

Invertible phases of matter with spatial symmetry

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We propose a general formula for the group of invertible topological phases on a space Y , possibly equipped with the action of a group G . Our formula applies to arbitrary symmetry types. When Y is Euclidean space and G a crystallographic group, the term ‘topological crystalline phases’ is sometimes used for these phases of matter.

In previous work [FH], recalled in §1 below, we determine the homotopy type of the space of invertible field theories with a fixed symmetry type. This result is a theorem about *field theories* in the framework of the Axiom System for field theory introduced by Segal in the 1980’s. It has wide applicability: invertible field theories enter quantum field theory and string theory in many different ways. In condensed matter theory our theorem can be used to classify invertible phases of matter (on Euclidean space), but only accepting standard unproved assertions about effective low energy field theories of discrete models. In this note we combine this theorem with a few more basic principles (§2) to offer a general formula for the abelian group of invertible topological phases of matter on a topological space Y equipped with the action of a group G . *Time* does not appear: Y models *space*, not *spacetime*. We motivate and present the formula in Ansatz 2.1 and Ansatz 3.3; the formula depends on a symmetry type but not on a dimension. As evidence we compute some illustrative examples and compare to known results. (See Example 2.3 and Example 3.5.) The pedagogical aspirations of this note are realized in §4, where we briefly explain some computational techniques in Borel equivariant homotopy theory, and in §5, where we illustrate via a

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specific example — a half-turn in 3-space — which we attack using three different methods.

The idea that invertible phases comprise a generalized *homology* group on space was suggested by Alexei Kitaev; he works with lattice models to motivate the particular homology theory. There are discussions of special cases of the problem we treat here in [SHFH, TE, HSHH]. The recent paper [SXG] uses a spectral sequence to compute the group of phases, as do we in §4.2, §5.3, but the generalized homology theory is not specified and physical arguments are used to compute differentials. We thank Lukasz Fidkowski, Mike Hermele, and Ashvin Vishwanath for bringing the specific example treated in §5 and the general problem to our attention, as well as for a very informative email correspondence.¹

1 Recollection of [FH]

Let d be the dimension of space. The *symmetry type* of a Wick-rotated relativistic field theory in spacetime dimension $d + 1$ is described by a pair (H, ρ) . The topological group H is the colimit of a sequence of compact Lie groups H_{d+1} , each sitting in a group extension

$$(1.1) \quad 1 \longrightarrow K \longrightarrow H_{d+1} \xrightarrow{\rho_{d+1}} O_{d+1}$$

in which the image of ρ_{d+1} is either O_{d+1} (symmetry type with time-reversal) or SO_{d+1} (no time-reversal). Then $\rho: H \rightarrow O$ is the stabilization of ρ_{d+1} as $d \rightarrow \infty$; see [FH, §2]. The subgroup K is the group of internal symmetries — those which act trivially on *spacetime* — and is independent of d . (If we break relativistic invariance, there is a slightly larger group which acts trivially on *space*; see [FH, Remark 9.32].) The homomorphism ρ determines a rank zero virtual real vector bundle $W \rightarrow BH$, the stabilization of rank zero virtual bundles over BH_{d+1} , and there is a corresponding Thom spectrum

$$(1.2) \quad MTH = \text{Thom}(BH; -W)$$

of the virtual vector bundle $-W \rightarrow BH$. Let $I\mathbb{Z}$ be the Anderson dual to the sphere spectrum and

$$(1.3) \quad E = E_{(H, \rho)} = \Sigma^2 I\mathbb{Z}^{MTH}$$

¹Debray [D] extends the ideas in this paper and carries out many computations to check the ansätze.

the spectrum of maps $MTH \rightarrow \Sigma^2 I\mathbb{Z}$. Then the main outcome² of [FH] is an identification of

$$(1.4) \quad E_{-d}(\text{pt}) \cong E^d(\text{pt}) \cong [MTH, \Sigma^{d+2} I\mathbb{Z}]$$

as the group of deformation classes of invertible reflection positive extended field theories in $d + 1$ dimensions with symmetry type (H, ρ) . Computations for various (H, ρ) may be found in [FH, §§9–10] as well as [Ka, KTTW, C, BC, GPW].

2 Invertible phases on a space

We imagine that invertible topological phases can be localized in space, possibly with noncompact support; satisfy some locality properties; and are equipped with a pushforward under proper continuous maps. Since $E_0(\text{pt})$ is the group of invertible phases in $0 + 1$ dimensions — that is, phases on a point — we posit the following.

Ansatz 2.1. Let Y be a locally compact topological space. Then the group of invertible topological phases on Y of symmetry type (H, ρ) is the Borel-Moore homology group $E_{0,BM}(Y)$.

If Y is the complement in a finite CW complex $\bar{Y}$ of a subcomplex $Y_0 \subset \bar{Y}$, then Borel-Moore homology reduces to relative homology: $E_{0,BM}(Y) \cong E_0(\bar{Y}, Y_0)$. Thus on Euclidean d -space we have

$$(2.2) \quad E_{0,BM}(\mathbb{E}^d) \cong E_0(S^d, \text{pt}) \cong E_{-d}(\text{pt}),$$

which recovers (1.4). If Y is compact, then $E_{0,BM}(Y) \cong E_0(Y)$.

Example 2.3 (Phases on a torus). Let $Y = (S^1)^{\times d}$ be the d -dimensional torus. After suspension Y is homotopy equivalent to a wedge of spheres, from

²This statement is left as a conjecture in that paper; what is proved from various ansätze is an identification of the *torsion* subgroup with isomorphism classes of invertible *topological* theories. The entire group (1.4) is also proved to be the group of isomorphism classes of “continuous” invertible theories; see [FH, §5.4].

which

$$(2.4) \quad E_0(Y) \cong \bigoplus_{i=0}^d E_{-i}(\mathrm{pt})^{\oplus \binom{d}{i}}.$$

For example, if $d = 2$ and we consider fermionic theories ($H = \mathrm{Spin}$), then

$$(2.5) \quad E_0(S^1 \times S^1) \cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z});$$

the summands correspond to theories supported on a point, on the 1-cells (figure eight), and on the 2-cell, respectively. We remark that all classes are represented by free fermions: (2.5) is also isomorphic to $KO^0(S^1 \times S^1)$. See [R] for a discussion of the physics of this example.

Remark 2.6 (Invertible phases on a compact smooth manifold).

A compact smooth d -manifold Y with boundary has a Spanier-Whitehead dual $D(Y/\partial Y) \simeq \mathrm{Thom}(Y; -TY) \simeq \Sigma^{-TY}Y$, according to [A], and so

$$(2.7) \quad \begin{aligned} E_0(Y, \partial Y) &\cong [S^0, E \wedge Y/\partial Y] \\ &\cong [MTH, \Sigma^2 I\mathbb{Z} \wedge Y/\partial Y] \\ &\cong [\Sigma^{\mathbb{R}^d - TY}(Y) \wedge MTH, \Sigma^{d+2} I\mathbb{Z}] \\ &\cong [\Sigma^{\mathbb{R}^d - TY}(Y), \Sigma^d E], \end{aligned}$$

where $\mathbb{R}^d \rightarrow Y$ is the trivial vector bundle with fiber $\mathbb{R}^d$. This last group is a twisted E -cohomology group of Y ; the twisting is trivialized by an E -orientation of Y .

The third line of (2.7) may be regarded as deformation classes of invertible field theories of symmetry type (H, ρ) with a background scalar field valued in Y , or rather in a twist of Y if Y is not E -oriented. This field theory interpretation was used in [TE] to study special cases.

3 Invertible phases on a G -space

It is natural consider a compact Lie group G is acting on a locally compact space Y and model *equivariant phases* on Y .³ For this there is a choice to make and so far simply working with *Borel equivariant* homotopy theory

³We allow noncompact groups acting with compact isotropy subgroups, i.e., topological stacks with compact Lie group stabilizers [FHT, A.2.2]. Example 2.3 is of this type: $(S^1)^{\times d}$ is isomorphic to the quotient stack $\mathbb{E}^d//\mathbb{Z}^d$.

seems to work. We therefore work in the category of Borel G -equivariant spectra. See [FH, §6] for an introduction and for notation explanation. We write $[-, -]^{hG}$ for the abelian group of homotopy classes of Borel equivariant maps between G -spectra.

As evidence in favor of Borel equivariant spectra, consider the case when Y^d is a closed manifold and G acts trivially on Y . Interpret the last line of (2.7) as twisted E -cohomology; replace E -cohomology by Borel equivariant E -cohomology; use the fact that the Borel G -equivariant cohomology of Y is the nonequivariant E -cohomology of the Borel construction $EG \times_G Y$; then since G acts trivially on Y , the Borel construction reduces to $EG \times_G Y \cong BG \times Y$; hence the Borel equivariant version of (2.7) is

$$(3.1) \quad [\Sigma^{\mathbb{R}^d - TY}(Y) \wedge MTH \wedge BG_+, \Sigma^{d+2}I\mathbb{Z}] \cong \tilde{E}_0(Y),$$

where $\tilde{E}$ is the spectrum (1.3) for the symmetry type $(H \times G, \rho \times e)$ obtained from (H, ρ) by taking the Cartesian product with G as an internal symmetry. This is the expected answer.

Denote the Borel equivariant homology of a G -space Y as

$$(3.2) \quad E_0^{hG}(Y) = [S^0, E \wedge Y_+]^{hG},$$

where on the right hand side E is regarded as a G -spectrum with trivial G -action.

Ansatz 3.3. Let Y be a locally compact topological space equipped with the action of a compact Lie group G . Then the group of invertible topological phases on Y of symmetry type (H, ρ) is the Borel-Moore equivariant homology group $E_{0, BM}^{hG}(Y)$.

Remark 3.4. Whereas Borel equivariant E -cohomology is the E -cohomology of the Borel construction, Borel equivariant E -homology (3.2) is *not* the E -homology of the Borel construction.

Example 3.5 (Euclidean symmetries with a fixed point). Suppose $Y = \mathbb{E}^d$ and G is a group of isometries which fixes a point $p \in \mathbb{E}^d$. Use p as a basepoint to identify the affine space $\mathbb{E}^d$ with the vector space $\mathbb{R}^d$; then the action is described by a homomorphism $\lambda: G \rightarrow O_d$. Let S^λ denote the associated representation sphere: the one point compactification of $\mathbb{R}^d$ with basepoint the new point at infinity and inherited G -action. Then Ansatz 3.3

computes the group of invertible phases:

$$\begin{aligned}
 (3.6) \quad E_0^{hG}(S^d, \infty) &\cong [S^0, E \wedge S^\lambda]^G \\
 &\cong [S^{-\lambda}, E]^G \\
 &\cong [\Sigma^{d-\lambda}(BG) \wedge MTH, \Sigma^{d+2}I\mathbb{Z}] \\
 &\cong [\mathrm{Thom}(BH \times BG; -W + \underline{\mathbb{R}}^d - V_\lambda), \Sigma^{d+2}I\mathbb{Z}],
 \end{aligned}$$

where $V_\lambda \rightarrow BG$ is associated to λ . (The isomorphism (3.6) is a special case of (2.7).) The last expression in (3.6) is the group of invertible phases in d space dimensions of the symmetry type $(H \times G, \rho \times \lambda)$. For $H = SO$ (bosonic theories) this reduces to the “crystalline equivalence principle” of [TE] in dimensions $d \leq 1$ for which we can replace MSO by $H\mathbb{Z}$. (Note that (3.6) includes a twist for symmetries which reverse orientation.)

4 Computational techniques in Borel equivariant theory

We offer a brief exposition of computational methods, relying on [FH, §6] and the references therein for background on equivariant stable homotopy theory.

4.1 Reduction to nonequivariant computations. The evaluation of the Borel equivariant maps between G -spectra can often be reduced to the computation of non-equivariant maps by the following devices.

(A) When M is a G -spectrum and N is an ordinary spectrum, regarded as a G -spectrum with trivial action one has

$$(4.1) \quad [M, N]^{hG} = [EG_+ \wedge_G M, N].$$

(B) (*Adams isomorphism*). When M has trivial G -action, N is a G -spectrum, and T is a finite free G -CW-complex, the transfer map

$$(4.2) \quad [M, (N \wedge T_+ \wedge S^{\mathfrak{g}})_{hG}] \rightarrow [M, N \wedge T_+]^{hG}$$

is an isomorphism. Here $S^{\mathfrak{g}}$ is the one point compactification of the Lie algebra of G and

$$(N \wedge S^{\mathfrak{g}})_{hG} = EG_+ \wedge_G (N \wedge S^{\mathfrak{g}}).$$

(C) Atiyah duality identifies the Spanier-Whitehead dual of a closed manifold M with the Thom complex M^{-TM} . When $W \subset G$ is a closed subgroup this implies that the Spanier-Whitehead dual of the homogeneous

space G/W is the Thom spectrum $G_+ \wedge_W S^{-\mathfrak{g}/\mathfrak{w}}$, in which $\mathfrak{g} = \text{Lie } G$ and $\mathfrak{w} = \text{Lie } W$.

(D) When $W \subset G$ is a closed subgroup one has an isomorphism

$$[M \wedge G_+/W, N]^{hG} = [M, N]^{hW}$$

from which, using Atiyah duality, one deduces an isomorphism

$$[M, N \wedge G/W_+]^{hG} = [M, N \wedge S^{\mathfrak{g}/\mathfrak{w}}]^{hW}.$$

Remark 4.3. In (B) when N is the suspension spectrum of a G -space X then $(N \wedge S^{\mathfrak{g}})_{hG}$ is the suspension spectrum of the Thom complex

$$\text{Thom}(EG \times_G X, \mathfrak{g}).$$

Computations in Borel equivariant homotopy theory can be made using the above rules, augmented with knowledge of the effect of the maps

$$(4.4) \quad [M, N \wedge G/(W_1)_+]^{hG} \rightarrow [M, N \wedge G/(W_2)_+]^{hG}$$

$$(4.5) \quad [M \wedge G/(W_2)_+, N]^{hG} \rightarrow [M \wedge G/(W_1)_+, N]^{hG}$$

induced by an equivariant map

$$G/W_1 \rightarrow G/W_2.$$

Remark 4.6. In the extended example in §5, the group G is cyclic of order 2 and the only map whose effect need be worked out is

$$G \rightarrow G/G.$$

When M and N have trivial G -action, the maps (4.4) and (4.5) are identified, using the rules above, with the maps

$$\begin{aligned} [M, N] &\rightarrow [M \wedge BG_+, N] \\ [M \wedge BG_+, N] &\rightarrow [M, N] \end{aligned}$$

induced by the transfer map $BG_+ \rightarrow S^0$ and the map $S^0 \rightarrow BG_+$ associated to a choice of point in BG .

4.2 Equivariant Atiyah-Hirzebruch spectral sequence. To motivate the construction assume G is a finite group and Z a pointed G -space. Let $L' \subset G$ be a subgroup and suppose $f: G/L' \times S^{p-1} \rightarrow Z$ is a continuous G -equivariant map for some positive integer p . The mapping cone of f is the union $W = Z \cup_f (G/L' \times D^p)$ which attaches an equivariant p -cell to the space Z . From the equivariant cofibration sequence

$$(4.7) \quad Z \longrightarrow W \longrightarrow W/Z \simeq G/L' \times (D^p, S^{p-1})$$

we obtain a boundary map in equivariant homology:

$$(4.8) \quad \partial: E_k^{hG}(W, Z) \longrightarrow E_{k-1}^{hG}(Z).$$

By excision and (4.1) the domain is isomorphic to $E^{p-k}(BL')$, which by (1.4) is interpreted as a group of topological phases in spatial dimension $p - k$. If $E = E_{(H, \rho)}$ as in (1.3), then these theories have symmetry type $(H \times L', \rho \times e)$. Suppose Z is obtained from a subcomplex $Z' \subset Z$ by attaching an equivariant $(p - 1)$ -cell $G/L \times D^{p-1}$, and compose (4.8) with the quotient map

$$(4.9) \quad E_{k-1}^{hG}(Z) \longrightarrow E_{k-1}^{hG}(Z, Z') \cong E_{k-1}^{hG}(G/L \times S^{p-1}) \cong E^{p-k}(BL).$$

If the composite is nonzero, which means the boundary of the p -cell attached in (4.7) intersects the $(p - 1)$ -cell in (4.9), then since the stabilizer subgroup can only increase by taking the boundary, we must have $L' \subset L$. The composite $E^{p-k}(BL') \rightarrow E^{p-k}(BL)$ is the *transfer*, the pushforward along the finite cover $BL' \rightarrow BL$ with fiber L'/L .

The Atiyah-Hirzebruch spectral sequence is obtained by filtering a G -CW complex by its skeleta and systematizing the argument above. Suppose Y is the complement of a subcomplex $Y_0 \subset \bar{Y}$ of a finite G -CW complex. Then the E^1 -page of the spectral sequence is the *Bredon homology* of $(\bar{Y}, Y_0)$ with coefficients in the covariant functor on the orbit category of G with values in $\mathbb{Z}$ -graded abelian groups whose component in degree q at G/L is $E^{-q}(BL)$, which is the language used to describe the systematization of the previous paragraph. In degree $-q$ the coefficient group is the group of invertible topological phases of symmetry type $(H \times L, \rho \times e)$ in (spatial) dimension q ; see (3.1). This is the E^1 -page contribution of an equivariant p -cell $e^p \times G/L$. The spectral sequence converges to an associated graded of $E_0^{hG}(\bar{Y}, Y_0)$.

The differential d^1 is the composition of the usual equivariant cellular boundary map with a transfer map, the latter nontrivial in case the stabilizer group L of a $(p-1)$ -cell e is strictly larger than the stabilizer group L' of a p -cell e' whose boundary rel the $(p-2)$ -skeleton maps with nontrivial degree to e . Assume G is finite. The transfer⁴ $E^{-d}(BL') \rightarrow E^{-d}(BL)$ has a field-theoretic interpretation as a map from $(d+1)$ -dimensional theories of H -manifolds equipped with a principal L' -bundle to $(d+1)$ -dimensional theories of H -manifolds equipped with a principal L -bundle. If M is a manifold (bordism) equipped with a principal L -bundle $P \rightarrow M$, then a section of the associated fiber bundle $P/L' \rightarrow M$ with fiber L/L' is equivalent to a reduction of $P \rightarrow M$ to structure group $L' \subset L$. The evaluation of the transfer of F on (M, P) is the (tensor) product over sections of $P/L' \rightarrow M$ of the values of the theory F . In general sections only exist locally, so we must use the extended locality of these field theories to compute the transfer.

We remark that there is a similar spectral sequence if G is a compact Lie group. See [SXG] for further information about the Atiyah-Hirzebruch spectral sequence in this context.

5 Fermionic phases on $\mathbb{E}^3$ with a half-turn

By way of illustration we now turn to the classification of phases on $\mathbb{E}^3$ which are symmetric with respect to the involution $(x, y, z) \mapsto (x, -y, -z)$. The one point compactification of $\mathbb{E}^3$ is the equivariant sphere $S^{1+2\sigma}$, where σ is the real sign representation. The symmetry type (H, ρ) has H the infinite Spin group, and in this case we may identify MTSpin with MSpin . Applying Ansatz 3.3 in the form (3.6), we determine the group of equivariant phases to be

$$(5.1) \quad [\text{MSpin}, \Sigma^2 I_{\mathbb{Z}} \wedge S^{1+2\sigma}]^{h\mathbb{Z}/2}.$$

We compute this group in three ways.

5.1 First method. Apply (3.6) with $d = 3$ and $\lambda = 1 + 2\sigma$ to compute (5.1) as

$$(5.2) \quad [\Sigma^{2-2\sigma} \mathbb{R}P^\infty \wedge \text{MSpin}, \Sigma^5 I\mathbb{Z}] \cong [\Sigma^{2-2\sigma} \mathbb{R}P^\infty, \Sigma ko\langle 0 \cdots 4 \rangle].$$

Here we use the Anderson-Brown-Peterson [ABP] decomposition of MSpin , in which the leading term is ko and higher terms do not appear since $\Sigma^5 I\mathbb{Z}$

⁴We assume an inclusion $L' \subset L$; an inclusion into a conjugate gLg^{-1} is then composition with an automorphism.

has vanishing homotopy groups above dimension 5; we also use the Anderson self-duality of ko (with a shift of 4) [HS]. Note $\Sigma^{2-2\sigma}\mathbb{RP}_+^\infty$ is the Thom spectrum $\text{Thom}(\mathbb{RP}^\infty; \underline{\mathbb{R}}^2 - L^{\oplus 2})$, where $L \rightarrow \mathbb{RP}^\infty$ is the tautological real line bundle. Let U be the Thom class of $\underline{\mathbb{R}}^2 - L^{\oplus 2} \rightarrow \mathbb{RP}^\infty$, $\bar{U}$ its mod 2 reduction, and $a \in H^1(\mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z})$ the generator. The right hand side of (5.2) can be computed from the nonequivariant Atiyah-Hirzebruch *cohomology* spectral sequence

$$(5.3) \quad \begin{aligned} E_2^{p,q} &\cong H^p(\Sigma^{2-2\sigma}\mathbb{RP}^\infty; ko\langle 0 \cdots 4 \rangle^q(\text{pt})) \\ &\implies [\Sigma^{2-2\sigma}\mathbb{RP}^\infty, \Sigma^{p+q} ko\langle 0 \cdots 4 \rangle]. \end{aligned}$$

The contributions in total degree 1 come from $E_2^{2,-1} \cong \mathbb{Z}/2\mathbb{Z} \cdot \bar{U}a^2$ and $E_2^{3,-2} \cong \mathbb{Z}/2\mathbb{Z} \cdot \bar{U}a^3$, which are killed respectively by $d_2(\bar{U}) = Sq^2(\bar{U})$ from $E_2^{0,0}$ and $d_2(\bar{U}a) = Sq^2(\bar{U}a)$ from $E_2^{1,-1}$. (Observe $Sq^k(\bar{U}) = \bar{U}w_k(\underline{\mathbb{R}}^2 - L^{\oplus 2})$.) Thus the group (5.1) of phases vanishes in this case.

5.2 Second method. Decompose $S^{1+2\sigma}$ into pieces of fixed isotropy and make use of the methodology described in §4.1. The first step is to write

$$S^{1+2\sigma} = S^1 \wedge S^{2\sigma}$$

and

$$[M\text{Spin}, \Sigma^2 I\mathbb{Z} \wedge S^{1+2\sigma}]^{h\mathbb{Z}/2} = [M\text{Spin}, \Sigma^3 I\mathbb{Z} \wedge S^{2\sigma}]^{h\mathbb{Z}/2}.$$

Now $S^{2\sigma}$ is the unreduced suspension of the unit sphere $S(2\sigma) \subset \mathbb{R}^{2\sigma}$ so there is a cofibration sequence of pointed $\mathbb{Z}/2$ -spaces (or spectra)

$$S(2\sigma)_+ \rightarrow S^0 \rightarrow S^{2\sigma}$$

and an exact sequence

$$(5.4) \quad \begin{aligned} [M\text{Spin}, \Sigma^3 I\mathbb{Z} \wedge S(2\sigma)_+]^{h\mathbb{Z}/2} &\rightarrow [M\text{Spin}, \Sigma^3 I\mathbb{Z}]^{h\mathbb{Z}/2} \\ &\rightarrow [M\text{Spin}, \Sigma^3 I\mathbb{Z} \wedge S^{2\sigma}]^{h\mathbb{Z}/2} \\ &\rightarrow [M\text{Spin}, \Sigma^4 I\mathbb{Z} \wedge S(2\sigma)_+]^{h\mathbb{Z}/2} \\ &\rightarrow [M\text{Spin}, \Sigma^4 I\mathbb{Z}]^{h\mathbb{Z}/2}. \end{aligned}$$

We will check that

$$(5.5) \quad [M\text{Spin}, \Sigma^3 I\mathbb{Z} \wedge S(2\sigma)_+]^{h\mathbb{Z}/2} \rightarrow [M\text{Spin}, \Sigma^3 I\mathbb{Z}]^{h\mathbb{Z}/2}$$

is an epimorphism and

$$(5.6) \quad [\mathrm{MSpin}, \Sigma^4 I\mathbb{Z} \wedge S(2\sigma)_+]^{h\mathbb{Z}/2} \rightarrow [\mathrm{MSpin}, \Sigma^4 I\mathbb{Z}]^{h\mathbb{Z}/2}$$

is a monomorphism, from which we deduce

$$[\mathrm{MSpin}, \Sigma^3 I\mathbb{Z} \wedge S^{2\sigma}]^{h\mathbb{Z}/2} = 0.$$

This implies that there is only one phase on $\mathbb{E}^3$ — the trivial phase — which is symmetric with respect to the involution $(x, y, z) \mapsto (x, -y, -z)$.

To evaluate (5.5) and (5.6) note the orbit space $S(2\sigma)/(Z/2)$ is just $\mathbf{RP}^1 = S^1$ so from the Adams isomorphism (4.2) we have

$$[\mathrm{MSpin}, \Sigma^k I\mathbb{Z} \wedge S(2\sigma)_+]^{h\mathbb{Z}/2} \approx [\mathrm{MSpin}, \Sigma^k I\mathbb{Z} \wedge \mathbf{RP}_+^1].$$

The composition

$$[\mathrm{MSpin}, \Sigma^3 I\mathbb{Z} \wedge S(2\sigma)_+]^{h\mathbb{Z}/2} \rightarrow [\mathrm{MSpin}, \Sigma^3 I\mathbb{Z}]^{h\mathbb{Z}/2} \rightarrow [\mathrm{MSpin}, \Sigma^3 I\mathbb{Z}]$$

is the map induced by the transfer map of spectra

$$(5.7) \quad \mathbf{RP}_+^1 \rightarrow S^0.$$

A choice of base point in $\mathbf{RP}^1$ gives a weak equivalence

$$(5.8) \quad \mathbf{RP}_+^1 \xrightarrow{\approx} S^1 \vee S^0.$$

Since $\mathbf{RP}^1$ is path connected, the homotopy class of this map is independent of this choice.

The following can be proved using standard methods.

Proposition 5.9. With respect to the decomposition (5.8) the transfer map

$$\mathbf{RP}_+^1 \rightarrow S^0$$

has components

$$\begin{aligned} \eta : S^1 &\rightarrow S^0 \\ 2 : S^0 &\rightarrow S^0. \end{aligned}$$

in which $\eta \in \pi_1 S^0 = \mathbb{Z}/2$ is the non-trivial element. □

Using the fact that the Atiyah-Bott-Shapiro map $\mathrm{MSpin} \rightarrow ko$ is an equivalence up to dimension 8, and the isomorphisms

$$\begin{aligned} [\mathrm{MSpin}, \Sigma^k I\mathbb{Z} \wedge S(2\sigma)_+]^{h\mathbb{Z}/2} &\approx [\mathrm{MSpin}, \Sigma^k I\mathbb{Z} \wedge \mathbf{RP}_+^1] \\ &\approx [\mathrm{MSpin}, \Sigma^{k+1} I\mathbb{Z}] \oplus [\mathrm{MSpin}, \Sigma^k I\mathbb{Z}] \\ [\mathrm{MSpin}, \Sigma^k I\mathbb{Z}]^{h\mathbb{Z}/2} &\approx [\mathrm{MSpin} \wedge B\mathbb{Z}/2_+, \Sigma^k I\mathbb{Z}] \\ &\approx [\mathrm{MSpin} \wedge B\mathbb{Z}/2, \Sigma^k I\mathbb{Z}] \oplus [\mathrm{MSpin}, \Sigma^k I\mathbb{Z}] \end{aligned}$$

one extracts the following table of values

k	$[\mathrm{MSpin}, \Sigma^k I\mathbb{Z}]$	$[\mathrm{MSpin}, \Sigma^k I\mathbb{Z} \wedge S(2\sigma)_+]^{h\mathbb{Z}/2}$	$[\mathrm{MSpin}, \Sigma^k I\mathbb{Z}]^{h\mathbb{Z}/2}$
0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
1	0	$\mathbb{Z}/2$	0
(5.10) 2	$\mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$
3	$\mathbb{Z}/2$	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$
4	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}/8$
5	0		0

as well as the fact that multiplication by the non-zero element $\eta \in \pi_1 S^0$ is the non-trivial map

$$[\mathrm{MSpin}, \Sigma^k I\mathbb{Z}] \rightarrow [\mathrm{MSpin}, \Sigma^{k-1} I\mathbb{Z}]$$

when $k = 4$ or 3 .

By Remark 4.6, homomorphisms

$$\begin{aligned} [\mathrm{MSpin}, \Sigma^k I\mathbb{Z} \wedge (\mathbb{Z}/2)_+]^{h\mathbb{Z}/2} &\rightarrow [\mathrm{MSpin}, \Sigma^k I\mathbb{Z}]^{h\mathbb{Z}/2} \\ [\mathrm{MSpin}, \Sigma^k I\mathbb{Z}]^{h\mathbb{Z}/2} &\rightarrow [\mathrm{MSpin} \wedge (\mathbb{Z}/2)_+, \Sigma^k I\mathbb{Z}]^{h\mathbb{Z}/2} \end{aligned}$$

induced by the map $\mathbb{Z}/2 \rightarrow \mathrm{pt}$ can be identified with the maps

$$\begin{aligned} [\mathrm{MSpin}, \Sigma^k I\mathbb{Z}] &\rightarrow [\mathrm{MSpin} \wedge B\mathbb{Z}/2_+, \Sigma^k I\mathbb{Z}] \\ [\mathrm{MSpin} \wedge B\mathbb{Z}/2_+, \Sigma^k I\mathbb{Z}] &\rightarrow [\mathrm{MSpin}, \Sigma^k I\mathbb{Z}] \end{aligned}$$

induced by the transfer map $B\mathbb{Z}/2_+ \rightarrow S^0$, and the inclusion map $S^0 \rightarrow B\mathbb{Z}/2_+$ associated to a choice of point in $B\mathbb{Z}/2$. The effect of the transfer map is given by the following table

k	$[\mathrm{MSpin}, \Sigma^k I\mathbb{Z}]$	— transfer $\twoheadrightarrow$	$[\mathrm{MSpin}, \Sigma^k I\mathbb{Z}]^{n_{\mathbb{Z}/2}/2}$
0	$\mathbb{Z}$	$[2]$	$\mathbb{Z}$
1	0		0
2	$\mathbb{Z}/2$	$[0 \ 1]^T$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$
3	$\mathbb{Z}/2$	$[0 \ 1]^T$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$
4	$\mathbb{Z}$	$[2 \ 1]^T$	$\mathbb{Z} \oplus \mathbb{Z}/8$
5	0		0

With these values, and Proposition 5.9 the map

$$[\mathrm{MSpin}, \Sigma^3 I\mathbb{Z} \wedge S(2\sigma)_+]^{h\mathbb{Z}/2} \rightarrow [\mathrm{MSpin}, \Sigma^3 I\mathbb{Z}]^{h\mathbb{Z}/2}$$

becomes

$$(5.11) \quad \mathbb{Z} \oplus \mathbb{Z}/2 \xrightarrow{\begin{bmatrix} * & 1 \\ 1 & 0 \end{bmatrix}} \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

which is indeed an epimorphism, while

$$[\mathrm{MSpin}, \Sigma^4 I\mathbb{Z} \wedge S(2\sigma)_+]^{h\mathbb{Z}/2} \rightarrow [\mathrm{MSpin}, \Sigma^4 I\mathbb{Z}]^{h\mathbb{Z}/2}$$

becomes

$$\mathbb{Z} \xrightarrow{\begin{bmatrix} 2 \\ * \end{bmatrix}} \mathbb{Z} \oplus \mathbb{Z}/8$$

which is a monomorphism.

5.3 Third method. A lecture by Mike Hermele based on [HSHH] suggested to us that the equivariant Atiyah-Hirzebruch homology spectral sequence has a physical interpretation in this context; here we describe how this spectral sequence plays out to kill the relevant group. See also [SXG] for many worked examples using this spectral sequence. We refer to §4.2 for an exposition of the equivariant Atiyah-Hirzebruch spectral sequence. In the case of equivariant phases on $\mathbb{E}^3$, we use the equivariant cell decomposition

$$S^{1+2\sigma} = S^1 \cup \mathbb{Z}/2 \times e^2 \cup \mathbb{Z}/2 \times e^3$$

of the one-point compactification of $\mathbb{E}^3$, the appropriate representation sphere. Using the table (5.10), the spectral sequence works out to be

	0	1	2	3
0				
-1		$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
-2		$\mathbb{Z} \oplus \mathbb{Z}/8$	$\mathbb{Z}$	$\mathbb{Z}$

$d^1: \mathbb{Z}/2 \oplus \mathbb{Z}/2 \rightarrow \mathbb{Z}/2$ (horizontal arrow from column 1 to 2)
 $d^1: \mathbb{Z}/2 \rightarrow \mathbb{Z}/2$ (horizontal arrow from column 2 to 3)
 $d^2: \mathbb{Z}/2 \oplus \mathbb{Z}/2 \rightarrow \mathbb{Z}$ (diagonal arrow from column 1 to 2, row -2)
 $d^1: \mathbb{Z} \oplus \mathbb{Z}/8 \rightarrow \mathbb{Z}$ (horizontal arrow from column 1 to 2)
 $d^1: \mathbb{Z} \rightarrow \mathbb{Z}$ (horizontal arrow from column 2 to 3)

The $(1, -d)$ entry is the group of invertible $(d+1)$ -dimensional fermionic phases with internal symmetry group $\mathbb{Z}/2\mathbb{Z}$ (the stabilizer group of the 1-cell), and the $(p, -d)$ entry for $p=2, 3$ is the group of invertible $(d+1)$ -dimensional fermionic phases. The group (3.2) of interest is the homology in degree 0.

Claim 5.12. The spectral sequence scorecard in degree 0 is:

- (i) the differential $d^1: E_{2,-1}^1 \rightarrow E_{1,-1}^1$ hits a $\mathbb{Z}/2\mathbb{Z}$ -subgroup;
- (ii) $d^1: E_{2,-2}^1 \rightarrow E_{1,-2}^1$ is injective; and
- (iii) the differential $d^2: E_{3,-2}^2 \rightarrow E_{1,-1}^2$ is onto the remaining $\mathbb{Z}/2\mathbb{Z}$.

Proof. The group $E_{1,-1}^1$ of invertible topological phases of spin 2-manifolds X equipped with a double cover $Q \rightarrow X$ may be described in terms of partition functions. Recall that a spin structure on a closed 2-manifold X gives a quadratic refinement q_X of the intersection pairing on $H^1(X; \mathbb{Z}/2\mathbb{Z})$, and q_X has an Arf invariant $\text{Arf}(q_X) \in \mathbb{Z}/2\mathbb{Z}$. The equivalence class of a double cover $Q \rightarrow X$ lives in $H^1(X; \mathbb{Z}/2\mathbb{Z})$. The four possible partition functions are 1, $(-1)^{\text{Arf}(q_X)}$, $(-1)^{\text{Arf}(q_X+Q)}$, and $(-1)^{q_X(Q)}$. A more precise version of (i) is: the first differential $d^1: E_{2,-1}^1 \rightarrow E_{1,-1}^1$ maps the second of these, which is a theory on spin manifolds without a double cover, onto the last of these.⁵ We can compute that from the transfer as follows. Let the target 2-groupoid for these extended field theories be the Morita category of central simple complex superalgebras equipped with a $\mathbb{Z}/2\mathbb{Z}$ -action. The four

⁵which Mike Hermele calls a “fermionic AKLT state”

theories evaluate on a point respectively to $\mathbb{C}$ with trivial involution, the Clifford algebra $A = \text{Cliff}_1^{\mathbb{C}}$ with trivial involution, the algebra A with nontrivial involution, and $\mathbb{C}$ with nontrivial involution. The transfer maps the second of these to $A \otimes A$ with the involution exchanging the factors, and this is Morita equivalent to $\mathbb{C}$ with nontrivial involution. This proves (i). Claim (ii) is straightforward: the differential $d^1: E_{2,-2}^1 \rightarrow E_{1,-2}^1$ does not involve a transfer, so reduces to the cellular differential. The differential in (iii) is induced by the transfer (5.7), and was worked out in (5.11). $\square$

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