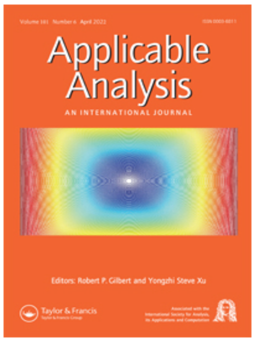

Applicable Analysis
An International Journal

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/gapa20

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APPLICABLE ANALYSIS
https://doi.org/10.1080/00036811.2021.1965583

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Uniqueness for an inverse coefficient problem for a one-dimensional time-fractional diffusion equation with non-zero boundary conditions

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ABSTRACT

We consider initial boundary value problems for one-dimensional diffusion equation with time-fractional derivative of order $\alpha \in (0,1)$ which are subject to non-zero Neumann boundary conditions. We prove the uniqueness for an inverse coefficient problem of determining a spatially varying potential and the order of the time-fractional derivative by Dirichlet data at one end point of the spatial interval. The imposed Neumann conditions are required to be within the correct Sobolev space of order α. Our proof is based on a representation formula of solution to an initial boundary value problem with non-zero boundary data. Moreover, we apply such a formula and prove the uniqueness in the determination of boundary value at another end point by Cauchy data at one end point.

ARTICLE HISTORY
Received 9 April 2021
Accepted 26 July 2021

COMMUNICATED BY
E. Francini

KEYWORDS
Inverse coefficient problem; fractional diffusion equation; uniqueness

AMS SUBJECT CLASSIFICATIONS
35R30; 35R11

1. Introduction

We consider the following initial boundary value problem for a one-dimensional time-fractional diffusion equation:

$$\begin{cases} d_t^\alpha u(x,t) = \partial_x^2 u(x,t) + p(x)u(x,t), & 0 < x < 1,\ 0 < t < T, \\ \partial_x u(0,t) = 0, \quad \partial_x u(1,t) = g(t), & 0 < t < T, \\ u(x,0) = 0, \quad 0 < x < 1. \end{cases} \tag{1}$$

Here and henceforth let $\partial_x = \frac{\partial}{\partial x}$, $\partial_x^2 = \frac{\partial^2}{\partial x^2}$, and we define for absolutely continuous g on $[0,T]$

$$d_t^\alpha g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\mathrm{d}g}{\mathrm{d}s}(s)\,\mathrm{d}s, \quad 0 < t < T,$$

that is, the fractional derivative of order α, $0 < \alpha < 1$, and of Caputo type (see, e.g., Podlubny [1]). The first equation in (1) is a time-fractional diffusion equation of subdiffusion type modelling, for example, anomalous diffusion in heterogeneous media. For some applications , see, e.g., Metzler and Klafter [2].

In this article, we are concerned with the question of uniqueness for the inverse problem:

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Let $g = g(t)$ be given for $0 < t < T$. Given data $u(0,t)$ for $0 < t < T$ or $u(1,t)$ for $0 < t < T$, does they uniquely determine $\alpha \in (0,1)$ and *$p(x)$, $0 < x < 1$?*

In place of (1), we can also consider

$$\begin{cases} d_t^\alpha u(x,t) = \partial_x^2 u(x,t) + p(x)u(x,t), & 0 < x < 1,\ 0 < t < T, \\ \partial_x u(0,t) = \partial_x u(1,t) = 0, & 0 < t < T, \\ u(x,0) = a(x), & 0 < x < 1. \end{cases} \tag{2}$$

Uniqueness for this type of inverse problem for (2) with $\alpha = 1$, that is, for the initial boundary value problem for the heat equation, was considered by, for example, Murayama [3], Suzuki and Murayama [4]. For the case with $0 < \alpha < 1$, we refer to Cheng, Nakagawa, Yamamoto and Yamazaki [5], Li, Zhang, Jia and Yamamoto [6]. Also see Jin and Rundell [7], Jing and Peng [8], Jing and Yamamoto [9], and survey chapters Li, Liu and Yamamoto [10], Li and Yamamoto [11], Liu, Li and Yamamoto [12]. Both for the cases of $\alpha = 1$ and $0 < \alpha < 1$, the uniqueness for (2) requires a quite strong condition to be imposed for the initial value $a(x)$.

On the other hand, for the inverse problem for (1) with a zero initial value but $g \not\equiv 0$, we refer to Pierce [13] who proved the uniqueness for $\alpha = 1$ with the quite mild assumption $g \not\equiv 0$.

For fixed $\alpha \in (1,2)$, Wei and Yan [14] established the uniqueness in determining $p(x)$ with $g \in C^2[0,T]$ imposing additional conditions.

For the inverse problem for (1) with $0 < \alpha < 1$, see Rundell and Yamamoto [15]. The purpose of this article is to complete [15] within a weaker class of solutions in suitable Sobolev space in time. For the case of $1 < \alpha < 2$, we can argue in a similar manner but we concentrate on the case $0 < \alpha < 1$.

For the mathematical formulations, we need to introduce function spaces and relevant operators; all functions considered are assumed to be real-valued. Let $L^2(0,1)$ be a usual Lebesgue space and let $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the scalar product and the norm respectively in $L^2(0,1)$, and let $\langle \cdot, \cdot \rangle_X$ be the scalar product in other Hilbert spaces X when we so specify.

We define the fractional Sobolev space $H^\alpha(0,T)$ on the interval $(0,T)$ (see, e.g.[16], Chapter VII) with the norm in $H^\alpha(0,T)$:

$$\|u\|_{H^\alpha(0,T)} := \left(\|u\|^2_{L^2(0,T)} + \int_0^T \int_0^T \frac{|u(t)-u(s)|^2}{|t-s|^{1+2\alpha}}\, dt\, ds \right)^{\frac{1}{2}}.$$

We further define the Banach spaces

$$H_\alpha(0,T) := \begin{cases} \{u \in H^\alpha(0,T);\ u(0) = 0\}, & \frac{1}{2} < \alpha < 1, \\ \left\{ v \in H^{\frac{1}{2}}(0,T);\ \int_0^T \frac{|v(t)|^2}{t}\, dt < \infty \right\}, & \alpha = \frac{1}{2}, \\ H^\alpha(0,T), \quad 0 < \alpha < \frac{1}{2} \end{cases}$$

with the following norm:

$$\|v\|_{H_\alpha(0,T)} = \begin{cases} \|v\|_{H^\alpha(0,T)}, & 0 < \alpha < 1,\ \alpha \ne \frac{1}{2}, \\ \left(\|v\|^2_{H^{\frac{1}{2}}(0,T)} + \int_0^T \frac{|v(t)|^2}{t}\, dt \right)^{\frac{1}{2}}, & \alpha = \frac{1}{2}. \end{cases}$$

We define the Abel (Riemann–Liouville) fractional integral operator

$$J^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds, \quad 0 < t < T,\ 0 < \alpha < 1.$$

Henceforth by $x \sim y$, we mean that there exists a constant $C > 0$ such that $C^{-1}y \le x \le Cy$ for all quantities x, y under consideration.

In Gorenflo, Luchko and Yamamoto [17], Kubica, Ryszewska and Yamamoto [18] (Theorem 2.1), it is proved that J^α is an isomorphism between $L^2(0,T)$ and $H_\alpha(0,T)$. We define

$$\partial_t^\alpha g = (J^\alpha)^{-1} g \quad \text{for } g \in H_\alpha(0,T) = J^\alpha L^2(0,T).$$

Then also by Theorem 2.5 in [18], we see

$$\begin{cases} \|\partial_t^\alpha g\|_{L^2(0,T)} \sim \|g\|_{H_\alpha(0,T)}, \quad g \in H_\alpha(0,T), \\ \partial_t^\alpha g = d_t^\alpha g \quad \text{if } g \in W^{1,1}(0,T) \text{ satisfies } g(0) = 0 \text{ and } t^{\alpha-1}\frac{\mathrm{d}g}{\mathrm{d}t} \in L^\infty(0,T). \end{cases} \tag{3}$$

In other words, ∂_t^α is an extension of the Caputo derivative d_t^α to $H_\alpha(0,T)$.

Thus throughout this article, in place of (1) we consider

$$\begin{cases} \partial_t^\alpha u(x,t) = \partial_x^2 u(x,t) + p(x)u(x,t), \quad 0 < x < 1,\ 0 < t < T, \\ \partial_x u(0,t) = 0, \quad \partial_x u(1,t) = g(t),\ 0 < t < T, \\ u \in H_\alpha(0,T;L^2(0,1)). \end{cases} \tag{4}$$

We assume

$$p,\ q \le 0, \quad p, q \not\equiv 0 \quad \text{on } [0,1], \quad p, q \in C[0,1]. \tag{5}$$

Then we can prove

Proposition 1.1: *Let $g \in H_\alpha(0,T)$ and let $0 < \alpha < 1$. Then there exists a unique solution $u_{p,\alpha} = u_{p,\alpha}(x,t) \in H_\alpha(0,T;L^2(0,1)) \cap L^2(0,T;H^2(0,1))$ solving (4).*

In (4), we interpret $u(x,\cdot) \in H_\alpha(0,T)$ as an initial condition: if $\alpha > \frac{1}{2}$, then the Sobolev embedding yields $H_\alpha(0,T;L^2(0,1)) \subset H^\alpha(0,T;L^2(0,1)) \subset C([0,T];L^2(0,1))$ and so this means that u satisfies the initial condition in a usual sense. However for $\alpha < \frac{1}{2}$, the time regularity does not admit such a usual initial condition and alternatively the third equation in (4) is required. For the class of solutions with the H_α-regularity in t, it is sufficient to assume the same regularity in t for boundary data $g(t)$, that is, $g \in H_\alpha(0,T)$. Moreover for $\alpha > \frac{1}{2}$, the condition means that $g(0) = 0$, which is a natural compatibility condition at $x = 0$ and $t = 0$. We emphasize that since the order of time derivative appearing in the equation is up to $\alpha < 1$, it is natural to work within 'α-time differentiability', and not in the C^1 nor H^1-class.

We can relax the condition on the signs of p, q in (5), but for simplicity of the arguments, we keep the condition $p, q \le 0, \not\equiv 0$ on $[0,T]$.

For the initial boundary value problems with the zero boundary values, we refer to Gorenflo, Luchko and Yamamoto [17], Kian and Yamamoto [19], Kubica, Ryszewska and Yamamoto [18], Kubica and Yamamoto [20], Luchko [21], Sakamoto and Yamamoto [22]. On the other hand, for initial boundary value problems with non-zero boundary data, there are not many works and we refer only to Yamamoto [23] in the case of less regular boundary data, and one can consult the references therein. On the other hand, the proof of Proposition 1.1 can be done directly, thanks to the one-dimensionality, and see Section 2.

Now we are ready to state the main result of this article.

Theorem 1.2: *We assume (5) and* $0 < \alpha, \beta < 1$,

$$g \in H_{\max\{\alpha,\beta\}}(0,T), \quad g \not\equiv 0 \text{ in } (0,T). \tag{6}$$

Then either $u_{p,\alpha}(0,t) = u_{q,\beta}(0,t)$ for $0 < t < T$ or $u_{p,\alpha}(1,t) = u_{q,\beta}(1,t)$ for $0 < t < T$, yields

$$\alpha = \beta, \quad p(x) = q(x),\ 0 < x < 1.$$

By the regularity shown in Proposition 1.1 and the trace theorem, we notice that the data $u_{p,\alpha}(0,t)$, etc. can make sense in $L^2(0,T)$. We stress that the condition $g \not\equiv 0$ in (6) for the boundary input is quite generous.

In the multidimensional spatial cases, our approach does not work. Our method relies on the inverse spectral problem, data for which are closely related to the Dirichlet-to-Neumann maps. As for works on Dirichlet-to-Neumann maps, we refer to Kian, Oksanen, Soccorsi and Yamamoto [24], Li, Imanuvilov and Yamamoto [25] for example. The formulation of inverse problems in terms of Dirichlet-to-Neumann maps requires many measurements, in general. As for other types of inverse problem in general dimensions with a single measurement, see Kian, Li, Liu and Yamamoto [26].

The article is composed of four sections. In Section 2, we prove Proposition 1.1 and a key representation formula of the solution $u_{p,\alpha}$ to (4). Section 3 is devoted to the proof of Theorem 1.2 on the basis of the representation formula in Section 2. In Section 4, we provide one application of the representation formula to prove the uniqueness in determining a boundary value at $x = 1$ by Cauchy data at $x = 0$.

2. Proof of Proposition 1.1 and a representation formula

2.1. Proof of Proposition 1.1

Recalling (5) we define an operator A_p in $L^2(0,1)$ by

$$\begin{cases} A_p w(x) = -\frac{d^2w}{dx^2}(x) - p(x)w(x), \quad 0 < x < 1, \\ \mathcal{D}(A_p) = \left\{ w \in H^2(0,1); \quad \frac{dw}{dx}(0) = \frac{dw}{dx}(1) = 0 \right\}. \end{cases}$$

Then A_p possesses eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots$. Let φ_n, $n \in \mathbb{N}$ be the associated unique eigenfunction for λ_n: $\varphi_n \in \mathcal{D}(A_p)$ satisfies $A_p\varphi_n = \lambda_n\varphi_n$ in $(0,1)$ and we make the normalisation $\varphi_n(1) = 1$. Moreover, it is known that $\langle \varphi_n, \varphi_m \rangle := \int_\Omega \varphi_n(x)\varphi_m(x)\, dx = 0$ for $n \neq m$ and we set the associated norming constants as

$$\rho_n := \|\varphi_n\|^2, \quad n \in \mathbb{N}.$$

We define

$$\begin{cases} v(x,t) = u_{p,\alpha}(x,t) - \frac{x^2}{2} g(t), \\ f(x,t) = -\frac{x^2}{2}\partial_t^\alpha g(t) + g(t) + \frac{x^2}{2} p(x) g(t), \quad 0 < x < 1,\ 0 < t < T. \end{cases} \tag{7}$$

Then (4) is equivalent to

$$\begin{cases} \partial_t^\alpha v(x,t) = \partial_x^2 v(x,t) + p(x)v(x,t) + f(x,t), \quad 0 < x < 1,\ 0 < t < T, \\ \partial_x v(0,t) = \partial_x v(1,t) = 0, \quad 0 < t < T, \\ v \in H_\alpha(0,T;L^2(0,1)). \end{cases} \tag{8}$$

Since $g \in H_\alpha(0,T)$, we see that $v \in H_\alpha(0,T;L^2(0,1))$ if and only if $u \in H_\alpha(0,T;L^2(0,1))$.

From $g \in H_\alpha(0,T)$ and $p \in C[0,1]$, it follows that $f \in L^2(0,T;L^2(0,1))$. Thus it is sufficient to prove the unique existence of solution $v \in H_\alpha(0,T;L^2(0,1)) \cap L^2(0,T;H^2(0,1))$ to (8). This follows from [18, 22] for example. We note that in [18, 22], the zero Dirichlet boundary condition is considered and the case of the zero Neumann boundary condition can be treated in the same way. Thus the proof of Proposition 1.1 is complete.

2.2. The representation formula.

For $\gamma_1, \gamma_2 > 0$, we define the two parameter Mittag–Leffler function:

$$E_{\gamma_1,\gamma_2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\gamma_1 k + \gamma_2)}, \quad z \in \mathbb{C}.$$

This is an entire function of order 1 in $z \in \mathbb{C}$ (e.g. [1, 27]). Then

Proposition 2.1 (representation formula): *Let $0 < \alpha < 1$, p satisfy (5) and $g \in H_\alpha(0,T)$. Then*

$$u_{p,\alpha}(x,t) = \sum_{n=1}^{\infty} \frac{1}{\rho_n} \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) g(s)\, \mathrm{d}s \right) \varphi_n(x) \tag{9}$$

in $H_\alpha(0,T; L^2(0,1)) \cap L^2(0,T; H^2(0,1))$.

Proof of Proposition 2.1.: By [22] for example, we have the representation

$$v(x,t) = \sum_{n=1}^{\infty} \frac{1}{\rho_n} \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) \langle f(\cdot,s), \varphi_n \rangle \, \mathrm{d}s \right) \varphi_n(x) \tag{10}$$

in $H_\alpha(0,T; L^2(0,1)) \cap L^2(0,T; H^2(0,1))$. Here we note Equations (7) and (8).

We set ${}_0C^1[0,T] := \{h \in C^1[0,T];\ h(0) = 0\}$.

First we prove (9) for $g \in {}_0C^1[0,T]$. We have to calculate the right-hand side of (10).

$$\begin{aligned}
&\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) \langle f(\cdot,s), \varphi_n \rangle \, \mathrm{d}s \\
&\quad = -\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) \partial_s^\alpha g(s)\, \mathrm{d}s \left\langle \frac{x^2}{2}, \varphi_n \right\rangle \\
&\qquad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) g(s)\, \mathrm{d}s \left\langle 1 + \frac{x^2}{2} p,\ \varphi_n \right\rangle.
\end{aligned} \tag{11}$$

We set

$$S := \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) \partial_s^\alpha g(s)\, \mathrm{d}s.$$

For $g \in {}_0C^1[0,T]$, by (3) we see that $\partial_s^\alpha g$ coincides with $d_s^\alpha g$:

$$\partial_s^\alpha g(s) = d_s^\alpha(s) = \frac{1}{\Gamma(1-\alpha)} \int_0^s (s-\xi)^{-\alpha} \frac{\mathrm{d}g}{\mathrm{d}\xi}(\xi)\, \mathrm{d}\xi.$$

Therefore, change of the order of integration yields

$$\begin{aligned}
S &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) \left(\int_0^s (s-\xi)^{-\alpha} \frac{\mathrm{d}g}{\mathrm{d}\xi}(\xi)\, \mathrm{d}\xi \right) \mathrm{d}s \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\mathrm{d}g}{\mathrm{d}\xi}(\xi) \left(\int_\xi^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha)(s-\xi)^{-\alpha}\, \mathrm{d}s \right) \mathrm{d}\xi \\
&= \int_0^t \frac{\mathrm{d}g}{\mathrm{d}\xi}(\xi) \frac{1}{\Gamma(1-\alpha)} \left(\int_0^{t-\xi} \eta^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \eta^\alpha)(t-\xi-\eta)^{-\alpha}\, \mathrm{d}\eta \right) \mathrm{d}\xi.
\end{aligned}$$

For the last equality we used the change of variables $s \to \eta$ by $\eta = t - s$. Moreover,

$$\frac{1}{\Gamma(1-\alpha)} \int_0^{t-\xi} \eta^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \eta^\alpha)(t-\xi-\eta)^{-\alpha} d\eta = E_{\alpha,1}(-\lambda_n(t-\xi)^\alpha)$$

(e.g. formula (1.100) (p . 25) in [1]). Hence, again applying integration by parts, we obtain

$$\begin{aligned} S &= \int_0^t \frac{\mathrm{d}g}{\mathrm{d}\xi}(\xi) E_{\alpha,1}(-\lambda_n(t-\xi)^\alpha)\, \mathrm{d}\xi \\ &= \left[g(\xi) E_{\alpha,1}(-\lambda_n(t-\xi)^\alpha) \right]_{\xi=0}^{\xi=t} - \int_0^t g(\xi) \frac{\mathrm{d}}{\mathrm{d}\xi} E_{\alpha,1}(-\lambda_n(t-\xi)^\alpha)\, \mathrm{d}\xi. \end{aligned}$$

Now, by the definition of the Mittag–Lefller function in view of the power series, the termwise differentiation yields

$$\frac{\mathrm{d}}{\mathrm{d}\xi} E_{\alpha,1}(-\lambda_n(t-\xi)^\alpha) = \lambda_n (t-\xi)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t-\xi)^\alpha), \quad 0 < \xi < t < T. \tag{12}$$

Therefore, using $g(0) = 0$ by $g \in {}_0C^1[0, T]$, we have

$$S = g(t) - \int_0^t \lambda_n (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha) g(s)\, \mathrm{d}s.$$

Substituting this into the above, we obtain

$$\begin{aligned} &\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha) \langle f(\cdot,s), \varphi_n \rangle s\, \mathrm{d}s = -g(t) \left\langle \frac{x^2}{2}, \varphi_n \right\rangle \\ &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha) g(s)\, \mathrm{d}s \left(\left\langle \lambda_n \frac{x^2}{2}, \varphi_n \right\rangle + \left\langle 1 + \frac{x^2}{2} p, \varphi_n \right\rangle \right). \end{aligned}$$

Here by integration by parts, we calculate

$$\begin{aligned} \left\langle \lambda_n \frac{x^2}{2}, \varphi_n \right\rangle + \left\langle \frac{x^2}{2} p, \varphi_n \right\rangle &= \left\langle \lambda_n \varphi_n + p\varphi_n, \frac{x^2}{2} \right\rangle \\ &= \left\langle -\frac{\mathrm{d}^2\varphi_n}{\mathrm{d}x^2}, \frac{x^2}{2} \right\rangle = \left[-\frac{\mathrm{d}\varphi_n}{\mathrm{d}x}(x) \frac{x^2}{2} \right]_{x=0}^{x=1} + \int_0^1 x \frac{\mathrm{d}\varphi_n}{\mathrm{d}x}(x) \mathrm{d}x \\ &= [x\varphi_n(x)]_{x=0}^{x=1} - \int_0^1 \varphi_n(x)\, \mathrm{d}x = 1 - \langle \varphi_n, 1 \rangle. \end{aligned}$$

Hence

$$\left\langle \lambda_n \frac{x^2}{2}, \varphi_n \right\rangle + \left\langle 1 + \frac{x^2}{2} p, \varphi_n \right\rangle = 1 - \langle \varphi_n, 1 \rangle + \langle 1, \varphi_n \rangle = 1,$$

so that

$$\begin{aligned} &\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha) \langle f(\cdot,s), \varphi_n \rangle\, \mathrm{d}s \\ &= -\left\langle \frac{x^2}{2}, \varphi_n \right\rangle g(t) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha) g(s)\, \mathrm{d}s. \end{aligned}$$

Substituting this into (10), since $\{\frac{1}{\sqrt{\rho_n}}\varphi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis in $L^2(0,1)$, we see

$$v(x,t) = -\sum_{n=1}^{\infty}\frac{1}{\rho_n}\left\langle\frac{x^2}{2},\,\varphi_n\right\rangle g(t)\varphi_n(x) + \sum_{n=1}^{\infty}\frac{1}{\rho_n}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)g(s)\,\mathrm{d}s\varphi_n(x)$$
$$= -\frac{x^2}{2}g(t) + \sum_{n=1}^{\infty}\frac{1}{\rho_n}\left(\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)g(s)\,\mathrm{d}s\right)\varphi_n(x).$$

Since $u = v + \frac{x^2}{2}g(t)$, we have proved (9) for $g \in {}_0C^1[0,T]$.

Next we have to prove (9) for $g \in H_\alpha(0,T)$. In Equations (7) and (10), we write $f := f_g$ and $v := v_g$ respectively in order to specify the dependence on g. Since ${}_0C^1[0,T] := \{h \in C^1[0,T];\ h(0) = 0\}$ is dense in $H_\alpha(0,T)$ (e.g. Lemma 2.2 in [18]), for each $g \in H_\alpha(0,T)$, we can find a sequence $g_\ell \in {}_0C^1[0,T]$, $\ell \in \mathbb{N}$ such that $g_\ell \longrightarrow g$ in $H_\alpha(0,T)$ as $\ell \longrightarrow \infty$. Then, since $\partial_t^\alpha g_\ell \longrightarrow \partial_t^\alpha g$ in $L^2(0,T)$ (e.g. Theorem 2.4 in [18]), it follows that $f_{g_\ell} \longrightarrow f_g$ in $L^2(0,T)$. Therefore, applying the well-posedness for the initial boundary value problem (e.g.[17], Theorem 4.1 in [18, 22]), we see that $v_{g_\ell} \longrightarrow v_g$ in $H_\alpha(0,T;L^2(0,1)) \cap L^2(0,T;H^2(0,1))$.

As we already proved, for $g_\ell \in {}_0C^1[0,T]$ we have

$$\sum_{n=1}^{\infty}\frac{1}{\rho_n}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)g_\ell(s)\,\mathrm{d}s\varphi_n - \frac{x^2}{2}g_\ell(t) \longrightarrow v_g \tag{13}$$

in the space $H_\alpha(0,T;L^2(0,1)) \cap L^2(0,T;H^2(0,1))$.

On the other hand, let $h \in L^2(0,T)$. Then one can prove by the asymptotic behaviour of φ_n for large $n \in \mathbb{N}$ (e.g. Section 2 of Chapter 1 of Levitan and Sargsjan [28]), that there exists a constant $\rho_0 > 0$ such that

$$\rho_n \ge \rho_0 \quad \text{for all } n \in \mathbb{N}. \tag{14}$$

Henceforth $C > 0$ denotes generic constants which are independent of n and choices of $h, g, t \in (0,T)$. Let $\psi \in C_0^\infty((0,1)\times(0,T))$. Then by integration by parts

$$\langle\varphi_n, \psi(\cdot,s)\rangle = \frac{1}{\lambda_n}\langle\lambda_n\varphi_n,\,\psi(\cdot,s)\rangle = \frac{1}{\lambda_n}\langle A_p\varphi_n,\,\psi(\cdot,s)\rangle = \frac{1}{\lambda_n}\langle\varphi_n, A_p\psi(\cdot,s)\rangle.$$

Therefore,

$$\left\langle\sum_{n=1}^{\infty}\frac{1}{\rho_n}\left(\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)h(s)\,\mathrm{d}s\right)\varphi_n,\ \psi\right\rangle_{L^2((0,1)\times(0,T))}$$
$$= \sum_{n=1}^{\infty}\frac{1}{\rho_n}\left\langle\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)h(s)\,\mathrm{d}s,\ \frac{1}{\lambda_n}\langle\varphi_n,\,A_p\psi(\cdot,t)\rangle_{L^2(0,1)}\right\rangle_{L^2(0,T)}.$$

Hence, also by (14) and the Cauchy–Schwarz inequality, we have

$$\left|\left\langle\sum_{n=1}^{\infty}\frac{1}{\rho_n}\left(\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)h(s)\,\mathrm{d}s\right)\varphi_n,\ \psi\right\rangle_{L^2((0,1)\times(0,T))}\right|$$
$$\le C\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\left\|\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)h(s)\,\mathrm{d}s\right\|_{L^2(0,T)}\|A_p\psi\|_{L^2(0,T;L^2(0,1))}\frac{\|\varphi_n\|}{\rho_n}$$

$$\le C\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\|s^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n s^{\alpha})*h\|_{L^2(0,T)}.$$

Here and henceforth we set $(g_1*g_2)(t):=\int_0^t g_1(t-s)g_2(s)\,\mathrm{d}s$. By a bound of $E_{\alpha,\alpha}(-\lambda_n s^{\alpha})$ (e.g. Theorem 1.6 (p . 35) in [1]), we have $|E_{\alpha,\alpha}(-\lambda_n s^{\alpha})|\le C$ for all $n\in\mathbb{N}$ and $s>0$. Hence, Young's inequality yields

$$\|s^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n s^{\alpha})*h\|_{L^2(0,T)}\le\|s^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n s^{\alpha})\|_{L^1(0,T)}\|h\|_{L^2(0,T)}\le C\|h\|_{L^2(0,T)}.$$

Since $C^{-1}n^2\le\lambda_n\le Cn^2$ for all $n\in\mathbb{N}$ (e.g. [28]), we can obtain

$$\left|\left\langle\sum_{n=1}^{\infty}\frac{1}{\rho_n}\left(\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^{\alpha})h(s)\,\mathrm{d}s\right)\varphi_n,\ \psi\right\rangle_{L^2((0,1)\times(0,T))}\right|$$
$$\le C\sum_{n=1}^{\infty}\frac{1}{n^2}\|h\|_{L^2(0,T)}\le C\|h\|_{L^2(0,T)}$$

for all $\psi\in C_0^{\infty}((0,1)\times(0,T))$.

Therefore, setting $h:=g-g_\ell$, we see that

$$\sum_{n=1}^{\infty}\frac{1}{\rho_n}\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^{\alpha})g_\ell(s)\,\mathrm{d}s\,\varphi_n\ -\frac{x^2}{2}g_\ell(t)$$
$$\longrightarrow\sum_{n=1}^{\infty}\frac{1}{\rho_n}\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^{\alpha})g(s)\,\mathrm{d}s\,\varphi_n\ -\frac{x^2}{2}g(t)\quad\text{in }(C_0^{\infty}((0,1)\times(0,T)))'$$

as $\ell\to\infty$.

In view of (13), the convergence is in $H_\alpha(0,T;L^2(0,1))\cap L^2(0,T;H^2(0,1))$, and both limits in (13) and the above must coincide. Hence,

$$v_g(x,t)=\sum_{n=1}^{\infty}\frac{1}{\rho_n}\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^{\alpha})g(s)\,\mathrm{d}s\,\varphi_n\ -\frac{x^2}{2}g(t)$$

in $H_\alpha(0,T;L^2(0,1))\cap L^2(0,T;H^2(0,1))$. Since $u_{p,\alpha}(x,t)=v_g(x,t)+\frac{x^2}{2}g(t)$ by (7), the proof of Proposition 2.1 is complete. ■

We conclude this section with the following lemma.

Lemma 2.2: *Let $K_{p,\alpha}(x,t)$ be defined by*

$$K_{p,\alpha}(x,t):=\sum_{n=1}^{\infty}\frac{\varphi_n(x)}{\rho_n}\int_0^t s^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n s^{\alpha})\,\mathrm{d}s=\sum_{n=1}^{\infty}\frac{\varphi_n(x)}{\lambda_n\rho_n}(1-E_{\alpha,1}(-\lambda_n t^{\alpha}))$$

for all $x\in[0,1]$ and $t\in[0,T]$.

Then,

(i) *The series is uniformly convergent in $x\in[0,1]$ and $t\in[0,T]$, and $K_{p,\alpha}(x,\cdot)\in L^{\infty}(0,\infty)$ and is analytic in $t>0$ for all $x\in[0,1]$.*

(ii)

$$\int_0^\xi u_{p,\alpha}(x,t)\,\mathrm{d}t = (K_{p,\alpha}(x,\cdot) * g)(\xi) \quad \textit{for all } x \in [0,1] \textit{ and } \xi \in [0,T].$$

Proof of (i).: In view of (12), we have

$$\int_0^t s^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n s^\alpha)\,\mathrm{d}s = \frac{1}{\lambda_n}\int_t^0 \frac{\mathrm{d}}{\mathrm{d}s}(E_{\alpha,1}(-\lambda_n s^\alpha))\,\mathrm{d}s = \frac{1}{\lambda_n}(1 - E_{\alpha,1}(-\lambda_n t^\alpha)). \tag{15}$$

Hence,

$$K_{p,\alpha}(x,t) = \sum_{n=1}^\infty \frac{1}{\lambda_n \rho_n}(1 - E_{\alpha,1}(-\lambda_n t^\alpha))\varphi_n(x), \quad 0 \le x \le 1,\ t > 0.$$

From Theorem 1.6 (p.35) in [1], we know that there exist constants $C > 0$ and $\theta_0 > 0$ such that

$$|E_{\alpha,1}(-\lambda_n z^\alpha)| \le C \quad \text{for all } n \in \mathbb{N} \text{ and } z \in \Sigma := \{z \in \mathbb{C};\ |\mathrm{Arg}\, z| < \theta_0\}.$$

We fix a small $\delta > 0$ arbitrarily. Since $\|\varphi_n\|_{H^\theta(0,1)} \le C\|A_p^{\frac{\theta}{2}}\varphi_n\|_{L^2(0,1)}$ with $0 < \theta < 2$, applying the Sobolev embedding Theorem and recalling $\rho_n = \|\varphi_n\|^2_{L^2(0,1)}$, we have

$$\|\varphi_n\|_{C[0,1]} \le C\|\varphi_n\|_{H^{\frac{1}{2}+\delta}(0,1)} \le C\|A_p^{\frac{1}{4}+\frac{\delta}{2}}\varphi_n\|_{L^2(0,1)} = C\lambda_n^{\frac{1}{4}+\frac{\delta}{2}}\sqrt{\rho_n}.$$

Hence, by (14), we obtain

$$\left|\frac{1}{\rho_n\lambda_n}(1 - E_{\alpha,1}(-\lambda_n z^\alpha))\varphi_n(x)\right| \le \frac{C}{\lambda_n\sqrt{\rho_n}}\lambda_n^{\frac{1}{4}+\frac{\delta}{2}}, \quad 0 \le x \le 1,\ z \in \Sigma,$$

and so

$$\sum_{n=1}^\infty \frac{1}{\lambda_n\rho_n}|(1 - E_{\alpha,1}(-\lambda_n z^\alpha))\varphi_n(x)| \le C\sum_{n=1}^\infty \frac{1}{\lambda_n^{\frac{3}{4}-\frac{\delta}{2}}} \le C\sum_{n=1}^\infty \frac{1}{n^{\frac{3}{2}-\delta}} < \infty, \quad 0 \le x \le 1,\ z \in \Sigma. \tag{16}$$

Here we used $\lambda_n \sim n^2$ (e.g.[28]). Since $E_{\alpha,1}(-\lambda_n z^\alpha)$ is analytic in $z \in \Sigma$, we can complete the proof of (i). ■

Proof of (ii).: Since the series in (9) is convergent in $L^2(0,T;H^2(0,1))$, by $H^2(0,1) \subset C[0,1]$, we see that

$$u_{p,\alpha}(x,t) = \int_0^t \left(\sum_{n=1}^\infty \frac{1}{\rho_n}(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)\varphi_n(x)\right) g(s)\,\mathrm{d}s$$

is convergent in $L^2(0,T;C[0,T])$. Therefore,

$$\int_0^\xi u_{p,\alpha}(x,t)\,\mathrm{d}t = \int_0^\xi \left\{\int_0^t \left(\sum_{n=1}^\infty \frac{1}{\rho_n}(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)\varphi_n(x)\right) g(s)\,\mathrm{d}s\right\} \mathrm{d}t$$

for all fixed $x \in [0,1]$. Exchanging the orders of the integrals and changing the variables $t \to \eta$: $\eta = t - s$, we obtain

$$\int_0^\xi \left(\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)g(s)\,\mathrm{d}s\right)\mathrm{d}t$$
$$= \int_0^\xi \left(\int_s^\xi (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)\,\mathrm{d}t\right) g(s)\,\mathrm{d}s = \int_0^\xi \left(\int_0^{\xi-s} \eta^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\eta^\alpha)\,\mathrm{d}\eta\right) g(s)\,\mathrm{d}s.$$

Hence by (15), we have verified (ii) and the proof of Lemma 2.2 is complete. ■

3. Proof of Theorem 1.2

We recall (5).

Let

$$\begin{cases} A_q w(x) = -\dfrac{d^2 w}{dx^2}(x) - q(x)w(x), \quad 0 < x < 1, \\ \mathcal{D}(A_q) = \left\{ w \in H^2(0,1);\ \dfrac{dw}{dx}(0) = \dfrac{dw}{dx}(1) = 0 \right\}. \end{cases}$$

We let $0 < \mu_1 < \mu_2 < \cdots$, denote all the eigenvalues of the operator A_q and let ψ_n, $n \in \mathbb{N}$ be the corresponding eigenfunction for μ_n, that is $\psi_n \in \mathcal{D}(A_q)$ satisfies $A_q\psi_n = \mu_n\psi_n$ in $(0,1)$ and we take the normalisation of the eigenfunctions to be $\psi_n(1) = 1$. Given this, we set $\sigma_n := \|\psi_n\|^2$, for $n \in \mathbb{N}$. Similarly to the analysis of Lemma 2.2, we define

$$K_{q,\beta}(1,t) = \sum_{n=1}^{\infty} \frac{1}{\sigma_n} \int_0^t s^{\beta-1} E_{\beta,\beta}(-\mu_n s^\beta)\, ds, \quad t > 0.$$

It is sufficient to prove the theorem with data $u_{p,\alpha}(1,t) = u_{q,\beta}(1,t)$, $0 < t < T$. For the other case at $x = 0$, replacing the conditions $\varphi_n(1) = \psi_n(1) = 1$ by $\varphi_n(0) = \psi_n(0) = 1$, we can repeat the whole argument and thus omit the details for this case.

Since $\int_0^\xi u_{p,\alpha}(1,t)\, dt = \int_0^\xi u_{q,\beta}(1,t)\, dt$ by $u_{p,\alpha}(1,t) = u_{q,\beta}(1,t)$ for $0 < t < T$, in view of Lemma 2.2, we see

$$(K_{p,\alpha}(1,\cdot) * g)(\xi) = (K_{q,\beta}(1,\cdot) * g)(\xi), \quad 0 < \xi < T,$$

that is,

$$((K_{p,\alpha} - K_{q,\beta})(1,\cdot) * g)(t) = 0, \quad 0 < t < T.$$

Since $g \not\equiv 0$, we apply the Titchmarsh convolution theorem (e.g.[29]), so that there exists $t_0 > 0$ such that

$$K_{p,\alpha}(1,t) = K_{q,\beta}(1,t), \quad 0 < t < t_0.$$

Lemma 2.2 implies that $K_{p,\alpha}(1,t)$ and $K_{q,\beta}(1,t)$ are analytic in $t > 0$, and so

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n\rho_n}(1 - E_{\alpha,1}(-\lambda_n t^\alpha)) = \sum_{n=1}^{\infty} \frac{1}{\mu_n\sigma_n}(1 - E_{\beta,1}(-\mu_n t^\beta)), \quad t > 0.$$

By the asymptotics of $E_{\alpha,1}(-\eta)$ and $E_{\beta,1}(-\eta)$ for large $\eta > 0$ (e.g. Theorem 1.4 (pp . 33–34) in [1]), we have

$$E_{\alpha,1}(-\lambda_n t^\alpha) = \frac{1}{\Gamma(1-\alpha)} \frac{1}{\lambda_n t^\alpha} + O\left(\frac{1}{t^{2\alpha}}\right)$$

and

$$E_{\beta,1}(-\mu_n t^\beta) = \frac{1}{\Gamma(1-\beta)} \frac{1}{\mu_n t^\beta} + O\left(\frac{1}{t^{2\beta}}\right)$$

for all large $t > 0$. Hence

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n\rho_n} - \frac{1}{\Gamma(1-\alpha)} \sum_{n=1}^{\infty} \frac{1}{\lambda_n\rho_n} \frac{1}{\lambda_n t^\alpha} + O\left(\frac{1}{t^{2\alpha}}\right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{\mu_n \sigma_n} - \frac{1}{\Gamma(1-\beta)} \sum_{n=1}^{\infty} \frac{1}{\mu_n \sigma_n} \frac{1}{\mu_n t^{\beta}} + O\left(\frac{1}{t^{2\beta}}\right)$$

for large $t > 0$. Letting $t \to \infty$, we obtain

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n \rho_n} = \sum_{n=1}^{\infty} \frac{1}{\mu_n \sigma_n}.$$

Assume that $\alpha > \beta$. Then

$$-\frac{1}{\Gamma(1-\alpha)} \sum_{n=1}^{\infty} \frac{1}{\lambda_n \rho_n} \frac{1}{\lambda_n t^{\alpha-\beta}} + O\left(\frac{1}{t^{2\alpha-\beta}}\right) = -\frac{1}{\Gamma(1-\beta)} \sum_{n=1}^{\infty} \frac{1}{\mu_n \sigma_n} \frac{1}{\mu_n} + O\left(\frac{1}{t^{\beta}}\right)$$

for large $t > 0$. Letting $t \to \infty$, we obtain

$$\frac{1}{\Gamma(1-\beta)} \sum_{n=1}^{\infty} \frac{1}{\mu_n^2 \sigma_n} = 0.$$

Since $\sigma_n = \|\psi_n\|^2 > 0$, this is impossible. Hence $\alpha \le \beta$. By an entirely similar argument we see that $\alpha < \beta$ is impossible and so conclude that $\alpha = \beta$.

Now we move to complete the proof of the theorem. We see

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n \rho_n} E_{\alpha,1}(-\lambda_n t^{\alpha}) = \sum_{n=1}^{\infty} \frac{1}{\mu_n \sigma_n} E_{\alpha,1}(-\mu_n t^{\alpha}), \quad t > 0. \tag{17}$$

Now we can argue similarly to [5]. Using

$$\left| \frac{1}{\lambda_n \rho_n} E_{\alpha,1}(-\lambda_n t^{\alpha}) \right| \le \frac{C}{\lambda_n}, \quad n \in \mathbb{N},\ t > 0,$$

we see that the series in (17) are convergent uniformly in $[0, \infty)$. Therefore we can take the Laplace transforms termwise to have

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n \rho_n} \int_0^{\infty} e^{-\zeta t} E_{\alpha,1}(-\lambda_n t^{\alpha})\, \mathrm{d}t = \sum_{n=1}^{\infty} \frac{1}{\mu_n \sigma_n} \int_0^{\infty} e^{-\zeta t} E_{\alpha,1}(-\mu_n t^{\alpha})\, \mathrm{d}t, \quad \zeta > 0.$$

By formula (1.80) (p . 21) in [1], we obtain

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n \rho_n} \frac{\zeta^{\alpha-1}}{\zeta^{\alpha} + \lambda_n} = \sum_{n=1}^{\infty} \frac{1}{\mu_n \sigma_n} \frac{\zeta^{\alpha-1}}{\zeta^{\alpha} + \mu_n}, \quad \zeta > 0.$$

Dividing by $\zeta^{\alpha-1}$ and setting $\eta = \zeta^{\alpha}$, we have

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n \rho_n} \frac{1}{\eta + \lambda_n} = \sum_{n=1}^{\infty} \frac{1}{\mu_n \sigma_n} \frac{1}{\eta + \mu_n}, \quad \eta > 0. \tag{18}$$

Since $\lambda_n \sim n^2$ and $\mu_n \sim n^2$ for large $n \in \mathbb{N}$, we see that both sides of (18) are convergent uniformly in any compact set in $\mathbb{C} \setminus (\{-\lambda_n\}_{n\in\mathbb{N}} \cup \{-\mu_n\}_{n\in\mathbb{N}})$ and are analytic in $\mathbb{C} \setminus (\{-\lambda_n\}_{n\in\mathbb{N}} \cup \{-\mu_n\}_{n\in\mathbb{N}})$.

Assume that $\lambda_m \notin \{\mu_n\}_{n\in\mathbb{N}}$ for $m \in \mathbb{N}$. Then we can choose a small circle C_m centred at $-\lambda_m$ and $\{-\mu_n\}_{n\in\mathbb{N}}$ is not included in the disk centred at $-\lambda_n$ bounded by C_m. Integrating on C_m and applying the Cauchy theorem, we have

$$\frac{2\pi\sqrt{-1}}{\lambda_m \rho_m} = 0,$$

which is impossible. Hence $\lambda_m \in \{\mu_n\}_{n\in\mathbb{N}}$ for each m. Similarly $\mu_m \in \{\lambda_n\}_{n\in\mathbb{N}}$ for each $m \in \mathbb{N}$. Therefore

$$\lambda_n = \mu_n, \quad n \in \mathbb{N}. \tag{19}$$

By (18), we have

$$\sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n \rho_n} - \frac{1}{\lambda_n \sigma_n} \right) \frac{1}{\eta + \lambda_n} = 0, \quad \eta \in \mathbb{C} \setminus \{-\lambda_n\}_{n\in\mathbb{N}}.$$

Again integrating on C_m, we obtain

$$\frac{2\pi\sqrt{-1}}{\lambda_n} \left(\frac{1}{\rho_n} - \frac{1}{\sigma_n} \right) = 0,$$

that is,

$$\rho_n = \sigma_n, \quad n \in \mathbb{N}. \tag{20}$$

Now, using (19) and (20), we apply the Gel'fand–Levitan theory (e.g. [30]), and we can obtain $p(x) = q(x)$ for $0 < x < 1$. The application is similar to [3–5], and so we omit the details. Thus the proof of Theorem 1.2 is complete.

4. Application of the representation formula

The representation formula Proposition 2.1 is useful for qualitative analyses of fractional equations. Here we explain one application.

We let $0 < \alpha < 1$ and we fix $p \in C[0,1]$, ≤ 0 on $[0,1]$. Let

$$\begin{cases} \partial_t^\alpha u(x,t) = \partial_x^2 u(x,t) + p(x)u(x,t), & 0 < x < 1,\ 0 < t < T, \\ u(0,t) = \partial_x u(0,t) = 0, & 0 < t < T, \\ u \in H_\alpha(0,T; L^2(0,1)). \end{cases} \tag{21}$$

Then we are interested in the question: can we conclude $u(x,t) = 0$ for $0 < x < 1$ and $0 < t < T$?

This is a kind of unique continuation property under the assumption $u \in H_\alpha(0,T;L^2(0,1))$ which can be interpreted as that an initial value of u is zero. This kind of unique continuation was proved by Cheng et al. [31] for $\alpha = \frac{1}{2}$, Lin and Nakamura [32] for $\alpha \in (0,1)$ and Lin and Nakamura [33] for $\alpha \in (0,1) \cup (1,2)$ for general time-fractional partial differential equations. Their proofs are based on the techniques of pseudo-differential operators.

For $\alpha = 1$, we can prove the unique continuation without any information of initial conditions, and the corresponding unique continuation is proved for a one-dimensional time-fractional equation by Li and Yamamoto [34]. More precisely, if u is in a suitable class and satisfies

$$\begin{cases} \partial_t^\alpha u(x,t) = \partial_x^2 u(x,t), & 0 < x < 1,\ 0 < t < T, \\ u(0,t) = \partial_x u(0,t) = 0, & 0 < t < T, \end{cases}$$

then $u(x,t) = 0$ for $0 < x < 1$ and $0 < t < T$.

However, such unique continuation not requiring any initial conditions, is not known for general case in multidimension.

In this section, for the one-dimensional case (21), we provide a simpler proof than [31–33], which relies on the representation formula Proposition 2.1.

Proposition 4.1: *Let $u \in H_\alpha(0,T;L^2(0,1)) \cap L^2(0,T;H^2(0,1))$ satisfy (21) and $\partial_x u(1,\cdot) \in H_\alpha(0,T)$. Then $u(x,t) = 0$, $0<x<1$, $0<t<T$.*

By the definition of $H_\alpha(0,T)$ given in Section 1, if $0 < \alpha < \frac{1}{2}$, then $H_\alpha(0,T) = H^\alpha(0,T)$ and in (21) the condition $u \in H_\alpha(0,T;L^2(0,1))$ does not require anything for the behaviour of the solution u near $t = 0$. In other words, we need not pose any conditions at $t = 0$ to u.

For $\frac{1}{2} < \alpha < 1$, the condition $\mu \in H_\alpha(0,T;L^2(0,1))$ requires $\mu(.,0) = 0$ in the sense of trace.

It seems that we can remove a condition $\partial_x u(1,\cdot) \in H_\alpha(0,T)$, but we here omit the details.

Proof: We set $g := \partial_x u(1,\cdot) \in H_\alpha(0,T)$. Then u satisfies (4) and $u(0,t) = 0$, $0<t<T$. Lemma 2.2 (ii) implies

$$(K_{p,\alpha}(0,\cdot) * g)(t) = 0, \quad 0 < t < T.$$

By the Titchmarsh theorem on the convolution (e.g.[29]), there exist $t_1, t_2 \geq 0$ such that

$$\begin{cases} t_1 + t_2 = T, \\ K_{p,\alpha}(0,t) = 0, \quad 0 \leq t \leq t_1, \\ g(t) = 0, \quad 0 \leq t \leq t_2. \end{cases} \tag{22}$$

Assuming that $t_1 > 0$, we will derive a contradiction, which proves $t_1 = 0$, that is, $g = 0$ in $(0,T)$. The argument is similar to the proof of Theorem 1.2.

The analyticity of $K_{p,\alpha}(0,t)$ in $t>0$ yields $K_{p,\alpha}(0,t) = 0$ for all $t>0$.

Since $\lim_{t\to\infty} E_{\alpha,1}(-\lambda_n t^\alpha) = 0$ (e.g. Theorem 1.6 (p . 35) in [1]), we have

$$\lim_{t\to\infty} K_{p,\alpha}(0,t) = \sum_{n=1}^{\infty} \frac{\varphi_n(0)}{\lambda_n \rho_n} = 0.$$

Hence

$$\sum_{n=1}^{\infty} \frac{\varphi_n(0)}{\lambda_n \rho_n} E_{\alpha,1}(-\lambda_n t^\alpha) = 0, \quad t > 0.$$

This series is convergent in $L^\infty(0,\infty)$ and so we can take the Laplace transform term by term. In view of formula (1.80) (p . 21) in [1], we obtain

$$\sum_{n=1}^{\infty} \frac{\varphi_n(0)}{\lambda_n \rho_n} \frac{z^{\alpha-1}}{z^\alpha + \lambda_n} = 0, \quad \operatorname{Re} z > 0.$$

Then, dividing by $z^{\alpha-1}$ and setting $\eta = z^\alpha$, we have

$$\sum_{n=1}^{\infty} \frac{\varphi_n(0)}{\lambda_n \rho_n} \frac{1}{\eta + \lambda_n} = 0, \quad \operatorname{Re} \eta > 0.$$

Similarly to (16), we can verify

$$\sum_{n=1}^{\infty} \frac{\varphi_n(0)}{\lambda_n \rho_n} < \infty,$$

and so we can continue analytically in η as much as possible to obtain

$$\sum_{n=1}^{\infty} \frac{\varphi_n(0)}{\lambda_n \rho_n} \frac{1}{\eta + \lambda_n} = 0, \quad \eta \in \mathbb{C} \setminus \{-\lambda_n\}_{n \in \mathbb{N}}.$$

Choosing a small circle Γ_1 centred at $-\lambda_1$ such that the interior of the disk bounded by Γ_1 does not contain $-\lambda_n$ with $n \geq 2$ and integrating on Γ_1, in terms of the Cauchy theorem, we see

$$\frac{\varphi_1(0)}{\lambda_1 \rho_1} 2\pi\sqrt{-1} = 0,$$

that is, $\varphi_1(0) = 0$. Since $\frac{d^2\varphi_1}{dx^2}(x) + (p(x) + \lambda_1)\varphi_1(x) = 0, 0 < x < 1$ and $\frac{d\varphi_1}{dx}(0) = 0$, we have $\varphi_1(x) = 0$ for all $0 < x < 1$, which is impossible.

Then we can conclude that $t_1 = 0$. By (22), we reach $g(t) = 0$ for $0 < t < T$. Thus the proof of Proposition 4.1 is complete. ■

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

The first author was supported in part by the National Science Foundation through award DMS-2111020. The second author was supported by Grant-in-Aid for Scientific Research (S) 15H05740 and Grant-in-Aid (A) 20H00117 of Japan Society for the Promotion of Science and by The National Natural Science Foundation of China (no. 11771270, 91730303). This paper has been supported by the RUDN University Strategic Academic Leadership Program .

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