DYNAMICS OF THRESHOLD SOLUTIONS FOR ENERGY CRITICAL NLS WITH INVERSE SQUARE POTENTIAL

KAI YANG, CHONGCHUN ZENG, AND XIAOYI ZHANG

ABSTRACT. We consider the focusing energy critical NLS with inverse square potential in dimension $d = 3, 4, 5$ with the details given in $d = 3$ and remarks on results in other dimensions. Solutions on the energy surface of the ground state are characterized. We prove that solutions with kinetic energy less than that of the ground state must scatter to zero or belong to the stable/unstable manifolds of the ground state. In the latter case they converge to the ground state exponentially in the energy space as $t \to \infty$ or $t \to -\infty$. (In 3-dim without radial assumption, this holds under the compactness assumption of non-scattering solutions on the energy surface.) When the kinetic energy is greater than that of the ground state, we show that all radial H^1 solutions blow up in finite time, with the only two exceptions in the case of 5-dim which belong to the stable/unstable manifold of the ground state. The proof relies on the detailed spectral analysis, local invariant manifold theory, and a global Virial analysis.

1. Introduction

Let $a \in (-\frac{1}{4}, 0)$ and $\mathcal{L}_a = -\Delta + \frac{a}{|x|^2}$, we consider the initial value problem

$$(\mathrm{NLS}_a) \qquad \begin{cases} (i\partial_t - \mathcal{L}_a)u + |u|^4 u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^3, \\ u(0,x) = u_0 \in \dot{H}^1(\mathbb{R}^3), \end{cases}$$

for $u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$. Here the space $\dot{H}^1(\mathbb{R}^3)$ is the usual Sobolev space whose norm is given by $\|\nabla f\|_2$. For a in the above range, the sharp Hardy's inequality implies that the bilinear form $\langle \mathcal{L}_a f, f\rangle$ is positive definite and thus defines an equivalent norm $\|\sqrt{\langle \mathcal{L}_a f, f\rangle}\|_2 = \|\mathcal{L}_a^{\frac{1}{2}} f\|_2$. We use $\dot{H}^1_a(\mathbb{R}^3)$ to denote the Hilbert space $\dot{H}^1(\mathbb{R}^3)$ equipped with this equivalent norm.

The solution appearing in this paper is always a strong solution, by which we mean a function u obeys the integral equation

$$u(t) = e^{-it\mathcal{L}_a} u_0 + i\int_0^t e^{-i(t-s)\mathcal{L}_a} |u(s)|^4 u(s)\, ds,$$

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and lies in a certain spacetime space, for instance $u \in C_t\dot{H}^1_x \cap L^{10}_{t,loc}L^{10}_x$. Constructing such solution via Strichartz methodology imposes further constrains on a: $a > -\frac{1}{4} + \frac{1}{25}$ as shown in [16]. We do not record the local theory here but would like to point out that as in the classical case, the boundedness of the spacetime norm $L^{10}_{t,x}(I \times \mathbb{R}^3)$ enables us to extend the solution beyond I and if $I = \mathbb{R}$, solution scatters. Therefore we define

$$S_I(u) = \iint_{I\times\mathbb{R}^3} |u(t,x)|^{10}\,dxdt,$$

as the scattering size of u. For a given solution u, we can repeatedly apply the local wellposedness to extend the solution to its maximal lifespan

$$(-T_*(u), T^*(u)).$$

On the interval of existence, the solution preserves its energy

$$E(u(t)) = \int_{\mathbb{R}^3} \tfrac{1}{2}|\nabla u(t,x)|^2 + \tfrac{a}{2|x|^2}|u(t,x)|^2 - \frac{1}{6}|u(t,x)|^6 dx.$$

NLS_a is referred to as energy critical as the natural scaling of the equation $u(t,x) \to \lambda^{-\frac{1}{2}} u(\frac{t}{\lambda^2}, \frac{x}{\lambda})$ also keeps the energy invariant.

In the preceding work [15, 16], the authors developed the fundamental analysis involving the operator $\mathcal{L}_a$ and used such to understand the scattering solutions of energy critical problem in both defocusing and focusing case. In [16], they proved the scattering for all finite energy solutions in the defocusing case in three dimensions and developed the crucial variational analysis of the ground state in the focusing case. Completion of the augment in multi-dimensions and focusing case was done by the first author in [28] and [29].

Let us be more specific on the focusing case. In d dimensions and for $a > -(\frac{d-2}{2})^2$, the ground state soliton is the unique (up to symmetries of the equation) positive solution of static NLS_a:

$$\mathcal{L}_a W = |W|^{\frac{4}{d-2}} W. \tag{1.1}$$

It was computed in [16] that

$$W(x) = [d(d-2)\beta^2]^{\frac{d-2}{4}} \left(\frac{|x|^{\beta-1}}{1+|x|^{2\beta}} \right)^{\frac{d-2}{2}}, \ \beta = \sqrt{1 + (\tfrac{2}{d-2})^2 a}. \tag{1.2}$$

Moreover, for $a \in (-(\frac{d-2}{2})^2, 0]$, W has the variational characterization which says W realizes the best constant in the sharp Sobolev inequality, see for instance, [1, 2, 16, 27]. While for positive a, the problem become very tricky as the best constant can not be realized except in the radially symmetric case. We will address that case elsewhere and only focus on the case of negative a in this paper.

We record the following scattering result which shows the ground state plays a role of scattering threshold.

Theorem 1.1 ([16, 28, 29]). *Let $3 \le d \le 6$ and $0 > a > -\left(\frac{d-2}{2}\right)^2 + \left(\frac{d-2}{d+2}\right)^2$. Let $u_0 \in \dot{H}^1(\mathbb{R}^d)$ satisfy $\|u_0\|_{\dot{H}^1_a} < \|W\|_{\dot{H}^1_a}$ and $E(u) < E(W)$. Then there exists a unique global solution u to d-dimensional NLS$_a$:*

$$(i\partial_t - \mathcal{L}_a)u = -|u|^{\frac{4}{d-2}}u, \; u(0,x) = u_0,$$

satisfying $\|u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(\mathbb{R}\times\mathbb{R}^d)} < C(\|u_0\|_{\dot{H}^1_a})$ in the following two scenarios: (1) $d = 4, 5, 6$; (2) $d = 3$ and u_0 is spherically symmetric.

The unavailability of the result in three dimensions is ultimately due to the absence of the same scattering result for 3d quintic focusing NLS except for the spherically symmetric case. Without the radial assumption, this remains as an open problem in 3d as of now. The direct impact is the lack of compactness of non-scattering solutions on the energy surface of $E(W)$ in three dimensions. We will take the compactness as an assumption when necessary and build part of our conditional result upon it.

Our goal in this paper is to characterize solutions on the energy surface of $E(W)$. Such problem was originated by Merle-Duyckaerts for the focusing energy critical nonlinear Schrödinger and wave equation in their seminal work [9, 10]. We are also aware of the recent progress in [26] on the same topic in the nonradial case. For focusing energy critical NLS, the ground state is given by the smooth bounded function

$$W_0(x) = \left(1 + \frac{|x|^2}{d(d-2)}\right)^{-\frac{d-2}{2}},$$

which was also proved to be the minimal energy non-scattering solution in the earlier work [8, 14, 17], except for $d = 3$ within the class of radial data. The result in [9] demonstrated the existence of two solutions $W_0^{\pm}$ exponentially decaying to the ground state W_0 on the energy surface and classified all radial solutions as either symmetry transformations of $W_0, W_0^{\pm}$, scattering solutions, or blowup solutions in both time directions. While our work is largely motivated by [9], the presence of the non-perturbative singular potential $\frac{a}{|x|^2}$ makes substantial differences. It breaks the translation symmetry of the equation and, at the same time, creates nontrivial singularity at the origin. Indeed, the fact that $\frac{a}{|x|^2}$ scales the same way as the Laplacian operator indicates the non-perturbative nature of this operator, making it impossible to treat the linearized problem around W as a compact perturbation to any well-understood linear problem. As another example of such impact, we see the ground state W, which is also a stationary solution of NLS$_a$, becomes singular at the origin thus fails to belong to the full range Strichartz spaces while the free linear solutions always do [4]. As a consequence, so far even the local well-posedness of NLS$_a$ has not been established for a close to $-\frac{1}{4}$.

On the other hand, despite the disadvantage caused by the potential, the breaking of the translation symmetry also brings certain benefits one can take advantage of. Indeed, it has been shown in [16, 28, 29] that the non-scattering solution on the energy surface of $E(W)$ can only concentrate around the origin instead of at any other places. Moreover, the lack of translation symmetry also indicates the manifold created by W and the symmetries on the energy surface is d-dimension less than that in the translation invariant case. Ultimately, we are able to piece all these and the delicate spectrum analysis together to obtain the classification of solutions on the energy surface of $E(W)$ without the radial assumption.

Naturally, we need to further restrict the range of a to ensure better regularity of W. To avoid the complexity brought up by the laborious numerology, we choose to work in dimension three even though the scattering theory in this dimension is still incomplete. Extending the 3d results to dimensions four and five is straightforward we will make a remark after each of our theorems. In the rest of higher dimensions, while most the argument can still go through, the rough nonlinearity indeed causes technical problems, for instance, in proving the Lipschitz continuity in the Strichartz spaces, a property we rely heavily on to construct the local stable/unstable manifold. Similar issue had been handled in [19, 20] in the case of NLS without potential. We will address the high dimension problem elsewhere.

Before stating the results, we introduce some notations. For $\theta, \mu \in \mathbb{S}^1 \times \mathbb{R}^+$, we use $\mathbf{T}_{\theta,\mu}$ and $\mathbf{g}_{\theta,\mu}$ to denote the symmetries transformation:

$$\mathbf{g}_{\theta,\mu} f(x) = e^{i\theta}\mu^{-\frac{1}{2}} f(\tfrac{x}{\mu});\ \mathbf{T}_{\theta,\mu} u(t,x) = e^{i\theta}\mu^{-\frac{1}{2}} u(\tfrac{t}{\mu^2}, \tfrac{x}{\mu}).$$

Our first result is the existence and uniqueness of solutions converging exponentially to W.

Theorem 1.2. *Let $a \in (-\frac{1}{4} + \frac{4}{25}, 0)$. There exist $\dot{H}^1(\mathbb{R}^3)$ solutions W^+ and W^- to NLS_a such that*

$$\lim_{t\to\infty} \|W^{\pm}(t) - W\|_{\dot{H}^1} \le Ce^{-ct},\ \|W^-\|_{\dot{H}^1_a} < \|W\|_{\dot{H}^1_a},\ \|W^+\|_{\dot{H}^1_a} > \|W\|_{\dot{H}^1_a},$$

for some $C, c > 0$. They are also unique in this class up to time translation. Moreover,

$$W^{\pm} \in \dot{H}^1_{rad}(\mathbb{R}^3), \quad E(W^{\pm}) = E(W),$$
$$\int_{-\infty}^0 \int_{\mathbb{R}^3} |W^-(t,x)|^{10} dxdt < \infty,\ W^{\pm} - W \in L^2(\mathbb{R}^3).$$

Remark 1.3. *1. In dimension $d = 4, 5$, the same statement holds for $0 > a > -(\frac{d-2}{2})^2 + (\frac{2(d-2)}{d+2})^2$ with the $L^{10}_{t,x}$ norm being replaced by $L^{\frac{2(d+2)}{d-2}}_{t,x}$. In particular, in dimension five where $W \in L^2(\mathbb{R}^5)$, $T_*(W^+) < \infty$. See Section 7 for details.*

2. These solutions $W^{\pm}$ correspond to the two branches of the 1-dim stable manifold of W in $\dot{H}^1(\mathbb{R}^3)$, which is a smooth curve tangent to the linear stable direction at W. The steady state W also has a 1-dim unstable manifold, given by $\overline{W^{\pm}}$ in this case, which satisfies the same properties in the reversed time direction.

The next result is to characterize solutions on the energy surface of $E(W)$. For the reason that was just stated, we impose the following assumption in one part of the result.

Assumption 1.4. *The trajectory of $\{u(t)\}$ is precompact modular scaling on I, i.e, there exists $\lambda(t)$ such that $\{\lambda(t)^{-\frac{1}{2}}u(t, \frac{x}{\lambda(t)}), t \in I\}$ is precompact in $\dot{H}^1(\mathbb{R}^3)$.*

We have the following

Theorem 1.5. *Let $a \in (-\frac{1}{4} + \frac{4}{25}, 0)$. Let $u \in \dot{H}^1(\mathbb{R}^3)$ be a solution of NLS_a satisfying $E(u) = E(W)$. We have*

a) If $\|u_0\|_{\dot{H}^1_a} = \|W\|_{\dot{H}^1_a}$, there exist θ, μ such that $u(t, x) = \mathbf{g}_{\theta,\mu}W$.

b) If $\|u_0\|_{\dot{H}^1_a} < \|W\|_{\dot{H}^1_a}$, then u must be a global solution. Suppose $S_{\mathbb{R}}(u) = \infty$, then u conforms into one of the following two cases:

b.1) $S_{[0,\infty)}(u) = \infty$. If moreover u satisfies Assumption 1.4 with $I = [0, \infty)$, there exist θ, μ, T such that $u(t, x) = \mathbf{T}_{\theta,\mu}W^-(t + T, x)$.

b.2) $S_{(-\infty,0]}(u) = \infty$. If moreover u satisfies Assumption 1.4 with $I = (-\infty, 0]$, then $u(t, x) = \mathbf{T}_{\theta,\mu}\overline{W^-(-t + T, x)}$ for some θ, μ, T.

c) If $\|u_0\|_{\dot{H}^1_a} > \|W\|_{\dot{H}^1_a}$, $u \in L^2(\mathbb{R}^3)$, and u is radially symmetric, then $T_*(u) + T^*(u) < \infty$, i.e. u blows up both forward and backward in time.

Remark 1.6. *1). Statement b) in Theorem 1.5 becomes* **unconditional** *in four and five dimensions and in three dimensions with radial initial data.*

2.) In four dimensions, c) can be stated in the same way. In five dimensions, the conclusion in c) should be "either $T_*(u) + T^*(u) < \infty$, or there exist θ, μ, T such that u equals one of the two solutions $\mathbf{T}_{\theta,\mu}W^+(t + T)$ and $\mathbf{T}_{\theta,\mu}\overline{W^+(-t + T)}$".

In the rest of the introduction we outline the main steps in the proof.

The analysis starts with linearizing NLS_a around W, from which we obtain a linear Hamiltonian PDE $u_t = iE''(W)u$ in the Hilbert space $\dot{H}^1_a(\mathbb{R}^3)$ with the symplectic structure i and the Hamiltonian given by the Hessian $E''(W)$ of the nonlinear energy $E(u)$. Considering W is a constrained minimizer of the energy which is invariant under the phase rotation and scaling, we first prove that the quadratic form defined by $E''(W)$ has 1-dim negative direction and a 2-dim kernel based the spherical harmonics expansion and careful study on the spatial asymptotics of the resulted ODEs. Incorporating the last piece of the puzzle, i.e. the absence of the generalized kernel, we find the operator $iE''(W)$ fits right into the general framework developed in recent work [21] which immediately gives us the exponential trichotomy of $iE''(W)$. Namely, the operator $iE''(W)$ has a 1-dim stable subspace, 1-dim unstable subspace, and 1 codim-2 center subspace containing the 2-dim kernel where the linear flow has at most quadratic growth as $|t| \to \infty$. These results are summarized in Proposition 3.3 in Section 3 and lays the foundation of the local nonlinear analysis of NLS_a.

Based on the linear analysis of $iE''(W)$, in Section 4 we establish a local coordinate near the manifold $\{\mathbf{g}_{\theta,\mu}W\}$ generated by W and the symmetries. In particular, the evolution of the modulation parameter μ representing the corresponding spatial scaling would turn out to be crucial in the nonlinear analysis.

Having the exponential trichotomy decomposition from Section 3, the classical invariant manifold theory hints at the existence and uniqueness of locally invariant 1-dim stable, 1-dim unstable, and codim-2 center manifolds, see for example, [6, 12, 24]. To fit NLS_a into the Lyapunov-Perron framework, we have to develop a Strichartz type space-time estimate for the linearized operator $iE''(W)$ with singular variable coefficients. Fortunately, treating the terms with variable coefficients as perturbations, a space-time estimate with mild temporal growth obtained by iterating a local-in-time estimate turns out to be sufficient for our construction of the local 1-dim stable manifold in Section 5. Its two branches are exactly $W^{\pm}$.

With the local structure being clearly established, our next step is to classify those one sided global but non-scattering solutions by proving they decay exponentially to W in $\dot{H}^1(\mathbb{R}^3)$. Actually from the dynamical system point of view based on the saddle structure near the manifold $\{\mathbf{g}_{\theta,\mu}W\}$, such statement is rather intuitive if the solution stays in the neighborhood of this manifold[1], which leaves us with precluding the solution running away or traveling into and out of small neighborhoods. It is where the global Virial analysis comes into play. While this part of the argument is largely guided by the work in [9], there are several new inputs making the proof more streamlined in the global Virial analysis.

In Section 6, we give the derivative estimate of Virial using the distance function $\mathbf{d}(u(t))$, which is shown to be the right quantity linking the Virial identity and the distance between u and the manifold from the variational characterization of the ground state in Section 4. Solutions on the energy surface with less kinetic energy than the ground state are characterized in Section 7, where the proof of b) in Theorem 1.5 can be found. It has been proved in the radial case and anticipated in the general case that the trajectory of such solution enjoys the precompactness after modular scaling parameter $\lambda(t)$, a property we rely heavily on in controlling the error in the Virial estimate. By properly adjusting $\lambda(t)$ (see Appendix for details), we can unify the choice of both $\lambda(t)$ and the modulation parameter $\mu(t)$ thus combine the full strength of the compactness and modulation estimates toward getting the exponential decay. The solutions on the energy surface with greater kinetic energy are considered in Section 8, where the proof of Theorem 1.5 c) can be found. Such solutions do not have compactness, instead, we add the additional L^2 and radial assumption to control the error

[1] In a forthcoming paper, we will show the exponential decay simply by assuming that the solution with energy $E(W)$ always stays in the neighborhood of the manifold.

and to avoid the solution evacuating to very low frequencies. We move some of the technical estimates in the main body to the Appendix.

2. Preliminaries

Notations: For easy reference, we include the often used notations into the following table:

$\mathcal{L}_a = -\Delta + \frac{a}{\|x\|^2}$	$\\|f\\|_{\dot H^1_a} = \\|\sqrt{\mathcal{L}_a} f\\|_2$
$\mathbf{d}(f) = \big\| \\|f\\|^2_{\dot H^1_a} - \\|W\\|^2_{\dot H^1_a} \big\|$	$\mathbf{g}_\mu f(x) = f_{[\mu]}(x) = \mu^{-\frac12} f(\frac{x}{\mu})$
$\mathbf{g}_{\theta,\mu} f(x) = f_{[\theta,\mu]} = e^{i\theta}\mu^{-\frac12} f(\frac{x}{\mu})$	$\mathbf{T}_{\theta,\mu} u(t,x) = e^{i\theta}\mu^{-\frac12} u(\frac{t}{\mu^2}, \frac{x}{\mu})$
$\langle x\rangle = \sqrt{1+\|x\|^2}$	$\beta = \sqrt{1+4a}$
$\\|f\\|_r = \\|f\\|_{L^r(\mathbb{R}^3)}$	$\\|f\\|_{\dot H^{1,r}} = \\|\sqrt{\mathcal{L}_a} f\\|_r$

Space, inner product: Throughout this paper, we shall use $\langle\cdot,\cdot\rangle$ to denote the duality parity between a Hilbert space and its dual space. $\dot H^1_a(\mathbb{R}^3)$ is the space of all complex functions endowed with the inner product $\Re\langle \mathcal{L}_a f, g\rangle = \Re\int \bar g \mathcal{L}_a f dx$ for any two complex functions. Occasionally, we also view $\dot H^1_a(\mathbb{R}^3)$ as a two dimensional real valued function space and use the notation $(\dot H^1_a)^2$. The same remark also applies to the Sobolev space $\dot H^1(\mathbb{R}^3)$.

Variational property of the ground state W. The following lemma says W is the extremizer in sharp Sobolev embedding from which one can also get the coercivity of energy.

Lemma 2.1 ([16])**.** *Let $a \in (-\frac14, 0)$ and $f \in \dot H^1(\mathbb{R}^3)$. Then*

$$\|f\|_6 \le \tfrac{\|W\|_6}{\|W\|_{\dot H^1_a}} \|f\|_{\dot H^1_a}.$$

The equality holds if and only if $f(x) = \alpha W(\lambda x)$ for some $\alpha \in \mathbb{C}$ and $\lambda > 0$. Moreover, if $\|f\|_{\dot H^1_a} \le \|W\|_{\dot H^1_a}$, then

$$\frac13 \|f\|^2_{\dot H^1_a} \le E(f) \le \frac12 \|f\|^2_{\dot H^1_a}. \tag{2.1}$$

Strichartz estimate of $e^{-it\mathcal{L}_a}$. We record the following linear estimate with the double endpoints estimate being given in the recent work [31].

Lemma 2.2 ([4, 31])**.** *Let $a > -\frac14$. Let the pair of numbers (p,q), $(\tilde p, \tilde q)$ satisfy*

$$\tfrac{2}{q} + \tfrac{3}{r} = \tfrac{2}{\tilde q} + \tfrac{2}{\tilde r} = \tfrac32, \; 2 \le q, \tilde q \le \infty.$$

Then the solution $u(t,x) : I \times \mathbb{R}^3 \to \mathbb{C}$ to the equation

$$(i\partial_t - \mathcal{L}_a) u = f$$

satisfy

$$\|u\|_{L^q_t L^r_x(I)} \lesssim \|u(t_0)\|_2 + \|f\|_{L^{\tilde q'}_t L^{\tilde r'}_x(I)}$$

for any $t_0 \in I$.

3. Spectral analysis for the linearized operator around ground state W

In order to study the dynamic structure of NLS_a near the ground state W, we write the equation for $v = u - W$ in the following vector form:

$$\partial_t v = \mathcal{L}(v) + R(v). \tag{3.1}$$

Here in the matrix form, the operator $\mathcal{L}$ can be written as

$$\mathcal{L} = \begin{pmatrix} 0 & \mathcal{L}_a - W^4 \\ -\mathcal{L}_a + 5W^4 & 0 \end{pmatrix}$$

and the nonlinearity is

$$R(v) = i|v + W|^4(v + W) - iW^5 - 5iW^4 v_1 + W^4 v_2.$$

The linearized equation inherits the Hamiltonian structure from the nonlinear one,

$$\mathcal{L} = JL, \quad -i \sim J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{ and } L = \begin{pmatrix} \mathcal{L}_a - 5W^4 & 0 \\ 0 & \mathcal{L}_a - W^4 \end{pmatrix}$$

where J is the symplectic structure and L is the Hessian of the energy. Our first step is to understand the diagonal operator L which will be further used to decode the operator $\mathcal{L}$ through Proposition 3.3.

Before stating the result, we first record several facts for the operator $L : (\dot{H}^1)^2 \to (\dot{H}^{-1})^2$, which is bounded and symmetric. Note W is the ground state solution, we have

$$(\mathcal{L}_a - W^4)W = 0, \ (\mathcal{L}_a - 5W^4)W = -4W^5 < 0,$$

which implies

$$\langle LW, W \rangle < 0.$$

Let W_1 be the generator of scaling symmetry, i.e.

$$W_1 = -\frac{d}{d\lambda} W_{[\lambda]}\Big|_{\lambda=1} = x \cdot \nabla W + \frac{1}{2} W.$$

It is easy to check that

$$(\mathcal{L}_a - 5W^4)W_1 = 0.$$

In the following lemma we will show that the three directions: W_1, iW, W are the only non-positive directions of L.

Proposition 3.1. *There exist $c, C > 0$ such that the quadratic form $Q(v) = \langle Lv, v \rangle$ on $(\dot{H}^1)^2$ satisfies*

$$c\|v\|^2_{\dot{H}^1_a} \le Q(v) \le C\|v\|^2_{\dot{H}^1_a}, \quad \forall v \in X_+,$$

where $X_+ \subset (\dot{H}^1)^2$ is the codim-3 closed subspace

$$X_+ = \{v \in (\dot{H}^1)^2 \mid \langle \mathcal{L}_a W, v \rangle = \langle \mathcal{L}_a W_1, v \rangle = \langle \mathcal{L}_a(iW), v \rangle = 0\}.$$

As a corollary, L has one dimensional negative direction and

$$\ker L = \operatorname{span}\{W_1, iW\}.$$

Moreover,

$$\ker(JL)^2 = \ker(JL) = \ker L.$$

Proof. The upper bound of $Q(v)$ follows directly from Hölder inequality and that $W \in L^6(\mathbb{R}^3)$. We will show the lower bound of $Q(v)$ by identifying the null and negative directions for each component in L.

We first consider the operator $\mathcal{L}_a - 5W^4$ and show that there is only one negative direction in the sense that for any real scalar valued function $v \in \dot{H}^1_a(\mathbb{R}^3)$ and

$$\langle \mathcal{L}_a v, W\rangle = 0, \tag{3.2}$$

we have

$$\langle (\mathcal{L}_a - 5W^4)v, v\rangle \geq 0. \tag{3.3}$$

Indeed, we will see that this is an implication of the fact that W is the constrained maximizer. Let $M = \langle \mathcal{L}_a W, W\rangle$ (which also equals $\int_{\mathbb{R}^3} W^6 dx$ from the ground state equation). For any $v \in \dot{H}^1_a(\mathbb{R}^3)$ obeying (3.2), by taking $\mu(s) = \frac{M^{\frac{1}{2}}}{(M+s^2\langle \mathcal{L}_a v, v\rangle)^{\frac{1}{2}}}$, the trajectory defined by

$$l(s) = \mu(s)(W + sv)$$

always obeys

$$\langle \mathcal{L}_a l(s), l(s)\rangle = M.$$

It can be computed that

$$\mu(0) = 1; \mu'(0) = 0; \mu''(0) = -M^{-1}\langle \mathcal{L}_a v, v\rangle,$$

and

$$l(0) = W,\ l_s(0) = v,\ l_{ss}(0) = -M^{-1}\langle \mathcal{L}_a v, v\rangle W.$$

From here and noting W is the constrained maximizer from Lemma 2.1:

$$\|W\|_6^6 = \sup_{\langle \mathcal{L}_a w, w\rangle = M} \int_{\mathbb{R}^3} |w(x)|^6 dx, \tag{3.4}$$

we have

$$\begin{aligned}
0 &\geq \frac{d^2}{ds^2}\int_{\mathbb{R}^3} |l(s)|^6 dx \,\big|_{s=0} \\
&= 30\int_{\mathbb{R}^3} l(0)^4 l_s(0)^2 dx + 6\int_{\mathbb{R}^3} l(0)^5 l_{ss}(0) dx \\
&= 30\int_{\mathbb{R}^3} W^4 v^2 dx - 6M^{-1}\langle \mathcal{L}_a v, v\rangle \int_{\mathbb{R}^3} W^6 dx \\
&= -6\int_{\mathbb{R}^3} (\mathcal{L}_a - 5W^4)v \cdot v dx = -6\langle (\mathcal{L}_a - 5W^4)v, v\rangle
\end{aligned}$$

(3.3) is proved.

Next we investigate the null direction of L and it is more convenient to work in L^2 setting instead of $\dot{H}^1_a$ setting. The operator $\mathcal{L}_a - 5W^4$ having only one negative direction in $\dot{H}^1_a(\mathbb{R}^3)$ implies $\mathcal{L}_a^{-\frac{1}{2}}(\mathcal{L}_a - 5W^4)\mathcal{L}_a^{-\frac{1}{2}}$ has only one negative direction in $L^2(\mathbb{R}^3)$. Easily we can write

$$\mathcal{L}_a^{-\frac{1}{2}}(\mathcal{L}_a - 5W^4)\mathcal{L}_a^{-\frac{1}{2}} = I - 5\mathcal{L}_a^{-\frac{1}{2}}W^4\mathcal{L}_a^{-\frac{1}{2}} := I - K.$$

We have the following result for K:

Claim 3.2. $K: \ L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ *is a compact operator.*

Postponing the proof for the moment, using this claim we know that $I-K$ has at most finitely many eigenvalues in $(-\infty, \frac{1}{2}]$ which can be ordered as

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$$

counting multiplicity.

From the previous discussion and recall that

$$(I-K)\mathcal{L}_a^{\frac{1}{2}}W_1 = 0,$$

we know

$$\lambda_1 < 0, \text{ and } \lambda_2 = 0.$$

Our goal now is to show $\lambda_3 > 0$. Note as $I - K$ is symmetric we can choose eigenfunctions as the orthonormal basis of $L^2(\mathbb{R}^3)$ and evaluate the L^2 bilinear form $\langle (I-K)u, u\rangle$. Switching back to $\dot{H}^1_a$ setting, we immediately get the desired estimate for $\mathcal{L}_a - 5W^4$:

$$\langle (\mathcal{L}_a - 5W^4)u, u\rangle \ge \lambda_3 \|u\|^2_{\dot{H}^1_a}, \ \forall u \perp_{\mathcal{L}_a} W, W_1. \tag{3.5}$$

Therefore it remains to show $\lambda_3 > 0$ or the kernel of $I - K$ is only one-dimensional in $L^2(\mathbb{R}^3)$. This is equivalent to showing the kernel of $\mathcal{L}_a - 5W^4$ is one dimensional in $\dot{H}^1(\mathbb{R}^3)$.

Consider the equation

$$(\mathcal{L}_a - 5W^4)u = 0,$$

we write u in the spherical harmonic expansion:

$$u(r,\theta) = \sum_{j=0}^{\infty} f_j(r)Y_j(\theta).$$

Here, $Y_j(\theta)$ is the jth spherical harmonics and $\{Y_j(\theta)\}_{j=0}^{\infty}$ form an orthonormal basis of $L^2(\mathbb{S}^2)$. Recall that

$$-\Delta_{\mathbb{S}^2}Y_j(\theta) = \mu_j Y_j(\theta), \ j = 0,1,2,\cdots$$
$$0 = \mu_0 < \mu_1 \le \mu_2 \le \cdots \ \to \infty, \ Y_0 = 1, \mu_1 = 2.$$

In spherical harmonic expansion, we have

$$(\mathcal{L}_a - 5W^4)u = -\sum_{j=0}^{\infty} \big((\partial_{rr} + \frac{2}{r}\partial_r - \frac{a+\mu_j}{r^2} + 5W^4)f_j(r)\big)Y_j(\theta).$$

Therefore we can discuss the contribution to the kernel from each spherical harmonic starting from $j = 0$.

Case 1. $j = 0$.

As $Y_0 = 1$, the kernel function in this mode must be a spherically symmetric function $u(r)$ satisfying

$$(\mathcal{L}_a - 5W^4)u = 0,$$

which in the radial coordinate, takes the form

$$u_{rr} + \frac{2}{r}u_r + 5W^4 u - \frac{a}{r^2}u = 0. \tag{3.6}$$

Suppose u is a solution independent of the known radial solution W_1, from Abel's theorem, we have

$$u_r W_1 - (W_1)_r u = \frac{C}{r^2}. \tag{3.7}$$

In the small neighborhood of $r = 0$, $W_1 \neq 0$, we can divide both sides of (3.7) by W_1^2 and obtain,

$$\left(\frac{u}{W_1}\right)_r = \frac{C}{r^2 W_1^2},\ 0 < r < \varepsilon.$$

Recalling $W_1(r) = O(r^{\frac{\beta-1}{2}})$ as $r \to 0^+$, integrating the above equation from r to ε, we have

$$u(r) = O(r^{-\frac{1}{2}(1+\beta)}),\ \text{as } r \to 0^+,$$

which is certainly not an $\dot{H}^1$ function. Therefore W_1 is the unique radial kernel.

Case 2. $\{j \in \mathbb{N}, \mu_j = 2\}$.

In this case, we assume there exists a function in the form of $G(r)Y_j(\theta)$ associated to the jth spherical harmonics in the kernel. Writing Laplacian operator in spherical coordinate, we have

$$0 = (\mathcal{L}_a - 5W^4)(G(r)Y_j(\theta)) = (\mathcal{L}_{a+2} - 5W^4)G(r) \cdot Y_j(\theta),$$

which implies

$$G(r) \in \ker(\mathcal{L}_{a+2} - 5W^4). \tag{3.8}$$

Our first goal toward getting a contradiction is to show positivity of G. To this end, we take any $v \in \dot{H}^1(\mathbb{R}^3)$ in the spherical harmonic expansion

$$v := \sum_{j=0}^{\infty} v_j(r)Y_j(\theta),$$

and evaluate

(3.9)

$$\langle(\mathcal{L}_{a+2} - 5W^4)v, v\rangle = \sum_{j=0}^{\infty}\langle(\mathcal{L}_{a+2} - 5W^4)v_j(r), v_j(r)\rangle + \sum_{j=1}^{\infty}\mu_j \int_{\mathbb{R}^3} \frac{|v_j(x)|^2}{|x|^2}dx > 0$$

As from (3.3), the first summand can be estimated

$$\begin{aligned}\langle(\mathcal{L}_{a+2}-5W^4)v_j(r),v_j(r)\rangle &= \langle(\mathcal{L}_{a+2}-5W^4)v_j(r)\cdot Y_1(\theta),v_j(r)Y_1(\theta)\rangle\\ &= \langle(\mathcal{L}_{a}-5W^4)(v_j(r)Y_1(\theta)),v_j(r)Y_1(\theta)\rangle\\ &\geq 0. \end{aligned}\tag{3.10}$$

We then know that $\mathcal{L}_{a+2}-5W^4$ is non-negative, which together with (3.8) implies that 0 is the first eigenvalue. Hence,

$$G(r)>0.$$

We now turn to looking at the equation of G and $-W'$(keeping in mind that $W'<0$),

$$-G''-\frac{2}{r}G'+\frac{a+2}{r^2}G-5W^4G=0,\tag{3.11}$$

$$-W'''-\frac{2}{r}W''+\frac{a+2}{r^2}W'-\frac{2a}{r^3}W-5W^4W'=0.\tag{3.12}$$

Computing $[(3.11)\cdot r^2W'-(3.12)\cdot r^2G]$, we obtain

$$r^2W'''G+2rW''G-r^2W'G''-2rW'G'+\frac{2a}{r}WG=0,$$

which can be further written into

$$\frac{d}{dr}\big[r^2(W''G-W'G')\big]+\frac{2a}{r}WG=0.\tag{3.13}$$

Recall the asymptotics of W and G from (1.2) and Lemma 9.1 in Appendix:

$$\begin{cases}\text{As } r\to 0^+,\ G(r)=O(r^{-\frac{1}{2}+\frac{1}{2}\sqrt{9+4a}}),\ -W'=O(r^{-\frac{3}{2}+\frac{\beta}{2}})\\ \text{As } r\to \infty,\ G(r)=O(r^{-\frac{1}{2}-\frac{1}{2}\sqrt{9+4a}}),\ -W'=O(r^{-\frac{3}{2}-\frac{\beta}{2}}),\end{cases}\tag{3.14}$$

we have

$$-W'>G \text{ as } r\to 0^+,\ -W'>G \text{ as } r\to\infty.$$

Let

$$r_0=\sup\{r>0\mid -W'>G \text{ on } (0,r)\}.$$

Possibly by replacing G by CG for some $C>0$ sufficiently large, it holds for some $r_0\in(0,\infty)$. We have

$$(W'+G)(r_0)=0,\ (W'+G)(r)<0,\ \forall r\in(0,r_0).$$

Hence $(W''+G')(r_0)\geq 0$ and thus

$$(W''G-W'G')(r_0)\geq 0.\tag{3.15}$$

Using this and the positivity of G, we integrate (3.13) over (r_0,r) to obtain

$$(W''G-W'G')(r)>0,\ \forall r\in(r_0,\infty).\tag{3.16}$$

Dividing both sides by G^2, we have

$$\frac{d}{dr}\left(\frac{W'}{G}(r)\right)>0,\ \forall r\in(r_0,\infty)$$

which in view of (3.14), contradicts with the asymptotics

$$\lim_{r\to\infty}\frac{W'}{G}(r)=-\infty$$

for any $a\in(-\frac{1}{4},0)$. Therefore there is no nontrivial kernel function of $\mathcal{L}_a-5W^4$ associated to the jth spherical harmonics for all j satisfying $\mu_j=2$.

Case 3. $\{j\in\mathbb{N},\mu_j>2\}$.

In this case, we take any function in the form of $G(r)Y_j(\theta)$, $G\neq 0$ and compute

$$\mathcal{L}_a(G(r)Y_j(\theta))=\mathcal{L}_{a+2}G(r)\cdot Y_j(\theta)+\frac{\mu_j-2}{r^2}G(r)Y_j(\theta).$$

Using (3.10) we immediately get

$$\begin{aligned}&\langle\mathcal{L}_a(G(r)Y_j(\theta)),G(r)Y_j(\theta)\rangle\\=&\langle(\mathcal{L}_{a+2}-5W^4)G(r),G(r)\rangle+(\mu_j-2)\int_{\mathbb{R}^3}\frac{|G(x)|^2}{|x|^2}dx>0.\end{aligned}$$

This shows there is no kernel function of $\mathcal{L}_a-5W^4$ associated to jth spherical harmonics for those j such that $\mu_j>2$.

The positivity of λ_3 is finally proved, and we end the discussion on the operator $\mathcal{L}_a-5W^4$.

Based on the results on $\mathcal{L}_a-5W^4$, we can get the result for $\mathcal{L}_a-W^4$ quickly. Let $\tilde\lambda_1\leq\tilde\lambda_2\leq\ldots$ denote the eigenvalues of $\mathcal{L}_a^{-\frac{1}{2}}(\mathcal{L}_a-W^4)\mathcal{L}_a^{-\frac{1}{2}}$. From

$$\langle(\mathcal{L}_a-5W^4)u,u\rangle<\langle(\mathcal{L}_a-W^4)u,u\rangle,$$

we obtain $\lambda_j<\tilde\lambda_j$, $j=1,2,\ldots$. Therefore $\tilde\lambda_2>\lambda_2=0$ and $\tilde\lambda_1=0$ due to $\operatorname{span}\{W\}=\ker(\mathcal{L}_a-W^4)$. This immediately implies

$$\langle(\mathcal{L}_a-W^4)u,u\rangle\geq\tilde\lambda_2\|u\|_{\dot H_a^1}^2,\quad\forall\text{ real }u\perp_{\mathcal{L}_a}W.$$

Combining the two parts together, we proved the estimate for $Q(v)$.

We turn to briefly proving the last statement regarding the generalized kernel. Suppose there exists a nontrivial $\dot H^1$ function $v\notin\ker(JL)$ such that

$$(JL)^2v=0.$$

Then v satisfies

$$JLv=c_1W_1+c_2iW,$$

for some real number c_1,c_2 such that $c_1c_2\neq 0$. Note JL is a bounded operator from $(\dot H^1)^2$ to $(\dot H^{-1})^2$, we immediately get a contradiction since $W_1,iW\notin\dot H^{-1}(\mathbb{R}^3)$ as shown in the following. Take a sequence of $\dot H^1$ function with uniform norm:

$$\psi_N(x)=N^{-\frac{1}{2}}\psi(x/N),\ \psi(r)=\begin{cases}1, & \frac{1}{2}<r\leq 1\\ 0, & r\geq 2,\ r\leq 1/4.\end{cases}$$

It is easy to see both $\int_{\mathbb{R}^3} W\psi_N(x)dx$ and $\int_{\mathbb{R}^3} W_1\psi_N(x)dx$ diverge as $N\to\infty$ by using the asymptotic estimate

$$W(r), W_1(r) = O(r^{-\frac{1}{2}-\frac{\beta}{2}}),\ \text{as } r\to\infty.$$

Therefore there is no generalized kernel for JL.

Finally we complete the proof by verifying the Claim 3.2. Indeed, note as

$$\mathcal{L}_a^{-\frac{1}{2}}:\ L^2(\mathbb{R}^3)\to \dot{H}^1(\mathbb{R}^3),\ \mathcal{L}_a^{-\frac{1}{2}}:\ \dot{H}^{-1}(\mathbb{R}^3)\to L^2(\mathbb{R}^3)$$

are both bounded and the embedding $L^{\frac{6}{5}}(\mathbb{R}^3)\hookrightarrow \dot{H}^{-1}(\mathbb{R}^3)$ is continuous, it suffices to show

$$W^4\mathcal{L}_a^{-\frac{1}{2}}:\ L^2(\mathbb{R}^3)\to L^{\frac{6}{5}}(\mathbb{R}^3)$$

is a compact operator. Taking a bounded sequence f_n in $L^2(\mathbb{R}^3)$ and a sufficiently small number $\varepsilon>0$, we estimate

$$\begin{aligned}
\||\nabla|^{\varepsilon}(W^4\mathcal{L}_a^{-\frac{1}{2}}f_n)\|_{\frac{6}{5}} &\le \||\nabla|^{\varepsilon}W^4\mathcal{L}_a^{-\frac{1}{2}}f_n\|_{\frac{6}{5}} + \|W^4|\nabla|^{\varepsilon}\mathcal{L}_a^{-\frac{1}{2}}f_n\|_{\frac{6}{5}}\\
&\le \|W\|_6^3\||\nabla|^{\varepsilon}W\|_6\|\mathcal{L}_a^{-\frac{1}{2}}f_n\|_6 + \|W\|^4_{\frac{12}{2-\varepsilon}}\||\nabla|^{\varepsilon}\mathcal{L}_a^{-\frac{1}{2}}f_n\|_{\frac{6}{1+2\varepsilon}}\\
&\lesssim \|W\|_6^3\|\nabla W\|_{\frac{6}{3-2\varepsilon}}\|f_n\|_2 + \|W\|^4_{\frac{12}{2-\varepsilon}}\|f_n\|_2 \lesssim 1.
\end{aligned}$$

And

$$\|\chi_{>R}W^4\mathcal{L}_a^{-\frac{1}{2}}f_n\|_{\frac{6}{5}} \le \|\chi_{>R}W^4\|_{\frac{3}{2}}\|\mathcal{L}_a^{-\frac{1}{2}}f_n\|_6 \lesssim R^{-c}$$

for some positive number c. The compactness of $W^4\mathcal{L}_a^{-\frac{1}{2}}$ is proved, hence the Claim 3.2. Proposition 3.1 is finally proved. □

In view of Proposition 3.1, we are able to apply Theorem 2.1 in [21] to obtain the following

Proposition 3.3. *The flow e^{tJL} is a well-defined operator and there exist closed subspaces E^u, E^s and E^c such that*

a) $\dim E^u = \dim E^s = 1$.

b) $e^{tJL}(E^{u,s,c}) = E^{u,s,c}$.

c) $\langle Lu, u\rangle = 0,\ \forall u\in E^{u,s}$, *and*

$$E^c = \{u\in(\dot{H}^1)^2; \langle Lu, v\rangle = 0,\ \forall v\in E^u\oplus E^s\}.$$

d) $\left|e^{tJL}|_{E^c}\right| \le C(1+|t|), \forall t\in\mathbb{R}$.

e) $E^c = \ker L\oplus E^e$ *and* $(\dot{H}^1)^2 = E^u\oplus E^s\oplus\ker L\oplus E^e$ *and*

$$L\sim\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&L_e\end{pmatrix}, JL\sim\begin{pmatrix}\lambda&0&0&0\\0&-\lambda&0&0\\0&0&0&A_{0e}\\0&0&0&A_e\end{pmatrix}$$

and

$$L_e\ge\epsilon>0,\ \langle L_e e^{tA_e}u, e^{tA_e}v\rangle = \langle L_e u, v\rangle.$$

Remark 3.4. *In the rest of the paper, we will assume $V^{\pm}$ is the eigenfunction taken from the E^u and E^s:*

$$JLV^{\pm} = \pm e_0 V^{\pm},\ e_0 > 0;\ \text{and } \langle LV^+, V^- \rangle = 1.$$

We claim that $V^{\pm} \in L^2(\mathbb{R}^3)$. Indeed, writing $V^{\pm} = V_1 \pm iV_2$, we have

$$\begin{cases} (\mathcal{L}_a - W^4)V_2 = e_0 V_1, \\ (\mathcal{L}_a - 5W^4)V_1 = -e_0 V_2, \end{cases}$$

which clearly implies

$$e_0 \int_{\mathbb{R}^3} |V^{\pm}|^2 dx = 4 \int_{\mathbb{R}^3} W^4 V_1 V_2 dx \lesssim \|W\|_6^4 \|V_1\|_6 \|V_2\|_6 \lesssim 1.$$

A more precise analysis on $V^{\pm}$ much as in Lemma 9.1 can be used to show that they decay exponentially in $|x|$ for sufficiently large $|x|$.

4. Modulation analysis

In this section, we perform the modulation analysis for solutions in the small neighborhood of the manifold $\{\mathbf{g}_{\theta,\mu} W\}$. On energy surface of the ground state, the distance to this manifold is controlled by

$$\mathbf{d}(f) = \big| \|f\|_{\dot{H}_a^1}^2 - \|W\|_{\dot{H}_a^1}^2 \big|,$$

as shown in the following result. The same result in the case of NLS can be found in [1, 2, 27].

Proposition 4.1. *Assume that $f \in \dot{H}_a^1(\mathbb{R}^3)$ and $E(f) = E(W)$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that when*

$$\mathbf{d}(f) < \delta, \quad \inf_{\theta \in \mathbb{S}^1, \mu > 0} \|f - \mathbf{g}_{\theta,\mu} W\|_{\dot{H}_a^1} < \varepsilon.$$

Proof. We argue by contradiction. Suppose the claim does not hold, then there must exist $\varepsilon_0 > 0$ and a sequence of $\dot{H}^1(\mathbb{R}^3)$ functions $\{f_n\}$ such that

$$E(f_n) = E(W),\ \mathbf{d}(f_n) \to 0, \tag{4.1}$$

but

$$\inf_{\theta \in \mathbb{S}^1, \mu > 0} \|f_n - \mathbf{g}_{\theta,\mu} W\|_{\dot{H}_a^1} > \varepsilon_0. \tag{4.2}$$

Replacing f_n by $f_n \cdot \frac{\|W\|_{\dot{H}_a^1}}{\|f_n\|_{\dot{H}_a^1}}$, we may assume

$$\|f_n\|_{\dot{H}_a^1} = \|W\|_{\dot{H}_a^1},\ \|f_n\|_6 \to \|W\|_6,\ \inf_{\theta \in \mathbb{S}^1, \mu > 0} \|f_n - \mathbf{g}_{\theta,\mu} W\|_{\dot{H}_a^1} > \varepsilon_0. \tag{4.3}$$

Applying Lemma 9.2 to $\{f_n\}$ we obtain

$$f_n = \sum_{j=1}^{J} \phi_n^j + r_n^J,$$

for each $J \in \{1, \cdots, J^*\}$ with the stated properties. In particular, from the $\dot{H}^1_a$ decoupling in Lemma 9.2 and (4.3) we have

(4.4)
$$\|W\|^2_{\dot{H}^1_a} = \lim_{n\to\infty}\Big(\sum_{j=1}^{J}\|\phi^j_n\|^2_{\dot{H}^1_a} + \|r^J_n\|^2_{\dot{H}^1_a}\Big) = \sum_{j=1}^{J}\|\phi^j\|^2_{X^j} + \lim_{n\to\infty}\|r^J_n\|^2_{\dot{H}^1_a}.$$

Here $\|\cdot\|_{X^j} = \|\cdot\|_{\dot{H}^1_a}$ if $x^j_n \equiv 0$ and $\|\cdot\|_{X^j} = \|\cdot\|_{\dot{H}^1}$ if $\frac{|x^j_n|}{\lambda^j_n} \to \infty$. As (4.4) holds for any J, we take a limit and get

$$\sum_{j=1}^{J^*}\|\phi^j\|^2_{X^j} \le \|W\|^2_{\dot{H}^1_a}. \tag{4.5}$$

On the other hand, using the decoupling in $L^6(\mathbb{R}^3)$, (4.3) and the sharp Sobolev embedding, we have

$$\|W\|_6^6 = \lim_{n\to\infty}\|f_n\|_6^6 = \sum_{j=1}^{J^*}\|\phi^j\|_6^6 \le \sum_{j=1}^{J^*}\|\phi^j\|^6_{\dot{H}^1_a}\cdot\frac{\|W\|_6^6}{\|W\|^6_{\dot{H}^1_a}},$$

which implies

$$\|W\|^6_{\dot{H}^1_a} \le \sum_{j=1}^{J^*}\|\phi^j\|^6_{\dot{H}^1_a}. \tag{4.6}$$

This together with (4.5) gives

$$\Big(\sum_{j=1}^{J^*}\|\phi^j\|^2_{X^j}\Big)^3 \le \sum_{j=1}^{J^*}\|\phi^j\|^6_{\dot{H}^1_a}.$$

Note also for $a<0$, $\|\phi\|_{\dot{H}^1_a} < \|\phi\|_{\dot{H}^1}$, this obviously implies that

$$J^*=1, \quad x^1_n \equiv 0, \text{ and } \limsup_{n\to\infty}\|r^1_n\|_6 = 0.$$

Therefore, (4.5) and (4.6) imply $\|\phi^1\|_{\dot{H}^1_a} = \|W\|_{\dot{H}^1_a}$, $\|\phi^1\|_6 = \|W\|_6$. Moreover

$$f_n = (\lambda_n)^{-\frac12}\phi^1\big(\frac{x}{\lambda_n}\big) + r^1_n, \text{ and } \|r^1_n\|_{\dot{H}^1_a} \to 0$$

follow from (4.4). Hence $\phi^1 = \mathbf{g}_{\theta_0,\mu_0}W$ for some θ_0, μ_0. This contradicts to the last inequality in (4.3). □

This together with implicit function theorem gives:

Lemma 4.2. *There exist $\delta_0, \varepsilon_0 > 0$ such that for any $f \in \dot{H}^1_a(\mathbb{R}^3)$ satisfying $E(f) = E(W)$ and $\mathbf{d}(f) < \delta_0$, there exists a unique pair $(\theta,\mu) \in \mathbb{S}^1 \times \mathbb{R}^+$ such that*

$$\mathbf{g}^{-1}_{\theta,\mu}f \perp \{iW, W_1\} \text{ and } \|f - \mathbf{g}_{\theta,\mu}W\|_{\dot{H}^1_a} < \varepsilon_0.$$

Moreover, the decomposition

$$\mathbf{g}^{-1}_{\theta,\mu}f = W + \alpha W + v, \ v \perp \{iW, W, W_1\}, \tag{4.7}$$

obeys

$$|\alpha| \sim \|v\|_{\dot H^1_a} \sim \|\mathbf{g}_{\theta,\mu}^{-1} f - W\|_{\dot H^1_a} \sim \mathbf{d}(f).$$

Proof. We prove this lemma in several steps.

Step 1. We first focus on the neighborhood of W. Define two functionals J_0, J_1: $\mathbb{S}^1 \times \mathbb{R}^+ \times \dot H^1_a(\mathbb{R}^3) \to \mathbb{R}$:

$$J_0(\theta,\mu,h) = \langle h, \mathbf{g}_{\theta,\mu}(iW)\rangle_{\dot H^1_a}, \qquad J_1(\theta,\mu,h) = \langle h, \mathbf{g}_{\theta,\mu}(W_1)\rangle_{\dot H^1_a}.$$

It is easy to check that J_0, J_1 are linear in h and C^1 in θ, μ. Moreover,

$$J_0(0,1,W) = J_1(0,1,W) = 0, \quad \frac{\partial(J_0,J_1)}{\partial(\theta,\mu)}\Big|_{(0,1,W)} = \begin{pmatrix} -\|W\|^2_{\dot H^1_a} & 0 \\ 0 & \|W_1\|^2_{\dot H^1_a} \end{pmatrix}.$$

Therefore the Implicit Function Theorem assures the existence of $r_1, r_2 > 0$ and a C^1 mapping γ: $\dot H^1_a(\mathbb{R}^3) \supset B_{r_1}(W) \to B_{r_2}\big((0,1)\big) \subset \mathbb{S}^1 \times \mathbb{R}^+$ such that for any $h \in B_{r_1}(W)$,

$$(J_0,J_1)(\theta,\mu,h) = 0,\ (\theta,\mu) \in B_{r_2}\big((0,1)\big) \text{ if and only if } (\theta,\mu) = \gamma(h),$$

which is also equivalent to $\mathbf{g}_{\theta,\mu}^{-1} h \perp \{iW, W_1\}$. Moreover, due to this orthogonality,

$$\|h - \mathbf{g}_{\theta,\mu}W\|_{\dot H^1_a} = \inf_{(\theta',\mu') \subset B_{r_2}\left((0,1)\right)} \|h - \mathbf{g}_{\theta',\mu'}W\|_{\dot H^1_a}.$$

Step 2. We show the global uniqueness of the above pair (θ,μ) for small $\varepsilon_0 > 0$. Suppose the uniqueness is not true, then there exist $\{f_n\} \subset \dot H^1(\mathbb{R}^3)$ and $\{(\theta_n,\mu_n)\}$, $\{(\tilde\theta_n,\tilde\mu_n)\} \subset \mathbb{S}^1 \times \mathbb{R}^+$ such that, for any n, $(\theta_n,\mu_n) \neq (\tilde\theta_n,\tilde\mu_n)$

$$\mathbf{g}^{-1}_{\theta_n,\mu_n} f_n,\ \mathbf{g}^{-1}_{\tilde\theta_n,\tilde\mu_n} f_n \perp \{iW, W_1\},\ \ \|f_n - \mathbf{g}_{\theta_n,\mu_n}W\|_{\dot H^1_a},\ \|f_n - \mathbf{g}_{\tilde\theta_n,\tilde\mu_n}W\|_{\dot H^1_a} < \frac{1}{n}.$$

This implies

$$\|\mathbf{g}_{\theta_n - \tilde\theta_n, \mu_n/\tilde\mu_n} W - W\|_{\dot H^1_a} < \frac{2}{n}.$$

Recall

Claim 4.3. *Let $\{\theta_n,\mu_n\}$ be such that* $\lim_{n\to\infty} \|\mathbf{g}_{\theta_n,\mu_n}W - W\|_{\dot H^1_a} = 0$. *Then* $\lim_{n\to\infty}(\theta_n,\mu_n) = (0,1)$.

The proof of this claim is a simple contradiction argument so we skip it. This contradicts the local uniqueness from which the global uniqueness follows.

Step 3. We prove the comparison with $\mathbf{d}(f)$ under the assumption $E(f) = E(W)$. From Proposition 4.1, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $f \in \dot H^1_a(\mathbb{R}^3)$ satisfying $E(f) = E(W)$ and $\mathbf{d}(f) < \delta$, it holds

$$\inf_{(\theta',\mu') \in \mathbb{S}^1 \times \mathbb{R}^+} \|f - \mathbf{g}_{\theta',\mu'}W\|_{\dot H^1_a} < \varepsilon.$$

By Step 1, such f can be written in the form of (4.7). From the scaling invariance of energy, without loss of generality, we may consider $\theta = 0, \mu = 1$ only. By expanding the energy functional around W, we have

$$\begin{aligned} E(W) =& E(f) = E(W + \alpha W + v) \\ &= E(W) + \frac{1}{2}\langle E''(W)(\alpha W + v), \alpha W + v\rangle + O(\|\alpha W + v\|^3_{\dot{H}^1_a}) \\ &= E(W) + \frac{1}{2}\alpha^2 Q(W) + \frac{1}{2}Q(v) + O(|\alpha|^3 + \|v\|^3_{\dot{H}^1_a}). \end{aligned}$$

Here we have used the orthogonality to drop the cross term $\langle Lv, W\rangle$. This together with the ellipticity of L on $\{W, iW, W_1\}^\perp$ from Lemma 2.1 and $Q(W) < 0$ gives

$$\|v\|^2_{\dot{H}^1_a} \sim -\alpha^2 Q(W) + O(|\alpha|^3 + \|v\|^3_{\dot{H}^1_a}).$$

As indicated from Step 1, if δ_0 is sufficiently small, $|\alpha|$ and $\|v\|_{\dot{H}^1_a}$ are sufficiently small accordingly, therefore we can view the cubic term as perturbation and obtain

$$|\alpha| \sim \|v\|_{\dot{H}^1_a}.$$

Finally, note also

$$\begin{aligned} \mathbf{d}(f) &= \left|\|W + \alpha W + v\|^2_{\dot{H}^1_a} - \|W\|^2_{\dot{H}^1_a}\right| \\ &= \left|(2\alpha + \alpha^2)\|W\|^2_{\dot{H}^1_a} + \|v\|^2_{\dot{H}^1_a}\right| = 2\|W\|^2_{\dot{H}^1_a}|\alpha| + O(\alpha^2 + \|v\|^2_{\dot{H}^1_a}). \end{aligned} \tag{4.8}$$

We conclude

$$\mathbf{d}(f) \sim |\alpha| \sim \|v\|_{\dot{H}^1_a} \sim \|\alpha W + v\|_{\dot{H}^1_a} = \|\mathbf{g}^{-1}_{\theta,\mu} f - W\|_{\dot{H}^1_a}.$$

□

For the rest of this section, we assume $u(t)$ is a solution of (NLS_a) on the time interval I satisfying

$$E(u) = E(W),\ \mathbf{d}(u(t)) < \delta_0,\ \forall t \in I.$$

From Lemma 4.2, there exists a unique pair $(\theta(t), \mu(t))$ for each $t \in I$ such that we can decompose

$$\mathbf{g}_{\theta(t),\mu(t)} u(t) = W + \alpha(t) W + \tilde{u}(t) := W + v(t),\ \textit{and}\ \tilde{u}(t) \perp \{W, iW, W_1\} \tag{4.9}$$

with $|\alpha|$ and $\|\tilde{u}(t)\|_{\dot{H}^1_a}$ comparable to $\mathbf{d}(u(t))$. Our next goal is to obtain the temporal derivative estimates on the modulation parameters $\theta(t)$ and $\mu(t)$.

Before stating the result, we prepare a set of estimates which are needed in analyzing the modulation equation. This is where we have to trade the range of a for a better integrability of the ground state W.

Lemma 4.4. *Let $a > -\frac{1}{4} + \frac{4}{25}$. Then for any real function $R \in L^{\frac{6}{5}}(\mathbb{R}^3)$ and $v \in \dot{H}^1_a(\mathbb{R}^3)$, we have the following bound*

$$|\langle R, W\rangle_{\dot{H}^1_a}| \lesssim \|R\|_{\frac{6}{5}}, \ |\langle v, W\rangle_{\dot{H}^1_a}| + |\langle x\cdot\nabla v, W\rangle_{\dot{H}^1_a}| \lesssim \|v\|_{\dot{H}^1_a},$$
$$|\langle \mathcal{L}_a v, W\rangle_{\dot{H}^1_a}| + |\langle W^4 v, W\rangle_{\dot{H}^1_a}| \lesssim \|v\|_{\dot{H}^1_a}.$$

Here the implicit constants depend only on W. The same set of estimates also hold when W is replaced by W_1.

Proof. It is straightforward to verify that under the constraint of a, $W, W_1 \in \dot{H}^1(\mathbb{R}^3)\cap\dot{H}^{1,\frac{30}{11}}(\mathbb{R}^3)$, which by embedding, implies $W, W_1 \in L^6(\mathbb{R}^3)\cap L^{30}(\mathbb{R}^3)$. Based on these bounds we can estimate

$$|\langle R, W\rangle_{\dot{H}^1_a}| = |\langle R, \mathcal{L}_a W\rangle| = |\langle R, W^5\rangle| \le \|R\|_{\frac{6}{5}}\|W\|_{30}^5 \lesssim \|R\|_{\frac{6}{5}},$$
$$|\langle v, W\rangle_{\dot{H}^1_a}| \lesssim \|v\|_{\dot{H}^1_a}\|W\|_{\dot{H}^1_a} \lesssim \|v\|_{\dot{H}^1_a},$$
$$\langle x\nabla v, W\rangle_{\dot{H}^1_a} = \int_{\mathbb{R}^3} x\nabla v W^5 dx = -3\int_{\mathbb{R}^3} vW^5 dx - 5\int_{\mathbb{R}^3} vW^4 x\nabla W\, dx,$$
$$|\langle x\nabla v, W\rangle_{\dot{H}^1_a}| \lesssim \|v\|_6\|W\|_6^5 + \|v\|_6\|W\|_6^4\|x\nabla W\|_6 \lesssim \|v\|_{\dot{H}^1_a},$$
$$|\langle \mathcal{L}_a v, W\rangle_{\dot{H}^1_a}| = |\langle \mathcal{L}_a v, W^5\rangle| \lesssim \|\nabla v\|_2\|\nabla W\|_{\frac{30}{11}}\|W\|_{30}^4 \lesssim \|v\|_{\dot{H}^1_a},$$
$$|\langle W^4 v, W\rangle_{\dot{H}^1_a}| = |\langle W^4 v, W^5\rangle| \lesssim \|v\|_6\|W\|_6^4\|W\|_{30}^5 \lesssim \|v\|_{\dot{H}^1_a}.$$

Finally as $\mathcal{L}_a W_1 = 5W^4 W_1$ and W, W_1 are both smooth functions with the same asymptotic behaviors as $|x| \to 0$ and $|x| \to \infty$, we have the same set of estimates when W is replaced by W_1. The lemma is proved. □

We are ready to state the following

Lemma 4.5. *The modulation parameters in the decomposition (4.9) obey*

$$|\alpha(t)| \sim \|v(t)\|_{\dot{H}^1_a} \sim \|\tilde{u}(t)\|_{\dot{H}^1_a} \sim \mathbf{d}(u(t)), \tag{4.10}$$

$$|\alpha'(t)| + |\theta'(t)| + \left|\frac{\mu'(t)}{\mu(t)}\right| \lesssim \mu^2(t)\mathbf{d}(u(t)). \tag{4.11}$$

All the implicit constants are time independent.

Proof. Estimate (4.10) follows directly from Lemma 4.2 so we only focus on (4.11). Recall

$$u_{[\theta(t),\mu(t)]}(t,x) = e^{i\theta(t)}\mu(t)^{-\frac{1}{2}}u(t, x/\mu(t)).$$

From the equation of u and letting $y = x\mu(t)$ we deduce the equation for $u_{[\theta(t),\mu(t)]}(t,y)$(for simplicity we drop the t dependence in θ, μ in subscript):

$$\begin{aligned}(i\partial_t - \mu^2(t)\mathcal{L}_a)u_{[\theta,\mu]} + \theta'(t)u_{[\theta,\mu]} + i\frac{\mu'(t)}{\mu(t)}(y\nabla_y + \frac{1}{2})u_{[\theta,\mu]}\\ = -\mu^2(t)|u_{[\theta,\mu]}|^4 u_{[\theta,\mu]}.\end{aligned} \tag{4.12}$$

Introducing the change of variable in time: $t \to s$ and $ds = \mu^2(t)dt$. Then in the (s, y) variable, (4.12) becomes

$$(4.13) \qquad (i\partial_s - \mathcal{L}_a)u_{[\theta,\mu]} + \theta_s u_{[\theta,\mu]} + i\frac{\mu_s}{\mu}(y\nabla_y + \frac{1}{2})u_{[\theta,\mu]} = -|u_{[\theta,\mu]}|^4 u_{[\theta,\mu]}.$$

Inserting the orthogonal decomposition from (4.9) in s, y variable: $u_{[\theta,\mu]}(s, y) = W(y) + v(s, y)$, we obtain the equation for $v := v_1 + iv_2$:

$$\partial_s v + (-\mathcal{L}_a + W^4)v_2 + i(\mathcal{L}_a - 5W^4)v_1 - i\theta_s(v + W) + \frac{\mu_s}{\mu}W_1 = -\frac{\mu_s}{\mu}(y\nabla_y v + \frac{1}{2}v) + R(v).$$

Here $R(v)$ is the high order error

$$R(v) = i|W + v|^4(W + v) - iW^5 - 5iW^4 v_1 + W^4 v_2$$

and obeys the estimate

$$(4.14) \qquad \|R(v)\|_{\frac{6}{5}} \lesssim \|v\|^2_{\dot{H}^1_a} + \|v\|^5_{\dot{H}^1_a} \lesssim \mathbf{d}(u(s))^2.$$

Finally inserting $v(s, y) = \alpha(s)W(y) + \tilde{u}(s, y)$, we obtain the equation for $\tilde{u} = \tilde{u}_1 + \tilde{u}_2$:

(4.15)

$$\partial_s \tilde{u} + \alpha_s W - i\theta_s W + \frac{\mu_s}{\mu}W_1 + (-\mathcal{L}_a + W^4)\tilde{u}_2 + i(\mathcal{L}_a - 5W^4)\tilde{u}_1 - 4i\alpha W^5$$

$$(4.16) \qquad = R(v) + i\theta_s v - \frac{\mu_s}{\mu}(y\nabla v + \frac{1}{2}v).$$

As $\tilde{u} \perp \{W, iW, W_1\}$, we can obtain the estimates of $\alpha_s, \theta_s, \mu_s/\mu$ simply by pairing the equation with these three directions in $\dot{H}^1_a(\mathbb{R}^3)$. All the extra terms can be bounded by using Lemma 4.4 for both real and imaginary parts as showing below.

First, we note (4.16) on the right side of the equation (4.15) only contribute the high order error. We have

$$\begin{aligned} &|\langle (4.16), W\rangle| + |\langle (4.16), iW\rangle| + |\langle (4.16), W_1\rangle| \\ &\lesssim \mathbf{d}(u(s))\big(\mathbf{d}(u(s)) + |\theta_s| + |\mu_s|/\mu\big) := \mathcal{E}(s). \end{aligned}$$

Taking inner product between (4.15) and W, iW and W_1 in $\dot{H}^1_a(\mathbb{R}^3)$ respectively yields

$$(4.17) \qquad \alpha_s \|W\|^2_{\dot{H}^1_a} = \langle (\mathcal{L}_a - W^4)\tilde{u}_2, W\rangle_{\dot{H}^1_a} + O(\mathcal{E}(s)).$$

$$(4.18) \qquad \theta_s \|W\|^2_{\dot{H}^1_a} = \langle (\mathcal{L}_a - 5W^4)\tilde{u}_1, W\rangle_{\dot{H}^1_a} - \alpha\langle 4W^5, W\rangle_{\dot{H}^1_a} + O(\mathcal{E}(s)).$$

$$(4.19) \qquad \frac{\mu_s}{\mu}\|W_1\|^2_{\dot{H}^1_a} = \langle (\mathcal{L}_a - W^4)\tilde{u}_2, W_1\rangle_{\dot{H}^1_a} + O(\mathcal{E}(s)).$$

Applying Lemma 4.4 we are able to control all terms on the left sides and obtain

$$|\alpha_s| + |\theta_s| + |\mu_s/\mu| \lesssim \mathbf{d}(u(s)).$$

Changing back to t variable we proved (4.11). The lemma is proved. □

5. Construction of local stable solutions

In this section, we show the existence and uniqueness of the solution converging exponentially to the ground state W.

We start by proving several linear estimates of the flow e^{tJL} in the Strichartz space. The way of doing it is to use the Strichartz estimate for $e^{it\mathcal{L}_a}$ and treat the W-related terms as perturbations. To this end, we define the Strichartz space over a time interval I:

$$\dot{S}^1(I) = L_t^\infty \dot{H}_a^1 \cap L_t^5 \dot{H}^{1,\frac{30}{11}}(I \times \mathbb{R}^3).$$

The Sobolev norm $\|\cdot\|_{\dot{H}^{1,p}}$ will be estimated mostly by the operator $(\mathcal{L}_a)^{\frac{1}{2}}$ due to the equivalence of Sobolev norms developed earlier in [15]. The specific version we will be using is the following:

Lemma 5.1 ([15])**.** *Let $a > -\frac{1}{4} + \frac{4}{25}$. Then for any $p \in [\frac{30}{29}, \frac{30}{11}]$ and $f \in C_c^\infty(\mathbb{R}^3)$, we have*

$$\|\nabla f\|_p \sim \|(\mathcal{L}_a)^{\frac{1}{2}} f\|_p.$$

Our first estimate is about the homogeneous flow on the central space E^c given in Proposition 3.3.

Lemma 5.2. *Let $u_0 \in E^c$ and $u(t,x) = e^{tJL}u_0$, then for any time $t > 0$,*

$$\|u(\pm t)\|_{\dot{H}^1} \lesssim \langle t \rangle \|u_0\|_{\dot{H}^1}, \tag{5.1}$$

$$\|u(\pm t)\|_2 \lesssim \|u_0\|_2 + \langle t \rangle^2 \|u_0\|_{\dot{H}^1}, \tag{5.2}$$

$$\|u\|_{\dot{S}^1([-T,T])} \lesssim \langle T \rangle^2 \|u_0\|_{\dot{H}^1}. \tag{5.3}$$

Remark 5.3. *Since $e^{\pm e_0 t}V^\pm = e^{tJL}V^\pm$, as a corollary of the this lemma, we have $V^\pm \in \dot{H}^{1,\frac{30}{11}}(\mathbb{R}^3)$.*

Proof. For simplicity we only focus on the estimate for positive times. Recall from Proposition 3.3, $E^c = \ker L \oplus E^e$, and

$$JL|_{\ker L \oplus E^e} = \begin{pmatrix} 0 & A_{0e} \\ 0 & A_e \end{pmatrix},$$

we have the expression of the linear flow

$$e^{tJL}|_{\ker L \oplus E^e} = \begin{pmatrix} I & \int_0^t A_{0e} e^{\tau A_e} d\tau \\ 0 & e^{tA_e} \end{pmatrix}.$$

Hence for any $u_0 \in E^c$ and $u_0 = u_0^k + u_0^e$, with $u_0^k \in \ker L$, $u_0^e \in E^e$; we can write

$$\begin{aligned} u(t) = e^{tJL}u_0 &= \begin{pmatrix} I & \int_0^t A_{0e} e^{\tau A_e} d\tau \\ 0 & e^{tA_e} \end{pmatrix} \begin{pmatrix} u_0^k \\ u_0^e \end{pmatrix} \\ &= \begin{pmatrix} u_0^k + \int_0^t A_{0e} e^{\tau A_e} u_0^e d\tau \\ e^{tA_e} u_0^e \end{pmatrix}. \end{aligned}$$

From Proposition 3.3 again, the second row is under control due to the ellipticity and the invariance of L_e:

$$\|e^{tA_e}u_0^e\|_{\dot H^1}\sim \|u_0^e\|_{\dot H^1}\lesssim \|u_0\|_{\dot H^1}.$$

Plugging this estimate and using the boundedness of A_{0e} on $\dot H^1(\mathbb{R}^3)$ we have

$$\|u(t)\|_{\dot H^1}\le \|e^{tA_e}u_0^e\|_{\dot H^1}+\|u_0^k\|_{\dot H^1}+\int_0^t\|A_{0e}e^{\tau A_e}u_0^e\|_{\dot H^1}d\tau\lesssim \langle t\rangle\|u_0\|_{\dot H^1}.$$

(5.1) is proved.

To prove the L^2 bound (5.2), we use the equation of u

$$iu_t=\mathcal{L}_a u-W^4(5u_1+iu_2). \tag{5.4}$$

Multiplying both sides by $\bar u$, taking the imaginary part and integrating over $[0,t]\times\mathbb{R}^3$ gives

$$\|u(t)\|_2^2-\|u_0\|_2^2=8\int_0^t\int_{\mathbb{R}^3}W^4u_1u_2dxds\lesssim t\|W\|_6^4\|u\|^2_{L_t^\infty\dot H^1([0,t])}$$

which together with the $\dot H^1$ estimate from (5.1) yields (5.2).

We turn to the estimate (5.3). Take a small number η we partition$[0,T]$ into

$$[0,T]=\bigcup_{j=0}^N I_j,\ \text{with } I_j=[j\eta,(j+1)\eta],\ j\le N-1; I_N=[N\eta,T].$$

On each interval I_j, by the Strichartz estimate of $e^{it\mathcal{L}_a}$ in [4], we obtain

$$\begin{aligned}\|u\|_{\dot S^1(I_j)}&\lesssim \|u(j\eta)\|_{\dot H^1}+\|\mathcal{L}_a^{\frac12}(W^4(5u_1+iu_2))\|_{L_t^1L_x^2(I_j)}\\&\lesssim \langle j\eta\rangle\|u_0\|_{\dot H^1}+\sum_{i=1}^2\|W^4\nabla u_i\|_{L_t^1L_x^2(I_j)}+\|\nabla W W^3u\|_{L_t^1L_x^2(I_j)}\\&\lesssim \langle j\eta\rangle\|u_0\|_{\dot H^1}+\eta^{\frac45}(\|W\|_{30}^4\|\nabla u\|_{L_t^5L_x^{\frac{30}{11}}(I_j)}+\|W\|_{30}^3\|\nabla W\|_{\frac{30}{11}}\|u\|_{L_t^5L_x^{30}(I_j)})\\&\lesssim \langle j\eta\rangle\|u_0\|_{\dot H^1}+\eta^{\frac45}\|u\|_{\dot S^1(I_j)}.\end{aligned}$$

Taking η sufficiently small, we obtain

$$\|u\|_{\dot S^1([j\eta,(j+1)\eta))}\lesssim \langle j\eta\rangle\|u_0\|_{\dot H^1}.$$

Summing in j we obtain (5.3). □

Next we prove the estimate for the inhomogeneous term.

Lemma 5.4. *Let $f\in E^c$ and*

$$v(t,x)=\int_0^t e^{(t-s)JL}f(s)ds,\ w(t,x)=\int_t^T e^{(t-s)JL}f(s)ds,$$

then

$$\|v\|_{\dot S^1([0,T])}+\|w\|_{\dot S^1([0,T])}\lesssim \langle T\rangle^2\|f\|_{L_t^1\dot H^1([0,T])}. \tag{5.5}$$

Proof. We only prove the estimate for v as the other one is similar. Again, we partition the interval $[0,T]$ into subintervals as in Lemma 5.2 and apply the Strichartz estimate on $I_j = [j\eta,(j+1)\eta]$ to $v(t,x)$ which solves

$$iv_t = \mathcal{L}_a v - (5W^4 v_1 + iW^4 v_2) + if.$$

We have

$$\begin{aligned}\|v\|_{\dot S^1(I_j)} &\lesssim \|v(j\eta)\|_{\dot H^1} + \|5W^4 v_1 + iW^4 v_2\|_{L^1_t \dot H^1(I_j)} + \|f\|_{L^1_t \dot H^1(I_j)} \\ &\lesssim \|v(j\eta)\|_{\dot H^1} + \eta^{\frac{4}{5}}\|v\|_{\dot S^1(I_j)} + \|f\|_{L^1_t \dot H^1(I_j)}\end{aligned}$$

Taking η small enough and using (5.1) from Lemma 5.2, we have

$$\|v(t)\|_{\dot H^1} \le \int_0^t \|e^{(t-s)JL} f(s)\|_{\dot H^1} ds \lesssim \int_0^t \langle t\rangle \|f(s)\|_{\dot H^1} ds \le \langle t\rangle \|f\|_{L^1_s \dot H^1([0,t])}.$$

From here, we continue the estimate of v and obtain

$$\|v\|_{\dot S^1(I_j)} \lesssim \langle j\eta\rangle \|f\|_{L^1_t \dot H^1([0,(j+1)\eta))}$$

Summing in j we obtain (5.5). □

We are now ready to state the following theorem which we will prove by analyzing the linearized equation (3.1) around the ground state W.

Theorem 5.5. *There exists $C>0$ depending only on equation (3.1) such that, for any $\lambda \in (0, e_0]$ and $y_0^- \in (-\delta,\delta)$ where $\delta = \frac{1}{C}\min(\lambda,\lambda^4)$, there exists a unique solution to (3.1):*

$$v_t = JLv + R(v)$$

satisfying

$$v(0) = y_0^- V^- + y_0^+ V^+ + v^c(0), \ \text{and} \ \|v(t)\|_{\dot H^1_a} \le C\delta e^{-\lambda t}. \tag{5.6}$$

Moreover,

$$\begin{cases} \|v(t)\|_2 \le C\delta e^{-\lambda t}, \ |y_0^+| + \|v^c(0)\|_{\dot H^1_a} \le C|y_0^-|^2, \\ \|v\|_{\dot S^1([t,\infty))} \le C^2 \delta e^{-e_0 t}. \end{cases} \tag{5.7}$$

For any y_0, $\tilde y_0$ such that $y_0^- \tilde y_0^- > 0$ and $|y_0^-|, |\tilde y_0^-| < \delta$, the corresponding solutions $v(t,x)$ and $\tilde v(t,x)$ obey $v(t) = \tilde v(t+T)$ for some $T = T(y_0, \tilde y_0)$.

Proof. As from Proposition 3.3, $(\dot H^1_a)^2 = E^s \oplus E^u \oplus E^c$, we can decompose

$$v = y^+ V^+ + y^- V^- + v^c \tag{5.8}$$

with $y^\pm = \langle LV^\mp, v\rangle$ and $v^c = v - y^- V^- - y^+ V^+$. Using the invariance of JL on E^u, E^s and E^c, we reduce the problem to the following system

$$\begin{cases} \dot y^- &= -e_0 y^- + R^-(v) \\ \dot y^+ &= e_0 y^+ + R^+(v) \\ \frac{\partial}{\partial t} v^c &= JLv^c + R^c(v). \end{cases}$$

Here, $R^{\pm}(v)$ and $R^c(v)$ are defined similarly as $y^{\pm}$ and v^c. Due to the lack of exponential decay in the unstable and center directions of the linear flow e^{tJL} as $t \to +\infty$, by Duhamel, exponential decaying solutions must satisfy

$$\begin{cases} y^-(t) &= e^{-e_0 t} y_0^- + \int_0^t e^{-e_0(t-s)} R^-(v(s))ds \\ y^+(t) &= -\int_t^\infty e^{e_0(t-s)} R^+(v(s))ds \\ v^c(t) &= \int_t^\infty e^{JL(t-s)} R^c(v(s))ds. \end{cases} \tag{5.9}$$

Our goal is to show that the above right sides define a contraction

$$(\tilde{y}^{\pm}, \tilde{v}^c) = F(y^{\pm}, v^c)$$

on the ball defined by

$$\begin{aligned} B_{\delta,\lambda} = \{(y^{\pm}, v^c) \in C^0([0,\infty)) \times \dot{S}^1([0,\infty)) \mid &\sup_{t\geq 0} e^{\lambda t} |y^{\pm}(t)| \leq 2\delta; \\ &\sup_{t\geq 0} e^{\lambda t} \|v^c\|_{\dot{S}^1([t,\infty))} \leq 2\delta\}. \end{aligned}$$

It is easy to see $B_{\delta,\lambda}$ increases in δ and decreases in λ.

We define another ball $\tilde{B} = B_{\delta,\lambda} \cap \{v(t,x) : \sup_{t\geq 0} e^{\lambda t}\|v(t)\|_2 \leq 2\delta\}$. We will show later that the solution obtained in $B_{\delta,\lambda}$ also belongs to $\tilde{B}$, from which we immediately prove the L^2 regularity in (5.7).

Taking $(y^{\pm}, v^c)$ from $B_{\delta,\lambda}$, we first reproduce the same bounds on $F(y^{\pm}, v^c)$ by using the equations (5.9).

To estimate $\tilde{y}^-(t)$, we first recall that V^- is the eigenfunction of JL associated to the eigenvalue $-e_0 < 0$, which allows us to estimate

$$\begin{aligned} &|R^-(v(s))| = \left|\langle LV^+, R(v(s))\rangle\right| = \left|\langle -e_0 JV^+, R(v(s))\rangle\right| \\ \lesssim &\|V^+\|_6 \|R(v(s))\|_{\frac{6}{5}} \lesssim \|W\|_6^3 \|v\|_6^2 + \|v\|_6^5 \\ \lesssim &|y^+(s)|^2 + |y^-(s)|^2 + \|v^c(s)\|_{\dot{H}^1}^2 + |y^+(s)|^5 + |y^-(s)|^5 + \|v^c(s)\|_{\dot{H}^1}^5 \\ \lesssim &(2\delta)^2 e^{-2\lambda s}. \end{aligned}$$

Inserting this to the first equation in (5.9) we have

$$\begin{aligned} e^{\lambda t}|\tilde{y}^-(t)| &\leq e^{(\lambda - e_0)t}|y_0^-| + e^{\lambda t}\int_0^t e^{-(t-s)e_0}|R^-(v(s))|ds \\ &\leq |y_0^-| + C(2\delta)^2 \int_0^t e^{(\lambda-e_0)(t-s)} e^{-2\lambda s} ds \\ &\leq |y_0^-| + 4C\delta^2/\lambda \leq 2\delta. \end{aligned}$$

The estimate of $\tilde{y}^+(t)$ is similar. Indeed, arguing in the same way as for $R^-(v(s))$, we have

$$\begin{aligned} &|R^+(v(s))| \lesssim \delta^2 e^{-2\lambda s}, \\ &e^{\lambda t}|\tilde{y}^+(t)| \lesssim \delta^2 \int_t^\infty e^{(\lambda+e_0)(t-s)} e^{-2\lambda s} ds \lesssim \delta^2 e^{-\lambda t} \leq 2\delta \end{aligned} \tag{5.10}$$

for the same choice of δ.

We now turn to the estimate of $\tilde{v}^c(t,x)$ and we start by stating a nonlinear estimate which will be used multiple times.

Claim 5.6. *For v defined in (5.8) and $\{y^\pm, v^c\} \in B_{\delta,\lambda}$, we have*
$$\|R^c(v(s))\|_{L_t^1\dot{H}^1([T_0,T_0+T_1])} \lesssim \delta^2 e^{-2\lambda T_0}\langle T_1\rangle.$$

Indeed, from the expression of v in (5.8), it is straightforward to check
$$\|\nabla v\|_{L_t^5 L_x^{\frac{30}{11}}([T_0,T_0+T_1])} \lesssim \delta e^{-\lambda T_0}. \tag{5.11}$$
Applying this estimate and using Sobolev embedding, we immediately get
$$\begin{aligned}
\|R(v(s))\|_{L_t^1\dot{H}^1([T_0,T_0+T_1])} &\lesssim \sum_{i=0}^{3} \|W^i v^{5-i}\|_{L_t^1\dot{H}^1([T_0,T_0+T_1])} \\
&\lesssim \sum_{i=0}^{3} T_1^{\frac{i}{5}} \|\nabla W\|^{i}_{\frac{30}{11}} \|\nabla v\|^{5-i}_{L_t^5 L_x^{\frac{30}{11}}([T_0,T_0+T_1])} \\
&\lesssim \sum_{i=0}^{3} T_1^{\frac{i}{5}} \|\nabla v\|^{5-i}_{L_t^5 L_x^{\frac{30}{11}}([T_0,T_0+T_1])}.
\end{aligned} \tag{5.12}$$
Inserting (5.11) into (5.12), we proved the Claim 5.6.

We are ready to estimate $\tilde{v}^c$ on $[T,\infty)$. By triangle inequality, we have
$$\begin{aligned}
\|\tilde{v}^c\|_{\dot{S}^1([T,\infty))} &= \left\| \int_t^\infty e^{(t-s)JL} R^c(v(s))ds \right\|_{\dot{S}^1([T,\infty))} \\
&\leq \sum_{N\geq 1, N\in\mathbb{N}} \left\| \int_t^{T+N} e^{(t-s)JL} R^c(v(s))ds \right\|_{\dot{S}^1([T+N-1,T+N])} \\
&\quad + \sum_{N\geq 1, N\in\mathbb{N}} \left\| \int_{T+N}^{\infty} e^{(t-s)JL} R^c(v(s))ds \right\|_{\dot{S}^1([T+N-1,T+N])} \\
&:= I + II.
\end{aligned}$$
To estimate I, we use time translated version of (5.5) in Lemma 5.4 and get
$$\begin{aligned}
I &\leq \sum_{N\geq 1, N\in\mathbb{N}} \|R^c(v(s))\|_{L_t^1\dot{H}^1([T+N-1,T+N])} \\
&\lesssim \sum_{N\geq 1, N\in\mathbb{N}} \delta^2 e^{-2\lambda(T+N-1)} \leq \frac{1}{\lambda} C\delta^2 e^{-2\lambda T}.
\end{aligned}$$
To estimate II, we further partition the integral into
$$\begin{aligned}
II &\leq \sum_{N\geq 1} \sum_{M\geq N+1} \left\| \int_{T+M-1}^{T+M} e^{(t-s)JL} R^c(v(s))ds \right\|_{\dot{S}^1([T+N-1,T+N])} \\
&\leq \sum_{N\geq 1, M\geq N+1} \int_{T+M-1}^{T+M} \|e^{(t-s)JL} R^c(v(s))\|_{\dot{S}^1([T+N-1,T+N])} ds.
\end{aligned}$$

Note $|t-s| \le M$, applying Lemma 5.2 we obtain

$$II \le \sum_{N\ge 1, M\ge N+1} \int_{T+M-1}^{T+M} M^2 \|R^c(v(s))\|_{\dot H^1} ds,$$

from which we sum in N and use Claim 5.6 to continue

$$\begin{aligned} II &\le \sum_{M\ge 2} M^3 \|R^c(v(s))\|_{L^1_t \dot H^1([T+M-1,T+M])} \\ &\lesssim \sum_{M\ge 2} M^3 \delta^2 e^{-2\lambda(T+M-1)} \le \frac{C}{\lambda^4}\delta^2 e^{-2\lambda T}. \end{aligned}$$

Collecting the estimates for I and II, we obtain

$$\sup_{T\ge 0} e^{\lambda T} \|\tilde v_c\|_{\dot S^1([T,\infty))} \le \sup_{T\ge 0} e^{\lambda T}(I+II) \le C(\frac{1}{\lambda^4}+\frac{1}{\lambda})\delta^2 \le 2\delta. \tag{5.13}$$

This shows that the map $(\tilde y^{\pm}, \tilde v^c)$ defined by the right side of (5.9) maps $B_{\delta,\lambda}$ to itself. Due to the polynomial form of the nonlinearity, following the similar argument we can easily show the map is a contraction on $B_{\delta,\lambda}$ with a Lipschitz constant $\frac{1}{2}$, hence the existence and uniqueness of the solution to (5.9) in $B_{\delta,\lambda}$ is proved.

Next we show that $(\tilde y^{\pm}, \tilde v^c) \in \tilde B$ if $(y^{\pm}, v^c) \in \tilde B$ and it suffices to estimate the L^2 norm only. Using the estimate of $\tilde y^{\pm}$ and the fact that $V^{\pm} \in L^2(\mathbb{R}^3)$ from Remark 3.4, we further reduce the matter to showing $\|\tilde v^c(t)\|_2 \lesssim \delta e^{-\lambda t}$. Taking the L^2 norm on the expression of $\tilde v^c$ and using the L^2 linear estimate from (5.2) we have

$$\|\tilde v^c(t)\|_2 \lesssim \int_t^\infty \|R^c v(s)\|_2 ds + \int_t^\infty \langle t-s\rangle^2 \|R^c(v(s))\|_{\dot H^1} ds.$$

From here we partition the integral into pieces and arguing in the same way as above. The only missing piece is $\|R^c v(s)\|_{L^1_t L^2_x}$ on a unit time interval which can be done easily

$$\begin{aligned} \|R^c(v)\|_{L^1_t L^2_x([t+N-1,t+N])} &\lesssim \sum_{i=0}^{3} \|W\|_{10}^i \|v\|^{5-i}_{L^{10}_t L^{10}_x([t+N-1,t+N])} \\ &\lesssim \|v\|^2_{\dot S^1([t+N-1,t+N])} + \|v\|^5_{\dot S^1([t+N-1,t+N])}. \end{aligned}$$

The rest of the argument will be similar, we omit the details. This proves the L^2 estimate in (5.7).

To see the quadratic estimate (5.7), we note by repeating the same argument, the solution map is contractive on a smaller ball $B_{|y_0^-|,e_0}$. This together with the uniqueness in $B_{\delta,\lambda}$ implies that the constructed solution must lie in $B_{|y_0^-|,e_0}$. From here we apply the estimate in analogue with (5.10) and (5.13) with δ being replaced by $|y_0^-|$ and λ by e_0, we immediately obtain

$$|y_0^+| + \|v^c(0)\|_{\dot H^1_a} \lesssim |y_0^-|^2.$$

In the above we prove the existence of the stable solution with $y^-(0) = y_0^-$ which is unique in $B_{\delta,\lambda}$. The stronger uniqueness of such solution in the set of functions characterized by (5.6) is a simple consequence of the following Lemma 5.7 and the above uniqueness.

To complete the proof of Theorem 5.5, for any y_0^- and $\tilde{y}_0^-$ satisfying $|y_0^-|, |\tilde{y}_0^-| < \delta$ and $y_0^- \tilde{y}_0^- > 0$, let $v(t)$ and $\tilde{v}(t)$ be the corresponding exponentially decaying solutions. From the continuity and decay of $\tilde{v}(t)$ in $\dot{H}_a^1(\mathbb{R}^3)$, we know there must exist a time T such that $\tilde{y}^-(T) = \langle LV^+, \tilde{v}(T)\rangle = y_0^-$, from the uniqueness we conclude that $v(t) = \tilde{v}(t+T)$. □

The following lemma gives the exponential decay of the Strichartz norm from the exponential decay of the $\dot{H}_a^1$ norm.

Lemma 5.7. *Assume $v(t)$ is a solution to (3.1) satisfying that, for some $\lambda > 0$,*

$$\|v(t)\|_{\dot{H}_a^1} \lesssim e^{-\lambda t}, \quad t \geq 0,$$

then

$$\|v\|_{\dot{S}^1([t,\infty))} \lesssim e^{-\lambda t}, \quad t \geq 0. \tag{5.14}$$

Proof. Let $T_0 > 0$ be sufficiently large. It suffices to prove the estimate (5.14) for all $t \geq T_0$ as the estimate for $t \in [0, T_0)$ follows from the estimate of $\|v\|_{\dot{S}^1([T_0,\infty))}$ and the standard local estimate on $[t, T_0)$. Let $\tau \geq T_0$ and η be a small number to be chosen later. Applying the Strichartz estimate on the interval $[\tau, \tau+\eta]$ and using the similar nonlinear estimate as in (5.12), we obtain

$$\|v\|_{\dot{S}^1([\tau,\tau+\eta])} \leq C\|v(\tau)\|_{\dot{H}_a^1} + C\sum_{i=0}^{4} \eta^{\frac{i}{5}} \|v\|^{5-i}_{\dot{S}^1([\tau,\tau+\eta])}$$

for some constant C independent of v and η. Recall that $\|v(\tau)\|_{\dot{H}_a^1} \lesssim e^{-\lambda\tau} \leq e^{-\lambda T_0}$, for η sufficiently small and T_0 sufficiently large, the standard continuity argument gives

$$\|v\|_{\dot{S}^1([\tau,\tau+\eta])} \leq 2C\|v(\tau)\|_{\dot{H}_a^1} \lesssim e^{-\lambda\tau}.$$

The estimate of $\|v\|_{\dot{S}^1([t,\infty))}$ then comes from partitioning the interval $[t, \infty)$ and adding up the estimate on each subinterval. The proof is complete. □

Lemma 5.7 together with Theorem 5.5 finally gives rise to the following result, which characterizes all solutions decaying exponentially to the ground state:

Corollary 5.8. *There exist exactly two solutions (up to time translation) $W_\pm$ of NLS_a satisfying*

$$\begin{cases} \|W_\pm - W\|_{H^1} \leq Ce^{-e_0 t}, \ \forall t \geq 0. \\ \|W_+(0)\|_{\dot{H}_a^1} > \|W\|_{\dot{H}_a^1}, \quad \|W_-(0)\|_{\dot{H}_a^1} < \|W\|_{\dot{H}_a^1}. \end{cases}$$

Moreover, if a solution $u(t,x)$ of NLS_a satisfying

$$\|u(t)-W\|_{\dot H^1_a}\le Ce^{-\lambda t},\ \forall t\ge 0$$

for any $C>0$ and $\lambda\in(0,e_0]$, there must exist unique $T^{\pm}$ such that

$$\begin{cases} u(t)=W_+(t+T^+) & \text{if } \|u\|_{\dot H^1_a}>\|W\|_{\dot H^1_a},\\ u(t)=W_-(t+T^-) & \text{if } \|u\|_{\dot H^1_a}<\|W\|_{\dot H^1_a}.\end{cases}$$

We remark that this Corollary does not tell us the behavior of $W_\pm$ for $t<0$, we will discuss this problem in Section 7 and Section 8, and complete the picture of the dynamics of all solutions on the energy surface.

6. Global analysis-Virial

In the previous sections, we develop the modulation analysis which enables us to control the solution near the two dimensional manifold generated by the symmetry transformations applied to W. When the solution is away from the manifold, we use the monotonicity formula arising from Virial to control the solution. To this end, in this section we establish Virial estimates by incorporating the modulation estimates developed in Section 4.

Let $\phi(x)$ be a smooth radial function such that

$$\phi(x)=\begin{cases}|x|^2, & |x|\le 1;\\ 0, & |x|>2,\end{cases}\quad \textit{and } \phi_R(x)=R^2\phi(\frac{x}{R}).$$

Moreover, we can choose ϕ such that the radial derivative satisfies

$$\phi''(r)\le 2. \tag{6.1}$$

From such ϕ we define the truncated Virial

$$V_R(t)=\int_{\mathbb{R}^3}\phi_R(x)|u(t,x)|^2dx.$$

For a solution $u(t)$ of NLS$_a$ with $E(u)=E(W)$, the time derivatives of $V_R(t)$ are computed as

$$\begin{aligned}\partial_t V_R(t)&=2\mathbf{Im}\int_{\mathbb{R}^3}\overline{u(t)}\,\nabla u(t)\cdot\nabla\phi_R dx;\\ \partial_{tt}V_R(t)&=4\mathbf{Re}\int_{\mathbb{R}^3}(\phi_R)_{jk}(x)u_j(t)\bar u_k(t)\,dx-\frac{4}{3}\int_{\mathbb{R}^3}(\Delta\phi_R)|u(t)|^6\,dx\\ &\quad-\int_{\mathbb{R}^3}(\Delta^2\phi_R)|u(t)|^2\,dx+4a\int_{\mathbb{R}^3}\tfrac{x}{|x|^4}\nabla\phi_R|u(t)|^2\,dx\\ &=16(\|W\|^2_{\dot H^1_a}-\|u(t)\|^2_{\dot H^1_a})+A_R(u(t)),\end{aligned}$$

where

$$\begin{aligned}A_R(u(t))\quad=\quad&\int_{|x|>R}(4\partial_{rr}\phi_R-8)|\nabla u(t)|^2dx+\int_{|x|>R}(-\frac{4}{3}\Delta\phi_R+8)|u(t)|^6dx\\ &-\int_{\mathbb{R}^3}\Delta^2\phi_R|u(t)|^2dx+\int_{|x|>R}(\frac{4a}{|x|^4}x\nabla\phi_R|u(t)|^2-\frac{8a|u(t)|^2}{|x|^2})dx.\end{aligned}$$

As seen in Section 4, $\mathbf{d}(u(t))$ plays a role of measuring the distance between $u(t)$ and the manifold, we then rewrite $\partial_{tt}V_R(t)$ into

$$\partial_{tt}V_R(t) = \begin{cases} 16\ \mathbf{d}(u(t)) + A_R(u(t)), & \text{if } \|u(t)\|_{\dot H^1_a} < \|W\|_{\dot H^1_a}; \\ -16\ \mathbf{d}(u(t)) + A_R(u(t)), & \text{if } \|u(t)\|_{\dot H^1_a} > \|W\|_{\dot H^1_a}. \end{cases} \tag{6.2}$$

The rest of this section is devoted to giving proper estimates on $\partial_t V_R(t)$ and $A_R(t)$. We start with the following elementary lemma which shows how they are rescaled under the transformation of symmetries.

Lemma 6.1. *For any $(\theta,\mu) \in \mathbb{S}^1 \times \mathbb{R}^+$, we have the following scaling relations:*

$$\begin{aligned} \partial_t V_R(t) &= \mu^{-2} \cdot 2\mathbf{Im} \int_{\mathbb{R}^3} \nabla \phi_{\mu R} \cdot \nabla u_{[\theta,\mu]}\ \overline{u_{[\theta,\mu]}} dx. \\ A_R(u(t)) &= A_{\mu R}(u(t)_{[\theta,\mu]}). \end{aligned}$$

In addition,

$$A_R(W) = 0.$$

The verification of this Lemma is straightforward so we skip it. This Lemma together with the modulation analysis from Section 4 yields:

Lemma 6.2 (Virial estimate)**.** *Let $u(t)$ be an $\dot H^1_a$-solution of NLS_a with $E(u) = E(W)$. For those t satisfying $\mathbf{d}(u(t)) < \delta_0$, let*

$$u(t)_{[\theta(t),\mu(t)]} = W + v(t)$$

be the orthogonal decomposition of $u(t)$ given by Lemma 4.5 with the corresponding bounds. We have

$$|\partial_t V_R(t)| \lesssim R^2 \mathbf{d}(u(t)), \tag{6.3}$$

$$A_R(u(t)) \lesssim \int_{|x|>R} \big(|u(t,x)|^6 + \frac{|u(t,x)|^2}{|x|^2}\big) dx \tag{6.4}$$

(6.5)

$$|A_R(u(t))| \lesssim \begin{cases} \int_{|x|>R} \big(|\nabla u(t)|^2 + |u(t)|^6 + \frac{|u(t)|^2}{|x|^2}\big) dx. \\ [(\mu(t)R)^{-\frac{\beta}{2}} \mathbf{d}(u(t)) + \mathbf{d}(u(t))^2], \quad \text{if } \mathbf{d}(u(t)) < \delta_0 \text{ and } |\mu(t)R| \gtrsim 1, \end{cases}$$

where the constants are independent of $\mathbf{d}(u)$, R and $\|u\|_{\dot H^1_a}$.

Proof. We first estimate $\partial_t V_R(t)$. From Hölder inequality and Sobolev embedding, we have

$$|\partial_t V_R(t)| \le \|\nabla u\|_2 \|u\|_6 \|\nabla \phi_R\|_3 \lesssim R^2 \|\nabla \phi\|_3 \|\nabla u\|_2^2 \lesssim R^2(\mathbf{d}(u(t)) + \|W\|^2_{\dot H^1_a}).$$

This proves the bound in the case of $\mathbf{d}(u(t)) \ge \delta_0$ if the implicit constant is allowed to depend on δ_0. To get the bound in the rest of the case

$\mathbf{d}(u(t)) < \delta_0$, we use Lemma 6.1 with (θ, μ) being given by $(\theta(t), \mu(t))$ and the decomposition to get

$$\begin{aligned}
|\partial_t V_R(t)| &= \left|\mu(t)^{-2} \cdot 2\mathbf{Im} \int_{\mathbb{R}^3} \nabla\phi_{\mu(t)R} \nabla(W + v(t))(W + \bar{v}(t))dx\right| \\
&= \left|\mu(t)^{-2} \cdot 2\mathbf{Im} \int_{\mathbb{R}^3} \nabla\phi_{\mu(t)R} (\nabla W \bar{v}(t) + \nabla v(t) W + \nabla v(t)\bar{v}(t))dx\right| \\
&\le \mu(t)^{-2} \|\nabla\phi_{\mu(t)R}\|_3 (\|\nabla W\|_2 \|\nabla v\|_2 + \|\nabla v\|_2^2) \\
&\lesssim R^2 \mathbf{d}(u(t)).
\end{aligned}$$

We now turn to estimating $A_R(u(t))$. Using (6.1), we can throw away the first non-positive term in the expression of $A_R(u(t))$ and estimate the rest three terms to get (6.4). The same direct estimate also gives the first line in (6.5). To get the second bound when $\mathbf{d}(u(t)) < \delta_0$ and $\mu(t)R \gtrsim 1$, we recall $W(x) = O(|x|^{-\frac{1}{2}-\frac{1}{2}\beta})$ for $|x| \gtrsim 1$. This together with the decomposition and Lemma 6.1 yields

$$\begin{aligned}
|A_R(u(t))| = |A_{\mu(t)R}(u(t)_{[\theta(t),\mu(t)]})| &= |A_{\mu(t)R}(W + v(t)) - A_{\mu(t)R}(W)| \\
&\lesssim \|\nabla W\|_{L^2(|x| \ge \mu(t)R)} \|\nabla v(t)\|_2 + \|\nabla v(t)\|_2^2 \\
&+ \|v\|_6 (\|W\|^5_{L^6(|x|\ge\mu(t)R)} + \|v(t)\|_6^5) + \|W/|x|\|_{L^2(|x|\ge\mu(t)R)} \|\nabla v(t)\|_2 \\
&\lesssim (\mu(t)R)^{-\frac{\beta}{2}} \mathbf{d}(u(t)) + (\mathbf{d}(u(t)))^2.
\end{aligned}$$

Lemma 6.2 is proved. □

7. Exponential convergence in the sub-critical case

In this section, we focus on characterizing the non-scattering solutions on the energy surface of $E(W)$ when the kinetic energy is less than that of the ground state W. The main result is the following

Theorem 7.1. *Let u be a solution of NLS_a satisfying*

$$E(u) = E(W), \|u_0\|_{\dot{H}^1_a} < \|W\|_{\dot{H}^1_a}, \|u\|_{S([0,\infty))} = \infty. \tag{7.1}$$

Then there exist $\theta \in \mathbb{S}^1$, $\mu > 0$ and a unique time $T = T(u)$ such that

$$u(t, x) = e^{i\theta} \mu^{\frac{1}{2}} W_-(\mu^2 t + T, \mu x). \tag{7.2}$$

In the opposite time direction, u exists globally and obeys $\|u\|_{S((-\infty,0])} < \infty$.

We note first that (7.2) comes directly from

$$\|u(\mu^{-2}t)_{[\theta,\mu]} - W\|_{\dot{H}^1_a} \le Ce^{-ct}, \forall\, t \ge 0, \tag{7.3}$$

satisfied by the solution $u(\mu^{-2}t)_{[\theta,\mu]}$ and Corollary 5.8.

Therefore throughout the rest of this section we will only focus on the proof of (7.3). We start by discussing properties of solutions obeying (7.1).

7.1. **Properties of solutions satisfying** (7.1)**.** From (7.1), we know u is non-scattering at the minimal energy $E(W)$. The minimality induces the compactness at least in the radial case, as was proved in an earlier work [16, 28, 29]. In the non-radial case and dimension $d = 3$, the compactness is still unavailable we will take (7.4) as an assumption and build our conditional result upon it. Results in dimensions four and five become unconditional. More specifically, there exists $\lambda(t) : [0, \infty) \to \mathbb{R}^+$ such that

$$\{u(t)_{[\lambda(t)]}\}_{t\in[0,\infty)} \textit{ is precompact in } \dot{H}^1_a(\mathbb{R}^3) (\textit{or } \dot{H}^1(\mathbb{R}^3)). \tag{7.4}$$

The first step of proving such statement is to take an arbitrary sequence $\{u(t_n)\}$ and show that there exist λ_n such that $\{\mathbf{g}_{\lambda_n} u(t_n)\}$ is precompact in $\dot{H}^1(\mathbb{R}^3)$. While this had been achieved in [16, 28, 29], it is not entirely clear from here how to jump to the continuous choice of $\lambda(t)$. Here we provide a point of view through which we are able to make the choice of continuous $\lambda(t)$ explicitly and more quantitatively.

Let $\psi : [0, \infty) \to [0, 1)$ be a smooth function such that

$$\psi(0) = 0, \ \lim_{s\to+\infty} \psi(s) = 1 \ \textit{ and } \ \psi'(s) > 0.$$

Define the weighted norm

$$V(R, u) = \int_{\mathbb{R}^3} \psi(|x|/R)|\nabla u|^2 dx.$$

Then for any nontrivial function $u \in \dot{H}^1(\mathbb{R}^3)$, we can easily check that

$$\partial_R V(R, u) < 0, \ V(0, u) = \int_{\mathbb{R}^3} |\nabla u|^2 dx,$$
$$V(\infty, u) = 0, \text{ and } V(R, u_{[\mu]}) = V(R/\mu, u).$$

Due to the monotonicity of V, for any $u \in \dot{H}^1(\mathbb{R}^3)$, there exists a unique $\Lambda(u)$ such that $V(1, u_{[\Lambda]}) = \frac{1}{2}$. Clearly $\Lambda : \dot{H}^1(\mathbb{R}^3) \to \mathbb{R}^+$ is smooth. We have the following lemma.

Lemma 7.2. *Suppose sequences $\{u_n\} \subset \dot{H}^1(\mathbb{R}^3)$ and $\{\lambda_n\} \subset \mathbb{R}^+$ satisfy that $\mathbf{g}_{\lambda_n} u_n$ converges in $\dot{H}^1(\mathbb{R}^3)$. Then $\{\mathbf{g}_{\Lambda(u_n)} u_n\}$ also converges in $\dot{H}^1(\mathbb{R}^3)$.*

Proof. Let

$$\lim_{n\to\infty} \mathbf{g}_{\lambda_n} u_n = \phi \textit{ in } \dot{H}^1(\mathbb{R}^3). \tag{7.5}$$

Let $\Lambda_0 = \Lambda(\phi)$. Then from scaling we have

$$V(1/\Lambda_0, \phi) = V(1, \mathbf{g}_{\Lambda_0}\phi) = V(1, \mathbf{g}_{\Lambda(u_n)} u_n) = V(\lambda_n/\Lambda(u_n), \mathbf{g}_{\lambda_n} u_n)$$

which clearly implies $\lambda_n/\Lambda(u_n) \to 1/\Lambda_0$ as $n \to \infty$ by using the strong convergence (7.5). Hence

$$\mathbf{g}_{\Lambda(u_n)} u_n \to \mathbf{g}_{\Lambda_0}\phi \text{ in } \dot{H}^1(\mathbb{R}^3).$$

□

Therefore for any solution $u(t)$ whose orbit is precompact modular scaling in $\dot{H}^1(\mathbb{R}^3)$, we can take $\lambda(t) = \Lambda(u(t))$ as an underlying choice of scaling parameter which is subject to further mollification in this section. On the one hand, the precompactness of $u(t)$ up to the rescaling has some crucial implications on this $\lambda(t)$. On the other hand, there is certain freedom in the choice of this scaling function $\lambda(t)$ and we will refine our choice to help us to prove Theorem 7.1.

Firstly the compactness implies directly that there exists $C(\varepsilon) > 0$ such that

$$\int_{|x|>\frac{C(\varepsilon)}{\lambda(t)}} |\nabla u(t)|^2 + |u(t)|^6 + \frac{|u(t)|^2}{|x|^2} dx < \varepsilon. \tag{7.6}$$

Secondly we recall that the scaling parameter $\lambda(t)$ obeys

$$\lim_{t\to\infty} \lambda(t)\sqrt{t} = \infty. \tag{7.7}$$

Indeed, if this is not true, there exists a sequence of time $t_n \to \infty$ such that $\lambda^2(t_n)t_n \to c < \infty$, as a result

$$\lim_{n\to\infty} \lambda(t_n) \to 0. \tag{7.8}$$

Let

$$v_n(t) = \mathbf{g}_{\lambda(t_n)} u\big(t_n + \tfrac{t}{\lambda^2(t_n)}\big),$$

we have

$$v_n(0) = \mathbf{g}_{\lambda(t_n)} u(t_n),\ \ v_n(-t_n\lambda^2(t_n)) = \mathbf{g}_{\lambda(t_n)} u(0). \tag{7.9}$$

From the compactness, there exists a subsequence and $v_0 \in \dot{H}^1_a(\mathbb{R}^3)$ such that $v_n(0) \to v_0$ in $\dot{H}^1_a(\mathbb{R}^3)$. Let $v(t, x)$ be the solution of NLS_a with data v_0. The standard local theory implies $v_n(-t_n\lambda^2(t_n)) \to v(-c) \neq 0$ in $\dot{H}^1_a(\mathbb{R}^3)$, which immediately contradicts with $\mathbf{g}_{\lambda(t_n)} u(0) \rightharpoonup 0$ weakly in $\dot{H}^1_a(\mathbb{R}^3)$ from (7.8). Next, we have the following

Lemma 7.3 (Almost constancy)**.** *Let u be the solution satisfying (7.1), then there exist $\delta > 0$ and $0 < c < C < \infty$ such that for any $\tau \geq 0$, on the interval*

$$I_\tau := [\tau, \tau + \frac{\delta}{\lambda^2(\tau)}],$$

we have

$$c \leq \frac{\lambda(\tau_1)}{\lambda(\tau_2)} \leq C,\ \ \textit{for any } \tau_1, \tau_2 \in I_\tau. \tag{7.10}$$

Proof. We argue by contradiction. Suppose (7.10) fails, there must exist two sequences of times $0 < t_n < s_n < \infty$ and $(s_n - t_n)\lambda^2(t_n) \to 0$ but

$$\frac{\lambda(t_n)}{\lambda(s_n)} + \frac{\lambda(s_n)}{\lambda(t_n)} \to \infty. \tag{7.11}$$

Define the scaled solution

$$v_n(t,x) = \mathbf{g}_{\lambda(t_n)} u\big(t_n + \frac{t}{\lambda^2(t_n)}\big) \text{ and } \gamma_n := \lambda^2(t_n)(s_n - t_n). \tag{7.12}$$

We have

$$\begin{cases} v_n(0) = \mathbf{g}_{\lambda(t_n)} u(t_n), \\ v_n(\gamma_n) = \mathbf{g}_{\lambda(t_n)} u(s_n) = \mathbf{g}_{\lambda(t_n)/\lambda(s_n)} \mathbf{g}_{\lambda(s_n)} u(s_n), \\ \gamma_n \to 0. \end{cases} \tag{7.13}$$

From the first equation in (7.13) and the compactness, we know there exist a subsequence and $v_0 \in \dot{H}^1_a(\mathbb{R}^3)$ such that $v_n(0) \to v_0$ in $\dot{H}^1_a(\mathbb{R}^3)$. This together with the standard local theory implies

$$\lim_{n\to\infty} v_n(\gamma_n) = v_0 \neq 0 \text{ in } \dot{H}^1_a(\mathbb{R}^3).$$

In addition, the second expression in (7.13) together with (7.11) and the compactness along the sequence $\{s_n\}$ imply

$$v_n(\gamma_n) \rightharpoonup 0 \textit{ weakly in } \dot{H}^1_a(\mathbb{R}^3),$$

after passing to a subsequence if necessary. We get a contradiction. Lemma 7.3 is then proved. □

The next observation on $\lambda(t)$ is that $\lambda(t)$ is basically comparable to $\mu(t)$ given by Proposition 4.1 when the solution $u(t)$ is close to the manifold.

Lemma 7.4. *Let u be the solution of NLS$_a$ on the time interval I satisfying (7.4). Suppose $\mathbf{d}(u(t)) < \delta_0$ on I hence $u(t)$ is subject to the orthogonal decomposition $\mathbf{g}_{\theta(t),\mu(t)} u(t) = W + v(t)$. Then there exist constants $0 < c < C < \infty$ such that*

$$c < \frac{\lambda(t)}{\mu(t)} < C, \ \forall t \in I.$$

Proof. We argue by contradiction. Suppose this is not true, there must exist a sequence of times $t_n \in I$ such that

$$\frac{\mu(t_n)}{\lambda(t_n)} \to 0 \quad or \quad \frac{\mu(t_n)}{\lambda(t_n)} \to \infty. \tag{7.14}$$

From the compactness we can extract a subsequence and $V \in \dot{H}^1_a(\mathbb{R}^3)$ such that

$$\mathbf{g}_{\lambda(t_n)} u(t_n) \to V \text{ in } \dot{H}^1_a(\mathbb{R}^3),$$

which along with $\mathbf{d}(u(t)) < \delta_0$ implies

$$\|V\|^2_{\dot{H}^1_a} \longleftarrow \|u(t_n)\|^2_{\dot{H}^1_a} = \|W\|^2_{\dot{H}^1_a} - \mathbf{d}(u(t_n)) > \|W\|^2_{\dot{H}^1_a} - \delta_0. \tag{7.15}$$

On the other hand, along the same sequence, we apply the symmetry $\mathbf{g}_{-\theta(t_n),\frac{\lambda(t_n)}{\mu(t_n)}}$ on both sides of the orthogonal decomposition

$$\mathbf{g}_{\theta(t_n),\mu(t_n)} u(t_n) = W + v(t_n), \tag{7.16}$$

to obtain

$$\mathbf{g}_{\lambda(t_n)}u(t_n) = \mathbf{g}_{-\theta(t_n),\frac{\lambda(t_n)}{\mu(t_n)}}(W + v(t_n)).$$

Passing to a subsequence if necessary and taking weak limit on both sides, using (7.14) we have

$$\mathbf{g}_{-\theta(t_n),\frac{\lambda(t_n)}{\mu(t_n)}}v(t_n) \rightharpoonup V \text{ weakly in } \dot{H}^1_a(\mathbb{R}^3).$$

This together with (4.10) shows

$$\|V\|_{\dot{H}^1_a} \leq \liminf_{n\to\infty} \|\mathbf{g}_{-\theta(t_n),\frac{\lambda(t_n)}{\mu(t_n)}}v(t_n)\|_{\dot{H}^1_a} \lesssim \mathbf{d}(u(t_n)) < \delta_0,$$

which contradicts (7.15) and completes the proof of this lemma. □

Next we show that such precompactness implies that $u(t)$ keeps getting closer to the manifold $\{\mathbf{g}_{\theta,\mu}W\}$.

Lemma 7.5. *Let u be the solution of NLS$_a$ satisfying (7.1). Then there exists a sequence of time $t_n \to \infty$ such that $\mathbf{d}(u(t_n)) \to 0$.*

Proof. Let $C(\varepsilon)$ be the function defined in (7.6). Then from (7.7), for any $\varepsilon > 0$, there exists $T_0 = T_0(\varepsilon) > 0$ such that when $t > T_0$, $\lambda(t)t^{\frac{1}{2}} > C(\varepsilon)/\varepsilon^{\frac{1}{2}}$. Therefore on the time interval $[T_0, T]$ we have

$$(\varepsilon T)^{\frac{1}{2}} > \frac{C(\varepsilon)}{\lambda(t)},\ \forall t \in [T_0, T].$$

Take $R = (\varepsilon T)^{\frac{1}{2}}$ and apply Lemma 6.2 for $t \in [T_0, T]$ we obtain

$$\begin{aligned}
|\partial_t V_R(t)| &\lesssim R^2 = \varepsilon T,\\
|A_R(u(t))| &\lesssim \int_{|x|>R} \big(|\nabla u(t)|^2 + |u(t)|^{2^*} + \frac{|u(t)|^2}{|x|^2}\big)dx\\
&\lesssim \int_{|x|>\frac{C(\varepsilon)}{\lambda(t)}} \big(|\nabla u(t)|^2 + |u(t)|^{2^*} + \frac{|u(t)|^2}{|x|^2}\big)dx\\
&\leq \varepsilon,
\end{aligned}$$

These two estimates together with (6.2) and (6.5) give

$$\partial_{tt}V_R(t) \geq 16\mathbf{d}(u(t)) - C\varepsilon.$$

Integrating in t over $[T_0, T]$ and dividing by T we have

$$\frac{1}{T}\int_{T_0}^{T} \mathbf{d}(u(t))dt \lesssim \frac{\varepsilon(T - T_0) + R^2}{T} \lesssim \varepsilon,$$

which immediately gives

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{d}(u(t))dt = 0,$$

by first taking $T \to \infty$ then $\varepsilon \to 0$. The convergence of $\mathbf{d}(u(t))$ along a sequence of time is proved. □

Lemma 7.3 implies that $\lambda(t)$ has a change of $C\lambda(t)$ on the interval with the length $O(1/\lambda^2(t))$. Therefore it is intuitive to imagine

$$|\lambda'(t)| \leq C\lambda^3(t). \tag{7.17}$$

Lemma 7.4 implies that we can replace $\lambda(t)$ by $\mu(t)$ on the interval where $\mathbf{d}(u(t)) < \delta_0$. From (7.17) and the derivative estimate for $\mu(t)$ in Lemma 4.5, it is reasonable to expect

$$\frac{|\lambda'(t)|}{\lambda^3(t)} \leq \begin{cases} C, & \text{when } \mathbf{d}(u(t)) > \delta_0; \\ C\mathbf{d}(u(t)), & \text{when } \mathbf{d}(u(t)) \leq \delta_0, \end{cases} \tag{7.18}$$

from which we may further modifying the constant to guarantee

$$\frac{|\lambda'(t)|}{\lambda^3(t)} \leq C\mathbf{d}(u(t)),\ \forall t \geq 0. \tag{7.19}$$

In fact we can modify $\lambda(t)$ such that it is differentiable almost everywhere and

$$\left|\frac{1}{\lambda^2(a)} - \frac{1}{\lambda^2(b)}\right| \leq C\int_a^b \mathbf{d}(u(t))dt,\ \forall [a,b] \subset [0,\infty). \tag{7.20}$$

See Lemma 9.3 in the Appendix.

We will revisit this estimate later when we prove the uniform lower bound for $\lambda(t)$. Now we turn to considering the distance function $\mathbf{d}(u(t))$ with the goal of proving the exponential decay of $\mathbf{d}(u(t))$. We start by showing

Lemma 7.6 (Integral estimate of $\mathbf{d}(u(t))$)**.** *Let u be the solution of NLS_a satisfying (7.4), then there exists $C > 0$ such that for any $[a,b] \subset [0,\infty)$,*

$$\int_a^b \mathbf{d}(u(t))dt \leq C \sup_{t\in[a,b]} \frac{1}{\lambda(t)^2}[\mathbf{d}(u(a)) + \mathbf{d}(u(b))]. \tag{7.21}$$

Proof. Estimate (7.21) is scaling invariant, by rescaling the solution, we only need to prove the estimate with additional assumption $\min_{t\in[a,b]} \lambda(t) = 1$. In this case, (7.21) can be proved by applying the Fundamental Theorem of Calculus to

$$\partial_{tt}V_R(t) \geq 8\mathbf{d}(u(t)),\ \textit{and}\ |\partial_t V_R(t)| \lesssim \mathbf{d}(u(t)),\ t \in [a,b] \tag{7.22}$$

for some properly chosen R. Indeed, the second estimate in (7.22) follows directly from (6.3) once R is chosen. To fix this R and control $\partial_{tt}V_R(t)$, we use the fact $\lambda(t) \geq 1$, (6.5) and the compactness, in particular (7.6), of u to get

$$|A_R(u(t))| \leq 8\mathbf{d}(u(t))$$

for some $R = R(\delta_0)$ in both of the two cases $\mathbf{d}(u(t)) \geq \delta_0$ and $\mathbf{d}(u(t)) < \delta_0$. The estimate on $\partial_{tt}V_R(t)$ follows then quickly from the expression (6.2). (7.21) is proved. □

The major obstacle of translating the integration estimate to the point-wise decay of $\mathbf{d}(u(t))$ is the uniform lower bound of $\lambda(t)$. We will show this is indeed the case knowing $\mathbf{d}(u(t))$ converges to 0 along a sequence of time, a result that can be deduced again from Virial analysis. We prove these results in the following two lemmas.

Lemma 7.7. *Let u be the solution of NLS_a satisfying (7.1), there exists a constant $c > 0$ such that*

$$\inf_{t\in[0,\infty)} \lambda(t) \geq c.$$

Proof. Let the sequence t_n be determined by Lemma 7.5 such that $\mathbf{d}(u(t_n)) \to 0$ as $n \to \infty$. There exists sufficiently large N such that

$$C(\mathbf{d}(u(t_N)) + \mathbf{d}(u(t_m))) \leq \frac{1}{10}, \ \forall m \geq N.$$

where C is the constant in (7.20). It suffices for us to get the upper bound of $\frac{1}{\lambda^2(t)}$ on $[t_N, \infty)$.

Take any $\tau \in [t_N, \infty)$ and any $m \geq N$ such that $\tau \in [t_N, t_m]$. Applying (7.20) on $[t_N, \tau]$ and Lemma 7.6, we estimate

$$\begin{aligned}
\left|\frac{1}{\lambda^2(\tau)} - \frac{1}{\lambda^2(t_N)}\right| &\leq C \int_{t_N}^{\tau} \mathbf{d}(u(t))dt \\
&\leq C \int_{t_N}^{t_m} \mathbf{d}(u(t))dt \\
&\leq C \sup_{t\in[t_N,t_m]} \frac{1}{\lambda^2(t)} \times \big(\mathbf{d}(u(t_N)) + \mathbf{d}(u(t_m))\big) \\
&\leq \frac{1}{10} \sup_{t\in[t_N,t_m]} \frac{1}{\lambda^2(t)}.
\end{aligned}$$

Therefore from triangle inequality we have

$$\frac{1}{\lambda^2(\tau)} \leq \frac{1}{10}\left(\sup_{t\in[t_N,t_m]} \frac{1}{\lambda^2(t)}\right) + \frac{1}{\lambda^2(t_N)}, \ \forall\, \tau \in [t_N, \infty).$$

Taking supremum in τ on $[t_N, t_m]$ yields

$$\sup_{\tau\in[t_N,t_m]} \frac{1}{\lambda^2(\tau)} \leq \frac{2}{\lambda^2(t_N)} \text{ and thus } \sup_{\tau\in[t_N,\infty)} \frac{1}{\lambda^2(\tau)} \leq \frac{2}{\lambda^2(t_N)}$$

by letting $m \to \infty$. The uniform bound for $\frac{1}{\lambda(\tau)}$ comes from this and the boundedness on the closed interval $[0, t_N]$. Lemma 7.7 is proved. □

Finally we are ready to prove Theorem 7.1.

7.2. **Proof of Theorem 7.1.**

Proof. The key of the estimate is to show $\mathbf{d}(u(t)) \to 0$ and in the orthogonal decomposition

$$\mathbf{g}_{\theta(t),\mu(t)}u(t) = W + \alpha(t)W + v(t), \tag{7.23}$$

all the parameters converges exponentially to their limits.

We start by considering $\mathbf{d}(u(t))$ for which we can use Lemma 7.6 and Lemma 7.7 to get

$$\int_t^{t_n} \mathbf{d}(u(s))ds \le C[\mathbf{d}(u(t)) + \mathbf{d}(u(t_n))]. \tag{7.24}$$

Here $\{t_n\}$ is the sequence in Lemma 7.5, along which $\mathbf{d}(u(t_n)) \to 0$. Taking $t_n \to \infty$ in (7.24) gives immediately

$$\int_t^{\infty} \mathbf{d}(u(s))ds \le C\mathbf{d}(u(t)),\ \forall t \ge 0,$$

which together with Grönwall's inequality yields

$$\int_t^{\infty} \mathbf{d}(u(s))ds \le Ce^{-ct}, \tag{7.25}$$

for some $c, C > 0$.

Now before proving the convergence of $\mathbf{d}(u(t))$ we go back to (7.20) and consider the convergence of $\lambda(t)$. Combining the estimates (7.20) and (7.25) we immediately see that $\frac{1}{\lambda^2(t)}$ converges as $t \to \infty$. Therefore, there exists λ_∞ such that $\lim_{t\to\infty} \lambda(t) = \lambda_\infty \in (0, \infty]$ from the lower bound estimate Lemma 7.7. To preclude the possibility that $\lambda_\infty = \infty$, we argue by contradiction. Assuming this is the case, i.e.

$$\lim_{t\to\infty} \frac{1}{\lambda(t)} = 0, \tag{7.26}$$

and recalling $\mathbf{d}(u(t_n)) \to 0$, for any $\varepsilon > 0$, there must exist $N_0 \in \mathbb{N}$ such that

$$\frac{1}{\lambda(t)} < \varepsilon,\ \forall t \ge t_{N_0}\ \textit{and}\ \ \mathbf{d}(u(t_n)) < \varepsilon,\ \forall n \ge N_0. \tag{7.27}$$

Taking any $t_* \ge t_{N_0}$ and applying (7.20), (7.21) we obtain

$$\begin{aligned}\left|\frac{1}{\lambda^2(t_*)} - \frac{1}{\lambda^2(t_n)}\right| &\le C\left|\int_{t_*}^{t_n} \mathbf{d}(u(t))dt\right| \le C\int_{t_{N_0}}^{t_n} \mathbf{d}(u(t))dt \\ &\le C \sup_{t\in[t_{N_0},t_n]} \frac{1}{\lambda^2(t)}[\mathbf{d}(u(t_n)) + \mathbf{d}(u(t_{N_0}))].\end{aligned}$$

Letting $n \to \infty$ we have

$$\frac{1}{\lambda^2(t_*)} \le C \sup_{t\in[t_{N_0},\infty)} \frac{1}{\lambda^2(t)}\mathbf{d}(u(t_{N_0})) \le C\varepsilon \sup_{t\in[t_{N_0},\infty)} \frac{1}{\lambda^2(t)}.$$

Choosing $C\varepsilon \leq \frac{1}{2}$ and taking supremum in t_* over $[t_{N_0}, \infty)$, we obtain $\frac{1}{\lambda(t)} = 0$ for all $t \geq t_{N_0}$, which is a contradiction. Therefore we conclude that

$$\lim_{t\to\infty} \lambda(t) = \lambda_\infty \in (0, \infty). \tag{7.28}$$

Next we turn to proving the convergence

$$\lim_{t\to\infty} \mathbf{d}(u(t)) = 0. \tag{7.29}$$

Again we argue by contradiction. If this is not true, there must exist a subsequence in n (which we still use the same notation) and a constant $c \in (0, \delta_0)$ such that $\max_{[t_n, t_{n+1}]} \mathbf{d}(u(t)) \geq c$. Therefore we can find $\tau_n \in (t_n, t_{n+1})$ such that

$$\mathbf{d}(u(\tau_n)) = c \textit{ and } \mathbf{d}(u(t)) \leq c, \ \forall t \in [t_n, \tau_n]. \tag{7.30}$$

Applying Lemma 4.5, integrating α' over $[t_n, \tau_n]$ and using the fact that $\mu = \lambda$, (7.28) and (7.25) we have

$$\begin{aligned} |\alpha(t_n) - \alpha(\tau_n)| &\leq \int_{t_n}^{\tau_n} |\alpha'(s)| ds \leq C \int_{t_n}^{\tau_n} \frac{|\alpha'(s)|}{\mu^2(s)} ds \\ &\leq C \int_{t_n}^{\tau_n} \mathbf{d}(u(s)) ds \leq Ce^{-ct_n}. \end{aligned} \tag{7.31}$$

As from Lemma 4.5, $\alpha(t) \sim \mathbf{d}(u(t))$ for $t = t_n, \tau_n$. Taking $n \to \infty$ in (7.31) gives $\alpha(\tau_n) \to 0$, which contradicts with (7.30). Therefore (7.29) is proved.

Due to (7.29), orthogonal decomposition remains valid for all large enough $t \geq T_0$. In particular, this implies $\mu(t) = \lambda(t) \to \lambda_\infty$ and $\alpha(t) \sim \mathbf{d}(u(t)) \to 0$. Combining these estimate and repeating the same estimate in (7.31) over the interval $[t, \tau]$ we have

$$|\alpha(t) - \alpha(\tau)| \leq Ce^{-ct}, \ \forall t \geq T_0.$$

which by taking $\tau \to \infty$ gives rise to

$$|\alpha(t)| \leq Ce^{-ct}, \ \forall t \geq T_0.$$

From here we apply Lemma 4.5 again to get

$$\|v(t)\|_{\dot{H}^1_a} + \mathbf{d}(u(t)) \leq Ce^{-cT}, \forall t \geq T_0. \tag{7.32}$$

Finally, from the derivatives estimate of μ', θ' in Lemma 4.5, using the boundedness of $\mu(t)$ and (7.32), we know that there exists $\theta_\infty \in \mathbb{S}^1$ such that

$$|\theta(t) - \theta_\infty| + |\mu(t) - \lambda_\infty| \leq Ce^{-ct}, \ \forall t \geq T_0.$$

Therefore finally, we have

$$\begin{aligned} \|\mathbf{g}_{\theta_\infty,\lambda_\infty} u(t) - W\|_{\dot{H}^1_a} &= \|u(t) - \mathbf{g}^{-1}_{\theta_\infty,\lambda_\infty} W\|_{\dot{H}^1_a} \\ &\leq \|\mathbf{g}_{\theta(t),\mu(t)} u(t) - W\|_{\dot{H}^1_a} + \|(\mathbf{g}^{-1}_{\theta(t),\mu(t)} - \mathbf{g}^{-1}_{\theta_\infty,\lambda_\infty}) W\|_{\dot{H}^1_a} \\ &\leq \alpha(t)\|W\|_{\dot{H}^1_a} + \|v(t)\|_{\dot{H}^1_a} + C(|\theta(t) - \theta_\infty| + |\mu(t) - \lambda_\infty|) \\ &\leq Ce^{-ct}, \ \forall t \geq T_0, \end{aligned}$$

which by incorporating the finite bound on closed interval $[0, T_0]$ and changing the notation give rise to (7.3) in Theorem 7.1.

To complete the proof of Theorem 7.1, we shall prove $\|W_-\|_{\dot{S}^1((-\infty,0])} < \infty$ by contradiction. Assume $\|W_-\|_{\dot{S}^1((-\infty,0])} = \infty$ all the above results apply to $W_-(t,x)$ for $t \in (-\infty, 0]$. In particular, Lemma 7.6 and Lemma 7.7 imply

$$\int_a^b \mathbf{d}(W_-(t))dt \le C\big(\mathbf{d}(W_-(a)) + \mathbf{d}(W_-(b))\big), \quad \forall a, b \in \mathbb{R}.$$

Since $\lim_{|t|\to\infty} \mathbf{d}(W_-(t)) = 0$, we obtain $\int_{-\infty}^{\infty} \mathbf{d}(W_-(t))dt = 0$, which implies $W_- \equiv W$. It is a contradiction and the proof is complete. □

Remark 7.8. *1. The above argument verifies b.1) in the statement of Theorem 1.5. By time reversal symmetry, we immediately have b.2). The only missing piece is to show all the solutions on the energy surface with less kinetic energy than that of the ground state must be global solution. There a contradiction argument together with the uniform control on the kinetic energy (2.1) and the compactness of the solution leads to the conclusion, see [14] or [9] for details.*

2. Except for the compact assumption, the analysis in this chapter is not dimension sensitive and can be extended easily to all dimensions $d \ge 4$.

8. Exponential convergence in the super-critical case

In this section, we characterize solutions of NLS_a on the energy surface of $E(W)$ if the kinetic energy is greater than that of the ground state. Being different from Section 7, such solutions do not automatically obey the compactness. We thus add additional spatial decay and symmetry requirement to get a proper control on the solution. Our result is the following

Theorem 8.1. *Let u be a solution to NLS_a satisfying*

$$E(u) = E(W), \ \|u\|_{\dot{H}^1_a} > \|W\|_{\dot{H}^1_a}, \ \text{and } u \in H^1_{rad}(\mathbb{R}^3), \tag{8.1}$$

then the maximal lifespan of u must be finite.

We start by pointing out some of the implications from the symmetry and regularity assumptions.

Lemma 8.2. *Suppose u is the solution in Theorem 8.1. Then we have the following*

1) On the interval I where $\mathbf{d}(u(t)) < \delta_0$, there exists $c > 0$ such that $\mu(t)$ appearing in the orthogonal decomposition given in Lemma 4.2

$$\mathbf{g}_{\theta(t),\mu(t)} u(t) = W + v(t) \tag{8.2}$$

satisfies

$$\mu(t) \ge c, \ \forall t \in I. \tag{8.3}$$

2) There exists $R_0 = R_0(\delta_0, W, \|u\|_2)$ such that when $R \ge R_0$,

$$A_R(u(t)) \le \mathbf{d}(u(t)). \tag{8.4}$$

Proof. We first prove (8.3). Taking L^2 norm on both sides of (8.2) and using $\|v(t)\|_6 \lesssim \|v\|_{\dot H^1_a} \le C\delta_0$ from Lemma 4.5 we have

$$\mu(t)\|u(t)\|_2 \ge \|W + v(t)\|_{L^2(|x|\le 1)}$$
$$\ge \|W\|_{L^2(|x|\le 1)} - C\|v(t)\|_6 \ge \|W\|_{L^2(|x|\le 1)} - C\delta_0.$$

Inequality (8.3) then follows from the mass conservation. It is worthwhile to note that in this step that we do not need the radial symmetry.

We turn to proving (8.4). We first recall the decay estimate for the radial function in three dimensions:

$$|x|^2|u(x)|^2 \lesssim \|u\|_2\|\nabla u\|_2,$$

which can be proved by using the Fundamental Theorem of Calculus and Hardy's inequality. Inserting this decay estimate into the interpolation, we have

$$\|u\|_{L^6(|x|\ge R)} \le \|u\|_2^{\frac{1}{3}}\|u\|_{L^\infty(|x|\ge R)}^{\frac{2}{3}} \le R^{-\frac{2}{3}}\|u\|_2^{\frac{2}{3}}\|\nabla u\|_2^{\frac{1}{3}}. \tag{8.5}$$

This estimate together with the first bound in (6.4) gives

$$A_R(u(t)) \lesssim \int_{|x|>R} |u(t,x)|^6 dx + \int_{|x|>R} \frac{|u(t,x)|^2}{|x|^2} dx$$
$$\lesssim R^{-4}\|u\|_2^4(\mathbf{d}(u(t)) + \|W\|_{\dot H^1}^2) + R^{-2}\|u\|_2^2.$$

By taking R large enough depending on $\|u\|_2$, δ_0 and W, we immediately have (8.4) in the case of $\mathbf{d}(u(t)) \ge \delta_0$. In the remaining case when $\mathbf{d}(u(t)) < \delta_0$, (8.4) follows directly from (8.3) and the second estimate of (6.5). The lemma is proved. □

We are ready to prove Theorem 8.1.

Proof. We argue by contradiction. Assume $u(t)$ exists for all $t \in [0,\infty)$, our goal is to show there must be some $(\theta,\mu) \in \mathbb{S}^1 \times \mathbb{R}^+$ such that

$$u(t,x) = e^{-i\theta}\mu^{\frac{1}{2}}W_+(\mu^2 t + T, \mu x). \tag{8.6}$$

As will be explained later this together with the fact that $W_+ \notin L^2(\mathbb{R}^3)$ in three dimensions from Corollary 5.8 immediately yields a contradiction.

Like in Section 7, the key in proving (8.6) is to show $\mathbf{d}(u(t)) \to 0$ and in the orthogonal decomposition

$$\mathbf{g}_{\theta(t),\mu(t)}u(t) = W + \alpha(t)W + \tilde u(t) := W + v(t), \tag{8.7}$$

all the parameters converge exponentially to their limits.

We first establish the integral estimate for $\mathbf{d}(u(t))$, which again will follow from the Virial analysis. For $R \ge R_0$, we apply (8.4) to get

$$\partial_{tt}V_R(t) = -16\mathbf{d}(u(t)) + A_R(u(t)) \le -15\mathbf{d}(u(t)), \tag{8.8}$$

hence $\partial_t V_R(t)$ decreases on $[0,\infty)$. This further implies that

$$\partial_t V_R(t) > 0, \ \forall t \ge 0. \tag{8.9}$$

Indeed, if this is not true, as $\partial_t V_R(t)$ is decreasing, there must exist $t_0 > 0$ such that

$$\partial_t V_R(t) < \partial_t V_R(t_0) < 0, \ \forall t > t_0,$$

which obviously contradicts with the uniform bound $V_R(t) \geq 0$.

Using the positivity of $\partial_t V_R(t)$ together with the estimate of it from (6.3), we integrate (8.8) over $[t, T]$ to get

$$\int_t^T \mathbf{d}(u(s))ds \leq \partial_t V_R(t) - \partial_t V_R(T) \leq \partial_t V_R(t) \leq C\mathbf{d}(u(t)), \quad \forall t \geq 0.$$

Taking $T \to \infty$ we obtain

$$\int_t^\infty \mathbf{d}(u(s))ds \leq C\mathbf{d}(u(t)), \ \forall t \geq 0. \tag{8.10}$$

As a direct implication, there exists a sequence $\{t_n\} \subset (0, \infty)$ such that $\lim_{n\to\infty} \mathbf{d}(u(t_n)) = 0$. Therefore we can perform the decomposition (8.7) in the neighborhood of t_n for large n. We claim that

$$\mu(t_n) \lesssim 1. \tag{8.11}$$

Indeed, if this is not true, passing to a subsequence, we have $\mu(t_{n_k}) \to \infty$. Along this subsequence we use Hölder and (8.7) to estimate

$$\begin{aligned} V_R(t_{n_k}) &= \int_{|x|\leq\varepsilon} \phi_R(x)|u(t_{n_k})|^2 dx + \int_{|x|>\epsilon} \phi_R(x)|u(t_{n_k})|^2 dx \\ &\leq \varepsilon^2 \|u(t_{n_k})\|_2^2 + R^4 \|u(t_{n_k})\|_{L^6(|x|>\varepsilon)}^{\frac{1}{3}} \\ &\lesssim \varepsilon^2 + R^4 \|\mathbf{g}_{[\theta(t_{n_k}),\mu(t_{n_k})]} u(t_{n_k})\|_{L^6(|x|\geq\varepsilon\mu(t_{n_k}))}^{\frac{1}{3}} \\ &\lesssim \varepsilon^2 + R^4 \|W\|_{L^6(|x|\geq\varepsilon\mu(t_{n_k}))}^{\frac{1}{3}} + \|v(t_{n_k})\|_6^{\frac{1}{3}}. \end{aligned}$$

Taking $n_k \to \infty$ then $\varepsilon \to 0$, we obtain $\lim_{n_k\to\infty} V_R(t_{n_k}) = 0$ which contradicts (8.9).

Next, we prove that

$$\lim_{t\to\infty} \mathbf{d}(u(t)) = 0. \tag{8.12}$$

We argue by contradiction. If this is not true, there must exist $c \in (0, \delta_0)$, a subsequence in $\{t_n\}$ (for which we use the same notation) and another sequence τ_n such that

$$\tau_n \in (t_n, t_{n+1}), \ \mathbf{d}(u(\tau_n)) = c, \ \mathbf{d}(u(t)) \in (0, c], \ \forall t \in [t_n, \tau_n]. \tag{8.13}$$

Take any $t \in [t_n, \tau_n]$, we use the derivative estimate from Lemma 4.5 and (8.10) to obtain

$$\left| \frac{1}{\mu(t_n)^2} - \frac{1}{\mu(t)^2} \right| \lesssim \int_{t_n}^t \left| \frac{\mu'(t)}{\mu(t)^3} \right| dt \lesssim \int_{t_n}^\infty \mathbf{d}(u(t))dt \to 0. \tag{8.14}$$

This together with the control from (8.3) and (8.11) implies

$$\mu(t) \sim 1, \ \forall t \in [t_n, \tau_n]. \tag{8.15}$$

Inserting this to the estimate of $\alpha(t)$ we have

$$|\alpha(t_n) - \alpha(\tau_n)| \leq \int_{t_n}^{\tau_n} |\alpha'(t)|dt \lesssim \int_{t_n}^{\tau_n} \frac{|\alpha'(t)|}{\mu^2(t)} \lesssim \int_{t_n}^{\infty} \mathbf{d}(u(t)) \to 0 \tag{8.16}$$

as $n \to \infty$. We get a contradiction as $\alpha(t_n) \sim \mathbf{d}(u(t_n)) \to 0$, but $\alpha(\tau_n) \sim \mathbf{d}(u(\tau_n)) \sim 1$ from (8.13). The convergence of $\mathbf{d}(u(t))$ in (8.12) is proved.

Given (8.12), we can perform the decomposition for all $t \geq T_0$ and repeat the same argument as in (8.11) to show $\mu(t) \sim 1$. The exponential convergence of all the parameters follows from the same argument in Section 7. We will not repeat here. Therefore (8.6) follows from Corollary 5.8. However, $W \notin L^2(\mathbb{R}^3)$ together with $W_\pm - W \in L^2(\mathbb{R}^3)$(by (5.7)) contradicts with $u(t) \in H^1_{rad}(\mathbb{R}^3)$ and thus Theorem 8.1 is proved. □

Remark 8.3. *Theorem 8.1 verifies Theorem 1.5 c). As also seen from the proof, the statement in dimension four is the same after a notational change. In dimension five, due to the fact $W^+ \in L^2$, any solution obeying (8.1) conforms into one of the three scenarios: blowing up both forward and backward in time; coinciding with W^+ up to symmetries or $\bar{W}^+$ up to symmetries. This justifies the remark after Theorem 1.5.*

9. Appendix

Lemma 9.1 (Asymptotic behavior of $G(r)$)**.** *Let W be the ground state in (1.2) and $G(x) = G(|x|) \in \dot{H}^1_{rad}(\mathbb{R}^3)$ solving*

$$G'' + \frac{2}{r}G' - \frac{a+2}{r^2}G + 5W^4G = 0. \tag{9.1}$$

Then

$$As\ r \to 0^+,\ G(r) = O(r^{-\frac{1}{2}+\frac{1}{2}\sqrt{9+4a}}), \tag{9.2}$$

$$As\ r \to \infty,\ G(r) = O(r^{-\frac{1}{2}-\frac{1}{2}\sqrt{9+4a}}). \tag{9.3}$$

Proof. We prove the two asymptotics separately. Near 0, we introduce the new variable

$$s = r^\beta,\ \beta = \sqrt{1+4a},$$

and rewrite the equation (9.1) into

$$G_{ss} + \frac{\beta+1}{\beta s}G_s - \frac{a+2}{\beta^2 s^2}G + \frac{15}{(1+s^2)^2}G = 0. \tag{9.4}$$

It is easy to see 0 is the regular-singular point for this ODE with analytic coefficients, therefore there must exist two linear independent solutions in the form of power series:

$$G_+(s) = s^{\alpha_+}\sum_{n=0}^{\infty} a_n s^n, \quad a_0 = 1;$$

$$G_-(s) = s^{\alpha_-}\sum_{n=0}^{\infty} b_n s^n, \quad b_0 = 1.$$

Here, s^{α_+} and s^{α_-} with

$$\alpha_\pm = \frac{1}{\beta}(-\frac{1}{2} \pm \frac{1}{2}\sqrt{9+4a})$$

are solutions to Cauchy-Euler equation

$$G_{ss} + \frac{\beta+1}{\beta s}G_s - \frac{a+2}{\beta^2 s^2}G = 0.$$

Clearly

$$G_\pm(s) = O(s^{\alpha_\pm}) \text{ as } s \to 0^+.$$

General solutions to (9.4) are

$$c_+G_+(s) + c_-G_-(s).$$

Since our $G(x) = G(|x|) \in \dot{H}^1_{rad}(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$, clearly it must hold $c_- = 0$ and thus we obtain the desired asymptotics of G near 0 after we change the variable back to r.

For the asymptotic behavior near infinity, we can reduce the issue into a similar situation by introducing the change of variable

$$s = r^{-\beta}.$$

Equation (9.1) in variable s is

$$G_{ss} + \frac{\beta-1}{\beta s}G_s - \frac{a+2}{\beta^2 s^2}G + \frac{15}{(1+s^2)^2}G = 0. \tag{9.5}$$

From a similar analysis, it has two linear independent solutions of order $O(s^{\frac{1}{\beta}(\frac{1}{2}\pm\frac{1}{2}\sqrt{9+4a})})$ near $s = 0$. Going back to r variable and using $G(r) \in \dot{H}^1(\mathbb{R}^3)$, we are able to select the right asymptotics

$$G(r) = O(r^{-\frac{1}{2}-\frac{1}{2}\sqrt{9+4a}})$$

as $r \to \infty$. The Lemma is proved. □

Lemma 9.2 ($\dot{H}^1_a$ linear profile decomposition)**.** *Let $\{f_n\}$ be a bounded sequence in $\dot{H}^1_a(\mathbb{R}^3)$. After passing to a subsequence, there exist $J^* \in \{0, 1, 2, \cdots\} \cup \{\infty\}$, $\{\phi^j\}_{j=1}^{J^*} \subset \dot{H}^1_a(\mathbb{R}^3)$, $\{(\lambda_n^j, x_n^j)\}_{j=1}^{J^*} \subset \mathbb{R}^+ \times \mathbb{R}^3$ such that for every $0 \le J \le J^*$, we have the decomposition*

$$f_n = \sum_{j=1}^{J} \phi_n^j + r_n^J, \;\; \phi_n^j = (\lambda_n^j)^{-\frac{1}{2}} \phi^j\big(\frac{x - x_n^j}{\lambda_n^j}\big) := g_n^j \phi^j, \;\; r_n^J \in \dot{H}^1_a(\mathbb{R}^3)$$

satisfying

$$
\begin{aligned}
&\lim_{J\to J^*}\limsup_{n\to\infty}\|r_n^J\|_6=0;\\
&\lim_{n\to\infty}\Big(\|f_n\|_{\dot H_a^1}^2-\sum_{j=1}^J\|\phi_n^j\|_{\dot H_a^1}^2-\|r_n^J\|_{\dot H_a^1}^2\Big)=0,\ \forall J; \qquad (9.6)\\
&\lim_{n\to\infty}\Big(\|f_n\|_6^6-\sum_{j=1}^J\|\phi_n^j\|_6^6-\|r_n^J\|_6^6\Big)=0,\ \forall J.
\end{aligned}
$$

Moreover, for all $j\neq k$, we have the asymptotic orthogonality property

$$\lim_{n\to\infty}\left(\left|\frac{\lambda_n^j}{\lambda_n^k}\right|+\left|\frac{\lambda_n^k}{\lambda_n^j}\right|+\frac{|x_n^j-x_n^k|^2}{\lambda_n^j\lambda_n^k}\right)=0.$$

Finally we may also assume for each j, either $|x_n^j|/\lambda_n^j\to\infty$ or $x_n^j\equiv 0$, therefore

$$\|\phi_n^j\|_{\dot H_a^1}\to\|\phi^j\|_{X^j}=\begin{cases}\|\phi^j\|_{\dot H^1} & \text{as } \frac{|x_n^j|}{\lambda_n^j}\to\infty\\ \|\phi^j\|_{\dot H_a^1} & \text{as } x_n^j\equiv 0.\end{cases} \qquad (9.7)$$

Proof. We use a classical $\dot H^1$ linear profile decomposition developed in the work of Gérard in [11] as a blackbox to prove this Lemma. For a slight different form we will be using, we refer the readers to see [18]. As $\{f_n\}$ is also a bounded sequence in $\dot H^1(\mathbb{R}^3)$, from [11] we obtain a decomposition which enjoys all the properties in Lemma 9.2 except (9.6) and (9.7). Convergence (9.7) is a result quoted directly from Lemma 3.3 in [16]. Therefore the proof of Lemma 9.2 is reduced to only proving the $\dot H_a^1$ decoupling (9.6) by using all the other statements in this Lemma. Before proving (9.6), we record two properties also coming from the classical result. The first one is what appears in [11] in the position of (9.6), the decoupling in $\dot H^1(\mathbb{R}^3)$:

$$\lim_{n\to\infty}\Big(\|f_n\|_{\dot H^1}^2-\sum_{j=1}^J\|\phi^j\|_{\dot H^1}^2-\|r_n^J\|_{\dot H^1}^2\Big)=0. \qquad (9.8)$$

The second one is the weak convergence

$$(g_n^j)^{-1}r_n^J\rightharpoonup 0,\quad \textit{weakly in } \dot H^1(\mathbb{R}^3),\ \forall 1\le j\le J. \qquad (9.9)$$

In view of (9.8) and the expression of $\dot H_a^1$-norm, we further reduce the matter to proving

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}\frac{1}{|x|^2}\Big(|f_n|^2-\sum_{j=1}^J|\phi_n^j|^2-|r_n^J|^2\Big)dx=0. \qquad (9.10)$$

To see (9.10), we use the decomposition to write

$$|f_n|^2 - \sum_{j=1}^{J} |\phi_n^j|^2 - |r_n^J|^2 = \sum_{j\neq k} \phi_n^j \bar{\phi}_n^k + 2\Re \sum_{j=1}^{J} r_n^J \bar{\phi}_n^j,$$

and estimate the contribution to (9.10) from each above term. To estimate the cross term, we write

$$\int_{\mathbb{R}^3} \frac{\phi_n^j(x)\bar{\phi}_n^k(x)}{|x|^2} dx = \int_{\mathbb{R}^3} \phi^j(y) \frac{(g_n^j)^{-1} g_n^k \bar{\phi}^k(y)}{|y + x_n^j/\lambda_n^j|^2} dy := A$$

and discuss the convergence in two cases. Note here by density argument, we may assume ϕ^j, $\phi^k \in C_c^\infty(\mathbb{R}^3)$.

In the first case where $\left|\log \frac{\lambda_n^j}{\lambda_n^k}\right| \to \infty$, we use Hardy's inequality to obtain

$$\begin{aligned}
|A| &\le \min\Bigg(\Big\| \frac{\phi^j}{|y + x_n^j/\lambda_n^j|^{\frac{3}{4}}} \Big\|_2 \Big\| \frac{(g_n^j)^{-1} g_n^k \phi^k}{|y + x_n^j/\lambda_n^j|^{\frac{5}{4}}} \Big\|_2, \ \Big\| \frac{\phi^j}{|y + x_n^j/\lambda_n^j|^{\frac{5}{4}}} \Big\|_2 \Big\| \frac{(g_n^j)^{-1} g_n^k \phi^k}{|y + x_n^j/\lambda_n^j|^{\frac{3}{4}}} \Big\|_2 \Bigg) \\
&\lesssim \min\Big((\lambda_n^j/\lambda_n^k)^{\frac{1}{4}} \|\phi^j\|_{\dot H^{\frac{3}{4}}} \|\phi^k\|_{\dot H^{\frac{5}{4}}}, \ (\lambda_n^j/\lambda_n^k)^{-\frac{1}{4}} \|\phi^j\|_{\dot H^{\frac{5}{4}}} \|\phi^k\|_{\dot H^{\frac{3}{4}}} \\
&\lesssim \min\big((\lambda_n^j/\lambda_n^k)^{\frac{1}{4}}, \ (\lambda_n^j/\lambda_n^k)^{-\frac{1}{4}} \big) \to 0, \ \textit{as } n \to \infty.
\end{aligned}$$

In the second case where $\lambda_n^j \sim \lambda_n^k$, the orthogonality condition guarantees $\frac{|x_n^j - x_n^k|^2}{\lambda_n^j \lambda_n^k} \to \infty$ as $n \to \infty$. Going back to the expression A, this means the support of ϕ^j and $(g_n^j)^{-1} g_n^k \bar{\phi}^k$ do not overlap, hence $A = 0$ for sufficiently large n.

We turn to estimating $\int_{\mathbb{R}^3} \frac{r_n^J(x)\bar{\phi}_n^j(x)}{|x|^2} dx$, which by changing of variables, can be written as

$$\int_{\mathbb{R}^3} \frac{r_n^J(x)\bar{\phi}_n^j(x)}{|x|^2} dx = \int_{\mathbb{R}^3} \frac{(g_n^j)^{-1} r_n^J(y)}{|y + x_n^j/\lambda_n^j|^2} \bar{\phi}^j(y) dy. \tag{9.11}$$

By density argument we may assume $\phi^j \in C_c^\infty(\mathbb{R}^3/\{0\})$. Recall as part of the classical result, for each j either $x_n^j \equiv 0$ or $\frac{|x_n^j|}{\lambda_n^j} \to \infty$. In the first case, we immediately have (9.11) $\to 0$ from the weak convergence (9.9) and the property of ϕ^j. In the case when $\frac{|x_n^j|}{\lambda_n^j} \to \infty$, assuming $\text{supp}(\phi^j) \subset \{|x| \le R\}$, we can estimate

$$(9.11) \le |x_n^j/\lambda_n^j - R|^{-2} \|g_n^j r_n^J\|_6 \|\phi^j\|_{\frac{6}{5}} \to 0,$$

as $n \to \infty$. Combining all the pieces together we prove (9.10), hence end the proof of Lemma 9.2. □

The following lemma is concerned with the modification of the scaling size function $\lambda(t)$ in Section 7. Let $u(t)$ be a solution to NLS_a and $\lambda \in C^0([0,\infty), \mathbb{R}^+)$ satisfying (7.1) and (7.4).

Lemma 9.3. *(Modification of $\lambda(t)$ in Section 7.) There exist $0 < C_1 < C_2$ and a function $\tilde{\lambda} \in C^0\big([0, \infty), \mathbb{R}^+\big)$ such that it satisfies (7.20) and $\tilde{\lambda}'$ exists almost everywhere and*

$$\frac{\lambda(t)}{\tilde{\lambda}(t)} \in (C_1, C_2), \quad \forall t \in [0, \infty) \tag{9.12}$$

The above property (9.12) means that the new $\tilde{\lambda}$ also satisfies (7.4).

Proof. The proof of this lemma is pure technicality and we divide it into several steps. Let δ_0 be the constant given in Lemma 4.2.

Step 1. Let

$$A_l = \{t \in (0, \infty) \mid \mathbf{d}(u(t)) > \frac{2}{3}\delta_0\}, \quad A_s = \{t \in (0, \infty) \mid \mathbf{d}(u(t)) \le \frac{2}{3}\delta_0\}.$$

Since A_l is open, it must be the disjoint union of at most countably many intervals

$$A_l = \cup_n I_n, \quad I_n = (a_n, b_n), \ a_n < b_n < \infty, \ I_n \cap I_m = \emptyset, \forall m \neq n,$$

where all $b_n < \infty$ is due to Lemma 7.5. Recall $\mu(t)$, $t \in A_s$, is the function given in Lemma 4.2. According to Lemma 7.4,

$$\frac{\lambda(a_n)}{\mu(a_n)}, \ \frac{\mu(a_n)}{\lambda(a_n)}, \ \frac{\lambda(b_n)}{\mu(b_n)}, \ \frac{\mu(b_n)}{\lambda(b_n)},$$

are bounded uniformly in n. Therefore there exist linear functions $l_n(t)$, $t \in I_n$, bounded uniformly in n such that

$$\lambda_1(t) = \begin{cases} l_n(t)\lambda(t), & t \in I_n \\ \mu(t), & t \in A_s \end{cases}$$

is continuous. Apparently $\frac{\lambda_1(t)}{\lambda(t)}$ has positive upper and lower bounds. Therefore $\lambda_1(t)$ satisfies (7.4) and all the subsequent properties. Moreover $\lambda_1(t)$ is C^1 in the interior of A_s and in particular satisfies

$$|\lambda_1'(t)| \le C\lambda_1(t)^3 \mathbf{d}(u(t)), \quad \text{if } \mathbf{d}(u(t)) < \frac{2}{3}\delta_0.$$

Step 2. Similarly, let

$$B_l = \{t \in (0, \infty) \mid \mathbf{d}(u(t)) > \frac{\delta_0}{3}\}, \quad B_s = \{t \in (0, \infty) \mid \mathbf{d}(u(t)) \le \frac{\delta_0}{3}\}.$$

Since B_l is open, it must be the disjoint union of at most countably many intervals

$$B_l = \cup_n J_n, \quad J_n = (a_n', b_n'), \ a_n' < b_n' < \infty, \ J_n \cap J_m = \emptyset, \forall m \neq n.$$

We classify the intervals in B_l into two categories by singling out

$$\Lambda_l = \{n \mid \exists\, t \in J_n, \ \text{s.t. } \mathbf{d}(u(t)) \ge \frac{2}{3}\delta_0\}.$$

We shall only modify λ_1 in such intervals.

For any $n \in \Lambda_l$, let

$$t_{n*} = \inf\{t \in J_n \mid \mathbf{d}(u(t)) \geq \frac{2}{3}\delta_0\} > a_n', \quad t_n^* = \sup\{t \in J_n \mid \mathbf{d}(u(t)) \geq \frac{2}{3}\delta_0\} < b_n'.$$

Clearly $\mathbf{d}(u(t_{n*})) = \mathbf{d}(u(t_n^*)) = \frac{2}{3}\delta_0$. Define $t_0 = t_{n*}$ and

$$t_{j+1} = t_j + \frac{\varepsilon\delta}{\lambda_1(t_j)^2}, \ j = 0, 1, 2 \dots, \quad k_n = \min\{j \mid t_j > t_n^*\},$$

where δ is given by Lemma 7.3 for λ_1 and ε is chosen from the next claim.

Claim. There exists $\varepsilon \in (0, 1]$ such that $k_n < \infty$ and $t_{k_n} \in [t_{k_n-1}, b_n')$.

In fact, λ_1 is continuous on $\overline{J_n}$ and thus λ_1^{-2} has a positive lower bound, so obviously $k_n < \infty$. To see $t_{k_n} < b_n'$, we argue by contradiction. If this is not true, there must exist a sequence $\varepsilon_m \to 0+$ and intervals such that

$$b_n' \in [t_{k_n-1}, t_{k_n-1} + \frac{\varepsilon_m\delta}{\lambda_1^2(t_{k_n-1})}],$$

which clearly implies $|b_n' - t_n^*| \leq \frac{\varepsilon_m\delta}{\lambda_1^2(t_{k_n-1})}$. This together with (4.10), (4.11) and Lemma 7.3 gives

$$\begin{aligned} |\alpha(b_n') - \alpha(t_n^*)| &\leq \sup_{t\in[t_n^*, b_n']} |\alpha'(t)||b_n' - t_n^*| \leq \sup_{t\in[t_n^*, b_n']} |\alpha'(t)| \frac{\varepsilon_m\delta}{\lambda_1^2(t_{k_n-1})} \\ &\leq C \sup_{t\in[t_n^*, b_n']} \frac{|\alpha'(t)|}{\lambda_1^2(t)} \varepsilon_m\delta \leq C \sup_{t\in[t_n^*, b_n']} \mathbf{d}(u(t))\varepsilon_m\delta \leq C\delta_0\varepsilon_m\delta \end{aligned}$$

where $\lambda_1 = \mu$ for $t \in [t_n^*, b_n']$ was also used. On the other hand, from (4.8), we can estimate

$$|\mathbf{d}(u(b_n')) - \mathbf{d}(u(t_n^*))| \leq 2\|W\|_{\dot{H}_a^1}^2 |\alpha(b_n') - \alpha(t_n^*)| + C\delta_0^2 \leq C\delta_0\varepsilon_m\delta + C\delta_0^2 \leq \frac{1}{10}\delta_0.$$

This contradicts with the value of $\mathbf{d}$ on these two points: $\mathbf{d}(u(t_n^*)) = \frac{2}{3}\delta_0$ and $\mathbf{d}(u(b_n')) = \frac{1}{3}\delta_0$. The claim is proved.

We are ready to start the final modification of $\lambda_1(t)$ on J_n with $n \in \Lambda_l$. For any integer $j \in [0, k_{n-1}]$, there exist constant $\sigma_{n,j,1}$ and $\sigma_{n,j,2}$ such that the function defined by

$$\psi_{n,j}(t) = (\sigma_{n,j,1}t + \sigma_{n,j,2})^{-\frac{1}{2}},$$

satisfies

$$\psi_{n,j}(t_j) = \lambda_1(t_j), \ \psi_{n,j}(t_{j+1}) = \lambda_1(t_{j+1}).$$

Since $\lambda_1(t_j) \sim \lambda_1(t_{j+1})$ and $\psi_{n,j}$ is monotonic, its boundary condition implies

$$\psi_{n,j} \sim \lambda_1 \ \text{ on } [t_j, t_{j+1}].$$

One may compute explicitly

$$|\psi_{n,j}'(t)|/\psi_{n,j}(t)^3 = \frac{1}{2}|\sigma_{n,j,1}| = \frac{|\lambda_1(t_{j+1})^{-2} - \lambda_1(t_j)^{-2}|}{2(t_{j+1} - t_j)} \lesssim \frac{\lambda_1(t_j)^{-2}}{t_{j+1} - t_j} = (\varepsilon\delta)^{-1}.$$

Define

$$\tilde{\lambda}(t) = \begin{cases} \psi_{n,j}(t), & t \in [t_j, t_{j+1}] \subset J_n \subset B_l,\ n \in \Lambda_l,\ 0 \le j < k_n, \\ \lambda_1(t), & \text{otherwise.} \end{cases}$$

Clear $\tilde{\lambda}$ satisfies (7.20) as locally it is equal to $\psi_{n,j}$ or μ both of which satisfy (7.20). The construction also ensures $\lambda(t) \sim \tilde{\lambda}(t)$. □

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(K. Yang) School of Mathematics, SouthEast University, Nanjing, P.R. China 211189

E-mail address: `yangkai99sk@gmail.com, kaiyang@seu.eud.cn`

(C. Zeng) School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332

E-mail address: `zengch@math.gatech.edu`

(X. Zhang) Department of Mathematics, University of Iowa, Iowa City, IA 52242

E-mail address: `xiaoyi-zhang@uiowa.edu`