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John K. Hunter, Jingyang Shu and Qingtian Zhang

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GLOBAL SOLUTIONS OF A SURFACE QUASIGEOSTROPHIC FRONT EQUATION

JOHN K. HUNTER, JINGYANG SHU AND QINGTIAN ZHANG

We consider a nonlinear, spatially nonlocal initial value problem in one space dimension on $\mathbb{R}$ that describes the motion of surface quasigeostrophic (SQG) fronts. We prove that the initial value problem has a unique local smooth solution under a convergence condition on the multilinear expansion of the nonlinear term in the equation, and, for sufficiently smooth and small initial data, we prove that the solution is global.

1. Introduction

In this paper, we prove the existence of global small, smooth solutions of the initial value problem

$$\begin{cases} \varphi_t(x,t)+\displaystyle\int_{\mathbb{R}}[\varphi_x(x,t)-\varphi_x(x+\zeta,t)]\left\{\frac{1}{|\zeta|}-\frac{1}{\sqrt{\zeta^2+[\varphi(x,t)-\varphi(x+\zeta,t)]^2}}\right\}\mathrm{d}\zeta=2\log|\partial_x|\varphi_x(x,t), \\ \varphi(x,0)=\varphi_0(x), \end{cases} \tag{1-1}$$

where $\varphi:\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}$ is defined for $x\in\mathbb{R},\ t\in\mathbb{R}_+$, and

$$L=\log|\partial_x| \tag{1-2}$$

is the Fourier multiplier operator with symbol $\log|\xi|$. Our main result is stated in Theorem 5.1.

This initial value problem describes front solutions of the surface quasigeostrophic (SQG) equation

$$\theta_t+u\cdot\nabla\theta=0,\quad u=(-\Delta)^{-1/2}\nabla^{\perp}\theta, \tag{1-3}$$

where $(-\Delta)^{-1/2}$ is a fractional inverse Laplacian on $\mathbb{R}^2$ and $\nabla^{\perp}=(-\partial_y,\partial_x)$. The SQG equation arises as a description of quasigeostrophic flows confined to a surface [Lapeyre 2017; Pedlosky 1987]. After the

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incompressible Euler equation, it is the most physically important member of a family of two-dimensional active scalar problems for θ with a divergence-free transport velocity $u = (-\Delta)^{-\alpha/2}\nabla^{\perp}\theta$ and $0 < \alpha \leq 2$. The case $\alpha = 2$ gives the vorticity-stream function formulation of the incompressible Euler equation [Majda and Bertozzi 2002], while $\alpha = 1$ gives the SQG equation.

The SQG equation is also of interest from an analytical perspective because it has similar features to the three-dimensional incompressible Euler equation [Constantin et al. 1994]; in both cases, the question of singularity formation in smooth solutions remains open. The SQG equation has global weak solutions [Marchand 2008; Resnick 1995], and, as for the Euler equation, nonunique weak solutions of the SQG initial value problem may be constructed by convex integration [Buckmaster et al. 2019; Isett and Ma 2021]. The SQG equation also has a nontrivial family of global smooth solutions [Castro et al. 2020].

By SQG front solutions, we mean piecewise-constant solutions of (1-3) with

$$\theta(x, y, t) = \begin{cases} \theta_+ & \text{if } y > \varphi(x, t), \\ \theta_- & \text{if } y < \varphi(x, t), \end{cases}$$

where θ_+ and θ_- are distinct constants, in which θ has a jump discontinuity across a front located at $y = \varphi(x, t)$ with $x \in \mathbb{R}$; in (1-1), the jump is normalized to $\theta_+ - \theta_- = 2\pi$. We assume that the front is a graph and do not consider questions related to the breaking or filamentation of the front.

We contrast these front solutions with SQG patches, in which

$$\theta(x, y, t) = \begin{cases} \theta_+ & \text{if } (x, y) \in \Omega(t), \\ 0 & \text{if } (x, y) \notin \Omega(t), \end{cases}$$

where $\Omega(t) \subset \mathbb{R}^2$ is a bounded, simply connected region with smooth boundary. Contour dynamics equations for the motion of patches in SQG, Euler, and generalized SQG (with arbitrary values of $0 < \alpha \leq 2$) are straightforward to write down, although they require an appropriate regularization of a locally nonintegrable singularity in the Green's function of $(-\Delta)^{\alpha/2}$ when $0 < \alpha \leq 1$. Local well-posedness of the contour dynamics equations for SQG and generalized SQG patches is proved in [Córdoba et al. 2018; Gancedo 2008], and generalized SQG patches in the more locally singular regime $0 < \alpha < 1$ are studied in [Chae et al. 2012; Khor and Rodrigo 2021a; 2021b].

The boundary of a vortex patch in the Euler equation remains globally smooth in time [Bertozzi and Constantin 1993; Chemin 1993; 1998], but this question remains open for SQG patches. Splash singularities cannot occur in a smooth boundary of an SQG patch [Gancedo and Strain 2014], while numerical results suggest the formation of complex, self-similar singularities in a single patch [Scott and Dritschel 2014; 2019] and a curvature blow up when two patches touch [Córdoba et al. 2005]. Singularity formation in the boundary of generalized SQG patches has been proved in the presence of a rigid boundary when α is sufficiently close to 2 [Gancedo and Patel 2021; Kiselev et al. 2016; 2017], and a class of nontrivial global smooth solutions for SQG patches is constructed in [Castro et al. 2016a; 2016b; Gómez-Serrano 2019].

When $0 < \alpha < 1$, it is straightforward to derive contour dynamics equations for fronts in the same way as one does for patches. In that case, [Córdoba et al. 2019] proves the global well-posedness of the initial-value problem on $\mathbb{R}$ for small, smooth generalized SQG fronts.

When $1 \le \alpha \le 2$, additional problems arise in the formulation of contour dynamics equations for fronts as a result of the slow decay of the Green's function and the lack of compact support of θ. Front equations, including (1-1), are derived by a regularization procedure in [Hunter and Shu 2018], and a detailed derivation of (1-1) from the SQG equation is given in [Hunter et al. 2020]. Unlike the front equations with $\alpha \neq 1$, the SQG front equation requires both "ultraviolet" and "infrared" regularization in the front equation to account for the failure of both local and global integrability of the SQG Green's function $G(r) = 1/r$ on $\mathbb{R}$. This failure leads to the logarithmic derivatives in (1-1), rather than the fractional derivatives that occur for generalized SQG fronts with $\alpha \neq 1$.

In the case of spatially periodic fronts with $x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, one can write down front equations directly by using the Green's function of $(-\Delta)^{\alpha/2}$ on the cylinder $\mathbb{T} \times \mathbb{R}$. Local well-posedness for spatially periodic SQG front-type equations is proved in [Rodrigo 2005] for C^∞-solutions by a Nash–Moser method and in [Fefferman and Rodrigo 2011] for analytic solutions by a Cauchy–Kowalewski method. Almost sharp fronts, across which θ is continuous, are studied in [Córdoba et al. 2004; Fefferman et al. 2012; Fefferman and Rodrigo 2012; 2015].

The local well-posedness in Sobolev spaces of a cubically nonlinear approximation of (1-1) for spatially periodic solutions is proved in [Hunter et al. 2018]. In this paper, we consider the fully nonlinear equation (1-1) on $\mathbb{R}$. The problem on $\mathbb{R}$ differs from the problem on $\mathbb{T}$ in two respects. First, the logarithmic multiplier $\log|\xi|$ is unbounded at low frequencies, which does not occur on $\mathbb{T}$ when $\xi \in \mathbb{Z} \setminus \{0\}$ is discrete and nonzero. Second, the linearized equation on $\mathbb{R}$ provides dispersive decay, which allows us to get global solutions for sufficiently small, smooth initial data. In this paper, we do not attempt to obtain a sharp regularity result for these solutions.

The general strategy for proving the global existence of small solutions of dispersive equations is to prove an energy estimate together with a dispersive decay estimate. Energy estimates for (1-1) in the usual H^s-Sobolev spaces lead to a logarithmic loss of derivatives [Hunter and Shu 2018]. However, as shown in [Hunter et al. 2018] for spatially periodic solutions of the cubic approximation, we can obtain good energy estimates in suitably weighted H^s-spaces by paralinearizing the equation and using the linear dispersive term to control the logarithmic loss of derivatives from the nonlinear term.

The proof of the dispersive estimates is more delicate. The linear part of the equation provides $t^{-1/2}$ decay for the L^∞-norm of the solution, but this is not sufficient to close the global energy estimates for the full equation, since the $O(t^{-1})$ contribution from the cubically nonlinear term is not integrable in time. We therefore need to analyze the nonlinear dispersive behavior in more detail. We do this by the method of space-time resonances introduced by Germain, Masmoudi and Shatah [Germain 2010; Germain et al. 2009; 2012], together with estimates for weighted L^∞_ξ-norms — the so-called Z-norms — developed by Ionescu and his collaborators [Córdoba et al. 2019; Deng et al. 2017a; 2017b; Ionescu and Pausader 2013; Ionescu and Pusateri 2015; 2016; 2018].

Our Z-norm estimates in Section 8 involve a detailed frequency-space analysis. The most difficult part is the estimate of the cubically nonlinear terms. In most regions of frequency space, these terms are nonresonant, and we can use integration by parts in either the spatial or temporal frequency variables to estimate the corresponding oscillatory integrals. In regions of space-time resonances, we use the method of modified scattering to account for the nonlinear, long-time asymptotics of the solutions [Ionescu and Pusateri 2014; Ozawa 1991].

In [Córdoba et al. 2019], where the authors prove global well-posedness of the initial-value problem for the generalized SQG front equation with $0 < \alpha < 1$, the linearized equation $\varphi_t = \partial_x |\partial_x|^{1-\alpha} \varphi$ has a scaling invariance, with dispersion relation $\tau = \xi |\xi|^{1-\alpha}$, and it commutes with the vector field $x\partial_x + (2-\alpha)t\partial_t$. This commutation provides a key ingredient in the dispersive estimates. The SQG equation considered here corresponds to the limiting case $\alpha = 1$, and its linearized dispersion relation is $\tau = 2\xi \log|\xi|$. The linearized equation $\varphi_t = 2\log|\partial_x|\varphi_x$ is not scale-invariant, but it has a combined scaling-Galilean invariance and commutes with the scaling-Galilean vector field $\mathcal{S} = (x+2t)\partial_x + t\partial_t$, which we use to obtain dispersive estimates.

This paper is organized as follows. In Section 2, we collect some fundamental facts and estimates that we use later. In Section 3, we expand and paralinearize the nonlinear terms in the evolution equation. In Section 4, we derive weighted energy estimates and prove a local well-posedness result in Theorem 4.1. In Section 5, we state the global existence result in Theorem 5.1. Finally, in Sections 6–8 we carry out the three key steps in the proof of global existence: linear dispersive estimates, scaling-Galilean estimates, and nonlinear dispersive estimates.

2. Preliminaries

2A. ***Paradifferential calculus.*** In this section, we state several lemmas for Fourier multiplier operators that follow from the Weyl paradifferential calculus. Further discussion of the Weyl calculus and paraproducts can be found in [Bahouri et al. 2011; Chemin 1998; Hörmander 1985; Taylor 2000].

We denote the Fourier transform of $f : \mathbb{R} \to \mathbb{C}$ by $\hat{f} : \mathbb{R} \to \mathbb{C}$, where $\hat{f} = \mathcal{F}f$ is given by

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{i\xi x}\, \mathrm{d}\xi, \qquad \hat{f}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} f(x) e^{-i\xi x}\, \mathrm{d}x.$$

For $s \in \mathbb{R}$, we denote by $H^s(\mathbb{R})$ the space of Schwartz distributions f with $\|f\|_{H^s} < \infty$, where

$$\|f\|_{H^s} = \left[\int_{\mathbb{R}} (1+|\xi|^2)^s |\hat{f}(\xi)|^2\, \mathrm{d}\xi \right]^{1/2}.$$

Throughout this paper, we use $A \lesssim B$ to mean there is a constant C such that $A \le CB$, and $A \gtrsim B$ to mean there is a constant C such that $A \ge CB$. We use $A \approx B$ to mean that $A \lesssim B$ and $B \lesssim A$.

Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth function such that

$$\chi \text{ is supported in the interval } \left\{\xi \in \mathbb{R} \;\middle|\; |\xi| \le \tfrac{1}{10}\right\}, \text{ and } \chi(\xi) = 1 \text{ on } \left\{\xi \in \mathbb{R} \;\middle|\; |\xi| \le \tfrac{3}{40}\right\}. \tag{2-1}$$

If f is a Schwartz distribution and $a : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is a symbol, then we define a Weyl paraproduct $T_a f$ by

$$\mathcal{F}[T_a f](\xi) = \int_{\mathbb{R}} \chi\left(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2} \right) \tilde{a}\left(\xi - \eta, \frac{\xi+\eta}{2} \right) \hat{f}(\eta)\, \mathrm{d}\eta, \tag{2-2}$$

where $\tilde{a}(\xi, \eta)$ denotes the partial Fourier transform of $a(x, \eta)$ with respect to x. For $r_1, r_2 \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we define a normed symbol space by

$$\mathcal{M}_{(r_1,r_2)} = \{a : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \mid \|a\|_{\mathcal{M}_{(r_1,r_2)}} < \infty\},$$
$$\|a\|_{\mathcal{M}_{(r_1,r_2)}} = \sup_{(x,\xi)\in\mathbb{R}^2} \left\{ \sum_{\alpha=0}^{r_1} \sum_{\beta=0}^{r_2} (1+|\xi|)^\beta |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \right\}. \tag{2-3}$$

The following lemma is proved in Appendix B.

Lemma 2.1. *Let $s \in \mathbb{R}$. If $a \in \mathcal{M}_{(1,1)}$ and $f \in H^s(\mathbb{R})$, then $T_a f \in H^s(\mathbb{R})$ and*

$$\|T_a f\|_{H^s} \lesssim \|a\|_{\mathcal{M}_{(1,1)}} \|f\|_{H^s}.$$

Next, we prove some commutator estimates. We denote by $\log_+ |\partial_x|$ the Fourier multiplier with symbol

$$\log_+ |\xi| = \begin{cases} \log|\xi| & \text{for } |\xi| > 1, \\ 0 & \text{for } |\xi| \le 1. \end{cases}$$

Lemma 2.2. *Let $s \in \mathbb{R}$. Suppose that $f \in H^s(\mathbb{R})$, $a \in \mathcal{M}_{(2,1)}$, and $b \in \mathcal{M}_{(1,2)}$. Then*

$$\|[\partial_x, T_a] f\|_{H^s} \lesssim \|a\|_{\mathcal{M}_{(2,1)}} \|f\|_{H^s}, \tag{2-4}$$
$$\|[\log_+ |\partial_x|, T_a] f\|_{H^s} \lesssim \|a\|_{\mathcal{M}_{(2,1)}} \|f\|_{H^{s-1}}, \tag{2-5}$$
$$\|[x, T_b] f\|_{H^s} \lesssim \|b\|_{\mathcal{M}_{(1,2)}} \|f\|_{H^s}, \tag{2-6}$$
$$\|x T_b f - T_{xb} f\|_{H^s} \lesssim \|b\|_{\mathcal{M}_{(1,2)}} \|f\|_{H^s}. \tag{2-7}$$

Proof. (1) We have $[\partial_x, T_a] = T_{\partial_x a}$, so (2-4) follows from Lemma 2.1.

(2) Next, we prove (2-5). By the definition (2-2) of the Weyl paraproduct, we have for $\xi \neq 0$ that

$$\begin{aligned} \mathcal{F}[\log_+ |\partial_x| T_a v](\xi) &= \log_+ |\xi| \int_{\mathbb{R}} \chi\left(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\right) \tilde{a}\left(\xi-\eta, \frac{\xi+\eta}{2}\right) \hat{v}(\eta)\, d\eta \\ &= \int_{\mathbb{R}} \log_+ |\xi-\eta+\eta| \chi\left(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\right) \tilde{a}\left(\xi-\eta, \frac{\xi+\eta}{2}\right) \hat{v}(\eta)\, d\eta. \end{aligned} \tag{2-8}$$

If (ξ, η) belongs to the support of $\chi(|\xi-\eta|^2/(1+|\xi+\eta|^2))$, then we claim that

$$\left|\frac{\xi-\eta}{\eta}\right| \le \frac{17}{18} \quad \text{when } |\eta| \ge 2. \tag{2-9}$$

To prove this claim, we observe that

$$\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2} \le \frac{1}{10}$$

implies that

$$9\left|\frac{\xi-\eta}{\eta} - \frac{2}{9}\right|^2 \le \frac{40}{9} + \frac{1}{\eta^2} \le \frac{169}{36},$$

and it follows that

$$\left|\frac{\xi-\eta}{\eta}\right| \le \left|\frac{\xi-\eta}{\eta} - \frac{2}{9}\right| + \frac{2}{9} \le \frac{17}{18}.$$

We introduce a smooth cutoff function $\iota(\eta)$ supported in $\{|\eta| \le 3\}$ with $\iota(\eta) = 1$ on $\{|\eta| \le 2\}$. In view of (2-9), when $|\eta| > 2$ we can use

$$\log|\xi - \eta + \eta| = \log|\eta| + \log\left|1 + \frac{\xi - \eta}{\eta}\right|,$$

and we obtain from (2-8) that, for $|\xi| > 1$,

$$\begin{aligned}\mathcal{F}[\log_+|\partial_x| T_a f](\xi) = \int_{\mathbb{R}} (1 - \iota(\eta))\left[\log|\eta| + \log\left|1 + \frac{\xi - \eta}{\eta}\right|\right]\chi\left(\frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2}\right)\tilde{a}\left(\xi - \eta, \frac{\xi + \eta}{2}\right)\hat{f}(\eta)\,\mathrm{d}\eta \\ + \log_+|\xi| \int_{\mathbb{R}} \iota(\eta)\chi\left(\frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2}\right)\tilde{a}\left(\xi - \eta, \frac{\xi + \eta}{2}\right)\hat{f}(\eta)\,\mathrm{d}\eta.\end{aligned}$$

We also have

$$\mathcal{F}[T_a \log_+|\partial_x| f](\xi) = \int_{\mathbb{R}} \log_+|\eta|\chi\left(\frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2}\right)\tilde{a}\left(\xi - \eta, \frac{\xi + \eta}{2}\right)\hat{f}(\eta)\,\mathrm{d}\eta.$$

By taking the difference of the previous two equations, we get

$$\begin{aligned}&\mathcal{F}[\log_+|\partial_x| T_a f](\xi) - \mathcal{F}[T_a \log_+|\partial_x| f](\xi) \\ &\quad = \int_{\mathbb{R}} (1 - \iota(\eta))\left[\log\left|1 + \frac{\xi - \eta}{\eta}\right|\right]\chi\left(\frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2}\right)\tilde{a}\left(\xi - \eta, \frac{\xi + \eta}{2}\right)\hat{f}(\eta)\,\mathrm{d}\eta \\ &\qquad + \int_{\mathbb{R}} \iota(\eta)(\log_+|\xi| - \log_+|\eta|)\chi\left(\frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2}\right)\tilde{a}\left(\xi - \eta, \frac{\xi + \eta}{2}\right)\hat{f}(\eta)\,\mathrm{d}\eta. \quad (2\text{-}10)\end{aligned}$$

The integrand in the first integral on the right-hand side of (2-10) is supported on

$$\left\{(\xi, \eta)\ \middle|\ |\eta| > 2,\ \frac{|\xi - \eta|}{|\eta|} < \frac{17}{18}\right\}.$$

Thus, if $\mathcal{P}(\xi, \eta)$ is a smooth cutoff function supported in a small neighborhood of this set and equal to 1 on the set, then the first integral can be written as

$$\int_{\mathbb{R}} \mathcal{P}(\xi, \eta)\left[\frac{\eta}{\xi - \eta}\log\left|1 + \frac{\xi - \eta}{\eta}\right|\right]\chi\left(\frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2}\right)(\xi - \eta)\tilde{a}\left(\xi - \eta, \frac{\xi + \eta}{2}\right)\left[\frac{1 - \iota(\eta)}{\eta}\hat{f}(\eta)\right]\mathrm{d}\eta.$$

We define

$$\tilde{A}(\zeta_1, \zeta_2) = \frac{2\zeta_2 - \zeta_1}{2i\zeta_1}\log\left|1 + \frac{2\zeta_1}{2\zeta_2 - \zeta_1}\right|\widetilde{\partial_1 a}(\zeta_1, \zeta_2)\mathcal{P}\left(\zeta_2 + \frac{\zeta_1}{2}, \zeta_2 - \frac{\zeta_1}{2}\right),$$

so that

$$A(x, \zeta_2) = \frac{\partial_x^{-1}}{2}(2\zeta_2 + i\partial_x)\log|1 - 2i\partial_x(2\zeta_2 + i\partial_x)^{-1}|\mathcal{P}\left(\zeta_2 - \frac{i\partial_x}{2}, \zeta_2 + \frac{i\partial_x}{2}\right)\partial_x a(x, \zeta_2).$$

Then the first integral on the right-hand-side of (2-10) can be written in terms of a paradifferential operator with symbol A as

$$\int_{\mathbb{R}} \chi\left(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\right)\tilde{A}\left(\xi-\eta, \frac{\xi+\eta}{2}\right)\left[\frac{1-\iota(\eta)}{\eta}\hat{f}(\eta)\right] d\eta = \mathcal{F}[T_A g](\xi), \quad g = \mathcal{F}^{-1}\left[\frac{1-\iota}{\eta}\hat{f}\right].$$

By Lemma 2.1, we have

$$\|T_A g\| \lesssim \|A\|_{\mathcal{M}_{(1,1)}} \|g\|_{H^s} \lesssim \|A\|_{\mathcal{M}_{(1,1)}} \|f\|_{H^{s-1}}.$$

Because of the cutoff function $\mathcal{P}$, we see that the support of $\tilde{A}(\zeta_1, \zeta_2)$ is contained in

$$|2\zeta_2 - \zeta_1| > 4, \quad \left|\frac{2\zeta_1}{2\zeta_2-\zeta_1}\right| < \frac{17}{18}.$$

So $\partial_1 a(\,\cdot\,, \zeta_2) \mapsto A(\,\cdot\,, \zeta_2)$ is a zeroth-order pseudodifferential operator. By carrying out a dyadic decomposition and using Bernstein's inequality [Bahouri et al. 2011], we obtain that

$$\|A\|_{\mathcal{M}_{(1,1)}} \lesssim \|a\|_{\mathcal{M}_{(2,1)}}.$$

It follows that the first term on the right-hand side of (2-10) satisfies the estimate (2-5).

For the second term on the right-hand side of (2-10), the cutoff functions χ, ι ensure that $|\xi| < 6$, $|\eta| < 3$. Therefore we have the H^s-estimate

$$\begin{aligned}
&\left\|(1+|\xi|^2)^{s/2}\int_{\mathbb{R}} \iota(\eta)(\log_+|\xi| - \log_+|\eta|)\chi\left(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\right)\tilde{a}\left(\xi-\eta, \frac{\xi+\eta}{2}\right)\hat{f}(\eta)\, d\eta\right\|_{L^2_\xi} \\
&\quad\lesssim \left\|(1+|\xi|^2)^{s/2}\log_+|\xi|\int_{\mathbb{R}} \chi\left(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\right)\tilde{a}\left(\xi-\eta, \frac{\xi+\eta}{2}\right)[\iota(\eta)\hat{f}(\eta)]\, d\eta\right\|_{L^2_\xi} \\
&\qquad\qquad + \left\|(1+|\xi|^2)^{s/2}\int_{\mathbb{R}} \chi\left(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\right)\tilde{a}\left(\xi-\eta, \frac{\xi+\eta}{2}\right)[\iota(\eta)\log_+|\eta|\hat{f}(\eta)]\, d\eta\right\|_{L^2_\xi} \\
&\quad\lesssim \left\|\int_{\mathbb{R}} \chi\left(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\right)\tilde{a}\left(\xi-\eta, \frac{\xi+\eta}{2}\right)[\iota(\eta)\hat{f}(\eta)]\, d\eta\right\|_{L^2_\xi} \\
&\qquad\qquad + \left\|\int_{\mathbb{R}} \chi\left(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\right)\tilde{a}\left(\xi-\eta, \frac{\xi+\eta}{2}\right)[\iota(\eta)\log_+|\eta|\hat{f}(\eta)]\, d\eta\right\|_{L^2_\xi} \\
&\quad= \|T_a g\|_{L^2} + \|T_a h\|_{L^2},
\end{aligned}$$

where

$$g = \mathcal{F}^{-1}[\iota\hat{f}], \quad h = \mathcal{F}^{-1}[\iota\log_+|\eta|\hat{f}],$$

and Lemma 2.1 implies that the second term also satisfies (2-5).

(3) To prove (2-6), we compute that

$$\begin{aligned}
&\mathcal{F}[[x, T_b]f](\xi) \\
&\quad= i\partial_\xi \widehat{T_b f}(\xi) - \widehat{T_b(xf)}(\xi) \\
&\quad= i\int_{\mathbb{R}} \partial_\xi\left[\chi\left(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\right)\tilde{b}\left(\xi-\eta, \frac{\xi+\eta}{2}\right)\right]\hat{f}(\eta)\, d\eta - i\int_{\mathbb{R}} \chi\left(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\right)\tilde{b}\left(\xi-\eta, \frac{\xi+\eta}{2}\right)\partial_\eta\hat{f}(\eta)\, d\eta.
\end{aligned}$$

We rewrite the first integral above as

$$\begin{aligned}
&\int_{\mathbb{R}} \partial_\xi \left[\chi\left(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\right)\tilde{b}\left(\xi-\eta, \frac{\xi+\eta}{2}\right)\right]\hat{f}(\eta)\,\mathrm{d}\eta \\
&= \int_{\mathbb{R}} (\partial_{\xi_1}+\partial_{\xi_2})\left[\chi\left(\frac{|\xi_1-\eta|^2}{1+|\xi_2+\eta|^2}\right)\tilde{b}\left(\xi_1-\eta, \frac{\xi_2+\eta}{2}\right)\right]\bigg|_{\xi_1=\xi_2=\xi}\hat{f}(\eta)\,\mathrm{d}\eta \\
&= \int_{\mathbb{R}} (2\partial_{\xi_2}-\partial_\eta)\left[\chi\left(\frac{|\xi_1-\eta|^2}{1+|\xi_2+\eta|^2}\right)\tilde{b}\left(\xi_1-\eta, \frac{\xi_2+\eta}{2}\right)\right]\bigg|_{\xi_1=\xi_2=\xi}\hat{f}(\eta)\,\mathrm{d}\eta \\
&= \int_{\mathbb{R}} 2\partial_{\xi_2}\left[l\chi\left(\frac{|\xi_1-\eta|^2}{1+|\xi_2+\eta|^2}\right)\tilde{b}\left(\xi_1-\eta, \frac{\xi_2+\eta}{2}\right)\right]\bigg|_{\xi_1=\xi_2=\xi}\hat{f}(\eta)+\left[\chi\left(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\right)\tilde{b}\left(\xi-\eta, \frac{\xi+\eta}{2}\right)\right]\partial_\eta\hat{f}(\eta)\,\mathrm{d}\eta.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathcal{F}[x, T_b]f &= 2i\int_{\mathbb{R}} \partial_{\xi_2}\left[\chi\left(\frac{|\xi_1-\eta|^2}{1+|\xi_2+\eta|^2}\right)\tilde{b}\left(\xi_1-\eta, \frac{\xi_2+\eta}{2}\right)\right]\bigg|_{\xi_1=\xi_2=\xi}\hat{f}(\eta)\,\mathrm{d}\eta \\
&= 2i\int_{\mathbb{R}} \frac{2|\xi-\eta|^2(\xi+\eta)}{[1+|\xi+\eta|^2]^2}\chi'\left(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\right)\tilde{b}\left(\xi-\eta, \frac{\xi+\eta}{2}\right)\hat{f}(\eta)\,\mathrm{d}\eta \\
&\qquad + i\int_{\mathbb{R}} \chi\left(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\right)\widetilde{\partial_2 b}\left(\xi-\eta, \frac{\xi+\eta}{2}\right)\hat{f}(\eta)\,\mathrm{d}\eta. \qquad (2\text{-}11)
\end{aligned}$$

From (2-9), in the support of the cutoff function χ we have

$$\tfrac{1}{18}|\eta| \le |\xi| \le \tfrac{35}{18}|\eta| \quad \text{when } |\eta| > 2, \qquad \text{and} \qquad |\xi| < 6 \quad \text{when } |\eta| < 2.$$

Thus, the first integral on the right-hand-side of (2-11) satisfies

$$\begin{aligned}
&\left\|(1+|\xi|^2)^{s/2}\int_{\mathbb{R}} \frac{2|\xi-\eta|^2(\xi+\eta)}{[1+|\xi+\eta|^2]^2}\chi'\left(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\right)\tilde{b}\left(\xi-\eta, \frac{\xi+\eta}{2}\right)\hat{f}(\eta)\,\mathrm{d}\eta\right\|_{L^2_\xi} \\
&\quad \lesssim \left\|\int_{\mathbb{R}} \frac{2|\xi-\eta|^2(\xi+\eta)}{[1+|\xi+\eta|^2]^2}\chi'\left(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\right)\tilde{b}\left(\xi-\eta, \frac{\xi+\eta}{2}\right)[\iota(\eta)\hat{f}(\eta)]\,\mathrm{d}\eta\right\|_{L^2_\xi} \\
&\qquad + \left\|\int_{\mathbb{R}} \frac{2|\xi-\eta|^2(\xi+\eta)}{[1+|\xi+\eta|^2]^2}\chi'\left(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\right)\tilde{b}\left(\xi-\eta, \frac{\xi+\eta}{2}\right)[(1-\iota(\eta))(1+|\eta|^2)^{s/2}\hat{f}(\eta)]\,\mathrm{d}\eta\right\|_{L^2_\xi}.
\end{aligned}$$

These terms can be expressed in terms of a Weyl pseudodifferential operator B^w in (B-1) with symbol

$$B(x,\xi) = \frac{4\xi\,\partial_x}{(1+4\xi^2)^2}\chi'\left(\frac{-\partial_x^2}{1+4\xi^2}\right)b(x,\xi).$$

Using Theorem B.2 and Bernstein's inequality, we then get that

$$\begin{aligned}
\left\|(1+|\xi|^2)^{s/2}\int_{\mathbb{R}} \frac{2|\xi-\eta|^2(\xi+\eta)}{[1+|\xi+\eta|^2]^2}\chi'\left(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\right)\tilde{b}\left(\xi-\eta, \frac{\xi+\eta}{2}\right)\hat{f}(\eta)\,\mathrm{d}\eta\right\|_{L^2_\xi} &\lesssim \|B\|_{\mathcal{M}_{(1,1)}}\|f\|_{H^s} \\
&\lesssim \|b\|_{\mathcal{M}_{(1,1)}}\|f\|_{H^s}.
\end{aligned}$$

The second integral on the right-hand-side of (2-11) is the paraproduct $\mathcal{F}[T_{\partial_2 b} f]$. By using Lemma 2.1 and the previous estimate, we then obtain (2-6).

(4) To prove (2-7), we compute that

$$\begin{aligned}
&\mathcal{F}(xT_b f - T_{xb} f)\\
&= i\int_{\mathbb{R}} \partial_\xi\left[\chi\left(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\right)\tilde{b}\left(\xi-\eta, \frac{\xi+\eta}{2}\right)\right]\hat{f}(\eta)\,\mathrm{d}\eta - i\int_{\mathbb{R}}\left[\chi\left(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\right)\partial_1\tilde{b}\left(\xi-\eta, \frac{\xi+\eta}{2}\right)\right]\hat{f}(\eta)\,\mathrm{d}\eta\\
&= i\int_{\mathbb{R}}\left[\partial_\xi\chi\left(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\right)\tilde{b}\left(\xi-\eta, \frac{\xi+\eta}{2}\right) + \frac{1}{2}\chi\left(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\right)\partial_2\tilde{b}\left(\xi-\eta, \frac{\xi+\eta}{2}\right)\right]\hat{f}(\eta)\,\mathrm{d}\eta.
\end{aligned}$$

The first term satisfies

$$\left\|\int_{\mathbb{R}}\partial_\xi\chi\left(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\right)\tilde{b}\left(\xi-\eta, \frac{\xi+\eta}{2}\right)\hat{f}(\eta)\,\mathrm{d}\eta\right\|_{H^s} \lesssim \|b\|_{\mathcal{M}_{(1,1)}}\|f\|_{H^s},$$

and the second term satisfies

$$\left\|\int_{\mathbb{R}}\chi\left(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\right)\partial_2\tilde{b}\left(\xi-\eta, \frac{\xi+\eta}{2}\right)\hat{f}(\eta)\,\mathrm{d}\eta\right\|_{H^s} \lesssim \|b\|_{\mathcal{M}_{(1,2)}}\|f\|_{H^s},$$

which proves (2-7). □

Finally, writing $D = -i\partial_x$, we give an expansion of the operator $|D| = |\partial_x|$ acting on paraproducts; see [Li 2019].

Lemma 2.3. *Let $s \in \mathbb{R}$, $s \geq 2$. If $a \in \mathcal{M}_{(3,1)}$ and $f \in H^s(\mathbb{R})$, then*

$$|D|^s T_a f = T_a|D|^s f + sT_{Da}|D|^{s-2}Df + \mathcal{R},$$

where $\mathcal{R}$ satisfies

$$\|\mathcal{R}\|_{L^2} \lesssim \|a\|_{\mathcal{M}_{(3,1)}}\|f\|_{H^{s-2}(\mathbb{R})},$$

and Da means that the differential operator D acts on the function $x \mapsto a(x,\xi)$ for fixed ξ.

Proof. By the definition of the Weyl paraproduct

$$\begin{aligned}
\mathcal{F}(|D|^s T_a f)(\xi) &= |\xi|^s\int_{\mathbb{R}}\chi\left(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\right)\tilde{a}\left(\xi-\eta, \frac{\xi+\eta}{2}\right)\hat{f}(\eta)\,\mathrm{d}\eta\\
&= \int_{\mathbb{R}}|\xi-\eta+\eta|^s\chi\left(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\right)\tilde{a}\left(\xi-\eta, \frac{\xi+\eta}{2}\right)\hat{f}(\eta)\,\mathrm{d}\eta,
\end{aligned}$$

where $\tilde{a}$ denotes the partial Fourier transform of a in the first variable. The low-frequency part satisfies the remainder estimate, so it can be absorbed into $\mathcal{R}$, and we only need to consider the high-frequency part with $|\eta| > 2$. In that case, (2-9) is satisfied on the support of $\chi(|\xi-\eta|^2/(1+|\xi+\eta|^2))$. Define $b(x) = (1+x)^s - 1 - sx$. Then

$$|\xi-\eta+\eta|^s = |\eta|^s\left|1+\frac{\xi-\eta}{\eta}\right|^s = |\eta|^s\left[1+s\frac{\xi-\eta}{\eta}+b\left(\frac{\xi-\eta}{\eta}\right)\right].$$

In the expression for $\mathcal{F}[|D|^s T_a f]$, we get

$$\mathcal{F}[|D|^s T_a f](\xi) = \int_{\mathbb{R}} |\eta|^s \left[1 + s\frac{\xi-\eta}{\eta} + b\left(\frac{\xi-\eta}{\eta}\right)\right] \chi\left(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\right) \tilde{a}\left(\xi-\eta, \frac{\xi+\eta}{2}\right) \hat{f}(\eta)\, \mathrm{d}\eta.$$

Then we only need to estimate

$$\int_{\mathbb{R}} |\eta|^2 b\left(\frac{\xi-\eta}{\eta}\right) \chi\left(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\right) \tilde{a}\left(\xi-\eta, \frac{\xi+\eta}{2}\right) [(1-\iota(\eta))|\eta|^{s-2} \hat{f}(\eta)]\, \mathrm{d}\eta. \tag{2-12}$$

Define the symbol A by

$$\tilde{A}(\zeta_1, \zeta_2) = \left|\frac{2\zeta_2 - \zeta_1}{2}\right|^2 \left(1 - \iota\left(\frac{2\zeta_2-\zeta_1}{2}\right)\right) b\left(\frac{2\zeta_1}{2\zeta_2-\zeta_1}\right) \tilde{a}(\zeta_1, \zeta_2).$$

Then (2-12) can be viewed as a paradifferential operator with symbol A. By considering the supports of χ, ι and using Bernstein's inequality, we see that

$$\|A\|_{\mathcal{M}_{(1,1)}} \lesssim \|a\|_{\mathcal{M}_{(3,1)}}.$$

The result then follows by applying Lemma 2.1 to (2-12). □

2B. ***Fourier multipliers.*** Let $\psi : \mathbb{R} \to [0, 1]$ be a smooth function supported in $\left[-\frac{8}{5}, \frac{8}{5}\right]$ and equal to 1 in $\left[-\frac{5}{4}, \frac{5}{4}\right]$. For any $k \in \mathbb{Z}$, we define

$$\begin{aligned} &\psi_k(\xi) = \psi(\xi/2^k) - \psi(\xi/2^{k-1}), \quad \psi_{\le k}(\xi) = \psi(\xi/2^k), \quad \psi_{\ge k}(\xi) = 1 - \psi(\xi/2^{k-1}), \\ &\tilde{\psi}_k(\xi) = \psi_{k-1}(\xi) + \psi_k(\xi) + \psi_{k+1}(\xi), \end{aligned} \tag{2-13}$$

and denote by P_k, $P_{\le k}$, $P_{\ge k}$, and $\widetilde{P}_k$ the Fourier multiplier operators with symbols ψ_k, $\psi_{\le k}$, $\psi_{\ge k}$, and $\tilde{\psi}_k$, respectively. Notice that $\psi_k(\xi) = \psi_0(\xi/2^k)$, $\tilde{\psi}_k(\xi) = \tilde{\psi}_0(\xi/2^k)$.

It is easy to check that

$$\|\psi_k\|_{L^2} \approx 2^{k/2}, \quad \|\psi_k'\|_{L^2} \approx 2^{-k/2}. \tag{2-14}$$

We will need the following interpolation lemma, whose proof can be found in [Ionescu and Pusateri 2016].

Lemma 2.4. *For any $k \in \mathbb{Z}$ and $f \in L^2(\mathbb{R})$, we have*

$$\|\widehat{P_k f}\|_{L^\infty}^2 \lesssim \|P_k f\|_{L^1}^2 \lesssim 2^{-k} \|\hat{f}\|_{L^2_\xi} [2^k \|\partial_\xi \hat{f}\|_{L^2_\xi} + \|\hat{f}\|_{L^2_\xi}].$$

We will also use an estimate for multilinear Fourier multipliers proved in [Ionescu and Pusateri 2015]. Before stating the estimate, we introduce some notation.

We define a norm on symbols $\kappa : \mathbb{R}^d \to \mathbb{C}$ by

$$\|\kappa\|_{S^\infty} = \|\mathcal{F}^{-1}\kappa\|_{L^1},$$

and define the symbol class

$$S^\infty = \{\kappa : \mathbb{R}^d \to \mathbb{C} \mid \kappa \text{ continuous and } \|\kappa\|_{S^\infty} < \infty\}. \tag{2-15}$$

Given $\kappa \in S^\infty$, we define a multilinear operator M_κ acting on Schwartz functions $f_1, \dots, f_m \in \mathcal{S}(\mathbb{R})$ by

$$M_\kappa(f_1, \dots, f_m)(x) = \int_{\mathbb{R}^m} e^{ix(\xi_1+\cdots+\xi_m)} \kappa(\xi_1, \dots, \xi_m) \hat{f}_1(\xi_1) \cdots \hat{f}_m(\xi_m) \, \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_m.$$

Lemma 2.5. (i) *If $\kappa_1, \kappa_2 \in S^\infty$, then $\kappa_1\kappa_2 \in S^\infty$.*

(ii) *Suppose that $1 \le p_1, \dots, p_m \le \infty$, $1 \le p \le \infty$, satisfy*

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = \frac{1}{p}.$$

If $\kappa \in S^\infty$, then

$$\|M_\kappa\|_{L^{p_1}\times\cdots\times L^{p_m} \to L^p} \lesssim \|\kappa\|_{S^\infty}.$$

(iii) *Assume $p, q, r \in [1, \infty]$ satisfy*

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1,$$

and $m \in S^\infty_{\eta_1,\eta_2} L^\infty_\xi$. Then, for any $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$, and $h \in L^r(\mathbb{R})$,

$$\left\| \int_{\mathbb{R}^2} m(\eta_1, \eta_2, \xi) \hat{f}(\eta_1) \hat{g}(\eta_2) \hat{h}(\xi - \eta_1 - \eta_2) \, \mathrm{d}\eta_1 \, \mathrm{d}\eta_2 \right\|_{L^\infty_\xi} \lesssim \|m\|_{S^\infty_{\eta_1,\eta_2} L^\infty_\xi} \|f\|_{L^p} \|g\|_{L^q} \|h\|_{L^r}.$$

In particular, using interpolation, we can estimate the S^∞-norm of a symbol $m(\eta_1, \eta_2)$ in C_c^∞ by

$$\|m\|_{S^\infty} \lesssim \|m\|_{L^1}^{1/4} \|\partial_{\eta_i}^2 m\|_{L^1}^{1/2} \|\partial_{\eta_1}^2 \partial_{\eta_2}^2 m\|_{L^1}^{1/4}, \quad \text{where } i = 1, 2. \tag{2-16}$$

3. Reformulation of the equation

3A. ***Expansion of the equation.*** In this section, we expand the nonlinearity in the SQG front equation

$$\varphi_t(x,t) + \int_{\mathbb{R}} [\varphi_x(x,t) - \varphi_x(x+\zeta,t)] \left\{ \frac{1}{|\zeta|} - \frac{1}{\sqrt{\zeta^2 + [\varphi(x,t) - \varphi(x+\zeta,t)]^2}} \right\} \mathrm{d}\zeta = 2 \log|\partial_x| \varphi_x(x,t) \tag{3-1}$$

for fronts with small slopes $|\varphi_x| \ll 1$. As we will show, (3-1) can be rewritten as

$$\varphi_t(x,t) - \sum_{n=1}^\infty \frac{c_n}{2n+1} \partial_x \int_{\mathbb{R}^{2n+1}} \boldsymbol{T}_n(\boldsymbol{\eta}_n) \hat{\varphi}(\eta_1, t) \hat{\varphi}(\eta_2, t) \cdots \hat{\varphi}(\eta_{2n+1}, t) e^{i(\eta_1+\eta_2+\cdots+\eta_{2n+1})x} \, \mathrm{d}\boldsymbol{\eta}_n = 2 \log|\partial_x| \varphi_x(x,t), \tag{3-2}$$

where $\boldsymbol{\eta}_n = (\eta_1, \eta_2, \dots, \eta_{2n+1})$, and

$$\boldsymbol{T}_n(\boldsymbol{\eta}_n) = \int_{\mathbb{R}} \frac{\prod_{j=1}^{2n+1} (1 - e^{i\eta_j \zeta})}{|\zeta|^{2n+1}} \, \mathrm{d}\zeta, \quad c_n = \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2} - n\right) \Gamma(n+1)}. \tag{3-3}$$

We remark that $c_n = O(n^{-1/2})$ as $n \to \infty$.

In fact, if we expand the nonlinearity in (3-1) around $\varphi_x(x,t)=0$, we find that

$$\begin{aligned}\int_{\mathbb{R}}\left[\frac{\varphi_x(x,t)-\varphi_x(x+\zeta,t)}{|\zeta|}-\frac{\varphi_x(x,t)-\varphi_x(x+\zeta,t)}{\sqrt{\zeta^2+(\varphi(x,t)-\varphi(x+\zeta,t))^2}}\right]\mathrm{d}\zeta \\ =-\sum_{n=1}^{\infty}c_n\int_{\mathbb{R}}\frac{[\varphi_x(x,t)-\varphi_x(x+\zeta,t)]\cdot[\varphi(x,t)-\varphi(x+\zeta,t)]^{2n}}{|\zeta|^{2n+1}}\,\mathrm{d}\zeta \\ =-\sum_{n=1}^{\infty}\frac{c_n}{2n+1}\partial_x\int_{\mathbb{R}}\left[\frac{\varphi(x,t)-\varphi(x+\zeta,t)}{|\zeta|}\right]^{2n+1}\mathrm{d}\zeta.\end{aligned}$$

Writing

$$f_n(x)=\int_{\mathbb{R}}\left[\frac{\varphi(x)-\varphi(x+\zeta)}{|\zeta|}\right]^{2n+1}\mathrm{d}\zeta,\quad \varphi(x)=\int_{\mathbb{R}}\hat{\varphi}(\eta)e^{i\eta x}\,\mathrm{d}\eta,$$

we have

$$f_n(x)=\int_{\mathbb{R}^{2n+1}}\boldsymbol{T}_n(\boldsymbol{\eta}_n)\hat{\varphi}(\eta_1)\hat{\varphi}(\eta_2)\cdots\hat{\varphi}(\eta_{2n+1})e^{i(\eta_1+\eta_2+\cdots+\eta_{2n+1})x}\,\mathrm{d}\boldsymbol{\eta}_n,$$

which gives (3-2).

Isolating the lowest-degree nonlinear term in (3-2), which is cubic, we can also write (3-1) as

$$\begin{aligned}\varphi_t(x,t)+\frac{1}{6}\partial_x\int_{\mathbb{R}^3}\boldsymbol{T}_1(\eta_1,\eta_2,\eta_3)\hat{\varphi}(\eta_1,t)\hat{\varphi}(\eta_2,t)\hat{\varphi}(\eta_3,t)e^{i(\eta_1+\eta_2+\eta_3)x}\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\eta_3+\mathcal{N}_{\geq 5}(\varphi)(x,t) \\ =2\log|\partial_x|\varphi_x(x,t),\end{aligned}\tag{3-4}$$

where $\mathcal{N}_{\geq 5}(\varphi)$ denotes the nonlinear terms of quintic degree or higher:

$$\mathcal{N}_{\geq 5}(\varphi)(x,t)=-\sum_{n=2}^{\infty}\frac{c_n}{2n+1}\partial_x\int_{\mathbb{R}^{2n+1}}\boldsymbol{T}_n(\boldsymbol{\eta}_n)\hat{\varphi}(\eta_1,t)\hat{\varphi}(\eta_2,t)\cdots\hat{\varphi}(\eta_{2n+1},t)e^{i(\eta_1+\eta_2+\cdots+\eta_{2n+1})x}\,\mathrm{d}\boldsymbol{\eta}_n.\tag{3-5}$$

Equation (3-4) will be used in Section 8 in order to carry out nonlinear dispersive estimates, where the main difficulty is controlling the slowest decay in time caused by the lowest-degree, cubic nonlinearity.

In Appendix A, we evaluate the integrals in (3-3) and show that we can write (3-2) in the alternative form

$$\varphi_t+\partial_x\left\{\sum_{n=1}^{\infty}\sum_{\ell=1}^{2n+1}(-1)^{\ell+1}d_{n,\ell}\varphi^{2n-\ell+1}\partial_x^{2n}\log|\partial_x|\varphi^{\ell}\right\}=2\log|\partial_x|\varphi_x,\tag{3-6}$$

where the constants $d_{n,\ell}$ are given in (A-4). We will not use (3-6) in this paper since it makes sense classically only for C^∞-solutions and does not make explicit the fact that, owing to a cancellation of derivatives, the nonlinear flux in (3-6) involves at most logarithmic derivatives of φ. However, we remark that if the quintic and higher-order terms in (3-6) are neglected, then the equation becomes

$$\varphi_t+\tfrac{1}{2}\partial_x\left\{\varphi^2\log|\partial_x|\varphi_{xx}-\varphi\log|\partial_x|(\varphi^2)_{xx}+\tfrac{1}{3}\log|\partial_x|(\varphi^3)_{xx}\right\}=2\log|\partial_x|\varphi_x,$$

which is the cubic approximation for the front equation that is derived in [Hunter and Shu 2018] and analyzed in [Hunter et al. 2018].

3B. ***Paralinearization of the equation.*** In this section, we paralinearize the SQG front equation (3-2) and put it in a form that allows us to make weighted energy estimates. This form extracts a nonlinear term $\log|\partial_x|(T_{B^{\log}[\varphi]}\varphi)$ from the flux that is responsible for the logarithmic loss of derivatives in the dispersionless equation.

We use Weyl paradifferential calculus to decompose the nonlinearity in (3-1). In the following, we use $C(n,s)$ to denote a positive constant depending only on n and s, which may change from line to line, and $L=\log|\partial_x|$ is defined in (1-2).

Proposition 3.1. *Suppose that $\varphi(\cdot,t)\in H^s(\mathbb{R})$ with $s\ge 4$ and $\|\varphi_x\|_{W^{3,\infty}}+\|\log|\partial_x|\varphi_x\|_{W^{2,\infty}}$ is sufficiently small. Then* (3-1) *can be written as*

$$\varphi_t+\partial_x T_{B^0[\varphi]}\varphi+\mathcal{R}(\varphi)=\log|\partial_x|[(2-T_{B^{\log}[\varphi]})\varphi]_x, \tag{3-7}$$

where the symbols $B^0[\varphi]$ and $B^{\log}[\varphi]$ are given by the following multilinear expansions in φ_x:

$$B^{\log}[\varphi](\cdot,\xi)=\sum_{n=1}^{\infty}B_n^{\log}[\varphi](\cdot,\xi),\qquad B^0[\varphi](\cdot,\xi)=\sum_{n=1}^{\infty}B_n^0[\varphi](\cdot,\xi),$$

$$B_n^{\log}[\varphi](\cdot,\xi)=-\mathcal{F}_\zeta^{-1}\left\{2c_n\int_{\mathbb{R}^{2n}}\delta\left(\zeta-\sum_{j=1}^{2n}\eta_j\right)\prod_{j=1}^{2n}\left[i\eta_j\hat{\varphi}(\eta_j)\chi\left(\frac{(2n+1)\eta_j}{\xi}\right)\right]\mathrm{d}\hat{\boldsymbol{\eta}}_n\right\}, \tag{3-8}$$

$$B_n^0[\varphi](\cdot,\xi)=\mathcal{F}_\zeta^{-1}\left\{2c_n\int_{\mathbb{R}^{2n}}\delta\left(\zeta-\sum_{j=1}^{2n}\eta_j\right)\prod_{j=1}^{2n}\left[i\eta_j\hat{\varphi}(\eta_j)\chi\left(\frac{(2n+1)\eta_j}{\xi}\right)\right]\int_{[0,1]^{2n}}\log\left|\sum_{j=1}^{2n}\eta_j s_j\right|\mathrm{d}\hat{\mathbf{s}}_n\,\mathrm{d}\hat{\boldsymbol{\eta}}_n\right\}.$$

Here, c_n is given by (3-3), *δ is the delta-distribution, χ is the cutoff function in* (2-1), *$\hat{\boldsymbol{\eta}}_n=(\eta_1,\eta_2,\dots,\eta_{2n})$, and $\hat{\mathbf{s}}_n=(s_1,\dots,s_{2n})$. The operators $T_{B^{\log}[\varphi]}$ and $T_{B^0[\varphi]}$ are self-adjoint and their symbols satisfy the estimates*

$$\begin{aligned}\|B^{\log}[\varphi]\|_{\mathcal{M}_{(j,2)}}&\lesssim\sum_{n=1}^{\infty}C(n,s)|c_n|\|\varphi_x\|_{W^{j,\infty}}^{2n},\qquad j=2,3,\\ \|B^0[\varphi]\|_{\mathcal{M}_{(2,2)}}&\lesssim\sum_{n=1}^{\infty}C(n,s)|c_n|(\|\log|\partial_x|\varphi_x\|_{W^{2,\infty}}^{2n}+\|\varphi_x\|_{W^{2,\infty}}^{2n}),\end{aligned} \tag{3-9}$$

while the remainder term $\mathcal{R}$ satisfies

$$\|\mathcal{R}(\varphi)\|_{H^s}\lesssim\|\varphi\|_{H^s}\left\{\sum_{n=1}^{\infty}C(n,s)|c_n|(\|\varphi_x\|_{W^{3,\infty}}^{2n}+\|\log|\partial_x|\varphi_x\|_{W^{2,\infty}}^{2n})\right\}, \tag{3-10}$$

where the constants $C(n,s)$ have at most exponential growth in n.

Proof. We define

$$f_n(x)=\int_{\mathbb{R}^{2n+1}}\boldsymbol{T}_n(\boldsymbol{\eta}_n)\hat{\varphi}(\eta_1)\hat{\varphi}(\eta_2)\cdots\hat{\varphi}(\eta_{2n+1})e^{i(\eta_1+\eta_2+\cdots+\eta_{2n+1})x}\,\mathrm{d}\boldsymbol{\eta}_n.$$

In view of (3-2) and the commutator estimate (2-5), we only need to prove that

$$-\sum_{n=1}^{\infty}\frac{c_n}{2n+1}\partial_x f_n(x)=\partial_x T_{B^0[\varphi]}\varphi+\partial_x[(T_{B^{\log}[\varphi]})\log|\partial_x|\varphi]+\mathcal{R},$$

where $\mathcal{R}$ satisfies (3-10), and to do this it suffices to prove for each n that

$$\frac{c_n}{2n+1}\partial_x f_n(x) = -\partial_x T_{B_n^0[\varphi]}\varphi - \partial_x[(T_{B_n^{\log}[\varphi]})\log|\partial_x|\varphi] + \mathcal{R}_n,$$
$$\|\mathcal{R}_n\|_{H^s} \lesssim C(n,s)|c_n|(\|\varphi_x\|_{W^{2,\infty}}^{2n} + \|\log|\partial_x|\varphi_x\|_{W^{2,\infty}}^{2n})\|\varphi\|_{H^s}.$$

By symmetry, we can assume that $|\eta_{2n+1}|$ is the largest frequency in the expression of f_n. Then

$$\begin{aligned}
&\frac{c_n}{2n+1}\partial_x f_n(x)\\
&= c_n\partial_x \int_{\substack{|\eta_{2n+1}|\ge|\eta_j|\\ \text{for all } j=1,\dots,2n}} \boldsymbol{T}_n(\boldsymbol{\eta}_n)\hat{\varphi}(\eta_1)\hat{\varphi}(\eta_2)\cdots\hat{\varphi}(\eta_{2n+1})e^{i(\eta_1+\eta_2+\cdots+\eta_{2n+1})x}\,\mathrm{d}\boldsymbol{\eta}_n\\
&= c_n\partial_x \int_{\mathbb{R}}\int_{\substack{|\eta_j|\le|\eta_{2n+1}|\\ \text{for all } j=1,\dots,2n}} \boldsymbol{T}_n(\boldsymbol{\eta}_n)\hat{\varphi}(\eta_1)\hat{\varphi}(\eta_2)\cdots\hat{\varphi}(\eta_{2n})e^{i(\eta_1+\eta_2+\cdots+\eta_{2n})x}\,\mathrm{d}\hat{\boldsymbol{\eta}}_n\hat{\varphi}(\eta_{2n+1})e^{ix\eta_{2n+1}}\,\mathrm{d}\eta_{2n+1}. \quad (3\text{-}11)
\end{aligned}$$

To proceed, we split the above integral into two parts corresponding to the lower and higher frequencies of η_{2n+1}. Define $\boldsymbol{U}_n(\boldsymbol{\eta}_n) = \boldsymbol{T}_n(\boldsymbol{\eta}_n)\chi(\eta_{2n+1})$ and $\boldsymbol{\Lambda}_n(\boldsymbol{\eta}_n) = \boldsymbol{T}_n(\boldsymbol{\eta}_n) - \boldsymbol{U}_n(\boldsymbol{\eta}_n)$. For the lower-frequency part, we have

$$\begin{aligned}
&\partial_x \int_{\mathbb{R}}\int_{\substack{|\eta_j|\le|\eta_{2n+1}|\\ \text{for all } j=1,\dots,2n}} \boldsymbol{U}_n(\boldsymbol{\eta}_n)\hat{\varphi}(\eta_1)\hat{\varphi}(\eta_2)\cdots\hat{\varphi}(\eta_{2n})e^{i(\eta_1+\eta_2+\cdots+\eta_{2n})x}\,\mathrm{d}\hat{\boldsymbol{\eta}}_n\hat{\varphi}(\eta_{2n+1})e^{ix\eta_{2n+1}}\,\mathrm{d}\eta_{2n+1}\\
&= \partial_x \int_{\mathbb{R}}\int_{\substack{|\eta_j|\le|\eta_{2n+1}|\\ \text{for all } j=1,\dots,2n}} \chi(\eta_{2n+1})\int_{\mathbb{R}} \frac{\prod_{j=1}^{2n+1}(1-e^{i\eta_j\zeta})}{|\zeta|^{2n+1}}\,\mathrm{d}\zeta\,\hat{\varphi}(\eta_1)\hat{\varphi}(\eta_2)\cdots\hat{\varphi}(\eta_{2n})\\
&\qquad\qquad\cdot e^{i(\eta_1+\eta_2+\cdots+\eta_{2n})x}\,\mathrm{d}\hat{\boldsymbol{\eta}}_n\hat{\varphi}(\eta_{2n+1})e^{ix\eta_{2n+1}}\,\mathrm{d}\eta_{2n+1}\\
&= \partial_x\int_{\mathbb{R}} \boldsymbol{A}_n(x,\eta_{2n+1})\hat{\varphi}(\eta_{2n+1})e^{ix\eta_{2n+1}}\,\mathrm{d}\eta_{2n+1},
\end{aligned}$$

where the symbol $\boldsymbol{A}_n$ is defined by

$$\boldsymbol{A}_n(x,\eta_{2n+1}) = \int_{\substack{|\eta_j|\le|\eta_{2n+1}|\\ \text{for all } j=1,\dots,2n}} \chi(\eta_{2n+1})\int_{\mathbb{R}} \frac{\prod_{j=1}^{2n+1}(1-e^{i\eta_j\zeta})}{|\zeta|^{2n+1}}\,\mathrm{d}\zeta\;\hat{\varphi}(\eta_1)\hat{\varphi}(\eta_2)\cdots\hat{\varphi}(\eta_{2n})e^{i(\eta_1+\eta_2+\cdots+\eta_{2n})x}\,\mathrm{d}\hat{\boldsymbol{\eta}}_n.$$

Using an L^2-boundedness theorem for pseudodifferential operators [Boulkhemair 1999, Theorem 1.1], Lemma 2.5, and the compact support of the cutoff functions, we obtain

$$\begin{aligned}
&\left\|\partial_x\int_{\mathbb{R}} \boldsymbol{A}_n(x,\eta_{2n+1})\hat{\varphi}(\eta_{2n+1})e^{ix\eta_{2n+1}}\,\mathrm{d}\eta_{2n+1}\right\|_{L^2}\\
&\qquad\lesssim \sum_{i,j\le 1}\|\partial_x^i\partial_{\eta_{2n+1}}^j\boldsymbol{A}_n\|_{L^\infty}\|\varphi\|_{L^2}\\
&\qquad\lesssim \left\|\frac{\mathbb{I}_n(\hat{\boldsymbol{\eta}}_n,\eta_{2n+1})\chi(\eta_{2n+1})}{\prod_{j=1}^{2n}(1+|\eta_j|)}\prod_{j=1}^{2n}(1+|\eta_j|)\int_{\mathbb{R}}\frac{\prod_{j=1}^{2n+1}(1-e^{i\eta_j\zeta})}{|\zeta|^{2n+1}}\,\mathrm{d}\zeta\right\|_{S^\infty_{\hat{\eta}_n}L^\infty_{\eta_{2n+1}}}\|\varphi\|_{W^{1,\infty}}^{2n}\|\varphi\|_{L^2},
\end{aligned}$$

where

$$\mathbb{I}_n(\hat{\boldsymbol{\eta}}_n,\eta_{2n+1}) = \begin{cases}1 & \text{if } |\eta_j|\le|\eta_{2n+1}| \text{ for } j=1,\dots,2n,\\ 0 & \text{otherwise.}\end{cases} \quad (3\text{-}12)$$

Thus, the lower-frequency part satisfies the estimate (3-10), and this term can be absorbed in $\mathcal{R}$ in (3-7).

Next, we consider the higher-frequency part in (3-11), which we write as

$$c_n\partial_x\int_{\mathbb{R}}\int_{\substack{|\eta_j|\le|\eta_{2n+1}|\\ \text{for all } j=1,\dots,2n}} \boldsymbol{\Lambda}_n(\boldsymbol{\eta}_n)\prod_{j=1}^{2n}\left\{\chi\left(\frac{(2n+1)\eta_j}{\eta_{2n+1}}\right)+\left[1-\chi\left(\frac{(2n+1)\eta_j}{\eta_{2n+1}}\right)\right]\right\}\hat{\varphi}(\eta_j) \cdot e^{i(\eta_1+\eta_2+\cdots+\eta_{2n})x}\,\mathrm{d}\hat{\boldsymbol{\eta}}_n\hat{\varphi}(\eta_{2n+1})e^{ix\eta_{2n+1}}\,\mathrm{d}\eta_{2n+1}. \tag{3-13}$$

We expand the product in the above integral, and consider two cases depending on whether a term in the expansion contains only factors of χ or contains at least one factor $1-\chi$. In the first case, the frequency η_{2n+1} is much larger than all of the other frequencies, and we can extract a logarithmic derivative acting on the highest frequency; in the second case at least one other frequency is comparable to η_{2n+1}, and we get a remainder term by distributing derivatives on comparable frequencies.

Case I: When we take only factors of χ in the expansion of the product in (3-13), we get the integral

$$c_n\partial_x\int_{\mathbb{R}}\int_{\substack{|\eta_j|\le|\eta_{2n+1}|\\ \text{for all } j=1,\dots,2n}} \boldsymbol{\Lambda}_n(\boldsymbol{\eta}_n)\prod_{j=1}^{2n}\chi\left(\frac{(2n+1)\eta_j}{\eta_{2n+1}}\right)\hat{\varphi}(\eta_j)e^{i(\eta_1+\eta_2+\cdots+\eta_{2n})x}\,\mathrm{d}\hat{\boldsymbol{\eta}}_n\hat{\varphi}(\eta_{2n+1})e^{ix\eta_{2n+1}}\,\mathrm{d}\eta_{2n+1}. \tag{3-14}$$

From (3-3), we can write $\boldsymbol{\Lambda}_n=[1-\chi(\eta_{2n+1})]\boldsymbol{T}_n$ as an integral with respect to $\boldsymbol{s}_n=(s_1,s_2,\dots,s_{2n+1})$,

$$\begin{aligned}
\boldsymbol{\Lambda}_n(\boldsymbol{\eta}_n)&=-(1-\chi(\eta_{2n+1}))\int_{\mathbb{R}}\operatorname{sgn}\zeta\int_{[0,1]^{2n+1}}\prod_{j=1}^{2n+1}i\eta_je^{i\eta_js_j\zeta}\,\mathrm{d}\boldsymbol{s}_n\,\mathrm{d}\zeta\\
&=2(-1)^n(1-\chi(\eta_{2n+1}))\left(\prod_{j=1}^{2n+1}\eta_j\right)\int_{[0,1]^{2n+1}}\frac{1}{\sum_{j=1}^{2n+1}\eta_js_j}\,\mathrm{d}\boldsymbol{s}_n\\
&=2(1-\chi(\eta_{2n+1}))\left(\prod_{j=1}^{2n}(i\eta_j)\right)\int_{[0,1]^{2n}}\log\left|1+\sum_{j=1}^{2n}\frac{\eta_j}{\eta_{2n+1}}s_j\right|-\log\left|\sum_{j=1}^{2n}\frac{\eta_j}{\eta_{2n+1}}s_j\right|\,\mathrm{d}\hat{\boldsymbol{s}}_n\\
&=2(1-\chi(\eta_{2n+1}))\log|\eta_{2n+1}|\cdot\prod_{j=1}^{2n}(i\eta_j)-2\left(\prod_{j=1}^{2n}(i\eta_j)\right)\int_{[0,1]^{2n}}\log\left|\sum_{j=1}^{2n}\eta_js_j\right|\,\mathrm{d}\hat{\boldsymbol{s}}_n\\
&\qquad+(1-\chi(\eta_{2n+1}))\left(\prod_{j=1}^{2n}(i\eta_j)\right)\int_{[0,1]^{2n}}\log\left|1+\sum_{j=1}^{2n}\frac{\eta_j}{\eta_{2n+1}}s_j\right|\,\mathrm{d}\hat{\boldsymbol{s}}_n.
\end{aligned}$$

Substitution of this expression into (3-14) gives the three terms

$$c_n\partial_x\int_{\mathbb{R}}\int_{\substack{|\eta_j|\le|\eta_{2n+1}|\\ \text{for all } j=1,\dots,2n}} \boldsymbol{\Lambda}_n^{\log}(\boldsymbol{\eta}_n)\prod_{j=1}^{2n}\chi\left(\frac{(2n+1)\eta_j}{\eta_{2n+1}}\right)\hat{\varphi}(\eta_j)e^{i(\eta_1+\eta_2+\cdots+\eta_{2n})x}\,\mathrm{d}\hat{\boldsymbol{\eta}}_n\hat{\varphi}(\eta_{2n+1})e^{ix\eta_{2n+1}}\,\mathrm{d}\eta_{2n+1}, \tag{3-15}$$

$$c_n\partial_x\int_{\mathbb{R}}\int_{\substack{|\eta_j|\le|\eta_{2n+1}|\\ \text{for all } j=1,\dots,2n}} \boldsymbol{\Lambda}_n^{0}(\boldsymbol{\eta}_n)\prod_{j=1}^{2n}\chi\left(\frac{(2n+1)\eta_j}{\eta_{2n+1}}\right)\hat{\varphi}(\eta_j)e^{i(\eta_1+\eta_2+\cdots+\eta_{2n})x}\,\mathrm{d}\hat{\boldsymbol{\eta}}_n\hat{\varphi}(\eta_{2n+1})e^{ix\eta_{2n+1}}\,\mathrm{d}\eta_{2n+1}, \tag{3-16}$$

$$c_n\partial_x\int_{\mathbb{R}}\int_{\substack{|\eta_j|\le|\eta_{2n+1}|\\ \text{for all } j=1,\dots,2n}}\mathbf{\Lambda}_n^{\le -1}(\boldsymbol{\eta}_n)(\boldsymbol{\eta}_n)\prod_{j=1}^{2n}\chi\left(\frac{(2n+1)\eta_j}{\eta_{2n+1}}\right)\hat{\varphi}(\eta_j)e^{i(\eta_1+\eta_2+\dots+\eta_{2n})x}\,\mathrm{d}\hat{\boldsymbol{\eta}}_n\hat{\varphi}(\eta_{2n+1})e^{ix\eta_{2n+1}}\,\mathrm{d}\eta_{2n+1}, \tag{3-17}$$

where

$$\mathbf{\Lambda}_n^{\log}(\boldsymbol{\eta}_n)=2(1-\chi(\eta_{2n+1}))\log|\eta_{2n+1}|\cdot\prod_{j=1}^{2n}(i\eta_j),$$

$$\mathbf{\Lambda}_n^{0}(\boldsymbol{\eta}_n)=-2(1-\chi(\eta_{2n+1}))\left(\prod_{j=1}^{2n}(i\eta_j)\right)\int_{[0,1]^{2n}}\log\left|\sum_{j=1}^{2n}\eta_j s_j\right|\mathrm{d}\hat{\boldsymbol{s}}_n,$$

$$\mathbf{\Lambda}_n^{\le -1}(\boldsymbol{\eta}_n)=2(1-\chi(\eta_{2n+1}))\left(\prod_{j=1}^{2n}(i\eta_j)\right)\int_{[0,1]^{2n}}\log\left|1+\sum_{j=1}^{2n}\frac{\eta_j}{\eta_{2n+1}}s_j\right|\mathrm{d}\hat{\boldsymbol{s}}_n.$$

We claim that the terms (3-15) and (3-16) can be rewritten as

$$-\partial_x T_{B_n^{\log}[\varphi]}\log_+|\partial_x|\varphi+\mathcal{R}_1\quad\text{and}\quad -\partial_x T_{B_n^{0}[\varphi]}\varphi+\mathcal{R}_2, \tag{3-18}$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ satisfy the estimate (3-10). Indeed,

$$\mathcal{F}[\partial_x T_{B_n^{\log}[\varphi]}\log_+|\partial_x|\varphi](\xi)$$
$$=-2c_n i\xi\int_{\mathbb{R}}\chi\left(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\right)\log_+|\eta|\int_{\mathbb{R}^{2n}}\delta\left(\xi-\eta-\sum_{j=1}^{2n}\eta_j\right)\prod_{j=1}^{2n}\left[i\eta_j\hat{\varphi}(\eta_j)\chi\left(\frac{2(2n+1)\eta_j}{\xi+\eta}\right)\right]\mathrm{d}\hat{\boldsymbol{\eta}}_n\hat{\varphi}(\eta)\,\mathrm{d}\eta,$$

while the Fourier transform of (3-15) is

$$2c_n i\xi\int_{\mathbb{R}}\int_{\substack{|\eta_j|\le|\eta_{2n+1}|\\ \text{for all } j=1,\dots,2n}}\delta\left(\xi-\sum_{j=1}^{2n+1}\eta_j\right)(1-\chi(\eta_{2n+1}))\log|\eta_{2n+1}|$$
$$\cdot\prod_{j=1}^{2n}\left[\chi\left(\frac{(2n+1)\eta_j}{\eta_{2n+1}}\right)(i\eta_j)\hat{\varphi}(\eta_j)\right]\mathrm{d}\hat{\boldsymbol{\eta}}_n\hat{\varphi}(\eta_{2n+1})\,\mathrm{d}\eta_{2n+1}.$$

The difference of the above two integrals is

$$2c_n i\xi\int_{\mathbb{R}^{2n+1}}\boldsymbol{\delta}\left(\xi-\sum_{j=1}^{2n+1}\eta_j\right)\log|\eta_{2n+1}|$$
$$\cdot\left[\mathbb{I}_n(\hat{\boldsymbol{\eta}}_n,\eta_{2n+1})\prod_{j=1}^{2n}\chi\left(\frac{(2n+1)\eta_j}{\eta_{2n+1}}\right)(i\eta_j)\hat{\varphi}(\eta_j)(1-\chi(\eta_{2n+1}))\right.$$
$$\left.-\chi\left(\frac{|\xi-\eta_{2n+1}|^2}{1+|\xi+\eta_{2n+1}|^2}\right)\mathbf{1}_{|\eta_{2n+1}|>1}\prod_{j=1}^{2n}\left(i\eta_j\hat{\varphi}(\eta_j)\chi\left(\frac{2(2n+1)\eta_j}{\xi+\eta_{2n+1}}\right)\right)\right]\mathrm{d}\hat{\boldsymbol{\eta}}_n\hat{\varphi}(\eta_{2n+1})\,\mathrm{d}\eta_{2n+1}, \tag{3-19}$$

where $\mathbb{I}_n$ is given by (3-12).

When $\boldsymbol{\eta}_n$ satisfies

$$|\eta_j|\le\frac{1}{40}\frac{1}{2n+1}|\eta_{2n+1}|\quad\text{for } j=1,2,\dots,2n, \tag{3-20}$$

we have $\mathbb{I}_n = 1$ and $\chi((2n+1)\eta_j/\eta_{2n+1}) = 1$. In addition, since $\xi = \sum_{j=1}^{2n+1} \eta_j$, we have

$$\frac{|\xi - \eta_{2n+1}|^2}{1+|\xi + \eta_{2n+1}|^2} \le \frac{|\xi - \eta_{2n+1}|}{|\xi + \eta_{2n+1}|} = \frac{\left|\sum_{j=1}^{2n} \eta_j\right|}{\left|\sum_{j=1}^{2n} \eta_j + 2\eta_{2n+1}\right|} \le \frac{\frac{1}{40}|\eta_{2n+1}|}{\left(2-\frac{1}{40}\right)|\eta_{2n+1}|} = \frac{1}{79} < \frac{3}{40},$$

$$\frac{2(2n+1)|\eta_j|}{|\xi + \eta_{2n+1}|} \le \frac{\frac{1}{20}|\eta_{2n+1}|}{\left(2-\frac{1}{40}\right)|\eta_{2n+1}|} = \frac{2}{79} < \frac{3}{40},$$

so

$$\chi\left(\frac{|\xi - \eta_{2n+1}|^2}{1+|\xi + \eta_{2n+1}|^2}\right) = 1, \quad \chi\left(\frac{2(2n+1)\eta_j}{\xi + \eta_{2n+1}}\right) = 1.$$

Therefore the integrand of (3-19) is supported outside the set (3-20), and there exists $j_1 \in \{1, \dots, 2n\}$, such that

$$|\eta_{j_1}| > \frac{1}{40}\frac{1}{2n+1}|\eta_{2n+1}|.$$

Since $|\eta_{2n+1}|$ is the largest frequency, we see that $|\eta_{j_1}|$ and $|\eta_{2n+1}|$ are comparable in the error term. Therefore, the H^s-norm of (3-19) is bounded by

$$\|\varphi\|_{H^s} C(n,s)|c_n|(\|\varphi_x\|_{W^{3,\infty}}^{2n} + \|L\varphi_x\|_{W^{2,\infty}}^{2n}).$$

It follows that (3-15) can be written as in (3-18). A similar calculation applies to (3-16).

Next, we estimate the symbols $B_n^{\log}[\varphi]$ and $B_n^0[\varphi]$. First, we notice that they are real-valued, so that $T_{B_n^{\log}[\varphi]}$ and $T_{B_n^0[\varphi]}$ are self-adjoint. Again, without loss of generality, we assume $|\eta_{2n}| = \max_{1\le j\le 2n} |\eta_j|$ and observe that

$$\begin{aligned}\int_{[0,1]^{2n}} &\log\left|\sum_{j=1}^{2n} \eta_j s_j\right| d\hat{\boldsymbol{s}}_n \\ &= \log|\eta_{2n}| + \int_{[0,1]^{2n-1}} \left\{\left(\sum_{j=1}^{2n-1} \frac{\eta_j}{\eta_{2n}} s_j\right) \log\left|1 + \frac{1}{\sum_{j=1}^{2n-1}(\eta_j/\eta_{2n})s_j}\right| + \log\left|1+\frac{1}{\sum_{j=1}^{2n-1}(\eta_j/\eta_{2n})s_j}\right| - 1\right\} d\boldsymbol{s}_{n-1} \\ &= \log|\eta_{2n}| + O(1).\end{aligned}$$

Thus, using Young's inequality, we obtain from (3-8) the estimate (3-9), where the constants $C(n,s)$ have at most exponential growth in n.

To estimate the third term (3-17), we observe that on the support of the functions $\chi((2n+1)\eta_j/\eta_{2n+1})$, we have

$$\frac{|\eta_j|}{|\eta_{2n+1}|} \le \frac{1}{10(2n+1)}.$$

Since $s_j \in [0, 1]$, a Taylor expansion gives

$$|\boldsymbol{\Lambda}_n^{\le -1}(\boldsymbol{\eta}_n)| \lesssim \frac{\left[\prod_{j=1}^{2n} |\eta_j|\right]\left[\sum_{j=1}^{2n} |\eta_j|\right]}{|\eta_{2n+1}|}.$$

Therefore the H^s-norm of (3-17) is bounded by $C(n,s)|c_n|\|\varphi\|_{H^s}\|\varphi_x\|_{W^{2,\infty}}^{2n}$, where $C(n,s)$ has at most exponential growth in n.

Case II: When there is at least one factor of the form $1-\chi$ in the expansion of the product in the integral (3-13), we get a term of the form

$$f_{(n)}(x) = c_n \partial_x \int_{\mathbb{R}} \int_{\substack{|\eta_j| \le |\eta_{2n+1}| \\ \text{for all } j=1,\dots,2n}} \boldsymbol{\Lambda}_n(\boldsymbol{\eta}_n) \prod_{k=1}^{\ell} \left[1 - \chi\left(\frac{(2n+1)\eta_{j_k}}{\eta_{2n+1}} \right) \right] \prod_{k=\ell+1}^{2n} \chi\left(\frac{(2n+1)\eta_{j_k}}{\eta_{2n+1}} \right) \cdot \left(\prod_{j=1}^{2n} \hat{\varphi}(\eta_j) \right) e^{i(\eta_1+\eta_2+\cdots+\eta_{2n})x} \, \mathrm{d}\hat{\boldsymbol{\eta}}_n \hat{\varphi}(\eta_{2n+1}) e^{ix\eta_{2n+1}} \, \mathrm{d}\eta_{2n+1}, \quad (3\text{-}21)$$

where $1 \le \ell \le 2n$ is an integer, and $\{j_k : k = 1, \dots, 2n\}$ is a permutation of $\{1, \dots, 2n\}$.

We know $1-\chi((2n+1)\eta_{j_1}/(\eta_{2n+1}))$ is compactly supported on

$$\frac{|\eta_{j_1}|}{|\eta_{2n+1}|} \ge \frac{3}{40(2n+1)}.$$

By assumption, η_{2n+1} has the largest absolute value, so

$$\frac{3}{40(2n+1)} |\eta_{2n+1}| \le |\eta_{j_1}| \le |\eta_{2n+1}|,$$

meaning that the frequencies $|\eta_{j_1}|$ and $|\eta_{2n+1}|$ are comparable.

Without loss of generality, we assume that $|\eta_{j_1}| \le |\eta_{j_2}| \le \cdots \le |\eta_{j_{2n}}| \le |\eta_{2n+1}|$, define $\eta_{j_{2n+1}} = \eta_{2n+1}$, and, using (3-3), split the integral for $\boldsymbol{\Lambda}_n$ into three parts:

$$\boldsymbol{\Lambda}_n(\boldsymbol{\eta}_n) = \boldsymbol{\Lambda}_n^{\text{low}}(\boldsymbol{\eta}_n) + \sum_{k=1}^{2n} \boldsymbol{\Lambda}_n^{\text{med},(k)}(\boldsymbol{\eta}_n) + \boldsymbol{\Lambda}_n^{\text{high}}(\boldsymbol{\eta}_n),$$

where

$$\boldsymbol{\Lambda}_n^{\text{low}}(\boldsymbol{\eta}_n) = [1-\chi(\eta_{2n+1})] \int_{|\eta_{2n+1}\zeta|<2} \frac{\prod_{j=1}^{2n+1}(1-e^{i\eta_j\zeta})}{\zeta^{2n+1}} \operatorname{sgn}\zeta \, \mathrm{d}\zeta, \quad (3\text{-}22)$$

$$\boldsymbol{\Lambda}_n^{\text{med},(k)}(\boldsymbol{\eta}_n) = [1-\chi(\eta_{2n+1})] \int_{2/|\eta_{j_{k+1}}| \le |\zeta| \le 2/|\eta_{j_k}|} \frac{\prod_{j=1}^{2n+1}(1-e^{i\eta_j\zeta})}{\zeta^{2n+1}} \operatorname{sgn}\zeta \, \mathrm{d}\zeta, \quad (3\text{-}23)$$

$$\boldsymbol{\Lambda}_n^{\text{high}}(\boldsymbol{\eta}_n) = [1-\chi(\eta_{2n+1})] \int_{|\eta_{j_1}\zeta|>2} \frac{\prod_{j=1}^{2n+1}(1-e^{i\eta_j\zeta})}{\zeta^{2n+1}} \operatorname{sgn}\zeta \, \mathrm{d}\zeta. \quad (3\text{-}24)$$

To estimate (3-22), we notice that

$$|\boldsymbol{\Lambda}_n^{\text{low}}(\boldsymbol{\eta}_n)| \le \prod_{k=1}^{2n+1} |\eta_k| \cdot \int_{|\eta_{2n+1}\zeta|<2} \left(\prod_{k=1}^{2n+1} \frac{|1-e^{i\eta_{j_k}\zeta}|}{|\eta_{j_k}\zeta|} \right) \mathrm{d}\zeta \le C(n,s) \left(\prod_{k=1}^{2n} |\eta_{j_k}| \right).$$

For each $1 \le k \le 2n$, we consider two cases. If $k \ne 2n$, we estimate (3-23) as

$$\begin{aligned}
|\boldsymbol{\Lambda}_n^{\text{med},(k)}(\boldsymbol{\eta}_n)| &\le \prod_{\ell=1}^{k} |\eta_{j_\ell}| \cdot \int_{2/|\eta_{j_{k+1}}| \le |\zeta| \le 2/|\eta_{j_k}|} \left(\prod_{\ell=1}^{k} \frac{|1-e^{i\eta_{j_\ell}\zeta}|}{|\eta_{j_\ell}\zeta|} \right) \cdot \frac{\prod_{\ell=k+1}^{2n+1} |1-e^{i\eta_{j_\ell}\zeta}|}{|\zeta|^{2n+1-k}} \, \mathrm{d}\zeta \\
&\le 2^{2n+1-k} \prod_{\ell=1}^{k} |\eta_{j_\ell}| \cdot \int_{2/|\eta_{j_{k+1}}| \le |\zeta| \le 2/|\eta_{j_k}|} |\zeta|^{-2n-1+k} \, \mathrm{d}\zeta \\
&\le \frac{2}{2n-k} (|\eta_{j_k}|^{2n-k} + |\eta_{j_{k+1}}|^{2n-k}) \prod_{\ell=1}^{k} |\eta_{j_\ell}| \le 2 \left(\prod_{k=1}^{2n} |\eta_{j_k}| \right).
\end{aligned}$$

If $k = 2n$, we have

$$|\mathbf{\Lambda}_n^{\mathrm{med},(k)}(\boldsymbol{\eta}_n)| \le 2\prod_{\ell=1}^{2n}|\eta_{j_\ell}|\cdot\int_{2/|\eta_{j_{2n+1}}|\le|\zeta|\le 2/|\eta_{j_{2n}}|}\frac{1}{|\zeta|}\,\mathrm{d}\zeta = 4\prod_{\ell=1}^{2n}|\eta_{j_\ell}|\cdot\log\left|\frac{\eta_{j_{2n+1}}}{\eta_{j_{2n}}}\right| \le C(n,s)\prod_{\ell=1}^{2n}|\eta_{j_\ell}|,$$

where the last line follows from the fact that $|\eta_{j_{2n}}|$ and $|\eta_{j_{2n+1}}|$ are comparable.

As for (3-24), we have

$$\begin{aligned}|\mathbf{\Lambda}_n^{high}(\boldsymbol{\eta}_n)| &\le |\eta_{j_1}|\int_{|\eta_{j_1}\zeta|>2}\left(\prod_{k=2}^{2n+1}\frac{|1-e^{i\eta_{j_k}\zeta}|}{|\zeta|}\right)\cdot\frac{|1-e^{i\eta_{j_1}\zeta}|}{|\eta_{j_1}\zeta|}\,\mathrm{d}\zeta\\ &\le 2^{2n}|\eta_{j_1}|\int_{|\eta_{j_1}\zeta|>2}\frac{\mathrm{d}\zeta}{|\zeta|^{2n}} \le \frac{4}{2n-1}\left(\prod_{k=1}^{2n}|\eta_{j_k}|\right).\end{aligned}$$

Collecting these estimates, we find that

$$|\mathbf{\Lambda}_n(\boldsymbol{\eta}_n)| \le C(n,s)\left(\prod_{k=1}^{2n}|\eta_{j_k}|\right).$$

Using the L^2-boundedness theorem for pseudodifferential operators, we can bound the H^s-norm of $f_{(n)}$ in (3-21) by

$$\|f_{(n)}\|_{H^s} \lesssim \sum_{j,k=0,1}\|\partial_x^j\partial_{\eta_{2n+1}}^k\boldsymbol{P}_n\|_{L^\infty_{x,\eta_{2n+1}}}\|\varphi\|_{H^s},$$

where

$$\begin{aligned}\boldsymbol{P}_n(x,\eta_{2n+1}) = \left(i\sum_{j=1}^{2n+1}\eta_j\right)\int_{\mathbb{R}^{2n}}\mathbb{I}_n(\hat{\boldsymbol{\eta}}_n,\eta_{2n+1})\boldsymbol{T}_n(\boldsymbol{\eta}_n)\prod_{k=1}^{\ell}\left[1-\chi\left(\frac{(2n+1)\eta_{j_k}}{\eta_{2n+1}}\right)\right]&\\ \cdot\prod_{k=\ell+1}^{2n}\chi\left(\frac{(2n+1)\eta_{j_k}}{\eta_{2n+1}}\right)\prod_{j=1}^{2n}\hat{\varphi}(\eta_j)e^{i(\eta_1+\eta_2+\cdots+\eta_{2n})x}\,\mathrm{d}\hat{\boldsymbol{\eta}}_n&.\end{aligned}$$

Considering the support of the cutoff functions, we therefore have

$$\|f_{(n)}\|_{H^s} \lesssim \|\varphi\|_{H^s}\left(\sum_{n=1}^{\infty}C(n,s)|c_n|\|\varphi_x\|_{W^{2,\infty}}^{2n}\right).$$

So we have proved that the equation can be written as

$$\varphi_t + \partial_x T_{B^0[\varphi]}\varphi + \mathcal{R}(\varphi) = 2\log|\partial_x|\varphi_x - [(T_{B^{\log}[\varphi]})\log_+|\partial_x|\varphi]_x.$$

Then the proposition follows by the commutator estimate (2-5) and the fact that $\mathbf{1}_{|\xi|<1}\partial_x\log|\partial_x|$ is bounded from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$. □

4. Energy estimates and local well-posedness

In this section, we prove a local well-posedness result for the initial value problem (1-1), together with a criterion for the continuation of solutions, which is given in the next theorem. For simplicity, we consider only integer norms with $s \in \mathbb{N}$, and we do not seek a result with optimal regularity. We recall that $L = \log|\partial_x|$ denotes the operator in (1-2), and, from Theorem B.3, there exists a constant $\upsilon > 0$ such that

$$\|T_a\|_{L^2\to L^2} \le \upsilon \|a\|_{\mathcal{M}_{(1,1)}} \quad \text{for all } a \in \mathcal{M}_{(1,1)}. \tag{4-1}$$

Theorem 4.1. *Let $s \ge 7$ be an integer and $\upsilon > 0$ the uniform constant in* (4-1). *There exists a constant $\widetilde{C} > 0$, depending only on s, such that the following statements hold. Suppose $0 < C_0 < \min\{\frac{1}{4}, \frac{1}{4\upsilon}\}$, $C_M > 0$ are constants and $\varphi_0 \in H^s(\mathbb{R})$ satisfies*

$$\|B^{\log}[\varphi_0]\|_{\mathcal{M}_{(1,1)}} < C_0, \qquad \sum_{n=1}^{\infty} \widetilde{C}^n |c_n| (\|\varphi_{0x}\|^{2n}_{W^{3,\infty}} + \|L\varphi_{0x}\|^{2n}_{W^{2,\infty}}) < C_M,$$

where c_n is defined in (3-3) *and $B^{\log}[\varphi_0]$ is defined in* (3-8). *Then there exists a maximal time of existence $0 < T_{\max} \le \infty$, depending only on C_0, C_M, and s such that the initial value problem* (1-1) *has a unique solution with $\varphi \in C([0, T_{\max}); H^s(\mathbb{R}))$ that satisfies*

$$\|B^{\log}[\varphi(t)]\|_{\mathcal{M}_{(1,1)}} < 2C_0, \quad \sum_{n=1}^{\infty} \widetilde{C}^n |c_n| (\|\varphi_x(t)\|^{2n}_{W^{3,\infty}} + \|L\varphi_x(t)\|^{2n}_{W^{2,\infty}}) < 2C_M \quad \textit{for all } t \in [0, T_{\max}).$$

If $T_{\max} < \infty$, then

$$\lim_{t\uparrow T_{\max}} \|B^{\log}[\varphi(\cdot, t)]\|_{\mathcal{M}_{(1,1)}} = 2C_0 \quad \textit{or} \quad \lim_{t\uparrow T_{\max}} \sum_{n=1}^{\infty} \widetilde{C}^n |c_n| (\|\varphi_x(t)\|^{2n}_{W^{3,\infty}} + \|L\varphi_x(t)\|^{2n}_{W^{2,\infty}}) = 2C_M. \tag{4-2}$$

For any $0 < T < T_{\max}$, the solution map $U : H^s(\mathbb{R}) \to C([0, T]; H^s(\mathbb{R}))$ defined by $U : \varphi_0 \mapsto \varphi$ is continuous. Moreover, if $0 \le r < s - 1$ is an integer and $\varphi, \psi \in C([0, T]; H^s(\mathbb{R}))$ are solutions of (1-1) *with initial data $\varphi(\cdot, 0) = \varphi_0$, $\psi(\cdot, 0) = \psi_0$, then*

$$\|\varphi(\cdot, t) - \psi(\cdot, t)\|_{H^r} \le C\|\varphi_0 - \psi_0\|_{H^r} \quad \textit{for all } t \in [0, T], \tag{4-3}$$

where C is a constant depending on $\|\varphi_0\|_{H^{\tilde{r}}}$, $\|\psi_0\|_{H^{\tilde{r}}}$, C_0, C_M, $\widetilde{C}$, and $\tilde{r}$, with $\tilde{r} = \max(r + 2, 5)$.

This local well-posedness theorem is sufficient to continue solutions so long as the criterion in (4-2) is not satisfied, which allows us to obtain global solutions with small initial data. The range of C_0 is not optimal and may be extended by more careful estimates. However, our proof method is not suitable for large data because it depends on a multilinear expansion of the nonlinear term. We will not address the question of local well-posedness for large data in the present paper.

As noted in the Introduction, standard H^s-estimates for (1-1) do not close, so we introduce homogeneous and nonhomogeneous weighted energies

$$E^{(s)}[\varphi](t) = \int_{\mathbb{R}} |D|^s\varphi(x, t) \cdot (2 - T_{B^{\log}[\varphi]})^{2s+1} |D|^s\varphi(x, t)\, dx, \qquad \widetilde{E}^{(s)}[\varphi](t) = \sum_{j=0}^{s} E^{(j)}[\varphi](t). \tag{4-4}$$

The solutions we construct satisfy $\|B^{\log}[\varphi]\|_{\mathcal{M}_{(1,1)}} < \frac{1}{2\upsilon}$, so $(2 - T_{B^{\log}[\varphi]})$ is a positive, self-adjoint operator on L^2, and these weighted energies are equivalent to the standard H^s-energies.

Theorem 4.1 is proved in the following subsections, where we use F to denote an increasing, continuous, positive function, which might change from line to line.

4A. ***Linearized equation and energy estimates.*** We begin by studying a linearization of (3-7). Given functions $\varphi_0(x)$, $u(x,t)$, and $\Upsilon(x,t)$, we consider the linear initial value problem

$$\varphi_t + \partial_x T_{B^0[u]}\varphi + \Upsilon(x,t) = L[(2 - T_{B^{\log}[u]})\varphi]_x, \quad \varphi(x,0) = \varphi_0(x), \tag{4-5}$$

and define linearized energies for this equation by

$$E_u^{(s)}[\varphi](t) = \int_{\mathbb{R}} |D|^s\varphi(x,t)\cdot(2 - T_{B^{\log}[u]})^{2s+1}|D|^s\varphi(x,t)\,\mathrm{d}x, \quad \widetilde{E}_u^{(s)}[\varphi](t) = \sum_{j=0}^{s} E_u^{(j)}[\varphi](t). \tag{4-6}$$

In order to derive energy estimates for (4-5), we first state a lemma.

Lemma 4.2. *Let $s \in \mathbb{N}$ and $T > 0$. Suppose that $u \in C([0,T]; W^{3,\infty}(\mathbb{R}))$ with $u_t \in C([0,T]; W^{2,\infty}(\mathbb{R}))$, and $\psi \in C^1([0,T]; L^2(\mathbb{R}))$. Then*

$$\partial_t(2 - T_{B^{\log}[u]})^s\psi = (2 - T_{B^{\log}[u]})^s\psi_t - s(2 - T_{B^{\log}[u]})^{s-1}T_{\partial_t B^{\log}[u]}\psi + \mathcal{R}_2(\psi,u),$$

where the remainder term satisfies

$$\|\mathcal{R}_2(\psi,u)\|_{H^k} \lesssim \|\psi\|_{H^k}\left\{\sum_{n=1}^{\infty} C(n,s)|c_n|(\|u_x\|_{W^{1,\infty}}^{2n} + \|u_{xt}\|_{W^{1,\infty}}^{2n})\right\} \quad \text{for all } k \in \mathbb{N},$$

for constants $C(n,s)$ with at most exponential growth in n.

Proof. Since s is an integer, we can calculate the time derivative as

$$\begin{aligned}\partial_t(2 - T_{B^{\log}[u]})^s\psi &= (2 - T_{B^{\log}[u]})^s\psi_t - T_{\partial_t B^{\log}[u]}(2 - T_{B^{\log}[u]})^{s-1}\psi \\ &\quad - (2 - T_{B^{\log}[u]})T_{\partial_t B^{\log}[u]}(2 - T_{B^{\log}[u]})^{s-2}\psi - \cdots - (2 - T_{B^{\log}[u]})^{s-1}T_{\partial_t B^{\log}[u]}\psi.\end{aligned}$$

By Lemma 2.1, the equivalence of symbol norms in (B-2), and the symbol estimates in Proposition 3.1, we get that

$$\begin{aligned}\|[T_{\partial_t B^{\log}[u]}, (2 - T_{B^{\log}[u]})]f\|_{H^k} &\lesssim \|f\|_{H^k}\|\partial_t B^{\log}[u]\|_{\mathcal{M}_{(1,1)}}\|B^{\log}[u]\|_{\mathcal{M}_{(1,1)}} \\ &\lesssim \|f\|_{H^k}\left\{\sum_{n=1}^{\infty} C(n,s)|c_n|(\|u_x\|_{W^{1,\infty}}^{2n} + \|u_{xt}\|_{W^{1,\infty}}^{2n})\right\}.\end{aligned}$$

Taking $f = (2 - T_{B^{\log}[u]})^{s-2}\psi, (2 - T_{B^{\log}[u]})^{s-3}\psi, \ldots, (2 - T_{B^{\log}[u]})\psi$ and applying the above estimate repeatedly, we obtain the conclusion. □

We then get energy estimates and an existence result for the linearized initial value problem (4-5).

Proposition 4.3. *Let $s \geq 2$ be an integer, let $C_M > 0$, $T > 0$ be constants, and $0 < C_0 < \frac{1}{4\upsilon}$, where υ is a positive constant from Theorem B.3. Then there exists a constant $\widetilde{C} > 0$, depending only on s, such that the following statements hold. Suppose that*

$$\varphi_0 \in H^s(\mathbb{R}), \quad \Upsilon \in L^\infty([0,T]; H^s(\mathbb{R})), \quad u \in L^\infty([0,T]; W^{4,\infty}(\mathbb{R})),$$

with $u_t, Lu_x \in L^\infty([0,T]; W^{2,\infty}(\mathbb{R}))$, and for all $t \in [0,T]$

$$\|B^{\log}[u(t)]\|_{\mathcal{M}_{(1,1)}} < 2C_0, \quad \sum_{n=1}^{\infty} \widetilde{C}^n|c_n|(\|u_x(t)\|_{W^{3,\infty}}^{2n} + \|Lu_x(t)\|_{W^{2,\infty}}^{2n} + \|u_{tx}(t)\|_{W^{1,\infty}}^{2n}) < 2C_M.$$

Then the initial value problem (4-5) *has a unique solution* $\varphi \in C([0,T]; H^s(\mathbb{R}))$. *Moreover*, *the linearized energy* (4-6) *satisfies*

$$\widetilde{E}_u^{(s)}[\varphi](t) \le \widetilde{E}_u^{(s)}[\varphi_0] + \int_0^t F(\|u_x(\tau)\|_{W^{3,\infty}} + \|u_{tx}(\tau)\|_{W^{1,\infty}} + \|Lu_x(t)\|_{W^{2,\infty}}) \cdot \{(\|u_x(\tau)\|_{W^{3,\infty}} + \|u_{tx}(\tau)\|_{W^{1,\infty}} + \|Lu_x(t)\|_{W^{2,\infty}})^2 \|\varphi(\tau)\|_{H^s}^2 + \|\Upsilon(\tau)\|_{H^s} \|\varphi(\tau)\|_{H^s}\}\, \mathrm{d}\tau, \tag{4-7}$$

where F is an increasing, *continuous*, *positive function such that*

$$F(\|u_x(\tau)\|_{W^{3,\infty}} + \|u_{tx}(\tau)\|_{W^{1,\infty}} + \|Lu_x(t)\|_{W^{2,\infty}}) \approx \sum_{n=0}^{\infty} \widetilde{C}^n |c_n| (\|u_x\|_{W^{3,\infty}}^{2n} + \|u_{tx}\|_{W^{1,\infty}}^{2n} + \|Lu_x(\tau)\|_{W^{2,\infty}}^{2n}). \tag{4-8}$$

Proof. The existence and uniqueness of solutions of (4-5) follow from energy estimates and a duality argument, which we outline briefly.

We write the linearized equation in (4-5) as $A\varphi = -\Upsilon$, where the operator A is given by

$$A = \partial_t - \partial_x L(2 - T_{B^{\log}[u]}) + \partial_x T_{B^0[u]}.$$

The formal adjoint of A is

$$A^* = -\partial_t + (2 - T_{B^{\log}[u]})\partial_x L - T_{B^0[u]}\partial_x.$$

For $T > 0$, we define a space $\mathcal{E}$ of test functions by

$$\mathcal{E} = \{w \in C^\infty([0,T]; H^\infty(\mathbb{R})) \mid w(x,T) = 0\}$$

and for $w \in \mathcal{E},\ f \in A^*\mathcal{E}$ consider the equation $A^* w = f$, or

$$-\partial_t w + (2 - T_{B^{\log}[u]})\partial_x L w - T_{B^0[u]}\partial_x w = f.$$

Applying the operator $T_{1/(2-B^{\log}[u])}$ to this equation, we get that

$$-T_{1/(2-B^{\log}[u])}\partial_t w + \partial_x L w - T_{B^0[u]/(2-B^{\log}[u])}\partial_x w = T_{1/(2-B^{\log}[u])} f + \mathcal{R}', \tag{4-9}$$

where, in view of the commutator estimate in Theorem B.6, the remainder term $\mathcal{R}'$ satisfies

$$\|\mathcal{R}'\|_{L^2} \lesssim \left(\left\| \frac{1}{2 - B^{\log}[u]} \right\|_{\mathcal{M}_{(2,2)}} + \|B^0[u]\|_{\mathcal{M}_{(2,2)}} \right) \|w\|_{L^2}.$$

Multiplying (4-9) by w, integrating with respect to x over $\mathbb{R}$, and rewriting the result, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} w T_{1/(2-B^{\log}[u])} w \,\mathrm{d}x + \int_{\mathbb{R}} w [T_{B^0[u]/(2-B^{\log}[u])}, \partial_x] w \,\mathrm{d}x = \int_{\mathbb{R}} \{ w T_{\partial_t(1/(2-B^{\log}[u]))} w \,\mathrm{d}x - 2 w T_{1/(2-B^{\log}[u])} f - 2 w \mathcal{R}' \} \,\mathrm{d}x.$$

By the assumptions on u, the $\mathcal{M}_{(1,1)}$-norm of the symbol $1/(2 - B^{\log}[u])$ is positive and bounded away from zero. In fact, since $\|B^{\log}[u]\|_{\mathcal{M}_{(1,1)}} < 2C_0$ and $\|2 - B^{\log}[u]\|_{\mathcal{M}_{(1,1)}} > 2 - 2C_0$ for $C_0 < \frac{1}{4}$, we get from the definition in (2-3) of the $\mathcal{M}_{(1,1)}$-norm that

$$\begin{aligned}\left\|\frac{1}{(2-B^{\log}[u])}\right\|_{\mathcal{M}_{(1,1)}} &\le \left\|\frac{1}{(2-B^{\log}[u])}\right\|_{L^\infty_{x,\xi}}+\left\|\frac{1}{(2-B^{\log}[u])^2}\right\|_{L^\infty_{x,\xi}}\|B^{\log}[u]\|_{\mathcal{M}_{(1,1)}}+2\left\|\frac{1}{(2-B^{\log}[u])^3}\right\|_{L^\infty_{x,\xi}}\|B^{\log}[u]\|^2_{\mathcal{M}_{(1,1)}}\\ &< \frac{1}{2-2C_0}+\frac{2C_0}{(2-2C_0)^2}+\frac{2(2C_0)^2}{(2-2C_0)^3}=\frac{1-C_0+2C_0^2}{2(1-C_0)^3}<\frac{1+\frac{1}{8}}{2\left(\frac{3}{4}\right)^3}=\frac{4}{3}.\end{aligned}$$

Since

$$\frac{1}{2-B^{\log}[u]}=\frac{1}{2}+\frac{B^{\log}[u]}{2(2-B^{\log}[u])},$$

we also have

$$\left\|\frac{1}{(2-B^{\log}[u])}\right\|_{\mathcal{M}_{(1,1)}}\ge\frac{1}{2}-\frac{1}{2}\left\|\frac{1}{(2-B^{\log}[u])}\right\|_{\mathcal{M}_{(1,1)}}\|B^{\log}[u]\|_{\mathcal{M}_{(1,1)}}>\frac{1}{2}-\frac{1}{2}\cdot\frac{4}{3}\cdot\frac{1}{2}=\frac{1}{6}.$$

Thus, the integral $\int_{\mathbb{R}} wT_{1/(2-B^{\log}[u])}w\,\mathrm{d}x$ is equivalent to $\|w\|^2_{L^2}$. Integrating the previous equation over the time interval $[t, T]$, where $0\le t\le T$, and using the commutator estimate in Lemma 2.3, we obtain

$$\|w(t)\|_{L^2}\lesssim\int_t^T\left\|\frac{1}{2-B^{\log}[u]}\right\|_{\mathcal{M}_{(2,2)}}\|w(s)\|_{L^2}+\|B^0[u]\|_{\mathcal{M}_{(2,2)}}\|w(s)\|_{L^2}+\left\|\frac{1}{2-B^{\log}[u]}\right\|_{\mathcal{M}_{(2,2)}}\|f(s)\|_{L^2}\,\mathrm{d}s.$$

Then, by Gronwall's inequality,

$$\|w(t)\|_{L^2}\lesssim\int_0^T e^{\int_s^t\|1/(2-B^{\log}[u])\|_{\mathcal{M}_{(2,2)}}+\|B^0[u]\|_{\mathcal{M}_{(2,2)}}\,\mathrm{d}\tau}\left\|\frac{1}{2-B^{\log}[u]}\right\|_{\mathcal{M}_{(2,2)}}\|f(s)\|_{L^2}\,\mathrm{d}s\le C\int_0^T\|f(s)\|_{L^2}\,\mathrm{d}s,$$

where C is a constant related to u. It follows that

$$\|w(t)\|^2_{L^2}\le C\int_0^T\|A^*w(s)\|^2_{L^2}\,\mathrm{d}s\quad\text{for every } w\in\mathcal{E}\text{ and all } t\in[0,T],\tag{4-10}$$

which implies that A^* is one-to-one on $\mathcal{E}$.

The distributional form of (4-5) is

$$\int_0^T(\varphi, A^*w)_{L^2}\,\mathrm{d}t+(\varphi_0, w(0))_{L^2}=\int_0^T(-\Upsilon, w)_{L^2}\,\mathrm{d}t\quad\text{for all } w\in\mathcal{E}.$$

We define a linear form $\mathfrak{L}: A^*\mathcal{E}\to\mathbb{R}$ by

$$\mathfrak{L}(A^*w)=\int_0^T(-\Upsilon, w)_{L^2}\,\mathrm{d}t-(\varphi_0, w(0))_{L^2}.$$

Using the Cauchy–Schwarz inequality and (4-10), we have

$$|\mathfrak{L}(A^*w)|^2\lesssim(T\|\Upsilon\|^2_{L^2(0,T;L^2)}+\|\varphi_0\|^2_{L^2})\|A^*w\|^2_{L^2(0,T;L^2)},$$

and $\mathfrak{L}$ extends to a continuous form on $L^2(0, T; L^2)$ by the Hahn–Banach theorem. The Riesz representation theorem and the density of $A^*\mathcal{E}$ in $L^2(0, T; L^2)$ then imply that there exists a unique $\varphi \in L^2(0, T; L^2)$ such that

$$\mathfrak{L}(A^*w) = \int_0^T (\varphi(t), A^*w(t))_{L^2}\,\mathrm{d}t.$$

Thus, we have proved the existence of a unique weak solution of (4-5) in $L^2(0, T; L^2)$. The H^s-regularity of solutions can be obtained by applying the above process to the equation for $\partial_x^s\varphi$ with appropriately weighted L^2-norms.

To derive the energy estimates, we apply the operator $|D|^s$ to the equation in (4-5) to get

$$|D|^s\varphi_t + \partial_x|D|^s T_{B^0[u]}\varphi + |D|^s\Upsilon = \partial_x L|D|^s[(2 - T_{B^{\log}[u]})\varphi]. \tag{4-11}$$

Using Lemma 2.3 and Proposition 3.1, we find that

$$\begin{aligned}|D|^s[(2 - T_{B^{\log}[u]})\varphi] &= 2|D|^s\varphi - |D|^s(T_{B^{\log}[u]}\varphi)\\ &= 2|D|^s\varphi - T_{B^{\log}[u]}|D|^s\varphi + sT_{\partial_x B^{\log}[u]}|D|^{s-2}\varphi_x + \mathcal{R}_2,\end{aligned}$$

where

$$\|\partial_x\mathcal{R}_2\|_{L^2} \lesssim \left(\sum_{n=1}^{\infty} C(n, s)|c_n|\|u_x\|_{W^{3,\infty}}^{2n}\right)\|\varphi\|_{H^{s-1}}.$$

Thus, after absorbing a low-frequency part into the remainder, we can write the right-hand side of (4-11) as

$$\partial_x L|D|^s[(2 - T_{B^{\log}[u]})\varphi] = \partial_x \log_+|\partial_x|[(2 - T_{B^{\log}[u]})|D|^s\varphi + sT_{\partial_x B^{\log}[u]}|D|^{s-2}\varphi_x] + \mathcal{R}_3,$$

where

$$\|\mathcal{R}_3\|_{L^2} \lesssim \left(\sum_{n=1}^{\infty} C(n, s)|c_n|\|u_x\|_{W^{3,\infty}}^{2n}\right)\|\varphi\|_{H^{s-1}}.$$

Applying $(2-T_{B^{\log}[u]})^s$ to (4-11) and using Lemma 2.2 to commute $(2 - T_{B^{\log}[u]})^s$ with ∂_x and $\log_+|\partial_x|$, we obtain that

$$\begin{aligned}&(2 - T_{B^{\log}[u]})^s|D|^s\varphi_t + (2 - T_{B^{\log}[u]})^s\partial_x|D|^s T_{B^0[\varphi]}\varphi + (2 - T_{B^{\log}[u]})^s|D|^s\Upsilon\\ &\quad= (2 - T_{B^{\log}[u]})^s\partial_x \log_+|\partial_x|[(2 - T_{B^{\log}[u]})|D|^s\varphi + sT_{\partial_x B^{\log}[u]}|D|^{s-2}\varphi_x] + (2 - T_{B^{\log}[u]})^s\mathcal{R}_3\\ &\quad= \log_+|\partial_x|\{(2 - T_{B^{\log}[u]})^s\partial_x[(2 - T_{B^{\log}[u]})|D|^s\varphi + sT_{\partial_x B^{\log}[u]}|D|^{s-2}\varphi_x]\} + \mathcal{R}_4\\ &\quad= \log_+|\partial_x|\{(2 - T_{B^{\log}[u]})^{s+1}|D|^s\varphi_x - (s+1)(2 - T_{B^{\log}[u]})^s T_{\partial_x B^{\log}[u]}|D|^s\varphi\} + \mathcal{R}_4\\ &\quad= \partial_x L\{(2 - T_{B^{\log}[u]})^{s+1}|D|^s\varphi\} + \mathcal{R}_5,\end{aligned} \tag{4-12}$$

where $\|\mathcal{R}_4\|_{L^2} + \|\mathcal{R}_5\|_{L^2} \lesssim \left(\sum_{n=1}^{\infty} C(n, s)|c_n|\|u_x\|_{W^{3,\infty}}^{2n}\right)\|\varphi\|_{H^s}$.

By Lemma 4.2, with $\psi = |D|^s\varphi$, the time derivative of $E_u^{(s)}(t)$ in (4-6) is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_u^{(s)}[\varphi](t) = &-\int_{\mathbb{R}}(2s+1)|D|^s\varphi\cdot(2 - T_{B^{\log}[u]})^{2s}T_{\partial_t B^{\log}[u]}|D|^s\varphi\,\mathrm{d}x\\ &+ 2\int_{\mathbb{R}}|D|^s\varphi\cdot(2 - T_{B^{\log}[u]})^{2s+1}|D|^s\varphi_t\,\mathrm{d}x + \int_{\mathbb{R}}\mathcal{R}_2\cdot|D|^s\varphi\,\mathrm{d}x.\end{aligned} \tag{4-13}$$

We will estimate each of the terms on the right-hand side of (4-13), where the second term requires the most work.

The first term on the right-hand side of (4-13) can be estimated by

$$\left|\int_{\mathbb{R}}(2s+1)|D|^s\varphi\cdot(2-T_{B^{\log}[u]})^{2s}T_{\partial_t B^{\log}[u]}|D|^s\varphi\,\mathrm{d}x\right|\lesssim\left(\sum_{n=1}^{\infty}C(n,s)|c_n|(\|u_x\|_{W^{1,\infty}}^{2n}+\|u_{tx}\|_{W^{1,\infty}}^{2n})\right)\|\varphi\|_{H^s}^2.$$

Using Lemma 4.2, we can estimate the third term on the right-hand side of (4-13) by

$$\int_{\mathbb{R}}\mathcal{R}_2\cdot|D|^s\varphi\,\mathrm{d}x\lesssim\left(\sum_{n=1}^{\infty}C(n,s)|c_n|(\|u_x\|_{W^{2,\infty}}^{2n}+\|u_{tx}\|_{W^{1,\infty}}^{2n})\right)\|\varphi\|_{H^s}\|\varphi\|_{H^{s-1}}.$$

To estimate the second term on the right-hand side (4-13), we multiply (4-12) by $(2-T_{B^{\log}[u]})^{s+1}|D|^s\varphi$, integrate the result with respect to x, and use the self-adjointness of $(2-T_{B^{\log}[u]})^{s+1}$, which gives

$$\int_{\mathbb{R}}|D|^s\varphi\cdot(2-T_{B^{\log}[u]})^{2s+1}|D|^s\varphi_t\,\mathrm{d}x=\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV},$$

where

$$\begin{aligned}\mathrm{I}&=-\int_{\mathbb{R}}|D|^s\varphi\cdot(2-T_{B^{\log}[u]})^{2s+1}|D|^s\partial_x T_{B^0[\varphi]}\varphi\,\mathrm{d}x,\\ \mathrm{II}&=\int_{\mathbb{R}}(2-T_{B^{\log}[u]})^{s+1}|D|^s\varphi\cdot\partial_x L(2-T_{B^{\log}[u]})^{s+1}|D|^s\varphi\,\mathrm{d}x,\\ \mathrm{III}&=\int_{\mathbb{R}}(2-T_{B^{\log}[u]})^{s+1}|D|^s\varphi\cdot\mathcal{R}_5\,\mathrm{d}x,\\ \mathrm{IV}&=\int_{\mathbb{R}}|D|^s\varphi\cdot(2-T_{B^{\log}[u]})^{2s+1}|D|^s\Upsilon\,\mathrm{d}x.\end{aligned}$$

We have $\mathrm{II}=0$, since $\partial_x L$ is skew-symmetric, and

$$\mathrm{III}\lesssim\left(\sum_{n=1}^{\infty}C(n,s)|c_n|\|u_x\|_{W^{3,\infty}}^{2n}\right)\|\varphi\|_{H^s}^2,\qquad\mathrm{IV}\lesssim F(\|u_x\|_{W^{3,\infty}})\|\varphi\|_{H^s}\|\Upsilon\|_{H^s},$$

since $\|\mathcal{R}_5\|_{L^2}\lesssim\big(\sum_{n=1}^{\infty}C(n,s)|c_n|\|u_x\|_{W^{3,\infty}}^{2n}\big)\|\varphi\|_{H^s}$ and $(2-T_{B^{\log}[u]})^{s+1}$ is bounded on L^2.

Term I estimate: We write $\mathrm{I}=-\mathrm{I}_a+\mathrm{I}_b$, where

$$\begin{aligned}\mathrm{I}_a&=\int_{\mathbb{R}}|D|^s\varphi\cdot(2-T_{B^{\log}[u]})^{2s+1}\partial_x T_{B^0[u]}|D|^s\varphi\,\mathrm{d}x,\\ \mathrm{I}_b&=\int_{\mathbb{R}}|D|^s\varphi\cdot(2-T_{B^{\log}[u]})^{2s+1}\partial_x[T_{B^0[u]},|D|^s]\varphi\,\mathrm{d}x.\end{aligned}$$

By a commutator estimate and (3-9), the second integral satisfies

$$|\mathrm{I}_b|\lesssim\left(\sum_{n=1}^{\infty}C(n,s)|c_n|(\|u_x\|_{W^{2,\infty}}^{2n}+\|Lu_x\|_{W^{2,\infty}}^{2n})\right)\|\varphi\|_{H^s}^2.$$

To estimate the first integral, we write it as

$$\mathrm{I}_a=\mathrm{I}_{a_1}-\mathrm{I}_{a_2},$$

where

$$\mathrm{I}_{a_1}=\int_{\mathbb{R}}|D|^s\varphi\cdot[(2-T_{B^{\log}[u]})^{2s+1},\partial_x](T_{B^0[u]}|D|^s\varphi)\,\mathrm{d}x,$$
$$\mathrm{I}_{a_2}=\int_{\mathbb{R}}|D|^s\varphi_x\cdot(2-T_{B^{\log}[u]})^{2s+1}(T_{B^0[u]}|D|^s\varphi)\,\mathrm{d}x.$$

Term I_{a_1} estimate: A Kato–Ponce type commutator estimate and (3-9) gives

$$|\mathrm{I}_{a_1}|\lesssim\left(\sum_{n=1}^{\infty}C(n,s)|c_n|(\|u_x\|_{W^{2,\infty}}^{2n}+\|Lu_x\|_{W^{1,\infty}}^{2n})\right)\|\varphi\|_{H^s}^2.$$

Term I_{a_2} estimate: We have

$$\begin{aligned}\mathrm{I}_{a_2}&=\int_{\mathbb{R}}(T_{B^0[u]}|D|^s\varphi)\cdot\{\partial_x((2-T_{B^{\log}[u]})^{2s+1}|D|^s\varphi)-[\partial_x,(2-T_{B^{\log}[u]})^{2s+1}]|D|^s\varphi\}\,\mathrm{d}x\\&=-\int_{\mathbb{R}}\partial_x(T_{B^0[u]}|D|^s\varphi)\cdot(2-T_{B^{\log}[u]})^{2s+1}|D|^s\varphi\,\mathrm{d}x\\&\qquad\qquad-\int_{\mathbb{R}}(T_{B^0[u]}|D|^s\varphi)\cdot[\partial_x,(2-T_{B^{\log}[u]})^{2s+1}]|D|^s\varphi\,\mathrm{d}x\\&=-\int_{\mathbb{R}}(T_{B^0[u]}|D|^s\varphi_x+[\partial_x,T_{B^0[u]}]|D|^s\varphi)\cdot(2-T_{B^{\log}[u]})^{2s+1}|D|^s\varphi\,\mathrm{d}x\\&\qquad\qquad-\int_{\mathbb{R}}(T_{B^0[u]}|D|^s\varphi)\cdot[\partial_x,(2-T_{B^{\log}[u]})^{2s+1}]|D|^s\varphi\,\mathrm{d}x.\end{aligned}\tag{4-14}$$

Using commutator estimates and (3-9), we get that

$$\|[\partial_x,T_{B^0[u]}]|D|^s\varphi\|_{L^2}\lesssim\left(\sum_{n=1}^{\infty}C(n,s)|c_n|(\|u_x\|_{W^{2,\infty}}^2+\|Lu_x\|_{W^{2,\infty}}^2)\right)\|\varphi\|_{H^s},$$
$$\|[\partial_x,(2-T_{B^{\log}[u]})^{2s+1}]|D|^s\varphi\|_{L^2}\lesssim\left(\sum_{n=1}^{\infty}C(n,s)|c_n|(\|u_x\|_{W^{2,\infty}}^{2n})\right)\|\varphi\|_{H^s},$$
$$\|\partial_x[(2-T_{B^{\log}[u]})^{2s+1},T_{B^0[u]}]|D|^s\varphi\|_{L^2}\lesssim\left(\sum_{n=1}^{\infty}C(n,s)|c_n|(\|u_x\|_{W^{2,\infty}}^{2n}+\|Lu_x\|_{W^{2,\infty}}^{2n})\right)\|\varphi\|_{H^s}.$$

Since $T_{B^0[u]}$ is self-adjoint, we can rewrite (4-14) as

$$\mathrm{I}_{a_2}=-\mathrm{I}_{a_2}+\mathcal{R}_6,$$

with

$$|\mathcal{R}_6|\lesssim\left(\sum_{n=1}^{\infty}C(n,s)|c_n|(\|u_x\|_{W^{2,\infty}}^{2n}+\|Lu_x\|_{W^{2,\infty}}^{2n})\right)\|\varphi\|_{H^s}^2,$$

and we conclude that

$$|\mathrm{I}_{a_2}|\lesssim\left(\sum_{n=1}^{\infty}C(n,s)|c_n|(\|u_x\|_{W^{2,\infty}}^{2n}+\|Lu_x\|_{W^{2,\infty}}^{2n})\right)\|\varphi\|_{H^s}^2.$$

This completes the estimate of the terms on the right-hand side of (4-13). Collecting the above estimates and using the interpolation inequalities for $E_u^{(0)}$ and $E_u^{(s)}$, we obtain (4-7).

Finally, by Proposition 3.1, we observe that the coefficients $C(n,s) > 0$ grow at most exponentially with n. Thus, there exists a sufficiently large constant $\widetilde{C}(s) > 0$ such that $C(n,s) \le \widetilde{C}(s)^n$. The series in (4-8) then converges whenever $\|u_x\|_{W^{3,\infty}} + \|u_{tx}\|_{W^{1,\infty}} + \|Lu_x(\tau)\|_{W^{2,\infty}}$ is sufficiently small, and we can choose F to be an increasing, continuous, positive function that satisfies (4-8). ☐

In particular, setting $u = \varphi$ in Proposition 4.3, and using the continuity of φ in time, we get the following a priori estimate for (1-1).

Proposition 4.4. *Let $s \ge 5$ be an integer and let $0 < C_0 < \frac{1}{4\upsilon}$, where υ is a positive constant from Theorem B.3. Let $C_M > 0$ be a constant. Then there exist constants $\widetilde{C} > 0$, depending only on s, and $T > 0$ such that the following statements hold. If $\varphi_0 \in H^s(\mathbb{R})$ satisfies*

$$\|B^{\log}[\varphi_0]\|_{\mathcal{M}_{(1,1)}} < C_0, \qquad \sum_{n=1}^{\infty} \widetilde{C}^n |c_n| (\|\partial_x \varphi_0\|_{W^{3,\infty}}^{2n} + \|L\partial_x\varphi_0\|_{W^{2,\infty}}^{2n}) < C_M,$$

then the solution $\varphi \in C([0,T]; H^s(\mathbb{R}))$ of (1-1) with initial data $\varphi(\cdot,0) = \varphi_0$ satisfies

$$\begin{aligned} \widetilde{E}^{(s)}[\varphi](t) \le \widetilde{E}^{(s)}[\varphi_0] + \int_0^t (\|\varphi_x(\tau)\|_{W^{3,\infty}} + \|L\varphi_x(\tau)\|_{W^{2,\infty}})^2 \\ \cdot F(\|\varphi_x(\tau)\|_{W^{3,\infty}} + \|L\varphi_x(\tau)\|_{W^{2,\infty}}) \widetilde{E}^{(s)}[\varphi](\tau)\,\mathrm{d}\tau, \\ \|B^{\log}[\varphi(t)]\|_{\mathcal{M}_{(1,1)}} < 2C_0, \qquad \sum_{n=1}^{\infty} \widetilde{C}^n |c_n| (\|\varphi_x(t)\|_{W^{3,\infty}}^{2n} + \|L\varphi_x(t)\|_{W^{2,\infty}}^{2n}) < 2C_M \end{aligned} \tag{4-15}$$

for all $t \in [0,T]$. In (4-15), $\widetilde{E}^{(s)}$ is defined in (4-4), and $F(\cdot)$ is an increasing, continuous, nonnegative function such that

$$F(\|\varphi_x\|_{W^{3,\infty}} + \|L\varphi_x\|_{W^{2,\infty}}) \approx \sum_{n=0}^{\infty} \widetilde{C}^n |c_n| (\|\varphi_x\|_{W^{3,\infty}}^{2n} + \|L\varphi_x\|_{W^{2,\infty}}^{2n}).$$

4B. ***Iteration scheme.*** Given a function u satisfying the conditions in Proposition 4.3 and $\varphi_0 \in H^s(\mathbb{R})$, we define a map $\boldsymbol{G} : u \mapsto \varphi$, where $\varphi \in L^\infty([0,T]; H^s(\mathbb{R}))$ is the solution of the initial value problem

$$\varphi_t + \partial_x T_{B^0[u]}\varphi + \mathcal{R}(u) = L[(2 - T_{B^{\log}[u]})\varphi]_x, \quad \varphi(x,0) = \varphi_0(x),$$

with the same remainder $\mathcal{R}(\cdot)$ as the one in (3-7).

We then construct a sequence $\{\varphi^{(i)}\}$ of approximate solutions of (1-1) by

$$\varphi^{(0)}(x,t) = \varphi_0(x), \quad \varphi^{(i)} = \boldsymbol{G}(\varphi^{(i-1)}) \quad \text{for } i \in \mathbb{N}. \tag{4-16}$$

For sufficiently small $T > 0$, we will prove that this sequence is bounded in $L^\infty([0,T]; H^s(\mathbb{R}))$ and Cauchy in $L^\infty([0,T]; H^3(\mathbb{R}))$, which implies that its limit is a local solution of the initial value problem (1-1).

To begin with, we identify a bounded subset X_T of $L^\infty([0,T]; H^s(\mathbb{R}))$ that $\boldsymbol{G}$ maps into itself.

Proposition 4.5 (boundedness). *Let $s \ge 6$. Suppose that $0 < C_0 < \min\{\frac{1}{4}, \frac{1}{4\upsilon}\}$, $C_M > 0$, $\bar{C} > 0$ are constants and $\varphi_0 \in H^s(\mathbb{R})$ satisfies*

$$\|B^{\log}[\varphi_0]\|_{\mathcal{M}_{(1,1)}} < C_0, \qquad \sum_{n=1}^{\infty} \widetilde{C}^n |c_n| (\|\varphi_{0x}\|_{W^{3,\infty}}^{2n} + \|L\varphi_{0x}\|_{W^{2,\infty}}^{2n}) < C_M, \qquad \|\varphi_0\|_{H^s} \le \bar{C}.$$

Define

$$X_T = \Big\{ u \in L^{\infty}([0,T]; H^s(\mathbb{R})) \;\Big|\; u_t,\, Lu_x \in L^{\infty}([0,T]; W^{2,\infty}(\mathbb{R})), \\ \|u\|_{L^{\infty}([0,T];H^s(\mathbb{R}))} \le 2\cdot \frac{(2+2C_0)^{s+1/2}}{\min\{1,(2-2C_0)^{s+1/2}\}}\overline{C}, \\ \sum_{n=1}^{\infty} \widetilde{C}^n |c_n| (\|u_x(t)\|^{2n}_{W^{3,\infty}} + \|Lu_x(t)\|^{2n}_{W^{2,\infty}} + \|u_{tx}(t)\|^{2n}_{W^{1,\infty}}) < 2C_M, \\ \|B^{\log}[u(t)]\|_{\mathcal{M}_{(1,1)}} < 2C_0 \text{ for all } t\in[0,T],\; u(0)=\varphi_0 \Big\}.$$

Then there exists $T > 0$ such that $\boldsymbol{G} : X_T \to X_T$.

Proof. Taking $\Upsilon = \mathcal{R}(u)$ in Proposition 4.3 and using (3-10), we obtain that

$$\widetilde{E}^{(s)}_u[\varphi](t) \le \widetilde{E}^{(s)}_u[\varphi_0] + \int_0^t (\|u_x(\tau)\|_{W^{3,\infty}} + \|u_{tx}(\tau)\|_{W^{1,\infty}} + \|Lu_x(\tau)\|_{W^{2,\infty}})^2 \\ \cdot F(\|u_x(\tau)\|_{W^{3,\infty}} + \|u_{tx}(\tau)\|_{W^{1,\infty}} + \|Lu_x(\tau)\|_{W^{2,\infty}})(\|\varphi(\tau)\|^2_{H^s} + \|u(\tau)\|^2_{H^s})\,\mathrm{d}\tau,$$

where F is a positive continuous function. Since $u \in X_T$, by Sobolev embedding, there exists a constant $C_1 > 0$, which depends only on $\widetilde{C}$, $\overline{C}$, C_0, C_M, such that

$$\|u_x(\tau)\|_{W^{3,\infty}} + \|u_{tx}(\tau)\|_{W^{1,\infty}} + \|Lu_x(\tau)\|_{W^{2,\infty}} \le C_1.$$

Since $\|\varphi_0\|_{H^s} \le \overline{C}$, we have $\widetilde{E}^{(s)}_u[\varphi_0] \le (2+2\upsilon C_0)^{2s+1}\overline{C}^2$, and

$$\min\{1,(2-2\upsilon C_0)^{2s+1}\}\|\varphi(\cdot,t)\|^2_{H^s} \le \widetilde{E}^{(s)}_u[\varphi](t) \le (2+2\upsilon C_0)^{2s+1}\upsilon\|\varphi(\cdot,t)\|^2_{H^s}, \tag{4-17}$$

where υ is the constant in (4-1).

Writing $\widetilde{E}^{(s)}_u[\varphi](t) = \widetilde{E}^{(s)}_u(t)$, we therefore get that

$$\widetilde{E}^{(s)}_u(t) \le (2+2\upsilon C_0)^{2s+1}\overline{C}^2 \\ + \int_0^t C_1^2 F(C_1)\Big[\max\{1,(2-2\upsilon C_0)^{-2s-1}\}\widetilde{E}^{(s)}_u(\tau) + 4\cdot\frac{(2+2\upsilon C_0)^{2s+1}}{\min\{1,(2-2\upsilon C_0)^{2s+1}\}}\overline{C}^2\Big]\,\mathrm{d}\tau.$$

Thus, when

$$T < \frac{1}{4C_1^2F(C_1)}\min[1,(2-2\upsilon C_0)^{2s+1}],$$

we have

$$\|\widetilde{E}^{(s)}_u(t)\|_{L^{\infty}(0,T)} \le (2+2\upsilon C_0)^{2s+1}\overline{C}^2 + \tfrac{1}{4}\|\widetilde{E}^{(s)}_u(t)\|_{L^{\infty}(0,T)} + (2+2\upsilon C_0)^{2s+1}\overline{C}^2,$$

and it follows that

$$\|\widetilde{E}^{(s)}_u(t)\|_{L^{\infty}(0,T)} \le \tfrac{8}{3}\cdot(2+2\upsilon C_0)^{2s+1}\overline{C}^2.$$

Then, from (4-17), we get that

$$\|\varphi(t)\|_{L^{\infty}(0,T;H^s(\mathbb{R}))} \le 2\cdot\frac{(2+2\upsilon C_0)^{s+1/2}}{\min\{1,(2-2\upsilon C_0)^{s+1/2}\}}\overline{C}.$$

Using the equation for φ, we see that $\|\varphi_t(t)\|_{H^{s'}}$ is bounded for any $0 < s' < 5$, and $\|\varphi_{tt}(t)\|_{H^{s''}}$ is bounded for any $0 < s'' < 4$. Thus $\|\varphi(t)\|_{H^{s'}}$ and $\|\varphi_t(t)\|_{H^{s''}}$ are continuous in t. By Sobolev embedding and the symbol estimates in Proposition 3.1,

$$\|\partial_t B^{\log}[\varphi]\|_{\mathcal{M}_{(j,2)}} \lesssim \sum_{n=1}^{\infty} C(n,s)|c_n|\|\varphi_x\|_{W^{j,\infty}}^{2n-1}\|\varphi_{tx}\|_{W^{j,\infty}} \le C_2(C_M, C_0, s), \quad j = 2, 3,$$

for some constant $C_2(C_M, C_0, s) > 0$ depending on C_M, C_0, and s. Therefore,

$$\|B^{\log}[\varphi(t)]\|_{\mathcal{M}_{(1,1)}} \le \|B^{\log}[\varphi(0)]\|_{\mathcal{M}_{(1,1)}} + \int_0^t \|B^{\log}[\varphi(\tau)]\|_{\mathcal{M}_{(1,1)}}\, \mathrm{d}\tau \le C_0 + tC_2(C_M, C_0, s),$$

and by taking $T < C_0/(C_2(C_M, C_0, s))$, we obtain $\|B^{\log}[\varphi(t)]\|_{\mathcal{M}_{(1,1)}} < 2C_0$.

Next, for $t \in [0, T]$, we observe that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \widetilde{C}^n|c_n|(\|\varphi_x(t)\|_{W^{3,\infty}}^{2n} + \|L\varphi_x(t)\|_{W^{2,\infty}}^{2n} + \|\varphi_{tx}(t)\|_{W^{1,\infty}}^{2n}) \\
&\quad\le \sum_{n=1}^{\infty} \widetilde{C}^n|c_n|\bigg[(\|\varphi_x(0)\|_{W^{3,\infty}} + \int_0^t \|\varphi_{tx}(\tau)\|_{W^{3,\infty}}\, \mathrm{d}\tau)^{2n} + (\|L\varphi_x(0)\|_{W^{2,\infty}} \\
&\qquad\qquad + \int_0^t \|L\varphi_{tx}(\tau)\|_{W^{2,\infty}}\, \mathrm{d}\tau)^{2n} + \bigg(\|\varphi_{tx}(0)\|_{W^{1,\infty}} + \int_0^t \|\varphi_{ttx}(\tau)\|_{W^{1,\infty}}\, \mathrm{d}\tau\bigg)^{2n}\bigg] \\
&\quad\le \sum_{n=1}^{\infty} \widetilde{C}^n|c_n|(\|\varphi_x(0)\|_{W^{3,\infty}}^{2n} + \|L\varphi_x(0)\|_{W^{2,\infty}}^{2n} + \|\varphi_{tx}(0)\|_{W^{1,\infty}}^{2n}) + TC_3(C_M, C_0, s)
\end{aligned}$$

for some constant $C_3(C_M, C_0, s) > 0$ depending on C_M, C_0 and s. Since

$$\sum_{n=1}^{\infty} \widetilde{C}^n|c_n|(\|\varphi_x(0)\|_{W^{3,\infty}}^{2n} + \|L\varphi_x(0)\|_{W^{2,\infty}}^{2n} + \|\varphi_{tx}(0)\|_{W^{1,\infty}}^{2n}) < C_M,$$

we obtain that when $T < C_M/(C_3(C_M, C_0, s))$,

$$\sum_{n=1}^{\infty} \widetilde{C}^n|c_n|(\|\varphi_x(t)\|_{W^{3,\infty}}^{2n} + \|L\varphi_x(t)\|_{W^{2,\infty}}^{2n} + \|\varphi_{tx}(t)\|_{W^{1,\infty}}^{2n}) \le 2C_M.$$

By combining the above arguments, we find that for

$$T = \min\left\{\frac{1}{4C_1^2F(C_1)}\min[1, (2-2\upsilon C_0)^{2s+1}], \frac{C_0}{C_2(C_M, C_0, s)}, \frac{C_M}{C_3(C_M, C_0, s)}\right\}$$

we have that $\boldsymbol{G} : X_T \to X_T$. □

Next, we prove that $\boldsymbol{G}$ is a contraction with respect to a low norm.

Proposition 4.6 (contraction). *For sufficiently small $T > 0$, the map $\boldsymbol{G} : X_T \to X_T$ defined above is a contraction with respect to $\|\cdot\|_{L_t^\infty H_x^3}$.*

Proof. For $u, v \in X_T$, let φ and ψ be solutions of the equations

$$\varphi_t + \partial_x T_{B^0[u]}\varphi + \mathcal{R}(u) = L[(2 - T_{B^{\log}[u]})\varphi]_x,$$
$$\psi_t + \partial_x T_{B^0[v]}\psi + \mathcal{R}(v) = L[(2 - T_{B^{\log}[v]})\psi]_x,$$

with the same initial data. Taking the difference of these equations, we get

$$\begin{aligned}&(\varphi-\psi)_t+\partial_x T_{B^0[u]}(\varphi-\psi)\\&\qquad=L[(2-T_{B^{\log}[u]})(\varphi-\psi)]_x+\partial_x(T_{B^0[v]}-T_{B^0[u]})\psi-L[(T_{B^{\log}[u]}-T_{B^{\log}[v]})\psi]_x+\mathcal{R}(v)-\mathcal{R}(u).\end{aligned}$$

Applying Proposition 4.3 with

$$\Upsilon=-\partial_x(T_{B^0[v]}-T_{B^0[u]})\psi+L[(T_{B^{\log}[u]}+T_{B^{\log}[v]})\psi]_x+\mathcal{R}(u)-\mathcal{R}(v),$$

we obtain that, for $k\le 3$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}|D|^k(\varphi-\psi)(2-T_{B^{\log}[u]})^{2k+1}|D|^k(\varphi-\psi)\,\mathrm{d}x\\&\qquad\lesssim(\|u_x\|_{W^{3,\infty}}+\|u_{tx}\|_{W^{1,\infty}}+\|Lu_x\|_{W^{2,\infty}})^2F_1(\|u_x\|_{W^{3,\infty}}+\|u_{tx}\|_{W^{1,\infty}}+\|Lu_x\|_{W^{2,\infty}})\|\varphi-\psi\|_{H^k}^2\\&\qquad\quad+F_2(\|u_x\|_{W^{3,\infty}}+\|u_{tx}\|_{W^{1,\infty}}+\|Lu_x\|_{W^{2,\infty}}+\|v_x\|_{W^{3,\infty}}+\|v_{tx}\|_{W^{1,\infty}}+\|Lv_x\|_{W^{2,\infty}})\\&\qquad\qquad\cdot\|u-v\|_{H^4}\|\varphi-\psi\|_{H^k}\|\psi_x\|_{W^{k,\infty}},\end{aligned}$$

where F_1, F_2 are positive, continuous functions. Since $\varphi=\psi$ at $t=0$, we have, by Gronwall's inequality,

$$\|(\varphi-\psi)(t)\|_{H^k}\lesssim\int_0^t e^{\int_\tau^t F_1(\cdots)(s)\,\mathrm{d}s}F_2(\cdots)(\tau)\|(u-v)(\,\cdot\,,\tau)\|_{H^3}\|\psi_x(\,\cdot\,,\tau)\|_{W^{k,\infty}}\,\mathrm{d}\tau,$$

where $(\cdots)$ denotes the same arguments as above.

By the H^s-energy estimate in Proposition 4.3 with $s\ge 5$, the function

$$e^{\int_\tau^t F_1(\cdots)(s)\,\mathrm{d}s}F_2(\cdots)(\tau)$$

is bounded by a positive constant $C(C_M, C_0, s)$ on $[0, T]$ for $k\le 3$. Thus, by taking $T<1/C(C_M, C_0, s)$, we deduce that

$$\|\varphi-\psi\|_{L^\infty(0,T;H^3)}\le\lambda\|u-v\|_{L^\infty(0,T;H^3)}$$

for some $0<\lambda<1$. □

Proposition 4.6 implies that for any $s\ge 6$ and $\varphi_0\in H^s(\mathbb{R})$ that satisfies the conditions in Proposition 4.5, the sequence $\{\varphi^{(i)}\}$ defined in (4-16), which share a common life span $[0, T]$ with T is only related to the constants C_M, C_0, s, is a Cauchy sequence with respect to $\|\cdot\|_{L_t^\infty H_x^3}$. In fact, for any $0<\epsilon<1$, there is a positive integer $N=\log_\lambda(1-\lambda)\epsilon/(3\overline{C})$ such that, for all $i>j>N$,

$$\begin{aligned}\|\varphi^{(i)}-\varphi^{(j)}\|_{L_t^\infty H_x^3}&\le\|\varphi^{(i)}-\varphi^{(i-1)}\|_{L_t^\infty H_x^3}+\cdots+\|\varphi^{(j+1)}-\varphi^{(j)}\|_{L_t^\infty H_x^3}\\&\le(\lambda^{i-N-1}+\cdots+\lambda^{j-N})\|\varphi^{(N+1)}-\varphi^{(N)}\|_{L_t^\infty H_x^3}\\&\le(\lambda^{i-N-1}+\cdots+\lambda^{j-N})\lambda^N\|\varphi^{(1)}-\varphi^{(0)}\|_{L_t^\infty H_x^3}\le\frac{3\overline{C}}{1-\lambda}\lambda^N<\epsilon.\end{aligned}$$

So $\varphi=\lim_{j\to\infty}\varphi^{(j)}$ exists in $L^\infty([0, T]; H^3(\mathbb{R}))$. From Proposition 4.5, the sequence is bounded in $L^\infty([0, T]; H^s(\mathbb{R}))$, and it follows by a weak compactness argument that $\varphi\in L^\infty([0, T]; H^s(\mathbb{R}))$. Hence, φ is a fixed point of $\boldsymbol{G}$ and a solution of (1-1).

4C. ***Continuity in time.*** Next, we prove that the solution just constructed is a continuous function of time with values in H^s. First, we notice that $\varphi_t \in L^\infty([0,T]; H^{s'}(\mathbb{R}))$ for any $s' < s-1$, which implies that $\varphi \in C([0,T]; H^{s'}(\mathbb{R}))$.

The equation is time-reversible and translation-invariant in time, so it suffices to prove that

$$\lim_{t\to 0+} \|\varphi(t)-\varphi(0)\|_{H^s} = 0.$$

Since $\|\varphi(t)\|_{H^s}$ is bounded on $[0,T]$ and $\varphi(t) \to \varphi(0)$ strongly in $H^{s'}$, the weak H^s-limit of any convergent subsequence is unique, and we see that $\varphi(t)$ converges to $\varphi(0)$ in the weak H^s-topology. To show convergence in the strong H^s-topology, we only need to prove the norm-convergence

$$\lim_{t\to 0+} \|\varphi(t)\|_{H^s} = \|\varphi(0)\|_{H^s}. \tag{4-18}$$

Writing the weighted energy in (4-4) as $\widetilde{E}^{(s)}(t) = \widetilde{E}^{(s)}[\varphi](t)$, we have from the a priori estimate in Proposition 4.4 that

$$\begin{aligned}
&\widetilde{E}^{(s)}(0) - \int_0^t (\|\varphi_x(\tau)\|_{W^{3,\infty}} + \|L\varphi_x(\tau)\|_{W^{2,\infty}})^2 F(\|\varphi_x(\tau)\|_{W^{3,\infty}} + \|L\varphi_x(\tau)\|_{W^{2,\infty}}) \widetilde{E}^{(s)}(\tau)\,\mathrm{d}\tau \\
&\lesssim \widetilde{E}^{(s)}(t) \lesssim \widetilde{E}^{(s)}(0) + \int_0^t (\|\varphi_x(\tau)\|_{W^{3,\infty}} + \|L\varphi_x(\tau)\|_{W^{2,\infty}})^2 F(\|\varphi_x(\tau)\|_{W^{3,\infty}} + \|L\varphi_x(\tau)\|_{W^{2,\infty}}) \widetilde{E}^{(s)}(\tau)\,\mathrm{d}\tau,
\end{aligned}$$

and by Gronwall's inequality,

$$\begin{aligned}
&\widetilde{E}^{(s)}(0) e^{-\int_0^t (\|\varphi_x(\tau)\|_{W^{3,\infty}} + \|L\varphi_x(\tau)\|_{W^{2,\infty}})^2 F(\|\varphi_x(\tau)\|_{W^{3,\infty}} + \|L\varphi_x(\tau)\|_{W^{2,\infty}})\,\mathrm{d}\tau} \\
&\qquad \le \widetilde{E}^{(s)}(t) \le \widetilde{E}^{(s)}(0) e^{\int_0^t (\|\varphi_x(\tau)\|_{W^{3,\infty}} + \|L\varphi_x(\tau)\|_{W^{2,\infty}})^2 F(\|\varphi_x(\tau)\|_{W^{3,\infty}} + \|L\varphi_x(\tau)\|_{W^{2,\infty}})\,\mathrm{d}\tau}.
\end{aligned}$$

Using the notation in (4-6), we define an equivalent weighted norm on H^s by

$$\|\varphi\|_{H^s_w} = (\widetilde{E}^{(s)}_{\varphi(0)}[\varphi](t))^{1/2}.$$

Then

$$\lim_{t\to 0+} \widetilde{E}^{(s)}_{\varphi(0)}(t) = \lim_{t\to 0+} \widetilde{E}^{(s)}(t) = \widetilde{E}^{(s)}_{\varphi(0)}(0),$$

so (4-18) holds, which proves that $\varphi \in C([0,T]; H^s(\mathbb{R}))$.

4D. ***Lipschitz continuous dependence on H^r.*** To prove the stability estimate (4-3), we suppose that $s \ge 7$ is an integer, and $\varphi, \psi \in C([0,T]; H^s(\mathbb{R}))$ are solutions of (1-1) with initial data φ_0 and ψ_0, respectively. Then

$$\begin{aligned}
\varphi_t + \partial_x T_{B^0[\varphi]}\varphi + \mathcal{R}(\varphi) &= L[(2 - T_{B^{\log}[\varphi]})\varphi]_x, \quad \varphi(0) = \varphi_0, \\
\psi_t + \partial_x T_{B^0[\psi]}\psi + \mathcal{R}(\psi) &= L[(2 - T_{B^{\log}[\psi]})\psi]_x, \quad \psi(0) = \psi_0.
\end{aligned}$$

Taking the difference of these equations, we obtain an equation for $\varphi - \psi$:

$$\begin{aligned}
&(\varphi-\psi)_t + \partial_x T_{B^0[\varphi]}(\varphi-\psi) + \mathcal{R}(\varphi) - \mathcal{R}(\psi) \\
&\qquad = L[(2 - T_{B^{\log}[\varphi]})(\varphi-\psi)]_x - \partial_x T_{B^0[\varphi]-B^0[\psi]}\psi - L[(T_{B^{\log}[\varphi]-B^{\log}[\psi]})\psi]_x.
\end{aligned} \tag{4-19}$$

Using (3-8), we compute that

$$
\begin{aligned}
&B_n^{\log}[\varphi](\cdot,\xi)-B_n^{\log}[\psi](\cdot,\xi)\\
&=-\mathcal{F}_\zeta^{-1}\biggl\{2c_n\int_{\mathbb{R}^{2n}}\delta\Bigl(\zeta-\sum_{j=1}^{2n}\eta_j\Bigr)\sum_{k=1}^{2n}\prod_{j=1}^{k-1}\Bigl[i\eta_j\hat{\varphi}(\eta_j)\chi\Bigl(\frac{(2n+1)\eta_j}{\xi}\Bigr)\Bigr]\Bigl[i\eta_k(\hat{\varphi}(\eta_k)-\hat{\psi}(\eta_k))\chi\Bigl(\frac{(2n+1)\eta_k}{\xi}\Bigr)\Bigr]\\
&\qquad\cdot\prod_{j=k+1}^{2n}\Bigl[i\eta_j\hat{\psi}(\eta_j)\chi\Bigl(\frac{(2n+1)\eta_j}{\xi}\Bigr)\Bigr]\,\mathrm{d}\hat{\boldsymbol{\eta}}_n\biggr\}.
\end{aligned}
$$

By an argument similar to the one in Proposition 3.1, we get the following symbol estimates:

$$
\|B_n^{\log}[\varphi]-B_n^{\log}[\psi]\|_{\mathcal{M}_{(j,2)}}\lesssim C(n,s)|c_n|\sum_{k=1}^{2n}\|\varphi_x\|_{W^{j,\infty}}^{k-1}\|\varphi_x-\psi_x\|_{W^{j,\infty}}\|\psi_x\|_{W^{j,\infty}}^{2n-k},\quad j=2,3,
$$

so that

$$
\|B^{\log}[\varphi]-B^{\log}[\psi]\|_{\mathcal{M}_{(j,2)}}\lesssim\sum_{n=1}^{\infty}C(n,s)|c_n|\sum_{k=1}^{2n}\|\varphi_x\|_{W^{j,\infty}}^{k-1}\|\varphi_x-\psi_x\|_{W^{j,\infty}}\|\psi_x\|_{W^{j,\infty}}^{2n-k},\quad j=2,3.
$$

Similarly, we have

$$
\begin{aligned}
&\|B^0[\varphi]-B^0[\psi]\|_{\mathcal{M}_{(2,2)}}\\
&\lesssim\sum_{n=1}^{\infty}C(n,s)|c_n|\sum_{k=1}^{2n}\bigl(\|L\varphi_x\|_{W^{2,\infty}}^{k-1}\|L(\varphi-\psi)_x\|_{W^{2,\infty}}\|L\psi_x\|_{W^{2,\infty}}^{2n-k}+\|\varphi_x\|_{W^{2,\infty}}^{k-1}\|(\varphi-\psi)_x\|_{W^{2,\infty}}\|\psi_x\|_{W^{2,\infty}}^{2n-k}\bigr),
\end{aligned}
$$

and

$$
\begin{aligned}
\|\mathcal{R}(\varphi)-\mathcal{R}(\psi)\|_{H^s}\lesssim\|\varphi-\psi\|_{H^s}\biggl\{&\sum_{n=1}^{\infty}C(n,s)|c_n|\sum_{k=1}^{2n}\bigl(\|L\varphi_x\|_{W^{2,\infty}}^{k-1}\|L(\varphi-\psi)_x\|_{W^{2,\infty}}\|L\psi_x\|_{W^{2,\infty}}^{2n-k}\\
&+\|\varphi_x\|_{W^{2,\infty}}^{k-1}\|(\varphi-\psi)_x\|_{W^{2,\infty}}\|\psi_x\|_{W^{2,\infty}}^{2n-k}\bigr)\biggr\}.
\end{aligned}
$$

We apply ∂_x^r to (4-19) for $\varphi-\psi$, and carry out weighted energy estimates for the homogeneous energies $E_\varphi^{(r)}[\varphi-\psi]$ defined by (4-4). We then get that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}E_\varphi^{(r)}[\varphi-\psi](t)\\
&\le(\|\varphi_x\|_{W^{3,\infty}}+\|\psi_x\|_{W^{3,\infty}})^2\mathcal{P}_1(\|\varphi\|_{H^r},\|\psi\|_{H^r})E_\varphi^{(r)}[\varphi-\psi](t)\\
&\quad+(\|\varphi_x\|_{W^{3,\infty}}+\|\psi_x\|_{W^{3,\infty}})\mathcal{P}_2(\|\varphi\|_{H^r},\|\psi\|_{H^r})\|L\psi(t)\|_{H^{r+1}}\|(\varphi-\psi)(t)\|_{H^r}\|(\varphi-\psi)(t)\|_{H^5},
\end{aligned}\tag{4-20}
$$

where $\mathcal{P}_1$ and $\mathcal{P}_2$ are positive polynomials. Here we have used the estimates

$$
\|\partial_x^{r+1}LT_{B^{\log}[\varphi]-B^{\log}[\psi]}\psi\|_{L^2}\lesssim\|\partial_x^{r+1}L\psi\|_{L^2}\sum_{n=1}^{\infty}C(n,s)|c_n|\sum_{k=1}^{2n}\|\varphi_x\|_{W^{3,\infty}}^{k-1}\|\varphi_x-\psi_x\|_{W^{2,\infty}}\|\psi_x\|_{W^{3,\infty}}^{2n-k},
$$

and

$$
\begin{aligned}
\|\partial_x^{r+1}T_{B^0[\varphi]-B^0[\psi]}\psi\|_{L^2}\lesssim\|\partial_x^{r+1}\psi\|_{L^2}&\sum_{n=1}^{\infty}C(n,s)|c_n|\sum_{k=1}^{2n}\bigl(\|L\varphi_x\|_{W^{2,\infty}}^{k-1}\|L(\varphi-\psi)_x\|_{W^{2,\infty}}\|L\psi_x\|_{W^{2,\infty}}^{2n-k}\\
&+\|\varphi_x\|_{W^{2,\infty}}^{k-1}\|(\varphi-\psi)_x\|_{W^{2,\infty}}\|\psi_x\|_{W^{2,\infty}}^{2n-k}\bigr).
\end{aligned}
$$

Thus, if $5 \le r < s-1$, Gronwall's inequality implies that

$$\widetilde{E}_{\varphi}^{(r)}[\varphi-\psi](t) \le \widetilde{E}_{\varphi}^{(r)}[\varphi-\psi](0)\exp\int_0^t (\|\varphi\|_{H^r}+\|\psi\|_{H^r})^2\mathcal{P}_1(\|\varphi\|_{H^r},\|\psi\|_{H^r}) + (\|\varphi\|_{H^r}+\|\psi\|_{H^r})\mathcal{P}_2(\|\varphi\|_{H^r},\|\psi\|_{H^r})\|L\psi\|_{H^{r+1}}\,\mathrm{d}t.$$

On the other hand, when $0 \le r < 5$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{E}_{\varphi}^{(r)}[\varphi-\psi](t)\\
&\le (\|\varphi\|_{H^5}+\|\psi\|_{H^5})^2\mathcal{P}_1(\|\varphi\|_{H^5},\|\psi\|_{H^5})\widetilde{E}_{\varphi}^{(r)}[\varphi-\psi](t)\\
&\qquad +\|B^0[\varphi]-B^0[\psi]\|_{L^2}\|\psi\|_{H^5}\|\varphi-\psi\|_{H^r}+\|B^{\log}[\varphi]-B^{\log}[\psi]\|_{L^2}\|L\psi\|_{H^5}\|\varphi-\psi\|_{H^r}\\
&\le \widetilde{E}_{\varphi}^{(r)}[\varphi-\psi](t)\{(\|\varphi\|_{H^5}+\|\psi\|_{H^5})^2\mathcal{P}_1(\|\varphi\|_{H^5},\|\psi\|_{H^5})+\|L\psi\|_{H^5}\mathcal{P}_2(\|\varphi\|_{H^5},\|\psi\|_{H^5})(\|\varphi\|_{H^5}+\|\psi\|_{H^5})\},
\end{aligned}$$

and Gronwall's inequality implies that

$$\widetilde{E}_{\varphi}^{(r)}[\varphi-\psi](t) \le \widetilde{E}_{\varphi}^{(r)}[\varphi-\psi](0)\exp\int_0^t \big\{(\|\varphi\|_{H^5}+\|\psi\|_{H^5})^2\mathcal{P}_1(\|\varphi\|_{H^5},\|\psi\|_{H^5}) + \|L\psi\|_{H^5}\mathcal{P}_2(\|\varphi\|_{H^5},\|\psi\|_{H^5})(\|\varphi\|_{H^5}+\|\psi\|_{H^5})\big\}\,\mathrm{d}t.$$

Here we have used the fact that $(E_{\varphi}^{(r)})^{1/2}$ and the H^r-norm are equivalent, as in (4-17).

In either case, the stability estimate (4-3) follows.

4E. *Continuous dependence on H^s.* Assume $s \ge 7$, and suppose that $\varphi \in C([0,T];H^s(\mathbb{R}))$ is a solution of (1-1) with initial data $\varphi(0)=\varphi_0$.

For $\epsilon > 0$, define a smoothing operator J_ϵ by

$$J_\epsilon f(x) = \mathcal{F}^{-1}[\nu(\epsilon\,\cdot\,)\hat{f}(\,\cdot\,)](x),$$

where $\nu(\xi)$ is a smooth bump function supported on $|\xi| \le 2$ and equal to 1 on $|\xi| \le 1$.

Let $\varphi^{(\epsilon)}$ be a smooth approximate solution for φ with initial data

$$\varphi^{(\epsilon)}(0) = \varphi_0^{(\epsilon)}, \quad \varphi_0^{(\epsilon)} = J_\epsilon\varphi_0,$$

so that $\|\varphi_0-\varphi_0^{(\epsilon)}\|_{H^s} \to 0$ as $\epsilon \to 0+$. Note that the smoothed initial data also satisfy the conditions

$$\|B^{\log}[J_\epsilon\varphi_0]\|_{\mathcal{M}_{(1,1)}} < C_0, \quad \sum_{n=1}^{\infty}\widetilde{C}^n|c_n|(\|\partial_x J_\epsilon\varphi_0\|_{W^{3,\infty}}^{2n}+\|L\partial_x J_\epsilon\varphi_0\|_{W^{2,\infty}}^{2n}) < C_M.$$

Since the life span T is only related to the constants C_0, C_M, s, the solutions $\varphi^{(\epsilon)}$ and φ share the common life span $[0,T]$.

We shall prove that

$$\sup_{t\in[0,T]}\|\varphi(t)-\varphi^{(\epsilon)}(t)\|_{H^s} \to 0 \quad \text{as } \epsilon \to 0+. \tag{4-21}$$

Taking $\psi = \varphi^{(\epsilon)}$ and $r = s$ in (4-20) in the above stability argument, we obtain that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} E_\varphi^{(s)}[\varphi - \varphi^{(\epsilon)}](t) \\
&\quad \le (\|\varphi\|_{H^s} + \|\varphi^{(\epsilon)}\|_{H^s})^2 \mathcal{P}_1(\|\varphi\|_{H^s}, \|\varphi^{(\epsilon)}\|_{H^s}) E_\varphi^{(s)}[\varphi - \varphi^{(\epsilon)}](t) \\
&\qquad\qquad + (\|\varphi\|_{H^s} + \|\varphi^{(\epsilon)}\|_{H^s}) \mathcal{P}_2(\|\varphi\|_{H^s}, \|\varphi^{(\epsilon)}\|_{H^s}) \|L\varphi^{(\epsilon)}\|_{H^{s+1}} \|(\varphi - \varphi^{(\epsilon)})(t)\|_{H^s} \|(\varphi - \varphi^{(\epsilon)})(t)\|_{H^5}.
\end{aligned}$$

Notice that

$$\begin{gathered}
\|L\varphi^{(\epsilon)}\|_{H^{s+1}} \lesssim |\epsilon^{-1}\log\epsilon| \cdot \|\varphi^{(\epsilon)}\|_{H^s}, \\
|\epsilon^{-1}\log\epsilon| \|(\varphi - \varphi^{(\epsilon)})(t)\|_{H^5} \lesssim |\epsilon^{-2}| \|(\varphi - \varphi^{(\epsilon)})(t)\|_{H^5} \lesssim \|(\varphi - \varphi^{(\epsilon)})(t)\|_{H^7}.
\end{gathered}$$

Therefore, since $s \ge 7$ and $\|\varphi^{(\epsilon)}\|_{H^s} \le \|\varphi\|_{H^s}$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} E_\varphi^{(s)}[\varphi - \varphi^{(\epsilon)}](t) \le \|\varphi\|_{H^s}^2 \mathcal{P}_1(\|\varphi\|_{H^s}, \|\varphi^{(\epsilon)}\|_{H^s}) E_\varphi^{(s)}[\varphi - \varphi^{(\epsilon)}](t)& \\
+ \|\varphi\|_{H^s}^2 \mathcal{P}_2(\|\varphi\|_{H^s}, \|\varphi^{(\epsilon)}\|_{H^s}) \|(\varphi - \varphi^{(\epsilon)})(t)\|_{H^s}^2&.
\end{aligned}$$

Thus, we obtain

$$E_\varphi^{(s)}[\varphi - \varphi^{(\epsilon)}](t) \le E_\varphi^{(s)}[\varphi - \varphi^{(\epsilon)}](0) \exp \int_0^t \|\varphi\|_{H^s}^2 (\mathcal{P}_1 + \mathcal{P}_2)(\|\varphi\|_{H^s}, \|\varphi^{(\epsilon)}\|_{H^s}) \,\mathrm{d}\tau, \tag{4-22}$$

which implies that (4-21) holds.

Finally, suppose that $\{\varphi_{0,n} \mid n \in \mathbb{N}\}$ is a sequence in $H^s(\mathbb{R})$ with $\varphi_{0,n} \to \varphi_0$ in $H^s(\mathbb{R})$ as $n \to \infty$, and $\varphi_n \in C([0, T_n]; H^s(\mathbb{R}))$ are solutions of (1-1) with initial data $\varphi_{0,n}$. By Sobolev embedding,

$$\begin{gathered}
\lim_{n\to\infty} \|B^{\log}[\varphi_{0,n}]\|_{\mathcal{M}(1,1)} = \|B^{\log}[\varphi_0]\|_{\mathcal{M}(1,1)} < C_0, \\
\lim_{n\to\infty} \sum_{k=1}^{\infty} \widetilde{C}^k |c_k| (\|\partial_x \varphi_{0,n}\|_{W^{3,\infty}}^{2k} + \|L\partial_x \varphi_{0,n}\|_{W^{2,\infty}}^{2k}) = \sum_{k=1}^{\infty} \widetilde{C}^k |c_k| (\|\partial_x \varphi_0\|_{W^{3,\infty}}^{2k} + \|L\partial_x \varphi_0\|_{W^{2,\infty}}^{2k}) < C_M.
\end{gathered}$$

Therefore, there exists $N_1 \in \mathbb{N}$ such that for $n > N_1$ we have

$$\|B^{\log}[\varphi_{0,n}]\|_{\mathcal{M}(1,1)} < C_0, \quad \sum_{k=1}^{\infty} \widetilde{C}^k |c_k| (\|\partial_x \varphi_{0,n}\|_{W^{3,\infty}}^{2k} + \|L\partial_x \varphi_{0,n}\|_{W^{2,\infty}}^{2k}) < C_M.$$

Hence, the φ_n share the same life span as φ for $n > N_1$.

Denote by $\varphi^{(\epsilon)}$ and $\varphi_n^{(\epsilon)}$ the solutions with initial data $J_\epsilon \varphi_0$ and $J_\epsilon \varphi_{0,n}$, respectively. Then

$$\|\varphi - \varphi_n\|_{H^s} \le \|\varphi - \varphi^{(\epsilon)}\|_{H^s} + \|\varphi^{(\epsilon)} - \varphi_n^{(\epsilon)}\|_{H^s} + \|\varphi_n^{(\epsilon)} - \varphi_n\|_{H^s}.$$

Fix $\delta > 0$. In view of (4-22) and the fact that the H^s-norms of $\|\varphi_n\|$ are uniformly bounded, by taking ϵ small enough, we can make

$$\sup_{t\in[0,T]} \|\varphi(t) - \varphi^{(\epsilon)}(t)\|_{H^s} < \tfrac{1}{3}\delta, \qquad \sup_{t\in[0,T]} \|\varphi_n^{(\epsilon)}(t) - \varphi_n(t)\|_{H^s} < \tfrac{1}{3}\delta \quad \text{for all } n \in \mathbb{N}.$$

The stability estimate (4-3) with $r = s$ implies that there exists $N_2 > N_1$ such that

$$\sup_{t\in[0,T]} \|\varphi^{(\epsilon)} - \varphi_n^{(\epsilon)}\|_{H^s} < \tfrac{1}{3}\delta \quad \text{for all } n > N_2.$$

It follows that

$$\sup_{t\in[0,T]} \|\varphi - \varphi_n\|_{H^s} < \delta \quad \text{for all } n > N_2,$$

and since $\delta > 0$ is arbitrary, we conclude that

$$\lim_{n\to\infty} \|\varphi - \varphi_n\|_{C([0,T];H^s)} = 0,$$

which proves the continuous dependence of the solution map on H^s.

5. Global solution for small initial data

Beginning with this section, we address the global well-posedness of (1-1) with small initial data. From now on, we fix the parameter values

$$s = 1200, \quad r = 1, \quad p_0 = 10^{-4}. \tag{5-1}$$

The front equation (1-1) is invariant under the transformation

$$x \mapsto \lambda(x + 2\log|\lambda|t), \quad t \mapsto \lambda t, \quad \varphi \mapsto \lambda\varphi.$$

The scaling-Galilean part of this transformation is generated by the vector field

$$\mathcal{S} = (x + 2t)\partial_x + t\partial_t, \tag{5-2}$$

and the linearized equation $\varphi_t = 2\log|\partial_x|\varphi_x$ commutes with $\mathcal{S}$ (see Lemma 7.1). We also introduce the notation

$$h(x,t) = e^{-2t\partial_x \log|\partial_x|}\varphi(x,t), \quad \hat{h}(\xi,t) = e^{-2it\xi\log|\xi|}\hat{\varphi}(\xi,t) \tag{5-3}$$

for the function h obtained by removing the action of the linearized evolution group on φ. When convenient, we write $h(\,\cdot\,,t) = h(t)$, $\varphi(\,\cdot\,,t) = \varphi(t)$.

Our global existence theorem is as follows.

Theorem 5.1. *Let s, r, p_0 be defined as in* (5-1). *There exists a constant $0 < \bar{\epsilon} \ll 1$ such that if $\varphi_0 \in H^s(\mathbb{R})$ satisfies*

$$\|\varphi_0\|_{H^s} + \|x\partial_x\varphi_0\|_{H^r} \le \epsilon_0$$

for some $0 < \epsilon_0 \le \bar{\epsilon}$, then there exists a unique global solution $\varphi \in C([0,\infty); H^s(\mathbb{R}))$ of (1-1). *Moreover, this solution satisfies*

$$\|\varphi(t)\|_{H^s} + \|\mathcal{S}\varphi(t)\|_{H^r} \lesssim \epsilon_0(t+1)^{p_0},$$

where $\mathcal{S}$ is the vector field defined in (5-2).

To prove this theorem, given the local existence theory, we only need to show that the criterion (4-2) in Theorem 4.1 is never satisfied. In particular, Lemma 5.3 below, together with the symbol estimates in Proposition 3.1, guarantees that the life-span of solutions can be extended to infinity.

In order to derive the global a priori estimates, we introduce the Z-norm of a function $f \in L^2(\mathbb{R})$, defined by

$$\|f\|_Z = \|(|\xi| + |\xi|^{r+4})\hat{f}(\xi)\|_{L^\infty_\xi}, \tag{5-4}$$

and prove the global bounds in Theorem 5.1 by use of the following bootstrap argument.

Proposition 5.2 (bootstrap). *Let $T > 1$ and suppose that $\varphi \in C([0, T]; H^s)$ is a solution of* (1-1), *where the initial data satisfies*

$$\|\varphi_0\|_{H^s} + \|x\partial_x\varphi_0\|_{H^r} \le \epsilon_0$$

for some $0 < \epsilon_0 \ll 1$. If there exists $\epsilon_0 \ll \epsilon_1 \lesssim \epsilon_0^{1/3}$ such that the solution satisfies

$$(t+1)^{-p_0}(\|\varphi(t)\|_{H^s} + \|\mathcal{S}\varphi(t)\|_{H^r}) + \|\varphi\|_Z \le \epsilon_1$$

for every $t \in [0, T]$, then the solution satisfies an improved bound

$$(t+1)^{-p_0}(\|\varphi(t)\|_{H^s} + \|\mathcal{S}\varphi(t)\|_{H^r}) + \|\varphi\|_Z \lesssim \epsilon_0.$$

We call the assumptions in Proposition 5.2 the *bootstrap assumptions*. To prove the proposition, we establish the following lemmas, most of whose proofs are deferred to the following sections. As before, we let L denote the Fourier multiplier (1-2).

Lemma 5.3 (sharp pointwise decay). *Under the bootstrap assumptions*,

$$\||\partial_x|^{r+2}\varphi_x(t)\|_{L^\infty} + \|L\varphi_x(t)\|_{L^\infty} \lesssim \epsilon_1(t+1)^{-1/2}.$$

Lemma 5.4 (scaling-Galilean estimate). *Under the bootstrap assumptions*,

$$(t+1)^{-p_0}\|\mathcal{S}\varphi(t)\|_{H^r} \lesssim \epsilon_0.$$

Lemma 5.5. *Under the bootstrap assumptions*,

$$(t+1)^{-p_0}(\|\varphi(t)\|_{H^s} + \|x\partial_x\varphi(t)\|_{H^r}) \lesssim \epsilon_0.$$

Proof. Recall the energy estimate (4-15)

$$\widetilde{E}^{(s)}(t) \lesssim \widetilde{E}^{(s)}(0)e^{\int_0^t F(\|\varphi_x(\tau)\|_{W^{3,\infty}} + \|L\varphi_x(\tau)\|_{W^{2,\infty}})(\|\varphi_x(\tau)\|_{W^{3,\infty}} + \|L\varphi_x(\tau)\|_{W^{2,\infty}})^2\,d\tau}.$$

From Lemma 5.3, we have

$$F(\|\varphi_x(\tau)\|_{W^{3,\infty}} + \|L\varphi_x(\tau)\|_{W^{2,\infty}}) \lesssim 1,$$
$$\|\varphi_x(\tau)\|_{W^{3,\infty}} + \|L\varphi_x(\tau)\|_{W^{2,\infty}} \lesssim (t+1)^{-1/2}\epsilon_1,$$

which implies that

$$\widetilde{E}^{(s)}(t) \lesssim \epsilon_0^2(t+1)^{C\epsilon_1^2}$$

for some constant C, so once $\epsilon_1^2 \ll p_0$, we have

$$(t+1)^{-p_0}\|\varphi\|_{H^s} \lesssim \epsilon_0.$$

Next, we observe that we can use $\|\mathcal{S}\varphi\|_{H^r}$ to control $\|x\partial_x h\|_{H^r}$. It follows from (5-3), the definition of $\mathcal{S}$, and (3-2) that

$$\begin{aligned}
\mathcal{F}_x[x\partial_x h](\xi) &= -\partial_\xi(\xi\hat{h}(\xi)) = -\hat{h}(\xi) - \xi\partial_\xi\hat{h}(\xi),\\
\xi\partial_\xi\hat{h}(\xi,t) &= \xi e^{-2it\xi\log|\xi|}(-2it(\log|\xi|+1)\hat{\varphi}(\xi,t) + \partial_\xi\hat{\varphi}(\xi,t))\\
&= e^{-2it\xi\log|\xi|}[\xi\partial_\xi\hat{\varphi}(\xi,t) - (2it\xi-1)\hat{\varphi}(\xi,t) - t\hat{\varphi}_t(\xi,t) - t\widehat{\mathcal{N}}(\xi,t) - \hat{\varphi}(\xi,t)]\\
&= e^{-2it\xi\log|\xi|}[-\widehat{\mathcal{S}\varphi}(\xi,t) - \hat{\varphi}(\xi,t) - t\widehat{\mathcal{N}}(\xi,t)],
\end{aligned} \tag{5-5}$$

where $\mathcal{N}$ denotes the nonlinear term in (3-2), which satisfies the estimate

$$\||\partial_x|^j\mathcal{N}\|_{L^2} \lesssim \sum_{n=1}^{\infty}(\|\varphi_x\|_{W^{3,\infty}}^{2n} + \|L\varphi_x\|_{W^{2,\infty}}^{2n})\|\varphi\|_{H^{j+2}} \quad \text{for all } j = 0,\dots,r. \tag{5-6}$$

By the bootstrap assumptions, Lemma 5.3, and Lemma 5.4 we then find that

$$(t+1)^{-p_0}\|x\partial_x h(t)\|_{H^r} \lesssim \epsilon_0,$$

and the same estimate holds for φ in view of (5-3). □

Lemma 5.6 (nonlinear dispersive estimate). *Under the bootstrap assumptions, the solution of* (1-1) *satisfies*

$$\|\varphi(t)\|_Z \lesssim \epsilon_0.$$

Proposition 5.2 then follows by combining Lemmas 5.3–5.6. Lemma 5.3 will be proved in Section 6, Lemma 5.4 will be proved in Section 7, and Lemma 5.6 will be proved in Section 8.

6. Linear dispersive estimate

In this section, we prove a dispersive estimate for the linearized evolution operator $e^{2t\partial_x\log|\partial_x|}$ defined in (5-3) and use it to prove Lemma 5.3. We recall that P_k and $\widetilde{P}_k$ are the frequency-localization operators with symbols ψ_k and $\tilde{\psi}_k$, respectively (see (2-13)).

Lemma 6.1. *For $t > 0$ and $f \in L^2$, we have the linear dispersive estimates*

$$\|e^{2t\partial_x\log|\partial_x|}P_k f\|_{L^\infty} \lesssim (t+1)^{-1/2}2^{k/2}\|\widehat{P_k f}\|_{L^\infty_\xi} + (t+1)^{-3/4}2^{-k/4}[\|P_k(x\partial_x f)\|_{L^2} + \|\widetilde{P}_k f\|_{L^2}]. \tag{6-1}$$

Proof. Using the inverse Fourier transform, we can write the solution as

$$e^{2t\partial_x\log|\partial_x|}P_k f = \int_{\mathbb{R}} e^{ix\xi+2i(\xi\log|\xi|)t}\psi_k(\xi)\hat{f}(\xi)\,\mathrm{d}\xi.$$

Since

$$\partial_\xi e^{ix\xi+2i(\xi\log|\xi|)t} = [ix + 2it(\log|\xi|+1)]e^{ix\xi+2i(\xi\log|\xi|)t}, \tag{6-2}$$

we can integrate by parts to get

$$
\begin{aligned}
&\|e^{2t\partial_x \log|\partial_x|} P_k f\|_{L^\infty} \\
&= \left\| \int_{\mathbb{R}} e^{ix\xi + 2i(\xi \log|\xi|)t} \hat{f}(\xi)\psi_k(\xi)\,\mathrm{d}\xi \right\|_{L^\infty} \\
&= \left\| \int_{\mathbb{R}} \frac{1}{ix+2it(\log|\xi|+1)} \partial_\xi e^{ix\xi+2i(\xi\log|\xi|)t} \hat{f}(\xi)\psi_k(\xi)\,\mathrm{d}\xi \right\|_{L^\infty} \\
&= \left\| \int_{\mathbb{R}} e^{ix\xi+2i(\xi\log|\xi|)t} \partial_\xi \left(\frac{1}{ix+2it(\log|\xi|+1)} \hat{f}(\xi)\psi_k(\xi) \right) \mathrm{d}\xi \right\|_{L^\infty} \\
&= \left\| \int_{\mathbb{R}} e^{ix\xi+2i(\xi\log|\xi|)t} \left(\frac{-2it}{\xi[ix+2it(\log|\xi|+1)]^2} \hat{f}(\xi)\psi_k(\xi) + \frac{1}{ix+2it(\log|\xi|+1)} \psi_k(\xi)\partial_\xi \hat{f}(\xi) \right.\right. \\
&\qquad\qquad \left.\left. + \frac{1}{ix+2it(\log|\xi|+1)} \hat{f}(\xi)\psi_k'(\xi) \right) \mathrm{d}\xi \right\|_{L^\infty}.
\end{aligned}
$$

(1) If $|ix+2it(\log|\xi|+1)| \gtrsim (t+1)$, we use (2-14) and get

$$
\begin{aligned}
\|e^{2t\partial_x \log|\partial_x|} P_k f\|_{L^\infty} &\lesssim \frac{1}{t+1} \int_{\mathbb{R}} |\xi^{-1} \hat{f}(\xi)\psi_k(\xi)| + |\psi_k(\xi)\partial_\xi \hat{f}(\xi)| + |\hat{f}(\xi)\psi_k'(\xi)|\,\mathrm{d}\xi \\
&\lesssim \frac{1}{t+1} [2^{-k/2} \|\widehat{P_k f}\|_{L^2_\xi} + 2^{-k/2} \|P_k \mathcal{F}^{-1}(\xi\partial_\xi \hat{f})\|_{L^2} + 2^{-k/2} \|\widetilde{P}_k f\|_{L^2}].
\end{aligned}
$$

Then (6-1) follows when $(t+1)^{-1} \lesssim 2^k$. Otherwise, when $t+1 \lesssim 2^{-k}$, we have

$$
\|e^{2t\partial_x \log|\partial_x|} P_k f\|_{L^\infty} \lesssim 2^k \|\widehat{P_k f}\|_{L^\infty_\xi} \lesssim (t+1)^{-1/2} 2^{k/2} \|\widehat{P_k f}\|_{L^\infty_\xi}.
$$

(2) Next we prove estimates for the case when $|ix+2it(\log|\xi|+1)| \ll (t+1)$. Let

$$
\xi_0^\pm = \pm e^{-1-x/2t}
$$

be the solutions of $x+2t(\log|\xi|+1) = 0$. Since ψ_k is supported in an annulus with radius around 2^k, we only need to consider the case when $|\xi_0^\pm| \approx 2^k$ and ψ_k is supported on the neighborhood of the stationary phase point $\xi_0^\pm$. We decompose the integral and estimate it as

$$
\left| \int_{\mathbb{R}} e^{ix\xi+2i(\xi\log|\xi|)t} \hat{f}(\xi)\psi_k(\xi)\,\mathrm{d}\xi \right| \lesssim \sum_{l \le k+N} [|J_l^+| + |J_l^-|],
$$

with

$$
J_l^\pm = \int_{\mathbb{R}} e^{ix\xi+2i(\xi\log|\xi|)t} \hat{f}(\xi)\psi_k(\xi)\mathbf{1}_\pm(\xi)\psi_l(\xi-\xi_0^\pm)\,\mathrm{d}\xi,
$$

where $\mathbf{1}_\pm$ is the indicator function supported on $\mathbb{R}_\pm$ and N is large enough that the support of ψ_k is covered by the set $\bigcup_{l \le k+N} \{\xi \mid \psi_l(\xi-\xi_0^\pm) = 1\}$.

When $2^l \le 2^{k/2}(t+1)^{-1/2}$, we have

$$
\sum_{2^l \le 2^{k/2}(t+1)^{-1/2}} |J_l^\pm| \lesssim \sum_{2^l \le 2^{k/2}(t+1)^{-1/2}} 2^l \|\widehat{P_k f}\|_{L^\infty} \le 2^{k/2}(t+1)^{-1/2} \|\widehat{P_k f}\|_{L^\infty}.
$$

When $2^{k/2}(t+1)^{-1/2} \le 2^l \le 2^{k+N}$, since $|\xi-\xi_0| \approx 2^l$ and $|\xi_0| \approx 2^k$, we get the estimate

$$x + 2t(\log|\xi| + 1) = 2t\log\left|\frac{\xi}{\xi_0}\right| \approx 2t\log\left|1 \pm \frac{2^l}{2^k}\right|.$$

Using (6-2) and integration by parts, we have

$$\begin{aligned}|J_l^{\pm}| &\lesssim \frac{2^{k-l}}{(t+1)}\int_{\mathbb{R}} \big\{\big(|\partial_\xi \hat{f}(\xi)| + 2^{-l}|\hat{f}(\xi)|\big)\psi_l(\xi-\xi_0^{\pm}) + |\hat{f}(\xi)\psi_k(\xi)\psi_l'(\xi-\xi_0^{\pm})|\big\}\,\mathrm{d}\xi \\ &\lesssim \frac{2^{k-l}}{(t+1)}\|\hat{f}\|_{L^\infty} + \frac{2^{k-l/2}}{(t+1)}\|\partial_\xi \hat{f}\|_{L^2}.\end{aligned}$$

Then we take the sum of J_l over $2^l \ge 2^{k/2}(t+1)^{-1/2}$ to get the estimates (6-1). □

Proof of Lemma 5.3. After splitting into high-frequency and low-frequency parts, it suffices to bound the terms

$$\Big\|\sum_{k\le 0} P_k L\varphi_x\Big\|_{L^\infty}, \quad \Big\|\sum_{k>0} P_k \partial_x^{3+\epsilon}\varphi\Big\|_{L^\infty}.$$

Take the function f in Lemma 6.1 to be $L\partial_x h$. Since $e^{2t\partial_x \log|\partial_x|}$ and P_k commute, and

$$x\partial_x^2 Lh = \partial_x(x\partial_x Lh) - \partial_x Lh = \partial_x[x\partial_x, L]h + \partial_x L(x\partial_x h) - \partial_x Lh = -\partial_x h + \partial_x L(x\partial_x h) - \partial_x Lh,$$

we have that

$$\begin{aligned}\|P_k L\partial_x\varphi\|_{L^\infty} &\lesssim (t+1)^{-1/2}\|\mathcal{F}(P_k L|\partial_x|^{3/2}\varphi)\|_{L^\infty_\xi} \\ &\quad + (t+1)^{-3/4}[2^{(3/4)k}\|P_k L(x\partial_x h)\|_{L^2} + 2^{(3/4)k}(1+|k|)\|P_k h\|_{L^2} + \|\widetilde{P}_k(|\partial_x|^{3/4}L\varphi)\|_{L^2}].\end{aligned}$$

It follows from (5-5) that

$$\|P_k(x\partial_x h)\|_{L^2} \lesssim \|P_k\varphi\|_{L^2} + \|P_k\mathcal{S}\varphi\|_{L^2} + t\|P_k\mathcal{N}\|_{L^2}.$$

We first observe that $k \le 0$ automatically leads to $(t+1)^{-1/4+p_0}2^{(3/4)k}|k| \lesssim 1$, and then we have

$$\begin{aligned}\|P_k L\partial_x\varphi\|_{L^\infty} &\lesssim (t+1)^{-1/2}2^{k/2}|k|\,\|\psi_k(\xi)|\xi|\hat{\varphi}(\xi)\|_{L^\infty_\xi} \\ &\quad + (t+1)^{-1/2-p_0}[\|\widetilde{P}_k\varphi\|_{L^2} + \|P_k\mathcal{S}\varphi\|_{L^2} + t\|P_k\mathcal{N}\|_{L^2}]. \qquad (6\text{-}3)\end{aligned}$$

Summing over for $k \le 0$, using (5-6), the bootstrap assumptions, and (6-3) in the corresponding ranges of k, and we obtain that

$$\Big\|\sum_{k\le 0} P_k L\varphi_x\Big\|_{L^\infty} \lesssim \epsilon_1(t+1)^{-1/2}.$$

To estimate $\|P_k|\partial_x|^{r+2}\varphi_x\|_{L^\infty}$, we take the function f in Lemma 6.1 to be $|\partial_x|^{r+2}h_x$ and obtain

$$\begin{aligned}\|P_k|\partial_x|^{r+2}\varphi_x\|_{L^\infty} &\lesssim (t+1)^{-1/2}\|\mathcal{F}(P_k|\partial_x|^{r+7/2}\varphi)\|_{L^\infty_\xi} \\ &\quad + (t+1)^{-3/4}[2^{-k/4}\|P_k(x|\partial_x|^{r+4}h)\|_{L^2} + \|\widetilde{P}_k(|\partial_x|^{r+7/4}\varphi_x)\|_{L^2}].\end{aligned}$$

Using

$$-x|\partial_x|^{r+4}h = [x\partial_x, \partial_x|\partial_x|^{r+2}]h + \partial_x|\partial_x|^{r+2}(x\partial_x h) = -(r+3)\partial_x|\partial_x|^{r+2}h + \partial_x|\partial_x|^{r+2}(x\partial_x h),$$

and (5-5), we get that

$$\begin{aligned}
&\|P_k|\partial_x|^{r+2}\varphi_x\|_{L^\infty} \\
&\quad\lesssim (t+1)^{-1/2}\|\psi_k(\xi)|\xi|^{r+7/2}\hat{\varphi}(\xi)\|_{L^\infty_\xi} \\
&\qquad + (t+1)^{-3/4}2^{(3/4)k}\big[\||\partial_x|^{r+2}\widetilde{P}_k\varphi\|_{L^2} + \||\partial_x|^{r+2}P_k\mathcal{S}\varphi\|_{L^2} + \||\partial_x|^{r+2}P_k\varphi\|_{L^2} + t\||\partial_x|^{r+2}P_k\mathcal{N}\|_{L^2}\big].
\end{aligned}$$

For $k \in \mathbb{Z}_+$ and $(t+1)^{-1/4+p_0}2^{(r+11/4)k} \lesssim 1$, we have

$$\begin{aligned}
\|P_k|\partial_x|^{r+2}\varphi_x\|_{L^\infty} &\lesssim (t+1)^{-1/2}2^{-k/2}\|\psi_k(\xi)|\xi|^{r+4}\hat{\varphi}(\xi)\|_{L^\infty_\xi} \\
&\quad + (t+1)^{-1/2-p_0}[\|\widetilde{P}_k\varphi\|_{L^2} + \|P_k\mathcal{S}\varphi\|_{L^2} + t\|P_k\mathcal{N}\|_{L^2}]. \qquad (6\text{-}4)
\end{aligned}$$

Finally, for $k \in \mathbb{Z}_+$ and $(t+1)^{-1/4+p_0}2^{(r+11/4)k} \gtrsim 1$, we have

$$\begin{aligned}
\|P_k\partial_x^{r+2}\varphi_x\|_{L^\infty} &\lesssim \||\xi|^{r+3}\psi_k(\xi)\hat{\varphi}(\xi)\|_{L^1_\xi} \lesssim \||\xi|^{r+3-s}\psi_k(\xi)\|_{L^2}\|\widetilde{P}_k\varphi\|_{H^s} \\
&\lesssim 2^{(r+3-s+1/2)k}\|\widetilde{P}_k\varphi\|_{H^s} \lesssim (t+1)^{-(s-7/2-r)(1-4p_0)/(11+4r)}\|\widetilde{P}_k\varphi\|_{H^s}. \qquad (6\text{-}5)
\end{aligned}$$

Summing over $k \in \mathbb{Z}_+$, using (5-6), the bootstrap assumptions, and (6-4)–(6-5) in the corresponding ranges of k, we obtain that

$$\Big\|\sum_{k>0} P_k|\partial_x|^{r+2}\varphi_x\Big\|_{L^\infty} \lesssim \epsilon_1(t+1)^{-1/2},$$

which completes the proof. □

7. Scaling-Galilean estimate

In this section, we prove the scaling-Galilean estimate in Lemma 5.4.

First, we summarize some commutator identities for the scaling-Galilean operator $\mathcal{S}$ defined in (5-2) and $L = \log|\partial_x|$. The straightforward proofs follow by use of the Fourier transform and are omitted.

Lemma 7.1. *Let $\varphi(x,t)$ be a Schwartz distribution on $\mathbb{R}^2$ such that $L\varphi(x,t)$ is a Schwartz distribution. Then*

$$[\mathcal{S}, \partial_x]\varphi = -\partial_x\varphi, \quad [\mathcal{S}, L]\varphi = -\varphi, \quad [\mathcal{S}, L\partial_x]\varphi = -\varphi_x - L\partial_x\varphi,$$
$$[\mathcal{S}, \partial_t]\varphi = -2\partial_x\varphi - \partial_t\varphi, \quad [\mathcal{S}, \partial_t - 2L\partial_x]\varphi = -\partial_t\varphi + 2L\partial_x\varphi.$$

Next, we prove a weighted energy estimate for $\mathcal{S}\varphi$.

Proof of Lemma 5.4. Applying $\mathcal{S}$ to (3-7) and using Lemma 7.1, we get

$$(\mathcal{S}\varphi)_t - 2L\partial_x(\mathcal{S}\varphi) + \partial_x T_{B^0[\varphi]}\mathcal{S}\varphi + L[T_{B^{\log}[\varphi]}\mathcal{S}\varphi]_x + \mathcal{S}\mathcal{R} = \text{commutators},$$

where the commutators are

$$\partial_x[\mathcal{S}, T_{B^0[\varphi]}]\varphi, \quad [\mathcal{S}, \partial_x]T_{B^0[\varphi]}\varphi, \quad [\mathcal{S}, L\partial_x](T_{B^{\log}[\varphi]}\varphi), \quad L\partial_x([\mathcal{S}, T_{B^{\log}[\varphi]}]\varphi).$$

The first commutator can be written as

$$\begin{aligned}[\mathcal{S}, T_{B^0[\varphi]}]\varphi &= [(x+2t)\partial_x + t\partial_t, T_{B^0[\varphi]}]\varphi \\ &= (x+2t)\partial_x T_{B^0[\varphi]}\varphi - T_{B^0[\varphi]}[(x+2t)\partial_x\varphi] + t\partial_t T_{B^0[\varphi]}\varphi - T_{B^0[\varphi]}(t\partial_t\varphi) \\ &= (x+2t)T_{\partial_x B^0[\varphi]}\varphi + [(x+2t), T_{B^0[\varphi]}]\partial_x\varphi + T_{t\partial_t B^0[\varphi]}\varphi \\ &= T_{(x+2t)\partial_x B^0[\varphi]}\varphi + (xT_{\partial_x B^0[\varphi]}\varphi - T_{x\partial_x B^0[\varphi]}\varphi) + [x, T_{B^0[\varphi]}]\partial_x\varphi + T_{t\partial_t B^0[\varphi]}\varphi.\end{aligned}$$

By the commutator estimates in Lemma 2.2 and Theorem B.4, we obtain for $0 \le k \le r$ that

$$\begin{aligned}\|\partial_x[\mathcal{S}, T_{B^0[\varphi]}]\varphi\|_{H^k} &\lesssim \|[x, T_{B^0[\varphi]}]\partial_x\varphi\|_{H^{k+1}} + \|xT_{\partial_x B^0[\varphi]}\varphi - T_{x\partial_x B^0[\varphi]}\varphi\|_{H^{k+1}} + \|T_{\mathcal{S}B^0[\varphi]}\varphi_x\|_{H^k} \\ &\lesssim \|B^0[\varphi]\|_{\mathcal{M}_{(1,2)}}\|\varphi\|_{H^{k+2}} + \|B^0[\varphi]\|_{\mathcal{M}_{(2,2)}}\|\varphi\|_{H^{k+1}} + \|\mathcal{S}B^0[\varphi]\|_{\mathcal{L}^2_1}\|\varphi_x\|_{W^{k+1,\infty}}.\end{aligned}$$

Using (3-9), together with Lemma 2.5 and similar estimates for $\|\mathcal{S}B^0[\varphi]\|_{\mathcal{L}^2_1}$, we find that

$$\|\partial_x[\mathcal{S}, T_{B^0[\varphi]}]\varphi\|_{H^k} \lesssim F(\|L\varphi_x\|_{W^{2,\infty}} + \|\varphi_x\|_{W^{2,\infty}})(\|L\varphi_x\|_{W^{2,\infty}} + \|\varphi_x\|_{W^{2,\infty}})\|\varphi_x\|_{W^{r+1,\infty}}(\|\mathcal{S}\varphi\|_{H^r} + \|\varphi\|_{H^s}).$$

Similarly, we have

$$\begin{aligned}&\|L\partial_x([\mathcal{S}, T_{B^{\log}[\varphi]}]\varphi)\|_{H^k} \\ &\qquad\lesssim F(\|L\varphi_x\|_{W^{2,\infty}} + \|\varphi_x\|_{W^{2,\infty}})(\|L\varphi_x\|_{W^{2,\infty}} + \|\varphi_x\|_{W^{2,\infty}})\|\varphi_x\|_{W^{r+1,\infty}}(\|\mathcal{S}\varphi\|_{H^r} + \|\varphi\|_{H^s}).\end{aligned}$$

By Lemma 7.1, Lemma 2.1, and (3-9), the second and third commutators satisfy

$$\begin{aligned}\|[\mathcal{S}, \partial_x]T_{B^0[\varphi]}\varphi\|_{H^k} &= \|T_{B^0[\varphi]}\varphi\|_{H^{k+1}} \\ &\lesssim F(\|L\varphi_x\|_{W^{2,\infty}} + \|\varphi_x\|_{W^{2,\infty}})(\|L\varphi_x\|_{W^{2,\infty}} + \|\varphi_x\|_{W^{2,\infty}})^2\|\varphi\|_{H^{k+1}}, \\ \|[\mathcal{S}, L\partial_x](T_{B^{\log}[\varphi]}\varphi)\|_{H^k} &= \|(T_{B^{\log}[\varphi]}\varphi)_x + L\partial_x(T_{B^{\log}[\varphi]}\varphi)\|_{H^k} \\ &\lesssim F(\|L\varphi_x\|_{W^{1,\infty}} + \|\varphi_x\|_{W^{1,\infty}})(\|L\varphi_x\|_{W^{1,\infty}} + \|\varphi_x\|_{W^{1,\infty}})^2(\|\varphi\|_{H^{k+1}} + \|L\varphi\|_{H^{k+1}}).\end{aligned}$$

Thus, the evolution equation for $\mathcal{S}\varphi$ can be written as

$$(\mathcal{S}\varphi)_t + \partial_x T_{B^0[\varphi]}\mathcal{S}\varphi + \mathcal{R}_{\mathcal{S}} = L[(2 - T_{B^{\log}[\varphi]})\mathcal{S}\varphi],$$

where the remainder $\mathcal{R}_{\mathcal{S}}$ satisfies

$$\|\mathcal{R}_{\mathcal{S}}\|_{H^k} \lesssim (\|\varphi_x\|_{W^{2,\infty}} + \|L\varphi_x\|_{W^{2,\infty}})^2(\|\mathcal{S}\varphi\|_{H^r} + \|\varphi\|_{H^{r+1}} + \|L\varphi\|_{H^{r+1}}).$$

As in (4-4), we define a weighted energy for $\mathcal{S}\varphi$ by

$$\begin{aligned}E^{(j)}_{\mathcal{S}}(t) &= \int_{\mathbb{R}} |D|^j\mathcal{S}\varphi(x,t)\cdot(2 - T_{B^{\log}[\varphi]})^{2j+1}|D|^j\mathcal{S}\varphi(x,t)\,\mathrm{d}x, \qquad j = 0, 1, \dots, r, \\ \widetilde{E}^{(r)}_{\mathcal{S}}(t) &= \sum_{j=0}^{r} E^{(j)}_{\mathcal{S}}(t),\end{aligned}$$

and repeat estimates similar to the ones in the proof of Proposition 4.3 to get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E^{(j)}_{\mathcal{S}}(t) &\lesssim F(\|L\varphi_x\|_{W^{2,\infty}} + \|\varphi_x\|_{W^{2,\infty}})(\|L\varphi_x\|_{W^{2,\infty}} + \|\varphi_x\|_{W^{3,\infty}})^2\|\mathcal{S}\varphi\|^2_{H^j} \\ &\qquad + (\|\varphi_x\|_{W^{2,\infty}} + \|L\varphi_x\|_{W^{2,\infty}})^2(\|\mathcal{S}\varphi\|_{H^r} + \|\varphi\|_{H^{r+1}} + \|L\varphi\|_{H^{r+1}})\|\mathcal{S}\varphi\|_{H^j}.\end{aligned}$$

Using Lemma 5.3 and the equivalence of $\widetilde{E}_{\mathcal{S}}^{(r)}$ and $\|\mathcal{S}\varphi\|_{H^r}^2$ when $\|2 - T_{B^{\log}[\varphi]}\|_{L^2\to L^2}$ is bounded away from zero, we find by integrating in t that

$$\widetilde{E}_{\mathcal{S}}^{(r)}(t) \lesssim \epsilon_0^2 (t+1)^{2p_0},$$

which proves the lemma. □

8. Nonlinear dispersive estimate

In this section, we prove the estimate

$$(|\xi| + |\xi|^{r+4})|\hat{\varphi}(\xi, t)| \lesssim \epsilon_0 \quad \text{for all } \xi \in \mathbb{R}, \tag{8-1}$$

which establishes Lemma 5.6 for the Z-norm $\|\varphi\|_Z$ defined in (5-4).

Using the interpolation result in Lemma 2.4, we first prove in Section 8A that the estimate (8-1) holds for sufficiently large and small $|\xi|$. In Section 8B, we introduce a logarithmic phase shift into the solution which is used later to absorb the effects of the space-time resonances. The main part of the section is a detailed analysis of the nonresonant and resonant interactions between different Fourier components of the solution, which is carried out in Sections 8C–8G. A detailed flow-chart of the cases considered is given in Figure 1.

To classify the cubic resonances between frequencies $\xi - \eta_1 - \eta_2$, η_1, η_2 into ξ, where $\xi, \eta_1, \eta_2 \in \mathbb{R}$, we introduce the phase

$$\Phi(\xi, \eta_1, \eta_2) = 2(\xi - \eta_1 - \eta_2)\log|\xi - \eta_1 - \eta_2| + 2\eta_1 \log|\eta_1| + 2\eta_2 \log|\eta_2| - 2\xi \log|\xi|. \tag{8-2}$$

The space resonances satisfy $\partial_{\eta_1}\Phi = \partial_{\eta_2}\Phi = 0$, which implies that the frequencies $\xi - \eta_1 - \eta_2$, η_1, η_2 have the same linearized group velocity. It is straightforward to check that the only space resonances are

$$(\xi - \eta_1 - \eta_2, \eta_1, \eta_2) = (-\xi, \xi, \xi),\ (\xi, -\xi, \xi), \text{ or } (\xi, \xi, -\xi), \tag{8-3}$$

$$(\xi - \eta_1 - \eta_2, \eta_1, \eta_2) = \left(\tfrac{1}{3}\xi, \tfrac{1}{3}\xi, \tfrac{1}{3}\xi\right). \tag{8-4}$$

The time resonances satisfy $\Phi = 0$, which implies that the time frequencies of $\xi - \eta_1 - \eta_2$, η_1, η_2 are in resonance with the time frequency of ξ. This condition is satisfied by the resonances (8-3), which are space-time resonances, but not by (8-4), which is a space resonance. There are additional time resonances of the form $\xi = \xi - \eta + \eta$, but they are not space resonances for $\xi \neq \eta$, so they require no further analysis.

8A. *Large and small frequencies.* When $|\xi| < (t+1)^{-p_0}$, Lemma 2.4, the bootstrap assumptions, Lemma 5.4, and the conservation of $\|\varphi\|_{L^2}$, imply that

$$\begin{aligned}|(|\xi| + |\xi|^{r+4})\hat{\varphi}(\xi, t)|^2 &\lesssim (|\xi| + |\xi|^{r+4})^2 |\xi|^{-1} \|\hat{\varphi}\|_{L^2_\xi} (|\xi| \|\partial_\xi \hat{\varphi}\|_{L^2_\xi} + \|\hat{\varphi}\|_{L^2_\xi}) \\ &\lesssim (|\xi| + |\xi|^{r+4}) \|\varphi\|_{L^2} (\|\mathcal{S}\varphi\|_{L^2} + \|\varphi\|_{L^2}) \lesssim \epsilon_0^2.\end{aligned}$$

Let $p_1 = 10^{-6}$. When $|\xi| \geq (t+1)^{p_1}$, similarly from Lemma 2.4 and the bootstrap assumptions we get

$$\begin{aligned}|(|\xi| + |\xi|^{r+4})\hat{\varphi}(\xi, t)|^2 &\lesssim \frac{(|\xi| + |\xi|^{r+4})^2}{|\xi|^{s+1}} \|\varphi\|_{H^s} (\|\mathcal{S}\varphi\|_{L^2} + \|\varphi\|_{L^2}) \\ &\lesssim |\xi|^{2r+7-s} (t+1)^{2p_0} \epsilon_0^2 \lesssim |\xi|^{2r+7-s+2p_0/p_1} \epsilon_0^2 \lesssim \epsilon_0^2,\end{aligned}$$

since $2r + 7 - s + 2p_0/p_1 < 0$ for the parameter values in (5-1).

Thus, we only need to consider the frequency range $(t+1)^{-p_0} \le |\xi| \le (t+1)^{p_1}$. In the following, we fix ξ in this range, and use $\mathfrak{d}$ to denote a smooth cutoff function such that

$$\begin{gathered}\mathfrak{d}(\xi, t) = 1 \text{ on } \{(\xi, t) \mid (t+1)^{-p_0} \le |\xi| \le (t+1)^{p_1}\},\\ \mathfrak{d}(\xi, t) \text{ is supported on a small neighborhood of } \{(\xi, t) \mid (t+1)^{-p_0} \le |\xi| \le (t+1)^{p_1}\}.\end{gathered} \tag{8-5}$$

8B. *Modified scattering.* Taking the Fourier transform of (3-4), we obtain that

$$\begin{aligned}\hat{\varphi}_t(\xi, t) + \frac{i\xi}{6} \iint_{\mathbb{R}^2} \boldsymbol{T}_1(\eta_1, \eta_2, \xi-\eta_1-\eta_2)\hat{\varphi}(\xi-\eta_1-\eta_2, t)\hat{\varphi}(\eta_1, t)\hat{\varphi}(\eta_2, t)\, \mathrm{d}\eta_1\, \mathrm{d}\eta_2 \\ + \widehat{\mathcal{N}_{\ge 5}(\varphi)}(\xi, t) = 2i\xi \log|\xi|\hat{\varphi}(\xi, t),\end{aligned} \tag{8-6}$$

where $\mathcal{N}_{\ge 5}(\varphi)$ is given by (3-5), and, from (A-3),

$$\begin{aligned}\boldsymbol{T}_1(\eta_1, \eta_2, \eta_3) = -\eta_1^2 \log|\eta_1| - \eta_2^2 \log|\eta_2| - \eta_3^2 \log|\eta_3| - (\eta_1+\eta_2+\eta_3)^2 \log|\eta_1+\eta_2+\eta_3| \\ + \{(\eta_1+\eta_2)^2 \log|\eta_1+\eta_2| + (\eta_1+\eta_3)^2 \log|\eta_1+\eta_3| + (\eta_2+\eta_3)^2 \log|\eta_2+\eta_3|\}.\end{aligned} \tag{8-7}$$

We can also write (8-6) in terms of $\hat{h} = e^{-2it\xi \log|\xi|}\hat{\varphi}$ defined in (5-3) as

$$\begin{aligned}\hat{h}_t(\xi, t) + \frac{i\xi}{6} \iint_{\mathbb{R}^2} \boldsymbol{T}_1(\eta_1, \eta_2, \xi-\eta_1-\eta_2) e^{it\Phi(\xi,\eta_1,\eta_2)} \hat{h}(\xi-\eta_1-\eta_2, t)\hat{h}(\eta_1, t)\hat{h}(\eta_2, t)\, \mathrm{d}\eta_1\, \mathrm{d}\eta_2 \\ + e^{-2it\xi \log|\xi|}\widehat{\mathcal{N}_{\ge 5}(\varphi)}(\xi, t) = 0,\end{aligned} \tag{8-8}$$

where Φ is defined in (8-2).

Nonlinearity leads to a cumulative frequency shift in the long-time behavior of the Fourier components of the solution due to space-time resonances of the form $\xi = \xi + \xi - \xi$. To account for this effect, we use the method of modified scattering and introduce a phase correction

$$\Theta(\xi, t) = \frac{\pi\xi|\xi|}{3}[\boldsymbol{T}_1(\xi, \xi, -\xi) + \boldsymbol{T}_1(\xi, -\xi, \xi) + \boldsymbol{T}_1(-\xi, \xi, \xi)] \int_0^t \frac{|\hat{\varphi}(\xi, \tau)|^2}{\tau+1}\, \mathrm{d}\tau. \tag{8-9}$$

This phase correction is generic in cubically nonlinear dispersive equations and grows logarithmically in time; see [Córdoba et al. 2019; Ifrim and Tataru 2015; 2016; Ionescu and Pusateri 2015]. We then let

$$\hat{v}(\xi, t) = e^{i\Theta(\xi,t)}\hat{h}(\xi, t).$$

Using (8-8) and (8-9), we find that

$$\hat{v}_t(\xi, t) = e^{i\Theta(\xi,t)}[\hat{h}_t(\xi, t) + i\Theta_t(\xi, t)\hat{h}(\xi, t)] = U(\xi, t) - e^{-2it\xi \log|\xi|}e^{i\Theta(\xi,t)}\widehat{\mathcal{N}_{\ge 5}(\varphi)}(\xi, t), \tag{8-10}$$

where

$$\begin{aligned}U(\xi, t) = e^{i\Theta(\xi,t)}\Bigg\{-\frac{i\xi}{6} \iint_{\mathbb{R}^2} \boldsymbol{T}_1(\eta_1, \eta_2, \xi-\eta_1-\eta_2) e^{it\Phi(\xi,\eta_1,\eta_2)} \hat{h}(\xi-\eta_1-\eta_2)\hat{h}(\eta_1)\hat{h}(\eta_2)\, \mathrm{d}\eta_1\, \mathrm{d}\eta_2 \\ + \frac{\pi i\xi|\xi|}{3}[\boldsymbol{T}_1(\xi, \xi, -\xi) + \boldsymbol{T}_1(\xi, -\xi, \xi) + \boldsymbol{T}_1(-\xi, \xi, \xi)]\frac{|\hat{h}(\xi, t)|^2\hat{h}(\xi, t)}{t+1}\Bigg\}.\end{aligned} \tag{8-11}$$

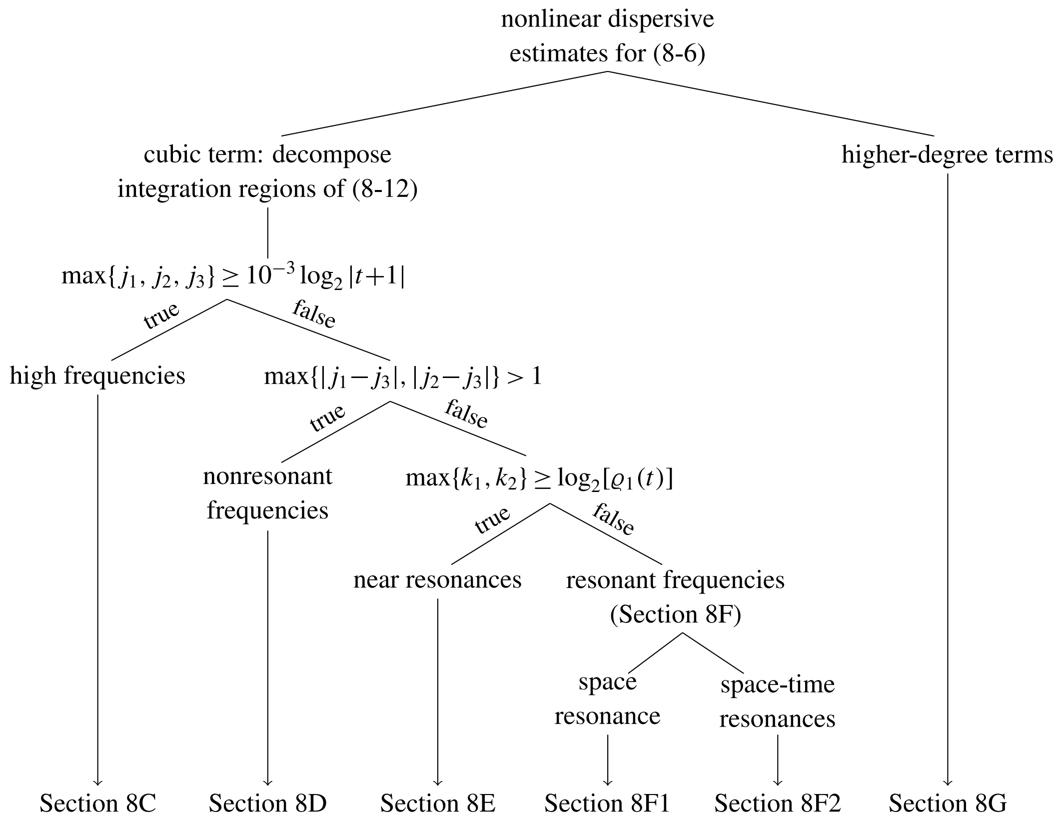

Figure 1. Plan of Section 8. Here $(j_1, j_2, j_3) \in \mathbb{Z}^3$ are the dyadic blocks in the cubic decomposition (8-12), and $\varrho_1(t)$ is defined in (8-23).

Then we get from (8-10) that

$$\begin{aligned}\|\varphi\|_Z &= \|(|\xi|+|\xi|^{r+4})\hat{\varphi}(\xi,t)\|_{L^\infty_\xi} = \|(|\xi|+|\xi|^{r+4})\hat{v}(\xi,t)\|_{L^\infty_\xi}\\ &\lesssim \int_0^t \{\|(|\xi|+|\xi|^{r+4})U(\xi,\tau)\|_{L^\infty_\xi} + \|(|\xi|+|\xi|^{r+4})\widehat{\mathcal{N}_{\geq 5}(\varphi)}(\xi,\tau)\|_{L^\infty_\xi}\}\,\mathrm{d}\tau.\end{aligned}$$

We will estimate the cubic terms in U in Sections 8C–8F and the higher-degree terms involving $\widehat{\mathcal{N}_{\geq 5}(\varphi)}$ in Section 8G. We do not need to consider the terms in U that involve the phase correction until we come to an analysis of the space-time resonances in Section 8F.

Suppressing the dependence of $\hat{h}(\xi,t)$ on t, we carry out a dyadic decomposition of the integral in the expression (8-11) for U, and write it as a sum over $(j_1, j_2, j_3) \in \mathbb{Z}^3$ of terms of the form

$$\iint_{\mathbb{R}^2} \boldsymbol{T}_1(\eta_1, \eta_2, \xi-\eta_1-\eta_2) e^{it\Phi(\xi,\eta_1,\eta_2)} \hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2. \tag{8-12}$$

Here, $h_j = P_j h$, where P_j is the Fourier multiplier with symbol ψ_j defined in (2-13). In the following subsections, we estimate this integral in various regions of frequency space.

In Section 8C, we estimate the integral for high frequencies (at least one of j_1, j_2, j_3 is large). In Section 8D, we estimate the integral for nonresonant frequencies, using oscillatory integral estimates with respect to the frequency variables together with multilinear estimates to get sufficient time decay.

In Section 8E, we consider frequencies that are close to the resonant frequencies. In that case, the bounds for the multilinear symbols are worse, so we cannot obtain sufficient time decay by the method used for the nonresonant frequencies. We resolve this issue by an additional dyadic decomposition centered at each resonant point and a refinement of the symbol estimates.

Finally, in Section 8F, we consider frequencies that are at the space resonance or space-time resonances. For the space resonance, we estimate the integral in a region about the space resonance point that shrinks in time, using an oscillatory integral estimate with respect to time and the equation to eliminate the time derivative of the solution. For the space-time resonances, we take advantage of the modified scattering phase correction and estimate the integral on shrinking regions about the space-time resonance points.

As a checklist for the complete discussion of all cases, the plan for the rest of Section 8 is displayed in Figure 1.

8C. *High frequencies.* When $\max\{j_1, j_2, j_3\} \geq 10^{-3}\log_2|t+1|$, we can estimate the nonlinear terms (8-12) by using Lemma 2.5, with the L^∞-norm placed on the lowest-derivative term. There are, in total, $r+6=13$ derivatives shared by three factors of φ. Thus, we can ensure that the term with least derivatives has at most four derivatives, with or without a logarithmic derivative.

To be more specific, introducing the cutoff function $\mathfrak{d}$ in (8-5), using Hölder's inequality, Sobolev embedding, and the bootstrap assumptions, we obtain the estimate

$$\begin{aligned}\Bigg\| \xi(|\xi|+|\xi|^{r+4})\mathfrak{d}(\xi,t)\iint_{\mathbb{R}^2} \boldsymbol{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)e^{it\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\Bigg\|_{L^\infty_\xi}&\\ \lesssim (t+1)^{(r+8-s)10^{-3}}\|\varphi_{\min}\|_{L^2}(\|\varphi_{\mathrm{med}}\|_{L^\infty}+\|L\partial_x\varphi_{\mathrm{med}}\|_{W^{r,\infty}})\|\varphi_{\max}\|_{H^s}&\\ \lesssim (t+1)^{(r+8-s)10^{-3}}\|\varphi_{j_1}\|_{H^s}\|\varphi_{j_2}\|_{H^s}\|\varphi_{j_3}\|_{H^s},&\end{aligned}$$

where max, med, min represent the maximum, median, and minimum of j_1, j_2, j_3. From (5-1), we have $(r+8-s)10^{-3} < -1.1$, so the right-hand-side is summable over j_1, j_2, j_3, and the sum is integrable for $t \in (0,\infty)$.

8D. *Nonresonant frequencies.* We now only need to consider when $\max\{j_1, j_2, j_3\} < 10^{-3}\log_2(t+1)$. The regions $|j_1 - j_3| > 1$ and $|j_2 - j_3| > 1$ correspond to nonresonant frequencies. Without loss of generality, we assume $|j_1 - j_3| > 1$.

Notice that by (8-2), we have

$$\partial_{\eta_1}\Phi = 2\log|\eta_1| - 2\log|\xi-\eta_1-\eta_2|. \tag{8-13}$$

Since $|\eta_1|$ and $|\xi-\eta_1-\eta_2|$ are in different dyadic blocks, we have

$$\big||\eta_1|-|\xi-\eta_1-\eta_2|\big| \gtrsim \max\{|\eta_1|, |\xi-\eta_1-\eta_2|\}.$$

Therefore, $|\partial_{\eta_1}\Phi| \gtrsim 1$.

After integrating by parts, we have

$$
\begin{aligned}
\iint_{\mathbb{R}^2} &\boldsymbol{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)e^{it\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2 \\
&= \iint_{\mathbb{R}^2} \frac{\boldsymbol{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{it\partial_{\eta_1}\Phi(\xi,\eta_1,\eta_2)}\partial_{\eta_1}e^{it\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2 \\
&= -W_1-W_2-W_3,
\end{aligned}
$$

where

$$
\begin{aligned}
W_1(\xi,t) &= \iint_{\mathbb{R}^2} \partial_{\eta_1}\left[\frac{\boldsymbol{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{it\partial_{\eta_1}\Phi(\xi,\eta_1,\eta_2)}\right]e^{it\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2, \\
W_2(\xi,t) &= \iint_{\mathbb{R}^2} \left[\frac{\boldsymbol{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{it\partial_{\eta_1}\Phi(\xi,\eta_1,\eta_2)}\right]e^{it\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\partial_{\eta_1}\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2, \\
W_3(\xi,t) &= \iint_{\mathbb{R}^2} \left[\frac{\boldsymbol{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{it\partial_{\eta_1}\Phi(\xi,\eta_1,\eta_2)}\right]e^{it\Phi(\xi,\eta_1,\eta_2)}\partial_{\eta_1}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2.
\end{aligned}
$$

In order to estimate these terms, we note from (8-7) that

$$
\begin{aligned}
\partial_{\eta_1}[\boldsymbol{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)] &= -2\big\{\eta_1\log|\eta_1|-(\eta_1+\eta_2)\log|\eta_1+\eta_2| \\
&\qquad +(\xi-\eta_1)\log|\xi-\eta_1|-(\xi-\eta_1-\eta_2)\log|\xi-\eta_1-\eta_2|\big\}, \\
\partial_{\eta_2}[\boldsymbol{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)] &= -2\big\{\eta_2\log|\eta_2|-(\eta_1+\eta_2)\log|\eta_1+\eta_2| \\
&\qquad +(\xi-\eta_2)\log|\xi-\eta_2|-(\xi-\eta_1-\eta_2)\log|\xi-\eta_1-\eta_2|\big\}.
\end{aligned}
\tag{8-14}
$$

Estimate of W_1: Since

$$
\|W_1\|_{L^\infty_\xi} \lesssim \|\mathcal{F}^{-1}(W_1)\|_{L^1}, \tag{8-15}
$$

it suffices to estimate the L^1_x-norm of

$$
\iiint_{\mathbb{R}^3} e^{i\xi x}\partial_{\eta_1}\left[\frac{\boldsymbol{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{it\partial_{\eta_1}\Phi(\xi,\eta_1,\eta_2)}\right]e^{it\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\xi.
$$

Notice that by (8-13)

$$
\partial_{\eta_1}\frac{\boldsymbol{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{\partial_{\eta_1}\Phi(\xi,\eta_1,\eta_2)} = \kappa_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)-\frac{\kappa_2(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{2},
$$

where

$$
\begin{aligned}
\kappa_1(\eta_1,\eta_2,\eta_3) &= \frac{\partial_{\eta_1}\boldsymbol{T}_1(\eta_1,\eta_2,\eta_3)-\partial_{\eta_3}\boldsymbol{T}_1(\eta_1,\eta_2,\eta_3)}{\log|\eta_1|-\log|\eta_3|}, \\
\kappa_2(\eta_1,\eta_2,\eta_3) &= \boldsymbol{T}_1(\eta_1,\eta_2,\eta_3)\frac{1/\eta_1+1/\eta_3}{(\log|\eta_1|-\log|\eta_3|)^2}.
\end{aligned}
$$

Making a change of variable $\eta_3=\xi-\eta_1-\eta_2$, we need to estimate the trilinear form

$$
\frac{1}{it}\iiint_{\mathbb{R}^3} e^{i\xi x}[\kappa_1(\eta_1,\eta_2,\eta_3)+\kappa_2(\eta_1,\eta_2,\eta_3)]\hat{\varphi}_{j_1}(\eta_1)\hat{\varphi}_{j_2}(\eta_2)\hat{\varphi}_{j_3}(\eta_3)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\eta_3,
$$

with symbol

$$
[\kappa_1(\eta_1,\eta_2,\eta_3)+\kappa_2(\eta_1,\eta_2,\eta_3)]\psi_{j_1}(\eta_1)\psi_{j_2}(\eta_2)\psi_{j_3}(\eta_3).
$$

According to Lemma 2.5, this trilinear operator is bounded on $L^2 \times L^2 \times L^\infty \to L^1$ by

$$
\begin{aligned}
&\|[\kappa_1(\eta_1,\eta_2,\eta_3)+\kappa_2(\eta_1,\eta_2,\eta_3)]\psi_{j_1}(\eta_1)\psi_{j_2}(\eta_2)\psi_{j_3}(\eta_3)\|_{S^\infty} \\
&\quad\lesssim \big(\|\partial_{\eta_1}\boldsymbol{T}_1(\eta_1,\eta_2,\eta_3)\tilde{\psi}_{j_1}(\eta_1)\tilde{\psi}_{j_2}(\eta_2)\tilde{\psi}_{j_3}(\eta_3)\|_{S^\infty} \\
&\qquad\qquad + \|\partial_{\eta_3}\boldsymbol{T}_1(\eta_1,\eta_2,\eta_3)\tilde{\psi}_{j_1}(\eta_1)\tilde{\psi}_{j_2}(\eta_2)\tilde{\psi}_{j_3}(\eta_3)\|_{S^\infty}\big)\cdot\left\|\frac{\psi_{j_1}(\eta_1)\psi_{j_2}(\eta_2)\psi_{j_3}(\eta_3)}{\log|\eta_1|-\log|\eta_3|}\right\|_{S^\infty} \\
&\qquad +\Bigg(\left\|\frac{\boldsymbol{T}_1(\eta_1,\eta_2,\eta_3)}{\eta_1}\tilde{\psi}_{j_1}(\eta_1)\tilde{\psi}_{j_2}(\eta_2)\tilde{\psi}_{j_3}(\eta_3)\right\|_{S^\infty} \\
&\qquad\qquad + \left\|\frac{\boldsymbol{T}_1(\eta_1,\eta_2,\eta_3)}{\eta_3}\tilde{\psi}_{j_1}(\eta_1)\tilde{\psi}_{j_2}(\eta_2)\tilde{\psi}_{j_3}(\eta_3)\right\|_{S^\infty}\Bigg)\cdot\left\|\frac{\psi_{j_1}(\eta_1)\psi_{j_2}(\eta_2)\psi_{j_3}(\eta_3)}{(\log|\eta_1|-\log|\eta_3|)^2}\right\|_{S^\infty}. \qquad (8\text{-}16)
\end{aligned}
$$

In the following lemmas, we prove S^∞-estimates for these symbols.

Lemma 8.1. *Suppose that* $|j_1-j_3|>1$. *Then for any* $m\in\mathbb{Z}_+$,

$$\left\|\frac{1}{(\log|\eta_1|-\log|\eta_3|)^m}\psi_{j_1}(\eta_1)\psi_{j_2}(\eta_2)\psi_{j_3}(\eta_3)\right\|_{S^\infty}\lesssim 1.$$

Proof. By the definition of the S^∞-norm (2-15) and the definition of ψ_k (2-13), we have that

$$
\begin{aligned}
&\left\|\frac{\psi_{j_1}(\eta_1)\psi_{j_2}(\eta_2)\psi_{j_3}(\eta_3)}{(\log|\eta_1|-\log|\eta_3|)^m}\right\|_{S^\infty} \\
&\quad=\left\|\iiint_{\mathbb{R}^3}\frac{\psi_{j_1}(\eta_1)\psi_{j_2}(\eta_2)\psi_{j_3}(\eta_3)}{(\log|\eta_1|-\log|\eta_3|)^m}e^{i(y_1\eta_1+y_2\eta_2+y_3\eta_3)}\,d\eta_1\,d\eta_2\,d\eta_3\right\|_{L^1} \\
&\quad=\iiint_{\mathbb{R}^3}\left|\iiint_{\mathbb{R}^3}\frac{\psi_0(2^{-j_1}\eta_1)\psi_0(2^{-j_2}\eta_2)\psi_0(2^{-j_3}\eta_3)}{(\log|\eta_1|-\log|\eta_3|)^m}e^{i(y_1\eta_1+y_2\eta_2+y_3\eta_3)}\,d\eta_1\,d\eta_2\,d\eta_3\right|dy_1\,dy_2\,dy_3\lesssim 1,
\end{aligned}
$$

where the last inequality comes from oscillatory integral estimates, using the facts that $|j_1-j_3|>1$ and the support of ψ_0 is $\left[-\frac{8}{5},-\frac{5}{8}\right]\cup\left[\frac{5}{8},\frac{8}{5}\right]$. ☐

For the estimates of the other symbols in (8-16), we have the following lemma.

Lemma 8.2. *For any* $j_1, j_2, j_3\in\mathbb{Z}$, *we have*

$$\|\partial_{\eta_1}\boldsymbol{T}_1(\eta_1,\eta_2,\eta_3)\tilde{\psi}_{j_1}(\eta_1)\tilde{\psi}_{j_2}(\eta_2)\tilde{\psi}_{j_3}(\eta_3)\|_{S^\infty}\lesssim 2^{\max\{j_2,j_3\}}, \qquad (8\text{-}17)$$

$$\|\boldsymbol{T}_1(\eta_1,\eta_2,\eta_3)\tilde{\psi}_{j_1}(\eta_1)\tilde{\psi}_{j_2}(\eta_2)\tilde{\psi}_{j_3}(\eta_3)\|_{S^\infty}\lesssim 2^{\max\{j_1,j_2,j_3\}+\min\{j_1,j_2,j_3\}}, \qquad (8\text{-}18)$$

and

$$\left\|\frac{\boldsymbol{T}_1(\eta_1,\eta_2,\eta_3)}{\eta_1}\tilde{\psi}_{j_1}(\eta_1)\tilde{\psi}_{j_2}(\eta_2)\tilde{\psi}_{j_3}(\eta_3)\right\|_{S^\infty}\lesssim 2^{\max\{j_2,j_3\}}. \qquad (8\text{-}19)$$

Furthermore, since $\boldsymbol{T}_1$ *is symmetric, we also have*

$$\|\partial_{\eta_3}\boldsymbol{T}_1(\eta_1,\eta_2,\eta_3)\tilde{\psi}_{j_1}(\eta_1)\tilde{\psi}_{j_2}(\eta_2)\tilde{\psi}_{j_3}(\eta_3)\|_{S^\infty}\lesssim 2^{\max\{j_1,j_2\}},$$

$$\left\|\frac{\boldsymbol{T}_1(\eta_1,\eta_2,\eta_3)}{\eta_3}\tilde{\psi}_{j_1}(\eta_1)\tilde{\psi}_{j_2}(\eta_2)\tilde{\psi}_{j_3}(\eta_3)\right\|_{S^\infty}\lesssim 2^{\max\{j_1,j_2\}}.$$

Proof. (1) We prove (8-17) first. Using inverse Fourier transform in (η_1, η_2, η_3), we obtain

$$
\begin{aligned}
&\mathcal{F}^{-1}[\partial_{\eta_1}\boldsymbol{T}_1(\eta_1,\eta_2,\eta_3)\tilde{\psi}_{j_1}(\eta_1)\tilde{\psi}_{j_2}(\eta_2)\tilde{\psi}_{j_3}(\eta_3)]\\
&=\iiint_{\mathbb{R}^3} e^{i(y_1\eta_1+y_2\eta_2+y_3\eta_3)}\partial_{\eta_1}\left[\int_{\mathbb{R}}\frac{\prod_{j=1}^3(1-e^{i\eta_j\zeta})}{|\zeta|^3}\,\mathrm{d}\zeta\right]\tilde{\psi}_{j_1}(\eta_1)\tilde{\psi}_{j_2}(\eta_2)\tilde{\psi}_{j_3}(\eta_3)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\eta_3\\
&=\iiint_{\mathbb{R}^3}\left[\int_{\mathbb{R}}\frac{-i\zeta e^{i\eta_1(\zeta+y_1)}(e^{iy_2\eta_2}-e^{i\eta_2(\zeta+y_2)})(e^{iy_3\eta_3}-e^{i\eta_3(\zeta+y_3)})}{|\zeta|^3}\,\mathrm{d}\zeta\right]\tilde{\psi}_{j_1}(\eta_1)\tilde{\psi}_{j_2}(\eta_2)\tilde{\psi}_{j_3}(\eta_3)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\eta_3\\
&=\int_{\mathbb{R}}\frac{-i\zeta}{|\zeta|^3}\cdot[\mathcal{F}^{-1}[\tilde{\psi}_{j_1}](y_1+\zeta)]\cdot[\mathcal{F}^{-1}[\tilde{\psi}_{j_2}](y_2)-\mathcal{F}^{-1}[\tilde{\psi}_{j_2}](\zeta+y_2)][\mathcal{F}^{-1}[\tilde{\psi}_{j_3}](y_3)-\mathcal{F}^{-1}[\tilde{\psi}_{j_3}](\zeta+y_3)]\,\mathrm{d}\zeta.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
|\mathcal{F}^{-1}[\tilde{\psi}_{j_1}](y_1+\zeta)| &= 2^{j_1}|\mathcal{F}^{-1}[\tilde{\psi}_0](2^{j_1}(y_1+\zeta))|,\\
|\mathcal{F}^{-1}[\tilde{\psi}_{j_2}](y_2)-\mathcal{F}^{-1}[\tilde{\psi}_{j_2}](\zeta+y_2)| &= 2^{j_2}|\mathcal{F}^{-1}[\tilde{\psi}_0](2^{j_2}y_2)-\mathcal{F}^{-1}[\tilde{\psi}_0](2^{j_2}(\zeta+y_2))|,\\
|\mathcal{F}^{-1}[\tilde{\psi}_{j_3}](y_3)-\mathcal{F}^{-1}[\tilde{\psi}_{j_3}](\zeta+y_3)| &= 2^{j_3}|\mathcal{F}^{-1}[\tilde{\psi}_0](2^{j_3}y_3)-\mathcal{F}^{-1}[\tilde{\psi}_0](2^{j_3}(\zeta+y_3))|,
\end{aligned}
$$

and that

$$
\begin{aligned}
\int_{\mathbb{R}}|\mathcal{F}^{-1}[\tilde{\psi}_0](2^{j_1}(y_1+\zeta))|\,\mathrm{d}y_1 &\lesssim 2^{-j_1},\\
\int_{\mathbb{R}}|\mathcal{F}^{-1}[\tilde{\psi}_{j_2}](2^{j_2}y_2)-\mathcal{F}^{-1}[\tilde{\psi}_{j_2}](2^{j_2}(\zeta+y_2))|\,\mathrm{d}y_2 &\lesssim \min\{2^{-j_2},|\zeta|\},\\
\int_{\mathbb{R}}|\mathcal{F}^{-1}[\tilde{\psi}_{j_3}](2^{j_3}y_3)-\mathcal{F}^{-1}[\tilde{\psi}_{j_3}](2^{j_3}(\zeta+y_3))|\,\mathrm{d}y_3 &\lesssim \min\{2^{-j_3},|\zeta|\}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\|\mathcal{F}^{-1}[\partial_{\eta_1}\boldsymbol{T}_1(\eta_1,\eta_2,\eta_3)\tilde{\psi}_{j_1}(\eta_1)\tilde{\psi}_{j_2}(\eta_2)\tilde{\psi}_{j_3}(\eta_3)]\|_{L^1}\\
&\lesssim \int_{\mathbb{R}}\frac{1}{|\zeta|^2}2^{j_2+j_3}\min\{2^{-j_2},|\zeta|\}\min\{2^{-j_3},|\zeta|\}\,\mathrm{d}\zeta\\
&=2^{j_2+j_3}\bigg(\int_{|\zeta|>\max\{2^{-j_2},2^{-j_3}\}}\frac{1}{|\zeta|^2}2^{-j_2-j_3}\,\mathrm{d}\zeta\\
&\qquad\qquad+\int_{\min\{2^{-j_2},2^{-j_3}\}<|\zeta|<\max\{2^{-j_2},2^{-j_3}\}}\frac{1}{|\zeta|}\min\{2^{-j_2},2^{-j_3}\}\,\mathrm{d}\zeta+\int_{|\zeta|<\min\{2^{-j_2},2^{-j_3}\}}1\,\mathrm{d}\zeta\bigg)\\
&\lesssim 2^{\max\{j_2,j_3\}}.
\end{aligned}
$$

(2) Next, we prove (8-18) and (8-19). The estimate of (8-18) is similar to (8-17). We first use the inverse Fourier transform and write

$$
\begin{aligned}
&\mathcal{F}^{-1}[\boldsymbol{T}_1(\eta_1,\eta_2,\eta_3)\tilde{\psi}_{j_1}(\eta_1)\tilde{\psi}_{j_2}(\eta_2)\tilde{\psi}_{j_3}(\eta_3)]\\
&=\iiint_{\mathbb{R}^3} e^{i(y_1\eta_1+y_2\eta_2+y_3\eta_3)}\left[\int_{\mathbb{R}}\frac{\prod_{j=1}^3(1-e^{i\eta_j\zeta})}{|\zeta|^3}\,\mathrm{d}\zeta\right]\tilde{\psi}_{j_1}(\eta_1)\tilde{\psi}_{j_2}(\eta_2)\tilde{\psi}_{j_3}(\eta_3)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\eta_3\\
&=\iiint_{\mathbb{R}^3}\left[\int_{\mathbb{R}}\frac{(e^{iy_1\eta_1}-e^{i\eta_1(\zeta+y_1)})(e^{iy_2\eta_2}-e^{i\eta_2(\zeta+y_2)})(e^{iy_3\eta_3}-e^{i\eta_3(\zeta+y_3)})}{|\zeta|^3}\,\mathrm{d}\zeta\right]\tilde{\psi}_{j_1}(\eta_1)\tilde{\psi}_{j_2}(\eta_2)\tilde{\psi}_{j_3}(\eta_3)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\eta_3\\
&=\int_{\mathbb{R}}\frac{1}{|\zeta|^3}[\mathcal{F}^{-1}[\tilde{\psi}_{j_1}](y_1)-\mathcal{F}^{-1}[\tilde{\psi}_{j_1}](\zeta+y_1)]\cdot[\mathcal{F}^{-1}[\tilde{\psi}_{j_2}](y_2)-\mathcal{F}^{-1}[\tilde{\psi}_{j_2}](\zeta+y_2)]\\
&\qquad\qquad\cdot[\mathcal{F}^{-1}[\tilde{\psi}_{j_3}](y_3)-\mathcal{F}^{-1}[\tilde{\psi}_{j_3}](\zeta+y_3)]\,\mathrm{d}\zeta.
\end{aligned}
$$

Taking the L^1-norm, we obtain

$$
\begin{aligned}
&\|\mathcal{F}^{-1}[\boldsymbol{T}_1(\eta_1,\eta_2,\eta_3)\tilde{\psi}_{j_1}(\eta_1)\tilde{\psi}_{j_2}(\eta_2)\tilde{\psi}_{j_3}(\eta_3)]\|_{L^1}\\
&\qquad\lesssim \int_{\mathbb{R}} 2^{j_1+j_2+j_3}\frac{1}{|\zeta|^3}\min\{2^{-j_1},|\zeta|\}\min\{2^{-j_2},|\zeta|\}\min\{2^{-j_3},|\zeta|\}\,\mathrm{d}\zeta\\
&\qquad\lesssim \int_{|\zeta|>\max\{2^{-j_1},2^{-j_2},2^{-j_3}\}}\frac{1}{|\zeta|^3}\,\mathrm{d}\zeta+\int_{|\zeta|<\min\{2^{-j_1},2^{-j_2},2^{-j_3}\}}2^{j_1+j_2+j_3}\,\mathrm{d}\zeta\\
&\qquad\quad+\int_{\min\{2^{-j_1},2^{-j_2},2^{-j_3}\}<|\zeta|<\operatorname{med}\{2^{-j_1},2^{-j_2},2^{-j_3}\}}2^{\operatorname{med}\{j_1,j_2,j_3\}+\min\{j_1,j_2,j_3\}}\frac{1}{|\zeta|}\,\mathrm{d}\zeta\\
&\qquad\qquad+\int_{\operatorname{med}\{2^{-j_1},2^{-j_2},2^{-j_3}\}<|\zeta|<\max\{2^{-j_1},2^{-j_2},2^{-j_3}\}}2^{\min\{j_1,j_2,j_3\}}\frac{1}{|\zeta|^2}\,\mathrm{d}\zeta\\
&\qquad\lesssim 2^{2\min\{j_1,j_2,j_3\}}+2^{\max\{j_1,j_2,j_3\}+\operatorname{med}\{j_1,j_2,j_3\}}+2^{\max\{j_1,j_2,j_3\}+\min\{j_1,j_2,j_3\}}+2^{\min\{j_1,j_2,j_3\}+\operatorname{med}\{j_1,j_2,j_3\}}\\
&\qquad\lesssim 2^{\max\{j_1,j_2,j_3\}+\min\{j_1,j_2,j_3\}},
\end{aligned}
$$

which proves (8-18).

As for (8-19), we define

$$\tilde{\tilde{\psi}}_k(\eta)=\sum_{j=k-3}^{k+3}\psi_j(\eta).$$

Then it follows from the support of ψ_k and the fact that ψ_k forms a partition of unity that

$$
\begin{aligned}
&\frac{\boldsymbol{T}_1(\eta_1,\eta_2,\eta_3)}{\eta_1}\tilde{\psi}_{j_1}(\eta_1)\tilde{\psi}_{j_2}(\eta_2)\tilde{\psi}_{j_3}(\eta_3)\\
&\qquad\qquad=[\boldsymbol{T}_1(\eta_1,\eta_2,\eta_3)\tilde{\psi}_{j_1}(\eta_1)\tilde{\psi}_{j_2}(\eta_2)\tilde{\psi}_{j_3}(\eta_3)]\cdot\left[\frac{1}{\eta_1}\tilde{\tilde{\psi}}_{j_1}(\eta_1)\tilde{\tilde{\psi}}_{j_2}(\eta_2)\tilde{\tilde{\psi}}_{j_3}(\eta_3)\right].
\end{aligned}
$$

By Lemma 2.5, we have

$$
\begin{aligned}
&\left\|\frac{\boldsymbol{T}_1(\eta_1,\eta_2,\eta_3)}{\eta_1}\tilde{\psi}_{j_1}(\eta_1)\tilde{\psi}_{j_2}(\eta_2)\tilde{\psi}_{j_3}(\eta_3)\right\|_{S^\infty}\\
&\qquad\lesssim\|\boldsymbol{T}_1(\eta_1,\eta_2,\eta_3)\tilde{\psi}_{j_1}(\eta_1)\tilde{\psi}_{j_2}(\eta_2)\tilde{\psi}_{j_3}(\eta_3)\|_{S^\infty}\left\|\frac{\tilde{\tilde{\psi}}_{j_1}(\eta_1)\tilde{\tilde{\psi}}_{j_2}(\eta_2)\tilde{\tilde{\psi}}_{j_3}(\eta_3)}{\eta_1}\right\|_{S^\infty}.
\end{aligned}\tag{8-20}
$$

In view of (8-18), we only need to estimate the second term. To this end, we have

$$\left\|\frac{1}{\eta_1}\tilde{\tilde{\psi}}_{j_1}(\eta_1)\tilde{\tilde{\psi}}_{j_2}(\eta_2)\tilde{\tilde{\psi}}_{j_3}(\eta_3)\right\|_{S^\infty}=\left\|\int_{\mathbb{R}}\eta_1^{-1}\tilde{\tilde{\psi}}_{j_1}(\eta_1)e^{i\eta_1y_1}\,\mathrm{d}\eta_1\mathcal{F}^{-1}[\tilde{\tilde{\psi}}_{j_2}](y_2)\mathcal{F}^{-1}[\tilde{\tilde{\psi}}_{j_3}](y_3)\right\|_{L^1}\lesssim 2^{-j_1}.$$

Therefore, by (8-20) and considering all the possible relations between j_1, j_2, and j_3, we obtain (8-19). □

Applying the above lemmas to (8-16) and (8-15), we obtain

$$\|W_1\|_{L^\infty_\xi}\lesssim(t+1)^{-1}\big[\|\partial_x\varphi_{\max\{j_1,j_2\}}\|_{L^\infty}\|\varphi_{j_3}\|_{L^2}\|\varphi_{\min\{j_1,j_2\}}\|_{L^2}+\|\partial_x\varphi_{\max\{j_2,j_3\}}\|_{L^\infty}\|\varphi_{j_1}\|_{L^2}\|\varphi_{\min\{j_2,j_3\}}\|_{L^2}\big].$$

Since the two terms are symmetric in j_1 and j_3, it suffices to estimate one of them, as the other one is similar. We use Lemma 6.1 and get

$$\|\partial_x\varphi_{\max\{j_1,j_2\}}\|_{L^\infty} \lesssim (t+1)^{-1/2}\||\xi|^{3/2}\hat{h}_{\max\{j_1,j_2\}}\|_{L^\infty_\xi} + (t+1)^{-3/4}\big[\||\partial_x|^{3/4}P_{\max\{j_1,j_2\}}(x\partial_x h)\|_{L^2} + \||\partial_x|^{3/4}h_{\max\{j_1,j_2\}}\|_{L^2}\big].$$

Therefore,

$$\begin{aligned}\|W_1\|_{L^\infty_\xi} \lesssim{}& (t+1)^{-3/2}\big(\mathbf{1}_{\max\{j_1,j_2\}\le 0}2^{(1/2)\max\{j_1,j_2\}}\||\xi|\hat{h}_{\max\{j_1,j_2\}}\|_{L^\infty_\xi} \\ &\qquad + \mathbf{1}_{\max\{j_1,j_2\}>0}2^{(-3/2-r)\max\{j_1,j_2\}}\||\xi|^{r+3}\hat{h}_{\max\{j_1,j_2\}}\|_{L^\infty_\xi}\big)\|\varphi_{j_3}\|_{L^2}\|\varphi_{\min\{j_1,j_2\}}\|_{L^2} \\ &+ (t+1)^{-7/4}\big[\||\partial_x|^{3/4}P_{\max\{j_1,j_2\}}(x\partial_x h)\|_{L^2} + \||\partial_x|^{3/4}h_{\max\{j_1,j_2\}}\|_{L^2}\big]\|\varphi_{j_3}\|_{L^2}\|\varphi_{\min\{j_1,j_2\}}\|_{L^2} \\ &+ (t+1)^{-3/2}\big(\mathbf{1}_{\max\{j_2,j_3\}\le 0}2^{(1/2)\max\{j_2,j_3\}}\||\xi|\hat{h}_{\max\{j_2,j_3\}}\|_{L^\infty_\xi} \\ &\qquad + \mathbf{1}_{\max\{j_2,j_3\}>0}2^{(-3/2-r)\max\{j_2,j_3\}}\||\xi|^{r+3}\hat{h}_{\max\{j_2,j_3\}}\|_{L^\infty_\xi}\big)\|\varphi_{j_1}\|_{L^2}\|\varphi_{\min\{j_2,j_3\}}\|_{L^2} \\ &+ (t+1)^{-7/4}\big[\||\partial_x|^{3/4}P_{\max\{j_2,j_3\}}(x\partial_x h)\|_{L^2} + \||\partial_x|^{3/4}h_{\max\{j_2,j_3\}}\|_{L^2}\big]\|\varphi_{j_1}\|_{L^2}\|\varphi_{\min\{j_2,j_3\}}\|_{L^2} \\ \lesssim{}& (t+1)^{-3/2}\big(\mathbf{1}_{\max\{j_1,j_2\}\le 0}2^{(1/2)\max\{j_1,j_2\}} + \mathbf{1}_{\max\{j_1,j_2\}>0}2^{(-3/2-r)\max\{j_1,j_2\}}\big) \\ &\qquad \cdot\|h_{\max\{j_1,j_2\}}\|_Z\|\varphi_{j_3}\|_{L^2}\|\varphi_{\min\{j_1,j_2\}}\|_{L^2} \\ &+ (t+1)^{-7/4}\big[\||\partial_x|^{3/4}P_{\max\{j_1,j_2\}}(x\partial_x h)\|_{L^2} + \||\partial_x|^{3/4}h_{\max\{j_1,j_2\}}\|_{L^2}\big]\|\varphi_{j_3}\|_{L^2}\|\varphi_{\min\{j_1,j_2\}}\|_{L^2} \\ &+ (t+1)^{-3/2}\big(\mathbf{1}_{\max\{j_2,j_3\}\le 0}2^{(1/2)\max\{j_2,j_3\}} + \mathbf{1}_{\max\{j_2,j_3\}>0}2^{(-3/2-r)\max\{j_2,j_3\}}\big) \\ &\qquad \cdot\|h_{\max\{j_2,j_3\}}\|_Z\|\varphi_{j_1}\|_{L^2}\|\varphi_{\min\{j_2,j_3\}}\|_{L^2} \\ &+ (t+1)^{-7/4}\big[\||\partial_x|^{3/4}P_{\max\{j_2,j_3\}}(x\partial_x h)\|_{L^2} + \||\partial_x|^{3/4}h_{\max\{j_2,j_3\}}\|_{L^2}\big]\|\varphi_{j_1}\|_{L^2}\|\varphi_{\min\{j_2,j_3\}}\|_{L^2}.\end{aligned}$$

Estimate of W_2 and W_3: We rewrite W_2 as

$$\iint_{\mathbb{R}^2}\left[\frac{\boldsymbol{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{it\partial_{\eta_1}\Phi(\xi,\eta_1,\eta_2)(\xi-\eta_1-\eta_2)}\right]e^{it\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)[(\xi-\eta_1-\eta_2)\partial_{\eta_1}\hat{h}_{j_3}(\xi-\eta_1-\eta_2)]\,d\eta_1\,d\eta_2.$$

In view of the multilinear estimate Lemma 2.5, we need to estimate the S^∞-norm of the symbol

$$\frac{\boldsymbol{T}_1(\eta_1,\eta_2,\eta_3)}{(\log|\eta_1|-\log|\eta_3|)\eta_3}\psi_{j_1}(\eta_1)\psi_{j_2}(\eta_2)\tilde{\psi}_{j_3}(\eta_3).$$

In a similar way to the estimates of W_1, using Lemmas 8.1 and 8.2, we obtain

$$\|W_2\|_{L^\infty_\xi} \lesssim (t+1)^{-1}\|\partial_x\varphi_{\max\{j_1,j_2\}}\|_{L^\infty}\|\xi\partial_\xi\hat{h}_{j_3}\|_{L^2_\xi}\|\varphi_{\min\{j_1,j_2\}}\|_{L^2}.$$

Using Lemma 6.1, we have

$$\begin{aligned}\|W_2\|_{L^\infty_\xi} \lesssim{}& (t+1)^{-3/2}(\mathbf{1}_{\max\{j_1,j_2\}\le 0}2^{(1/2)\max\{j_1,j_2\}} + \mathbf{1}_{\max\{j_1,j_2\}>0}2^{(-3/2-r)\max\{j_1,j_2\}}) \\ &\qquad \cdot\|h_{\max\{j_1,j_2\}}\|_Z\|\xi\partial_\xi\hat{h}_{j_3}\|_{L^2_\xi}\|\varphi_{\min\{j_1,j_2\}}\|_{L^2} \\ &+ (t+1)^{-7/4}[\||\partial_x|^{3/4}P_{\max\{j_1,j_2\}}(x\partial_x h)\|_{L^2} + \||\partial_x|^{3/4}h_{\max\{j_1,j_2\}}\|_{L^2}]\|\xi\partial_\xi\hat{h}_{j_3}\|_{L^2_\xi}\|\varphi_{\min\{j_1,j_2\}}\|_{L^2}.\end{aligned}$$

Similarly, we have

$$\|W_3\|_{L^\infty_\xi} \lesssim (t+1)^{-3/2}(\mathbf{1}_{\max\{j_2,j_3\}\le 0}2^{(1/2)\max\{j_2,j_3\}}+\mathbf{1}_{\max\{j_2,j_3\}>0}2^{(-3/2-r)\max\{j_2,j_3\}}) \cdot\|h_{\max\{j_2,j_3\}}\|_Z\|\xi\partial_\xi\hat{h}_{j_1}\|_{L^2_\xi}\|\varphi_{\min\{j_2,j_3\}}\|_{L^2} +(t+1)^{-7/4}[\||\partial_x|^{3/4}P_{\max\{j_2,j_3\}}(x\partial_x h)\|_{L^2}+\||\partial_x|^{3/4}h_{\max\{j_2,j_3\}}\|_{L^2}]\|\xi\partial_\xi\hat{h}_{j_1}\|_{L^2_\xi}\|\varphi_{\min\{j_2,j_3\}}\|_{L^2}.$$

In conclusion, for nonresonant frequencies we have shown that

$$\begin{aligned}
&\left\|\xi(|\xi|+|\xi|^{r+4})\mathfrak{d}(\xi,t)\iint_{\mathbb{R}^2}T_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)e^{it\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\,d\eta_1\,d\eta_2\right\|_{L^\infty_\xi}\\
&\lesssim (t+1)^{(r+5)p_1}(\|W_1\|_{L^\infty_\xi}+\|W_2\|_{L^\infty_\xi}+\|W_3\|_{L^\infty_\xi})\\
&\lesssim (t+1)^{-3/2+(r+5)p_1}(\|\varphi_{j_1}\|_{L^2}\|\varphi_{\min\{j_2,j_3\}}\|_{L^2}+\|\xi\partial_\xi\hat{h}_{j_1}\|_{L^2_\xi}\|\varphi_{\min\{j_2,j_3\}}\|_{L^2})\\
&\qquad\cdot(\mathbf{1}_{\max\{j_2,j_3\}\le 0}2^{(1/2)\max\{j_2,j_3\}}+\mathbf{1}_{\max\{j_2,j_3\}>0}2^{(-3/2-r)\max\{j_2,j_3\}})\|h_{\max\{j_2,j_3\}}\|_Z\\
&\quad+(t+1)^{-3/2+(r+5)p_1}(\|\varphi_{j_3}\|_{L^2}\|\varphi_{\min\{j_1,j_2\}}\|_{L^2}+\|\xi\partial_\xi\hat{h}_{j_3}\|_{L^2_\xi}\|\varphi_{\min\{j_1,j_2\}}\|_{L^2})\\
&\qquad\cdot(\mathbf{1}_{\max\{j_1,j_2\}\le 0}2^{(1/2)\max\{j_1,j_2\}}+\mathbf{1}_{\max\{j_1,j_2\}>0}2^{(-3/2-r)\max\{j_1,j_2\}})\|h_{\max\{j_1,j_2\}}\|_Z\\
&\quad+(t+1)^{-7/4+(r+5)p_1}\Big[(\||\partial_x|^{3/4}P_{\max\{j_1,j_2\}}(x\partial_x h)\|_{L^2}+\||\partial_x|^{3/4}h_{\max\{j_1,j_2\}}\|_{L^2})\|\varphi_{j_3}\|_{L^2}\|\varphi_{\min\{j_1,j_2\}}\|_{L^2}\\
&\qquad+(\||\partial_x|^{3/4}P_{\max\{j_2,j_3\}}(x\partial_x h)\|_{L^2}+\||\partial_x|^{3/4}h_{\max\{j_2,j_3\}}\|_{L^2})\|\varphi_{j_1}\|_{L^2}\|\varphi_{\min\{j_2,j_3\}}\|_{L^2}\\
&\qquad+(\||\partial_x|^{3/4}P_{\max\{j_1,j_2\}}(x\partial_x h)\|_{L^2}+\||\partial_x|^{3/4}h_{\max\{j_1,j_2\}}\|_{L^2})\|\xi\partial_\xi\hat{h}_{j_3}\|_{L^2_\xi}\|\varphi_{\min\{j_1,j_2\}}\|_{L^2}\\
&\qquad+(\||\partial_x|^{3/4}P_{\max\{j_2,j_3\}}(x\partial_x h)\|_{L^2}+\||\partial_x|^{3/4}h_{\max\{j_2,j_3\}}\|_{L^2})\|\xi\partial_\xi\hat{h}_{j_1}\|_{L^2_\xi}\|\varphi_{\min\{j_2,j_3\}}\|_{L^2}\Big].
\end{aligned}$$

By the bootstrap assumptions and Lemma 5.5, the right-hand side is summable for j_1, j_2, j_3 and the sum is integrable for $t\in(0,\infty)$.

8E. *Close to resonance.* When

$$\max\{j_1,j_2,j_3\}<10^{-3}\log_2(t+1),\quad |j_3-j_2|\le 1,\ |j_3-j_1|\le 1, \tag{8-21}$$

in (8-12), we need to consider the following two cases:

(i) The frequencies η_1, η_2 and $\xi-\eta_1-\eta_2$ have the same sign. By the definition of the multiplier P_j and the cutoff function ψ, we can assume that

$$\tfrac{5}{8}2^{j_1}\le|\eta_1|\le\tfrac{8}{5}2^{j_1},\quad \tfrac{5}{8}2^{j_2}\le|\eta_2|\le\tfrac{8}{5}2^{j_2},\quad \tfrac{5}{8}2^{j_3}\le|\xi-\eta_1-\eta_2|\le\tfrac{8}{5}2^{j_3},$$

and therefore

$$\tfrac{5}{8}(2^{j_1}+2^{j_2}+2^{j_3})\le|\xi|\le\tfrac{8}{5}(2^{j_1}+2^{j_2}+2^{j_3}).$$

This corresponds to the region near the space resonance $\eta_1=\eta_2=\xi-\eta_1-\eta_2=\frac{1}{3}\xi$ in (8-4).

(ii) The frequencies η_1, η_2 and $\xi-\eta_1-\eta_2$ do not have the same sign. This corresponds to the region near the space-time resonances $(\eta_1,\eta_2)=(\xi,\xi)$, $(\xi,-\xi)$, or $(-\xi,\xi)$ in (8-3). Since the symbol $T_1'(\eta_1,\eta_2,\eta_3)$ is symmetric in η_1, η_2, and η_3, it suffices to consider (8-12) in the region near (ξ,ξ).

To estimate (8-12) in the region (8-21), we decompose the region further. Writing $(\xi_1,\xi_2,\xi_3)=(\xi,\xi,-\xi)$ or $\left(\frac{1}{3}\xi,\frac{1}{3}\xi,\frac{1}{3}\xi\right)$, we decompose (8-21) using the additional cutoff functions $\psi_{k_1}(\eta_1-\xi_1)$ and

$\psi_{k_2}(\eta_2-\xi_2)$. Since

$$\sum_{(k_1,k_2)\in\mathbb{Z}^2} \psi_{k_1}(\eta_1-\xi_1)\psi_{k_2}(\eta_2-\xi_2)=1,$$

we can write the integral (8-12) as

$$\iint_{\mathbb{R}^2} \boldsymbol{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)e^{it\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2) \cdot\left[\sum_{k_1=-\infty}^{\max\{j_1,j_3\}+1}\psi_{k_1}(\eta_1-\xi_1)\right]\cdot\left[\sum_{k_2=-\infty}^{\max\{j_2,j_3\}+1}\psi_{k_2}(\eta_2-\xi_2)\right]d\eta_1\,d\eta_2,$$

where

$$\left[\sum_{k_1=-\infty}^{\max\{j_1,j_3\}+1}\psi_{k_1}(\eta_1-\xi_1)\right]\cdot\left[\sum_{k_2=-\infty}^{\max\{j_2,j_3\}+1}\psi_{k_2}(\eta_2-\xi_2)\right]=1$$

on the support of $\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2)$. Thus, we need to consider

$$\iint_{\mathbb{R}^2} \boldsymbol{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)e^{it\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2) \cdot\psi_{k_1}(\eta_1-\xi_1)\psi_{k_2}(\eta_2-\xi_2)\,d\eta_1\,d\eta_2. \tag{8-22}$$

In this subsection, we restrict our attention to

$$k_1\ge\log_2[\varrho_1(t)] \quad\text{or}\quad k_2\ge\log_2[\varrho_1(t)],$$

where

$$\varrho_1(t)=(t+1)^{-0.49}. \tag{8-23}$$

The case of $k_1<\log_2[\varrho_1(t)]$ and $k_2<\log_2[\varrho_1(t)]$, related to the resonant frequencies, will be discussed in Section 8F.

Since these expressions are symmetric in η_1 and η_2, we assume without loss of generality that $j_1\ge k_1\ge\log_2[\varrho_1(t)]$. The other case can be discussed in a similar way.

Integrating by parts, we can write (8-22) as

$$\begin{aligned}\iint_{\mathbb{R}^2} &\frac{\boldsymbol{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{2it(\log|\eta_1|-\log|\xi-\eta_1-\eta_2|)}\partial_{\eta_1}e^{it\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2) \cdot\psi_{k_1}(\eta_1-\xi_1)\psi_{k_2}(\eta_2-\xi_2)\,d\eta_1\,d\eta_2\\ &=\frac{i}{2t}(V_1+V_2+V_3+V_4),\end{aligned}$$

where

$$V_1(\xi,t)=\iint_{\mathbb{R}^2}\partial_{\eta_1}\left[\frac{\boldsymbol{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{\log|\eta_1|-\log|\xi-\eta_1-\eta_2|}\right]e^{it\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2) \cdot\psi_{k_1}(\eta_1-\xi_1)\psi_{k_2}(\eta_2-\xi_2)\,d\eta_1\,d\eta_2,$$

$$V_2(\xi,t)=\iint_{\mathbb{R}^2}\left[\frac{\boldsymbol{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{\log|\eta_1|-\log|\xi-\eta_1-\eta_2|}\right]e^{it\Phi(\xi,\eta_1,\eta_2)}\partial_{\eta_1}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2) \cdot\psi_{k_1}(\eta_1-\xi_1)\psi_{k_2}(\eta_2-\xi_2)\,d\eta_1\,d\eta_2,$$

$$V_3(\xi,t)=\iint_{\mathbb{R}^2}\left[\frac{\boldsymbol{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{\log|\eta_1|-\log|\xi-\eta_1-\eta_2|}\right]e^{it\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\partial_{\eta_1}\hat{h}_{j_3}(\xi-\eta_1-\eta_2) \cdot\psi_{k_1}(\eta_1-\xi_1)\psi_{k_2}(\eta_2-\xi_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2,$$

$$V_4(\xi,t)=\iint_{\mathbb{R}^2}\left[\frac{\boldsymbol{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{\log|\eta_1|-\log|\xi-\eta_1-\eta_2|}\right]e^{it\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2) \cdot\partial_{\eta_1}\psi_{k_1}(\eta_1-\xi_1)\psi_{k_2}(\eta_2-\xi_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2.$$

Estimate of V_1: We first denote the symbol for V_1 as

$$\begin{aligned}
&m(\eta_1,\eta_2,\xi)\\
&=\frac{-2}{\log|\eta_1|-\log|\xi-\eta_1-\eta_2|}\cdot\big[\eta_1\log|\eta_1|-(\eta_1+\eta_2)\log|\eta_1+\eta_2|\\
&\qquad+(\xi-\eta_1)\log|\xi-\eta_1|-(\xi-\eta_1-\eta_2)\log|\xi-\eta_1-\eta_2|\big]\\
&\quad-\frac{\eta_1^{-1}+(\xi-\eta_1-\eta_2)^{-1}}{(\log|\eta_1|-\log|\xi-\eta_1-\eta_2|)^2}\cdot\big[-\eta_1^2\log|\eta_1|-\eta_2^2\log|\eta_2|-\eta_3^2\log|\eta_3|\\
&\qquad-(\eta_1+\eta_2+\eta_3)^2\log|\eta_1+\eta_2+\eta_3|+(\eta_1+\eta_2)^2\log|\eta_1+\eta_2|\\
&\qquad+(\eta_1+\eta_3)^2\log|\eta_1+\eta_3|+(\eta_2+\eta_3)^2\log|\eta_2+\eta_3|\big].
\end{aligned}$$

After the change of variables $\upsilon_i=\eta_i-\xi_i$, $i=1,2$, it suffices to estimate

$$\left\|\iint_{\mathbb{R}^2}m(\upsilon_1+\xi_1,\upsilon_2+\xi_2,\xi)e^{it\Phi(\xi,\upsilon_1+\xi_1,\upsilon_2+\xi_2)}\hat{h}_{j_1}(\upsilon_1+\xi_1)\hat{h}_{j_2}(\upsilon_2+\xi_2)\hat{h}_{j_3}(\xi_3-\upsilon_1-\upsilon_2) \cdot\psi_{k_1}(\upsilon_1)\psi_{k_2}(\upsilon_2)\,\mathrm{d}\upsilon_1\,\mathrm{d}\upsilon_2\right\|_{L^\infty_\xi}.$$

Using Lemma 2.5, we have

$$\|V_1\|_{L^\infty_\xi}\lesssim\|\chi^{k_1,k_2}_{j_1,j_3}(\upsilon_1,\upsilon_2,\xi)m(\upsilon_1+\xi_1,\upsilon_2+\xi_2,\xi)\|_{S^\infty_{\upsilon_1,\upsilon_2}L^\infty_\xi} \cdot\|\hat{\varphi}_{j_1}(\upsilon_1+\xi_1)\psi_{k_1}(\upsilon_1)\|_{L^2_{\upsilon_1}L^\infty_\xi}\|\hat{\varphi}_{j_2}(\upsilon_2+\xi_2)\psi_{k_2}(\upsilon_2)\|_{L^2_{\upsilon_2}L^\infty_\xi}\|\varphi_{j_3}\|_{L^\infty},$$

where

$$\chi^{k_1,k_2}_{j_1,j_3}(\upsilon_1,\upsilon_2,\xi)=\tilde{\psi}_{k_1}(\upsilon_1)\tilde{\psi}_{k_2}(\upsilon_2)\tilde{\psi}_{j_1}(\upsilon_1+\xi_1)\tilde{\psi}_{j_2}(\upsilon_2+\xi_2)\tilde{\psi}_{j_3}(\xi_3-\upsilon_1-\upsilon_2)\chi(\xi).$$

(i) If $(\xi_1,\xi_2,\xi_3)=\big(\frac{1}{3}\xi,\frac{1}{3}\xi,\frac{1}{3}\xi\big)$, since S^∞-norm is rotational and scaling invariant, setting $w_1=\upsilon_1$, $w_2=-2\upsilon_1-\upsilon_2$, and using (2-16), we have

$$\begin{aligned}
&\|\chi^{k_1,k_2}_{j_1,j_3}(\upsilon_1,\upsilon_2,\xi)m(\upsilon_1+\xi_1,\upsilon_2+\xi_2,\xi)\|_{S^\infty_{\upsilon_1,\upsilon_2}L^\infty_\xi}\\
&\qquad=\|\chi^{k_1,k_2}_{j_1,j_3}(w_1,-2w_1-w_2,\xi)m(w_1+\xi_1,-2w_1-w_2+\xi_2,\xi)\|_{S^\infty_{w_1,w_2}L^\infty_\xi}\\
&\qquad\lesssim\|\chi^{k_1,k_2}_{j_1,j_3}(w_1,-2w_1-w_2,\xi)m(w_1+\xi_1,-2w_1-w_2+\xi_2,\xi)\|^{1/4}_{L^1_{w_1w_2}L^\infty_\xi}\\
&\qquad\quad\cdot\|\partial^2_{w_1}[\chi^{k_1,k_2}_{j_1,j_3}(w_1,-2w_1-w_2,\xi)m(w_1+\xi_1,-2w_1-w_2+\xi_2,\xi)]\|^{1/2}_{L^1_{w_1w_2}L^\infty_\xi}\\
&\qquad\quad\cdot\|\partial^2_{w_1}\partial^2_{w_2}[\chi^{k_1,k_2}_{j_1,j_3}(w_1,-2w_1-w_2,\xi)m(w_1+\xi_1,-2w_1-w_2+\xi_2,\xi)]\|^{1/4}_{L^1_{w_1w_2}L^\infty_\xi}\\
&\qquad\lesssim(1+|j_1|)(2^{j_1}2^{j_1})^{1/4}(2^{-j_1-2k_1}2^{j_1})^{1/2}(2^{-j_1-3k_1-k_2}2^{j_1})^{1/4}\\
&\qquad=(1+|j_1|)\cdot 2^{j_1/2-(7k_1+k_2)/4}\le(1+|j_1|)\cdot 2^{j_1/2-k_1-k_2},
\end{aligned}$$

where we have used the estimate

$$\left|\frac{\chi_{j_1,j_3}^{k_1,k_2}}{\log\left|w_1+\frac{1}{3}\xi\right|-\log\left|\frac{1}{3}\xi+w_1+w_2\right|}\right|\lesssim 2^{j_1-k_1},$$

$$\left|\chi_{j_1,j_3}^{k_1,k_2}\partial_{w_1}^2\frac{1}{\log\left|w_1+\frac{1}{3}\xi\right|-\log\left|\frac{1}{3}\xi+w_1+w_2\right|}\right|\lesssim 2^{3(j_1-k_1)}2^{2(-2j_1+k_1)}=2^{-j_1-k_1},$$

$$\left|\chi_{j_1,j_3}^{k_1,k_2}\partial_{w_1}^2\partial_{w_2}^2\frac{1}{\log\left|w_1+\frac{1}{3}\xi\right|-\log\left|\frac{1}{3}\xi+w_1+w_2\right|}\right|\lesssim 2^{5(j_1-k_1)}2^{-2j_1}2^{2(-2j_1+k_1)}=2^{-j_1-3k_1}.$$

Therefore, using (2-14), (6-1), and (8-21), we obtain

$$\begin{aligned}\|V_1\|_{L^\infty_\xi}&\\ &\lesssim(1+|j_1|)2^{j_1/2-k_1-k_2-j_3}(t+1)^{-1}\|\hat\varphi_{j_1}(\upsilon_1+\xi_1)\psi_{k_1}(\upsilon_1)\|_{L^2_{\upsilon_1}L^\infty_\xi}\|\hat\varphi_{j_2}(\upsilon_2+\xi_2)\psi_{k_2}(\upsilon_2)\|_{L^2_{\upsilon_2}L^\infty_\xi}\|\partial_x\varphi_{j_3}\|_{L^\infty}\\ &\lesssim(1+|j_1|)2^{-j_1/2-k_1/2-k_2/2}\|\psi_{k_1}\hat\varphi_{j_1}\|_{L^\infty_\xi}\|\psi_{k_2}\hat\varphi_{j_2}\|_{L^\infty_\xi}\\ &\qquad\cdot\{(t+1)^{-3/2}\||\xi|^{3/2}\hat h_{j_3}^{k_1,k_2}\|_{L^\infty_\xi}+(t+1)^{-7/4}[\||\partial_x|^{3/4}P_{j_3}^{k_1,k_2}(x\partial_x h)\|_{L^2}+\||\partial_x|^{3/4}h_{j_3}^{k_1,k_2}\|_{L^2}]\}.\end{aligned}\tag{8-24}$$

(ii) If $(\xi_1,\xi_2,\xi_3)=(\xi,\xi,-\xi)$, we use (2-16) to obtain

$$\begin{aligned}&\|\chi_{j_1,j_3}^{k_1,k_2}(\upsilon_1,\upsilon_2,\xi)m(\upsilon_1+\xi_1,\upsilon_2+\xi_2,\xi)\|_{S^\infty_{\upsilon_1,\upsilon_2}L^\infty_\xi}\\ &\lesssim\|\chi_{j_1,j_3}^{k_1,k_2}(\upsilon_1,\upsilon_2,\xi)m(\upsilon_1+\xi_1,\upsilon_2+\xi_2,\xi)\|_{L^1_{\upsilon_1\upsilon_2}}^{1/4}\|\partial_{\upsilon_1}^2[\chi_{j_1,j_3}^{k_1,k_2}(\upsilon_1,\upsilon_2,\xi)m(\upsilon_1+\xi_1,\upsilon_2+\xi_2,\xi)]\|_{L^1_{\upsilon_1\upsilon_2}}^{1/2}\\ &\qquad\qquad\cdot\|\partial_{\upsilon_1}^2\partial_{\upsilon_2}^2[\chi_{j_1,j_3}^{k_1,k_2}(\upsilon_1,\upsilon_2,\xi)m(\upsilon_1+\xi_1,\upsilon_2+\xi_2,\xi)]\|_{L^1_{\upsilon_1\upsilon_2}}^{1/4}\\ &\lesssim(1+|j_1|)(2^{j_1+k_1}2^{j_1})^{1/4}(2^{-j_1-k_1}2^{j_1})^{1/2}(2^{-j_1-2k_1-k_2}2^{j_1})^{1/4}\\ &=(1+|j_1|)\cdot 2^{j_1/2-k_2},\end{aligned}$$

where we have used the estimates

$$\left|\frac{\chi_{j_1,j_3}^{k_1,k_2}}{\log|\upsilon_1+\xi|-\log|-\xi-\upsilon_1-\upsilon_2|}\right|\lesssim 2^{j_1-k_2},$$

$$\left|\chi_{j_1,j_3}^{k_1,k_2}\partial_{\upsilon_1}^2\frac{1}{\log|\upsilon_1+\xi|-\log|-\xi-\upsilon_1-\upsilon_2|}\right|\lesssim 2^{3(j_1-k_2)}2^{2(-2j_1+k_2)}=2^{-j_1-k_2},$$

$$\left|\chi_{j_1,j_3}^{k_1,k_2}\partial_{\upsilon_1}^2\partial_{\upsilon_2}^2\frac{1}{\log|\upsilon_1+\xi|-\log|-\xi-\upsilon_1-\upsilon_2|}\right|\lesssim 2^{5(j_1-k_2)}2^{-2j_1}2^{2(-2j_1+k_2)}=2^{-j_1-3k_2}.$$

Therefore, using (2-14), (6-1), and (8-21)

$$\begin{aligned}\|V_1\|_{L^\infty_\xi}&\lesssim(1+|j_1|)2^{j_1/2-j_3-k_2}(t+1)^{-1}\|\hat\varphi_{j_1}(\upsilon_1+\xi_1)\psi_{k_1}(\upsilon_1)\|_{L^2_{\upsilon_1}L^\infty_\xi}\|\hat\varphi_{j_2}(\upsilon_2+\xi_2)\psi_{k_2}(\upsilon_2)\|_{L^2_{\upsilon_2}L^\infty_\xi}\|\partial_x\varphi_{j_3}\|_{L^\infty}\\ &\lesssim(1+|j_1|)2^{-j_1/2+k_1/2-k_2/2}\|\psi_{k_1}\hat\varphi_{j_1}\|_{L^\infty_\xi}\|\psi_{k_2}\hat\varphi_{j_2}\|_{L^\infty_\xi}\\ &\qquad\cdot\{(t+1)^{-3/2}\||\xi|^{3/2}\hat h_{j_3}\|_{L^\infty_\xi}+(t+1)^{-7/4}[\||\partial_x|^{3/4}P_{j_3}(x\partial_x h)\|_{L^2}+\||\partial_x|^{3/4}h_{j_3}\|_{L^2}]\}.\end{aligned}\tag{8-25}$$

Estimates of V_2–V_4: The estimates for V_2–V_4 are similar to V_1. We omit the details here. The resulting estimates are as follows.

The symbol for V_2–V_4 can be estimated as

$$\left\| \chi_{j_1,j_3}^{k_1,k_2}(\upsilon_1,\upsilon_2,\xi)\frac{\boldsymbol{T}_1'(\xi_1+\upsilon_1,\xi_2+\upsilon_2,\xi_3-\upsilon_1-\upsilon_2)}{\log|\xi_1+\upsilon_1|-\log|\xi_3-\upsilon_1-\upsilon_2|}\right\|_{S^\infty_{\upsilon_1\upsilon_2}L^\infty_\xi} \lesssim (2^{j_1+k_1}2^{2j_1})^{1/4}(2^{-j_1-k_1}2^{2j_1})^{1/2}(2^{-j_1-2k_2-k_1}2^{2j_1})^{1/4} = (1+|j_1|)\cdot 2^{3j_1/2-k_2}.$$

We then have the following estimates:

$$\|V_2\|_{L^\infty_\xi} \lesssim (1+|j_1|)2^{-j_1/2+k_1/2-k_2}\|\eta_1\partial_{\eta_1}\hat{\varphi}_{j_1}(\eta_1)\|_{L^2_{\eta_1}}\|\psi_{k_2}\hat{\varphi}_{j_2}\|_{L^\infty_\xi} \cdot\{(t+1)^{-3/2}\||\xi|^{3/2}\hat{h}_{j_3}\|_{L^\infty_\xi}+(t+1)^{-7/4}[\||\partial_x|^{3/4}P_{j_3}(x\partial_x h)\|_{L^2}+\||\partial_x|^{3/4}h_{j_3}\|_{L^2}]\}, \tag{8-26}$$

$$\|V_3\|_{L^\infty_\xi} \lesssim (1+|j_1|)2^{-j_1/2+k_1/2-k_2}\|\eta_3\partial_{\eta_3}\hat{\varphi}_{j_3}(\eta_3)\|_{L^2_{\eta_3}}\|\psi_{k_2}\hat{\varphi}_{j_2}\|_{L^\infty_\xi} \cdot\{(t+1)^{-3/2}\||\xi|^{3/2}\hat{h}_{j_1}\|_{L^\infty_\xi}+(t+1)^{-7/4}[\||\partial_x|^{3/4}P_{j_1}(x\partial_x h)\|_{L^2}+\||\partial_x|^{3/4}h_{j_1}\|_{L^2}]\}, \tag{8-27}$$

$$\|V_4\|_{L^\infty_\xi} \lesssim (1+|j_1|)2^{j_1/2-k_2}\|\psi_{k_1}\hat{\varphi}_{j_1}\|_{L^\infty_\xi}\|\psi_{k_2}\hat{\varphi}_{j_2}\|_{L^\infty_\xi} \cdot\{(t+1)^{-3/2}\||\xi|^{3/2}\hat{h}_{j_3}\|_{L^\infty_\xi}+(t+1)^{-7/4}[\||\partial_x|^{3/4}P_{j_3}(x\partial_x h)\|_{L^2}+\||\partial_x|^{3/4}h_{j_3}\|_{L^2}]\}. \tag{8-28}$$

Finally, we sum over $\log_2[\varrho_1(t)]\le k_1, k_2\le \max\{j_1, j_3\}+1$, and combine the estimates (8-24)–(8-28) to get

$$\begin{aligned}&\left\|\xi(|\xi|+|\xi|^{r+4})\mathfrak{d}(\xi,t)\iint_{\mathbb{R}^2}\boldsymbol{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)e^{iAt\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\right.\\ &\qquad\left.\cdot\left[\sum_{k_1=\log_2[\varrho_1(t)]}^{\max\{j_1,j_3\}+1}\psi_{k_1}(\eta_1-\xi_1)\right]\cdot\left[\sum_{k_2=\log_2[\varrho_1(t)]}^{\max\{j_2,j_3\}+1}\psi_{k_2}(\eta_2-\xi_2)\right]\mathrm{d}\eta_1\,\mathrm{d}\eta_2\right\|_{L^\infty_\xi}\\ &\lesssim (1+|j_1|)[\max\{j_1,j_3\}-\log_2[\varrho_1(t)]]^2(t+1)^{(r+4)p_1}[\varrho_1(t)]^{-1}\\ &\quad\cdot[\||\xi|\psi_{k_1}\hat{\varphi}_{j_1}\|_{L^\infty_\xi}\||\xi|\psi_{k_2}\hat{\varphi}_{j_2}\|_{L^\infty_\xi}+\|\eta_1\partial_{\eta_1}\hat{\varphi}_{j_1}(\eta_1)\|_{L^2_{\eta_1}}\||\xi|\psi_{k_2}\hat{\varphi}_{j_2}\|_{L^\infty_\xi}+\||\xi|\psi_{k_1}\hat{\varphi}_{j_1}\|_{L^\infty_\xi}\|\eta_2\partial_{\eta_2}\hat{\varphi}_{j_2}(\eta_2)\|_{L^2_{\eta_2}}]\\ &\quad\cdot\{(t+1)^{-3/2}\||\xi|^{3/2}\hat{h}_{j_3}\|_{L^\infty_\xi}+(t+1)^{-7/4}[\||\partial_x|^{3/4}P_{j_3}(x\partial_x h)\|_{L^2}+\||\partial_x|^{3/4}h_{j_3}\|_{L^2}]\}.\end{aligned}$$

The right-hand side is summable with respect to j_1, j_2, j_3 under $|j_3-j_2|\le 1$ and $|j_3-j_1|\le 1$, since we can write

$$\||\xi|^{3/2}\hat{h}_j\|_{L^\infty_\xi}\lesssim(\mathbf{1}_{j\le 0}2^{j/2}+\mathbf{1}_{j>0}2^{(-r-3/2)j})\|h_j\|_Z,$$

and the resulting sum is integrable for $t\in(0,\infty)$ due to (8-23).

8F. ***Resonant frequencies.*** In this section, we estimate (8-22) in the region

$$|j_1-j_3|\le 1,\quad |j_2-j_3|\le 1,\quad k_1<\log_2[\varrho_1(t)],\quad k_2<\log_2[\varrho_1(t)],$$

after summing over (k_1, k_2).

If $m<\log_2\varrho_1(t)\le m+1$ for $m\in\mathbb{Z}$, then $-\infty<k_i\le m$ for $i=1,2$, and

$$\sum_{k_i=-\infty}^{m}\psi_{k_i}(\xi)\quad\text{is supported in }\left\{\xi\in\mathbb{R}\,\middle|\,|\xi|<\tfrac{8}{5}\cdot 2^m\right\}.$$

Thus, after summing over (k_1, k_2), we only need to consider (8-22) with a cutoff function in the integrand of the form

$$\mathfrak{b}(\xi, \eta_1, \eta_2, t) = \psi\left(\frac{\eta_1 - \xi_1}{\varrho(t)}\right) \cdot \psi\left(\frac{\eta_2 - \xi_2}{\varrho(t)}\right), \tag{8-29}$$

where the function $\varrho(t)$ is defined by

$$\varrho(t) = 2^m \quad \text{if } 2^m < \varrho_1(t) \le 2^{m+1}.$$

In particular, from (8-23), we have

$$\tfrac{1}{2}(t+1)^{-0.49} \le \varrho(t) < (t+1)^{-0.49}. \tag{8-30}$$

The point

$$(\xi_1, \xi_2) \in \left\{\left(\tfrac{1}{3}\xi, \tfrac{1}{3}\xi\right), (\xi, \xi), (\xi, -\xi), (-\xi, \xi)\right\}$$

is one of the resonance points in (8-3)–(8-4). We therefore need to estimate (8-22) with the cutoff function (8-29) replacing $\psi_{k_1}(\eta_1 - \xi_1)\psi_{k_2}(\eta_2 - \xi_2)$, in which case the integral is taken over one of the following four disjoint sets:

$$\begin{aligned}
A_1 &= \left\{(\eta_1, \eta_2) \;\middle|\; \left|\eta_1 - \tfrac{1}{3}\xi\right| < \tfrac{8}{5}\varrho(t),\ \left|\eta_2 - \tfrac{1}{3}\xi\right| < \tfrac{8}{5}\varrho(t)\right\},\\
A_2 &= \left\{(\eta_1, \eta_2) \;\middle|\; |\eta_1 - \xi| < \tfrac{8}{5}\varrho(t),\ |\eta_2 - \xi| < \tfrac{8}{5}\varrho(t)\right\},\\
A_3 &= \left\{(\eta_1, \eta_2) \;\middle|\; |(\eta_1 - \xi)| < \tfrac{8}{5}\varrho(t),\ |\eta_2 - (-\xi)| < \tfrac{8}{5}\varrho(t)\right\},\\
A_4 &= \left\{(\eta_1, \eta_2) \;\middle|\; |\eta_1 - (-\xi)| < \tfrac{8}{5}\varrho(t),\ |\eta_2 - \xi| < \tfrac{8}{5}\varrho(t)\right\}.
\end{aligned}$$

The regions A_1, A_2, A_3, A_4 are discs centered at $\left(\frac{1}{3}\xi, \frac{1}{3}\xi\right)$, (ξ, ξ), $(\xi, -\xi)$, and $(-\xi, \xi)$, respectively. The region A_1 corresponds to space resonances $\xi = \frac{1}{3}\xi + \frac{1}{3}\xi + \frac{1}{3}\xi$, while A_2, A_3, A_4 correspond to space-time resonances $\xi = \xi + \xi - \xi$.

8F1. *Space resonances.* When $(\eta_1, \eta_2) \in A_1$, we can expand $\boldsymbol{T}_1/\Phi$ around $\left(\xi, \frac{1}{3}\xi, \frac{1}{3}\xi\right)$ as

$$\frac{\boldsymbol{T}_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{\Phi(\xi, \eta_1, \eta_2)} = \left(\frac{1}{2} - \frac{2\log 2}{3\log 3}\right)\xi + O\left(\left|\eta_1 - \frac{\xi}{3}\right|^2 + \left|\eta_2 - \frac{\xi}{3}\right|^2\right). \tag{8-31}$$

For $m \in \mathbb{Z}$, let $t_m = 2^{-m/0.49} - 1$ denote the time such that $\log_2 \varrho_1(t_m) = m$, and for $t \in [0, \infty)$, let $M(t) \in \mathbb{Z}$ be the negative integer such that $M(t) < \log_2 \varrho_1(t) \le M(t) + 1$. Then $\varrho(t)$ and the cut-off function $\mathfrak{b}(\xi, \eta_1, \eta_2, t)$ in (8-29) are discontinuous at $t = t_m$. After writing

$$e^{i\tau\Phi(\xi,\eta_1,\eta_2)} = \frac{1}{i\Phi(\xi, \eta_1, \eta_2)}[\partial_\tau e^{i\tau\Phi(\xi,\eta_1,\eta_2)}],$$

and integrating by parts with respect to τ in each time interval between the time discontinuities, we get

$$\int_0^t i\xi \iint_{\mathbb{R}^2} \boldsymbol{T}_1(\eta_1, \eta_2, \xi-\eta_1-\eta_2) e^{i\tau\Phi(\xi,\eta_1,\eta_2)} \hat{h}_{j_1}(\eta_1, \tau)\hat{h}_{j_2}(\eta_2, \tau)\hat{h}_{j_3}(\xi-\eta_1-\eta_2, \tau)\mathfrak{b}(\xi, \eta_1, \eta_2, \tau)\, d\eta_1\, d\eta_2\, d\tau$$
$$= J_1 - \int_0^t J_2(\tau)\, d\tau,$$

where

$$J_1 = \iint_{\mathbb{R}^2} \xi \frac{\boldsymbol{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{\Phi(\xi,\eta_1,\eta_2)} \hat{h}_{j_1}(\eta_1,\tau)\hat{h}_{j_2}(\eta_2,\tau)\hat{h}_{j_3}(\xi-\eta_1-\eta_2,\tau) \cdot e^{i\tau\Phi(\xi,\eta_1,\eta_2)}\mathfrak{b}(\xi,\eta_1,\eta_2,\tau)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\Big|_{\tau=t_{M(t)}}^{\tau=t}$$
$$+ \sum_{m=M(t)+1}^{0} \iint_{\mathbb{R}^2} \xi \frac{\boldsymbol{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{\Phi(\xi,\eta_1,\eta_2)} \hat{h}_{j_1}(\eta_1,\tau)\hat{h}_{j_2}(\eta_2,\tau)\hat{h}_{j_3}(\xi-\eta_1-\eta_2,\tau) \cdot e^{i\tau\Phi(\xi,\eta_1,\eta_2)}\mathfrak{b}(\xi,\eta_1,\eta_2,\tau)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\Big|_{\tau=t_m}^{\tau=t_{m-1}},$$

$$J_2(\tau) = \xi \iint_{\mathbb{R}^2} \frac{\boldsymbol{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{\Phi(\xi,\eta_1,\eta_2)} e^{i\tau\Phi(\xi,\eta_1,\eta_2)} \cdot \partial_\tau[\hat{h}_{j_1}(\eta_1,\tau)\hat{h}_{j_2}(\eta_2,\tau)\hat{h}_{j_3}(\xi-\eta_1-\eta_2,\tau)]\mathfrak{b}(\xi,\eta_1,\eta_2,\tau)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2.$$

For J_1, we have from (8-31) that

$$\begin{aligned}
&\left|(|\xi|+|\xi|^{r+4})\iint_{\mathbb{R}^2}\mathfrak{b}(\xi,\eta_1,\eta_2,t)\xi\frac{\boldsymbol{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1,\tau)\hat{h}_{j_2}(\eta_2,\tau)\hat{h}_{j_3}(\xi-\eta_1-\eta_2,\tau)\cdot e^{i\tau\Phi(\xi,\eta_1,\eta_2)}\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\right| \\
&\lesssim \left|(|\xi|+|\xi|^{r+4})\iint_{\mathbb{R}^2}\mathfrak{b}(\xi,\eta_1,\eta_2,t)\xi^2\hat{h}_{j_1}(\eta_1,\tau)\hat{h}_{j_2}(\eta_2,\tau)\hat{h}_{j_3}(\xi-\eta_1-\eta_2,\tau)e^{i\tau\Phi(\xi,\eta_1,\eta_2)}\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\right| \\
&\qquad + (|\xi|+|\xi|^{r+4})\iint_{\mathbb{R}^2}\mathfrak{b}(\xi,\eta_1,\eta_2,t)[\varrho(\tau)]^2|\hat{h}_{j_1}(\eta_1,\tau)\hat{h}_{j_2}(\eta_2,\tau)\hat{h}_{j_3}(\xi-\eta_1-\eta_2,\tau)|\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2 \\
&\lesssim (\tau+1)^{2p_0+(r+3)p_1}\||\xi|\hat{h}_{j_1}\|_{L^\infty_\xi}\||\xi|\hat{h}_{j_2}\|_{L^\infty_\xi}\||\xi|\hat{h}_{j_3}\|_{L^\infty_\xi}([\varrho(\tau)]^2+[\varrho(\tau)]^4).
\end{aligned}$$

Notice that for A_1, the number of summations over j_1, j_2, j_3 in J_1 and over m are of the order $\log(t+1)$. Therefore, the right-hand side of this inequality is uniformly bounded for $\tau \geq 0$ after summing over j_1, j_2, j_3.

After taking the time derivative, the term J_2 can be written as a sum of three terms:

$$\xi\iint_{\mathbb{R}^2}\frac{\boldsymbol{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{\Phi(\xi,\eta_1,\eta_2)}e^{i\tau\Phi(\xi,\eta_1,\eta_2)}[\partial_\tau\hat{h}_{j_1}(\eta_1,\tau)\hat{h}_{j_2}(\eta_2,\tau)\hat{h}_{j_3}(\xi-\eta_1-\eta_2,\tau)]\mathfrak{b}(\xi,\eta_1,\eta_2,\tau)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2,$$
$$\xi\iint_{\mathbb{R}^2}\frac{\boldsymbol{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{\Phi(\xi,\eta_1,\eta_2)}e^{i\tau\Phi(\xi,\eta_1,\eta_2)}[\hat{h}_{j_1}(\eta_1,\tau)\partial_\tau\hat{h}_{j_2}(\eta_2,\tau)\hat{h}_{j_3}(\tau,\xi-\eta_1-\eta_2)]\mathfrak{b}(\xi,\eta_1,\eta_2,\tau)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2,$$
$$\xi\iint_{\mathbb{R}^2}\frac{\boldsymbol{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{\Phi(\xi,\eta_1,\eta_2)}e^{i\tau\Phi(\xi,\eta_1,\eta_2)}[\hat{h}_{j_1}(\eta_1,\tau)\hat{h}_{j_2}(\eta_2,\tau)\partial_\tau\hat{h}_{j_3}(\xi-\eta_1-\eta_2,\tau)]\mathfrak{b}(\xi,\eta_1,\eta_2,\tau)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2.$$

Notice that by (8-8),and the bootstrap assumptions and Lemma 5.3, we have

$$\begin{aligned}
\|\partial_t\hat{h}\|_{L^\infty_\xi} &\lesssim \left\|\xi\iint_{\mathbb{R}^2}\boldsymbol{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)e^{it\Phi(\xi,\eta_1,\eta_2)}\hat{h}(\xi-\eta_1-\eta_2)\hat{h}(\eta_1)\hat{h}(\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\right\|_{L^\infty_\xi} + \|\widehat{\mathcal{N}_{\geq 5}(\varphi)}\|_{L^\infty_\xi} \\
&\lesssim \|\partial_x\{\varphi^2\log|\partial_x|\varphi_{xx} - \varphi\log|\partial_x|(\varphi^2)_{xx} + \tfrac{1}{3}\log|\partial_x|(\varphi^3)_{xx}\}\|_{L^1} + \|\mathcal{N}_{\geq 5}(\varphi)\|_{L^1} \\
&\lesssim \|\varphi\|^2_{H^s}\cdot\sum_{j=0}^{\infty}(\|\varphi_x\|^{2j+1}_{W^{3,\infty}} + \|L\varphi_x\|^{2j+1}_{W^{3,\infty}}) \lesssim \epsilon_1^3(t+1)^{2p_0-1/2}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}|(|\xi|+|\xi|^{r+4})J_2(\tau)| &\lesssim \sum \|h_{\ell_1}\|_Z \|\partial_\tau \hat{h}_{\ell_2}\|_{L^\infty_\xi} \|h_{\ell_3}\|_Z [\varrho(\tau)]^2 \\ &\lesssim \epsilon_1^3 (\tau+1)^{p_0-1/2} [\varrho(\tau)]^2 \sum \|h_{\ell_1}\|_Z \|h_{\ell_3}\|_Z,\end{aligned}$$

where we sum over all permutations (ℓ_1, ℓ_2, ℓ_3) of (j_1, j_2, j_3) in the space resonance region A_1. Again, we notice that the number of summations is of order $\log(\tau+1)$, and the resulting sum is integrable for $\tau \in (1, \infty)$.

8F2. *Space-time resonances.* We now use modified scattering to consider the term in (8-11) given by

$$\begin{aligned}\iint_{A_2 \bigcup A_3 \bigcup A_4} \frac{i\xi}{6} \mathfrak{b}(\xi, \eta_1, \eta_2, t) \boldsymbol{T}_1(\eta_1, \eta_2, \xi-\eta_1-\eta_2) e^{it\Phi(\xi,\eta_1,\eta_2)} \hat{h}(\xi-\eta_1-\eta_2)\hat{h}(\eta_1)\hat{h}(\eta_2)\, \mathrm{d}\eta_1\, \mathrm{d}\eta_2 \\ - \frac{\pi i \xi|\xi|}{3(t+1)} [\boldsymbol{T}_1(\xi,\xi,-\xi) + \boldsymbol{T}_1(\xi,-\xi,\xi) + \boldsymbol{T}_1(-\xi,\xi,\xi)] |\hat{h}(t,\xi)|^2 \hat{h}(t,\xi). \quad (8\text{-}32)\end{aligned}$$

The estimates for A_2, A_3, and A_4 are similar, so we only present the details for the A_2 integral. The corresponding integral for A_2 in (8-32) can be decomposed into

$$\begin{aligned}\frac{i\xi}{6} \iint_{A_2} & e^{it\Phi(\xi,\eta_1,\eta_2)} \mathfrak{b}(\xi,\eta_1,\eta_2,t) \\ & \cdot \big[\boldsymbol{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2) \hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2) - \boldsymbol{T}_1(\xi,\xi,-\xi)|\hat{h}(\xi)|^2\hat{h}(\xi)\big]\, \mathrm{d}\eta_1\, \mathrm{d}\eta_2 \quad (8\text{-}33)\end{aligned}$$

and

$$\frac{i\xi}{6} \boldsymbol{T}_1(\xi,\xi,-\xi)|\hat{h}(t,\xi)|^2\hat{h}(t,\xi) \left[\iint_{A_2} e^{it\Phi(\xi,\eta_1,\eta_2)} \psi\left(\frac{\eta_1-\xi}{\varrho(t)}\right) \cdot \psi\left(\frac{\eta_2-\xi}{\varrho(t)}\right) \mathrm{d}\eta_1\, \mathrm{d}\eta_2 - \frac{2\pi|\xi|}{t+1}\right]. \quad (8\text{-}34)$$

The estimates for (8-33) are achieved by a Taylor expansion and (8-14)

$$\begin{aligned}&\bigg| (|\xi|+|\xi|^{r+4}) \frac{i\xi}{6} \iint_{A_2} e^{it\Phi(\xi,\eta_1,\eta_2)} \mathfrak{b}(\xi,\eta_1,\eta_2,t) [\boldsymbol{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2) \\ &\qquad\qquad - \boldsymbol{T}_1(\xi,\xi,-\xi)|\hat{h}(\xi)|^2\hat{h}(\xi)]\, \mathrm{d}\eta_1\, \mathrm{d}\eta_2 \bigg| \\ &\lesssim (|\xi|+|\xi|^{r+4})|\xi| \iint_{A_2} \big|\partial_{\eta_1}[\boldsymbol{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2)]|_{\eta_1=\eta_1'}(\xi-\eta_1)\big| \\ &\qquad + \big|\partial_{\eta_2}[\boldsymbol{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2)]|_{\eta_2=\eta_2'}(\xi-\eta_2)\big|\, \mathrm{d}\eta_1\, \mathrm{d}\eta_2 \\ &\lesssim (t+1)^{(r+3)p_1} \|\xi\hat{\varphi}_{j_1}\|_{L^\infty_\xi} \|\xi\hat{\varphi}_{j_2}\|_{L^\infty_\xi} \|\xi\hat{\varphi}_{j_3}\|_{L^\infty_\xi} [\varrho(t)]^3 + \sum \|\xi\hat{\varphi}_{\ell_1}\|_{L^\infty_\xi} \|\xi\hat{\varphi}_{\ell_2}\|_{L^\infty_\xi} \|\mathcal{S}\varphi_{\ell_3}\|_{H^r} [\varrho(t)]^{5/2},\end{aligned}$$

where (η_1', η_2') in the first inequality is some point on the line segment connecting (ξ, ξ) and (η_1, η_2), and the summation in the second inequality is over permutations (ℓ_1, ℓ_2, ℓ_3) of (j_1, j_2, j_3). Taking a summation over j_1, j_2, j_3 and using the estimates in the above subsections together with the time decay of $\varrho(t)$ in (8-30), we see that this term is integrable in time and is bounded by a constant multiple of ϵ_0^2.

As for (8-34), it suffices to estimate

$$\left|(|\xi|+|\xi|^{r+4})\frac{i\xi}{6}\boldsymbol{T}_1(\xi,\xi,-\xi)|\hat{h}(t,\xi)|^2\hat{h}(t,\xi)\left[\iint_{A_2} e^{it\Phi(\xi,\eta_1,\eta_2)}\psi\left(\frac{\eta_1-\xi}{\varrho(t)}\right)\cdot\psi\left(\frac{\eta_2-\xi}{\varrho(t)}\right)\mathrm{d}\eta_1\,\mathrm{d}\eta_2-\frac{2\pi|\xi|}{t}\right]\right|$$
$$\lesssim \|(|\xi|+|\xi|^{r+4})\hat{\varphi}(\xi)\|_{L^\infty_\xi}\||\xi|\hat{\varphi}(\xi)\|_{L^\infty_\xi}\|(|\xi|+|\xi|^3)\hat{\varphi}(\xi)\|_{L^\infty_\xi} \cdot\left\|\iint_{A_2} e^{it\Phi(\xi,\eta_1,\eta_2)}\psi\left(\frac{\eta_1-\xi}{\varrho(t)}\right)\cdot\psi\left(\frac{\eta_2-\xi}{\varrho(t)}\right)\mathrm{d}\eta_1\,\mathrm{d}\eta_2-\frac{2\pi|\xi|}{t}\right\|_{L^\infty_\xi}$$
$$\lesssim \|\varphi\|_Z^3\left\|\iint_{A_2} e^{it\Phi(\xi,\eta_1,\eta_2)}\psi\left(\frac{\eta_1-\xi}{\varrho(t)}\right)\cdot\psi\left(\frac{\eta_2-\xi}{\varrho(t)}\right)\mathrm{d}\eta_1\,\mathrm{d}\eta_2-\frac{2\pi|\xi|}{t}\right\|_{L^\infty_\xi}.$$

Writing $(\eta_1,\eta_2)=(\xi+\zeta_1,\xi+\zeta_2)$, we find from (8-2) that

$$\Phi(\xi,\eta_1,\eta_2)=-\frac{\zeta_1\zeta_2}{\xi}+O\left(\frac{\zeta_1^3+\zeta_2^3}{\xi^2}\right)=-\frac{\zeta_1\zeta_2}{\xi}+O([\varrho(t)]^3(t+1)^{2p_0}).$$

Since $p_0=10^{-4}$ and $\varrho(t)$ satisfies (8-30), the error term is integrable in time, so we now only need to estimate

$$J_3=\left\|\iint_{\mathbb{R}^2} e^{-it\zeta_1\zeta_2/\xi}\psi\left(\frac{\zeta_1}{\varrho(t)}\right)\cdot\psi\left(\frac{\zeta_2}{\varrho(t)}\right)\mathrm{d}\zeta_1\,\mathrm{d}\zeta_2-\frac{2\pi|\xi|}{t}\right\|_{L^\infty_\xi}. \tag{8-35}$$

Making the change of variables

$$\zeta_1=\sqrt{\frac{|\xi|}{t}}x_1,\quad \zeta_2=\sqrt{\frac{|\xi|}{t}}x_2$$

in (8-35) and using the fact that $|\xi|\le(t+1)^{p_1}$, we find that

$$J_3\le\frac{(t+1)^{p_1}}{t}\left\|\iint_{\mathbb{R}^2} e^{-ix_1x_2}\psi\left(\frac{\sqrt{|\xi|}}{\sqrt{t}\varrho(t)}x_1\right)\cdot\psi\left(\frac{\sqrt{|\xi|}}{\sqrt{t}\varrho(t)}x_1\right)\mathrm{d}x_1\,\mathrm{d}x_2-2\pi\right\|_{L^\infty_\xi}. \tag{8-36}$$

The integral identity

$$\int_{\mathbb{R}} e^{-ax^2-bx}\,\mathrm{d}x=\sqrt{\frac{\pi}{a}}e^{b^2/(4a)}\quad\text{for all } a,b\in\mathbb{C}\text{ with }\Re a>0$$

gives that

$$\iint_{\mathbb{R}^2} e^{-ix_1x_2}e^{-x_1^2/B^2}e^{-x_2^2/B^2}\,\mathrm{d}x_1\,\mathrm{d}x_2=\sqrt{\pi}B\int_{\mathbb{R}} e^{-x_2^2/B^2}e^{-B^2x_2^2/4}\,\mathrm{d}x_2=2\pi+O(B^{-1})\quad\text{as } B\to\infty,$$

and therefore

$$\iint_{\mathbb{R}^2} e^{-ix_1x_2}\psi\left(\frac{x_1}{B}\right)\psi\left(\frac{x_2}{B}\right)\mathrm{d}x_1\,\mathrm{d}x_2=2\pi+O(B^{-1/2})\quad\text{as } B\to\infty. \tag{8-37}$$

Using (8-37) with $B=\sqrt{t}\varrho(t)/\sqrt{|\xi|}=O(t^{0.01-p_0/2})$ in (8-36) then yields

$$J_3\lesssim\frac{(t+1)^{5p_1/4}}{t^{1.005}}.$$

Since $p_1 = 10^{-6}$, the right-hand side decays faster in time than $1/t$, which implies that (8-34) is integrable in time and bounded by a constant multiple of ϵ_0^3.

Putting all the above estimates together, we conclude that

$$\int_0^\infty \|(|\xi| + |\xi|^{r+4})U(\xi, t)\|_{L^\infty_\xi} \, \mathrm{d}t \lesssim \epsilon_0.$$

8G. ***Higher-degree terms.*** In this subsection, we prove that

$$\|(|\xi| + |\xi|^{r+4})\widehat{\mathcal{N}_{\geq 5}(\varphi)}\|_{L^\infty_\xi}$$

is integrable in time. We begin by proving an estimate for the symbol $\boldsymbol{T}_n$. We have

$$\begin{aligned}
&\mathcal{F}^{-1}[\boldsymbol{T}_n(\eta_1, \eta_2, \dots, \eta_{2n+1})\psi_{j_1}(\eta_1)\psi_{j_2}(\eta_2)\cdots\psi_{j_{2n+1}}(\eta_{2n+1})] \\
&\quad = \iiint_{\mathbb{R}^{2n+1}} e^{i(y_1\eta_1 + y_2\eta_2 + \cdots + y_{2n+1}\eta_{2n+1})} \left[\int_{\mathbb{R}} \frac{\prod_{j=1}^{2n+1}(1 - e^{i\eta_j\zeta})}{|\zeta|^{2n+1}} \, \mathrm{d}\zeta\right] \psi_{j_1}(\eta_1)\psi_{j_2}(\eta_2)\cdots\psi_{j_{2n+1}}(\eta_{2n+1}) \, \mathrm{d}\boldsymbol{\eta}_n \\
&\quad = \iiint_{\mathbb{R}^{2n+1}} \left[\int_{\mathbb{R}} \frac{(e^{iy_1\eta_1} - e^{i\eta_1(\zeta + y_1)})\cdots(e^{iy_{2n+1}\eta_{2n+1}} - e^{i\eta_{2n+1}(\zeta + y_{2n+1})})}{|\zeta|^{2n+1}} \, \mathrm{d}\zeta\right] \psi_{j_1}(\eta_1)\cdots\psi_{j_{2n+1}}(\eta_{2n+1}) \, \mathrm{d}\boldsymbol{\eta}_n \\
&\quad = \int_{\mathbb{R}} \frac{1}{|\zeta|^{2n+1}} \big[\mathcal{F}^{-1}[\psi_{j_1}](y_1) - \mathcal{F}^{-1}[\psi_{j_1}](\zeta + y_1)\big]\cdots\big[\mathcal{F}^{-1}[\psi_{j_{2n+1}}](y_{2n+1}) - \mathcal{F}^{-1}[\psi_{j_{2n+1}}](\zeta + y_{2n+1})\big] \, \mathrm{d}\zeta,
\end{aligned}$$

and it follows that

$$\begin{aligned}
&\|\mathcal{F}^{-1}[\boldsymbol{T}_n(\eta_1, \eta_2, \dots, \eta_{2n+1})\psi_{j_1}(\eta_1)\psi_{j_2}(\eta_2)\cdots\psi_{j_{2n+1}}(\eta_{2n+1})]\|_{L^1} \\
&\qquad \lesssim \int_{\mathbb{R}} 2^{j_1 + \cdots + j_{2n+1}} \frac{1}{|\zeta|^{2n+1}} \min\{2^{-j_1}, |\zeta|\}\min\{2^{-j_2}, |\zeta|\}\cdots\min\{2^{-j_{2n+1}}, |\zeta|\} \, \mathrm{d}\zeta.
\end{aligned}$$

Let $\ell_1, \ell_2, \dots, \ell_{2n+1}$ be a permutation of $j_1, j_2, \dots, j_{2n+1}$ satisfying $2^{-\ell_1} \leq 2^{-\ell_2} \leq \cdots \leq 2^{-\ell_{2n+1}}$. Then

$$\begin{aligned}
&\|\mathcal{F}^{-1}[\boldsymbol{T}_n(\eta_1, \eta_2, \dots, \eta_{2n+1})\psi_{j_1}(\eta_1)\psi_{j_2}(\eta_2)\cdots\psi_{j_{2n+1}}(\eta_{2n+1})]\|_{L^1} \\
&\quad \lesssim \int_{|\zeta| > 2^{-\ell_{2n+1}}} \frac{1}{|\zeta|^{2n+1}} \, \mathrm{d}\zeta + \int_{2^{-\ell_{2n}} < |\zeta| < 2^{-\ell_{2n+1}}} \frac{2^{\ell_1}}{|\zeta|^{2n}} \, \mathrm{d}\zeta \\
&\qquad\qquad + \cdots + \int_{2^{-\ell_1} < |\zeta| < 2^{-\ell_2}} \frac{2^{\ell_1 + \cdots + \ell_{2n}}}{|\zeta|} \, \mathrm{d}\zeta + \int_{|\zeta| < 2^{-\ell_1}} 2^{\ell_1 + \cdots + \ell_{2n+1}} \, \mathrm{d}\zeta \\
&\quad \lesssim 2^{\ell_2 + \cdots + \ell_{2n+1}}.
\end{aligned}$$

Therefore, by Lemma 2.5, we have

$$\|(|\xi| + |\xi|^{r+4})\widehat{\mathcal{N}_{\geq 5}(\varphi)}\|_{L^\infty_\xi} \lesssim (t+1)^{(r+4)p_1}\|\mathcal{N}_{\geq 5}(\varphi)\|_{L^1} \lesssim \|\varphi\|_{H^1}^2 \sum_{n=2}^\infty (\|\varphi_x\|_{L^\infty}^{2n-1} + \|L\varphi_x\|_{L^\infty}^{2n-1}).$$

Using the dispersive estimate Lemma 5.3, we see that the right-hand-side is integrable in t, which leads to

$$\int_0^\infty \|(|\xi| + |\xi|^{r+3})\widehat{\mathcal{N}_{\geq 5}(\varphi)}\|_{L^\infty_\xi} \, \mathrm{d}t \lesssim \epsilon_0.$$

This completes the proof of Theorem 5.1.

Appendix A: Alternative formulation of the SQG front equation

We first prove an algebraic identity that will be used in deriving (3-6).

Lemma A.1. *Let $N \geq 2$ be an integer. Then for any integer $1 \leq p \leq N-1$ and any $\eta_j \in \mathbb{R}$, $j = 1, 2, \dots, N$,*

$$\sum_{\ell=1}^{N} \sum_{1\leq m_1<m_2<\cdots<m_\ell \leq N} (-1)^\ell (\eta_{m_1} + \eta_{m_2} + \cdots + \eta_{m_\ell})^p = 0. \tag{A-1}$$

Proof. A general term in the expansion of left-hand-side of (A-1) is proportional to

$$\eta_1^{\alpha_1} \eta_2^{\alpha_2} \cdots \eta_N^{\alpha_N}, \tag{A-2}$$

where $\alpha_1, \alpha_2, \dots, \alpha_N$ are nonnegative integers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_N = p$. It suffices to show that the coefficients of the monomials (A-2) are zero. Let $1 \leq M \leq N-1$ denote the number of nonzero terms in the list $(\alpha_1, \alpha_2, \dots, \alpha_N)$. Using the multinomial theorem, we see that the coefficient of (A-2) is

$$\binom{p}{\alpha_1, \dots, \alpha_N} \cdot \sum_{j=0}^{N-M} (-1)^{M+j} \binom{N-M}{j} = \binom{p}{\alpha_1, \dots, \alpha_N} \cdot (-1)^M (1-1)^{N-M} = 0. \qquad \square$$

To compute $\boldsymbol{T}_n(\boldsymbol{\eta}_n)$ in (3-3), we first expand the product

$$\begin{aligned} \Re \prod_{j=1}^{2n+1} (1 - e^{i\eta_j \zeta}) &= 1 + \sum_{\ell=1}^{2n+1} \sum_{1\leq m_1<m_2<\cdots<m_\ell\leq 2n+1} (-1)^\ell \cos((\eta_{m_1} + \eta_{m_2} + \cdots + \eta_{m_\ell})\zeta) \\ &= \sum_{\ell=1}^{2n+1} \sum_{1\leq m_1<m_2<\cdots<m_\ell\leq 2n+1} (-1)^{\ell+1} [1 - \cos((\eta_{m_1} + \eta_{m_2} + \cdots + \eta_{m_\ell})\zeta)]. \end{aligned}$$

We replace the integral over $\mathbb{R}$ in (3-3) by an integral over $\mathbb{R} \setminus (-\epsilon, \epsilon)$, where $\epsilon \ll 1$, and decompose the expression for $\boldsymbol{T}_n$ into a sum of terms of the form

$$\begin{aligned} \int_{\epsilon<|\zeta|<\infty} \frac{1-\cos(\eta\zeta)}{|\zeta|^{2n+1}} \,\mathrm{d}\zeta &= \int_{\epsilon<|\zeta|\leq 1/|\eta|} \frac{1+\sum_{j=1}^n (-1)^j (\eta\zeta)^{2j}/(2j)! - \cos(\eta\zeta)}{|\zeta|^{2n+1}} \,\mathrm{d}\zeta + \int_{|\zeta|>1/|\eta|} \frac{1-\cos(\eta\zeta)}{|\zeta|^{2n+1}} \,\mathrm{d}\zeta \\ &\qquad - \sum_{j=1}^n \frac{(-1)^j \eta^{2j}}{(2j)!} \int_{\epsilon<|\zeta|\leq 1/|\eta|} \frac{1}{|\zeta|^{2n-2j+1}} \,\mathrm{d}\zeta \\ &= C_{n,1}\eta^{2n} - \sum_{j=1}^n \frac{(-1)^j \eta^{2j}}{(2j)!} \int_{\epsilon<|\zeta|\leq 1/|\eta|} \frac{1}{|\zeta|^{2n-2j+1}} \,\mathrm{d}\zeta + o(1), \end{aligned}$$

where

$$C_{n,1} = \int_{|\theta|\leq 1} \frac{1 + \sum_{j=1}^n (-1)^j (\theta)^{2j}/(2j)! - \cos(\theta)}{|\theta|^{2n+1}} \,\mathrm{d}\theta + \int_{|\theta|>1} \frac{1-\cos(\theta)}{|\theta|^{2n+1}} \,\mathrm{d}\theta$$

is some constant that depends only on n.

We have

$$\sum_{j=1}^n \frac{(-1)^j \eta^{2j}}{(2j)!} \int_{\epsilon<|\zeta|\leq 1/|\eta|} \frac{1}{|\zeta|^{2n-2j+1}} \,\mathrm{d}\zeta = C_{n,2}^{\epsilon} \eta^{2n} + \sum_{j=1}^{n-1} C_{n,3}^{j,\epsilon} \eta^{2j} + C_{n,4}\eta^{2n} \log|\eta|,$$

where

$$C^{\epsilon}_{n,2}=\sum_{j=1}^{n-1}\frac{(-1)^{j+1}}{(n-j)(2j)!}+2\frac{(-1)^{n+1}\log\epsilon}{(2n)!},\quad C^{j,\epsilon}_{n,3}=\frac{(-1)^{j}\epsilon^{2j-2n}}{(n-j)(2j)!},\quad C_{n,4}=2\frac{(-1)^{n+1}}{(2n)!}.$$

Thus, we conclude that

$$\int_{\epsilon<|\zeta|\le 1/|\eta|}\frac{1-\cos(\eta\zeta)}{|\zeta|^{2n+1}}\,\mathrm{d}\zeta=(C_{n,1}-C^{\epsilon}_{n,2})\eta^{2n}-\sum_{j=1}^{n-1}C^{j,\epsilon}_{n,3}\eta^{2j}-C_{n,4}\eta^{2n}\log|\eta|.$$

We use these results in the expression for $\boldsymbol{T}_n$ and take the limit as $\epsilon\to 0^{+}$. The singularity at $\epsilon=0$ does not enter into the final result because of the cancellation in Lemma A.1, and we find that

$$\boldsymbol{T}_n(\boldsymbol{\eta}_n)=2\frac{(-1)^{n+1}}{(2n)!}\sum_{\ell=1}^{2n+1}\sum_{1\le m_1<m_2<\cdots<m_\ell\le 2n+1}(-1)^{\ell}(\eta_{m_1}+\cdots+\eta_{m_\ell})^{2n}\log|\eta_{m_1}+\eta_{m_2}+\cdots+\eta_{m_\ell}|.\tag{A-3}$$

It follows that

$$f_n=2\frac{(-1)^{n}}{(2n)!}\sum_{\ell=1}^{2n+1}\binom{2n+1}{\ell}(-1)^{\ell}\varphi^{2n-\ell+1}\partial^{2n}\log|\partial|(\varphi^{\ell}).$$

Therefore, we conclude that

$$\begin{aligned}\int_{\mathbb{R}}\biggl[\frac{\varphi_x(x,t)-\varphi_x(x+\zeta,t)}{|\zeta|}&-\frac{\varphi_x(x,t)-\varphi_x(x+\zeta,t)}{\sqrt{\zeta^2+(\varphi(x,t)-\varphi(x+\zeta,t))^2}}\biggr]\mathrm{d}\zeta\\ &=-\sum_{n=1}^{\infty}\frac{2c_n(-1)^n}{\Gamma(2n+2)}\partial_x\left\{\sum_{\ell=1}^{2n+1}\binom{2n+1}{\ell}(-1)^{\ell}\varphi^{2n-\ell+1}(x,t)\partial_x^{2n}\log|\partial_x|(\varphi^{\ell}(x,t))\right\}\\ &=\sum_{n=1}^{\infty}\sum_{\ell=1}^{2n+1}(-1)^{\ell+1}d_{n,\ell}\partial_x\{\varphi^{2n-\ell+1}(x,t)\partial_x^{2n}\log|\partial_x|(\varphi^{\ell}(x,t))\},\end{aligned}$$

where

$$d_{n,\ell}=\frac{2\sqrt{\pi}}{\left|\Gamma\left(\frac{1}{2}-n\right)\right|\Gamma(\ell+1)\Gamma(2n+2-\ell)\Gamma(n+1)}>0.\tag{A-4}$$

Using this expansion in (3-1), we get (3-6).

Appendix B: Paradifferential calculus

In this appendix, we use the Weyl calculus [Lerner 2010] to prove some estimates for Weyl paraproducts.

B.1. ***Weyl operators.*** The Weyl quantization of a symbol $a:\mathbb{R}\times\mathbb{R}\to\mathbb{C}$ is the operator a^w defined by

$$(a^w f)(x)=\frac{1}{2\pi}\iint_{\mathbb{R}^2}e^{i(x-y)\xi}a\left(\frac{x+y}{2},\xi\right)f(y)\,\mathrm{d}y\,\mathrm{d}\xi=\int_{\mathbb{R}}\mathcal{F}_2^{-1}a\left(\frac{x+y}{2},x-y\right)f(y)\,\mathrm{d}y,$$

where $\mathcal{F}_i a$ denotes the Fourier transform of $a(x_1, x_2)$ with respect to with the i-th variable ($i = 1, 2$). The Fourier transform of $a^w f$ can be written as

$$\begin{aligned}\mathcal{F}(a^w f)(\xi) &= \frac{1}{2\pi} \iiint_{\mathbb{R}^3} e^{i(x-y)\eta - ix\xi} a\left(\frac{x+y}{2}, \eta\right) f(y)\, \mathrm{d}y\, \mathrm{d}\eta\, \mathrm{d}x \\ &= \int_{\mathbb{R}} \mathcal{F}_1^{-1} a\left(\xi - \eta, \frac{\xi+\eta}{2}\right) \hat{f}(\eta)\, \mathrm{d}\eta. \end{aligned} \tag{B-1}$$

For $m \in \mathbb{R}$, we have the symbol class

$$S^m_{1,0} = \left\{ a(x,\xi) \in C^\infty(\mathbb{R} \times \mathbb{R}) \;\Big|\; \sup_{\xi \in \mathbb{R}} \|\partial_\xi^\alpha \partial_x^\beta a(\,\cdot\,, \xi)\|_{L^\infty} \le C_{\alpha\beta}(1+|\xi|)^{m-|\alpha|} \text{ for all } \alpha, \beta \in \mathbb{N}_0 \right\}.$$

For integers $r_1, r_2 \ge 0$, we define a symbol norm by

$$M^m_{r_1,r_2}(a) = \max_{0 \le \alpha \le r_2} \sup_{\xi \in \mathbb{R}} \|(1+|\xi|)^{\alpha - m} \partial_\xi^\alpha a(\,\cdot\,, \xi)\|_{W^{r_1,\infty}},$$

and introduce a class of symbols with finite regularity

$$\Gamma^m_{r_1,r_2} = \{a : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \mid M^m_{r_1,r_2}(a) < \infty\}.$$

We note that if $\mathcal{M}_{(r_1,r_2)}$ is the symbol class defined in (2-3), then

$$\|(1+|\xi|)^{-m} a(x,\xi)\|_{\mathcal{M}_{(r_1,r_2)}} \approx M^m_{r_1,r_2}(a). \tag{B-2}$$

In particular, $\mathcal{M}_{(r_1,r_2)} = \Gamma^0_{r_1,r_2}$.

B.2. ***Paradifferential operators.*** Recall from Section 2 that $\chi : \mathbb{R} \to \mathbb{R}$ is a smooth function supported in the interval $\left\{\xi \in \mathbb{R} \;\middle|\; |\xi| \le \frac{1}{10}\right\}$ and equal to 1 on $\left\{\xi \in \mathbb{R} \;\middle|\; |\xi| \le \frac{3}{40}\right\}$. If $f : \mathbb{R} \to \mathbb{C}$ and $a : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is a symbol, then the Weyl paraproduct $T_a f$ in (2-2) is defined by

$$\mathcal{F}[T_a f](\xi) = \int_{\mathbb{R}} \chi\left(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\right) \tilde{a}\left(\xi - \eta, \frac{\xi+\eta}{2}\right) \hat{f}(\eta)\, \mathrm{d}\eta.$$

Introducing the notation

$$\sigma_a(\,\cdot\,, \zeta_2) = \mathcal{F}_1\left[\chi\left(\frac{|\zeta_1|^2}{1+4|\zeta_2|^2}\right) \tilde{a}(\zeta_1, \zeta_2)\right],$$

we can also write

$$\mathcal{F}[T_a f](\xi) = \int_{\mathbb{R}} \mathcal{F}_1^{-1} \sigma_a\left(\xi - \eta, \frac{\xi+\eta}{2}\right) \hat{f}(\eta)\, \mathrm{d}\eta.$$

Comparing this result with (B-1), we see that $T_a = \sigma_a^w$.

Lemma B.1. *If $a \in \Gamma^m_{r_1,r_2}$, then $\sigma_a \in \Gamma^m_{r_1,r_2}$ and $M^m_{r_1,r_2}(\sigma_a) \lesssim M^m_{r_1,r_2}(a)$.*

Proof. To prove that $\sigma_a \in \Gamma^m_{r_1,r_2}$, we write

$$\partial_{\zeta_2}^\alpha \partial_x^\beta \sigma_a(x, \zeta_2) = \sum_{i_1+i_2=\alpha} c_{i_1,i_2,\alpha} \mathcal{F}_{\zeta_1}\left[\partial_{\zeta_2}^{i_1} \chi\left(\frac{|\zeta_1|^2}{1+4|\zeta_2|^2}\right) \partial_{\zeta_2}^{i_2} \widetilde{\partial_x^\beta a}(\zeta_1, \zeta_2)\right],$$

where the $c_{i_1,i_2,\alpha}$ are multinomial coefficients.

For each term, by Young's inequality,

$$\begin{aligned}\left|(1+|\zeta_2|)^\alpha \mathcal{F}_{\zeta_1}\left[\partial_{\zeta_2}^{i_1}\chi\left(\frac{|\zeta_1|^2}{1+4|\zeta_2|^2}\right)\partial_{\zeta_2}^{i_2}\widetilde{\partial_x^\beta a}(\zeta_1,\zeta_2)\right]\right| \\ &= \left|\mathcal{F}_{\zeta_1}\left[(1+|\zeta_2|)^{i_1}\partial_{\zeta_2}^{i_1}\chi\left(\frac{|\zeta_1|^2}{1+4|\zeta_2|^2}\right)\right][(1+|\zeta_2|)^{i_2}\partial_{\zeta_2}^{i_2}\partial_x^\beta a(-x,\zeta_2)]\right| \\ &\le \left\|\mathcal{F}_{\zeta_1}\left[(1+|\zeta_2|)^{i_1}\partial_{\zeta_2}^{i_1}\chi\left(\frac{|\zeta_1|^2}{1+4|\zeta_2|^2}\right)\right](x,\zeta_2)\right\|_{L_x^1}\|(1+|\zeta_2|)^{i_2}\partial_{\zeta_2}^{i_2}\partial_x^\beta a(-x,\zeta_2)\|_{L_x^\infty}.\end{aligned}$$

Using Faà di Bruno's formula, a general term of $(1+|\zeta_2|)^{i_1}\partial_{\zeta_2}^{i_1}\chi(|\zeta_1|^2/(1+4|\zeta_2|^2))$ is a linear combination of the terms of the form

$$(1+|\zeta_2|)^{i_1}\chi^{(m_1+\cdots+m_{i_1})}\left(\frac{|\zeta_1|^2}{1+4|\zeta_2|^2}\right)\prod_{\ell=1}^{i_1}\left[\partial_{\zeta_2}^{\ell}\left(\frac{|\zeta_1|^2}{1+4|\zeta_2|^2}\right)\right]^{m_\ell},$$

where $m_\ell \in \mathbb{N}_0$ satisfies $\sum_{\ell=1}^{i_1}\ell m_\ell = i_1$.

Bernstein's inequality implies that

$$\left\|\mathcal{F}_{\zeta_1}\left[(1+|\zeta_2|)^{i_1}\partial_{\zeta_2}^{i_1}\chi\left(\frac{|\zeta_1|^2}{1+4|\zeta_2|^2}\right)\right]\right\|_{L_x^1} \lesssim \left\|\int_{\mathbb{R}} e^{-ix\zeta_1}\left((1+4|\zeta_2|^2)^{i_1/2}\partial_{\zeta_2}^{i_1}\chi\left(\frac{|\zeta_1|^2}{1+4|\zeta_2|^2}\right)\right)\mathrm{d}\zeta_1\right\|_{L_x^1} \lesssim 1,$$

since the middle term in this inequality is supported on the set $\{(\zeta_1,\zeta_2) \mid |\zeta_1| \lesssim \sqrt{1+4|\zeta_2|^2}\}$. Therefore, we have that

$$\sum_{\alpha\le r_2,\beta\le r_1}\|(1+|\zeta_2|)^\alpha\partial_{\zeta_2}^\alpha\partial_x^\beta\sigma_a(x,\zeta_2)\|_{L_x^\infty} \lesssim \sum_{\alpha\le r_2,\beta\le r_1}\sum_{i\le\alpha}\|(1+|\zeta_2|)^i\partial_{\zeta_2}^i\partial_x^\beta a(-x,\zeta_2)\|_{L_x^\infty} \lesssim (1+|\zeta_2|)^m,$$

so $M_{r_1,r_2}^m(\sigma_a) \lesssim M_{r_1,r_2}^m(a)$. □

B.3. *H^s estimates.* The next theorem follows from [Boulkhemair 1999, Theorem 1.2].

Theorem B.2. *Let $m \in \mathbb{R}$. If $a \in \Gamma_{1,1}^m$, then the Weyl operator $a^w : H^s(\mathbb{R}) \to H^{s-m}(\mathbb{R})$ with symbol a is bounded and its operator norm is bounded by $M_{1,1}^m(a)$.*

Using Lemma B.1 and the fact that $T_a = \sigma_a^w$, we then get the following estimate for Weyl paraproducts.

Theorem B.3. *If $a \in \Gamma_{1,1}^m$, then the Weyl paraproduct operator $T_a : H^s(\mathbb{R}) \to H^{s-m}(\mathbb{R})$ is bounded for all $m, s \in \mathbb{R}$, and*

$$\|T_a f\|_{H^{s-m}} \le \upsilon M_{1,1}^m(a)\|f\|_{H^s},$$

where $\upsilon > 0$ is a constant independent of a.

In particular, setting $m = 0$ and using the fact that $M_{1,1}^0(a) \approx \|a\|_{\mathcal{M}_{(1,1)}}$, we get Lemma 2.1.

B.4. L^∞-L^2 *estimates.* We also need some estimates in which we bound $\|T_a f\|_{L^2}$ by $\|f\|_{L^\infty}$.

Theorem B.4. *Let $p(\xi)=|\xi|^k$, $k\geq 0$ or $p(\xi)=|\xi|^k\log|\xi|$, $k\geq 1$. Assume $f\in L^\infty(\mathbb{R})$ with $p(\partial_x)\partial_x f\in L^\infty(\mathbb{R})$, and $a(x,\xi)$ is a function such that $\|a\|_{\mathcal{L}^2_1}<\infty$, where*

$$\|a\|_{\mathcal{L}^2_1}:=\sup_\xi(\|a(\,\cdot\,,\xi)\|_{L^2}+\|\partial_\xi a(\,\cdot\,,\xi)\|_{L^2}).$$

Then we have

$$\|p(\partial_x)T_a f\|_{L^2}\lesssim(\|f\|_{L^\infty}+\|p(\partial_x)\partial_x f\|_{L^\infty})\|a\|_{\mathcal{L}^2_1}.$$

Proof. Recall that

$$T_a f(x)=\sigma_a^w f(x)=\int_{\mathbb{R}}\mathcal{F}_2^{-1}\sigma_a\Big(\frac{x+y}{2},x-y\Big)f(y)\,\mathrm{d}y=\int_{\mathbb{R}}\chi\Big(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\Big)\tilde{a}\Big(\xi-\eta,\frac{\xi+\eta}{2}\Big)\hat{f}(\eta)\,\mathrm{d}\eta.$$

We split $T_a f$ into a low-frequency part

$$\int_{\mathbb{R}}\mathcal{F}_2^{-1}\sigma_a\Big(\frac{x+y}{2},x-y\Big)[\iota(i\partial_y)f(y)]\,\mathrm{d}y=\int_{\mathbb{R}}\iota(\eta)\chi\Big(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\Big)\tilde{a}\Big(\xi-\eta,\frac{\xi+\eta}{2}\Big)\hat{f}(\eta)\,\mathrm{d}\eta$$

and a high-frequency part

$$\int_{\mathbb{R}}\mathcal{F}_2^{-1}\sigma_a\Big(\frac{x+y}{2},x-y\Big)[(1-\iota(i\partial_y))f(y)]\,\mathrm{d}y=\int_{\mathbb{R}}(1-\iota(\eta))\chi\Big(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\Big)\tilde{a}\Big(\xi-\eta,\frac{\xi+\eta}{2}\Big)\hat{f}(\eta)\,\mathrm{d}\eta.$$

Here, the cutoff function ι is the same as the one defined in the proof of Lemma 2.2.

The integrand in the low-frequency part is supported in $|\xi|<6$, $|\eta|<2$. Thus, $|\xi+\eta|<10$ and $|\xi-\eta|<10$ on its support, so we can put a cutoff function $\iota\big(\frac{1}{5}(\xi+\eta)\big)\iota\big(\frac{1}{5}(\xi-\eta)\big)$ into the integral without changing its value:

$$\begin{aligned}\int_{\mathbb{R}}&\mathcal{F}_2^{-1}\sigma_a\Big(\frac{x+y}{2},x-y\Big)[\iota(i\partial_y)f(y)]\,\mathrm{d}y\\&=\int_{\mathbb{R}}\iota(\eta)\chi\Big(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\Big)\iota\Big(\frac{\xi+\eta}{5}\Big)\iota\Big(\frac{\xi-\eta}{5}\Big)\tilde{a}\Big(\xi-\eta,\frac{\xi+\eta}{2}\Big)\hat{f}(\eta)\,\mathrm{d}\eta.\end{aligned}$$

Therefore, defining $b(x,\xi)=\iota\big(\frac{1}{5}i\partial_x\big)\iota\big(\frac{2}{5}\xi\big)a(x,\xi)$, we have

$$\begin{aligned}\int_{\mathbb{R}}\mathcal{F}_2^{-1}\sigma_a\Big(\frac{x+y}{2},x-y\Big)[\iota(i\partial_y)f(y)]\,\mathrm{d}y&=\int_{\mathbb{R}}\iota(\eta)\chi\Big(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\Big)\tilde{b}\Big(\xi-\eta,\frac{\xi+\eta}{2}\Big)\hat{f}(\eta)\,\mathrm{d}\eta\\&=\int_{\mathbb{R}}\mathcal{F}_2^{-1}\sigma_b\Big(\frac{x+y}{2},x-y\Big)[\iota(i\partial_y)f(y)]\,\mathrm{d}y.\end{aligned}$$

So we obtain

$$\begin{aligned}\Big\|p(\partial_x)\int_{\mathbb{R}}\mathcal{F}_2^{-1}\sigma_b\Big(\frac{x+y}{2},x-y\Big)[\iota(i\partial_y)f(y)]\,\mathrm{d}y\Big\|_{L^2_x}&\lesssim\Big\|\int_{\mathbb{R}}\mathcal{F}_2^{-1}\sigma_b\Big(\frac{x+y}{2},x-y\Big)[\iota(i\partial_y)f(y)]\,\mathrm{d}y\Big\|_{L^2_x}\\&\lesssim\|\iota(i\partial_y)f(y)\|_{L^\infty_y}\Big\|\mathcal{F}_2^{-1}\sigma_b\Big(\frac{x+y}{2},x-y\Big)\Big\|_{L^1_yL^2_x}\\&=\|\iota(i\partial_y)f(y)\|_{L^\infty_y}\Big\|\mathcal{F}_2^{-1}\sigma_b\Big(x-\frac{z}{2},z\Big)\Big\|_{L^1_zL^2_x},\end{aligned}$$

where the last term satisfies

$$
\begin{aligned}
\left\|\mathcal{F}_2^{-1}\sigma_b\Big(x-\frac{z}{2},z\Big)\right\|_{L_z^1L_x^2} &= \left\|\mathcal{F}_\xi^{-1}\mathcal{F}_{\zeta_1}\Big[\chi\Big(\frac{|\zeta_1|^2}{1+4|\xi|^2}\Big)\tilde{b}(\zeta_1,\xi)\Big]\Big(x-\frac{z}{2},z\Big)\right\|_{L_z^1L_x^2}\\
&= \left\|\mathcal{F}_\xi^{-1}\mathcal{F}_{\zeta_1}\Big\{(1-\partial_\xi^2)^{1/2}\Big[\chi\Big(\frac{|\zeta_1|^2}{1+4|\xi|^2}\Big)\tilde{b}(\zeta_1,\xi)\Big]\Big\}\Big(x-\frac{z}{2},z\Big)\frac{1}{(1+z^2)^{1/2}}\right\|_{L_z^1L_x^2}\\
&\lesssim \left\|\mathcal{F}_\xi^{-1}\mathcal{F}_{\zeta_1}\Big\{(1-\partial_\xi^2)^{1/2}\Big[\chi\Big(\frac{|\zeta_1|^2}{1+4|\xi|^2}\Big)\tilde{b}(\zeta_1,\xi)\Big]\Big\}(x,z)\right\|_{L_z^2L_x^2}\\
&= \left\|(1-\partial_\xi^2)^{1/2}\Big[\chi\Big(\frac{|\zeta_1|^2}{1+4|\xi|^2}\Big)\tilde{b}(\zeta_1,\xi)\Big](\zeta_1,\xi)\right\|_{L_\xi^2L_{\zeta_1}^2} \lesssim \|b\|_{L_x^2H_\xi^1} \lesssim \|a\|_{\mathcal{L}_1^2}.
\end{aligned}
$$

For the high-frequency part, we make a dyadic decomposition of f, after which we mainly need to estimate

$$
\begin{aligned}
&\int_{\mathbb{R}}\mathcal{F}_2^{-1}\sigma_a\Big(\frac{x+y}{2},x-y\Big)[(1-\iota(i\partial_y))\psi_k(i\partial_y)f(y)]\,\mathrm{d}y\\
&\qquad = \int_{\mathbb{R}}(1-\iota(\eta))\chi\Big(\frac{|\xi-\eta|^2}{1+|\xi+\eta|^2}\Big)\tilde{a}\Big(\xi-\eta,\frac{\xi+\eta}{2}\Big)\psi_k(\eta)\hat{f}(\eta)\,\mathrm{d}\eta.
\end{aligned}
$$

When $|\eta|>2$, we have

$$
\tfrac{1}{18}|\eta|\le|\xi|\le\tfrac{35}{18}|\eta|,\quad \tfrac{1}{2}|\eta|\le|\xi+\eta|\le\tfrac{40}{9}|\eta|
$$

on the support of the cutoff function $\chi(|\xi-\eta|^2/(1+|\xi+\eta|^2))$. Therefore $|\eta|\approx|\xi+\eta|\approx|\xi|\approx 2^k$ on the support, and, since $|\eta|>2$, we only need to consider $k\ge 0$.

By the Hölder inequality and a change of coordinates,

$$
\begin{aligned}
&\left\|p(\partial_x)\int_{\mathbb{R}}\mathcal{F}_2^{-1}\sigma_a\Big(\frac{x+y}{2},x-y\Big)[(1-\iota(i\partial_y))f_k(y)]\,\mathrm{d}y\right\|_{L_x^2}\\
&\quad\lesssim 2^{-k}\|p(\partial_x)\partial_x f_k\|_{L^\infty}\left\|\mathcal{F}_\xi^{-1}\mathcal{F}_{\zeta_1}\Big[\chi\Big(\frac{|\zeta_1|^2}{1+4|\xi|^2}\Big)\tilde{a}(\zeta_1,\xi)\psi_k(\xi)\Big]\Big(\frac{x+y}{2},x-y\Big)\right\|_{L_y^1L_x^2}\\
&\quad\lesssim 2^{-k}\|p(\partial_x)\partial_x f\|_{L^\infty}\left\|\mathcal{F}_\xi^{-1}\mathcal{F}_{\zeta_1}\Big[\chi\Big(\frac{|\zeta_1|^2}{1+4|\xi|^2}\Big)\tilde{a}(\zeta_1,\xi)\psi_k(\xi)\Big]\Big(x+\frac{z}{2},z\Big)\right\|_{L_z^1L_x^2}.
\end{aligned}
$$

The last term satisfies

$$
\begin{aligned}
&\left\|\mathcal{F}_\xi^{-1}\mathcal{F}_{\zeta_1}\Big[\chi\Big(\frac{|\zeta_1|^2}{1+4|\xi|^2}\Big)\tilde{a}(\zeta_1,\xi)\psi_k(\xi)\Big]\Big(x+\frac{z}{2},z\Big)\right\|_{L_z^1L_x^2}\\
&= \left\|\mathcal{F}_\xi^{-1}\mathcal{F}_{\zeta_1}\Big\{(1-\partial_\xi^2)^{1/2}\Big[\chi\Big(\frac{|\zeta_1|^2}{1+4|\xi|^2}\Big)\tilde{a}(\zeta_1,\xi)\psi_k(\xi)\Big]\Big\}\Big(x+\frac{z}{2},z\Big)\frac{1}{(1+z^2)^{1/2}}\right\|_{L_z^1L_x^2}\\
&\lesssim \left\|\mathcal{F}_\xi^{-1}\mathcal{F}_{\zeta_1}\Big\{(1-\partial_\xi^2)^{1/2}\Big[\chi\Big(\frac{|\zeta_1|^2}{1+4|\xi|^2}\Big)\tilde{a}(\zeta_1,\xi)\psi_k(\xi)\Big]\Big\}(x,z)\right\|_{L_z^2L_x^2}
\end{aligned}
$$

$$\lesssim \left\| \partial_\xi \left[\chi\left(\frac{|\zeta_1|^2}{1+4|\xi|^2}\right)\tilde{a}(\zeta_1,\xi)\psi_k(\xi)\right]\right\|_{L^2_{\zeta_1}L^2_\xi} + \left\| \chi\left(\frac{|\zeta_1|^2}{1+4|\xi|^2}\right)\tilde{a}(\zeta_1,\xi)\psi_k(\xi)\right\|_{L^2_{\zeta_1}L^2_\xi}$$

$$\lesssim \left\| \frac{8|\xi||\zeta_1|^2}{(1+4|\xi|^2)^2}\chi'\left(\frac{|\zeta_1|^2}{1+4|\xi|^2}\right)\tilde{a}(\zeta_1,\xi)\psi_k(\xi)\right\|_{L^2_{\zeta_1}L^2_\xi} + \left\| \chi\left(\frac{|\zeta_1|^2}{1+4|\xi|^2}\right)\partial_\xi\tilde{a}(\zeta_1,\xi)\psi_k(\xi)\right\|_{L^2_{\zeta_1}L^2_\xi}$$
$$+ \left\| \chi\left(\frac{|\zeta_1|^2}{1+4|\xi|^2}\right)\tilde{a}(\zeta_1,\xi)\partial_\xi\psi_k(\xi)\right\|_{L^2_{\zeta_1}L^2_\xi} + \left\| \chi\left(\frac{|\zeta_1|^2}{1+4|\xi|^2}\right)\tilde{a}(\zeta_1,\xi)\psi_k(\xi)\right\|_{L^2_{\zeta_1}L^2_\xi}$$

$$\lesssim 2^{-k/2}\|a(x,\xi)\|_{L^2_x L^\infty_\xi} + 2^{k/2}\|\partial_\xi a(x,\xi)\|_{L^2_x L^\infty_\xi} + 2^{-k/2}\|a(x,\xi)\|_{L^2_x L^\infty_\xi} + 2^{k/2}\|a(x,\xi)\|_{L^2_x L^\infty_\xi}.$$

Summing these inequalities over $k \geq 0$, we obtain that

$$\left\| p(\partial_x)\int_{\mathbb{R}} \mathcal{F}_2^{-1}\sigma_a\left(\frac{x+y}{2}, x-y\right)[(1-\iota(i\partial_y))f(y)]\,\mathrm{d}y\right\|_{L^2_x} \lesssim \|p(\partial_x)\partial_x f\|_{L^\infty}\|a\|_{\mathcal{L}^2_1}.$$

The theorem then follows by combining the low- and high-frequency estimates. ☐

B.5. *Composition.* Finally, we state a commutator estimate for Weyl paraproducts. The composition of two symbols a and b is defined by

$$a\#b(x,\xi) = \iint_{\mathbb{R}^2} e^{-iy\eta}a(x,\xi+\eta)b(y+x,\xi)\,\mathrm{d}y\,\mathrm{d}\eta.$$

The following theorem is from [Lerner 2010, Theorem 2.3.7].

Theorem B.5 (composition). *Let $a_1 \in S^{m_1}_{1,0}$ and $a_2 \in S^{m_2}_{1,0}$. Then*

$$a_1\#a_2 - a_1a_2 - \frac{1}{2i}\{a_1,a_2\} \in S^{m_1+m_2-2}_{1,0},$$

where $\{a_1,a_2\} = \partial_\xi a_1\partial_x a_2 - \partial_\xi a_2\partial_x a_1$ is the Poisson bracket.

Using Theorem B.3, we therefore obtain the following estimate.

Theorem B.6. *Let $a \in \Gamma^{m_1}_{3,3}$, $b \in \Gamma^{m_2}_{3,3}$, and $f \in H^s(\mathbb{R})$. Then*

$$T_aT_bf = T_{ab}f + \frac{1}{2i}T_{\{a,b\}}f + \mathcal{R},$$

where $\{a,b\} = \partial_2 a\cdot\partial_1 b - \partial_2 b\cdot\partial_1 a$ is the Poisson bracket of a and b, and the remainder $\mathcal{R}$ satisfies

$$\|\mathcal{R}\|_{H^{s-(m_1+m_2-2)}} \lesssim M^{m_1}_{3,3}(a)M^{m_2}_{3,3}(b)\|f\|_{H^s}.$$

In addition,

$$\|[T_a,T_b]\|_{H^{s-(m_1+m_2-1)}} \lesssim M^{m_1}_{2,2}(a)M^{m_2}_{2,2}(b)\|f\|_{H^s}.$$

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JOHN K. HUNTER: jkhunter@ucdavis.edu
Department of Mathematics, University of California, Davis, CA, United States

JINGYANG SHU: jyshu@temple.edu
Department of Mathematics, Temple University, Philadelphia, PA, United States

QINGTIAN ZHANG: qingtian.zhang@mail.wvu.edu
Department of Mathematics, West Virginia University, Morgantown, WV, United States

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