mathcal{A}u(0),\mathcal{A}u_t(0)\rangle\\
&\quad -c^2\langle\mathcal{A}u(t),\mathcal{A}u_t(t)\rangle+c^2\|\mathcal{A}u_t\|^2_{L^2_t(\dot L^2)}\\
&\quad +\int_0^t\langle\nabla\sigma u_{tt}+\nabla(\mu u_t+\rho u-h)(s),\nabla u_{tt}(s)\rangle\,ds\\
&\le \frac{b_1}{2}\|\mathcal{A}u_t(0)\|^2_{L^2_t(\dot L^2)}+c^2\langle\mathcal{A}u(0),\mathcal{A}u_t(0)\rangle\\
&\qquad +\frac{b_1}{4}\|\mathcal{A}u_t(t)\|^2_{\dot L^2(\Omega)}+\frac{c^4}{b_1}\|\mathcal{A}u(t)\|^2_{\dot L^2(\Omega)}+c^2\|\mathcal{A}u_t\|^2_{L^2_t(\dot L^2)}\\
&\qquad +\|\nabla\sigma\|_{L^\infty(L^4)}\|u_{tt}\|_{L^2(L^4)}\|\nabla u_{tt}\|_{L^2_t(L^2)}+\frac{\epsilon}{2}\|\nabla u_{tt}\|^2_{L^2_t(L^2)}\\
&\qquad +\frac{1}{2\epsilon}\Big(\|\nabla\mu\|_{L^\infty_t(L^2)}\|u_t\|_{L^2(0,t;L^\infty(\Omega)}+\|\mu\|_{L^\infty(L^4)}\|\nabla u_t\|_{L^2(L^4)}\\
&\qquad\quad +\|\nabla\rho\|_{L^2_t(L^2)}\|u\|_{L^\infty_t(L^\infty)}+\|\rho\|_{L^2(L^4)}\|\nabla u\|_{L^\infty(L^4)}+\|\nabla h\|_{L^2_t(L^2)}\Big)^2.
\end{aligned}$$

Here we assume nondegeneracy

$$\sigma(x,t)\le\overline{\sigma}<\frac{b_2}{\Gamma(1-\alpha)T^\alpha}+1 \text{ for all } x\in\Omega\quad t\in(0,T)\,, \tag{52}$$

and a smallness condition on $\nabla\sigma$

$$\|\nabla\sigma\|_{L^\infty(L^4)}<\frac{\frac{b_2}{\Gamma(1-\alpha)T^\alpha}+1-\overline{\sigma}}{C_{H^1\to L^4}}. \tag{53}$$

Now choose $\epsilon<\frac{b_2}{\Gamma(1-\alpha)T^\alpha}+1-\underline{\sigma}-C_{H^1\to L^4}\|\nabla\sigma\|_{L^\infty(L^4)}$ to obtain, using Gronwall's Lemma,

$$\|\nabla u_{tt}\|^2_{L^2(L^2)}+\|\mathcal{A}u_t\|^2_{L^\infty_t(\dot L^2)}\quad\le C\Big(\|\mathcal{A}u_t(0)\|^2_{\dot L^2(\Omega)}+\|\nabla h\|^2_{L^2_t(\dot L^2)}\Big). \tag{54}$$

The required regularity on σ, μ, ρ, h, u_0, u_1, is, besides (52), (53)

$$\begin{aligned}&\mu\in L^\infty(0,T;H^1(\Omega))\,,\quad \rho\in L^2(0,T;H^1(\Omega))\,,\\ &h\in L^2(0,T;\dot H^1(\Omega))\,,\quad u_0,u_1\in\dot H^2(\Omega),\quad u_2=0\,.\end{aligned} \tag{55}$$

Proposition 3.5. *Under conditions (52), (53), (55), there exists a unique solution*

$$u\in U:=H^2(0,T;\dot H^1(\Omega))\cap W^{1,\infty}(0,T;\dot H^2(\Omega)) \tag{56}$$

to the initial boundary value problem (50), (51), and this solution satisfies the estimate (54).

Theorem 3.3. *For any $\alpha_2\in(0,1)$, $T>0$, $\kappa\in W^{1,4}(\Omega)$ there exists $R_0>0$ such that for any data $u_0,u_1\in\dot H^2(\Omega)$, $r\in L^2(0,T;\dot H^1(\Omega))$ satisfying*

$$\|\mathcal{A}u_1\|^2_{\dot L^2(\Omega)}+\|\mathcal{A}u_0\|^2_{\dot L^2(\Omega)}+\|\nabla r\|^2_{L^2(L^2)}\le R_0^2 \tag{57}$$

there exists a unique solution $u \in U$ of

$$
\begin{aligned}
&b_2\partial_t^{\alpha_2+2}u + (1-2\kappa u)u_{tt} + c^2\mathcal{A}u + b_1\mathcal{A}u_t = 2\kappa(u_t)^2 + r \text{ in } \Omega\times(0,T)\\
&\qquad u(0)=u_0, \quad u_t(0)=u_1, \quad u_{tt}=0 \text{ in } \Omega\,.
\end{aligned} \tag{58}
$$

To prove well-posedness of (10) in the FZ case, we apply proposition 3.5 with $h = 2\underline{\delta\kappa}(u\,u_{tt}+u_t^2)$, which is contained in $L^2(0,T;\dot{H}^1(\Omega))$ for $u\in U$ provided $\underline{\delta\kappa}\in L^3(\Omega)$.

Proposition 3.6. *Under the assumptions of Theorem 3.3, for any $\underline{\delta\kappa}\in L^3(\Omega)$ there exists a unique solution $z\in U$ of*

$$
\begin{aligned}
&b_2\partial_t^{\alpha_2+2}z + (1-2\kappa u)z_{tt} + c^2\mathcal{A}z + b_1\mathcal{A}z_t - 4\kappa u_t\, z_t - 2\kappa u_{tt}\, z\\
&\qquad = 2\underline{\delta\kappa}(u\,u_{tt}+u_t^2) \text{ in } \Omega\times(0,T)\\
&\qquad\quad z(0)=0, \quad z_t(0), \quad z_{tt}=0=0 \text{ in } \Omega\,,
\end{aligned} \tag{59}
$$

where $u\in U$ solves (58).

Fréchet differentiability follows from application of proposition 3.5 with $h = 2(\tilde\kappa-\kappa)(v\tilde u_{tt} + uv_{tt} + (\tilde u_t + u_t)v_t) + 2\kappa(vv_{tt}+v_t^2)$ where $v = G(\tilde\kappa) - G(\kappa)\in U$ and $u = G(\kappa)\in U$. Thus, for h to be contained in $L^2(0,T;\dot{H}^1(\Omega))$, it is again enough to asssume $\tilde\kappa - \kappa \in L^3(\Omega)$.

Theorem 3.4. *For any $\alpha_2\in(0,1)$, $T>0$, $\bar R>0$ there exists $R_0>0$ such that for any data $u_0,u_1\in\dot{H}^2(\Omega)$, $r\in L^2(0,T;\dot{H}^2(\Omega))$ satisfying (57), the parameter-to-state map $G: B_{\bar R}(0)\to U$ is well-defined according to Theorem 3.3. Moreover, it is Fréchet differentiable as an operator $G: B_{\bar R}(0)\to U$. Here $B_{\bar R}(0) = \{\kappa\in W^{1,4}(\Omega) : \|\kappa\|_{W^{1,4}(\Omega)}\le \bar R\}$.*

We now consider the linear problem in case $\alpha_1=\alpha_2=:\alpha$, $b_1 = b_2c^2+\delta$ with $\delta\ge 0$

$$
(1-\sigma)u_{tt} + c^2\mathcal{A}u + b_1\mathcal{A}\partial_t^\alpha u + b_2\partial_t^{\alpha+2}u + \mu u_t + \rho u = h \tag{60}
$$

in which the differential operator can partially be factorised as

$$
\begin{aligned}
&(1-\sigma)\partial_{tt} + c^2\mathcal{A} + (b_2c^2+\delta)\mathcal{A}\partial_t^\alpha + b_2\partial_t^{\alpha+2} + \mu\partial_t + \rho\,\mathrm{id}\\
&= \Big(\partial_{tt} + c^2\mathcal{A}\Big)\Big(b_2\partial_t^\alpha + \mathrm{id}\Big) - \sigma\partial_{tt} + \delta\mathcal{A}\partial_t^\alpha + \mu\partial_t + \rho\,\mathrm{id}.
\end{aligned}
$$

Thus, up to the "perturbation" terms containing σ, δ, μ, and ρ, the auxiliary function $\tilde u = b_2\partial_t^\alpha u + u$ satisfies a wave equation $\tilde u_{tt} + c^2\mathcal{A}\tilde u \doteq h$. Motivated by this fact, we multiply (60) with $\mathcal{A}\tilde u_t$ to obtain the energy identity

$$
\begin{aligned}
&\frac12\|\nabla(b_2\partial_t^\alpha u+u)_t(t)\|^2_{L^2(\Omega)} + \frac{c^2}{2}\|\mathcal{A}(b_2\partial_t^\alpha u+u)(t)\|^2_{\dot{L}^2(\Omega)}\\
&\qquad\qquad + \delta\int_0^t \langle\nabla\partial_t^\alpha u(s), \nabla(b_2\partial_t^\alpha u+u)_t(s)\rangle\,ds\\
&= \frac12\|\nabla(b_2\partial_t^\alpha u+u)_t(0)\|^2_{L^2(\Omega)} + \frac{c^2}{2}\|\mathcal{A}(b_2\partial_t^\alpha u+u)(0)\|^2_{\dot{L}^2(\Omega)}\\
&\qquad\qquad + \int_0^t\langle\nabla(h+\sigma u_{tt} - \mu u_t - \rho u)(s), \nabla(b_2\partial_t^\alpha u+u)_t(s)\rangle\,ds\,.
\end{aligned} \tag{61}
$$

The term containing δ can be nicely tackled by means of Lemma 3.1

$$\begin{aligned}\int_0^t \langle \nabla\partial_t^\alpha u(s), \nabla(b_2\partial_t^\alpha u+u)_t(s)\rangle\, ds \\ = \frac{b_2}{2}\|\nabla\partial_t^\alpha u(t)\|^2_{L^2(\Omega)} - \frac{b_2}{2}\|\nabla\partial_t^\alpha u(0)\|^2_{L^2(\Omega)} + \int_0^t \langle \nabla\partial_t^\alpha u(s), \nabla u_t(s)\rangle\, ds \\ \geq \frac{b_2}{2}\|\nabla\partial_t^\alpha u(t)\|^2_{L^2(\Omega)} + \frac{1}{2\Gamma(\alpha)t^{1-\alpha}}\|\nabla\partial_t^\alpha u\|^2_{L^2_t(L^2)},\end{aligned}$$

(which reflects the physical fact that δ is the diffusivity of sound and therefore the corresponding term models damping). However, in the term containing σ this is inhibited by the time-dependence of σ. Thus in case $\alpha_1=\alpha_2=:\alpha<1$ we have to restrict ourselves to the linearisation of the forward problem at $\kappa=0$ (where also $\mu=0$, $\rho=0$), where the above together with Young's inequality yields the energy estimate

(62)

$$\begin{aligned}\frac{1}{2}\|\nabla(b_2\partial_t^\alpha u+u)_t\|^2_{L^\infty_t(L^2)} + c^2\|\mathcal{A}(b_2\partial_t^\alpha u+u)\|^2_{L^\infty_t(\dot{L}^2)} \\ + \delta b_2\|\nabla\partial_t^\alpha u(t)\|^2_{L^\infty_t(L^2)} + \frac{\delta}{\Gamma(\alpha)t^{1-\alpha}}\|\nabla\partial_t^\alpha u\|^2_{L^2_t(L^2)} \\ \leq \|\nabla(b_2\partial_t^\alpha u+u)_t(0)\|^2_{L^2(\Omega)} + c^2\|\mathcal{A}(b_2\partial_t^\alpha u+u)(0)\|^2_{\dot{L}^2(\Omega)} + 2\|\nabla h\|^2_{L^1_t(L^2)}.\end{aligned}$$

Due to the identity $(\partial_t^\alpha u)_t(0)=\lim_{t\searrow 0}\frac{1}{\Gamma(1-\alpha)}t^{-\alpha}u_t(0)$, in order for the right hand side to be finite, we need to assume $u_t(0)=0$ here.

The PDE yields an estimate of $\tilde{u}_{tt}$ as follows

$$\|(b_2\partial_t^\alpha u+u)_{tt}\|_{L^\infty_t(L^2)} = \|h-c^2\mathcal{A}(b_2\partial_t^\alpha u+u)\|^2_{L^\infty_t(\dot{L}^2)}. \tag{63}$$

To extract temporal reguarity of u from regularity of $b_2\partial_t^\alpha u+u$ for $b_2>0$, we make use of regularity results of time fractional ODEs: $b_2\partial_t^\alpha u+u=\tilde{u}\in W^{k,\infty}(0,T;Z)$ implies $u\in W^{k+\alpha,\infty}(0,T;Z)$, due to the fact that I^α maps $L^\infty(0,T)$ to $C^{0,\alpha}(0,T)$; see [29, Corollary 2, p. 56]. Thus, provided $h\in L^1(0,T;\dot{H}^1(\Omega))\cap L^\infty(0,T;\dot{L}^2(\Omega))$ and the initial data $u_0\in\dot{H}^2(\Omega)$, $u_1=0$, $u_2\in\dot{L}^2(\Omega)$, we have that u exhibits the regularity

$$u\in U := W^{2+\alpha,\infty}(0,T;\dot{L}^2(\Omega))\cap W^{1+\alpha,\infty}(0,T;\dot{H}^1(\Omega))\cap W^{\alpha,\infty}(0,T;\dot{H}^2(\Omega)). \tag{64}$$

We thus obtain the following result in case of $\kappa=0$ on the (then linear) forward problem:

Proposition 3.7. *For any $\alpha\in(0,1)$, $T>0$, and for any data $u_0\in\dot{H}^2(\Omega)$, $u_2\in\dot{L}^2(\Omega)$, $r\in L^1(0,T;H_0^1(\Omega))\cap L^\infty(0,T;\dot{L}^2(\Omega))$ there exists a unique solution $u=G(0)\in U$ of*

$$\begin{aligned}b_2\partial_t^{\alpha+2}u+u_{tt}+c^2\mathcal{A}u+b_1\mathcal{A}\partial_t^\alpha u = r \ \text{in}\ \Omega\times(0,T)\\ u(0)=u_0,\quad u_t(0)=0,\quad u_{tt}=u_2\ \text{in}\ \Omega\,.\end{aligned}$$

We now consider the linearization of G at $\kappa=0$, which is defined by (10). To this end, we derive a weaker energy estimate by multiplying (60) with $\tilde{u}_t=(b_2\partial_t^\alpha u+u)_t$

to obtain the estimate

$$\begin{aligned}
&\frac{1}{2}\|(b_2\partial_t^\alpha u+u)_t\|^2_{L^\infty_t(\dot L^2)}+c^2\|\nabla(b_2\partial_t^\alpha u+u)\|^2_{L^\infty_t(L^2)}\\
&\quad+\delta b_2\|\partial_t^\alpha u(t)\|^2_{L^\infty_t(\dot L^2)}+\frac{\delta}{\Gamma(\alpha)t^{1-\alpha}}\|\partial_t^\alpha u\|^2_{L^2_t(\dot L^2)}\\
&\quad\le\|(b_2\partial_t^\alpha u+u)_t(0)\|^2_{\dot L^2(\Omega)}+c^2\|\nabla(b_2\partial_t^\alpha u+u)(0)\|^2_{L^2(\Omega)}+2\|h\|^2_{L^1_t(L^2)}.
\end{aligned}\tag{65}$$

The required regularity of $\underline{\delta\kappa}$ to guarantee $h=2\underline{\delta\kappa}(u\,u_{tt}+u_t^2)\in L^1(0,t;L^2(\Omega))$ for $u=G(0)\in U$ is obviously $\underline{\delta\kappa}\in L^\infty(\Omega)$.

Proposition 3.8. *Under the assumptions of Proposition 3.7, for any $\underline{\delta\kappa}\in L^\infty(\Omega)$ there exists a unique solution $z\in U_{lo}=W^{2+\alpha,\infty}(0,T;\dot H^{-1}(\Omega))\cap W^{1+\alpha,\infty}(0,T;\dot L^2(\Omega))\cap W^{\alpha,\infty}(0,T;\dot H^1(\Omega))$ of*

$$\begin{aligned}
&b_2\partial_t^{\alpha+2}z+z_{tt}+c^2\mathcal{A}z+b_1\mathcal{A}\partial_t^\alpha z=2\underline{\delta\kappa}(u\,u_{tt}+u_t^2)\ \textit{in}\ \Omega\times(0,T)\\
&z(0)=0,\quad z_t(0)=0,\quad z_{tt}=0\ \textit{in}\ \Omega\,.
\end{aligned}$$

Since we cannot establish well-definedness of $G(\tilde\kappa)$ for $\tilde\kappa\neq0$ we cannot prove Fréchet (actually not even directional) differentiability of G at $\kappa=0$, though; so z is only a formal linearization of G at $\kappa=0$ into the direction $\underline{\delta\kappa}$.

4. Injectivity of the linearised forward operator

The forward map is defined by $F(\kappa)=\mathrm{tr}_\Sigma u$, where $\mathrm{tr}_\Sigma v$ denotes the time trace of the space and time dependent function $v:(0,T)\times\Omega$ at the observation surface Σ (which may also just be a single point $\Sigma=\{x_0\}$) and u solves

$$\begin{aligned}
&u_{tt}+c^2\mathcal{A}u+Du=\kappa(x)(u^2)_{tt}+r\ \text{in}\ \Omega\times(0,T)\\
&u(0)=0,\quad u_t(0)=0\ \text{in}\ \Omega\,.
\end{aligned}\tag{66}$$

Its linearisation at $\kappa=0$ in direction $\underline{\delta\kappa}$ is $F'(0)\underline{\delta\kappa}=\mathrm{tr}_\Sigma z_0$, where z_0 solves

$$z_{tt}+c^2\mathcal{A}z+Dz=\underline{\delta\kappa}(u_0^2)_{tt},\tag{67}$$

where

$$u_{0,tt}+c^2\mathcal{A}u_0+Du_0=r.\tag{68}$$

Both PDEs (67), (68) come with homogeneous initial conditions.

As in the previous section, we consider the two damping models

$$D=b\mathcal{A}^\beta\partial_t^\alpha\qquad(\text{CH})\tag{69}$$

and

$$D=b_1\mathcal{A}\partial_t^{\alpha_1}+b_2\partial_t^{\alpha_2+2}\qquad(\text{FZ})\,.\tag{70}$$

The Laplace transformed solutions to the corresponding resolvent equation

$$\hat w(\lambda,s)=\frac{1}{\omega(\lambda,s)}\ \text{with}\ \omega(\lambda,s)=\begin{cases}s^2+b\lambda^\beta s^\alpha+c^2\lambda\ \text{for CH}\\ b_2s^{2+\alpha_2}+s^2+b_1\lambda s^{\alpha_1}+c^2\lambda\ \text{for FZ}\end{cases}\tag{71}$$

will play a crucial role in the proofs below.

Moroever, we will make use of the following results from [22].

Lemma 4.1 ([22, Lemma 4.1 and Remark 4.1]). *For $A, B > 0$, $\alpha \in (0,1)$ the function $\overline{\omega}$ defined by $\overline{\omega}(A,B;s) = s^2 + As^\alpha + B$ has precisely two complex-conjugate zeros which lie in the left hand complex plane. Moreover, for $\tilde{\lambda} \neq \lambda$ and $\omega(\lambda, s) := \overline{\omega}(b\lambda, c^2\lambda; s)$ the locations of the roots of $\omega(\lambda;\cdot)$ and of $\omega(\tilde{\lambda};\cdot)$ differ.*

From Lemma 4.1 we conclude that in case of CH, the function $\omega^{\text{CH}}(\lambda,\cdot)$ has precisely two complex-conjugate zeros $p_+^{\text{CH}}(\lambda)$, $p_-^{\text{CH}}(\lambda)$, which lie in the left hand complex plane.

For FZ, we first consider the particular parameter configuration (corresponding to vanishing viscosity of sound)

$$b_1 = b_2 c^2 \text{ and } \alpha_1 = \alpha_2 \tag{72}$$

in which we can factorise $\omega^{\text{FZ}}(\lambda, s) = (b_2 s^{\alpha_2} + 1)(s^2 + c^2\lambda)$ and get the roots

$$p_0^{\text{FZ}} = -\frac{1}{b_2} \text{ (only in case } \alpha_1 = \alpha_2 = 1), \qquad p_\pm^{\text{FZ}}(\lambda) = \pm ic\sqrt{\lambda}\,.$$

Note that p_0^{FZ} is independent of λ, but $p_\pm^{\text{FZ}}(\lambda)$ obviously allows to distinguish between different λ's. This distinction is possible in general, a fact that has already been shown for the CH case with $\beta = 1$ in see Lemma 4.1. As an additional result, that is not needed for the uniqueness proof but might be convenient for the computation of poles and residues, we state that the poles are single in certain cases.

Lemma 4.2. *The poles of $\hat{w}^{\text{CH}}$ and of $\hat{w}^{\text{FZ}}$ (except for p_0^{FZ} in case $\alpha_1 = \alpha_2 = 1$) differ for different λ. Moreover, in the case CH and in the case FZ with (72) the poles are single.*

Proof. For CH, let $f(z) = z^2 + c^2\lambda$, $g(z) = b\lambda^\beta z^\alpha$. Then for a sufficiently large $R > c^2\lambda$ let C_R be the circle radius R, centre at the origin. Then $|g(z)| < |f(z)|$ on C_R and so Rouché's theorem shows that $f(z)$ and $(f+g)(z)$ have the same number of roots, counted with multiplicity, within C_R. For f these are only at $z = \pm i\sqrt{\lambda}c$ so the same must be true of $f+g$ and so ω^{CH} has precisely one single root in the third and in the fourth quadrant, respectively.

Suppose now that $\hat{w}^{\text{CH}}$ has a pole at $re^{i\theta}$, where $\pi/2 < \theta < \pi$, for both λ_1 and λ_2. Then for $s = re^{i\theta}$

$$s^2 + b\lambda_1^\beta s^\alpha + c^2\lambda_1 = 0 \qquad s^2 + b\lambda_2^\beta s^\alpha + c^2\lambda_2 = 0$$

so that

$$\frac{b}{c^2}\frac{(\lambda_1^\beta - \lambda_2^\beta)}{(\lambda_1 - \lambda_2)} = -s^\alpha\,.$$

Now if $\lambda_1 \neq \lambda_2$ then the left hand side is positive and real and so $\alpha\theta = \pi$. This means that $\theta > \pi$, a contradiction.

In case of FZ, assuming that p is a pole of both $\hat{w}^{\text{FZ}}(\lambda_1,\cdot)$ and $\hat{w}^{\text{FZ}}(\lambda_2,\cdot)$ we have

$$0 = \omega(\lambda_1, p) - \omega(\lambda_2, p) = (\lambda_1 - \lambda_2)[b_1 p^{\alpha_1} + c^2]\,,$$

where due $\omega(\lambda_1, p) = 0$, the term in brackets $b_1 p^{\alpha_1} + c^2 = -\frac{p^2}{\lambda_1}(b_2 p^{\alpha_2} + 1) \neq 0$, hence $\lambda_1 = \lambda_2$. In the factorisable case (72) of FZ, obviously all roots are single. ☐

As in [21] (where we used the classical damping term $D = b\mathcal{A}\partial_t$), we assume that r has the form

$$r(x,t) = f(x)\chi''(t) + c^2\mathcal{A}f(x)\chi(t) + D[f(x)\chi(t)] \tag{73}$$

with some function f in the domain of $\mathcal{A}$ vanishing only on a set of measure zero and some twice differentiable function χ of time such that $(\chi^2)''(t_0) \neq 0$ for some $t_0 > 0$. With (73), the solution u_0 of equation (68) is clearly given by $u_0(x,t) = f(x)\chi(t)$, so that $\underline{d\kappa}(u_0^2)_{tt}$ can be written in the form

$$\left(\underline{d\kappa}(x)u_0^2(x,t)\right)_{tt} = \sum_{j=1}^{\infty} a_j \phi_j(x)\psi_j(t), \tag{74}$$

where a_j are the coefficients of $\underline{d\kappa}\cdot f$ with respect to the eigenfunction basis $(\phi_j)_{j\in\mathbb{N}}$, and $\psi_j = (\chi^2)''$.

We can rewrite equation (67) as

$$z_j''(t) + c^2\lambda_j z_j(t) + D_j z_j = a_j\psi_j(t)\,, \quad t>0\,, \qquad z_j(0)=0\,,\ z_j'(0)=0 \tag{75}$$

for all $j \in \mathbb{N}$, where

$$z_0(x,t) = \sum_{j=1}^{\infty} z_j(t)\phi_j(x)\,, \qquad D_j = \begin{cases} b\lambda_j^{\beta}\partial_t^{\alpha} \text{ for CH} \\ b_2\partial_t^{2+\alpha_2} + b_1\lambda_j\partial_t^{\alpha_1} \text{ for FZ.} \end{cases}$$

Applying the Laplace transform to both sides of (75) yields

$$\hat{z}_j(s) = \hat{w}_j(s)a_j\widehat{\psi}_j(s), \qquad \text{where } \ \hat{w}_j(s) = \frac{1}{\omega(\lambda_j,s)}\,, \qquad s\in\mathbb{C}\,, \tag{76}$$

and we have used homogeneity of the initial conditions.

Thus, assuming that $F'(0)\underline{d\kappa} = \mathrm{tr}_\Sigma z_0 = 0$ implies that

$$0 = \hat{z}_0(x_0,s) = \sum_{j=1}^{\infty} a_j\phi_j(x_0)\hat{w}_j(s)\widehat{\psi}_j(s)\,, \qquad \text{for all } s\in\mathbb{C}\,,\ x_0\in\Sigma\,.$$

Considering the residues at some pole p_m corresponding to the eigenvalue λ_m and using the fact that by Lemma 4.2, $\lim_{s\to p_m}(s-p_m)\hat{w}(\lambda_j,s) = 0$ for $j \neq m$ yields

$$\begin{aligned} 0 = \mathrm{Res}(\hat{z}_0(x_0;p_m)) &= \sum_{j=1}^{\infty} a_j\phi_j(x_0)\lim_{s\to p_m}(s-p_m)^{\ell_m}\hat{w}(\lambda_j,s)\widehat{\psi}_j(s) \\ &= \mathrm{Res}(\hat{w}_m;p_m)\widehat{\psi}_j(p_m)\sum_{k\in K_m} a_k\phi_k(x_0)\,. \end{aligned}$$

Here ℓ_m is the multiplicity of p_m as a root of $\omega(\lambda_m,\cdot)$ and $K_m \subseteq \mathbb{N}$ is an enumeration of the eigenspace basis $(\phi_k)_{k\in K_m}$ corresponding to the eigenvalue λ_m. To extract the coefficients a_k we assume that

$$\widehat{\psi}_k(p_m) \neq 0 \tag{77}$$

and that for all λ eigenvalue of $\mathcal{A}$ with eigenfunctions $(\phi_k)_{k\in K^\lambda}$, the restrictions of the eigenfunctions to the observation manifold are linear independent, that is, the following implication holds for any coefficient set $(b_k)_{k\in K^\lambda}$:

$$\left(\sum_{k\in K^\lambda} b_k\phi_k(x) = 0 \ \text{ for all } x\in\Sigma\right) \implies \left(b_k = 0 \text{ for all } k\in K^\lambda\right)\,. \tag{78}$$

From this we can conclude that $a_k = 0$ for all $k \in K_m$.

Now since $(u_0^2)_{tt}(t_0) = f(\chi^2)''(t_0)$ only vanishes on a set of measure zero and (74), we can conclude that $\underline{d\kappa} = 0$ almost everywhere.

Theorem 4.1. *Under the above assumptions* (74)*,* (77)*,* (78) *for all* $m \in \mathbb{N}$*,* $k \in K_m$*, the linearised derivative at* $\kappa = 0$*,* $F'(0)$ *is injective.*

Remark 4.1. In particular, (78) is satisfied in the spatially 1-dimensional case $\Sigma = \{x_0\}$, where all eigenvalues of $\mathcal{A}$ are single, i.e., $\#K_m = 1$ for all m, provided none of the eigenfunctions vanish at x_0; this can be achieved by taking x_0 on the boundary and where ϕ_j is subject to non-Dirichlet conditions.

To give an example in higher space dimensions, let us consider the unit disc Ω, where the eigenvalues and eigenfunctions of the negative Laplacian $\mathcal{A}$ with homogeneous Dirichlet boundary values are given in terms of the Bessel functions J_ℓ and their positive roots $\mu_{\ell,n}$ in polar coordinates $x = (r\cos\theta, r\sin\theta)$, $r \in [0,1]$, $\theta \in [0, 2\pi)$ as follows

$$\lambda_{\ell,n} = \mu_{\ell,n}^2 \,, \quad \phi_{\ell,n,1}(r,\theta) = J_\ell(\mu_{\ell,n} r)\cos(\ell\theta) \,, \quad \phi_{\ell,n,2}(r,\theta) = J_\ell(\mu_{\ell,n} r)\sin(\ell\theta) \,.$$

For a fixed eigenvalue λ of $\mathcal{A}$, the index set K^λ is therefore given by

$$K^\lambda = \{(\ell, n) \in \mathbb{N} \,:\, \mu_{\ell,n}^2 = \lambda\} = \{(\ell, n(\ell)) \,:\, \ell \in M^\lambda\} \tag{79}$$

since if $\lambda = \mu_{\ell,n}^2$ is an eigenvalue of $\mathcal{A}$ for any $\ell \in \mathbb{N}$, then the corresponding number $n(\ell)$ of the Bessel function root is clearly unique since these roots are single.

To satisfy (78), we select $r_* \in (0,1)$ in such a way that $\sqrt{\lambda} r_*$ is not a root of any Bessel function (which is actually the generic case) and choose the circle $\Sigma = \{(r_*\cos\theta, r_*\sin\theta),\ \theta \in [0, 2\pi)\}$ as observation surface. Indeed this can easily be seen to satisfy (78) as follows. The premise is written – in terms of the index set K^λ according to (79) specific to our setup, whose indexing also applies to the arbitrary coefficients b_k – as

$$\sum_{\ell \in M^\lambda} c_\ell \Big(b_{\ell,n(\ell),1}\cos(\ell\theta) + b_{\ell,n(\ell),2}\sin(\ell\theta) \Big) = 0 \quad \text{for all } \theta \in [0, 2\pi) \,,$$

where $c_\ell = J_\ell(\sqrt{\lambda} r_*) \neq 0$ for all $\ell \in \mathbb{N}_0$. Taking the $L^2(0, 2\pi)$ inner product with $\cos(j\theta)$ and $\sin(j\theta)$ (or simply using linear independence of the functions $\cos(\ell\theta)$, $\sin(\ell\theta)$, $\ell \in M^\lambda$), we conclude that $c_\ell b_{\ell,n(\ell),1} = 0$ and $c_\ell b_{\ell,n(\ell),2} = 0$ for all $\ell \in M^\lambda$, that is, due to the fact that $c_\ell \neq 0$, all the coefficients $b_{\ell,n(\ell),i}$, $\ell \in M^\lambda$, $i \in \{1, 2\}$ vanish. Alternatively one could choose Σ to be a diameter of the disc at an angle θ avoiding the zeros of $\cos(\ell\theta)$, $\sin(\ell\theta)$ and make use of the fact that the Bessel functions are linear independent.

This construction principle carries over to other geometries and higher space dimensions whenever the eigenfunctions allow for a separation of variables. Obvious examples for this are spheres or cuboids, where the eigenfunctions are composed of spherical harmonics and/or trigonometric functions. To see this, recall that (78) simply says that the eigenfunctions, when resticted to the observation surface, are linear independent. So if Σ is oriented along one of the directions of separability, one can make use of linear independence of the eigenfunction factors in the other direction. For an investigation on how to choose the observation location in a related problem using separable eigenfunctions, as well as numerical reconstruction results see [27].

A sufficient condition for (78) to hold is that there exist points $x_{0,\lambda,1}, \ldots x_{0,\lambda,N^\lambda} \in \Sigma$, $N^\lambda \geq \#K^\lambda$ such that the matrix $\phi_k(x_{0,\lambda,i})_{k \in K^\lambda, i \in \{1,\ldots,N^\lambda\}}$ has full rank $\#K^\lambda$. Accumulating this over the sequence of eigenvalues λ of $\mathcal{A}$ and bearing in mind the fact that in higher space dimensions their multiplicity can become arbitrarily large (e.g., in the cuboid example above), we conclude that the cardinality of Σ will typically have to be infinite; still this can allow Σ to consist of a countable discrete sequence of points.

5. Ill-posedness of the linearised inverse problem

As in the injectivity section, we consider the linear(ised at $\kappa = 0$) problem of recovering $\underline{\delta\kappa}(x)$ from time trace observations $h(t) = \mathrm{tr}_\Sigma z_0 = z_0(x_0, t_0),\ x_0 \in \Sigma$, where z_0 solves (67) with u_0 solving (68), both with homogeneous initial conditions.

Again, we assume that the excitation r has been chosen such that u_0 takes the form $u_0(x,t) = f(x)\chi(t)$ and employ the shorthand notation $\tilde{\chi} = (\chi^2)''$. Using the eigensystem of $\mathcal{A}$ we can then write

$$z_0(x,t) = \sum_{j=1}^{\infty} \phi(x) z_j(t)\,, \qquad \hat{z}_j(s) = \hat{w}(\lambda_j, s)\langle \underline{\delta\kappa} \cdot f, \phi_j \rangle \widehat{\tilde{\chi}}(s)$$

with $\hat{w}(\lambda_j, s)$ according to (71). As in the injectivity section we obtain (for simplicity in the 1-d case where all eigenvalues are single)

$$\mathrm{Res}(\hat{h}(p_m)) = \mathrm{Res}(\hat{w}_m; p_m)\widehat{\tilde{\chi}}(p_m)\langle \underline{\delta\kappa} \cdot f, \phi_m \rangle \phi_m(x_0)\,, \tag{80}$$

that is,

$$\langle \underline{\delta\kappa} \cdot f, \phi_m \rangle = \mathrm{Res}(\hat{h}(p_m)) \Big(\mathrm{Res}(\hat{w}_m; p_m)\widehat{\tilde{\chi}}(p_m)\phi_m(x_0)\Big)^{-1}\,. \tag{81}$$

By l'Hospital's rule we have

$$\begin{aligned}\mathrm{Res}(\hat{w}_m; p_m) &= \lim_{s\to p_m} \frac{s - p_m}{\omega(\lambda, s)} = \lim_{s\to p_m} \frac{1}{\omega'(\lambda, s)} \\ &= \begin{cases} \frac{1}{2p_m + \alpha b \lambda_j^\beta p_m^{\alpha-1}} & \text{for CH} \\ \frac{1}{(2+\alpha_2) b_2 p_m^{1+\alpha_2} + 2p_m + \alpha_1 b_1 \lambda_j p_m^{\alpha_1 - 1}} & \text{for FZ}\,. \end{cases}\end{aligned}$$

Thus, the factor multiplied with $\langle \underline{\delta\kappa} \cdot f, \phi_m \rangle$ in (81) only mildly grows with p_m.

Remark 5.1. In higher space dimensions, to resolve the equations

$$\mathrm{Res}(\hat{h}(p_m)) = \mathrm{Res}(\hat{w}_m; p_m)\widehat{\tilde{\chi}}(p_m) \sum_{k\in K_m} \langle \underline{\delta\kappa} \cdot f, \phi_k \rangle \phi_k(x_0)\,, \quad x_0 \in \Sigma$$

that replace (80) then, again condition (78) is obviously crucial. Referring to Remark 4.1, in the separable eigenfunction setting of, e.g., discs, balls, cubiods, etc., the fact that the eigenfunction factors along Σ are mutually orthogonal clearly aids numerical implementation and stability.

The major ill-posedness seems to lie in the evaluation of the residue of the observations $\mathrm{Res}(\hat{h}(p_m))$ at the poles p_m, from knowledge of $h(t)$ for $t > 0$, that is, from $\hat{h}(s) = \int_0^\infty e^{-st} h(t)\, dt$ for s with nonnegative real part (so that the integral defining the Laplace transform is well-defined). If these poles lie on the imaginary axis (wave equation), this is still well posed. The further left the poles lie, the more ill-posed this problem.

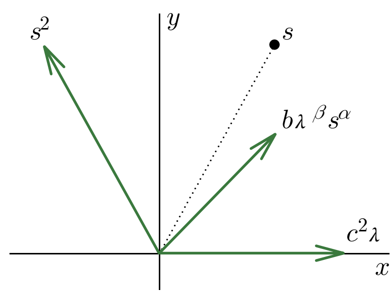
5.1. **Location of the poles of the relaxation functions.** Motivated by their role for the degree of ill-posedness of the inverse problem, we develop some further results – beyond those stated in Lemma 4.2 as essentials for our uniqueness proof – for each of the two models under consideration and also provide some computational results with plots of these poles for several parameter configurations.

The CH model: Poles of the CH model are the roots of the function

$$\omega_{\text{CH}}(s) := s^2 + b\lambda^\beta s^\alpha + c^2\lambda = 0\,.$$

We recall that according to Lemma 4.1, whose proof relies on Rouché's theorem, they lie in the left hand complex plane and this is easily shown by the following alternative argument.

Suppose s is a root in the first quadrant. Then $\mathbf{s}$, the line joining the origin to the point s can be split into a component in the direction of the positive real axis and one in the direction of the positive imaginary axis. Then since $\alpha \le 1$, $\mathbf{s}^\alpha$ has components in the same directions. Similarly, the vector $\mathbf{s}^2$ has a component parallel to the real axis and again one in the direction of the positive imaginary axis. Since $\lambda \ge 0$, the same is true of the vector $b\lambda^\beta\mathbf{s}^\alpha$. The third vector representing $c^2\lambda\mathbf{x}$ points along the real axis. However, the sum of these three vectors cannot add to zero contradicting the claim the root s lay in the first quadrant. An identical argument shows s cannot lie in the fourth quadrant and hence cannot lie in the right half plane.



For $b = 0$ the poles are along the imaginary axis and spaced exactly as the eigenvalue sequence $\{\lambda_n\}$ stretched by the factor c^2. As b increases so does the (negative) real component of the poles which follow a curve whose rough slope is determined by the ratio of b and c^2. The powers α and β also are factors that influence the skewness of the curve along which the poles align. The magnitude of the real and imaginary parts show the relative strengths of the damping and oscillation effects respectively in the equation.

The roots of $\omega_{\text{CH}}(s)$ are shown in Figure 1 with $b = 0.1$, $\lambda_n = n^2\pi^2$, and for both $c = 1$ and $c = 5$, as well as $\alpha = \frac{1}{2}$ and $\alpha = \frac{9}{10}$ and illustrate the above point.

We now provide some notes on how these poles were computed. For rational $\alpha = p/q$, $\omega_{\text{CH}}(s)$ can be written as $z^{2q} + Bz^p + C$ with $B = b\lambda^\beta$, $C = c^2\lambda$ and where $s = z^p$. Now the $2q^{\text{th}}$ degree polynomial can be represented as the characteristic polynomial of a $2q \times 2q$ matrix. Then the roots of this polynomial are calculated by computing the eigenvalues of the companion matrix. This gives a good approximation even for reasonably large q values but additional care must be taken, see, for example, [10]. Given now the values of $\{z_n\}$ for $\lambda \in \{\lambda_n\}$, one can recover $\{s_n\}$ from $s_n = z_n^p$. This is subject to considerable round-off error for even modest values of p. However it is usually sufficient as an initial approximation for Newton's method to then compute a more exact value of the roots of ω to desired accuracy. This is also successful for real α by first taking a rational approximation $\alpha \approx p/q$ for the initial approximation of the roots and then proceeding as above.

The FZ model: Poles of the fractional Zener model are the roots of the function

$$\omega_{\text{FZ}}(s) := b_2 s^{2+\alpha_2} + s^2 + b_1\lambda s^{\alpha_1} + c^2\lambda = 0.$$

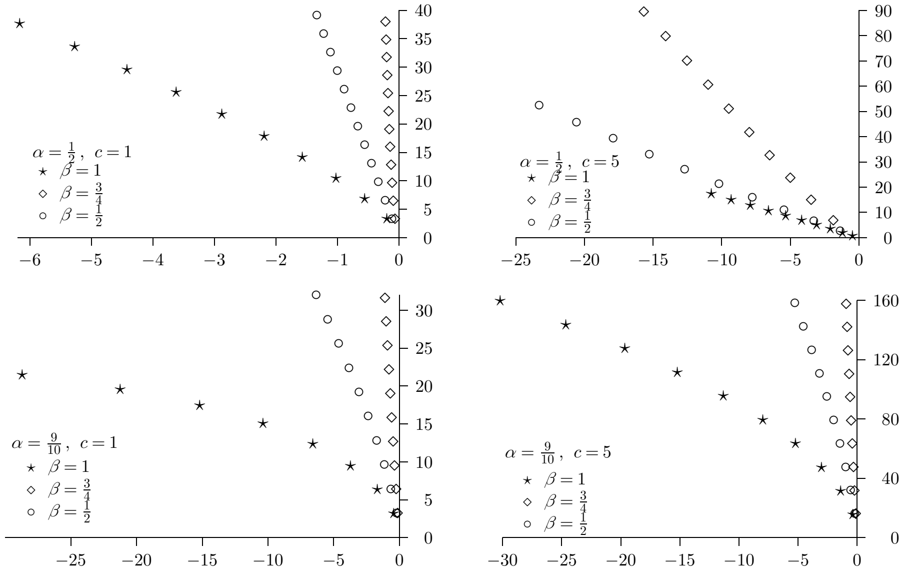


FIGURE 1. Roots of $\omega_{\text{CH}}(s)$ for various α, β, c values

There is a more complex relationship here and more constants whose value can affect the outcome. In the case that $\alpha_1 = \alpha_2 = \alpha$, we can re-write this as

$$\omega_{\text{FZ}}(s) := (b_2 s^\alpha + 1)(s^2 + c^2\lambda) + \delta\lambda s^\alpha = 0, \qquad \text{where } \delta := b_1 - c^2 b_2$$

and δ needs to be nonnegative, cf. [14, Section III.B]. If $\delta = 0$ then $\omega_{\text{FZ}}(s)$ factors. There will be two roots at $\pm i\sqrt{\lambda}\,c$ on the imaginary axis and a potential root coming from $s^\alpha + 1/b_2 = 0$. The latter only exists in case $\alpha = 1$ for otherwise writing $s = re^{i\theta}$ with $\theta \in (-\pi, \pi]$ we have that $\alpha\theta \in (-\pi, \pi)$ and therefore $\Im(s^\alpha + 1) = r^\alpha \sin(\alpha\theta) = 0$ implies $\alpha\theta = 0$, hence $\Re(s^\alpha + 1/b_2) = r^\alpha \cos(\alpha\theta) + 1/b_2 > 0$. In case $\alpha = 1$ we obviously have a root at $-1/b_2$, whose modulus, notably, does not increase with λ, as opposed to the two other complex conjugate roots of ω_{FZ}.

Clearly, physical reasoning leads us to the conclusion that in case of a nonnegative diffusivity of sound $\delta \geq 0$, all poles need to have nonpositive real part. However, the complex analysis arguments from [22, Lemma 4.1] via Rouché's theorem, using as a bounding function the dominant power part $f(z) = b_2 z^{2+\alpha_2} + c^2\lambda$, does not seem to directly carry over to the FZ case. This is basically due to the fact that we cannot say anything about the number of roots of the non-polynomial function f. Additionally, asymptotics in terms of powers of s will be much less effective here since, for small b_2 and/or α_2, the term s^2 will be de facto dominant even for relatively large magnitudes of s.

Therefore we have to take a different path to conclude that also in the FZ case, the poles lie in the left hand complex plane. We do so by means of energy estimates similar to those in section 3.2, which basically corresponds to the mentioned physical argument. As a (partial) counterpart to Lemma 4.1 in the CH case we state the following.

Lemma 5.1. *The roots of* $\omega_{\text{FZ}}(s)$ *with* $\alpha_1 = \alpha_2 = \alpha$ *and* $\delta := b_1 - c^2 b_2 \geq 0$ *lie in the left hand complex plane.*

Proof. We consider the following initial value problem for the relaxation equation

$$(82)\quad b_2\partial_t^{2+\alpha_2}w + w'' + b_1\lambda\partial_t^{\alpha_1}c^2\lambda w = 0\,, \quad w(0) = 0\,,\ w'(0) = 1\,, \quad w''(0) = 0\,.$$

The Laplace transform $\hat{w}$ of its solution satisfies

$$(b_2 s^{2+\alpha_2} + s^2 + b_1\lambda s^{\alpha_1}c^2\lambda)\hat{w}(s) = b_2 s^{\alpha_2} + 1$$

and therefore $\hat{w}(s) = \frac{b_2 s^{\alpha_2}+1}{\omega_{\text{FZ}}(s)}$. Now if $\alpha_1 = \alpha_2 = \alpha$ and $\delta := b_1 - c^2 b_2 \geq 0$, analogously to the proof of Proposition 3.7, we obtain an energy estimate for $\tilde{w} := b_2\partial_t^\alpha w + w$ by multiplying (82) with w'' and integrating with respect to time

$$\frac{1}{2}|\tilde{w}_t(t)|^2 + \frac{c^2\lambda}{2}|\tilde{w}(t)|^2 + \delta\lambda\Big(\frac{b_2}{2}|\partial_t^\alpha w(t)|^2 + \frac{1}{2\Gamma(\alpha)t^{1-\alpha}}\int_0^t |\partial_t^\alpha w(\tau)|^2\,d\tau\Big)$$
$$\leq \frac{1}{2}|\tilde{w}_t(0)|^2 + \frac{c^2\lambda}{2}|\tilde{w}(0)|^2$$

for all $t \geq 0$. This implies uniform boundedness

$$|\tilde{w}(t)| \leq \sqrt{(c^2\lambda_{\min})^{-1}|\tilde{w}_t(0)|^2 + |\tilde{w}(0)|^2} =: C$$

by a constant independent of λ. Taking Laplace transforms

$$|\hat{\tilde{w}}(s)| = \left|\int_0^\infty e^{-st}\tilde{w}(t)\,dt\right| \leq \int_0^\infty e^{-\Re(s)t}\,dt\; C = \frac{C}{\Re(s)}$$

for $\Re(s) > 0$, we see that $\hat{\tilde{w}}(s)$ cannot have any poles in the right half plane. Due to the identity $\hat{\tilde{w}}(s) = (b_2 s^\alpha + 1)\hat{w}(s) - b_2 s^{\alpha-1}w(0) = (b_2 s^\alpha + 1)\hat{w}(s) = \frac{(b_2 s^\alpha+1)^2}{\omega_{\text{FZ}}(s)}$ (where the numerator has no zeros in case $\alpha \in (0,1)$), the assertion follows. □

The effect of δ on the poles in the FZ model can also be assessed by means of the implicit function theorem, applied to the function

$$f(r,\theta;\delta) = \begin{pmatrix} b_2 r^{2+\alpha_2}\cos((2+\alpha_2)\theta) + r^2\cos(2\theta) + b_1\lambda r^{\alpha_1}\cos(\alpha_1\theta) + c^2\lambda \\ b_2 r^{2+\alpha_2}\sin((2+\alpha_2)\theta) + r^2\sin(2\theta) + b_1\lambda r^{\alpha_1}\sin(\alpha_1\theta) \end{pmatrix}$$

whose zeros are the magnitudes and arguments of the roots $s = re^{i\theta}$ of ω^{FZ}. Now

$$\frac{\partial f}{\partial(r,\theta)} = \begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \end{pmatrix}, \quad \frac{\partial f}{\partial\delta} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

and using Cramer's rule this yields

$$\frac{\partial r}{\partial\delta} = \frac{B_1C_2 - B_2C_1}{A_1B_2 - A_2B_1}, \quad \frac{\partial\theta}{\partial\delta} = \frac{C_1A_2 - C_2A_1}{A_1B_2 - A_2B_1},$$

where $b_1 = b_2c^2 + \delta$ and for (r,θ) satisfying $f(r,\theta;\delta) = 0$. Then

$$A_1 = (2+\alpha_2)b_2 r^{1+\alpha_2}\cos((2+\alpha_2)\theta) + 2r\cos(2\theta) + \alpha_1 b_1\lambda r^{\alpha_1-1}\cos(\alpha_1\theta)$$
$$= -\frac{1}{r}\Big(\alpha_2 r^2\cos(2\theta) + (2+\alpha_2-\alpha_1)b_1\lambda r^{\alpha_1}\cos(\alpha_1\theta) + (2+\alpha_2)c^2\lambda\Big)$$
$$A_2 = (2+\alpha_2)b_2 r^{1+\alpha_2}\sin((2+\alpha_2)\theta) + 2r\sin(2\theta) + \alpha_1 b_1\lambda r^{\alpha_1-1}\sin(\alpha_1\theta)$$
$$= -\frac{1}{r}\Big(\alpha_2 r^2\sin(2\theta) + (2+\alpha_2-\alpha_1)b_1\lambda r^{\alpha_1}\sin(\alpha_1\theta)\Big)$$
$$B_1 = -rA, \qquad B_2 = rA_1, \qquad C_1 = \lambda r^{\alpha_1}\cos(\alpha_1\theta), \qquad C_2 = \lambda r^{\alpha_1}\sin(\alpha_1\theta)\,.$$

This results in

$$A_1B_2-A_2B_1 = r(A_1^2+A_2^2)$$
$$B_1C_2-B_2C_1 = \lambda r^{\alpha_1}\big(\alpha_2 r^2\cos((2-\alpha_1)\theta)+(2+\alpha_2-\alpha_1)b_1\lambda r^{\alpha_1}+(2+\alpha_2)c^2\lambda\cos(\alpha_1\theta)\big)$$
$$C_1A_2-C_2A_1 = \lambda r^{\alpha_1-1}\big(-\alpha_2 r^2\sin((2-\alpha_1)\theta)+(2+\alpha_2)c^2\lambda\sin(\alpha_1\theta)\big)\,,$$

and therefore $\frac{\partial r}{\partial\delta}>0\,,\quad \frac{\partial\theta}{\partial\delta}>0$, for the case of the known roots $r=c\sqrt{\lambda}$, $\theta=\pm\pi/2$ at $\delta=0$, for $\alpha_1=\alpha_2$. That is, increasing δ tends to move the poles into the left hand complex plane, which is intuitive in view of its physical role as a diffusivity of sound.

Also here we have employed the method for numerically computing roots as described above. In particular we use this in order to illustrate the influence of $\delta>0$ on the behaviour of the roots, see Figure 2.

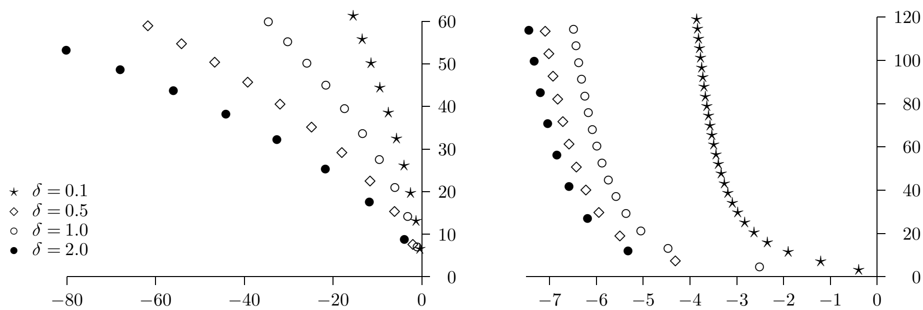


FIGURE 2. Roots of $\omega_{\rm FZ}(\mathbf{s})$ for various δ values with: left $\alpha=\mathbf{\frac{1}{2}}$, right $\alpha=\mathbf{\frac{9}{10}}$

The location of the poles can thus be computed from knowledge of the constants b_1, b_2, c, the exponents α and β as well as the values of λ_n. These poles are also obtainable from the time trace measurements as the zeros of the relaxation function which is the Laplace transform of this data g. Thus assuming the spectrum $\{\lambda_n\}_{n=1}^\infty$ of $\mathcal{A}$ was known it is perfectly reasonable that a least-squares fit could be made to determine the damping constants contained in the term D appearing in equation (3) and/or the wave speed c. While an ill-conditioned problem, it would be particularly feasible if the time trace data were measured at several points along an arc rather than at a single point. It is further conceivable that spectral information on the eigenvalues of $\mathcal{A}$ could be determined, in particular those of the low frequency. This in turn might be used to obtain knowledge on either a coefficient in $\mathcal{A}$ or on the domain Ω itself as there is geometrical information contained in the lowest few eigenvalues. See, for example, [12]. As noted earlier, the Laplace transform gives information on $\omega(s)$ on the right half plane. Using analytic continuation to recover the poles on the left half plane results in severe ill-conditioning and the more negative the real part the more extreme this becomes. The figures in this section indicate the effect various parameters have on this process.

6. RECONSTRUCTIONS OF κ

6.1. **Newton type methods for recovering κ.** From Theorems 3.1 and 3.3 we obtain well-definedness of the forward operator F by $F(\kappa)=\mathrm{tr}_\Sigma\circ G$, where $\mathrm{tr}_\Sigma v$

denotes the time trace at the observation surface Σ (which in one space dimension may also just be a single point $\Sigma = \{x_0\}$) in both CH and FZ model cases of the parameter-to-state map $G : \kappa \mapsto u$ where u solves (3) with (18) or (49) (the latter with $\alpha_1 = 1$). Hence the inverse problem under consideration can be stated as

$$F(\kappa) = g\,,$$

where g is the measured data (2) and we consider F as an operator $F : \mathcal{D}(\subseteq X) \to Y$. Here the domain $\mathcal{D}$ of F is defined as a ball with fixed radius in $X = W^{2,4}(\Omega) \cap W^{1,\infty}(\Omega)$ (CH) or $X = W^{1,4}(\Omega)$ (FZ) and additionally, the initial data and driving term are supposed to satisfy certain regularity and smallness conditions. For drawing this conclusion on the composite operator, in addition to the mentioned theorems on G, we use the fact that the trace operator tr_Σ is linear and well-defined on the spaces U_{lo} or U according to (35) or (56), respectively, and maps into $Y \subseteq C([0,T]; C(\Sigma))$. Typically we will have $Y = L^p(0,T,R^N)$, $N \in \mathbb{N} \cup \{\infty\}$ in case of Σ being a discrete set or $Y = L^p(0,T; L^q(\Sigma))$ in case of Σ being a surface.

From Theorems 3.2, 3.4 we additionally conclude Fréchet differentiability of F on $\mathcal{D}$. Thus we are in the position to formulate Newton's method which defines the iterate κ_{k+1} implicitely by the linearised problem

$$F'(\kappa_k)(\kappa_{k+1} - \kappa_k) = g - F(\kappa_k)\,,$$

or its frozen version

$$F'(\kappa_0)(\kappa_{k+1} - \kappa_k) = g - F(\kappa_k)\,,$$

which is known to save the computational effort of evaluating the derivative in each step – at the cost of yielding only linear convergence. This is the approach we are going to take in the numerical reconstructions to be shown in this section. Concerning solvability of the linearisation, we have commented on injectivity of $F'(\kappa_0)$ in case $\kappa_0 = 0$ in section 4. As pointed out in section 5, inversion must be expected to be ill-posed, though and also surjectivity of $F'(\kappa_0)$ is not likely to hold on the relatively large space Y. Therefore, we will rely on a regularized least squares variant

$$\kappa_{k+1} = \mathrm{argmin}_{\tilde{\kappa} \in \mathcal{D}} \|F(\kappa_k) + F'(\kappa_0)(\tilde{\kappa} - \kappa_k) - g\|_Y + \gamma\|\tilde{\kappa} - \kappa_0\|_X \tag{83}$$

of the frozen Newton method. A minimizer exists, since the cost function is weakly lower semicontinuous and has weakly compact sublevel sets in the space X, which in both CH and FZ cases is the dual of a separable space. Uniqueness of this minimizer follows easily from strict convexity of the cost function for $\gamma > 0$. In the injective setting of section 4 this remains valid also with $\gamma = 0$. Dispensing with uniqueness (that is, replacing "= argmin" by "$\subseteq$ argmin" above), we can also choose to completely skip the regularization term, since stabilization is already achieved by imposing the constraint $\tilde{\kappa} \in \mathcal{D}$. The formulation (83) also allows to deal with noisy data, cf., e.g., [13, 18, 19, 26].

We mention in passing that Proposition (3.8) still allows one to rigorously apply $F'(0)$ in the case $\alpha_1 = \alpha$ of CH as well; however, $F'(0)$ is only a formal linearization, not a true derivative. Moroever, in that case, well-definedness of $F(\kappa_k)$ is missing.

We now provide reconstruction results for κ in the CH model with $\beta = 1$.

The boundary conditions for our test cases were homogeneous Dirichlet at $x = 0$ and homogeneous Neumann at $x = 1$ with a nonhomogeneous driving term $r(x,t)$ with greater weight near $x = 1$. Thus the solution was small in the region near $x = 0$ in comparison to near $x = 1$ where the data $h(t) = u(1,t)$ was measured.

The consequence of this was that $\kappa(x)$ for x small was multiplied by terms that were also small in comparison to that at the rightmost endpoint and resulting in greater ill-conditioning of the inversion near $x = 0$. This will be apparent in each of the reconstructions to be shown below.

The data $h(t)$ was computed by the direct solver at the endpoint $x = 1$ and a sample of 50 points was taken to which uniformly distributed random noise was added as representing the actual data measurements that formed the overposed data. This was then filtered by a smoothing routine based on penalising the H^2 norm with regularisation parameter based on the estimate of the noise and then up-resolved to the working size for the inverse solver.

Discretisation of κ was done by means of a fixed set of 40 chapeau basis functions and we applied a regularised frozen Newton iteration, stopped by the discrepancy principle, for numerically solving the discretised inverse problem.

Figure 3 shows the reconstruction of a piecewise linear κ for the values $\alpha = 1$, $\alpha = 0.9$, $\alpha = 0.5$ and $\alpha = 0.25$. In each case the damping coefficient b was kept at $b = 0.1$ and the wave speed at $c = 1$. The $(L^\infty,\ L^2)$ norm difference for the final

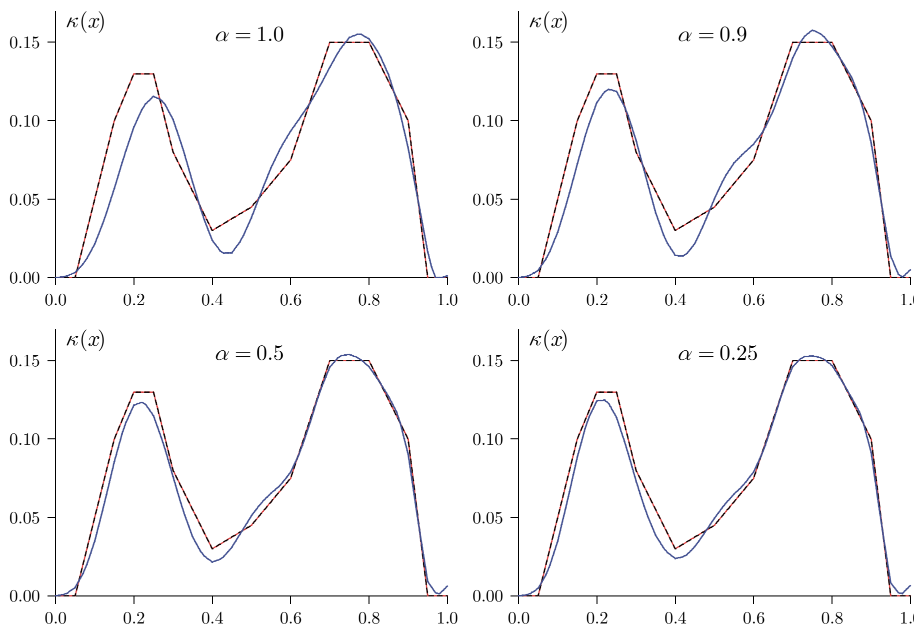


FIGURE 3. Reconstructions of $\kappa(x)$ for various α values. Noise = 0.1%.

versus the actual reconstruction were: $(0.109, 0.078)$, $(0.116, 0.084)$, $(0.184, 0.126)$, $(0.315, 0.191)$, respectively and show the increase in resolution possible with a decrease in α.

Note that the reconstructions of κ are clearly superior at the right hand endpoint due to imposed conditions as the wave is essentially transmitting information primarily from right to left but the amplitude is damped as it travels. The smaller the fractional damping the lesser is this effect which is also apparent from these figures.

The reconstructions naturally worsen with increasing noise levels as Figure 4 shows.

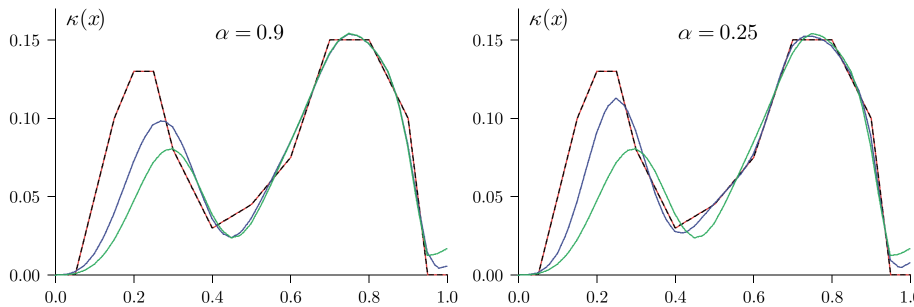


FIGURE 4. Reconstructions of $\kappa(x)$ for $\alpha = 0.25,\ 0.9$. Noise = 0.5% (blue) and 1% (green).

Figure 5 shows the singular values of the Jacobian matrix used in the (frozen) Newton method. The results confirm our findings from section 5.1, cf. Figure 1, namely the fact that the further away the (negative) real part of the poles is from the real axis, the more ill-posed the inverse problem of recovering information from them. This was pointed out at the end of section 5. Note that if the function κ can be well represented by a small number of basis functions then the dependence with respect to α will be fairly weak. On the other hand, if a large number of basis functions are needed for κ to be represented, then the dependence on α becomes much stronger although by this point the condition number of the Jacobian is already extremely high for all α and relatively few singular values are likely to be usable in any reconstruction with data subject to extremely small noise levels. The

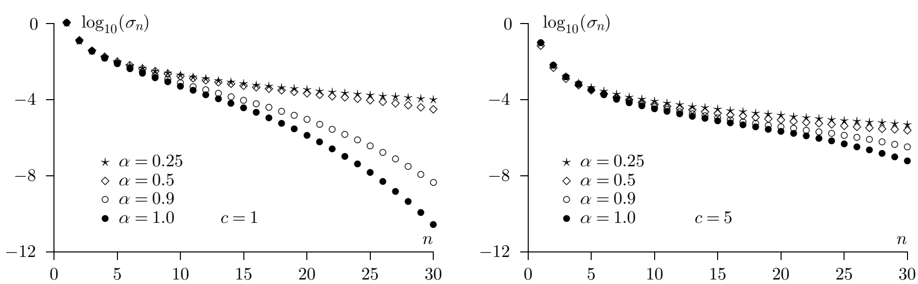


FIGURE 5. Singular values for various α values: left; $c = 1$, right; $c = 5$

effect of damping is to directly contribute to the ill-conditioning and thus it is clear that for fixed α and c this will increase as the coefficient b increases. The degree of ill-conditioning as a function of the wave speed c is less clear.

Figure 5 shows the singular values $\{\sigma_n\}$ of the Jacobian matrix for both $c = 1$ and $c = 5$. This illustrates the decay of the singular values and hence the level of ill-conditioning does depend on c but certainly not uniformly for all values of the fractional exponent α. For α near unity, that is damping approaches or is at the classical paradigm, there is a considerable increase in the smaller, high index singular values indicating the problem is much less ill-posed for larger wave speeds c. For the smaller index σ_n the ratio σ_n/σ_1 is almost the same indicating at most a weak effect due to the wave speed. Thus for a function $\kappa(x)$ requiring only a small number of basis functions the effect of wave speed is relatively minimal but this changes quite dramatically if a larger number of singular values are required. For α less than about one half the condition number σ_n/σ_1 becomes relatively independent of c – at least in the range indicated.

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