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PERSISTENCE OF SUPEROSCILLATIONS UNDER THE SCHRÖDINGER EQUATION

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Abstract. The goal of this paper is to provide new proofs of the persistence of superoscillations under the Schrödinger equation. Superoscillations were first put forward by Aharonov and have since received much study because of connections to physics, engineering, signal processing and information theory. An interesting mathematical question is to understand the time evolution of superoscillations under certain Schrödinger equations arising in physics. This paper provides an alternative proof of the persistence of superoscillations by some elementary convergence facts for sequence and series and some connections with certain polynomials and identities in combinatorics. The approach given opens new perspectives to establish persistence of superoscillations for the Schrödinger equation with more general potentials.

1. **Introduction.** Superoscillation is a phenomenon in which a signal which is globally band-limited can contain local segments that oscillate faster than its fastest Fourier components. An example of a function exhibiting this phenomenon is

$$F_N(x,a) = \left(\cos\frac{x}{N} + ia\sin\frac{x}{N}\right)^N = \sum_{k=0}^{N} C_k(N,a)e^{ix\left(1-\frac{2k}{N}\right)}$$

where N is a large integer and $a>1$ and $C_k(N,a) = \binom{N}{k}\left(\frac{1+a}{2}\right)^{N-k}\left(\frac{1-a}{2}\right)^k$. This function is periodic with period $N\pi$, but when expanded in its Fourier series, each individual component function has frequency $k_{k,N} = 1-\frac{2k}{N}$ which is bounded by 1. One can further show that $F_N(x,a)\to e^{iax}$ and that when $|x|\leq\sqrt{N}$ then $F_N(x,a)$ can be well approximated by e^{iax}. Superoscillations and the phenomenon was originally attributed to Yakir Aharonov, and then appeared in work by Michael Berry and similar concepts were known to Ingrid Daubechies.

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Superoscillations have come to play a role in quantum physics, in engineering through superresolutions and in signal processing and information theory. They also have been studied for purely mathematical reasons because of their interesting analytic properties and connections to approximation and harmonic analysis; the interested reader can consult [18, 16, 5, 9, 11, 12] for more details.

An important question in the area is to understand the evolution of a superoscillating function under a Schrödinger equation and determine if the superoscillating phenomenon persists under time evolution. In particular, one studies the following problem. Let $\psi_n(x,t)$ and $\psi(x,t)$ denote solutions to the following Schrödinger equations:

$$\begin{array}{rcl} i\frac{\partial \psi_n}{\partial t}(x,t) & = & \mathcal{H}(x,t)\psi_n(x,t) \\ \psi_n(x,0) & = & F_n(x,a) \end{array} \qquad \begin{array}{rcl} i\frac{\partial \psi}{\partial t}(x,t) & = & \mathcal{H}(x,t)\psi(x,t) \\ \psi(x,0) & = & e^{iax} \end{array}$$

where $\mathcal{H}(x,t)$ is an appropriate Hamiltonian for the system being studied and $F_n(x,a)$ is the superoscillating function above. Under what conditions do we have that $\lim_{n\to\infty} \psi_n(x,t) := \psi(x,t)$? In the following papers, [3, 4, 6, 19, 10, 5, 2, 8] versions of this question were studied when $\mathcal{H}(x,t)$ is the Laplacian, the quantum oscillator, and natural Hamiltonian's arising in physics. The method of proof was to cleverly connect the solutions to certain infinite order differential operators and their behavior on spaces of analytic functions. The convergence results then follow from continuity results for these operators on spaces of analytic functions. In this paper we take a different approach at studying these convergence results and obtain similar results.

Studying the results in literature around superoscillations point to the understanding of moments of the Green's functions. Let $\mathcal{K}$ be the Green's function associated with the one-dimensional linear Schrödinger problem:

$$\begin{array}{rcl} i\frac{\partial \mathcal{K}}{\partial t}(x,y,t) & = & \mathcal{H}(x,t)\mathcal{K}(x,y,t) \\ \mathcal{K}(x,y,0) & = & \delta(x-y). \end{array}$$

Then the *normalized moments* of the Green's function defined by

$$b_m(x,t) := \frac{1}{m!}\int_{\mathbb{R}} y^m \mathcal{K}(x,y,t)dy \quad x,t\in\mathbb{R} \quad m\in\mathbb{N}$$

play a fundamental role if we have $\lim_{n\to\infty}\psi_n(x,t) = \psi(x,t)$. Our main result is the following:

Theorem 1.1. *Let $x,t\in\mathbb{R}$. Suppose that $\sum_{m=0}^{\infty} m|b_m(x,t)|(2^3(1+a))^m = M_{x,t} < \infty$, then $\lim_{n\to\infty}\psi_n(x,t) = \psi(x,t)$. If we further have that $\sup_{x,t\in\mathbb{R}} M_{x,t} < \infty$, then we have for any $x,t\in\mathbb{R}$ that $\lim_{n\to\infty}\psi_n(x,t) = \psi(x,t)$.*

In Section 2 we collect some simple and direct computations that will be important in our analysis. These computations reduce to some basic manipulations and estimates of power series. In particular, we will point out connections to certain fundamental polynomials and identities that arise in combinatorics and approximation theory. However, from these estimates we are able to develop some tools that allow us to recover many of the results on superoscillations in the papers [6, 19, 10, 5, 2, 8, 14], by appealing to simple convergence facts of sequences and to avoid the connections to infinite order differential operators and spaces of entire functions. Further, we are able to obtain some information about rates of convergence of the superoscillating sequence to the given function. Our main theorem, Theorem 2.3, appears there and shows that if the Green's function associated to a linear Schrödinger equation possesses some appropriate decay as measured in moments then persistence of superoscillations will occur. This theorem is implicit in the work of [2], where something like this was done for a special case; our contribution is to extract out the general phenomenon and use it in the known instances of persistence of superoscillations. In Section 3 we show how the tools developed in Section 2, in particular Proposition 1, can be used to directly recover some mathematical facts about superoscillations in the literature. In Section 4, we compute explicitly the normalized moment of order m of the Green's function for the free particle Schrödinger equation, the harmonic oscillator, the Schrödinger equation with linear potential, the one with centrifugal potential and finally for the centripetal barrier oscillator. Appealing to Theorem 2.3 then gives another proof of the results in the literature surrounding superoscillations. The method we used should permit to get results with the Green's function for the Schrödinger equation with other potentials; but the computations are surely much more complicated.

The authors thank Juliette Leblond for pointing out reference [19] and Pamela Gorkin for suggesting an improvement of Lemma 2.2; both are also thanked for comments about an early draft of this paper.

2. **Preliminary results on superoscillating functions.** The first lemma we observe is purely combinatorial and is a direct computation from the definitions. It is here that we encounter the connection with Sterling numbers and some basics from approximation theory.

Lemma 2.1. *For* $1 \leq p \leq n$*,*

$$\mathcal{P}(n,a) = \sum_{k=0}^{n} C_k(n,a) \left(\frac{2k}{n}\right)^p = \sum_{l=1}^{p} s_n(p,l)(1-a)^l,$$

with

$$s_n(p,l) = \left(\frac{2}{n}\right)^{p-l} \prod_{j=0}^{l-1} \frac{n-j}{n} S(p,l),$$

and where $S(p,l)$ *is the Stirling number of second kind defined by*

$$S(p,l) = \frac{1}{l!} \sum_{i=0}^{l} (-1)^i \binom{l}{i} (l-i)^p.$$

Proof. We use the following well-known identity:

$$k^p = \sum_{l=1}^{p} S(p,l)(k)_l, \text{ where } (k)_l = k(k-1)(k-2)\cdots(k-l+1). \tag{1}$$

For an integer $n \geq 1$, and $1 \leq p \leq n$,

$$\begin{aligned}\mathcal{P}(n,a) &= \frac{2^p}{n^p}\sum_{k=0}^{n} C_k(n,a)k^p \\ &= \frac{2^p}{n^p}\sum_{k=0}^{n} C_k(n,a)\sum_{l=1}^{p} S(p,l)(k)_l \\ &= \sum_{l=1}^{p}\frac{2^p}{n^{p-l}}S(p,l)\sum_{k=l}^{n}\binom{n}{k}\frac{(k)_l}{n^l}\left(\frac{1-a}{2}\right)^k\left(\frac{1+a}{2}\right)^{n-k}.\end{aligned}$$

Using that $\frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$, one can get recursively that

$$\frac{(k)_l}{n^l}\binom{n}{k} = \binom{n-l}{k-l}\prod_{j=0}^{l-1}\frac{n-j}{n},$$

and we obtain that

$$\begin{aligned}\mathcal{P}(n,a) &= \sum_{l=1}^{p}\frac{2^p}{n^{p-l}}S(p,l)\sum_{k=l}^{n}\binom{n-l}{k-l}\prod_{j=0}^{l-1}\frac{n-j}{n}\left(\frac{1-a}{2}\right)^k\left(\frac{1+a}{2}\right)^{n-k} \\ &= 2^{p-n}\sum_{l=1}^{p}\frac{1}{n^{p-l}}S(p,l)\sum_{k=l}^{n}\binom{n-l}{k-l}\prod_{j=0}^{l-1}\frac{n-j}{n}(1-a)^k(1+a)^{n-k} \\ &= 2^{p-n}\sum_{l=1}^{p}\frac{(1-a)^l}{n^{p-l}}\prod_{j=0}^{l-1}\frac{n-j}{n}S(p,l)\sum_{k=l}^{n}\binom{n-l}{k-l}(1-a)^{k-l}(1+a)^{n-l-(k-l)} \\ &= 2^{p-n}\sum_{l=1}^{p}\frac{(1-a)^l}{n^{p-l}}\prod_{j=0}^{l-1}\frac{n-j}{n}S(p,l)\sum_{k=0}^{n-l}\binom{n-l}{k}(1-a)^k(1+a)^{n-l-k} \\ &= 2^{p-n}\sum_{l=1}^{p}\frac{(1-a)^l}{n^{p-l}}\prod_{j=0}^{l-1}\frac{n-j}{n}S(p,l)2^{n-l} \\ &= \sum_{l=1}^{p}\frac{2^{p-l}}{n^{p-l}}(1-a)^l\prod_{j=0}^{l-1}\frac{n-j}{n}S(p,l),\end{aligned}$$

which proves the lemma. □

We next collect an elementary estimate.

Lemma 2.2. *For any positive integers j and n we have:*

$$\left|\prod_{l=0}^{j-1}\frac{n-l}{n} - 1\right| \le \frac{2^{j-1}j}{n}. \tag{2}$$

Proof. We make the following computational observation:

$$\begin{aligned}
\prod_{l=0}^{j-1}\frac{n-l}{n}-1 &= \prod_{l=0}^{j-2}\frac{n-l}{n}\frac{n-j+1}{n}-1\\
&= \left(\prod_{l=0}^{j-2}\frac{n-l}{n}-1+1\right)\frac{n-j+1}{n}-1\\
&= \left(\prod_{l=0}^{j-2}\frac{n-l}{n}-1\right)\frac{n-j+1}{n}+\frac{n-j+1}{n}-1\\
&= \left(\prod_{l=0}^{j-2}\frac{n-l}{n}-1\right)\frac{n-j+1}{n}+\frac{-j+1}{n}.
\end{aligned}$$

This then implies that:

$$\left|\prod_{l=0}^{j-1}\frac{n-l}{n}-1\right| \le \frac{j-1}{n}+\left|\prod_{l=0}^{j-2}\frac{n-l}{n}-1\right|.$$

This can then be iterated and leads to:

$$\left|\prod_{l=0}^{j-1}\frac{n-l}{n}-1\right| \le \sum_{l=1}^{j-1}\frac{l}{n}=\frac{(j-1)j}{2n}.$$

□

Proposition 1. *Let $a\in\mathbb{R}$ with $a>1$. We have the following statements being true:*

(*i*) *Let $\{b_m\}$ be a sequence of complex numbers such that $\sum_{m=0}^{\infty} m|b_m|(2^3(1+a))^m$ converges. Then the sequence $\sum_{k=0}^{n}C_k(n,a)\sum_{m=0}^{\infty}b_m\left(1-\frac{2k}{n}\right)^m$ converges to $\sum_{m=0}^{\infty}b_m a^m$.*

(*ii*) *Let $P(z)=\sum_{l=0}^{N}p_l z^l$ be a holomorphic polynomial of degree N. The sequence $\sum_{k=0}^{n}C_k(n,a)e^{P\left(1-\frac{2k}{n}\right)}$ converges to $e^{P(a)}$.*

(*iii*) *Let $G(z)=\sum_{l=0}^{\infty}g_l z^l$ be an analytic function on the disc $D_{8(1+a)+\epsilon}(0)$. The sequence $\sum_{k=0}^{n}C_k(n,a)e^{G\left(\left(1-\frac{2k}{n}\right)\right)}$ converges to $e^{G(a)}$.*

Proof. The main idea behind the proof of (*i*) is to compare what we want for the limit with what is given. We are able to extract terms which obviously converge to 0 and others that need to be paired appropriately for the limit to converge. We will apply Lemma 2.1 and some direct estimates and computations in the course of the proof.

First observe the following:

$$
\begin{aligned}
&\sum_{k=0}^{n} C_k(n,a) \sum_{m=0}^{\infty} b_m \left(1-\frac{2k}{n}\right)^m \\
&\quad = \sum_{k=0}^{n} C_k(n,a) \left[\sum_{m=0}^{n} b_m \left(1-\frac{2k}{n}\right)^m + \sum_{m=n+1}^{\infty} b_m \left(1-\frac{2k}{n}\right)^m\right] \\
&\quad = \sum_{k=0}^{n} C_k(n,a) \sum_{m=0}^{n} b_m \left(1-\frac{2k}{n}\right)^m + \sum_{k=0}^{n} C_k(n,a) \sum_{m=n+1}^{\infty} b_m \left(1-\frac{2k}{n}\right)^m \\
&\quad = \sum_{m=0}^{n} b_m \sum_{k=0}^{n} C_k(n,a) \left(1-\frac{2k}{n}\right)^m + \sum_{m=n+1}^{\infty} b_m \sum_{k=0}^{n} C_k(n,a) \left(1-\frac{2k}{n}\right)^m \\
&\quad = A_n + B_n.
\end{aligned}
$$

Next, observe that

$$
\begin{aligned}
|B_n| &= \left| \sum_{m=n+1}^{\infty} b_m \sum_{k=0}^{n} C_k(n,a) \left(1-\frac{2k}{n}\right)^m \right| \\
&\leq \sum_{m=n+1}^{\infty} \frac{|b_m|}{2^n} \sum_{k=0}^{n} \binom{n}{k} (1+a)^{2n} = (1+a)^{2n} \sum_{m=n+1}^{\infty} |b_m|,
\end{aligned}
$$

where we used that $\sum_{m=n+1}^{\infty} |b_m|$ is the remainder term of the Taylor series expansion of the function $\sum_{m=0}^{\infty} b_m z^m$ (evaluated at $z=1$). Thus, $\lim_{n\to\infty} |B_n| = 0$. The condition on $\{b_m\}$ in the hypothesis is more than is needed to guarantee the convergence at this step. For $m \geq 1$, we have using the Binomial Theorem and by Lemma 2.1

$$
\begin{aligned}
&\sum_{k=0}^{n} C_k(n,a) \left(1-\frac{2k}{n}\right)^m \\
&\qquad = \sum_{j=0}^{m} \binom{m}{j} (-1)^j \sum_{k=0}^{n} C_k(n,a) \left(\frac{2k}{n}\right)^j \\
&\qquad = 1 + \sum_{j=1}^{m} \binom{m}{j} (-1)^j \sum_{k=0}^{n} C_k(n,a) \left(\frac{2k}{n}\right)^j \\
&\qquad = 1 + \sum_{j=1}^{m} \binom{m}{j} (-1)^j \left(\sum_{k=1}^{j} s_n(j,k)(1-a)^k\right) \\
&\qquad = 1 + \sum_{j=1}^{m} \binom{m}{j} (-1)^j \left(s_n(j,j)(1-a)^j + \sum_{k=1}^{j-1} s_n(j,k)(1-a)^k\right)
\end{aligned}
$$

$$
\begin{aligned}
&= 1 + \sum_{j=1}^{m} \binom{m}{j} (-1)^j \left(\prod_{l=0}^{j-1} \frac{n-l}{n} \right) (1-a)^j \\
&\quad + \sum_{j=1}^{m} \binom{m}{j} (-1)^j \sum_{k=1}^{j-1} s_n(j,k)(1-a)^k \\
&= 1 + \sum_{j=1}^{m} \binom{m}{j} (-1)^j \left(\prod_{l=0}^{j-1} \frac{n-l}{n} \right) (1-a)^j \\
&\quad + \sum_{j=0}^{m} \binom{m}{j} (-1)^j \sum_{k=1}^{j-1} (1-a)^k \left(\frac{2}{n} \right)^{j-k} \left(\prod_{l=0}^{k-1} \frac{n-l}{n} \right) S(j,k) \\
&= 1 + \sum_{j=1}^{m} \binom{m}{j} \left(\prod_{l=0}^{j-1} \frac{n-l}{n} \right) (a-1)^j \\
&\quad + \sum_{j=0}^{m} \binom{m}{j} (-1)^j \sum_{k=1}^{j-1} (1-a)^k \left(\frac{2}{n} \right)^{j-k} \left(\prod_{l=0}^{k-1} \frac{n-l}{n} \right) S(j,k),
\end{aligned}
$$

where

$$
I_n(m) := 1 + \sum_{j=1}^{m} \binom{m}{j} \left(\prod_{l=0}^{j-1} \frac{n-l}{n} \right) (a-1)^j \tag{3}
$$

$$
II_n(m) := \sum_{j=0}^{m} \binom{m}{j} (-1)^j \sum_{k=1}^{j-1} (1-a)^k \left(\frac{2}{n} \right)^{j-k} \left(\prod_{l=0}^{k-1} \frac{n-l}{n} \right) S(j,k). \tag{4}
$$

Next, observe that:

$$
\begin{aligned}
1 + \sum_{j=1}^{m} \binom{m}{j} \left(\prod_{l=0}^{j-1} \frac{n-l}{n} \right) (a-1)^j &= 1 + \sum_{j=1}^{m} \binom{m}{j} \left(\prod_{l=0}^{j-1} \frac{n-l}{n} - 1 \right) (a-1)^j \\
&\quad + \sum_{j=1}^{m} \binom{m}{j} (a-1)^j \\
&= a^m + \sum_{j=1}^{m} \binom{m}{j} \left(\prod_{l=0}^{j-1} \frac{n-l}{n} - 1 \right) (a-1)^j.
\end{aligned}
$$

We have that

$$
\begin{aligned}
\left| \sum_{m=0}^{n} b_m I_n(m) - \sum_{m=0}^{n} b_m a^m \right| &= \left| \sum_{m=0}^{n} b_m \left(1 + \sum_{j=1}^{m} \binom{m}{j} \left(\prod_{l=0}^{j-1} \frac{n-l}{n} \right) (a-1)^j - a^m \right) \right| \\
&= \left| \sum_{p=0}^{n} b_m \left(\sum_{j=1}^{m} \binom{m}{j} \left(\prod_{l=0}^{j-1} \frac{n-l}{n} - 1 \right) (a-1)^j \right) \right|.
\end{aligned}
$$

Thus, we get via simple estimates and Lemma 2.2 that

$$\begin{aligned}\left|\sum_{m=0}^{n} b_m I_n(m) - \sum_{m=0}^{n} b_m\right| &\leq \left|\sum_{m=0}^{n} b_m \left(\sum_{j=1}^{m}\binom{m}{j}\left(\prod_{l=0}^{j-1}\frac{n-l}{n}-1\right)(a-1)^j\right)\right| \\ &\leq \sum_{m=0}^{n}|b_m|\sum_{j=1}^{m}\binom{m}{j}\left|\prod_{l=0}^{j-1}\frac{n-l}{n}-1\right|(1+a)^j \\ &\leq \sum_{m=0}^{n}|b_m|\sum_{j=1}^{m}\binom{m}{j}\frac{(j-1)j}{2n}(1+a)^j \\ &\leq \frac{1}{2n}\sum_{m=1}^{n} m^2|b_m|(2+a)^m.\end{aligned}$$

By the conditions on the sequence $\{b_m\}$ we have that this last expression tends to 0 when n tends to ∞. It follows that

$$\lim_{n\to\infty}\left|\sum_{m=0}^{n} b_m I_n(m) - \sum_{m=0}^{n} b_m a^m\right| = \lim_{n\to\infty}\left|\sum_{m=0}^{n} b_m\left(I_n(m)-a^m\right)\right| = 0.$$

To get an upper bound of $II_n(m)$, we use again that $S(j,k)\leq k^{j-k}\binom{j}{k}$ to obtain that

$$\begin{aligned}|II_n(m)| &\leq \sum_{j=0}^{m}\binom{m}{j}\sum_{k=1}^{j-1}(1+a)^k\left(\frac{2}{n}\right)^{j-k}S(j,k) \\ &\leq \sum_{j=0}^{m}\binom{m}{j}\sum_{k=1}^{j-1}(1+a)^k\left(\frac{2k}{n}\right)^{j-k}\binom{j}{k} \\ &\leq \sum_{j=0}^{m}\binom{m}{j}\frac{2^{j-1}}{n}\sum_{k=1}^{j-1}(1+a)^k\left(\frac{k}{n}\right)^{j-k-1}k\binom{j}{k} \\ &\leq \sum_{j=0}^{m}\binom{m}{j}\frac{2^{j-1}(1+a)^{j-1}}{n}\sum_{k=1}^{j-1}j\binom{j-1}{k-1} \\ &\leq \sum_{j=0}^{m}\binom{m}{j}\frac{(2(1+a))^{j-1}}{n}\left(2^{j-1}-1\right) \\ &\leq \sum_{j=0}^{m}\binom{m}{j}j\frac{(4(1+a))^{j-1}}{n} \\ &\leq m\frac{(1+a)^m 2^{3m}}{n}.\end{aligned}$$

Thus, we get that

$$\begin{aligned}\left|\sum_{m=0}^{n} b_m II_n(m)\right| &\leq \sum_{m=0}^{n} |b_m| II_n(m)\\ &\leq \frac{1}{n}\sum_{m=0}^{n} m|b_m|(2^3(1+a))^m\\ &\leq \frac{1}{n}\sum_{m=0}^{\infty} m|b_m|(2^3(1+a))^m,\end{aligned}$$

which means that $\lim_{n\to\infty}\left|\sum_{m=0}^{n} b_m II_n(m)\right| = 0$ by the convergence of the series of general term $m|b_m|(2^3(1+a))^m$.

It follows that

$$\begin{aligned}&\left|\sum_{m=0}^{\infty} b_m \sum_{k=0}^{n} C_k(n,a)\left(1-\frac{2k}{n}\right)^m - \sum_{m=0}^{\infty} b_m a^m\right|\\ &\quad = \left|\sum_{m=0}^{n} b_m\left(I_n(m)-a^m\right) + \sum_{m=n+1}^{\infty} b_m a^m + \sum_{m=0}^{n} b_m II_n(m) + B_n\right|\\ &\quad \leq \left|\sum_{m=0}^{n} b_m\left(I_n(m)-a^m\right)\right| + \left|\sum_{m=n+1}^{\infty} b_m a^m\right| + \left|\sum_{p=0}^{n} b_m II_n(m)\right| + |B_n|\,.\end{aligned}$$

It is then clear that all these terms tend to zero as n tends to infinity giving us

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=0}^{n} C_k(n,a)\sum_{m=0}^{\infty} b_m\left(1-\frac{2k}{n}\right)^m &= \lim_{n\to\infty}\sum_{m=0}^{\infty} b_m\sum_{k=0}^{n} C_k(n,a)\left(1-\frac{2k}{n}\right)^m\\ &= \sum_{m=0}^{\infty} b_m a^m.\end{aligned}$$

Note that we also have that for any $\xi\in\mathbb{T}$ we have

$$\begin{aligned}&\lim_{n\to\infty}\sum_{k=0}^{n} C_k(n,a)\sum_{m=0}^{\infty} b_m\left(\xi\left(1-\frac{2k}{n}\right)\right)^m\\ &\quad = \lim_{n\to\infty}\sum_{m=0}^{\infty} b_m\sum_{k=0}^{n} C_k(n,a)\left(\xi\left(1-\frac{2k}{n}\right)\right)^m\\ &\quad = \sum_{m=0}^{\infty} b_m\left(\xi a\right)^m.\end{aligned}$$

The proof of (ii) follows immediately from (i) since one can use that $e^{P(z)}$ has a power series representation and can compute the coefficients for the power series. Indeed, one has that for $P(z)=\sum_{l=0}^{N} p_l z^l$ one can compute that:

$$e^{P(z)} = \sum_{m=0}^{\infty}\frac{P_m(p_1,\ldots,p_N)}{m!}z^m$$

where $P_m(p_1,\ldots,p_N)$ is a polynomial of degree m in the coefficients of the polynomial $P(z)$ (The case when $P(z)=p_1z+p_2z^2$ is especially interesting since the coefficients $P_m(p_1,p_2)$ are related to the expansion of z^m via Hermite polynomials; other cases should be equally interesting to explore!). Now since $e^{P(z)}$ is an entire function we have that for any $L\in\mathbb{R}$:

$$\sum_{m=0}^{\infty}\frac{|P_m(p_1,\ldots,p_N)|}{m!}L^m<\infty.$$

From this condition, and the analyticity of $e^{P(z)}$, one readily sees that:

$$\sum_{m=0}^{\infty}m\frac{|P_m(p_1,\ldots,p_N)|}{m!}(2^3(1+a))^m<\infty.$$

But, this then implies, by part (i) that

$$\begin{aligned}\sum_{k=0}^{n}C_k(n,a)e^{P\left(1-\frac{2k}{n}\right)} &= \sum_{k=0}^{n}C_k(n,a)\sum_{m=0}^{\infty}\frac{P_m(p_1,\ldots,p_N)}{m!}\left(1-\frac{2k}{n}\right)^m\\ &\to \sum_{m=0}^{\infty}\frac{P_m(p_1,\ldots,p_N)}{m!}a^m=e^{P(a)}\end{aligned}$$

by obvious choice of coefficients b_m.

We now turn case (iii) of general analytic functions. Suppose $G(z)=\sum_{l=0}^{\infty}g_lz^l$ satisfies the hypotheses of the proposition. We can expand $e^{G(z)}$ in a power series on the disc $D_{8(1+a)+\epsilon}(0)$ to have:

$$e^{G(z)}=\sum_{m=0}^{\infty}\frac{P_m(g_1,\ldots,g_m)}{m!}z^m$$

where $P_m(g_1,\ldots,g_m)$ is a polynomial in the first m coefficients of the power series expansion of G. Again the hypotheses imply that

$$\sum_{m=0}^{\infty}m\frac{|P_m(g_1,\ldots,g_m)|}{m!}(2^3(1+a))^m<\infty.$$

since the function $e^{G(z)}$ converges on the disc $D_{8(1+a)+\epsilon}(0)$, But, this then implies, by part (i) that

$$\begin{aligned}\sum_{k=0}^{n}C_k(n,a)e^{G\left(1-\frac{2k}{n}\right)} &= \sum_{k=0}^{n}C_k(n,a)\sum_{m=0}^{\infty}\frac{P_m(g_1,\ldots,g_m)}{m!}\left(1-\frac{2k}{n}\right)^m\\ &\to \sum_{m=0}^{\infty}\frac{P_m(g_1,\ldots,g_m)}{m!}a^m=e^{G(a)}\end{aligned}$$

by obvious choice of coefficients b_m. □

Remark 1. Note that the general estimates we proved above show that, if we denote $\psi_n:=\sum_{k=0}^{n}C_k(n,a)\sum_{m=0}^{\infty}b_m\left(1-\frac{2k}{n}\right)^m$ and $\psi:=\sum_{m=0}^{\infty}b_ma^m$ then is at least

$$|\psi_n-\psi|=O\left(\frac{1}{n}\right)$$

for the rate of convergence. One can of course do better when we have explicit information on the sequence $\{b_m\}$ and can exploit that to extract better convergence information.

Remark 2. The condition we have imposed upon $\{b_m\}$ is an artifact of the proof. It is a reasonable condition in that it reflects some information about the sequence of coefficients $\{b_m\}$ and the number $a > 1$. However, in many of the examples, since we are after some "uniformity" when determining the limit we typically have a stronger condition on $\{b_m\}$. For example a condition typical is that the coefficients satisfy

$$\lim_{m\to\infty} \sqrt[m]{|b_m|\, m!} = 0$$

which is saying that the function $b(z) = \sum_{m=0}^{\infty} b_m z^m$ is of infraexponential type.

We now state and prove the main result of this section. Recall that we are interested in the following problem. For the solutions $\psi_n(x,t)$ and $\psi(x,t)$ to the linear Schrödinger equations:

$$\begin{array}{rcl} i\frac{\partial \psi_n}{\partial t}(x,t) &=& \mathcal{H}(x,t)\psi(x,t) \\ \psi_n(x,0) &=& F_n(x,a) \end{array} \qquad \begin{array}{rcl} i\frac{\partial \psi}{\partial t}(x,t) &=& \mathcal{H}(x,t)\psi(x,t) \\ \psi(x,0) &=& e^{iax} \end{array}$$

when does $\lim_{n\to\infty} \psi_n(x,t) = \psi(x,t)$? We now demonstrate that when the moments of the Green's function satisfy certain decay, then this condition is sufficient to guarantee the desired convergence.

Let $\mathcal{K}$ be the Green's function associated with the one-dimensional linear Schrödinger problem:

$$\begin{array}{rcl} i\frac{\partial \mathcal{K}}{\partial t}(x,y,t) &=& \mathcal{H}(x,t)\mathcal{K}(x,y,t) \\ \mathcal{K}(x,y,0) &=& \delta(x-y). \end{array}$$

Define

$$b_m(x,t) := \frac{1}{m!}\int_{\mathbb{R}} y^m \mathcal{K}(x,y,t)dy \quad x,t\in\mathbb{R} \quad m\in\mathbb{N}.$$

The main result then shows that these normalized moments determine the convergence.

Theorem 2.3. *Let $x,t\in\mathbb{R}$. Suppose that $\sum_{m=0}^{\infty} m|b_m(x,t)|(2^3(1+a))^m = M_{x,t} < \infty$. Then $\lim_{n\to\infty} \psi_n(x,t) = \psi(x,t)$. If we further have that $\sup_{x,t\in\mathbb{R}} M_{x,t} < \infty$, then we have for any $x,t\in\mathbb{R}$ that $\lim_{n\to\infty} \psi_n(x,t) = \psi(x,t)$.*

Proof. Observe that by the definition of the Green's function and the two different equations being considered we have

$$\psi(x,t) = \int_{\mathbb{R}} e^{iay}\mathcal{K}(x,y,t)dy = \sum_{m=0}^{\infty} \frac{(ia)^m}{m!} \int_{\mathbb{R}} y^m \mathcal{K}(x,y,t)dt = \sum_{m=0}^{\infty} b_m(x,t)(ia)^m;$$

$$\begin{aligned}\psi_n(x,t) &= \int_{\mathbb{R}} F_n(y,a)\mathcal{K}(x,y,t)dy \\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{n} C_k(n,a)\left(i\left(1-\frac{2k}{n}\right)\right)^m \frac{1}{m!}\int_{\mathbb{R}} y^m\mathcal{K}(x,y,t)dy \\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{n} C_k(n,a)b_m(x,t)\left(i\left(1-\frac{2k}{n}\right)\right)^m.\end{aligned}$$

The interchange of the series and the integral is justified by the hypotheses on normalized moments of the Green's function. The form of $\psi(x,t)$ and $\psi_n(x,t)$ is exactly as in Proposition 1 and so then we have that $\lim_{n\to\infty} \psi_n(x,t) = \psi(x,t)$, giving the result. □

Remark 3. Note that the condition of convergence of the series with general term $m|b_m(x,t)|(2^3(1+a))^m$ is satisfied if

$$\lim_{m\to+\infty} \frac{|b_{m+1}|}{|b_m|} = \ell,$$

with $\ell < \frac{1}{2^3(1+a)}$ for a fixed $a > 1$.

Remark 4. The above theorem provides a sufficient condition on the Green's function $\mathcal{K}(x,y,t)$ and the Hamiltonian $\mathcal{H}(x,t)$ in terms of moments. It would be interesting to know if any condition of this type is also necessary for the desired conclusion. Additionally, we remark that a theorem of this type provides a "simple" test that one can preform to deduce when superoscillations persist and reduces to just verifying some decay conditions on the moments of the Green's function.

3. Time-dependent Schrödinger equation and superoscillations.

In this section, we recover the results in [6, 19, 10, 5, 2, 8, 14] by showing how the convergence results in Section 2, in particular Proposition 1, can be used to study the persistence of superoscillations under the evolution of the Schrödinger equation.

In the subsections below, we look at the problem:

$$i\frac{\partial\psi_n}{\partial t}(x,t) = \mathcal{H}(x,t)\psi_n(x,t) \qquad\qquad i\frac{\partial\psi}{\partial t}(x,t) = \mathcal{H}(x,t)\psi(x,t)$$

$$\psi_n(x,0) = F_n(x,a) = \sum_{k=0}^{n} C_k(n,a)e^{ix\left(1-\frac{2k}{n}\right)} \qquad\qquad \psi(x,0) = e^{iax}.$$

where $\mathcal{H}(x,t)$ is an appropriate Hamiltonian differential operator. We show how the results above let us deduce that $\lim_{n\to\infty} \psi_n(x,t) = \psi(x,t)$. One will use the appropriate solution operator to determine explicit formulas for $\psi_n(x,t)$ and $\psi(x,t)$ and then the convergence results will follow from the techniques developed in Section 2.

3.1. **The Hamiltonian $\mathcal{H} = -\Delta$.** In [10] the authors studied the following Schrödinger equation:

$$\begin{aligned} i\frac{\partial \psi_n}{\partial t}(x,t) &= -\Delta\psi_n(x,t) \\ \psi_n(x,0) &= F_n(x,a) = \sum_{k=0}^{n} C_k(n,a) e^{ix\left(1-\frac{2k}{n}\right)}. \end{aligned}$$

Using the Fourier Transform, they first prove:

Theorem 3.1 ([10, Theorem 1.1])**.** *The time evolution of these solutions takes the following form:*

$$\psi_n(x,t) = \sum_{k=0}^{n} C_k(n,a) e^{ix\left(1-\frac{2k}{n}\right)} e^{-it\left(1-\frac{2k}{n}\right)^2}.$$

The proof of this theorem uses the technique of taking the Fourier transform of the partial differential equation to turn it into a differential equation and then solves the differential equation using standard methods; we do not reproduce the argument here.

Notice that the form of the function $\psi_n(x,t)$ then falls under Proposition 1, part (ii) by taking $P(z) = xz - tz^2$ (here x and t are fixed and treated as the coefficients of the polynomial), and so we have:

Theorem 3.2 ([10, Corollary 3.4])**.** *For $a > 1$ and for all $x, t \in \mathbb{R}$ we have:*

$$\lim_{n\to\infty} \psi_n(x,t) = \lim_{n\to\infty} \sum_{k=0}^{n} C_k(n,a) e^{ix\left(1-\frac{2k}{n}\right)} e^{-it\left(1-\frac{2k}{n}\right)^2} = e^{iax - ia^2 t}.$$

3.2. **The quantum harmonic oscillator.** We next turn to the situation of the harmonic oscillator

$$\begin{aligned} i\frac{\partial \psi_n}{\partial t}(x,t) &= \frac{1}{2}\left(-\Delta + x^2\right)\psi_n(x,t) \\ \psi_n(x,0) &= F_n(x,a) = \sum_{k=0}^{n} C_k(n,a) e^{ix\left(1-\frac{2k}{n}\right)}. \end{aligned}$$

The persistence of superoscillations of this Schrödinger equation was studied in [5]; again we can recover the results in that paper by using the methods of Section 2. The authors of [5] first prove

Theorem 3.3 ([5, Theorem 5.3.2])**.** *The time evolution of these solutions takes the following form:*

$$\psi_n(x,t) = \frac{e^{-\frac{i}{2}x^2 \tan t}}{(\cos t)^{\frac{1}{2}}} \sum_{k=0}^{n} C_k(n,a) e^{i\frac{x}{\cos t}\left(1-\frac{2k}{n}\right)} e^{-\frac{i}{2}\left(1-\frac{2k}{n}\right)^2 \tan t}.$$

The proof of this theorem in [5] utilizes the Green's function for the Schrödinger operator to arrive at an explicit form for the solutions $\psi_n(x,t)$. By taking $P(z) = \frac{x}{\cos t} z - \frac{\tan t}{2} z^2$, noting that for fixed x and t that $\frac{e^{-\frac{i}{2}x^2 \tan t}}{(\cos t)^{\frac{1}{2}}}$ is a constant and so does not impact the convergence of the limit and observing that this form is exactly as appears in Proposition 1 we recover the following:

Theorem 3.4 ([5, Theorem 5.3.2, Second Half])**.** *For $a > 1$ and for all $x, t \in \mathbb{R}$ we have:*

$$\begin{aligned}\lim_{n\to\infty} \psi_n(x,t) &= \lim_{n\to\infty} \sum_{k=0}^{n} C_k(n,a) \frac{e^{-\frac{i}{2}x^2 \tan t}}{(\cos t)^{\frac{1}{2}}} e^{i\frac{x}{\cos t}\left(1-\frac{2k}{n}\right)} e^{-\frac{i}{2}\left(1-\frac{2k}{n}\right)^2 \tan t} \\ &= \frac{1}{(\cos t)^{\frac{1}{2}}} e^{-\frac{i}{2}(x^2+a^2)\tan t + \frac{iax}{\cos t}}.\end{aligned}$$

The proof of the convergence of $\psi_n(x,t)$ to $\psi(x,t)$ in [5] utilizes infinite order differential operators and their continuity properties on certain spaces of holomorphic functions; the argument we give here avoids that line of reasoning.

3.2.1. *Driven quantum harmonic oscillator and variable coefficient Schrödinger equations.* When one changes the Hamiltonian to be driven by a potential $f(t)$:

$$\begin{aligned} i\frac{\partial \psi_n}{\partial t}(x,t) &= \left(-\frac{\hbar^2}{2m}\Delta + \frac{m\omega^2}{2}x^2 + f(t)x\right)\psi_n(x,t) \\ \psi_n(x,0) &= F_n(x,a) = \sum_{j=0}^{n} C_j(n,a) e^{\frac{ipx}{\hbar}\left(1-\frac{2j}{n}\right)} \end{aligned}$$

the persistence of superoscillations was also studied in [5]; again we can recover the results in that paper by using the methods of Section 2. We instead recover the slightly more general result in [19] that considered the following linear Schrödinger equation with variable coefficients:

$$\begin{aligned} i\frac{\partial \psi_n}{\partial t}(x,t) &= \left(-a(t)\Delta + b(t)x^2 - ic(t)x\partial_x - id(t) - f(t)x + ig(t)\right)\psi_n(x,t) \\ \psi_n(x,0) &= F_n(x,a) = \sum_{k=0}^{n} C_k(n,a) e^{x\left(1-\frac{2k}{n}\right)}. \end{aligned}$$

It is clear that appropriate choices of functions $a(t), b(t), \ldots, g(t)$ recover the driven harmonic oscillator. However, more interestingly this Schrödinger equation allows for a slightly more general Hamiltonian with variable coefficients. The authors of [19] prove persistence of superoscillations; we recover their result with our methods.

Theorem 3.5 ([19, Theorem 2])**.** *The solution to the Schrödinger equation with initial condition $\psi(x,0) = e^{ax}$ takes the form:*

$$\psi(x,t) = \frac{e^{i(4\alpha\gamma-\beta^2)\frac{x^2}{4\gamma}}}{\sqrt{2\mu\gamma}} e^{i\left(\delta x + \frac{\beta x(a-\varepsilon)}{2\gamma}\right)} e^{-i\left(\frac{a^2-\varepsilon^2-4\kappa\gamma-2\varepsilon a}{4\gamma}\right)}.$$

With initial condition $\psi_n(x,0) = \sum_{k=0}^{n} C_k(n,a) e^{x\left(1-\frac{2k}{n}\right)}$, the solution to the Schrödinger equation takes the form:

$$\psi_n(x,t) = \frac{e^{i(4\alpha\gamma-\beta^2)\frac{x^2}{4\gamma}}}{\sqrt{2\mu\gamma}} \sum_{k=0}^{n} C_k(n,a) e^{-i\left(\frac{\left(1-\frac{2k}{n}\right)^2-\varepsilon^2-4\kappa\gamma-2\varepsilon\left(1-\frac{2k}{n}\right)}{4\gamma}\right)} e^{i\left(\delta x + \frac{\beta x\left(\left(1-\frac{2k}{n}\right)-\varepsilon\right)}{2\gamma}\right)}.$$

Moreover, for $a > 1$ and $x, t \in \mathbb{R}$

$$\lim_{n\to\infty} \psi_n(x,t) = \psi(x,t)$$

Here $\alpha(t), \beta(t), \gamma(t), \delta(t), \varepsilon(t)$ and $\kappa(t)$ are solutions to a Riccati-type system and μ is the solution to a certain characteristic equation associated to the variable coefficient Schrödinger equation (see [19] for the precise definition of these quantities).

The solutions $\psi_n(x,t)$ follow from the linearity of the Schrödinger equation and the explicit formula for the Green's function obtained in [19]. Based on the form of the solutions $\psi_n(x,t)$, it is clear that this falls under the framework in Proposition 1 and so the convergence of $\psi_n(x,t)$ to $\psi(x,t)$ follows.

3.3. **A type of generalized Schrödinger operator.** A related generalized Schrödinger equation was considered in [5] where they allow more derivatives in the Hamiltonian. They studied the persistence of super oscillations under the following Hamiltonian:

$$\begin{aligned} i\frac{\partial \psi_n}{\partial t}(x,t) &= (-1)^{p+1}\frac{\partial^p}{\partial x^p}\psi_n(x,t) \\ \psi_n(x,0) &= F_n(x,a) = \sum_{k=0}^{n} C_k(n,a)e^{ix\left(1-\frac{2k}{n}\right)}. \end{aligned}$$

when $p \in \mathbb{N}$ and prove the following theorem:

Theorem 3.6 ([5, Theorem 6.1.1, Theorem 6.1.4])**.** *The time evolution of these solutions takes the following form:*

$$\psi_n(x,t) = \sum_{k=0}^{n} C_k(n,a)e^{ix\left(1-\frac{2k}{n}\right)}e^{i^{k(p)}t\left(-i\left(1-\frac{2k}{n}\right)\right)^p}.$$

Moreover, we have that for $a>1$ and $x,t \in \mathbb{R}$ that

$$\lim_{n\to\infty}\psi_n(x,t) = e^{iax}e^{i^{k(p)}(-ia)^p t}$$

where $k(p)=1$ if p is even and $k(p)=0$ if p is odd.

The first half of the theorem is as in [5] and utilizes the Fourier transform to arrive at the solution. The convergence claim follows from Proposition 1 by choosing the appropriate (obvious) polynomial $P(z)$. We can also obtain the the results in [19, Theorems 3 and 4] that combine these generalized and variable coefficient Schrödinger equations, we do not state the results explicitly since they are an immediate blend of the results from these two previous subsections.

3.4. **The Schrödinger equation with the centrifugal potential.** The paper [14] shows that super oscillations are preserved under the Hamiltonian with centrifugal potential. Recall the following system with $x>0$ and $U>0$:

$$\begin{aligned} i\frac{\partial \psi_n}{\partial t}(x,t) &= \left(-\Delta + \frac{U}{x^2}\right)\psi_n(x,t) \\ \psi_n(x,0) &= F_n(x,a) = \sum_{k=0}^{n} C_k(n,a)e^{ix\left(1-\frac{2k}{n}\right)}. \end{aligned}$$

Let $\psi(x,t)$ be the solution to the same problem but with initial condition $\psi(x,0) = e^{iax}$. The following result gives a closed form for these functions:

Proposition 2 ([14, Proposition 2.3])**.** *The solution to the Cauchy problem for the centrifugal potential with initial data e^{iax} is given by:*

$$\begin{aligned} \psi(x,t) =& e^{\frac{ix^2}{2t}}\left(\frac{x^2}{2it}\right)^{-\beta}\sum_{k=0}^{\infty}\frac{1}{k!\Gamma\left(-2\beta+k+\frac{1}{2}\right)}\left(\frac{x^2}{2it}\right)^k \\ &\times \sum_{l=0}^{\infty}(-1)^l\frac{\Gamma\left(\left(l+\frac{1}{2}\right)+k-\beta\right)}{l!}\left((1-i)\,a\sqrt{t}\right)^l. \end{aligned}$$

Via linearity of the Schrödinger operator we have that the solutions $\psi_n(x,t)$ are given by:

$$\psi_n(x,t) = e^{\frac{ix^2}{2t}} \left(\frac{x^2}{2it}\right)^{-\beta} \sum_{k=0}^{\infty} \frac{1}{k!\Gamma\left(-2\beta+k+\frac{1}{2}\right)} \left(\frac{x^2}{2it}\right)^k \times$$
$$\sum_{k=0}^{n} C_k(n,a) \sum_{l=0}^{\infty} (-1)^l \frac{\Gamma\left(\left(l+\frac{1}{2}\right)+k-\beta\right)}{l!} \left((1-i)\left(1-\frac{2k}{n}\right)\sqrt{t}\right)^l.$$

The persistence of superoscillations is then the following result since the solutions $\psi_n(x,t)$ is amenable to analysis of the methods of Proposition 1.:

Theorem 3.7 ([14, Theorem 3.4])**.** *For $a > 1$ and $x > 0$ and $t \in \mathbb{R}$, $\lim_{n\to\infty} \psi_n(x,t) \to \psi(x,t)$.*

This proof can be deduced by exploiting the explicit form of the Green's function and checking certain estimates on the moments given in the next section.

3.5. **Schrödinger equation with linear potential.** We next demonstrate the persistence of superoscillations under the linear potential Schrödinger equation (with $U > 0$)

$$\begin{aligned} i\frac{\partial\psi_n}{\partial t}(x,t) &= \left(-\frac{1}{2}\frac{\partial^2}{\partial x^2} + Ux\right)\psi_n(x,t) \\ \psi_n(x,0) &= F_n(x,a) = \sum_{k=0}^{n} C_k(n,a) e^{ix\left(1-\frac{2k}{n}\right)}. \end{aligned}$$

We have an explicit formula for the Green's function of this partial differential equation:

$$\mathcal{K}(x,y,t) = \frac{1}{\sqrt{2\pi it}} e^{-\frac{iU^2t^3}{6} + iUty - \frac{\left(x-y+\frac{Ut^2}{2}\right)^2}{2it}}.$$

From this it is a straightforward exercise in calculus to demonstrate that the solution to this partial differential equation with initial condition $\psi(x,0) = e^{iax}$ is given by:

$$\psi(x,t) = e^{-i\left(\frac{U^2t^3}{6} + \frac{aUt^2}{6} + \frac{a^2t}{2} - ax\right)}.$$

By linearity we have that the solutions $\psi_n(x,t)$ take the form:

$$\psi_n(x,t) = \sum_{k=0}^{n} C_k(n,a) e^{-i\left(\frac{U^2t^3}{6} + \frac{\left(1-\frac{2k}{n}\right)Ut^2}{6} + \frac{\left(1-\frac{2k}{n}\right)^2 t}{2} - \left(1-\frac{2k}{n}\right)x\right)}.$$

Finally, by the form of these two solutions, and appealing to Proposition 1 we have the following theorem:

Theorem 3.8. *Let $a > 1$. Then the solutions to the equation:*

$$\begin{aligned} i\frac{\partial\psi_n}{\partial t}(x,t) &= \left(-\frac{1}{2}\frac{\partial^2}{\partial x^2} + Ux\right)\psi_n(x,t) \\ \psi_n(x,0) &= F_n(x,a) = \sum_{k=0}^{n} C_k(n,a) e^{ix\left(1-\frac{2k}{n}\right)} \end{aligned}$$

are given by:

$$\psi_n(x,t) = \sum_{k=0}^{n} C_k(n,a) e^{-i\left(\frac{U^2t^3}{6} + \frac{\left(1-\frac{2k}{n}\right)Ut^2}{6} + \frac{\left(1-\frac{2k}{n}\right)^2 t}{2} - \left(1-\frac{2k}{n}\right)x\right)}.$$

Moreover, for any $x, t \in \mathbb{R}$ we have that

$$
\begin{aligned}
\lim_{n\to\infty} \psi_n(x,t) &= \lim_{n\to\infty} \sum_{k=0}^{n} C_k(n,a) e^{-i\left(\frac{U^2t^3}{6} + \frac{\left(1-\frac{2k}{n}\right)Ut^2}{6} + \frac{\left(1-\frac{2k}{n}\right)^2 t}{2} - \left(1-\frac{2k}{n}\right)x\right)} \\
&= e^{-i\left(\frac{U^2t^3}{6} + \frac{aUt^2}{6} + \frac{a^2t}{2} - ax\right)} \\
&= \psi(x,t).
\end{aligned}
$$

This example appears to be new in the literature.

3.6. **Superoscillating tunneling waves and step potentials.** In this subsection we show how to recover the main result in [2]. We consider the following Schrödinger equation

$$
\begin{aligned}
i\frac{\partial \psi_n}{\partial t}(x,t) &= \left(-\frac{1}{2}\frac{\partial^2}{\partial x^2} + V_0\theta(x)\right)\psi_n(x,t) \\
\psi_n(x,0) &= F_n(x,a) = \sum_{k=0}^{n} C_k(n,a) e^{ix\left(1-\frac{2k}{n}\right)}
\end{aligned}
$$

where $\theta(x)$ is the step function that is 0 for x negative and 1 for $x > 0$ and V_0 is a (real) constant. The constant V_0 denotes the height of the barrier. Similarly we will be interested in the following problem:

$$
\begin{aligned}
i\frac{\partial \psi}{\partial t}(x,t) &= \left(-\frac{1}{2}\frac{\partial^2}{\partial x^2} + V_0\theta(x)\right)\psi(x,t) \\
\psi(x,0) &= e^{iax}.
\end{aligned}
$$

When $V_0 > a^2$ the limit function of ψ_n, which will turn out to be ψ will denote the wave that tunnels through the barrier. The main result in [2] is the following beautiful result:

Theorem 3.9 ([2, Theorem 4.4])**.** *The time evolution of these solutions takes the following form:*

$$
\psi_n(x,t) = \sum_{k=0}^{n} C_k(n,a) \sum_{m=0}^{\infty} B_m(x,t) \left(i\left(1-\frac{2k}{n}\right)\right)^m.
$$

Moreover, we have that for $a > 1$ and $x, t \in \mathbb{R}$ that

$$
\lim_{n\to\infty} \psi_n(x,t) = \sum_{m=0}^{\infty} B_m(x,t)(ia)^m = \int_{\mathbb{R}} e^{iax}\mathcal{K}(x,-it,y)dy.
$$

Here

$$
B_m(x,t) = \frac{1}{m!}\int_{\mathbb{R}} y^m \mathcal{K}(x,-it,y)dy
$$

where $\mathcal{K}(x,-it,y)$ is portion of the Green's function that is responsible for the transmission of the wave through the barrier.

The first half of the theorem is as in [2] and utilizes key information about the Green's function for the solution operator to this Schrödinger equation. The exact form of the Green's function can be found in [13] and is given by an explicit formula. The convergence claim follows from Proposition 1 by choosing $b_m = B_m(x,t)$ and then appealing to the estimates that $\{B_m(x,t)\}$ satisfies as in [2, pg. 13-14].

3.7. **Extensions to several variables.** In [6] the authors also show how to produce examples of superoscillating functions via taking the tensor product of one-variable superoscillating sequences. From [6, Theorem 3.4] we have that

$$f_n(x_1,\ldots,x_r)=\sum_{k=0}^{n}C_k(n,a)P\left(e^{ix_1(-i(1-2k/n)^p)},\ldots,e^{ix_r(-i(1-2k/n)^p)}\right)$$

where $P(u_1,\ldots,u_r)=\sum_{|\alpha|\leq h}a_\alpha u_1^{\alpha_1}\ldots u_r^{\alpha_r}$, with $a_\alpha\in\mathbb{C}$ converges to the value $P\left(e^{ix_1(-ia)^p},\ldots,e^{ix_r(-ia)^p}\right)$. As $f_n(x_1,\ldots,x_r)$ can be written as

$$\begin{aligned}f_n(x_1,\ldots,x_r) &= \sum_{|\alpha|\leq h}a_\alpha\sum_{k=0}^{n}C_k(n,a)e^{i(\alpha_1x_1+\ldots\alpha_rx_r)(-i(1-2k/n)^p)}\\ &= \sum_{|\alpha|\leq h}a_\alpha\sum_{k=0}^{n}C_k(n,a)e^{iP_\alpha(1-2k/n)}\end{aligned}$$

the result follows from Proposition 1 *(ii)* where $P_\alpha(z)=(\alpha_1x_1+\ldots+\alpha_rx_r)z^p$. Using similar methods one can likely recover the results in [15] about superoscillations in a uniform magnetic field.

3.8. **Generalized Schrödinger equations.** We now utilize part *(iii)* of Proposition 1. We are interested in generalized Schrödinger equations that arise from an analytic function G with some properties. This allows us to recover the result in [7]

Theorem 3.10 ([7, Theorem 6])**.** *Let $a\in\mathbb{R}$ with $a>1$. Let $G(z)$ be an analytic function on a disc with radius sufficiently large compared to a. Suppose that $G(ia)\in\mathbb{R}$ and $|G(ia)|\geq a$.*

$$\begin{aligned}i\tfrac{\partial\psi_n}{\partial t}(x,t) &= -G\left(\tfrac{d}{dz}\right)\psi_n(x,t)\\ \psi_n(x,0) &= F_n(x,a)=\sum_{k=0}^{n}C_k(n,a)e^{ix\left(1-\frac{2k}{n}\right)}.\end{aligned}$$

where $G\left(\dfrac{d}{dz}\right)=\displaystyle\sum_{m=0}^{\infty}g_m\frac{d^m}{dz^m}$. *Then the solution $\psi_n(x,t)$ is given by*

$$\psi_n(x,t)=\sum_{k=0}^{n}C_k(n,a)e^{-ix\left(1-\frac{2k}{n}\right)}e^{itG\left(i\left(1-\frac{2k}{n}\right)\right)}$$

and we have

$$\lim_{n\to\infty}\psi_n(x,t)=e^{itG(ia)}e^{iax}.$$

4. **Moments of Green's functions.** This section will compute explicit formulas for the *normalized moments* given by:

$$b_m(x,t):=\frac{1}{m!}\int_{-\infty}^{+\infty}y^m\mathcal{K}(x,y,t)dy$$

where $\mathcal{K}(x,y,t)$ is the corresponding Green's function for the Schrödinger equation with potential $\mathcal{H}(x,t)$. It is possible to give closed form expressions for the normalized moments of the Green's function for the free particle, the harmonic oscillator, the Schrödinger equation with linear potential, a centrifugal potential, and with a centripetal barrier potential. To establish the formulas, we used Maple 2016 and the Online Encyclopedia of Integer Sequences. We refer to [20] for the explicit expressions of the respective Green's functions.

In what follows, $\psi_n(x,t)$ and $\psi(x,t)$ denote the solution of the respective Schrödinger equation

$$\begin{array}{rcl} i\frac{\partial \psi_n}{\partial t}(x,t) & = & \mathcal{H}(x,t)\psi_n(x,t) \\ \psi_n(x,0) & = & F_n(x,a) \end{array} \qquad \begin{array}{rcl} i\frac{\partial \psi}{\partial t}(x,t) & = & \mathcal{H}(x,t)\psi(x,t) \\ \psi(x,0) & = & e^{iax} \end{array}$$

The results below show that $\lim_{n\to+\infty} \psi_n(x,t) = \psi(x,t)$ for x and t fixed by making use of Lemma 4.1 and Remark 3 to verify the hypothesis of Theorem 2.3. In what follows, we collect a fact that will be used exclusively in this section. Recall that he hypergeometric function Hypergeom$([a,b],[],Z)$, for a and b real numbers, is the function defined by the power series

$$\text{Hypergeom}([a,b],[],Z) = \sum_{j=0}^{+\infty} \frac{(a)_j(b)_j Z^j}{j!}, \ |Z|<1,$$

and we will male use of the following facts to prove that the conditions of Theorem 2.3 are satisfied for the normalized moments we compute.

Lemma 4.1. *The following statements hold:*

(*i*) $\frac{X^m}{m!}\textit{Hypergeom}\left(\left[-\frac{m}{2},-\frac{m}{2}+\frac{1}{2}\right],[],\frac{2Z}{X^2}\right) = \sum_{j=0}^{\lfloor m/2\rfloor} X^{m-2j}\frac{Z^j}{2^j(m-2j)!j!}$;

(*ii*) *The series* $\sum_{m\geq 0}\left(\sum_{j=0}^{\lfloor m/2\rfloor} X^{m-2j}\frac{Z^j}{2^j(m-2j)!j!}\right) m(2^3(1+a))^m$ *converges for fixed real numbers* X *and* Z.

Proof. (*i*) It is easy to see that $\frac{\left(-\frac{m}{2}\right)_j\left(-\frac{m}{2}+\frac{1}{2}\right)_j Z^j}{j!} = \frac{m!Z^j}{2^{2j}(m-2j)!j!}$ which leads to the equality claimed. For (*ii*), as $\left|\sum_{j=0}^{\lfloor m/2\rfloor} X^{m-2j}\frac{Z^j}{2^j(m-2j)!j!}\right| \leq \frac{m|X|^m|Z|^m}{(\lfloor m/4\rfloor)!^2}$, the series with general term $m\left|\sum_{j=0}^{\lfloor m/2\rfloor} X^{m-2j}\frac{Z^j}{2^j(m-2j)!j!}\right|(2^3(1+a))^m$ converges for X and Z fixed. □

4.1. **The free particle.** The Green's function for the free particle time-dependent Schrödinger equation $i\frac{\partial\psi(x,t)}{\partial t} = -\frac{1}{2}\partial_{x^2}\psi(x,t)$ is given by

$$\mathcal{K}_0(x,y,t) = \frac{1}{\sqrt{2\pi i t}}e^{-\frac{(x-y)^2}{2it}}.$$

One can obtain the following formula for the normalized moment of order m for m even:

$$\begin{aligned} b_m(x,t) &= \frac{1}{m!}\int_{-\infty}^{+\infty} y^m \mathcal{K}_0(x,y,t)dy \\ &= \frac{1}{\sqrt{2\pi it}}\frac{x^m}{m!}\text{Hypergeom}\left(\left[\frac{-m}{2},\frac{-m}{2}+\frac{1}{2}\right],[],\frac{2it}{x^2}\right) \\ &= \frac{1}{\sqrt{2\pi it}}\sum_{j=0}^{\lfloor m/2\rfloor}\frac{x^{m-2j}(it)^j}{2^j(m-2j)!j!}, \end{aligned}$$

where the second equality follows from Lemma 4.1 (i). It follows by Lemma 4.1 (ii) that the hypothesis of Theorem 2.3 are satisfied and for $x,t\in\mathbb{R}$ fixed, $\lim_{n\to\infty}\psi_n(x,t)=\psi(x,t)$.

4.2. **The harmonic oscillator.** The Green's function of the harmonic oscillator

$$i\frac{\partial\psi(x,t)}{\partial t}=\left(-\frac{\partial}{2\partial x^2}+Ux^2\right)\psi(x,t),\quad U>0$$

is given by

$$\mathcal{K}_1(x,y,t)=\sqrt{\frac{\alpha}{2\pi i\sin(\alpha t)}}e^{-\frac{\alpha}{2i\sin(\alpha t)}\left(x^2\cos(\alpha t)-2xy+y^2\cos(\alpha t)\right)}$$

with $\alpha=\sqrt{2U}$. Observe that via direct algebraic manipulations that it is possible to relate this Green's function to that for the free particle:

$$\begin{aligned} \mathcal{K}_1(x,y,t) &= \sqrt{\frac{\alpha}{2\pi i\sin(\alpha t)}}e^{-\frac{\alpha}{2i\sin(\alpha t)}(x^2\cos(\alpha t)-\frac{x^2}{\cos(\alpha t)}+\frac{x^2}{\cos(\alpha t)}-2xy+y^2\cos(\alpha t))} \\ &= \sqrt{\frac{\alpha}{2\pi i\sin(\alpha t)}}e^{-\frac{\alpha}{2i\sin(\alpha t)}\left(x^2\cos(\alpha t)-\frac{x^2}{\cos(\alpha t)}\right)}e^{-\frac{\alpha}{2i\sin(\alpha t)}\left(\frac{x^2}{\cos(\alpha t)}-2xy+y^2\cos(\alpha t)\right)} \\ &= e^{-\frac{\alpha}{2i\sin(\alpha t)}\left(x^2\cos(\alpha t)-\frac{x^2}{\cos(\alpha t)}\right)}\sqrt{\frac{\alpha}{2\pi i\sin(\alpha t)}}e^{-\frac{\alpha}{2i\sin(\alpha t)}\left(\frac{x}{\sqrt{\cos(\alpha t)}}-y\sqrt{\cos(\alpha t)}\right)^2} \\ &= e^{\frac{\alpha x^2\sin(2\alpha t)}{2i\cos(\alpha t)}}\mathcal{K}_0\left(\frac{x}{\sqrt{\cos(\alpha t)}},y\sqrt{\cos(\alpha t)},\frac{\sin(\alpha t)}{\alpha}\right). \end{aligned}$$

where we use in the last equality that $x^2\cos(\alpha t)-\frac{x^2}{\cos(\alpha t)}=-\frac{\sin^2(\alpha t)}{\cos(\alpha t)}x^2$. The normalized moment of order m is

$$\begin{aligned}
b_m(x,t) &= \frac{1}{m!}\int_{-\infty}^{+\infty} y^m \mathcal{K}_1(x,y,t)dy \\
&= \frac{1}{m!}e^{\frac{\alpha x^2\sin(\alpha t)}{2i\cos(\alpha t)}}\int_{-\infty}^{+\infty} y^m \mathcal{K}_0\left(\frac{x}{\sqrt{\cos(\alpha t)}}, y\sqrt{\cos(\alpha t)}, \frac{\sin(\alpha t)}{\alpha}\right)dy \\
&= \frac{1}{m!}\frac{e^{\frac{\alpha x^2\sin(\alpha t)}{2i\cos(\alpha t)}}}{(\cos(\alpha t))^{\frac{m+1}{2}}}\int_{-\infty}^{+\infty} v^m \mathcal{K}_0\left(\frac{x}{\sqrt{\cos(\alpha t)}}, v, \frac{\sin(\alpha t)}{\alpha}\right)dv \\
&= \frac{e^{\frac{\alpha x^2\sin(\alpha t)}{2i\cos(\alpha t)}}x^m}{m!(\cos(\alpha t))^{\frac{m+1}{2}}\left(\sqrt{\cos(\alpha t)}\right)^m} \\
&\quad \times \text{Hypergeom}\left(\left[\frac{-m}{2}, \frac{1-m}{2}\right], [], \frac{2i\cos(\alpha t)\sin(\alpha t)}{\alpha x^2}\right) \\
&= \frac{e^{\frac{\alpha x^2\sin(\alpha t)}{2i\cos(\alpha t)}}}{(\cos(\alpha t))^{m+\frac{1}{2}}}\frac{x^m}{m!}\text{Hypergeom}\left(\left[\frac{-m}{2}, \frac{1-m}{2}\right], [], \frac{2i\cos(\alpha t)\sin(\alpha t)}{\alpha x^2}\right)
\end{aligned}$$

By Lemma 4.1 (ii) with $X = x$ and $Z = \frac{i\cos(\alpha t)\sin(\alpha t)}{\alpha}$, for x and t fixed it follows $\sum_m m|b_m(x,t)|(2^3(1+a))^m$ converges, and the hypothesis of Theorem 2.3 then holds.

4.3. **The linear potential.** The Schrödinger equation with linear potential is

$$i\frac{\partial\psi(x,t)}{\partial t} = \left(-\frac{1}{2}\Delta - Ux\right)\psi(x,t), \quad U > 0,$$

and $\mathcal{K}_2(x,y,t) = \sqrt{\frac{1}{2\pi it}}e^{-\frac{iU^2t^3}{6}+iUty-\frac{1}{2it}\left(x-y+\frac{Ut^2}{2}\right)^2}$ is the corresponding Green's function. Note that simple algebraic manipulations provide:

$$\begin{aligned}
\mathcal{K}_2(x,y,t)dy &= \sqrt{\frac{1}{2\pi it}}e^{\frac{-Uti}{6}\left(Ut^2-6x+6x-6y\right)-\frac{1}{2it}\left(x-y+\frac{Ut^2}{2}\right)^2} \\
&= \sqrt{\frac{1}{2\pi it}}e^{\frac{-Uti}{6}\left(Ut^2-6x\right)}e^{-Uti(x-y)-\frac{1}{2it}\left(x-y+\frac{Ut^2}{2}\right)^2} \\
&= \sqrt{\frac{1}{2\pi it}}e^{\frac{-Uti}{6}\left(Ut^2-6x\right)}e^{\frac{-1}{2it}\left(-2Ut(x-y)+(x-y)^2+\frac{U^2t^4}{4}+(x-y)Ut^2\right)} \\
&= \sqrt{\frac{1}{2\pi it}}e^{\frac{-Uti}{6}\left(Ut^2-6x\right)}e^{\frac{-1}{2it}\left(x-\frac{Ut^2}{2}-y\right)^2} \\
&= \sqrt{\frac{1}{2\pi it}}e^{\frac{-Uti}{6}\left(Ut^2-6x\right)}\mathcal{K}_0\left(x-\frac{Ut^2}{2}, y, t\right).
\end{aligned}$$

One can deduce the following expression of normalized moments order m:

$$\begin{aligned}
b_m(x,t) &= \frac{1}{m!}\int_{-\infty}^{+\infty} y^m \mathcal{K}_2(x,y,t)dy \\
&= \frac{e^{\frac{-Uti}{6}\left(Ut^2-6x\right)}}{\sqrt{2\pi it}}\frac{\left(x-\frac{Ut^2}{2}\right)^m}{m!}\text{Hypergeom}\left(\left[\frac{-m}{2}, \frac{1-m}{2}\right], [], \frac{2it}{\left(x-\frac{Ut^2}{2}\right)^2}\right)
\end{aligned}$$

By Lemma 4.1 (ii) with $X = x - \frac{Ut^2}{2}$ and $Z = \frac{i\cos(\alpha t)\sin(\alpha t)}{\alpha}$, shows for x and t fixed that $\sum_m m|b_m(x,t)|(2^3(1+a))^m$ converges, and the hypothesis of Theorem 2.3 again are satisfied.

4.4. **The centrifugal potential.** The Green's function associated with the Schrödinger equation with centrifugal potential

$$i\frac{\partial\psi(x,t)}{\partial t} = \left(-\frac{1}{2}\Delta + \frac{U}{x^2}\right)\psi(x,t), \quad U > 0,$$

is

$$\mathcal{K}_3(x,y,t) = i^{2\beta-1/2}\frac{\sqrt{xy}}{t}e^{\frac{i(x^2+y^2)}{2t}}J_{-2\beta-1/2}\left(\frac{xy}{t}\right),$$

where $J_\alpha(r) = \frac{-1}{2i\pi}\left(\frac{r}{2}\right)^\alpha \int s^{\alpha-1}e^{-r^2s/4+1/s}ds$ is the Bessel function. The value $\beta_0 = \frac{-1+\sqrt{1+8U}}{4}$ is excluded since the Bessel function at $-2\beta_0 - 1/2$ diverges as x goes to 0. The normalized moment of order m is given by

$$\begin{aligned}\frac{1}{m!}\int_{-\infty}^{+\infty}\mathcal{K}_3(x,y,t)y^m dy &= \mathcal{F}_\beta(x,t)\frac{\Gamma\left(\frac{1}{2}+\frac{m}{2}-\beta\right)}{m!}\\ &\times\left(((-1+4\beta)t+ix^2)\left(-\frac{1+m}{2}+\beta\right)\text{WhittakerM}\left(\frac{1}{4}+\frac{m}{2}, -\beta+\frac{3}{4}, \frac{x^2i}{2t}\right)\right.\\ &\left.+4t\left(-\frac{1}{4}+\beta\right)\left(-1+\frac{m}{2}+\beta\right)\text{WhittakerM}\left(-\frac{3}{4}+\frac{m}{2}, -\beta+\frac{3}{4}, \frac{x^2i}{2t}\right)\right),\end{aligned}$$

where

$$\begin{aligned}\mathcal{F}_\beta(x,t) = \frac{2^{1/4}}{(-x)^{7/2}\Gamma(-2\beta+5/2)}ie^{\frac{ix^2}{4t}}\left(\frac{-i}{t}\right)^{-1/2+\beta}\left(\frac{ix^2}{t}\right)^{-1/4+\beta}\\ \cdot\left(x(-1)^m\left(\frac{-x}{t}\right)^{1/2-2\beta} + \sqrt{x}\sqrt{-x}\left(\frac{x}{t}\right)^{1/2-2\beta}\right),\end{aligned}$$

and WhittakerM(a,b,z) is the special function

$$\text{WhittakerM}(a,b,Z) = e^{-z/2}Z^{1/2+b}\text{Hypergeom}\left(\frac{1}{2}+b-a, 1+2b, Z\right).$$

Let $Z = \frac{x^2i}{2t}$, $a = \frac{1}{4}+\frac{m}{2}$, and $b = -\beta+\frac{3}{4}$. Observe that

$$\text{Hypergeom}\left(1-\beta-\frac{m}{2}, -2\beta+\frac{5}{2}, Z\right) = \sum_{k=0}^{+\infty}\frac{Z^k}{k!}\frac{\Gamma(1-\beta-\frac{m}{2}+k)\Gamma(-2\beta+\frac{5}{2})}{\Gamma(1-\beta-\frac{m}{2})\Gamma(-2\beta+\frac{5}{2}+k)},$$

and for any m, one has

$$\frac{\Gamma(1-\beta-\frac{m}{2}+k)}{\Gamma(1-\beta-\frac{m}{2})}\frac{\Gamma(-2\beta+\frac{5}{2})}{\Gamma(-2\beta+\frac{5}{2}+k)} \underset{k\to+\infty}{=} o\left((k/2)!\right).$$

It follows that

$$\sum_{k=0}^{+\infty}\frac{Z^k}{k!}\frac{\Gamma(1-\beta-\frac{m}{2}+k)}{\Gamma(1-\beta-\frac{m}{2})}\frac{\Gamma(-2\beta+\frac{5}{2})}{\Gamma(-2\beta+\frac{5}{2}+k)} = o\left(\sum_{k=0}^{+\infty}\frac{Z^k}{k!}(k/2)!\right),$$

and for x and t fixed, we have that

$$\text{WhittakerM}\left(\frac{1}{4}+\frac{m}{2},-\beta+\frac{3}{4},\frac{x^2 i}{2t}\right)=o\left(\sum_{k=0}^{+\infty}\frac{|x^2|^k}{2^k t^k k!}\,(k/2)!\right)$$

Further, for x and t fixed,

$$((-1+4\beta)t+ix^2)\left(-\frac{1+m}{2}+\beta\right)\text{WhittakerM}\left(\frac{1}{4}+\frac{m}{2},-\beta+\frac{3}{4},\frac{x^2 i}{2t}\right)=O(m^2),$$

and

$$4t\left(-\frac{1}{4}+\beta\right)\left(-1+\frac{m}{2}+\beta\right)\text{WhittakerM}\left(-\frac{3}{4}+\frac{m}{2},-\beta+\frac{3}{4},\frac{x^2 i}{2t}\right)=O(m^2),$$

which implies that

$$\begin{aligned} m|b_m(x,t)|(2^3(1+a))^m &= \frac{m(2^3(1+a))^m}{m!}\left|\int_{-\infty}^{+\infty}\mathcal{K}_3(x,y,t)y^m dy\right| \\ &= O\left(\frac{(m/2)!}{m!}m^3(2^3(1+a))^m\right). \end{aligned}$$

By Remark 3, it is easy to see that the hypotheses of Theorem 2.3 are satisfied and for $x,t\in\mathbb{R}$ fixed, $\lim_{n\to\infty}\psi_n(x,t)=\psi(x,t)$.

Remark 5. From what precedes, one gets that for x and t fixed, if $P_m(x,t)$ is a polynomial of degree 2 in x and $Z(x,t)$ is a function in x and t, then

$$P_m(x,t)\text{WhittakerM}\left(\frac{1}{4}+\frac{m}{2},-\beta+\frac{3}{4},Z(x,t)\right)=O(m^2).$$

The same occur when the first parameter of the Whittaker function is $-\frac{3}{4}+\frac{m}{2}$.

4.5. **The centripetal barrier potential.** The Green's function associated with the centripetal barrier oscillator given by

$$i\frac{\partial\psi(x,t)}{\partial t}=\left(-\frac{1}{2}\Delta+U\left(x-\frac{1}{x}\right)^2\right)\psi(x,t),\quad U>0,$$

is

$$\mathcal{K}_4(x,y,t)=i^{2\beta+3/2}\frac{2\alpha\sqrt{xy}}{\sin(2\alpha t)}e^{4i\alpha^2 t+i\alpha\cos(2\alpha t)\frac{(x^2+y^2)}{\sin(2\alpha t)}}J_{-2\beta-1/2}\left(\frac{2\alpha xy}{\sin(2\alpha t)}\right).$$

The function $\mathcal{K}_4(x,y,t)$ is related to the Green's function $\mathcal{K}_3(x,y,t)$ for the centrifugal potential as follows

$$\begin{aligned} \mathcal{K}_4(x,y,t) &= i^{2\beta+3/2}e^{4i\alpha^2 t}\frac{2\alpha\sqrt{xy}}{\sin(2\alpha t)}e^{i\alpha\cos(2\alpha t)\frac{(x^2+y^2)}{\sin(2\alpha t)}}J_{-2\beta-1/2}\left(\frac{2\alpha xy}{\sin(2\alpha t)}\right) \\ &= i^{2\beta+3/2}e^{4i\alpha^2 t}\frac{2\alpha\sqrt{x\cos(2\alpha t)y}\sqrt{\cos(2\alpha t)}}{\cos(2\alpha t)\sin(2\alpha t)}e^{i\alpha\cos^2(2\alpha t)\frac{(x^2+y^2)}{\cos(2\alpha t)\sin(2\alpha t)}} \\ &\quad\times J_{-2\beta-1/2}\left(\frac{2\alpha xy}{\sin(2\alpha t)}\right) \\ &= i^{2\beta+3/2}\sqrt{\cos(2\alpha t)}e^{4i\alpha^2 t}\widetilde{\mathcal{K}}_4(x,y,t). \end{aligned}$$

Focus on the term $\widetilde{\mathcal{K}}_4(x,y,t)$ which is the only term of $\mathcal{K}_4(x,y,t)$ depending on y. Let $T=\frac{\cos(2\alpha t)\sin(2\alpha t)}{2\alpha}$, $X=x\sqrt{\cos(2\alpha t)}$ and $Y=y\sqrt{\cos(2\alpha t)}$, then

$$\begin{aligned}
&\frac{1}{m!}\int_{-\infty}^{+\infty}\widetilde{\mathcal{K}}_4(x,y,t)y^m dy\\
&=\int_{-\infty}^{+\infty}\frac{2\alpha\sqrt{x\cos(2\alpha t)y}\sqrt{\cos(2\alpha t)}}{\cos(2\alpha t)\sin(2\alpha t)m!}e^{\frac{i\alpha\cos^2(2\alpha t)(x^2+y^2)}{\cos(2\alpha t)\sin(2\alpha t)}}J_{-2\beta-1/2}\left(\frac{2\alpha xy}{\sin(2\alpha t)}\right)y^m dy\\
&=\frac{\sqrt{\cos(2\alpha t)}}{m!}\int_{-\infty}^{+\infty}\frac{\sqrt{XY}}{T}e^{i\cos(2\alpha t)\frac{(X^2+Y^2)}{2T}}J_{-2\beta-1/2}\left(\frac{XY}{T}\right)\left(\frac{Y}{\sqrt{\cos(2\alpha t)}}\right)^m dY\\
&=\frac{1}{(\cos(2\alpha t))^{(m-1)/2}m!}\int_{-\infty}^{+\infty}\frac{\sqrt{XY}}{T}e^{\cos(2\alpha t)i\frac{(X^2+Y^2)}{2T}}J_{-2\beta-1/2}\left(\frac{XY}{T}\right)Y^m dY\\
&=\frac{\Gamma\left(\frac{1}{2}+\frac{m}{2}-\beta\right)}{(\cos(2\alpha t))^{(m-1)/2}m!}\mathcal{G}_\beta(X,T)\mathcal{H}_m(X,T),
\end{aligned}$$

where $\mathcal{H}_m(X,T)$ is given by

$$\begin{aligned}
\mathcal{H}_m(X,T)&=A(X,T)\text{WhittakerM}\left(\frac{1}{4}+\frac{m}{2},-\beta+\frac{3}{4},\frac{X^2}{2iT\cos(2\alpha t)}\right)\\
&\quad+B(X,T)\text{WhittakerM}\left(-\frac{3}{4}+\frac{m}{2},-\beta+\frac{3}{4},\frac{X^2}{2iT\cos(2\alpha t)}\right),
\end{aligned}$$

with

$$\begin{aligned}
A(X,T)&=(T(-1+4\beta)\cos(2\alpha t)+iX^2)(1+m-2\beta)\\
B(X,T)&=-T\,(-1+4\beta)\,(-2+m+2\beta)\cos(2\alpha t)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{G}_\beta(X,T)&=\frac{i}{2X^6\Gamma\left(-2\beta+\frac{5}{2}\right)}\left(X(-1)^m\left(-\frac{X}{T}\right)^{1/2-2\beta}+\left(\frac{X}{T}\right)^{1/2-2\beta}\sqrt{X}\sqrt{-X}\right)\\
&\quad\times\left(\frac{iX^2}{(T(2\cos(\alpha t)^2-1)}\right)^{3/4}\left(\frac{iX^2}{T\cos(2\alpha t)}\right)^{\beta}\left(\frac{-i\cos(2\alpha t)}{T}\right)^{-m/2+\beta}\\
&\quad\times 2^{1/4+m/2}e^{\frac{X^2(2\cos(2\alpha t)^2-1}{4iT\cos(2\alpha t)}}T^2\sqrt{-X}\sqrt{\frac{i-2i\cos(\alpha t)^2}{T}}.
\end{aligned}$$

By Remark 5, we obtain that

$$\mathcal{H}_m(X,T)=O(m^2),$$

which implies that

$$m(2^3(1+a))^m\mathcal{G}_\beta(X,T)\frac{\Gamma\left(\frac{1+m}{2}-\beta\right)}{(\cos(2\alpha t))^{\frac{m-1}{2}}m!}\mathcal{H}_m(X,T)=O\left(\frac{(\frac{m}{2})!}{m!}m^3(2^3(1+a))^m\right).$$

By Remark 3, it is easy to see that the hypotheses of Theorem 2.3 are satisfied and for $x,t\in\mathbb{R}$ fixed, and thus $\lim_{n\to\infty}\psi_n(x,t)=\psi(x,t)$.

5. **Concluding remarks.** As has been shown, precise knowledge about the Green's function for the resulting Schrödinger equation can be used to deduce the persistence of superoscillations. This is an essential ingredient in all the papers [1, 6, 19, 10, 5, 2, 8, 14] and suggests that one should seek out a suitable Green's function for more general Schrödinger operators. A viewpoint taken here is to exploit decay in the moments of the Green's function to deduce we have the desired convergence.

Some natural questions arise. How general is this procedure? Does it characterize the existence of superoscillations? Can anything be done in the case of nonlinear Schrödinger equations? Can one recover the results in [17] with a potential given by the Dirac comb using the approach from this paper?

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