

Fock space on $\mathbb{C}^\infty$ and Bose-Fock space [☆]Brett D. Wick ^a, Shengkun Wu ^{b,*}^a Department of Mathematics and Statistics, Washington University in St. Louis, MO 63130, USA^b College of Mathematics and Statistics, Chongqing University, Chongqing, 401331, PR China

ARTICLE INFO

Article history:

Received 14 March 2021

Available online 15 July 2021

Submitted by J. Bonet

Keywords:

Bose-Fock space

Fock space

Gibbs state

ABSTRACT

In this paper, we introduce the Fock space on $\mathbb{C}^\infty$ and obtain an isomorphism between the Fock space on $\mathbb{C}^\infty$ and Bose-Fock space. Based on this isomorphism, we obtain representations of some operators on the Bose-Fock space and answer a question in [2]. As a physical application, we study the Gibbs state.

© 2021 Elsevier Inc. All rights reserved.

1. Introduction

In [1], Bargman introduced the Fock space on $\mathbb{C}^n$ and discussed its connection with quantum mechanics. In [2], Berger and Coburn studied the operators on the Fock space on $\mathbb{C}^n$. In the last section of that paper, the authors asked a question: can the analysis in this paper be applied in the physically interesting case where $\mathbb{C}^n$ is replaced by an infinite-dimensional Hilbert space? However, in this paper, we will use $\mathbb{C}^\infty$ to replace $\mathbb{C}^n$ instead of infinite-dimensional Hilbert space. By this replacement, we will show that the Fock space on $\mathbb{C}^\infty$ is isomorphic to the Bose-Fock space. Then, we are going to generalize some conclusions in [2] and give a physical application.

The Bose-Fock space is used to describe the states of bosons in quantum mechanics, for the details of this space we refer to [4]. If $\mathcal{H}$ is a separable Hilbert space, the Full Fock space over $\mathcal{H}$ is the complete tensor algebra over $\mathcal{H}$:

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{k=0}^{\infty} (\otimes^k \mathcal{H})$$

[☆] Brett D. Wick's research is supported in part by a National Science Foundation DMS grants # 1560955 and # 1800057 and Australian Research Council – DP 190100970. Shengkun Wu's research is supported by CSC201906050022.

* Corresponding author.

E-mail addresses: wick@math.wustl.edu (B.D. Wick), shengkunwu@foxmail.com (S. Wu).

where $\otimes^k \mathcal{H}$ is the k th tensor power of $\mathcal{H}$ for $k \geq 1$ and $\otimes^0 \mathcal{H} = \mathbb{C}$. We define the projection on the Full Fock space over $\mathcal{H}$ by

$$P_+(u_1 \otimes \cdots \otimes u_k) = \frac{1}{k!} \sum_{\sigma} u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(k)},$$

where σ ranges over the group of permutations of k letters. The Bose-Fock space $\mathcal{F}_+(\mathcal{H})$ consists of all symmetric tensors, that is to say

$$\mathcal{F}_+(\mathcal{H}) = P_+ \bigoplus_{k=0}^{\infty} (\otimes^k \mathcal{H}).$$

It is easy to verify that if $\{h_j\}$ is an orthonormal basis for $\mathcal{H}$, then

$$\left\{ E_{\alpha} = \sqrt{\frac{k!}{\alpha!}} P_+(h_1^{\alpha_1} \otimes h_2^{\alpha_2} \otimes \cdots) : \alpha = (\alpha_1, \dots, \alpha_n, \dots), \sum \alpha_j = k, \quad k = 0, 1, 2, \dots \right\}$$

is an orthonormal basis for $\mathcal{F}_+(\mathcal{H})$, where the superscripts α_j denote tensor powers. For any $h \in \mathcal{H}$, the annihilation operator $a(h)$ and the creation operator $a^*(h)$ on the Fock space over $\mathcal{H}$ are given by

$$\begin{aligned} a(h)(h_1 \otimes h_2 \otimes \cdots \otimes h_n) &= n^{1/2} (h, h_1) h_2 \otimes h_3 \otimes \cdots \otimes h_n \\ a^*(h)(h_1 \otimes h_2 \otimes \cdots \otimes h_n) &= (n+1)^{1/2} h \otimes h_1 \otimes \cdots \otimes h_n. \end{aligned}$$

By definition, we know that the creation operators map $\otimes^n \mathcal{H}$ to $\otimes^{n+1} \mathcal{H}$ and the annihilation operators map $\otimes^n \mathcal{H}$ to $\otimes^{n-1} \mathcal{H}$. The annihilation operator $a_+(h)$ and the creation operator $a_+^*(h)$ on the Bose-Fock space are given by

$$a_+(f) = P_+ a(f) P_+ \quad \text{and} \quad a_+^*(f) = P_+ a^*(f) P_+,$$

then we have

$$P_+(h_1 \otimes h_2 \otimes \cdots \otimes h_n) = (n!)^{-1/2} a_+^*(h_1) a_+^*(h_2) \cdots a_+^*(h_n) \Omega, \quad (1.1)$$

where $\Omega = (1, 0, 0, \dots) \in \otimes^0 \mathcal{H}$. Next, we introduce the Weyl operators on the Bose-Fock space. Let

$$W(h) = \exp\{i\Phi(h)\}$$

be the Weyl operator, where

$$\Phi(h) = 2^{-1/2} \overline{(a_+(h) + a_+^*(h))}.$$

The Weyl algebra $\text{CCR}(\mathcal{H})$ is a C^* -algebra generated by

$$\{W(h) : h \in \mathcal{H}\},$$

where CCR stands for the canonical commutation relations, see [4, pg 10]. If H is an unbounded selfadjoint operator on $\mathcal{H}$, one can define H_n on $P_+(\otimes^k \mathcal{H})$ by setting $H_0 = I$ and

$$H_n(P_{\pm}(f_1 \otimes \cdots \otimes f_n)) = P_{\pm} \left(\sum_{i=1}^n f_1 \otimes f_2 \otimes \cdots \otimes H f_i \otimes \cdots \otimes f_n \right)$$

for all $f_i \in D(H)$, and then extending by continuity. The selfadjoint closure of this sum is called the second quantization of H and is denoted by $d\Gamma(H)$. Thus

$$d\Gamma(H) = \overline{\bigoplus_{n \geq 0} H_n}.$$

Let Gaussian measure $d\lambda_n$ on $\mathbb{C}^n$ be given by

$$d\lambda_n(z) = \frac{1}{(2\pi)^n} e^{-\frac{|z|^2}{2}} dz.$$

The Fock space on $\mathbb{C}^n$, denoted by $F^2(\mathbb{C}^n, d\lambda_n)$ or $F^2(\mathbb{C}^n)$, consists of all entire functions on $\mathbb{C}^n$ which are square-integrable with respect to $d\lambda_n$.

For any nonnegative integer k , let

$$e_k(w) = \sqrt{\frac{1}{2^k k!}} w^k, w \in \mathbb{C}.$$

We have $e_1(z)e_k(z) = \sqrt{k+1}e_{k+1}(z)$. Then the set $\{e_k\}$ is an orthonormal basis for $F^2(\mathbb{C}, d\lambda_1)$.

The Gaussian measure can be extended on $\mathbb{C}^\infty$, we denote it by $d\lambda_\infty$. $L^2(\mathbb{C}^\infty, d\lambda_\infty)$ consists of all square-integrable function on $\mathbb{C}^\infty$ with respect to the infinite dimensional Gaussian measure $d\lambda_\infty$. Let l_j be the complex linear functional such that for any $z = (z_1, \dots, z_j, \dots)$, we have

$$l_j(z) = z_j.$$

Let $e_k \circ l_j$ be a function on $\mathbb{C}^\infty$, such that

$$e_k \circ l_j(z) = e_k(z_j) = \sqrt{\frac{1}{2^k k!}} z_j^k.$$

The Fock space on $\mathbb{C}^\infty$ is defined to be a closed subspace of $L^2(\mathbb{C}^\infty, d\lambda_\infty)$ generated by the orthonormal set

$$\left\{ e_{\alpha_1} \circ l_1 \times e_{\alpha_2} \circ l_2 \times \dots \times e_{\alpha_m} \circ l_m \dots : \sum_m \alpha_m = k, \quad k = 0, 1, 2, \dots \right\}$$

and is denoted by $F^2(\mathbb{C}^\infty, d\lambda_\infty)$. In fact, $F^2(\mathbb{C}^n)$ can be regarded as a closed subspace of $F^2(\mathbb{C}^\infty, d\lambda_\infty)$, the embedding is given by

$$f(z_1, \dots, z_n) \rightarrow f(z_1, \dots, z_n, 0, 0, \dots), \text{ for any } f \in F^2(\mathbb{C}^n).$$

In fact, we have a sequence of embeddings

$$F^2(\mathbb{C}^1) \subset \dots \subset F^2(\mathbb{C}^n) \subset \dots \subset F^2(\mathbb{C}^\infty, d\lambda_\infty).$$

Let P_n be the projection from $F^2(\mathbb{C}^\infty, d\lambda_\infty)$ to $F^2(\mathbb{C}^n)$. Since the set of finite polynomials is dense in the Fock space on $\mathbb{C}^\infty$, we have

$$\bigcup_n F^2(\mathbb{C}^n, d\lambda_n) \text{ is dense in } F^2(\mathbb{C}^\infty, d\lambda_\infty).$$

We define an isomorphism B from $\mathcal{F}_+(\mathcal{H})$ to $F^2(\mathbb{C}^\infty, d\lambda_\infty)$ such that

$$Bc = c \text{ when } c \in \otimes^0 \mathcal{H} = \mathbb{C}$$

and

$$B\left[\sqrt{\frac{k!}{\alpha!}}P_+(h_1^{\alpha_1} \otimes \cdots \otimes h_n^{\alpha_n})\right] = e_{\alpha_1} \circ l_1 \cdots e_{\alpha_n} \circ l_n \quad (1.2)$$

when $(h_1^{\alpha_1} \otimes \cdots \otimes h_n^{\alpha_n}) \in \otimes^k \mathcal{H}$ with $\sum_{m=1}^n \alpha_m = k \neq 0$. We need to point out that the isomorphism B depends on the basis $\{h_j\}$. We call B as infinite Bargmann representation.

Let P denote the projection from $L^2(\mathbb{C}^\infty, d\lambda_\infty)$ to $F^2(\mathbb{C}^\infty, d\lambda_\infty)$. We define the Toeplitz operator with symbol $\phi \in L^2(\mathbb{C}^\infty, d\lambda_\infty)$ by

$$T_\phi h = P(\phi h)$$

for all $h \in F^2(\mathbb{C}^\infty, d\lambda_\infty)$ such that $\phi h \in L^2(\mathbb{C}^\infty, d\lambda_\infty)$. In Section 2, we are going to use some facts of infinite dimensional Gaussian measures to show that the Toeplitz operators on $F^2(\mathbb{C}^\infty, d\lambda_\infty)$ are unitary equivalent to the annihilation, creation and Weyl operators on the Bose-Fock space, which is the generalization of some conclusions in [2]. This equivalence can be used to translate some problems in the Bose-Fock space to $F^2(\mathbb{C}^\infty, d\lambda_\infty)$. In [5] and [6], the authors obtained some similar conclusions. In fact, they defined the Fock space on a metric space with some properties and obtained representations. However, their representation of a Weyl operator is a strong limit of a sequence of Toeplitz operators. $\mathbb{C}^\infty$ is a spacial case of such metric space, we will show that, in this situation, Weyl operators can be represented as Toeplitz operators. For the sake of completeness, we give all details here.

We need to point out that the reason that we care about the Fock space on $\mathbb{C}^\infty$ is that it gives us an application. Since $\bigcup_n F^2(\mathbb{C}^n, d\lambda_n)$ is dense in $F^2(\mathbb{C}^\infty, d\lambda_\infty)$, the problems in $F^2(\mathbb{C}^\infty, d\lambda_\infty)$ can be reduced to $F^2(\mathbb{C}^n, d\lambda_n)$. In Section 3, we will use this idea to discuss a problem in Quantum Statistical Mechanics. We will study the trace formula which will be applied to the Gibbs state. The Gibbs state of an operator A on the Bose-Fock space is

$$\omega(A) = \frac{\text{Tr}(e^{-\beta d\Gamma(H-\mu I)} A)}{\text{Tr}(e^{-\beta d\Gamma(H-\mu I)})},$$

where $e^{-\beta d\Gamma(H-\mu I)}$ is an operator and its definition will be given in Section 3. The Gibbs state is an important quantity in the Quantum Statistical Mechanics, in fact, it is just the trace of an operator on the Bose-Fock space, see [4]. Because we know the trace formula in the Fock space on $\mathbb{C}^n$, we can generalize the trace formula and apply it to the Gibbs state. In the theory of Many-Body Problems, some operators can be represented by the linear combination of products of annihilation and creation operators, see [9, Chapter 1]. So, it is important to study the product of annihilation and creation operators. Given

$$f^{(1)}, \dots, f^{(m)}, g^{(1)}, \dots, g^{(m)} \in \mathcal{H},$$

we will study the Gibbs state of the operator

$$a_+^*(f^{(1)}) \cdots a_+^*(f^{(m)}) a_+(g^{(1)}) \cdots a_+(g^{(m)}),$$

which is the generalization of [4, Proposition 5.2.28].

In Section 4, we will discuss the relationship between the Fock space on $\mathbb{C}^n$ and the Gaussian Harmonic analysis. In [11], the authors gave an isomorphism between the Fock-Sobolev space on $\mathbb{C}^n$ and the

Gauss-Sobolev space over $\mathbb{R}^n$. We will generalize this isomorphism in the infinite dimensional case. As an application, we study the boundedness of the annihilation and creation operators.

2. Annihilation, creation and Weyl operators

Let χ_k be the subspace of the Fock space on $\mathbb{C}^n$ generated by

$$\left\{ e_{\alpha_1} \circ l_1 \cdots e_{\alpha_m} \circ l_m \cdots : \sum_j \alpha_j = k \right\}.$$

Then, for any $h = \sum c_j h_j \in \mathcal{H}$, we have $Bh = \sum_j c_j e_1 \circ l_j$. By $P_+ h_m = h_m$ and (1.2), we have

$$Bh_m(z) = e_1(z_m).$$

Thus, we know that the isomorphism B maps $\mathcal{H}$ to χ_1 .

Next, we are going to show that the annihilation operators and creation operators are isomorphic to the Toeplitz operators with symbols in χ_1 .

Proposition 2.1. *For any $h \in \mathcal{H}$, let $a_+(h)$ and $a_+^*(h)$ be an annihilation operator and a creation operator, we have*

$$Ba_+^*(h)B^{-1} = T_{Bh} \quad \text{and} \quad Ba_+(h)B^{-1} = T_{\overline{Bh}}.$$

Proof. For any $h = \sum c_j h_j \in \mathcal{H}$, we have

$$Bh = \sum c_j e_1 \circ l_j.$$

Thus, for any $e_{\alpha_1} \circ l_1 \cdots e_{\alpha_m} \circ l_m \cdots$ with $\sum \alpha_j = k$, we have

$$\begin{aligned} & B^{-1}T_{Bh}e_{\alpha_1} \circ l_1 \cdots e_{\alpha_m} \circ l_m \cdots \\ &= B^{-1} \sum_j c_j e_1 \circ l_j \times e_{\alpha_1} \circ l_1 \cdots e_{\alpha_m} \circ l_m \cdots \\ &= B^{-1} \sum_j c_j \sqrt{\alpha_j + 1} e_{\alpha_1} \circ l_1 \cdots e_{\alpha_j+1} \circ l_j \cdots e_{\alpha_m} \circ l_m \cdots \\ &= \sum_j c_j \sqrt{\alpha_j + 1} \sqrt{\frac{(k+1)!}{\alpha_1! \cdots (\alpha_j+1)! \cdots}} P_+ \left(h_1^{\alpha_1} \otimes \cdots \otimes h_j^{\alpha_j+1} \otimes \cdots \right) \\ &= \sum_j c_j \sqrt{\frac{(k+1)!}{\alpha!}} P_+ \left(h_1^{\alpha_1} \otimes \cdots \otimes h_j^{\alpha_j+1} \otimes \cdots \right). \end{aligned}$$

On the other hand

$$\begin{aligned} a_+^*(h)B^{-1}e_{\alpha_1} \circ l_1 \cdots e_{\alpha_m} \circ l_m \cdots &= a_+^*(h) \sqrt{\frac{k!}{\alpha!}} P_+ (h_1^{\alpha_1} \otimes h_2^{\alpha_2} \otimes \cdots) \\ &= \sqrt{\frac{1}{\alpha!}} [a_+^*(h)a_+^*(h_1)^{\alpha_1} \cdots a_+^*(h_n)^{\alpha_n} \cdots] \Omega \quad (\text{by (1.1)}) \\ &= \sqrt{\frac{(k+1)!}{\alpha!}} P_+ (h \otimes h_1^{\alpha_1} \otimes \cdots h_n^{\alpha_n} \cdots) \end{aligned}$$

$$= \sqrt{\frac{(k+1)!}{\alpha!}} \sum c_j P_+(h_j \otimes h_1^{\alpha_1} \otimes \cdots h_n^{\alpha_n} \cdots),$$

which means that $Ba_+^*(h)B^{-1} = T_{Bh}$. Thus we have $Ba_+(h)B^{-1} = T_{\overline{Bh}}$ by taking adjoints. $\square$

We are going to give a representation of the Weyl operators, thus we need some facts about infinite dimensional Gaussian measure. For these facts about infinite dimensional Gaussian measure, we refer to [3, Chapter 2]. Because we need a particular theorem in [3], we give the details about the general theory.

A Borel probability measure γ on $\mathbb{R}^1$ is called Gaussian if it is either the Dirac measure δ_a at a point a or has density

$$p(\cdot, a, \sigma^2) : t \mapsto \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(t-a)^2}{2\sigma^2}\right)$$

with respect to the Lebesgue measure.

Let X be a locally convex space. Let X^* be the set of real linear continuous functionals on X . Let us denote by $\mathcal{E}(X, X^*)$ the minimal σ -field of subsets of Ω , with respect to which all functionals $f \in X^*$ are measurable. A probability measure γ defined on the σ -field $\mathcal{E}(X, X^*)$ is called Gaussian if, for any $f \in X^*$, the induced measure $\gamma \circ f^{-1}$ on $\mathbb{R}^1$ is Gaussian. Let

$$a_\gamma(f) = \int_X f(x) \gamma(dx).$$

We denote by X_γ^* the closure of the set

$$\{f - a_\gamma(f), f \in X^*\}$$

embedded into $L^2(\gamma)$, with respect to the norm of $L^2(\gamma)$. We define

$$\langle f, g \rangle_{X_\gamma^*} := \int_X [f(x) - a_\gamma(f)][g(x) - a_\gamma(g)] \gamma(dx)$$

For any $x \in X$, let

$$|h|_{X(\gamma)} = \sup \left\{ \langle l, h \rangle : l \in X_\gamma^*, \langle l, l \rangle_{X_\gamma^*} \leq 1 \right\}.$$

The space

$$X(\gamma) = \{h \in X : |h|_{X(\gamma)} < \infty\}$$

is called the Cameron-Martin space for X .

Lemma 2.2 ([3, Lemma 2.4.1]). *A vector x in X belongs to the Cameron-Martin space $X(\gamma)$ precisely when there exists $\hat{x} \in X_\gamma^*$ such that $f(x) = \langle \hat{x}, f \rangle_{X_\gamma^*}$ for any $f \in X^*$. In this case,*

$$|x|_{X(\gamma)} = \|\hat{x}\|_{L^2(\gamma)}.$$

Lemma 2.3 ([3, Lemma 2.4.4]). *Let γ be a Gaussian measure on a locally convex space X . If $x \in X(\gamma)$ and $\hat{x} \in X_\gamma^*$ satisfy*

$$f(x) = \langle \hat{x}, f \rangle_{X_\gamma^*}$$

for any $f \in X^*$, then the measures γ and $\gamma_x = \gamma(\cdot - x)$ are equivalent and the corresponding Radon-Nikodym density is given by the expression

$$d\gamma_x(z) = \exp\left(\hat{x}(z) - \frac{1}{2}|x|_{X(\lambda)}^2\right) d\gamma(z).$$

Next, we discuss a special case when $X = \mathbb{C}^\infty$ and $d\gamma = d\lambda_\infty$. Let $\Re l_k(z) = \Re z_k$ and $\Im l_k(z) = \Im z_k$. Then any real linear functional $f \in (\mathbb{C}^\infty)^*$ is a finite linear combination of $\Re l_k$ and $\Im l_k$, see [8, Theorem 4.3]. That is to say that there is a unique sequence $\{f_k = a_k + ib_k \in \mathbb{C}\}$ such that

$$f = \sum_{k=1}^n a_k \Re l_k + \sum_{k=1}^n b_k \Im l_k = \sum_{k=1}^n \Re f_k \Re l_k + \sum_{k=1}^n \Im f_k \Im l_k = \Re \sum_{k=1}^n \overline{f_k} l_k.$$

Thus, we have

$$a_{d\lambda_\infty}(f) = \int_{\mathbb{C}^\infty} f(z) d\lambda_\infty(z) = 0.$$

For any

$$f = \Re \sum_{k=1}^n \overline{f_k} l_k \quad \text{and} \quad g = \Re \sum_{k=1}^n \overline{g_k} l_k \in (\mathbb{C}^\infty)^*,$$

we have

$$\langle f, g \rangle_{(\mathbb{C}^\infty)^*_{d\lambda_\infty}} = \int_{\mathbb{C}^\infty} f(z) g(z) d\lambda_\infty(z) = \sum_{k=1}^n (\Re f_k \Re g_k + \Im f_k \Im g_k) = \Re \sum_{k=1}^n \overline{f_k} g_k. \quad (2.1)$$

Since $(\mathbb{C}^\infty)^*_{d\lambda_\infty}$ is the completion of $(\mathbb{C}^\infty)^*$ with respect to the inner product above, we know that, for any $f \in (\mathbb{C}^\infty)^*_{d\lambda_\infty}$, there is a sequence $\{f_k\}$ such that

$$f = \lim_{n \rightarrow \infty} \Re \sum_{k=1}^n \overline{f_k} l_k,$$

where the limit is taken in $(\mathbb{C}^\infty)^*_{d\lambda_\infty}$. We denote by $f = \Re \sum_{k=1}^\infty \overline{f_k} l_k$.

If z is in the Cameron-Martin space $\mathbb{C}^\infty(d\lambda_\infty)$ for $\mathbb{C}^\infty$, then we have

$$|z|_{\mathbb{C}^\infty(\gamma)} = \sup \left\{ |l(z)| : l \in (\mathbb{C}^\infty)^*, \langle l, l \rangle_{(\mathbb{C}^\infty)^*_{d\lambda_\infty}} \leq 1 \right\} < \infty.$$

Thus, by (2.1), we have

$$|z|_{\mathbb{C}^\infty(\gamma)} = \left(\sum_{k=1}^\infty |z_k|^2 \right)^{1/2} < \infty.$$

Thus, we write $|z|_{\mathbb{C}^\infty(\gamma)}$ as $|z|$. By (2.1), we know that $\Re \sum_{k=1}^n \overline{z_k} l_k$ is convergent in $L^2(\mathbb{C}^\infty, d\lambda_\infty)$, then we denote the limit by

$$\hat{z} = \Re \sum_{k=1}^\infty \overline{z_k} l_k. \quad (2.2)$$

Then for any $f = \Re \sum_{k=1}^n \overline{f_k} l_k$, we have

$$f(z) = \Re \sum_{k=1}^n \overline{f_k} z_k = \Re \langle f, \hat{z} \rangle_{(\mathbb{C}^\infty)^*_{d\lambda_\infty}}.$$

By Lemma 2.2, Lemma 2.3 and the argument above, we have the following Lemma.

Lemma 2.4. *Let $d\lambda_\infty$ be the Gaussian measure on $\mathbb{C}^\infty$.*

(1) *Let $\mathbb{C}^\infty(d\lambda_\infty)$ be the Cameron-Martin space for $\mathbb{C}^\infty$, then*

$$x = (x_1, \dots, x_k, \dots) \in \mathbb{C}^\infty(d\lambda_\infty) \quad \text{if and only if} \quad |x| = \left(\sum_{k=1}^{\infty} |x_k|^2 \right)^{1/2} < \infty.$$

(2) *For any $f, g \in (\mathbb{C}^\infty)^*_{d\lambda_\infty}$, there are sequences $(f_1, \dots, f_n, \dots)$ and $(g_1, \dots, g_n, \dots)$ with $(\sum_{k=1}^{\infty} |f_k|^2)^{1/2} < \infty$ and $(\sum_{k=1}^{\infty} |g_k|^2)^{1/2} < \infty$ such that*

$$f = \Re \sum_{k=1}^{\infty} \overline{f_k} l_k, \quad g = \Re \sum_{k=1}^{\infty} \overline{g_k} l_k \quad \text{and} \quad \langle g, f \rangle_{(\mathbb{C}^\infty)^*_{d\lambda_\infty}} = \Re \sum_{k=1}^{\infty} \overline{f_k} g_k.$$

(3) *For any $x = (x_1, \dots, x_k, \dots) \in \mathbb{C}^\infty(d\lambda_\infty)$, let $\hat{x} = \Re \sum_k \overline{x_k} l_k \in (\mathbb{C}^\infty)^*_{d\lambda_\infty}$, we have*

$$f(x) = \langle \hat{x}, f \rangle,$$

for any $f \in (\mathbb{C}^\infty)^*$. Moreover the measures $d\lambda_\infty$ and $d\lambda_{\infty, x} = d\lambda_\infty(\cdot - x)$ are equivalent and the corresponding Radon-Nikodym density is given by the expression

$$d\lambda_{\infty, x}(z) = \exp \left(\hat{x}(z) - \frac{1}{2} |x|_{\mathbb{C}^\infty(d\lambda_\infty)}^2 \right) d\lambda_\infty(z).$$

For any $x = (x_1, \dots, x_n, \dots) \in \mathbb{C}^\infty(d\lambda_\infty)$, we have $\sum_{k=1}^n \overline{x_k} l_k$ is convergent in $F^2(\mathbb{C}^\infty, d\lambda_\infty)$. Let $\lim_{n \rightarrow \infty} \sum_{k=1}^n \overline{x_k} l_k$ be the limit, we define

$$\langle x, z \rangle = \left[\lim_{n \rightarrow \infty} \sum_{k=1}^n \overline{x_k} l_k \right](z) \quad \text{for } z \in \mathbb{C}^\infty \text{ almost everywhere.} \quad (2.3)$$

Thus $\langle x, z \rangle$ is a function in $F^2(\mathbb{C}^\infty, d\lambda_\infty)$. By (2.1) and (2.2), we have

$$\Re \langle x, z \rangle = \hat{x}(z) \text{ almost everywhere and } \|\langle x, z \rangle\|_{(\mathbb{C}^\infty)^*_{d\lambda_\infty}} = 2|x| = 2\|\hat{x}\|_{\mathbb{C}^\infty(d\lambda_\infty)}. \quad (2.4)$$

We define the translation operator U_x on $L^2(\mathbb{C}^\infty, d\lambda_\infty)$ by

$$U_x f(z) = f(z - x) e^{\frac{1}{2} \langle x, z \rangle - \frac{1}{4} |x|^2}.$$

Remark 2.5. In the Fock space on $\mathbb{C}^n$, we can define the reproducing kernel for all $x^{(n)} = \{x_1, \dots, x_n\} \in \mathbb{C}^n$. The reproducing kernel is given by

$$K_{x^{(n)}}(z^{(n)}) = e^{\frac{1}{2} \sum_j \overline{x_j} z_j} = e^{\frac{1}{2} \langle x^{(n)}, z^{(n)} \rangle_{\mathbb{C}^n}}$$

for any $z^{(n)} \in \mathbb{C}^n$. Since we can't define the inner product of two points in $\mathbb{C}^\infty$, we can't define the reproducing kernel for all points in $\mathbb{C}^\infty$. However, for x in the Cameron-Martin space $\mathbb{C}^\infty(d\lambda_\infty)$ and $z \in \mathbb{C}^\infty$,

we can define $\langle x, z \rangle$ by the limit in $F^2(\mathbb{C}^\infty, d\lambda_\infty)$ as in (2.3). That is the reason that we need the Cameron-Martin space. So, the analogue of the normalized reproducing kernel in $F^2(\mathbb{C}^\infty, d\lambda_\infty)$ is given by

$$k_x(z) = e^{\frac{1}{2}\langle x, z \rangle - \frac{1}{4}|x|^2}.$$

On the other hand, by Lemma 2.4, we know that any $x \in \mathbb{C}^\infty(d\lambda_\infty)$ is a bounded sequence. Thus

$$\mathbb{C}^\infty(d\lambda_\infty) \subset \bigcup_N \left[B(0, N) \times \cdots \times B(0, N) \times \cdots \right] \subset \mathbb{C}^\infty,$$

where $B(0, N)$ is a ball in $\mathbb{C}$ with center 0 and radius N . Since $\left[B(0, N) \times \cdots \times B(0, N) \times \cdots \right]$ has measure 0 in $\mathbb{C}^\infty$ with respect to the Gaussian measure, we know that $\mathbb{C}^\infty(d\lambda_\infty)$ has measure 0.

If $x, z \in \mathbb{C}^\infty(d\lambda_\infty)$, we always require

$$\langle x, z \rangle = \lim_{n \rightarrow \infty} \sum_{k=1}^n \overline{x_k} z_k.$$

Since $\mathbb{C}^\infty(d\lambda_\infty)$ has measure 0, this requirement would't change the definition of $\langle x, z \rangle$ as a function in $F^2(\mathbb{C}^\infty, d\lambda_\infty)$.

We are going to study the operators $BW(h)B^{-1}$, so we need a lemma.

Lemma 2.6. *For any $x, y \in \mathbb{C}^\infty(d\lambda_\infty)$, we have*

$$U_y U_x = e^{-\frac{i}{2}\Im\langle x, y \rangle} U_{x+y} f(z) \quad \text{and} \quad \|U_x f\|_{L^2(\mathbb{C}^\infty, d\lambda_\infty)}^2 = \|f\|_{L^2(\mathbb{C}^\infty, d\lambda_\infty)}^2$$

for any $f \in L^2(\mathbb{C}^\infty, d\lambda_\infty)$. If $x^{(m)}$ converges to x in $\mathbb{C}^\infty(d\lambda_\infty)$, then $U_{x^{(m)}}$ converges to U_x in the strong operator topology on $F^2(\mathbb{C}^\infty, d\lambda_\infty)$. Moreover, U_x is a unitary operator on the Fock space over $\mathbb{C}^\infty$, thus $U_x f \in F^2(\mathbb{C}^\infty, d\lambda_\infty)$ when $f \in F^2(\mathbb{C}^\infty, d\lambda_\infty)$.

Proof. For any $f \in L^2(\mathbb{C}^\infty, d\lambda_\infty)$, by Lemma 2.4, we have

$$\begin{aligned} \|f\|_{L^2(\mathbb{C}^\infty, d\lambda_\infty)}^2 &= \int_{\mathbb{C}^\infty} |f(z)|^2 d\lambda_\infty \\ &= \int_{\mathbb{C}^\infty} |f(z-x)|^2 d\lambda_{\infty, x} \\ &= \int_{\mathbb{C}^\infty} |f(z-x)|^2 \exp\left(\hat{x}(z) - \frac{1}{2}|x|_{\mathbb{C}^\infty(d\lambda_\infty)}^2\right) d\lambda_\infty(z) \\ &= \int_{\mathbb{C}^\infty} |f(z-x)|^2 \exp\left(\Re\langle x, z \rangle - \frac{1}{2}|x|_{\mathbb{C}^\infty(d\lambda_\infty)}^2\right) d\lambda_\infty(z) \\ &= \int_{\mathbb{C}^\infty} |f(z-x) \exp\left(\frac{1}{2}\langle x, z \rangle - \frac{1}{4}|x|_{\mathbb{C}^\infty(d\lambda_\infty)}^2\right)|^2 d\lambda_\infty(z) \\ &= \int_{\mathbb{C}^\infty} |U_x f(z)|^2 d\lambda_\infty(z). \end{aligned}$$

Thus U_x is a unitary operator on $L^2(\mathbb{C}^\infty, d\lambda_\infty)$. For any $x, y \in \mathbb{C}^\infty(d\lambda_\infty)$, we have

$$\begin{aligned} U_y U_x f(z) &= U_y[f(z-x)e^{\frac{1}{2}\langle x, z \rangle - \frac{1}{4}|x|^2}] \\ &= f(z-x-y)e^{\frac{1}{2}\langle x, z-y \rangle - \frac{1}{4}|x|^2} e^{\frac{1}{2}\langle y, z \rangle - \frac{1}{4}|y|^2} \\ &= e^{-\frac{i}{2}\Im\langle x, y \rangle} U_{x+y} f(z). \end{aligned}$$

Thus $U_{-x} U_x = e^{-\frac{i}{2}\Im\langle x, x \rangle} U_{x-x} = I$, that is to say $U_{-x} = U_x^{-1} = U_x^*$.

For any $x^{(m)}, x \in \mathbb{C}^\infty(d\lambda_\infty)$ and $f \in F^2(\mathbb{C}^\infty, d\lambda_\infty)$, if $x^{(n)}$ converges to x in $\mathbb{C}^\infty(d\lambda_\infty)$, then

$$\begin{aligned} &\|U_{x^{(m)}} f - U_x f\|_{F^2(\mathbb{C}^\infty, d\lambda_\infty)} \\ &= \|e^{-\frac{i}{2}\Im\langle x^{(m)}, x \rangle} U_{x^{(m)}-x} f - f\|_{F^2(\mathbb{C}^\infty, d\lambda_\infty)} \\ &\leq \|(e^{-\frac{i}{2}\Im\langle x^{(m)}, x \rangle} - 1)U_{x^{(m)}-x} f\|_{F^2(\mathbb{C}^\infty, d\lambda_\infty)} + \|U_{x^{(m)}-x} f - f\|_{F^2(\mathbb{C}^\infty, d\lambda_\infty)} \\ &\leq |e^{-\frac{i}{2}\Im\langle x^{(m)}, x \rangle} - 1| \|f\|_{F^2(\mathbb{C}^\infty, d\lambda_\infty)} + \|U_{x^{(m)}-x} f - f\|_{F^2(\mathbb{C}^\infty, d\lambda_\infty)}. \end{aligned}$$

On one hand,

$$\lim_{m \rightarrow \infty} |\Im\langle x^{(m)}, x \rangle| = \lim_{m \rightarrow \infty} |\Im\langle x^{(m)} - x, x \rangle| \leq \lim_{m \rightarrow \infty} |x^{(m)} - x| |x| = 0.$$

On the other hand, we claim that if $y^{(m)} \rightarrow 0$ in $\mathbb{C}^\infty(d\lambda_\infty)$, then

$$\lim_{m \rightarrow \infty} \|U_{y^{(m)}} f - f\|_{F^2(\mathbb{C}^\infty, d\lambda_\infty)} = 0.$$

Thus, we obtain

$$\lim_{m \rightarrow \infty} \|U_{x^{(m)}} f - U_x f\|_{F^2(\mathbb{C}^\infty, d\lambda_\infty)} = 0.$$

Next, we prove the claim. If there is a constant $\epsilon > 0$, a function $g \in F^2(\mathbb{C}^\infty, d\lambda_\infty)$ and a sequence $y^{(m)}$ such that

$$\lim_{m \rightarrow \infty} |y^{(m)}| = 0 \quad \text{and} \quad \|U_{y^{(m)}} g - g\|_{F^2(\mathbb{C}^\infty, d\lambda_\infty)} > 3\epsilon$$

for any m . Since $g \in F^2(\mathbb{C}^\infty, d\lambda_\infty)$ there is a polynomial $p(z_1, \dots, z_j)$ such that

$$\|g - p\|_{F^2(\mathbb{C}^\infty, d\lambda_\infty)} < \epsilon.$$

Thus, we have

$$\|U_{y^{(m)}} p - p\|_{F^2(\mathbb{C}^\infty, d\lambda_\infty)} > \epsilon.$$

By the expression of the $U_{y^{(m)}}$ and (2.4) we have

$$U_{y^{(m)}} p(z) = p(z_1 - y_1^{(m)}, \dots, z_j - y_j^{(m)}) e^{\frac{1}{2}\langle y^{(m)}, z \rangle - \frac{1}{4}|y^{(m)}|^2} \quad \text{and} \quad \|\langle y^{(m)}, z \rangle\|_{(\mathbb{C}^\infty)^*_{d\lambda_\infty}} = 2|y^{(m)}| \rightarrow 0.$$

Thus, there is a subsequence $\{y^{(m_k)}\}$ such that $\langle y^{m_k}, z \rangle$ converges to 0 almost everywhere. Then $U_{y^{(m_k)}} p$ converges to p almost everywhere. On the other hand we have

$$\|U_{y^{(m_k)}}p\|_{L^2(\mathbb{C}^\infty, d\lambda_\infty)} = \|p\|_{L^2(\mathbb{C}^\infty, d\lambda_\infty)},$$

thus

$$\lim_{m \rightarrow \infty} \|U_{y^{(m_k)}}p - p\|_{F^2(\mathbb{C}^\infty, d\lambda_\infty)} = 0,$$

which is a contradiction. We have completed the proof of the claim.

In the last, we only need to prove that for any polynomial $p(z_1, \dots, z_j)$, we have $U_x p \in F^2(\mathbb{C}^\infty, d\lambda_\infty)$. For any $l \geq j$, let $x^{(l)} = (x_1, x_2, \dots, x_l, 0, 0, \dots)$, we have

$$U_{x^{(l)}}p(z_1, \dots, z_j) = p(z_1 - x_1, \dots, z_j - x_j) e^{\frac{1}{2} \sum_{m=1}^l \overline{x_m} z_m + \frac{1}{4} \sum_{m=1}^l |x_m|^2} \in F^2(\mathbb{C}^l, d\lambda_l).$$

Thus $U_{x^{(l)}}p \in F^2(\mathbb{C}^\infty, d\lambda_\infty)$ for any $l \geq j$. Send l to ∞ , and we have

$$U_x p \in F^2(\mathbb{C}^\infty, d\lambda_\infty).$$

Thus, U_x is a unitary operator from $F^2(\mathbb{C}^\infty, d\lambda_\infty)$ to $F^2(\mathbb{C}^\infty, d\lambda_\infty)$. $\square$

For any $x = (x_1, x_2, \dots) \in \mathbb{C}^\infty(d\lambda_\infty)$, since $\Re\langle x, z \rangle$ is a real-valued function, we have $T_{\Re\langle x, z \rangle}$ is an unbounded self-adjoint operator. Thus we can define $e^{iT_{\Re\langle x, z \rangle}}$ by functional calculus. Let $x^{(n)} = (x_1, \dots, x_n, 0, 0, \dots)$, then $U_{x^{(n)}}$ is an operator maps $F^2(\mathbb{C}^n, d\lambda_n)$ to $F^2(\mathbb{C}^n, d\lambda_n)$. In [2, Proposition 2 and pg 284], the authors have proved that

$$T_{e^{i\Re\langle x^{(n)}, z \rangle + \frac{|x^{(n)}|^2}{4}}} = e^{iT_{\Re\langle x^{(n)}, z \rangle}} = U_{-ix^{(n)}} \quad (2.5)$$

holds on the Fock space on $\mathbb{C}^n$. We can generalize this equality in the Fock space on $\mathbb{C}^\infty$.

Theorem 2.7. For any $h = \sum_{j=1}^\infty x_j h_j \in \mathcal{H}$, let $x = (x_1, \dots, x_n, \dots)$, we have

$$BW(h)B^{-1} = e^{iT_{\Re\langle x, z \rangle}} = U_{-ix} = T_{e^{i\Re\langle x, z \rangle + \frac{|x|^2}{4}}}$$

holds on the Fock space on $\mathbb{C}^\infty$. Moreover, the Weyl algebra is represented on $F^2(\mathbb{C}^\infty, d\lambda_\infty)$ as a Toeplitz algebra generated by

$$\left\{ T_{e^{i\Re\langle x, z \rangle + \frac{|x|^2}{4}}} : x \in \mathbb{C}^\infty(d\lambda_\infty) \right\}.$$

Proof. Since the Weyl algebra is generated by Weyl operators, we only need to prove the equality.

By the definition of the isomorphism B , we have $Bh = \sum_{j=1}^\infty x_j e_1 \circ l_j$. Since $\sum_j |x_j|^2 < \infty$, we have $x \in \mathbb{C}^\infty(d\lambda_\infty)$. Then $\langle x, z \rangle$ is well-defined and

$$\Re\langle x, z \rangle = \Re \sum_{j=1}^\infty x_j l_j = \sqrt{2} \Re \sum_{j=1}^\infty x_j e_j \circ l_j = 2^{-1/2} \overline{\left(\sum_{j=1}^\infty x_j e_j \circ l_j + \sum_{j=1}^\infty x_j e_j \circ l_j \right)}.$$

Moreover, by Proposition 2.1, we have $Ba_+^*(h)B^{-1} = T_{Bh}$ and $Ba_+(h)B^{-1} = T_{\overline{Bh}}$. Thus

$$BW(h)B^{-1} = B e^{i2^{-1/2} \overline{(a_+(h) + a_+^*(h))}} B^{-1} = e^{i2^{-1/2} (T_{\sum_{j=1}^\infty x_j e_1 \circ l_j} + T_{\sum_{j=1}^\infty x_j e_1 \circ l_j})} = e^{iT_{\Re\langle x, z \rangle}}.$$

Let $x^{(m)} = (x_1, \dots, x_m, 0, 0, \dots)$, by (2.5), we have

$$T_{e^{i\Re\langle x^{(m)}, z \rangle + \frac{|x^{(m)}|^2}{4}}} f = e^{iT_{\Re\langle x^{(m)}, z \rangle}} f = U_{-\overline{ix^{(m)}}} f$$

for any $m, n \in \mathbb{N}$ with $m \geq n$ and $f \in F^2(\mathbb{C}^n, d\lambda_n) \subseteq F^2(\mathbb{C}^m, d\lambda_m)$. For the operator $T_{e^{i\Re\langle x^{(m)}, z \rangle + \frac{|x^{(m)}|^2}{4}}}$, since $\lim_{m \rightarrow \infty} \|x^{(m)} - x\|_{\mathbb{C}^\infty(d\lambda_\infty)} = 0$, we have

$$\lim_{m \rightarrow \infty} \|\langle x^{(m)}, z \rangle - \langle x, z \rangle\|_{L^2(\mathbb{C}^\infty, d\lambda_\infty)} = 0.$$

There is a subsequence m_k such that

$$\lim_{k \rightarrow \infty} \langle x^{(m_k)}, z \rangle = \langle x, z \rangle \quad \text{almost everywhere.}$$

Thus we have

$$\lim_{k \rightarrow \infty} T_{e^{i\Re\langle x^{(m_k)}, z \rangle + \frac{|x^{(m_k)}|^2}{4}}} f = \lim_{k \rightarrow \infty} T_{e^{i\Re\langle x, z \rangle + \frac{|x|^2}{4}}} f.$$

Let $h^{(m)} = \sum_{j=1}^m x_j h_j$, by [4, Proposition 5.2.4], we have $W(h^{(m)})$ converges to $W(h)$ in the strong operator topology. Thus

$$\lim_{m \rightarrow \infty} e^{iT_{\Re\langle x^{(m)}, z \rangle}} f = e^{iT_{\Re\langle x, z \rangle}} f.$$

By Proposition 2.6, we have

$$\lim_{m \rightarrow \infty} U_{-\overline{ix^{(m)}}} f = U_{-\overline{ix}} f.$$

Then, we have

$$e^{iT_{\Re\langle x, z \rangle}} f = U_{-\overline{ix}} f = T_{e^{i\Re\langle x, z \rangle + \frac{|x|^2}{4}}} f$$

for any $f \in F^2(\mathbb{C}^n, d\lambda_n)$. We have completed the proof because $\bigcup_n F^2(\mathbb{C}^n, d\lambda_n)$ is dense in $F^2(\mathbb{C}^\infty, d\lambda_\infty)$. $\square$

3. The Gibbs state and trace formula

Next, we study the second quantization and the Gibbs state. Given an unbounded self-adjoint operator H on the Hilbert space $\mathcal{H}$, we have defined $d\Gamma(H)$, thus $e^{itd\Gamma(H)}$ is a unitary operator for any t . Let $U_t = e^{itH}$, then U_t is a unitary operator on $\mathcal{H}$. By [4, pg 8], we know that

$$e^{itd\Gamma(H)} P_+(f_1 \otimes \cdots \otimes f_n) = P_+(e^{itH} f_1 \otimes \cdots \otimes e^{itH} f_n). \quad (3.1)$$

Since $d\Gamma(H)$ is a self-adjoint operator, we know that $e^{itd\Gamma(H)}$ is a bounded operator.

Let β be a constant such that βH is a positive operator, then $\beta d\Gamma(H)$ is a positive operator. Similarly, the operator $e^{-\beta d\Gamma(H)}$ is given by

$$e^{-\beta d\Gamma(H)} P_+(f_1 \otimes \cdots \otimes f_n) = P_+(e^{-\beta H} f_1 \otimes \cdots \otimes e^{-\beta H} f_n). \quad (3.2)$$

Since $\beta d\Gamma(H)$ is a positive operator, we know that $e^{-\beta d\Gamma(H)}$ is a bounded operator.

Because there is no proof of (3.1) and (3.2) in [4], we give a simple proof here.

Let U_t be an operator such that

$$U_t P_+(f_1 \otimes \cdots \otimes f_n) = P_+(e^{itH} f_1 \otimes \cdots \otimes e^{itH} f_n).$$

It is easy to see that U_t is a one-parameter group of unitary operators and its generator is $d\Gamma(H)$, by [7, Theorem VIII.8] we have

$$U_t = e^{itd\Gamma(H)}.$$

The proof for (3.2) is similar, we only need to apply [7, Problem 39, pg 315].

Recall that, for any $j \in \mathbb{N}$, l_j is a function in $F^2(\mathbb{C}^\infty, d\lambda_\infty)$ such that $l_j(z) = z_j$. Since B is an isomorphism from $\mathcal{H}$ to χ_1 , we know that $Be^{itH}B^{-1}$ is an operator on χ_1 which is generated by $\{l_j\}$.

Theorem 3.1. *For any self-adjoint operator H on $\mathcal{H}$, we have*

$$Be^{itd\Gamma(H)}B^{-1}f(z) = f(Be^{itH}B^{-1}l_1(z), \dots, Be^{itH}B^{-1}l_n(z), \dots)$$

for any $f \in \bigcup_n F^2(\mathbb{C}^n, d\lambda_n)$. Let β be a constant such that βH is a positive operator, then we have

$$Be^{-\beta d\Gamma(H)}B^{-1}f(z) = f(Be^{-\beta H}B^{-1}l_1(z), \dots, Be^{-\beta H}B^{-1}l_n(z), \dots)$$

for any $f \in \bigcup_n F^2(\mathbb{C}^n, d\lambda_n)$.

Proof. We only need to prove the first conclusion, because the proof for the second is similar. Since

$$\begin{aligned} B\left[\sqrt{\frac{k!}{\alpha!}}P_+(h_1^{\alpha_1} \otimes h_2^{\alpha_2} \otimes \dots \otimes h_n^{\alpha_n})\right] &= e_{\alpha_1} \circ l_1 \dots e_{\alpha_n} \circ l_n \\ &= \sqrt{\frac{1}{2^k \alpha!}} l_1^{\alpha_1} \dots l_n^{\alpha_n} \end{aligned}$$

and $Bh_j(z) = e_1 \circ l_j = \sqrt{\frac{1}{2}}l_j$, we have

$$B\left[\sqrt{\frac{k!}{\alpha!}}P_+(h_1^{\alpha_1} \otimes h_2^{\alpha_2} \otimes \dots \otimes h_n^{\alpha_n})\right] = \sqrt{\frac{1}{\alpha!}}[Bh_1]^{\alpha_1} \dots [Bh_n]^{\alpha_n}.$$

Thus, for any $f_1, \dots, f_n \in \mathcal{H}$ we have

$$B\left[\sqrt{\frac{k!}{\alpha!}}P_+(f_1^{\alpha_1} \otimes \dots \otimes f_n^{\alpha_n})\right] = \sqrt{\frac{1}{\alpha!}}[Bf_1]^{\alpha_1} \dots [Bf_n]^{\alpha_n}. \quad (3.3)$$

For any monomial

$$\sqrt{\frac{1}{2^k \alpha!}}[l_1(z)]^{\alpha_1} \dots [l_n(z)]^{\alpha_n},$$

we have

$$\begin{aligned} &Be^{itd\Gamma(H)}B^{-1}\sqrt{\frac{1}{2^k \alpha!}}[l_1]^{\alpha_1} \dots [l_n]^{\alpha_n} \\ &= Be^{itd\Gamma(H)}B^{-1}B\left[\sqrt{\frac{k!}{\alpha!}}P_+(h_1^{\alpha_1} \otimes \dots \otimes h_n^{\alpha_n})\right] \\ &= B\left[\sqrt{\frac{k!}{\alpha!}}P_+([e^{itH}h_1]^{\alpha_1} \otimes \dots \otimes [e^{itH}h_n]^{\alpha_n})\right] \end{aligned}$$

$$\begin{aligned}
&= \sqrt{\frac{1}{\alpha!}} [Be^{itH} h_1]^{\alpha_1} \cdots [Be^{itH} h_n]^{\alpha_n} \quad (\text{by (3.3)}) \\
&= \sqrt{\frac{1}{\alpha!}} [Be^{itH} B^{-1} B h_1]^{\alpha_1} \cdots [Be^{itH} B^{-1} I h_n]^{\alpha_n} \\
&= \sqrt{\frac{1}{2^k \alpha!}} [Be^{itH} B^{-1} l_1]^{\alpha_1} \cdots [Be^{itH} B^{-1} l_n]^{\alpha_n}.
\end{aligned}$$

Then, for any polynomial p , we have

$$Be^{itd\Gamma(H)} B^{-1} p(z) = p(Be^{itH} B^{-1} l_1(z), \dots, Be^{itH} B^{-1} l_n(z), \dots).$$

For any positive integer n and $f \in F^2(\mathbb{C}^n, d\lambda_n)$, there is a sequence of polynomials $\{p_n\}$ such that

$$\lim_n p_n(z) = f(z) \quad \text{for any } z \in \mathbb{C}^n \quad \text{and} \quad \lim_n p_n = f \quad \text{in } F^2(\mathbb{C}^n, d\lambda_n).$$

Thus we have

$$Be^{itd\Gamma(H)} B^{-1} f(z) = f(Be^{itH} B^{-1} l_1(z), \dots, Be^{itH} B^{-1} l_n(z), \dots). \quad \square$$

Let μ be a real number, the Gibbs grand canonical equilibrium state is defined in terms of the generalized Hamiltonian $K_\mu = d\Gamma(H - \mu I)$ whenever

$$\exp\{-\beta d\Gamma(H - \mu I)\}$$

is trace-class. This latter property places a constraint on the possible values of μ .

Proposition 3.2 ([4, Proposition 5.2.27]). *Let H be a self-adjoint operator on the Hilbert space $\mathcal{H}$ and let $\beta \in \mathbb{R}$. The following conditions are equivalent:*

- (1) $e^{-\beta H}$ is of trace-class on $\mathcal{H}$ and $\beta(H - \mu I) > 0$,
- (2) $e^{-\beta d\Gamma(H - \mu I)}$ is of trace-class on $\mathcal{F}_+(\mathcal{H})$ for all $\mu \in \mathbb{R}$.

Let us now assume that $\exp\{-\beta K_\mu\}$ is of trace-class and then calculate the Gibbs state

$$\omega(A) = \frac{\text{Tr}(e^{-\beta d\Gamma(H - \mu I)} A)}{\text{Tr}(e^{-\beta d\Gamma(H - \mu I)})}$$

for any operator A on $\mathcal{F}_+(\mathcal{H})$ such that $e^{-\beta d\Gamma(H - \mu I)} A$ is of trace-class. By Proposition 3.2, we have $e^{-\beta(H - \mu I)}$ is a bounded operator. The Gibbs equilibrium state is important in the quantum physics, see [4]. As an application of the Fock space on $\mathbb{C}^\infty$, we are going to give a trace formula and apply it to the Gibbs equilibrium state.

For any $z \in \mathbb{C}^\infty$ and $x \in \mathbb{C}^\infty(d\lambda_\infty)$, let

$$K_x(z) = e^{\frac{1}{2}\langle x, z \rangle}.$$

Let $x^{(n)} = (x_1, \dots, x_n, 0, 0, \dots)$ and $z^{(n)} = (z_1, \dots, z_n, 0, 0, \dots)$, then $K_{x^{(n)}}(z^{(n)})$ is the reproducing kernel of $F^2(\mathbb{C}^n, d\lambda_n)$. Let $X^{(n)}$ be an operator on $F^2(\mathbb{C}^n, d\lambda_n)$, by [12, Proposition 4], we have

$$\text{Tr}(X^{(n)}) = \int_{\mathbb{C}^n} \langle X^{(n)} K_{x^{(n)}}, K_{x^{(n)}} \rangle d\lambda_n(x^{(n)}).$$

Theorem 3.3. *If X is an operator on $F^2(\mathbb{C}^\infty, d\lambda_\infty)$, then X is of trace-class if and only if*

$$|\lim_{n \rightarrow \infty} \int_{\mathbb{C}^n} \langle X K_{x^{(n)}}, K_{x^{(n)}} \rangle d\lambda_n(x^{(n)})| < \infty.$$

In that case, we have

$$\text{Tr}(X) = \lim_{n \rightarrow \infty} \int_{\mathbb{C}^n} \langle X K_{x^{(n)}}, K_{x^{(n)}} \rangle d\lambda_n(x^{(n)}).$$

Moreover, if A is an operator on the Bose-Fock space, then we have

$$\begin{aligned} & \text{Tr}(e^{-\beta d\Gamma(H-\mu I)} A) \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{C}^n} \int_{\mathbb{C}^\infty} (BAB^{-1} K_{x^{(n)}})(z) \overline{K_{x^{(n)}}(Be^{-\beta(H-\mu I)} B^{-1} l_1(z), \dots)} d\lambda_\infty(z) d\lambda_n(x^{(n)}). \end{aligned}$$

Proof. Let

$$E_n = \{e_{\alpha_1} \circ l_1 \times e_{\alpha_2} \circ l_2 \times \dots \times e_{\alpha_n} \circ l_n : \alpha_1, \dots, \alpha_n \geq 1\} \text{ when } n \geq 1 \text{ and } E_0 = \{\mathbb{I}\},$$

where $\mathbb{I}$ stands for the function which maps all $z \in \mathbb{C}^\infty$ to 1. It is easy to see that E_n is the set of basis of $F^2(\mathbb{C}^n, d\lambda_n) \ominus F^2(\mathbb{C}^{n-1}, d\lambda_n)$. Thus we know that $\bigcup_{n=0}^\infty E_n$ is a basis set of the Fock space on $\mathbb{C}^\infty$. If X is of trace-class, then

$$\text{Tr}(X) = \sum_{n=0}^\infty \sum_{e \in E_n} \langle X e, e \rangle = \lim_{m \rightarrow \infty} \sum_{n=0}^m \sum_{e \in E_n} \langle X e, e \rangle.$$

Since P_m is the projection from $F^2(\mathbb{C}^\infty, d\lambda_\infty)$ to $F^2(\mathbb{C}^m, d\lambda_m)$, we have $P_m X P_m$ is an operator from $F^2(\mathbb{C}^m, d\lambda_m)$ to $F^2(\mathbb{C}^m, d\lambda_m)$. Since $\bigcup_{n=0}^m E_n$ is a basis set of $F^2(\mathbb{C}^m, d\lambda_m)$, we have

$$\begin{aligned} \sum_{n=0}^m \sum_{e \in E_n} \langle X e, e \rangle &= \sum_{n=0}^m \sum_{e \in E_n} \langle P_m X P_m e, e \rangle = \text{Tr}(P_m X P_m) \\ &= \int_{\mathbb{C}^m} \langle P_m X P_m K_{x^{(m)}}, K_{x^{(m)}} \rangle d\lambda_m(x^{(m)}) \\ &= \int_{\mathbb{C}^m} \langle X K_{x^{(m)}}, K_{x^{(m)}} \rangle d\lambda_m(x^{(m)}). \end{aligned}$$

Thus

$$|\lim_{m \rightarrow \infty} \int_{\mathbb{C}^m} \langle X K_{x^{(m)}}, K_{x^{(m)}} \rangle d\lambda_m(x^{(m)})| = |\lim_{m \rightarrow \infty} \sum_{n=0}^m \sum_{e \in E_n} \langle X e, e \rangle| = |\text{Tr}(X)| < \infty.$$

On the other hand, if

$$|\lim_{n \rightarrow \infty} \int_{\mathbb{C}^n} \langle X K_{x^{(n)}}, K_{x^{(n)}} \rangle| < \infty,$$

then, by the argument above, we know that X is of trace-class.

The second conclusion follows from Theorem 3.1. $\square$

If $e^{-\beta d\Gamma(H-\mu I)}$ is of trace-class, by Proposition 3.2, we know that $\beta(H-\mu I)$ is a positive operator and $e^{-\beta H}$ is of trace-class on $\mathcal{H}$. We suppose that $\{\lambda_j : j = 1, \dots, n, \dots\}$ and $\{v_j : j = 1, \dots, n, \dots\}$ are eigenvalues and eigenvectors of $e^{-\beta(H-\mu I)}$ such that

$$e^{-\beta(H-\mu I)}v_j = \lambda_j v_j. \quad (3.4)$$

Thus $0 < \lambda_j < 1$. Since $e^{-\beta(H-\mu I)}$ is a self-adjoint operator, we suppose $\{v_j\}$ is a basis of $\mathcal{H}$.

Next, we are going to study the Gibbs state. Let B_H denote the isomorphism from the Bose-Fock space to the Fock space over $\mathbb{C}^\infty$ such that

$$B_H \left[\sqrt{\frac{k!}{\alpha!}} P_+(v_1^{\alpha_1} \otimes \dots \otimes v_n^{\alpha_n}) \right] (z) = e_{\alpha_1} \circ l_1 \dots e_{\alpha_n} \circ l_n,$$

where $k = \sum_m \alpha_m$. Because $B_H^{-1}l_j = \sqrt{2}B_H^{-1}e_1 \circ l_j = \sqrt{2}v_j$, we have

$$B_H e^{-\beta(H-\mu I)} B_H^{-1} l_j = \lambda_j l_j.$$

By the construction above and Theorem 3.3, we have the following corollary.

Corollary 3.4. *Let $\beta(H-\mu I)$ be a positive operator and $e^{-\beta(H-\mu I)}$ be of trace-class with eigenvalues $\{\lambda_k\}$ on $\mathcal{H}$. For any operator A , we have*

$$\begin{aligned} & \text{Tr}(e^{-\beta d\Gamma(H-\mu I)} A) \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{C}^n} \int_{\mathbb{C}^\infty} (B_H A B_H^{-1} K_{x^{(n)}})(z) \overline{K_{x^{(n)}}(\lambda_1 z_1, \dots, \lambda_n z_n, 0, 0, \dots)} d\lambda_\infty(z) d\lambda_n(x^{(n)}). \end{aligned}$$

In [4, Proposition 5.2.28], if $e^{-\beta d\Gamma(H-\mu I)}$ is of trace-class on the Bose-Fock space, we have the following formulas

$$\omega(W(f)) = \exp\{-\langle f, (1 + e^{-\beta(H-\mu I)})(1 - e^{-\beta(H-\mu I)})^{-1} f \rangle / 4\},$$

and

$$\omega(a_+^*(f)a_+(g)) = \langle g, e^{-\beta(H-\mu I)}(1 - e^{-\beta(H-\mu I)})^{-1} f \rangle.$$

Next, we are going to show that Theorem 3.3 implies these two formulas. Moreover, we will generalize the second one.

Corollary 3.5. *If $e^{-\beta d\Gamma(H-\mu I)}$ is of trace-class on the Bose-Fock space, then the Gibbs state of a Weyl operator $W(f)$ is given by*

$$\omega(W(f)) = \exp\{-\langle f, (1 + e^{-\beta(H-\mu I)})(1 - e^{-\beta(H-\mu I)})^{-1} f \rangle / 4\}.$$

Proof. Let $\{v_j\}$ be a basis as in (3.4), we suppose $f = \sum_j f_j v_j$. Let $y = (-i\overline{f_1}, \dots, -i\overline{f_n}, \dots) \in \mathbb{C}^\infty$, by Theorem 2.7 we have

$$B_H W(h) B_H^{-1} = U_y.$$

Then, by Corollary 3.4, we have

$$\begin{aligned} & \text{Tr}(e^{-\beta d\Gamma(H-\mu I)} W(f)) \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{C}^n} \int_{\mathbb{C}^\infty} (U_y K_{x(n)})(z) \overline{K_{x(n)}(\lambda_1 z_1, \dots, \lambda_n z_n, 0, 0, \dots)} d\lambda_\infty(z) d\lambda_n(x^{(n)}). \end{aligned}$$

By the definition of U_y , we have

$$\begin{aligned} & \int_{\mathbb{C}^\infty} (U_y K_{x(n)})(z) \overline{K_{x(n)}(\lambda_1 z_1, \dots, \lambda_n z_n, 0, 0, \dots)} d\lambda_\infty(z) \\ &= \int_{\mathbb{C}^\infty} e^{\frac{1}{2} \sum_{j=1}^n \overline{x_j}(z_j - y_j) + \frac{1}{2} \sum_{j=1}^\infty \overline{y_j} l_j(z) - \frac{1}{4} |y|^2} e^{\frac{1}{2} \sum_{j=1}^n x_j \lambda_j \overline{z_j}} d\lambda_\infty(z). \end{aligned}$$

Since

$$\int_{\mathbb{C}} e^{\frac{1}{2} \overline{y_j} l_j(z)} d\lambda_1(z_j) = 1 \text{ for any } j,$$

we have

$$\int_{\mathbb{C}^\infty} e^{\frac{1}{2} \sum_{j=n+1}^\infty \overline{y_j} l_j(z)} d\lambda_\infty(z) = \int_{\mathbb{C}^\infty} e^{\frac{1}{2} \sum_{j=m}^\infty \overline{y_j} l_j(z)} d\lambda_\infty(z)$$

for any integer m and

$$\begin{aligned} \lim_{m \rightarrow \infty} \left| \int_{\mathbb{C}^\infty} e^{\frac{1}{2} \sum_{j=m}^\infty \overline{y_j} l_j(z)} d\lambda_\infty(z) - 1 \right| &= \lim_{m \rightarrow \infty} \int_{\mathbb{C}^\infty} |e^{\frac{1}{2} \sum_{j=m}^\infty \overline{y_j} l_j(z)} - 1| d\lambda_\infty(z) \\ &\leq \lim_{m \rightarrow \infty} \|e^{\frac{1}{2} \sum_{j=m}^\infty \overline{y_j} l_j} - 1\|_{F^2(\mathbb{C}^\infty, d\lambda_\infty)} \\ &\leq \lim_{m \rightarrow \infty} \|e^{\frac{1}{4} \sum_{j=m}^\infty |y_j|^2} U_{y-y^{(m)}} 1 - 1\|_{F^2(\mathbb{C}^\infty, d\lambda_\infty)} \\ &= 0 \text{ (by Lemma 2.6)}. \end{aligned}$$

Thus

$$\int_{\mathbb{C}^\infty} e^{\frac{1}{2} \sum_{j=n+1}^\infty \overline{y_j} l_j(z)} d\lambda_\infty(z) = \lim_{m \rightarrow \infty} \int_{\mathbb{C}^\infty} e^{\frac{1}{2} \sum_{j=m}^\infty \overline{y_j} l_j(z)} d\lambda_\infty(z) = 1.$$

Then, we have

$$\begin{aligned} & \int_{\mathbb{C}^\infty} (U_y K_{x(n)})(z) \overline{K_{x(n)}(\lambda_1 z_1, \dots, \lambda_n z_n, 0, 0, \dots)} d\lambda_\infty(z) \\ &= \int_{\mathbb{C}^n} e^{\frac{1}{2} \sum_{j=1}^n \overline{x_j}(z_j - y_j) + \frac{1}{2} \sum_{j=1}^n \overline{y_j} z_j - \frac{1}{4} |y|^2} e^{\frac{1}{2} \sum_{j=1}^n x_j \lambda_j \overline{z_j}} d\lambda_n(z) \\ &= e^{\frac{1}{2} \sum_{j=1}^n \overline{x_j}(\lambda_j x_j - y_j) + \frac{1}{2} \sum_{j=1}^n \overline{y_j} \lambda_j x_j - \frac{1}{4} |y|^2}. \end{aligned}$$

Thus

$$\begin{aligned}
& \text{Tr}(e^{-\beta d\Gamma(H-\mu I)}W(f)) \\
&= \lim_{n \rightarrow \infty} \int_{\mathbb{C}^n} e^{\frac{1}{2} \sum_{j=1}^n \overline{x_j}(\lambda_j x_j - y_j) + \frac{1}{2} \sum_{j=1}^n \overline{y_j} \lambda_j x_j - \frac{1}{4} |y|^2} d\lambda_n(x^{(n)}) \\
&= \lim_{n \rightarrow \infty} \int_{\mathbb{C}^n} e^{\frac{1}{2} \sum_{j=1}^n \overline{x_j}(\lambda_j x_j - y_j) + \frac{1}{2} \sum_{j=1}^n \overline{y_j} \lambda_j x_j - \frac{1}{4} |y|^2} \frac{1}{(2\pi)^{2n}} e^{-\frac{1}{2} \sum_{j=1}^n |x_j|^2} d(x^{(n)}) \\
&= \lim_{n \rightarrow \infty} \int_{\mathbb{C}^n} e^{\frac{1}{2} \sum_{j=1}^n \overline{x_j}(-y_j) + \frac{1}{2} \sum_{j=1}^n \overline{y_j} \lambda_j x_j - \frac{1}{4} |y|^2} \frac{1}{(2\pi)^{2n}} e^{-\frac{1}{2} \sum_{j=1}^n (1-\lambda_j) |x_j|^2} d(x^{(n)}) \\
&= \lim_{n \rightarrow \infty} \frac{1}{\prod_{j=1}^n \sqrt{1-\lambda_j}} \int_{\mathbb{C}^n} e^{\frac{1}{2} \sum_{j=1}^n \frac{\overline{x_j}}{\sqrt{1-\lambda_j}}(-y_j) + \frac{1}{2} \sum_{j=1}^n \overline{y_j} \lambda_j \frac{x_j}{\sqrt{1-\lambda_j}} - \frac{1}{4} |y|^2} d\lambda_n(x^{(n)}) \\
&= \lim_{n \rightarrow \infty} \frac{1}{\prod_{j=1}^n \sqrt{1-\lambda_j}} e^{-\frac{1}{4} |y|^2 - \sum_{j=1}^n \frac{1}{2} \frac{\lambda_j}{1-\lambda_j} |y_j|^2} \\
&= \frac{1}{\prod_{j=1}^\infty \sqrt{1-\lambda_j}} e^{-\frac{1}{4} |y|^2 - \sum_{j=1}^\infty \frac{1}{2} \frac{\lambda_j}{1-\lambda_j} |y_j|^2} \\
&= \frac{1}{\prod_{j=1}^\infty \sqrt{1-\lambda_j}} e^{-\frac{1}{4} \sum_{j=1}^\infty \frac{1+\lambda_j}{1-\lambda_j} |y_j|^2},
\end{aligned}$$

where $\prod_{j=1}^\infty \sqrt{1-\lambda_j}$ is convergent because $\sum_j \lambda_j = \text{Tr}(e^{-\beta(H-\mu I)}) < \infty$ and $\lambda_j < 1$. By functional calculus, we have

$$\begin{aligned}
\left\langle f, (1 + e^{-\beta(H-\mu I)})(1 - e^{-\beta(H-\mu I)})^{-1} f \right\rangle &= \left\langle f, (1 + e^{-\beta(H-\mu I)})(1 - e^{-\beta(H-\mu I)})^{-1} \sum_j f_j v_j \right\rangle \\
&= \sum_j f_j \left\langle f, (1 + e^{-\beta(H-\mu I)})(1 - e^{-\beta(H-\mu I)})^{-1} v_j \right\rangle \\
&= \sum_j f_j \left\langle f, (1 + \lambda_j)(1 - \lambda_j)^{-1} v_j \right\rangle \\
&= \sum_j |f_j|^2 \frac{1 + \lambda_j}{1 - \lambda_j} = \sum_j |y_j|^2 \frac{1 + \lambda_j}{1 - \lambda_j}.
\end{aligned}$$

Thus, we have

$$\text{Tr}(e^{-\beta d\Gamma(H-\mu I)}W(f)) = \frac{1}{\prod_{j=1}^\infty \sqrt{1-\lambda_j}} e^{-\frac{1}{4} \left\langle f, (1 + e^{-\beta(H-\mu I)})(1 - e^{-\beta(H-\mu I)})^{-1} f \right\rangle}.$$

Since $W(f) = I$ when $f = 0$, we have

$$\text{Tr}(e^{-\beta d\Gamma(H-\mu I)}) = \frac{1}{\prod_{j=1}^\infty \sqrt{1-\lambda_j}}.$$

Then we have

$$\omega(W(f)) = \frac{\text{Tr}(e^{-\beta d\Gamma(H-\mu I)}W(f))}{\text{Tr}(e^{-\beta d\Gamma(H-\mu I)})} = e^{-\frac{1}{4} \left\langle f, (1 + e^{-\beta(H-\mu I)})(1 - e^{-\beta(H-\mu I)})^{-1} f \right\rangle}. \quad \square$$

Corollary 3.6. *If $e^{-\beta d\Gamma(H-\mu I)}$ is of trace-class on the Bose-Fock space, then*

$$\omega\left(a_+^*(f^{(1)})\cdots a_+^*(f^{(m)})a_+(g^{(1)})\cdots a_+(g^{(m)})\right) \\ = \left\langle \sqrt{m!}P_+\left(\tilde{f}^{(1)}\otimes\cdots\otimes\tilde{f}^{(m)}\right), \sqrt{m!}P_+\left(\tilde{g}^{(1)}\otimes\cdots\otimes\tilde{g}^{(m)}\right) \right\rangle_{\mathcal{F}_+(\mathcal{H})},$$

where

$$\tilde{f}^{(k)} = \frac{e^{-\beta(H-\mu I)}}{\sqrt{1-e^{-\beta(H-\mu I)}}}f^{(k)} \text{ and } \tilde{g}^{(k)} = \frac{1}{\sqrt{1-e^{-\beta(H-\mu I)}}}g^{(k)}.$$

Proof. Let $\tilde{x}^{(n)} = (\lambda_1 x_1, \dots, \lambda_n x_n, 0, \dots)$,

$$T_1 = a_+(f^{(1)})\cdots a_+(f^{(m)}) \text{ and } T_2 = a_+(g^{(1)})\cdots a_+(g^{(m)}).$$

Since

$$K_{x^{(n)}}(\lambda_1 z_1, \dots, \lambda_n z_n, 0, 0, \dots) = e^{\frac{1}{2}\sum_{j=1}^n \overline{x_j} \lambda_j z_j} = K_{\tilde{x}^{(n)}}(z),$$

we have

$$\begin{aligned} & \text{Tr}(e^{-\beta d\Gamma(H-\mu I)} T_1^* T_2) \text{ (by Corollary 3.4)} \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{C}^n} \int_{\mathbb{C}^\infty} (B_H T_1^* T_2 B_H^{-1} K_{x^{(n)}})(z) \overline{K_{x^{(n)}}(\lambda_1 z_1, \dots, \lambda_n z_n, 0, 0, \dots)} d\lambda_\infty(z) d\lambda_n(x^{(n)}) \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{C}^n} \int_{\mathbb{C}^\infty} (B_H T_1^* T_2 B_H^{-1} K_{x^{(n)}})(z) \overline{K_{\tilde{x}^{(n)}}(z)} d\lambda_\infty(z) d\lambda_n(x^{(n)}) \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{C}^n} \int_{\mathbb{C}^\infty} (B_H T_2 B_H^{-1} K_{x^{(n)}})(z) \overline{(B_H T_1 B_H^{-1} K_{\tilde{x}^{(n)}})(z)} d\lambda_\infty(z) d\lambda_n(x^{(n)}). \end{aligned}$$

By Proposition 2.1, we have

$$B_H T_1 B_H^{-1} = T_{\overline{B_H f^{(1)}}} \cdots T_{\overline{B_H f^{(m)}}} \text{ and } B_H T_2 B_H^{-1} = T_{\overline{B_H g^{(1)}}} \cdots T_{\overline{B_H g^{(m)}}}.$$

We suppose $g^{(k)} = \sum_j g_j^{(k)} v_j$ and $f^{(k)} = \sum_j f_j^{(k)} v_j$. For any polynomial p , we have

$$\begin{aligned} & \langle p, T_{\overline{B_H g^{(m)}}} K_{x^{(n)}} \rangle_{F^2(\mathbb{C}^\infty, d\lambda_\infty)} = \langle T_{B_H g^{(m)}} p, K_{x^{(n)}} \rangle_{F^2(\mathbb{C}^\infty, d\lambda_\infty)} \\ &= \sum_{j=1}^n g_j^{(m)} \frac{x_j}{\sqrt{2}} p(x^{(n)}) = \sum_{j=1}^n g_j^{(m)} \frac{x_j}{\sqrt{2}} \langle p, K_{x^{(n)}} \rangle_{F^2(\mathbb{C}^\infty, d\lambda_\infty)} \\ &= \langle p, \sum_{j=1}^n \overline{g_j^{(m)} \frac{x_j}{\sqrt{2}}} K_{x^{(n)}} \rangle_{F^2(\mathbb{C}^\infty, d\lambda_\infty)}, \end{aligned}$$

which means $(T_{\overline{B_H g^{(m)}}} K_{x^{(n)}})(z) = \sum_{j=1}^n \overline{g_j^{(m)} \frac{x_j}{\sqrt{2}}} K_{x^{(n)}}(z)$. Thus, we have

$$(B_H T_2 B_H^{-1} K_{x^{(n)}})(z) = \left(\sum_{j=1}^n \overline{g_j^{(1)} \frac{x_j}{\sqrt{2}}} \right) \cdots \left(\sum_{j=1}^n \overline{g_j^{(m)} \frac{x_j}{\sqrt{2}}} \right) K_{x^{(n)}}(z)$$

and

$$(B_H T_1 B_H^{-1} K_{\tilde{x}^{(n)}})(z) = \left(\sum_{j=1}^n \overline{f_j^{(1)} \frac{\lambda_j x_j}{\sqrt{2}}} \right) \cdots \left(\sum_{j=1}^n \overline{f_j^{(m)} \frac{\lambda_j x_j}{\sqrt{2}}} \right) K_{\tilde{x}^{(n)}}(z).$$

Then

$$\begin{aligned} & \text{Tr} \left(e^{-\beta d\Gamma(H-\mu I)} a_+^*(f^{(1)}) \cdots a_+^*(f^{(m)}) a_+(g^{(1)}) \cdots a_+(g^{(m)}) \right) \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{C}^n} \int_{\mathbb{C}^\infty} \prod_{k=1}^m \left(\sum_{j=1}^n \overline{g_j^{(k)} \frac{x_j}{\sqrt{2}}} \right) K_{x^{(n)}}(z) \overline{\prod_{k=1}^m \left(\sum_{j=1}^n \overline{f_j^{(k)} \frac{\lambda_j x_j}{\sqrt{2}}} \right) K_{\tilde{x}^{(n)}}(z)} d\lambda_\infty(z) d\lambda_n(x^{(n)}) \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{C}^n} \prod_{k=1}^m \left(\sum_{j=1}^n \overline{f_j^{(k)} \frac{\lambda_j x_j}{\sqrt{2}}} \right) \overline{\prod_{k=1}^m \left(\sum_{j=1}^n \overline{g_j^{(k)} \frac{x_j}{\sqrt{2}}} \right)} \int_{\mathbb{C}^\infty} K_{x^{(n)}}(z) \overline{K_{\tilde{x}^{(n)}}(z)} d\lambda_\infty(z) d\lambda_n(x^{(n)}) \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{C}^n} \prod_{k=1}^m \left(\sum_{j=1}^n \overline{f_j^{(k)} \frac{\lambda_j x_j}{\sqrt{2}}} \right) \overline{\prod_{k=1}^m \left(\sum_{j=1}^n \overline{g_j^{(k)} \frac{x_j}{\sqrt{2}}} \right)} e^{\frac{1}{2} \sum_{j=1}^n \lambda_j |x_j|^2} d\lambda_n(x^{(n)}) \\ &= \frac{1}{\prod_{j=1}^\infty \sqrt{1-\lambda_j}} \lim_{n \rightarrow \infty} \int_{\mathbb{C}^n} \prod_{k=1}^m \left(\sum_{j=1}^n \overline{\frac{\lambda_j f_j^{(k)}}{\sqrt{1-\lambda_j}} \frac{x_j}{\sqrt{2}}} \right) \overline{\prod_{k=1}^m \left(\sum_{j=1}^n \overline{\frac{g_j^{(k)}}{\sqrt{1-\lambda_j}} \frac{x_j}{\sqrt{2}}} \right)} d\lambda_n(x^{(n)}). \end{aligned}$$

By the proof of Corollary 3.5, we have

$$\text{Tr}(e^{-\beta d\Gamma(H-\mu I)}) = \frac{1}{\prod_{j=1}^\infty \sqrt{1-\lambda_j}},$$

thus

$$\begin{aligned} & \omega \left(a_+^*(f^{(1)}) \cdots a_+^*(f^{(m)}) a_+(g^{(1)}) \cdots a_+(g^{(m)}) \right) \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{C}^n} \prod_{k=1}^m \left(\sum_{j=1}^n \overline{\frac{\lambda_j f_j^{(k)}}{\sqrt{1-\lambda_j}} \frac{x_j}{\sqrt{2}}} \right) \overline{\prod_{k=1}^m \left(\sum_{j=1}^n \overline{\frac{g_j^{(k)}}{\sqrt{1-\lambda_j}} \frac{x_j}{\sqrt{2}}} \right)} d\lambda_n(x^{(n)}) \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{C}^\infty} \prod_{k=1}^m \left(\sum_{j=1}^n \overline{\frac{\lambda_j f_j^{(k)}}{\sqrt{1-\lambda_j}}} e_1 \circ l_j(x) \right) \overline{\prod_{k=1}^m \left(\sum_{j=1}^n \overline{\frac{g_j^{(k)}}{\sqrt{1-\lambda_j}}} e_1 \circ l_j(x) \right)} d\lambda_\infty(x) \\ &= \lim_{n \rightarrow \infty} \left\langle \prod_{k=1}^m \sum_{j=1}^n \overline{\frac{\lambda_j f_j^{(k)}}{\sqrt{1-\lambda_j}}} e_1 \circ l_j, \prod_{k=1}^m \sum_{j=1}^n \overline{\frac{g_j^{(k)}}{\sqrt{1-\lambda_j}}} e_1 \circ l_j \right\rangle_{F^2(\mathbb{C}^\infty, d\lambda_\infty)}. \end{aligned}$$

Let

$$\tilde{f}^{(k)} = \frac{e^{-\beta(H-\mu I)}}{\sqrt{1-e^{-\beta(H-\mu I)}}} f^{(k)} \text{ and } \tilde{g}^{(k)} = \frac{1}{\sqrt{1-e^{-(H-\mu I)}}} g^{(k)}.$$

By functional calculus, we have

$$B_H \tilde{f}^{(k)} = B_H \frac{e^{-\beta(H-\mu I)}}{\sqrt{1-e^{-\beta(H-\mu I)}}} f^{(k)} = B_H \sum_{j=1}^\infty \frac{\lambda_j f_j^{(k)}}{\sqrt{1-\lambda_j}} v_j = \sum_{j=1}^\infty \frac{\lambda_j f_j^{(k)}}{\sqrt{1-\lambda_j}} e_1 \circ l_j$$

and

$$B_H \tilde{g}^{(k)} = B_H \frac{1}{\sqrt{1 - e^{-\beta(H - \mu I)}}} g^{(k)} = B_H \sum_{j=1}^{\infty} \frac{g_j^{(k)}}{\sqrt{1 - \lambda_j}} v_j = \sum_{j=1}^{\infty} \frac{g_j^{(k)}}{\sqrt{1 - \lambda_j}} e_1 \circ l_j.$$

Let Q_n be a projection on χ_1 such that

$$Q_n \sum_{j=1}^{\infty} a_j e_1 \circ l_j = \sum_{j=1}^n a_j e_1 \circ l_j$$

for any $\sum_j a_j e_1 \circ l_j \in \chi_1$, further we denote $\hat{Q}_n = B_H^{-1} Q_n B_H$. Then, we have

$$\prod_{k=1}^m \sum_{j=1}^n \frac{\lambda_j f_j^{(k)}}{\sqrt{1 - \lambda_j}} e_1 \circ l_j = \prod_{k=1}^m Q_n \sum_{j=1}^{\infty} \frac{\lambda_j f_j^{(k)}}{\sqrt{1 - \lambda_j}} e_1 \circ l_j = \prod_{k=1}^m Q_n B_H \tilde{f}^{(k)} = \prod_{k=1}^m B_H \hat{Q}_n \tilde{f}^{(k)}$$

and

$$\prod_{k=1}^m \sum_{j=1}^n \frac{g_j^{(k)}}{\sqrt{1 - \lambda_j}} e_1 \circ l_j = \prod_{k=1}^m Q_n \sum_{j=1}^{\infty} \frac{g_j^{(k)}}{\sqrt{1 - \lambda_j}} e_1 \circ l_j = \prod_{k=1}^m Q_n B_H \tilde{g}^{(k)} = \prod_{k=1}^m B_H \hat{Q}_n \tilde{g}^{(k)}.$$

Thus

$$\begin{aligned} & \omega \left(a_+^*(f^{(1)}) \cdots a_+^*(f^{(m)}) a_+(g^{(1)}) \cdots a_+(g^{(m)}) \right) \\ &= \lim_{n \rightarrow \infty} \left\langle \prod_{k=1}^m \sum_{j=1}^n \frac{\lambda_j f_j^{(k)}}{\sqrt{1 - \lambda_j}} e_1 \circ l_j, \prod_{k=1}^m \sum_{j=1}^n \frac{g_j^{(k)}}{\sqrt{1 - \lambda_j}} e_1 \circ l_j \right\rangle_{F^2(\mathbb{C}^\infty, d\lambda_\infty)} \\ &= \lim_{n \rightarrow \infty} \left\langle \prod_{k=1}^m B_H \hat{Q}_n \tilde{f}^{(k)}, \prod_{k=1}^m B_H \hat{Q}_n \tilde{g}^{(k)} \right\rangle_{F^2(\mathbb{C}^\infty, d\lambda_\infty)} \\ &= \lim_{n \rightarrow \infty} \left\langle \sqrt{m!} P_+ \left(\hat{Q}_n \tilde{f}^{(1)} \otimes \cdots \otimes \hat{Q}_n \tilde{f}^{(m)} \right), \sqrt{m!} P_+ \left(\hat{Q}_n \tilde{g}^{(1)} \otimes \cdots \otimes \hat{Q}_n \tilde{g}^{(m)} \right) \right\rangle_{\mathcal{F}_+(\mathcal{H})} \\ &= \left\langle \sqrt{m!} P_+ \left(\tilde{f}^{(1)} \otimes \cdots \otimes \tilde{f}^{(m)} \right), \sqrt{m!} P_+ \left(\tilde{g}^{(1)} \otimes \cdots \otimes \tilde{g}^{(m)} \right) \right\rangle_{\mathcal{F}_+(\mathcal{H})}. \quad \square \end{aligned}$$

4. Fock-Sobolev spaces and Gaussian Harmonic analysis

Gaussian Harmonic analysis has been studied for a long time, see [10]. In this section, we introduce the Fock-Sobolev spaces and discuss its relationship with Gaussian Harmonic analysis. In [13], the author defined the Fock-Sobolev spaces. In [11], the authors used the relationship between the Fock-Sobolev spaces and the Gaussian Harmonic analysis to study the boundedness of an integral operator. We will show that, we have a similar conclusion in the infinite dimensional case. As an application of this relationship, we will study the boundedness of creation operators and annihilation operators.

For any $r \in \mathbb{N}$, the Fock Sobolev space on $\mathbb{C}^n$ consists of all $f \in F^2(\mathbb{C}^n)$ such that

$$\|f\|_{F^{2,r}(\mathbb{C}^n)} = \sum_{k=1}^r \left(\sum_{|\alpha|=k} \int |\partial^\alpha f(z)|^2 d\lambda_n(z) \right)^{1/2} < \infty. \quad (4.1)$$

We need to point out that, in some literature the norm of function in the Fock-Sobolev space is defined by

$$\|f\|_* = \sum_{k=1}^r \sum_{|\alpha|=k} \left(\int_{\mathbb{C}^n} |\partial^\alpha f(z)|^2 d\lambda_n(z) \right)^{1/2}.$$

$\|f\|_*$ and $\|f\|_{F^{2,r}(\mathbb{C}^n)}$ are equivalent for any n but not uniformly equivalent with respect to n . Since we need to generalize the Fock-Sobolev space to the infinite dimensional case, we use (4.1).

In the Fock space on $\mathbb{C}^n$, we have the following equality

$$T_{\overline{e_1(z_j)}} = \partial_{z_j}.$$

Since the Fock space on $\mathbb{C}^\infty$ is not an analytic function space, so we use the Toeplitz operator $T_{\overline{e_1(z_j)}}$ to define the Fock-Sobolev space on $\mathbb{C}^\infty$. Let the Fock-Sobolev space $F^{2,r}(\mathbb{C}^\infty)$ be the completion of finite polynomials with respect to the following norm

$$\|p\|_{F^{2,r}(\mathbb{C}^\infty)} = \sum_{k=1}^r \left(\sum_{j_1, \dots, j_k \geq 1} \int_{\mathbb{C}^\infty} |T_{\overline{e_1(z_{j_1})}} \cdots T_{\overline{e_1(z_{j_k})}} p(z)|^2 d\lambda_\infty(z) \right)^{1/2}.$$

To study the Fock-Sobolev space, we introduce the Gaussian Sobolev space. Let Gaussian measure $d\gamma_n$ on $\mathbb{R}^n$ be given by

$$d\gamma_n(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{2}} dx.$$

The Gaussian Measure can be extended to $\mathbb{R}^\infty$, we denote it by $d\gamma_\infty$. Let $L^2(\mathbb{R}^\infty, d\gamma_\infty)$ consist of all square-integrable function on $\mathbb{R}^\infty$ with respect to $d\gamma_\infty$. For any $k = 0, 1, \dots$, the Hermite polynomials H_k on the real line are defined by the formula

$$H_k(x) = \frac{(-1)^k}{\sqrt{k!}} \exp\left(\frac{x^2}{2}\right) \frac{d^k}{dx^k} \exp\left(-\frac{x^2}{2}\right).$$

By [3, Lemma 1.3.2 and Corollary 1.3.3], we have

$$H'_k(x) = \sqrt{k} H_{k-1}(x) \quad (4.2)$$

and $\{H_k\}$ is an orthonormal basis in the space $L^2(\mathbb{R}, \gamma_1)$. By [3, Lemma 2.5.1 and Example 2.3.5], we have

$$\{H_{\alpha_1}(x_1)H_{\alpha_2}(x_2) \cdots H_{\alpha_n}(x_n) : \sum \alpha_j = k, \quad k = 0, 1, 2, \dots\}$$

is a basis of $L^2(\mathbb{C}^\infty, d\lambda_\infty)$. Let I_k be the projection from $L^2(\mathbb{C}^\infty, d\lambda_\infty)$ to the subspace which is generated by

$$\{H_{\alpha_1}(x_1)H_{\alpha_2}(x_2) \cdots H_{\alpha_n}(x_n) : \sum \alpha_j = k\}.$$

By [3, Chapter 5], the Gaussian Sobolev class is the completion of all finite combination of Hermite polynomials with respect to the norm

$$\|f\|_{W^{2,r}(\mathbb{R}^\infty)} = \sum_{k=1}^r \left(\sum_{j_1, \dots, j_k \geq 1} \int_{\mathbb{C}^\infty} |\partial_{x_{j_1}} \cdots \partial_{x_{j_k}} f(x)|^2 d\gamma_\infty(x) \right)^{1/2}.$$

We also have

$$\|f\|_{W^{2,r}(\mathbb{R}^\infty)} \simeq \left\| \sum_{k=0}^{\infty} (1+k)^{r/2} I_k f \right\|_{L^2(\mathbb{R}^\infty, d\gamma_\infty)}. \quad (4.3)$$

The Gaussian Bargmann transform G is a unitary operator from $L^2(\mathbb{R}^\infty, d\gamma_\infty)$ to $F^2(\mathbb{C}^\infty, d\lambda_\infty)$ such that

$$G[H_{\alpha_1}(x_1)H_{\alpha_2}(x_2)\cdots H_{\alpha_n}(x_n)] = e_{\alpha_1}(z_1)\cdots e_{\alpha_n}(z_n).$$

Let Q_k be the projection from $F^2(\mathbb{C}^\infty, d\lambda_\infty)$ to χ_k . We have $GQ_kG^{-1} = I_k$. It is elementary that $T_{e_1(z_j)}e_{\alpha_j}(z_j) = \sqrt{\alpha_j}e_{\alpha_{j-1}}(z_j)$, thus by (4.2) and the definition of the Fock-Sobolev space and Gaussian Sobolev class we have

$$\|p\|_{F^{2,r}(\mathbb{C}^\infty)} = \|G^{-1}p\|_{W^{2,r}(\mathbb{R}^\infty)} \simeq \left\| \sum_{k=0}^{\infty} (1+k)^{r/2} Q_k p \right\|_{F^2(\mathbb{C}^\infty, d\lambda_\infty)}. \quad (4.4)$$

That is to say the Gaussian Bargmann transform G is also a unitary operator from $W^{2,r}(\mathbb{R}^\infty)$ to $F^{2,r}(\mathbb{C}^\infty, d\lambda_\infty)$.

Proposition 4.1. *For any $\phi \in \chi_1$ and $r \geq 1$, T_ϕ and T_ϕ^- are bounded from $F^{2,r}(\mathbb{C}^\infty)$ to $F^{2,r-1}(\mathbb{C}^\infty)$.*

Proof. Let $\phi = \sum_j c_j e_1(z_j)$. For any $e_{\alpha_1}(z_1)\cdots e_{\alpha_n}(z_n)\cdots$ with $\sum \alpha_j = k$, we have

$$\begin{aligned} & \|T_\phi e_{\alpha_1}(z_1)\cdots e_{\alpha_n}(z_n)\|_{F^2(\mathbb{C}^\infty, d\lambda_\infty)}^2 \\ &= \left\| \sum_j c_j e_1(z_j) e_{\alpha_1}(z_1)\cdots e_{\alpha_n}(z_n)\cdots \right\|_{F^2(\mathbb{C}^\infty, d\lambda_\infty)}^2 \\ &= \left\| \sum_j c_j \sqrt{\alpha_j + 1} e_{\alpha_1}(z_1)\cdots e_{\alpha_{j+1}}(z_j)\cdots \right\|_{F^2(\mathbb{C}^\infty, d\lambda_\infty)}^2 \\ &= \sum_j |c_j|^2 (\alpha_j + 1) \leq \|\phi\|^2 (k+1) \|e_{\alpha_1}(z_1)\cdots e_{\alpha_n}(z_n)\|_{F^2(\mathbb{C}^\infty, d\lambda_\infty)}^2. \end{aligned}$$

Thus, for any $p_k \in \chi_k$, we have

$$\|T_\phi p_k\|_{F^2(\mathbb{C}^\infty, d\lambda_\infty)}^2 \leq \|\phi\|_{F^2(\mathbb{C}^\infty, d\lambda_\infty)}^2 (k+1) \|p_k\|_{F^2(\mathbb{C}^\infty, d\lambda_\infty)}^2$$

and $T_\phi p_k \in \chi_{k+1}$. For any polynomial p , we have

$$\begin{aligned} \|T_\phi p\|_{F^{2,r-1}(\mathbb{C}^\infty)}^2 &= \left\| \sum_k T_\phi Q_k p \right\|_{F^{2,r-1}(\mathbb{C}^\infty)}^2 \\ &\simeq \left\| \sum_k (k+2)^{\frac{r-1}{2}} T_\phi Q_k p \right\|_{F^2(\mathbb{C}^\infty, d\lambda_\infty)}^2 \quad (\text{by (4.4)}) \\ &= \sum_k (k+2)^{r-1} \|T_\phi Q_k p\|_{F^2(\mathbb{C}^\infty, d\lambda_\infty)}^2 \\ &\leq \sum_k (k+2)^{r-1} \|\phi\|_{F^2(\mathbb{C}^\infty, d\lambda_\infty)}^2 (k+1) \|Q_k p\|_{F^2(\mathbb{C}^\infty, d\lambda_\infty)}^2 \\ &\lesssim \|\phi\|_{F^2(\mathbb{C}^\infty, d\lambda_\infty)}^2 \sum_k (k+1)^r \|Q_k p\|_{F^2(\mathbb{C}^\infty, d\lambda_\infty)}^2 \\ &\simeq \|\phi\|_{F^2(\mathbb{C}^\infty, d\lambda_\infty)}^2 \|p\|_{F^{2,r}(\mathbb{C}^\infty)}^2 \quad (\text{by (4.4)}), \end{aligned}$$

thus T_ϕ is bounded from $F^{2,r}(\mathbb{C}^\infty)$ to $F^{2,r-1}(\mathbb{C}^\infty)$. The proof for T_ϕ^- is similar. $\square$

References

- [1] V. Bargmann, On a Hilbert space of analytic functions and an associated integral transform, *Commun. Pure Appl. Math.* 14 (1961) 187–214.
- [2] C.A. Berger, L.A. Coburn, Toeplitz operators and quantum mechanics, *J. Funct. Anal.* 48 (1985) 273–299.
- [3] Vladimir L. Bogachev, *Gaussian Measure*, American Mathematical Society, 1998.
- [4] O. Bratteli, D. Robinson, *Operator Algebras and Quantum Statistical Mechanics II*, Springer-Verlag, Berlin/New York/Heidelberg, 1981.
- [5] J. Janas, K. Rudol, Toeplitz Operators on the Segal-Bargmann Space of Infinitely Many Variables, *Operator Theory: Advances and Applications*, vol. 43, 1990, pp. 217–228.
- [6] J. Janas, K. Rudol, Toeplitz operators in infinitely many variables, in: 15th OT Conference Proceedings, 1995, pp. 147–160.
- [7] Michael Reed, Barry Simon, *Methods of Modern Mathematical Physics, Vol. I*, Academic Press, New York, 1971.
- [8] H.H. Schaefer, *Topological Vector Spaces*, Springer-Verlag, Berlin-New York, 1971.
- [9] Franz Schwabl, *Advanced Quantum Mechanics*, Third Edition, Springer, 2005.
- [10] Wilfredo Urbina-Romero, *Gaussian Harmonic Analysis*, Springer, 2018.
- [11] B.D. Wick, S. Wu, Integral operators on Fock-Sobolev spaces via multipliers on Gauss-Sobolev spaces, [arXiv:2004.05231](https://arxiv.org/abs/2004.05231).
- [12] K. Zhu, *Analysis on Fock Spaces*, Springer, New York, 2012.
- [13] K. Zhu, Fock-Sobolev space and their Carleson measure, *J. Funct. Anal.* 263 (2012) 2483–2506.