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Hyperbolic rank rigidity for manifolds of $\frac{1}{4}$-pinched negative curvature

CHRIS CONNELL†, THANG NGUYEN‡ and RALF SPATZIER§

† *Department of Mathematics, Indiana University, Bloomington, IN 47405, USA*
(e-mail: connell@indiana.edu)
‡ *Courant Institute of Mathematical Sciences, New York University, New York, NY 10012, USA*
(e-mail: tnguyen@nyu.edu)
§ *Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA*
(e-mail: spatzier@umich.edu)

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Abstract. A Riemannian manifold M has higher hyperbolic rank if every geodesic has a perpendicular Jacobi field making sectional curvature -1 with the geodesic. If, in addition, the sectional curvatures of M lie in the interval $[-1, -\frac{1}{4}]$ and M is closed, we show that M is a locally symmetric space of rank one. This partially extends work by Constantine using completely different methods. It is also a partial counterpart to Hamenstädt's hyperbolic rank rigidity result for sectional curvatures ≤ -1, and complements well-known results on Euclidean and spherical rank rigidity.

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1. *Introduction*

We say that a closed Riemannian manifold M has *higher hyperbolic rank* if every geodesic $c(t)$ in M has a non-zero perpendicular Jacobi field $J(t)$ that spans a plane of sectional curvature -1 with $c'(t)$ for all $t \geq 0$ (where $J(t) \neq 0$).

Our notion of higher hyperbolic rank is *a priori* weaker than either the usual one, which requires that the Jacobi fields in question make curvature -1 for $t \in (-\infty, \infty)$, or else the version that uses parallel fields in place of Jacobi fields. In strict negative curvature these distinct formulations turn out to coincide (see Corollary 2.9). Actually, the techniques used in our proofs require us to introduce the notion of higher hyperbolic rank for positive time.

The main goal of this paper is the following hyperbolic rank rigidity result.

THEOREM 1.1. *Let M be a closed Riemannian manifold of higher hyperbolic rank and sectional curvatures K between $-1 \leq K \leq -\frac{1}{4}$. Then M is a rank-one locally symmetric space. In particular, if the pinching is strict then M has constant curvature -1.*

Constantine [**Con08**, Corollary 1] characterized constant curvature manifolds among those of non-positive curvature and higher hyperbolic rank under one of two conditions: odd dimension without further curvature restrictions besides Euclidean rank one; or even dimension, provided the sectional curvatures are pinched between $-(.93)^2$ and -1. He also showed that if one uses the stronger notion of parallel fields in place of Jacobi fields, then one may relax the lower curvature bound of -1, though the same pinching in even dimensions is still required. His method is rather different from ours, drawing on ergodicity results for the 2-frame flow of such manifolds. For $\frac{1}{4}$-pinched manifolds of negative curvature, however, ergodicity of the frame flow has been conjectured now for over 30 years, with no avenue for an approach in sight [**Bri82**, Conjecture 2.6]. To overcome this difficulty, we introduce entirely different methods for $\frac{1}{4}$-pinched metrics. These also allow us to include rank-one locally symmetric spaces in our classification above.

Both Constantine's result and ours are counterpoints to Hamenstädt's hyperbolic rank rigidity theorem [**Ham91b**], which follows.

THEOREM 1.2. (Hamenstädt) *Closed manifolds with sectional curvatures $K \leq -1$ and higher hyperbolic rank are locally symmetric spaces of real rank one.*

Compactness is truly essential in these results. Indeed, Connell found a counterexample amongst homogeneous manifolds of negative curvature whilst proving hyperbolic rank rigidity for such spaces under an additional condition [**Con02**].

Lin and Schmidt recently constructed non-compact manifolds of higher hyperbolic rank in [**LS16**] with both upper and lower curvature bounds -1 and curvatures arbitrarily pinched. In addition, their examples are not even locally homogeneous and every geodesic lies in a totally geodesic hyperbolic plane. In dimension three, Lin showed that finite-volume manifolds with higher hyperbolic rank always have constant curvature, without imposing any curvature properties [**LS16**].

The notion of higher hyperbolic rank is analogous to those of *higher Euclidean rank and spherical rank*. However, in the last two cases we are looking for parallel vector fields, not just Jacobi fields, along geodesics that make curvature 0 or 1 respectively. These versions of higher rank sometimes carry the designation *strong*. When 0, 1 or -1 are also extremal as values of sectional curvature, various rigidity theorems have been proved. In particular, we have the results of Ballmann and Burns-Spatzier in non-positive curvature where higher Euclidean rank manifolds are shown to be locally either Riemannian products or symmetric spaces (cf. [**Bal85**, **Bal95**, **BS87**], Eberlein and Heber [**EH90**] for certain non-compact manifolds and Watkins [**Wat13**] for manifolds without focal points). When the sectional curvatures are less than 1, and M has higher spherical rank, Shankar, Spatzier and Wilking showed that M is locally isometric to a compact rank-one symmetric space [**SSW05**]. Notably, there are counterexamples in the form of the Berger metrics for the analogous statements, replacing parallel fields by Jacobi fields in the definition of higher spherical rank (see [**SSW05**]).

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Thus the situation for closed manifolds is completely understood for upper curvature bounds, and we have full rigidity. For lower curvature bounds, the situation is more complicated. For one thing, there are many closed manifolds of non-negative curvature and higher Euclidean rank. The first examples, given by Heintze (private communication), were still homogeneous. More general, and in particular inhomogeneous, examples were constructed by Spatzier and Strake in [**SS90**]. For higher spherical rank and lower bound on the sectional curvature by 1, Schmidt, Shankar and Spatzier again proved local isometry to a sphere of curvature 1 if the spherical rank is at least $n-2>0$, n is odd, or if $n\neq 2, 6$ and M is a sphere [**SSS16**]. No counterexamples are known. If, in addition, M is Kähler of dimension at least four, then M is locally isometric to complex projective space with the Fubini–Study metric. In dimension three, Bettiol and Schmidt showed that higher rank implies local splitting of the metric, without any conditions on the curvature [**BS16**].

Let us outline our argument for Theorem 1.1 that occupies the remainder of this paper. All of our arguments hold for manifolds with sectional curvature bounds $-1\leq K<0$ for the first four sections, with the exception of Corollary 4.4 where we use strict $\frac{1}{4}$-pinching for the stable distribution to be C^1. We show that we may assume that every geodesic $c(t)$ has orthogonal parallel fields E with sectional curvature -1. The dimension of the latter vector space is called the *strong hyperbolic rank* of c. Following Constantine in [**Con08**, §5], strong rank agrees with the rank under lower sectional curvature bound -1 (cf. Proposition 2.6). We then show in §2 that the regular set $\mathcal{R}$ of unit tangent vectors v for which $rk^h(v)=rk^h(M)$ (cf. Definition 2.1) is dense and open. Additionally it has the property that if $v\in\mathcal{R}$ is recurrent then its stable and unstable manifolds also belong to $\mathcal{R}$. Next, in §3, we show that the distribution of parallel fields of curvature -1 is smooth on the regular set. Then in §4, we characterize these parallel fields in terms of unstable Jacobi fields of Lyapunov exponent 1 for bi-recurrent regular vectors. We use this to show that the slow unstable distribution extends to a smooth distribution on $\mathcal{R}$.

In §5, we prove the result under the stronger assumption of strict $\frac{1}{4}$-curvature pinching as the technicalities are significantly simpler and avoid the use of measurable normal forms from Pesin theory employed in §6. We are inspired here by arguments of Butler in [**But15**]. We construct a Kanai-like connection for which the slow and fast stable and unstable distributions are parallel. The construction is strongly motivated by a similar one from Benoist, Foulon and Labourie in [**BFL90**]. We use this to prove integrability of the slow unstable distribution. This distribution is also invariant under stable holonomy by an argument of Feres and Katok [**FK90**], and hence defines a distribution on $\partial\widetilde{M}$. As it is integrable and $\pi_1(M)$-invariant, we get a $\pi_1(M)$-invariant foliation on $\partial\widetilde{M}$, which is impossible thanks to an argument of Foulon [**Fou94**] (or the argument for [**Ham91b**, Corollary 4.4]).

Lastly, in §6, we treat the general case of non-strict $\frac{1}{4}$-curvature pinching. By a result of Connell [**Con03**], relying on Theorem 1.2 from the work of Hamenstädt, if M is not already a locally symmetric space, then the Lyapunov spectrum has non-uniform 2:1 resonance. Now we can use recent work of Melnick [**Mel17**] on normal forms to obtain a suitably invariant connection (cf. also Kalinin–Sadovskaya [**KS17**]). This allows us to prove integrability of the slow unstable distribution on almost every unstable manifold. As before, we can obtain a π_1-invariant foliation on $\partial\widetilde{M}$ and finish with the result of

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Foulon. This is technically more complicated, however, because we no longer have C^1 holonomy maps. Instead we adapt an argument of Feres and Katok, to show that stable holonomy maps almost everywhere preserve the tangencies of our slow unstable foliation. To this end, we show that the holonomy maps are differentiable with bounded derivatives, though not necessarily C^1, between good unstable manifolds. This allows us to obtain the desired holonomy invariance as in the strict $\frac{1}{4}$-pinching case to finish the proof of the main theorem.

In light of the above, in particular Theorem 1.1 as well as Constantine's results, we make the following conjecture.

CONJECTURE 1.3. *A closed manifold with sectional curvatures ≥ -1 and higher hyperbolic rank is isometric to a locally symmetric space of real rank one.*

Let us point out that the starting point of the proofs for upper and lower curvature bounds are radically different, although they share some common features. In the hyperbolic rank case in particular, for the upper curvature bound, we get control of the slow unstable foliation in terms of parallel fields. Hamenstädt used this to create Carnot metrics on the boundary with large conformal groups leading to the models of the various hyperbolic spaces. The lower curvature bound in comparison gives us control of the fast unstable distribution, which is integrable and does not apparently tell us anything about the slow directions. It is clear that the general case will be much more difficult, even if we assume that the metric has negative or at least non-positive curvature.

Finally let us note a consequence of Theorem 1.1 in terms of dynamics. Consider the geodesic flow g_t on the unit tangent bundle of a closed manifold M. For a geodesic $c \subset M$, the *maximal Lyapunov exponent* $\lambda_{\max}(c)$ for c is the biggest exponential growth rate of the norm of a Jacobi field $J(t)$ along c:

$$\lambda_{\max}(c) := \max_{J \text{ Jacobi for } c} \lim \frac{1}{t} \log \|J(t)\|.$$

Note that $\lambda_{\max}(c) \leq 1$ if the sectional curvatures of M are bounded below by -1, by Rauch's comparison theorem.

Given an ergodic g_t-invariant measure μ on the unit tangent bundle SM, $\lambda_{\max}(c)$ is constant μ-almost everywhere. In fact, it is just the maximal Lyapunov exponent in the sense of dynamical systems for g_t and μ (cf. §4).

COROLLARY 1.4. *Let M be a closed Riemannian manifold with sectional curvatures K between $-1 \leq K \leq -\frac{1}{4}$. Let μ be a probability measure of full support on the unit tangent bundle SM which is invariant and ergodic under the geodesic flow g_t. Suppose that the maximal Lyapunov exponent for g_t and μ is 1. Then M is a rank-one locally symmetric space.*

We supply a proof in §6. In fact, the reduction to Theorem 1.1 is identical to Constantine's in [**Con08**, §6], which in turn adapts an argument of Connell for upper curvature bounds [**Con03**].

2. *Definitions, semicontinuity and invariance on stable manifolds*

Let M be compact manifold of negative sectional curvature, and denote its unit tangent bundle by SM. We let $g_t : SM \to SM$ be the geodesic flow, and denote by $pt : SM \to M$ the footpoint map, i.e. $v \in T_{pt(v)}M$. For $v \in SM$, let c_v be the geodesic determined by v and let $v^\perp$ denote the perpendicular complement of v in $T_{pt(v)}M$.

Definition 2.1. The *hyperbolic rank* of a vector $v \in SM$, denoted by $rk^h(v)$, is the dimension of a maximal vector subspace of $v^\perp$ formed by the initial vectors of Jacobi fields that make curvature -1 with $g_t v$ for all $t \geq 0$. Moreover, the *hyperbolic rank of M*, denoted by $rk^h(M)$, is the infimum of $rk^h(v)$ over $v \in SM$.

We remark that when -1 is an extremal curvature the set in $v^\perp$ of initial vectors of such Jacobi fields is already a subspace.

LEMMA 2.2. *Let v be a unit vector recurrent under the geodesic flow. Suppose that $rk^h(v) > 0$. Then there is also an unstable or stable Jacobi field making curvature -1 with $g_t v$ for all $t \in \mathbb{R}$.*

Proof. Since $rk^h(v) > 0$, there is a Jacobi field $J(t)$ making curvature -1 with $g_t v$ for all $t \geq 0$. First assume that $J(t)$ is not stable. Decompose $J(t)$ into its stable and unstable components $J(t) = J^s(t) + J^u(t)$. Suppose $g_{t_n} v \to v$ with $t_n \to \infty$. Then, for a suitable subsequence of t_n, $J(t + t_n)/\|J(t_n)\|$ will converge to a Jacobi field $Y(t)$ along $c_v(t)$. Note then that $g_{t+t_n}(v) \to g_t v$ as $t_n \to \infty$. Moreover, for any $t \in \mathbb{R}$, $Y(t)$ is the limit of the vectors $J(t + t_n)$ which make curvature -1 with $g_{t+t_n}(v)$. Hence, $Y(t)$ also makes curvature -1 with $g_t v$ for any t. Also, $Y(t)$ is clearly unstable since $J^u(t) \not\equiv 0$.

If $J(t) = J^s(t)$ is stable, then the same procedure will produce a stable Jacobi field $Y(t)$ along $c(t)$ that makes curvature -1 with $g_t(v)$ for all $t \in \mathbb{R}$. □

LEMMA 2.3. *Suppose that $rk^h(M) > 0$. Then along every geodesic $c(t)$, we have an unstable Jacobi field that makes curvature -1 with $c(t)$ for all $t \in \mathbb{R}$. Similarly, there is a stable Jacobi field along $c(t)$ that makes curvature -1 with $c(t)$ for all $t \in \mathbb{R}$.*

Proof. Since the geodesic flow for M preserves the Liouville measure μ, μ-almost every unit tangent vector v is recurrent. By Lemma 2.2, the geodesics $c_v(t)$ have stable or unstable Jacobi fields along them that make curvature -1 with the geodesic for all $t \in \mathbb{R}$. As μ has full support in SM, such geodesics are dense and the same is true for any geodesic by taking limits.

Next we show that there are both stable and unstable Jacobi fields along any geodesic that make curvature -1 with the geodesic. Indeed, let $A^+ \subset SM$ be the set of unit tangent vectors v that have an unstable Jacobi field along $c_v(t)$ that make curvature -1 with $c_v(t)$. Similarly, define $A^- \subset SM$ as the set of unit tangent vectors v that have a stable Jacobi field along $c_v(t)$ that make curvature -1 with $c_v(t)$. Note that $A^- = -A^+$, and that $SM = A^+ \cup A^-$ by the proof shown above. Hence neither A^+ nor A^- can have measure 0 with respect to Liouville measure μ. Also, both A^+ and A^- are invariant under the geodesic flow g_t. Since g_t is ergodic with respect to μ, both A^+ and A^- must each have full measure. Now the claim is clear once again by taking limits. □

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Denote by $\Lambda(v, t)w$ the unstable Jacobi field along $g_t v$ with initial value $w \in v^\perp$. Then we let $\mathcal{E}(v) \subset v^\perp$ be the subspace of $v^\perp$ defined as follows: $w \in v^\perp$ belongs to $\mathcal{E}(v)$ if $\Lambda(v, t)w$ makes curvature -1 with $g_t v$ for all $t \geq 0$.

We define $\mathcal{R} = \{v \mid rk^h(v) = rk^h M\}$. We note that for $v \in \mathcal{R}$ and for all $u \in SM$, $\dim \mathcal{E}(v) \leq \dim \mathcal{E}(u)$.

LEMMA 2.4. *Suppose $v \in \mathcal{R}$ and $w \in \mathcal{E}(v)$. Then $\Lambda(v, t)w$ makes curvature -1 with $c_v(t)$ for all $t \in \mathbb{R}$ and $\mathcal{R}$ is invariant under the backward geodesic flow.*

Proof. First note that for $t \in \mathbb{R}$, the map $\Lambda(v, t) : v^\perp \to (g_t v)^\perp$, defined by $w \mapsto \Lambda(v, t)w$ is an isomorphism. We have by definition that $\Lambda(v, t)\mathcal{E}(g_{-t}v) \subset \mathcal{E}(v)$ for $t > 0$. Since $v \in \mathcal{R}$, we have $\dim \mathcal{E}(v) \leq \dim \mathcal{E}(g_{-t}v)$. Thus $\Lambda(v, t)\mathcal{E}(g_{-t}v) = \mathcal{E}(v)$ for $t > 0$. Therefore, for $w \in \mathcal{E}(v)$, the Jacobi field $\Lambda(v, t)w$ along c_v makes curvature -1 with $g_t v$ for all $t \in \mathbb{R}$. This immediately implies the last statement. □

Next, let $\widehat{\mathcal{E}}(v) \subset v^\perp$ be the subspace of $v^\perp$ defined as follows: $w \in v^\perp$ belongs to $\widehat{\mathcal{E}}(v)$ if the parallel vector field along $c_v(t)$ determined by w makes curvature -1 with $g_t v$ for all $t \in \mathbb{R}$ (not just $t \geq 0$ as in Definition 2.1). We have that $\widehat{\mathcal{E}}(v) \subset \mathcal{E}(v)$. Indeed if $E(t)$ is a parallel vector field along a geodesic $c(t)$ that makes curvature -1 with $c(t)$, then $e^t E(t)$ is an unstable Jacobi field that again makes curvature -1 with $c(t)$.

Definition 2.5. The *strong hyperbolic rank* $rk^{sh}(v)$ of v is the dimension of $\widehat{\mathcal{E}}(v)$. The *strong hyperbolic rank* $rk^{sh}(M)$ of M is the minimum of the strong hyperbolic ranks $rk^{sh}(v)$ over all $v \in SM$.

We use an argument of Constantine [**Con08**, §5] to prove the following.

PROPOSITION 2.6. *If M is a closed manifold with lower sectional curvature bound -1, $v \in \mathcal{R}$ and $w \in \mathcal{E}(v)$, then the parallel vector field determined by w along $c_v(t)$ makes curvature -1 for all $t \in \mathbb{R}$. Thus for all $v \in \mathcal{R}$, $rk^h(v) = rk^{sh}(v)$ and $\widehat{\mathcal{E}}(v) = \mathcal{E}(v)$.*

Proof. By Lemma 2.4, the unstable Jacobi field $\Lambda(v, t)w$ makes curvature -1 with $c_v(t)$ for all $t \in \mathbb{R}$. Then $\Lambda(v, t)w$ is a stable Jacobi field along $c_{-v}(t)$ still making curvature -1 with $c_{-v}(t)$. Hence the discussion in [**Con08**, §5] shows that $\Lambda(v, t)w = e^t E$ where E is parallel along $c_v(t)$ for all $t \in \mathbb{R}$. Clearly, E makes sectional curvature -1 with $c_v(t)$ as well. □

Note that $\mathcal{E}$ and $\widehat{\mathcal{E}}$ may not be continuous *a priori*. However, $\mathcal{E}$ and $\widehat{\mathcal{E}}$ are semicontinuous in the following sense.

LEMMA 2.7. *If v_n, $v \in SM$ and $v_n \to v$ as $n \to \infty$, then:*
(1) $\lim_{n\to\infty} \mathcal{E}(v_n) \subset \mathcal{E}(v)$ *and* $rk^h(v) \geq \limsup_{n\to\infty} rk^h(v_n)$*; and*
(2) $\lim_{n\to\infty} \widehat{\mathcal{E}}(v_n) \subset \widehat{\mathcal{E}}(v)$ *and* $rh^{sh}(v) \geq \limsup_{n\to\infty} rk^{sh}(v_n)$.
Here $\lim_{n\to\infty} \mathcal{E}(v_n)$ *simply denotes the set of all possible limit points of vectors in* $\mathcal{E}(v_n)$*, and similarly for* $\widehat{\mathcal{E}}$.

Proof. These claims are clear. □

We now define $\widehat{\mathcal{R}} = \{v \mid rk^{sh}(v) = rk^{sh} M\}$.

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LEMMA 2.8. *The sets $\mathcal{R}$ and $\widehat{\mathcal{R}}$ are both open with full measure and hence dense. Moreover, $\widehat{\mathcal{R}}$ is invariant under the geodesic flow.*

Proof. By Lemma 2.7, $\mathcal{R}$ is open. Since the geodesic flow is ergodic on SM with respect to Liouville measure and $\mathcal{R}$ is invariant under backward geodesic flow by Lemma 2.4, $\mathcal{R}$ has full measure. By Lemma 2.7, $\widehat{\mathcal{R}}$ is open and it is flow invariant by definition. Therefore the same argument applies. □

COROLLARY 2.9. *If M is a closed manifold with lower sectional curvature bound -1, then $rk^h(M) = rk^{sh}(M)$ and $\mathcal{R} \subset \widehat{\mathcal{R}}$.*

Proof. By Lemma 2.6, strong and weak rank agree on $\mathcal{R}$ which is an open dense set by Lemma 2.8. By Lemma 2.7, both weak and strong ranks can only go up outside $\mathcal{R}$. □

The next argument is well known and occurs in Constantine's work, for example.

As usual, we let $W^u(v)$ denote the (strong) unstable manifold of v under the geodesic flow, i.e. the vectors $w \in SM$ such that $d(g_t(v), g_t(w)) \to 0$ as $t \to -\infty$. We define the (strong) stable manifold $W^s(v)$ similarly for $t \to \infty$.

LEMMA 2.10. *If $v \in \widehat{\mathcal{R}}$ is backward recurrent under g_t, then $W^u(v) \subset \widehat{\mathcal{R}}$. If $v \in \widehat{\mathcal{R}}$ is forward recurrent under g_t, then $W^s(v) \subset \widehat{\mathcal{R}}$.*

Proof. Let $w \in W^u(v)$, then $g_{-t}w$ approximates $g_{-t}v$ when t large. On the other hand, since $\widehat{\mathcal{R}}$ is open, there is a neighborhood U of v in $\widehat{\mathcal{R}}$. Since v is backward recurrent, $g_{-t}v$ comes back to U and approximates v infinitely often. Thus there is t large that $g_{-t}w \in U \subset \widehat{\mathcal{R}}$. It follows that $w \in \widehat{\mathcal{R}}$ as $\widehat{\mathcal{R}}$ is invariant under the geodesic flow (cf. Lemma 2.8). The argument for the second statement of the lemma is similar. □

3. *Smoothness of hyperbolic rank*

Assume now that M has sectional curvature -1 as an extremal value, that is, either the sectional curvature $K \leq -1$ or $K \geq -1$. We want to prove smoothness of $\widehat{\mathcal{E}}$ on the regular set $\widehat{\mathcal{R}}$. Our arguments below are inspired by Ballmann, Brin and Eberlein's work [**Bal85**] and also [**Wat13**]. First let us recall the following lemma from [**SSS16**, Lemma 2.1].

LEMMA 3.1. *For $v \in S_pM$, the* Jacobi operator *$R_v : v^\perp \to v^\perp$ is defined by $R_v(w) = R(v, w)v$. Then w is an eigenvector of R_v with eigenvalue -1 if and only if $K(v, w) = -1$.*

While we do not use it here, let us mention [**SSS16**, Lemma 2.9] where smoothness of the eigenspace distribution of eigenvalue -1 is proved on a similarly defined regular set. Our situation is different as we characterize hyperbolic rank in terms of parallel transport of a vector, not just the vector. To this end, we define the following quadratic form. Let $E(t)$ and $W(t)$ be parallel fields along the geodesic $c_v(t)$, and set

$$\Omega_v^T(E(t), W(t)) = \int_{-T}^{T} \langle -E(t) - R_{g_t v}E(t), -W(t) - R_{g_t v}W(t) \rangle.$$

LEMMA 3.2. *The parallel field $E(t)$ belongs to the kernel of Ω_v^T if and only if $E(t)$ makes curvature -1 with $c_v(t)$ for $t \in [-T, T]$. In consequence, if $S < T$, then* $\ker \Omega_v^T \subset \ker \Omega_v^S$.

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Proof. If $E(t)$ makes curvature -1 with $c_v(t)$ for $t \in [-T, T]$, then $-E(t) - R_{g_t v} E(t) = 0$ by Lemma 3.1, and hence $E(t)$ is in the kernel of Ω_v^T.

Conversely, if $E(t)$ is in the kernel of Ω_v^T, let $W(t) = E(t)$. Since the integrand now is ≥ 0 for all $t \in [-T, T]$, $E(t) - R_{g_t v} E(t) = 0$ and hence $E(t)$ makes curvature -1 with $c_v(t)$, as claimed. □

Hence $\widehat{\mathcal{E}}(v)$ consists of the initial vectors of $\bigcap_T \ker \Omega_v^T$, which is the intersection of the descending set of vector subspaces $\ker \Omega_v^T$ as T increases. Hence there is a smallest number $T(v) < \infty$ such that $\widehat{\mathcal{E}}(v)$ consist of the initial vectors of $\ker \Omega_v^T$ for all $T > T(v)$.

PROPOSITION 3.3. *$\widehat{\mathcal{E}}$ is smooth on $\widehat{\mathcal{R}}$. In particular, $\widehat{\mathcal{E}}$ is smooth on $W^s(v)$ (respectively $W^u(v)$) where $v \in \widehat{\mathcal{R}}$ is forward (respectively backward) recurrent.*

Proof. Let $v \in \widehat{\mathcal{R}}$, and let $v_n \to v$. We may assume that $v_n \in \widehat{\mathcal{R}}$ since $\widehat{\mathcal{R}}$ is open. Note that $T(v_n) < T(v) + 1$ for all large enough n. Otherwise, we could find $rk^h M + 1$ many orthonormal parallel fields along c_{v_n} which make curvature -1 with $c_{v_n}(t)$ for $-T(v) - 1 < t < T(v) + 1$. Taking limits, we find $rk^h M + 1$ many orthonormal parallel fields along c_v which make curvature -1 with $c_{v_n}(t)$ for $-T(v) - 1 < t < T(v) + 1$. Therefore there exists a neighborhood $U \subset \widehat{\mathcal{R}}$ of v such that $T(u) < T(v) + 1$ for all $u \in U$. Since the quadratic forms $\Omega_w^{T(v)+1}$ are smooth on the neighborhood U of v, we see that the distribution is smooth on $\widehat{\mathcal{R}}$.

The last claim is immediate from smoothness on $\widehat{\mathcal{R}}$ and Lemma 2.10. □

4. *Maximal Lyapunov exponents and hyperbolic rank*

The geodesic flow $g_t : SM \to SM$ preserves the Liouville measure μ on SM, and is ergodic. Hence Lyapunov exponents are defined and constant almost everywhere with respect to μ. Recall that they measure the exponential growth rate of tangent vectors to SM under the derivative of g_t. As is well known, double tangent vectors to M correspond in a one-to-one way with Jacobi fields $J(t)$, essentially since $J(t)$ is uniquely determined by the initial condition $J(0)$, $J'(0)$. Moreover we have

$$(Dg_t)(J(0), J'(0)) = (J(t), J'(t)).$$

Thus we can work with Jacobi fields rather than double tangent vectors whenever convenient. We note that stable (respectively unstable) vectors for g_t correspond to Jacobi fields, which tend to 0 as $t \to \infty$ (respectively as $t \to -\infty$).

If $-1 \leq K \leq 0$, then all Lyapunov exponents of unstable Jacobi fields along the geodesic flow for any invariant measure are between 0 and 1, compare for example [**Bal95**, Ch. IV, Proposition 2.9]. Similarly, if $K \leq -1$, all Lyapunov exponents have absolute value at least 1. We want to understand the extremal case better. We suppose $K \geq -1$ throughout.

LEMMA 4.1. *Let $\widehat{\mathcal{E}}(v)^\perp$ be the orthocomplement (with respect to the Riemannian metric on M) of $\widehat{\mathcal{E}}(v)$ in $v^\perp$. Then $\Lambda(v, t)$ sends $\widehat{\mathcal{E}}(v)^\perp$ to $\widehat{\mathcal{E}}(g_t v)^\perp$.*

Proof. Indeed, let $E_1(t), \ldots, E_{n-1}(t)$ be a choice of parallel orthonormal fields along $g_t v$ and perpendicular to $g_t v$ such that $\{E_1(t), \ldots, E_k(t)\}$ forms a basis of $\widehat{\mathcal{E}}(g_t v)$. For any

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$w \in v^{\perp}$, the formula for an unstable Jacobi field becomes

$$\Lambda(v, t)w = \sum_i f_i(t)E_i(t).$$

Setting $a_{ij} = \langle R(g_t v, E_i(t))g_t v, E_j(t)\rangle$, the Jacobi equation is equivalent to

$$f_j''(t) + \sum_i a_{ij}(t) f_i(t) = 0.$$

Since $e^t E_i(t)$ is an unstable Jacobi field for $i \leq k$ and the $\{E_i(t)\}$ are orthonormal,

$$\langle R(g_t v, E_i(t))g_t v, E_j(t)\rangle = -\langle E_i(t), E_j(t)\rangle = -\delta_i^j$$

for all $i \leq k$ and any $j \leq n-1$. By the symmetries of the curvature tensor, $a_{ij} = a_{ji}$ and so we also have $a_{ji}(t) = a_{ij}(t) = -\delta_i^j$ for either $i \leq k$ or $j \leq k$. It follows that for all $t \in \mathbb{R}$ and all $i \leq k$

$$0 = f_i''(t) + \sum_j a_{ij}(t) f_j(t) = f_i''(t) - f_i(t).$$

Since $\Lambda(v, t)w$ is unstable, $\lim_{t\to-\infty} f_i(t) = 0$ for all i. If $w \in \widehat{\mathcal{E}}(v)^{\perp}$, then $f_i(0) = 0$ for all $i \leq k$. These two conditions together imply $f_i(t) = 0$ for all $t \in \mathbb{R}$ and $i \leq k$. Hence, $\Lambda(v, t)$ leaves $\widehat{\mathcal{E}}^{\perp}$ invariant. □

LEMMA 4.2. ([**Bal95**, Ch. IV, Proposition 2.9], [**Con03**, Lemma 2.3]) *$\|\Lambda(v, t)w\| \leq \|w\|e^t$ for all $t \geq 0$. The equality holds at a time $T \in \mathbb{R}$ if and only if the sectional curvature of the plane spanned by $\Lambda(v, t)w$ and $g_t v$ is -1 for all $0 \leq t \leq T$, if and only if $\Lambda(v, t)w = \|w\|e^t W(t)$ where $W(t)$ is parallel for all $0 \leq t \leq T$.*

Proof. By the Rauch Comparison theorem, $\|\Lambda(v, t)w\| \leq \|w\|e^t$ and $\|\Lambda'(v, t)w\| \leq \|\Lambda(v, t)w\|$ for all $t \geq 0$ (cf. [**Bal95**, Ch. IV, Proposition 2.9], which states a similar result for stable Jacobi fields). If equality holds at time $T > 0$ then $\|\Lambda(v, t)w\| = e^t\|w\|$ for all $0 \leq t \leq T$. Indeed, should $\|\Lambda(v, t_0)w\| < e^{t_0}\|w\|$ for some $0 < t_0 < T$, then we get a contradiction since

$$\begin{aligned}\|\Lambda(v, T)w\| &\leq \|\Lambda(\Lambda(v, t_0)w, T - t_0)\| \\ &\leq e^{T-t_0}\|\Lambda(v, t_0)w\| < e^{T-t_0}e^{t_0}\|w\| = e^T\|w\|.\end{aligned}$$

Therefore the vector field $W(t)$ for which $\Lambda(v, t)w = \|w\|e^t W(t)$ is a field of norm one. Hence, $\langle W(t), W'(t)\rangle = 0$ and we have

$$\begin{aligned}\|w\|e^t(1 + \|W'(t)\|^2)^{1/2} &= \|w\|\|(e^t W(t) + e^t W'(t))\| \\ &= \|\Lambda'(v, t)w\| \leq \|\Lambda(v, t)w\| = e^t\|w\|,\end{aligned}$$

by the estimate above on the derivative of the unstable Jacobi field. We see that $W' = 0$, i.e. W is parallel as desired. It now follows from the Jacobi equation that the sectional curvature between $W(t)$ and the geodesic is -1. □

By covering the unit tangent bundle with a countable base of open sets that generate the topology, and applying the ergodic theorem to the Liouville measure, there is a full

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measure set of unit tangent vectors that comes back to all its neighborhoods with positive frequency.

The argument in the next lemma is similar to that of [**BBE85**, Lemma 3.4] and [**Ham91a**, Proposition 1], but for the setting of a lower curvature bound of -1. We let $\mathcal{E}(v)^\perp$ denote the orthogonal complement of $\mathcal{E}(v)$ in $v^\perp$.

LEMMA 4.3. *Suppose $v \in \mathcal{R}$ returns with positive frequency to all its neighborhoods under g_t. Then, for $w \in \mathcal{E}(v)^\perp = \widehat{\mathcal{E}}(v)^\perp$, the unstable Jacobi field $\Lambda(v, t)w$ has Lyapunov exponent strictly smaller than 1. We have a similar statement for stable Jacobi fields and Lyapunov exponent -1.*

Proof. Let $T > 0$ be such that the dimension of parallel vector fields making curvature -1 with $g_t v$ for all $0 \le t \le T$ is $k = rk^h(M)$, i.e. $k = \dim \mathcal{E}(v)$ since $v \in \mathcal{R}$ (Corollary 2.9).

Pick $w_0 \in \mathcal{E}(v)^\perp$ that minimizes $\{\|w\|/\|\Lambda(v, T)w\| : w \in \mathcal{E}(v)^\perp\}$. By Lemma 4.2, we have that $\|\Lambda(v, T)w_0\| \le e^T \|w_0\|$. Suppose that we have the equality $\|\Lambda(v, T)w_0\| = e^T \|w_0\|$. Then, by Lemma 4.2, the parallel field of w_0 along $g_t v$ makes curvature -1 with $g_t v$ for all $0 \le t \le T$. Since $w_0 \in \mathcal{E}(v)^\perp$, the space of parallel fields making curvature -1 with $g_t v$ for all $0 \le t \le T$ has dimension at least $\dim \mathcal{E}(v) + 1 = k + 1$, which is a contradiction. Therefore $\|\Lambda(v, T)w_0\| < e^T \|w_0\|$.

Let $\epsilon > 0$ be such that $\|\Lambda(v, T)w_0\| = (1 - 2\epsilon)e^T \|w_0\|$. By continuity, we can choose a neighborhood $U \subset \mathcal{R}$ of v such that for all $u \in U$ and $w \in \mathcal{E}(u)^\perp$, we have the estimate $\|\Lambda(v, T)w\| \le (1 - \epsilon)e^T \|w\|$.

Since the $g_t v$ visit U with a positive frequency, there are $\delta > 0$ and $T_0 > 0$ such that for all $S > T_0$

$$|\{t \in [0, S] : g_t v \in U\}| > \delta S.$$

Now suppose that $w \in v^\perp \cap \mathcal{E}(v)^\perp$. Since $\mathcal{E}(v)^\perp = \widehat{\mathcal{E}}(v)^\perp$ on $\mathcal{R}$, we note by Lemma 4.1 that $\Lambda(v, S)w \in \mathcal{E}(g_S v)^\perp$ for all $S > 0$. Thus, $\|\Lambda(v, S)w\| \le e^S (1 - \epsilon)^{[\delta S/T]} \|w\|$. It follows that $\Lambda(v, t)w$ has Lyapunov exponent strictly smaller than 1. □

We remark that the argument in the last proof only provides the information that the unstable Jacobi fields come from parallel fields in forward time. This forces us to introduce both sets $\mathcal{R}$ and $\widehat{\mathcal{R}}$, and to use the equality of $\mathcal{E}$ and $\widehat{\mathcal{E}}$ on $\mathcal{R}$.

Recall that there is a contact form θ on SM invariant under the geodesic flow. Its exterior derivative $\omega = d\theta$ is a symplectic form on stable plus unstable distribution $E^s + E^u$. Observe that θ, and hence ω, are invariant under the geodesic flow, and thus every Oseledets space E_λ with Lyapunov exponent λ is ω-orthogonal to all $E_{\lambda'}$ unless $\lambda' = -\lambda$. Since ω is non-degenerate, ω restricted to $E_\lambda \times E_{-\lambda}$ is also non-degenerate for each λ. Note that $\widehat{\mathcal{E}}$ gives rise to unstable Jacobi fields with Lyapunov exponent 1.

This immediately gives the following.

COROLLARY 4.4. *Assume that the manifold has strict $\frac{1}{4}$-pinched sectional curvature. The maximal Lyapunov spaces E_1 can be extended to be a C^1 distribution E_1^u on the regular set $\widehat{\mathcal{R}}$. The orthogonal complement $(E_1^u)^\perp \cap E^s$ with respect to ω is defined and C^1 on the same set $\widehat{\mathcal{R}}$ and equals $\bigoplus_{-1<\lambda<0} E_\lambda$ almost everywhere. The analogous statements hold for E_{-1} (yielding E_1^s) and $E_{-1}^\perp \cap E^u = \bigoplus_{0<\lambda<1} E_\lambda$ almost everywhere as well.*

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We will call the spaces $E^s_{<1} := (E^u_1)^\perp \cap E^s$ and $E^u_{<1} := (E^s_1)^\perp \cap E^u$ the *extended slow stable and unstable* subspaces, respectively. Similarly we call E^s_1 and E^u_1 the *extended fast stable and unstable* subspaces.

Proof. On $\widehat{\mathcal{R}}$, $\widehat{\mathcal{E}}$ is defined and smooth. On $\mathcal{R}$, $\mathcal{E}$ is also defined and smooth. Moreover the distribution $\mathcal{E}$ agrees with $\widehat{\mathcal{E}}$ on $\mathcal{R}$.

We set $\Omega = \{v \in \mathcal{R},\ v$ is forward recurrent under $g_t v$ with positive frequency$\}$. By Lemma 4.3, E_1 agrees with the lift to unstable Jacobi fields of $\mathcal{E}$ on TSM, i.e. $w \in \mathcal{E}(v)$ is identified with the Jacobi field $\Lambda(v, t)w$. The set Ω has full measure. Hence E_1 extends smoothly on $\widehat{\mathcal{R}}$ to a distribution E^u_1.

Now take the orthogonal complement (with respect to the form ω) to E^u_1, $(E^u_1)^\perp \cap E^s$, on $\widehat{\mathcal{R}}$ in the stable distribution. Since the curvature is strictly $\frac{1}{4}$-pinched, E^s is C^1 [**HP75**]. As E^u_1 is even C^∞, $(E^u_1)^\perp \cap E^s$ is C^1. Since ω pairs Lyapunov spaces where defined almost everywhere on $\widehat{\mathcal{R}}$, $\bigoplus_{-1<\lambda<0} E_\lambda \subset (E^u_1)^\perp \cap E^s$. Since ω is non-degenerate, the dimension of the latter subspace is exactly $n - 1 - rk^h(M)$ everywhere on $\widehat{\mathcal{R}}$, and hence they agree.

A similar argument applies to E_{-1} and its perpendicular complement with respect to ω in the unstable subspace, where now we use vectors backward recurrent with positive frequency. □

5. *Slow stable spaces and integrability in strict* 1/4*-pinching*

Throughout this section, we will assume that the sectional curvature is strictly 1/4-pinched.

In the tangent bundle $T\,SM$ of the unit tangent bundle, consider the subset $T\widehat{\mathcal{R}} \subset T\,SM$, which is the union of tangent fibers of SM at points in $\widehat{\mathcal{R}}$. On $T\widehat{\mathcal{R}}$, there is a C^1 decomposition $E^s_1 + E^s_{<1} + E^0 + E^u_1 + E^u_{<1}$, where $E^{s/u}_1$, $E^{s/u}_{<1}$ denote the extended stable/unstable fast and slow Lyapunov exponent distributions, respectively, as defined in the last section.

We will define a special connection for which this decomposition is parallel, and use that to argue integrability of the slow unstable direction. Such connections were introduced by Kanai to study geodesic flows with smooth stable and unstable foliations in [**Kan88**]. Our particular construction is motivated by that of Benoist, Foulon and Labourie in [**BFL90**] where they classify contact Anosov flows with smooth Oseledets' decomposition. We refer to [**GHL04**, Definition 2.49 and Proposition 2.58] for the basic facts on affine connections we will need.

We recall the formula for the contact 1-form θ: $\theta_{(x,v)}(\xi) = \langle v, \xi^0\rangle$, where $(x, v) \in SM$ and $\xi \in T_{(x,v)}SM$, where $\xi^0 = d\ pt(\xi)$, and $\langle -, -\rangle$ denotes the Riemannian metric. Then the 2-form $d\theta$ has the property that $d\theta(E^u, E^u) = d\theta(E^s, E^s) = 0$ since $d\theta$ is invariant under the geodesic flow and shrinks E^s/E^u in forward/backward time respectively.

We let $\mathcal{X}$ denote the geodesic spray, i.e., the generator of the geodesic flow which is the vector field belonging to E_0 obtained by lifting unit tangent vectors of M to TSM horizontally. Note that $\theta(\mathcal{X}) = 1$ and $d\theta(\mathcal{X}, -) = 0$.

PROPOSITION 5.1. *There exists a unique connection ∇ on $T\widehat{\mathcal{R}}$ such that the following holds.*

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(1) *$\nabla\theta = 0$, $\nabla d\theta = 0$, and $\nabla E_0 \subset E_0$, $\nabla E_i^{s/u} \subset E_i^{s/u}$ for $i \in \{1, <1\}$.*

(2) *For any sections Z_1^s, $Z_{<1}^s$, Z_1^u, $Z_{<1}^u$ of E_1^s, $E_{<1}^s$, E_1^u, $E_{<1}^u$ respectively, we have for $i, j \in \{1, <1\}$*

$$\nabla_{Z_i^s} Z_j^u = p_{E_j^u}([Z_i^s, Z_j^u]),$$

$$\nabla_{Z_i^u} Z_j^s = p_{E_j^s}([Z_i^u, Z_j^s]),$$

$$\nabla_{\mathcal{X}} Z_i^{s/u} = [\mathcal{X}, Z_i^{s/u}],$$

where the $p_{E_j^{s/u}}$ are the projections to the $E_j^{s/u}$ subspaces.

In addition, ∇ is invariant under the geodesic flow g_t.

Proof. Suppose first that ∇ is a connection that satisfies the properties above. We note that $\nabla d\theta = 0$ is equivalent with $W d\theta(Y, Z) = d\theta(\nabla_W Y, Z) + d\theta(Y, \nabla_W Z)$ for any vector fields W, Y, Z. And $\nabla\theta = 0$ is equivalent with $\theta(\nabla_Y Z) = Y\theta(Z)$ for any vector fields Y, Z. Thus $\theta(\nabla_Y \mathcal{X}) = 0$ and $d\theta(\nabla_Y \mathcal{X}, Z) = 0$ for any vector fields Y, Z. It follows that $\nabla\mathcal{X} = 0$. Furthermore, given a C^1 function $f : \widehat{\mathcal{R}} \to \mathbb{R}$, we have $\nabla_Y(f\mathcal{X}) = Y(f)\mathcal{X}$ by the Leibniz rule.

Moreover, $\nabla_{Z_i^u} Z_j^u$ is uniquely determined by the condition $\nabla E_i^u \subset E_i^u$ and the equality

$$Z_i^u d\theta(Z_j^u, Z^s) = d\theta(\nabla_{Z_i^u} Z_j^u, Z^s) + d\theta(Z_j^u, \nabla_{Z_i^u} Z^s),$$

for arbitrary section Z^s of E^s and $i, j \in \{1, <1\}$. Similarly $\nabla_{Z_i^s} Z_j^s$ is uniquely determined. By linearity, $\nabla_Y Z$ is uniquely determined for all vector fields Y, Z. Conversely, we can use the above equations to define a connection. It is then easy to check that ∇ satisfies the properties of a connection on $\widehat{\mathcal{R}}$ (compare, for example [**GHL04**, Definition 2.49]). That ∇ is invariant under the geodesic flow g_t follows from the construction. Indeed, the slow and fast stable and unstable spaces are invariant under g_t, $(g_t)_*([Y, Z]) = [(g_t)_*Y, (g_t)_*Z]$ and $\mathcal{X}$ is invariant under g_t by definition. □

The next lemma is basically well known (compare, for example [**BFL90**, Lemma 2.5]). Since our connection is only defined on a dense open set and not necessarily bounded, we merely outline the proof. Since Liouville measure is ergodic for the geodesic flow g_t on SM, the Lyapunov exponents γ_i are defined and constant on a g_t-invariant full measure set Σ in $\widehat{\mathcal{R}}$. We can assume in addition that all $v \in \Sigma$ are forward and backward recurrent for g_t, and that the Oseledets decomposition $T_v\mathcal{R} = \bigoplus E_{\gamma_i}$ into Lyapunov subspaces E_{γ_i} is defined on Σ. Thus if $Z_i \in E_{\gamma_i}$, the forward and backward Lyapunov exponents are defined and equal to γ_i.

LEMMA 5.2. *Let $v \in \Sigma$. If K is a geodesic flow-invariant tensor and $Z_1, \ldots, Z_k$ are vectors in $T_v\widehat{\mathcal{R}}$ with $Z_i \in E_{\gamma_i}$, then $K(Z_1, \ldots, Z_k)$ is either zero or has Lyapunov exponent $\gamma_1 + \cdots + \gamma_k$.*

Proof. There is a neighborhood U of v and $C > 0$ such that $\|K(Y_1, \ldots, Y_k)\| \leq C\|Y_1\| \cdots \|Y_k\|$, for any vectors $Y_1, \ldots, Y_k$ with footpoints in the neighborhood U. Suppose that $K(Z_1, \ldots, Z_k) \neq 0$. If $g_t(v) \in U$ for some $t > 0$, then

$$\|D_v g_t K(Z_1, \ldots, Z_k)\| = \|K(D_v g_t Z_1, \ldots, D_v g_t Z_k)\| \leq C\|D_v g_t Z_1\| \cdots \|D_v g_t Z_k\|.$$

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Thus,

$$\frac{1}{t}\log(\|D_v g_t K(Z_1,\ldots,Z_k)\|) \le \frac{1}{t}(\log(C)+\log(\|D_v g_t Z_1\|)+\cdots+\log(\|D_v g_t Z_k\|)).$$

Since v is forward recurrent, there will be a sequence of times $t\to\infty$ with $g_t(v)\in U$. Thus the forward Lyapunov exponent of $K(Z_1,\ldots,Z_k)$ is at most $\gamma_1+\cdots+\gamma_k$. Hence $K(Z_1,\ldots,Z_k)$ cannot have non-zero components in E_γ if $\gamma>\gamma_1+\cdots+\gamma_k$.

Similarly, if $g_s(v)\in U$ for some $s<0$, then

$$\frac{1}{s}\log(\|D_v g_s K(Z_1,\ldots,Z_k)\|) \ge \frac{1}{s}(\log(C)+\log(\|D_v g_s Z_1\|)+\cdots+\log(\|D_v g_s Z_k\|)).$$

Since v is backward recurrent, arguing as above, the backward Lyapunov exponent of $K(Z_1,\ldots,Z_k)$ is at least $\gamma_1+\cdots+\gamma_k$. Hence $K(Z_1,\ldots,Z_k)$ cannot have non-zero components in E_γ if $\gamma<\gamma_1+\cdots+\gamma_k$. □

Recall that the connection ∇ is only C^1, and only defined on $\widehat{\mathcal{R}}$. This means that the torsion tensor is only a C^0-tensor, and the curvature tensor is not defined. However, slow and fast stable and unstable distributions are smooth on stable and unstable manifolds in $\widehat{\mathcal{R}}$. Hence the restriction of ∇ to stable or unstable manifolds is also smooth by the construction of ∇. In particular, the curvature tensor of ∇ restricted to stable or unstable manifolds is well defined.

COROLLARY 5.3. *The torsion and curvature tensors of* ∇ *restricted to the slow Lyapunov distributions* $E^{s/u}_{<1}$ *and also each stable/unstable space* $E^{s/u}$ *are zero.*

Proof. Since ∇ is geodesic flow invariant, so are the torsion and curvature tensors. In strict $\frac{1}{4}$-pinched manifolds, the ratio of any two Lyapunov exponents lies in $(\frac{1}{2},2)$. Thus this corollary follows at points of Σ, immediately from the previous lemma and the strict $\frac{1}{4}$-pinching condition. Since Σ is of full measure, and therefore dense, the statements hold everywhere on $\widehat{\mathcal{R}}$ by continuity. □

COROLLARY 5.4. *The slow unstable Lyapunov distribution* $E^u_{<1}$ *is integrable.*

Proof. By construction of ∇, the slow unstable Lyapunov distribution $E^u_{<1}$ is invariant under the parallel transport by ∇. Since ∇ is flat, parallel transport is independent of path. Thus we can choose canonical local parallel C^1 vector fields tangent to and spanning the distribution. On the other hand, since the restriction of torsion on unstable leaves is zero we have that the commutators of these vector fields are zero. By the Frobenius theorem for C^1 vector fields, [**Lan95**, Theorem 1.1 Ch. 6], the distribution is integrable. □

As usual, we will consider the $\pi_1(M)$-lifts of the stable and unstable manifolds by the same notation in $S\widetilde{M}$, and we will work in SM or $S\widetilde{M}$ as appropriate without further comment. Given $v\in S\widetilde{M}$, the map $\pi_v: W^u(v)\to\partial\widetilde{M}-\{c_v(-\infty)\}$, defined by $\pi_v(w)=c_w(\infty)$, is a C^1 diffeomorphism [**HP75**]. For $w\in g_tW^s(v)$, for some $t\in\mathbb{R}$, the *stable holonomy* is defined as

$$h_{v,w}=\pi_w^{-1}\circ\pi_v: W^u(v)-\{\pi_v^{-1}(c_w(-\infty))\}\to W^u(w)-\{\pi_w^{-1}(c_v(-\infty))\}.$$

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Note that $h_{v,w}(x)$ is simply the intersection of the weak stable manifold of x with $W^u(w)$. In particular, the stable holonomy maps are C^1. Indeed, the sectional curvatures of M are strictly $\frac{1}{4}$-pinched and hence the weak stable foliation is C^1 [**HP75**]. Moreover, the stable holonomy maps $h_{a,b}$ are C^1 with derivative bounded uniformly in $d_{S\widetilde{M}}(a, b)$ for $b \in \bigcup_t g^t W^s(a)$. This follows from the fact that the unstable foliation is uniformly transversal to the stable foliation, by compactness of SM. In fact, Hasselblatt [**Has94**, Corollary 1.7] showed that the derivative is even Hölder continuous.

We call a distribution *stable holonomy invariant* if it is invariant under (the derivative map of) all holonomies $h_{v,w}$ for all $v \in S\widetilde{M}$ and $w \in W^s(v)$. We will now adapt an argument by Feres and Katok [**FK90**, Lemma 4].

LEMMA 5.5. *The slow unstable spaces $E^u_{<1} \subset T\widehat{\mathcal{R}}$ are stable holonomy invariant.*

Proof. First consider $v \in \Sigma \cap \mathcal{R}$ and $w \in \bigcup_t g^t W^s(v) \cap \Sigma \cap \mathcal{R}$. The distance between $g^t v$ and $g^t w$ remains bounded in forward time. Hence the derivatives of the holonomy maps $h_{g^t v, g^t w}$ are uniformly bounded for all $t \geq 0$. If $u \in E^u_{<1}(v)$, then u has forward Lyapunov exponent $\lambda < 1$ since $v \in \Sigma$. It follows that the image vector $Dh_{v,w}(u)$ also has forward Lyapunov exponent $\lambda < 1$ and hence belongs to $E^u_{<1}(w)$. In particular, $Dh_{v,w} E^u_{<1}(v) \subset E^u_{<1}(w)$. By continuity of $Dh_{v,w}$ and of the extended slow space on $\widehat{\mathcal{R}}$, the same holds for all $v, w \in \widehat{\mathcal{R}}$. □

We follow ideas of Butler [**But15**] to derive the following.

COROLLARY 5.6. *The slow unstable distributions are trivial.*

Proof. By the strict $\frac{1}{4}$-pinching, the boundary $\partial\widetilde{M}$ of the universal cover admits a C^1 structure for which the projection maps from points or horospheres are C^1 [**HP75**]. By Lemma 5.5, the projection of the lifts of the slow unstable distribution is independent of the projection point on the horosphere. Using different horospheres we obtain a well-defined distribution on all of $\partial\widetilde{M}$. Note that this distribution is also invariant under $\pi_1(M)$.

By Corollary 5.4, this distribution is integrable and yields a C^1 foliation $\mathcal{F}$ on the boundary $\partial\widetilde{M}$ which is also $\pi_1(M)$-invariant. Since there is a hyperbolic element of $\pi_1(M)$ which acts with North–South dynamics on $\partial\widetilde{M}$, by Foulon [**Fou94**, Corollaire], the foliation generated by this distribution has to be trivial. □

We are now ready to finish the proof of our main result.

Proof of Theorem 1.1. (Strict $\frac{1}{4}$-pinching case): Since the slow unstable distribution is trivial, all unstable Jacobi fields belong to E^u_1. Hence all sectional curvatures are -1 on $\widehat{\mathcal{R}}$. Since $\widehat{\mathcal{R}}$ is open dense in SM, it follows that all sectional curvatures are -1. □

6. *Non-strictly $\frac{1}{4}$-pinched case*

In this section we extend the proof of the main theorem to the non-strictly $\frac{1}{4}$-pinched curvature case when the stable and unstable foliations are not necessarily C^1. This introduces two new difficulties: the Kanai connection may not be defined and the stable holonomy maps may not be C^1.

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6.1. *Measurable Kanai connections.* First, consider the set $\mathcal{O} \subset SM$ of vectors whose smallest positive Lyapunov exponent is $\frac{1}{2}$. If $\mathcal{O}$ has positive Liouville measure then [**Con03**, Theorem 1.3] implies that M is locally symmetric, and our theorem holds. Hence, we may assume $\mathcal{O}$ has measure 0 and there is a flow-invariant full measure set $\mathcal{P}$ and a $\nu > 0$ such that for all $v \in \mathcal{P}$ the unstable Lyapunov exponents satisfy $\frac{1}{2} + \nu < \chi_i^+(v) \leq 1$.

Note that, unlike in the strict $\frac{1}{4}$-pinched case, we cannot immediately use the vanishing of the torsion of the generalized Kanai connection established in Proposition 5.1. Indeed, the construction of the generalized Kanai connection used the fact that both stable and unstable distributions are C^1 on SM, which we do not *a priori* know in our case.

Instead, we replace the generalized Kanai connection with a similar one assembled from the flow-invariant system of measurable affine connections on unstable manifolds constructed by Melnick in [**Mel17**]. The connections are defined on whole unstable manifolds but they are only defined for unstable manifolds $W^u(v)$ for v in a set of full measure. Moreover, the transversal dependence is only measurable. Mark that we have switched from Melnick's usage of stable manifolds to unstable manifolds.

Following the notation in [**Mel17**], let $\mathcal{E}$ be the smooth tautological bundle over SM whose fiber at v is $W^u(v)$. We consider the cocycle F_v^t which is g^t restricted to $W^u(v)$. The ratio of maximal to minimal positive Lyapunov exponents lies in $[1, 2)$, and hence the integer r appearing in [**Mel17**, Theorem 3.13] is 1. In our notation, this theorem then reads as follows.

LEMMA 6.1. *There is a full measure flow-invariant set $\mathcal{U} \subset SM$ where there is a smooth flow-invariant flat connection ∇ on $TW^u(v)$ for $v \in \mathcal{U}$.*

Now we build a connection on vector fields tangent to the slow unstable distribution $E^u_{<1}$ on $W^u(v)$ for $v \in \mathcal{U}$. We emphasize that we do not assume integrability of the slow unstable distribution. We just construct a connection on sections of the vector bundle given by the slow unstable distribution. More specifically on the slow unstable distribution we have the following.

LEMMA 6.2. *On each unstable leaf $W^u(v)$ for v in a full measure flow-invariant subset $\mathcal{Q} \subset \mathcal{U} \cap \mathcal{P} \cap \widehat{\mathcal{R}}$, there exists a torsion-free and flow-invariant connection,*

$$\nabla^{<1} : TW^u(v) \times \Gamma^1(W^u(v), E^u_{<1}) \to E^u_{<1}$$

on $E^u_{<1}$. (Here $\Gamma^1(W^u(v), E^u_{<1})$ represents C^1 sections.) Moreover the restriction of the connection to $E^u_{<1}$ is torsion free.

Proof. Recall that the distribution $E^u_{<1}$ is smooth on $W^u(v)$ for $v \in \widehat{\mathcal{R}}$.

Given $X \in TW^u(v)$ and $Y \in \Gamma^1(W^u(v), E^u_{<1})$ we define the covariant derivative $\nabla_X^{<1} Y$ to be the vector in $E^u_{<1} \subset W^u(v)$ given by projection of the Melnick connection,

$$\nabla_X^{<1} Y := \mathrm{proj}_{E^u_{<1}} \nabla_X Y.$$

Note that this operator is $\mathbb{R}$-bilinear in X and Y, since projections are linear. For $f \in C^1(W^u(v))$, since scalar functions commute with projection we have

$$\nabla_{fX}^{<1} Y = \mathrm{proj}_{E^u_{<1}} f \nabla_X Y = f \,\mathrm{proj}_{E^u_{<1}} \nabla_X Y = f \nabla_X^{<1} Y$$

$$\nabla_X^{<1} fY = \mathrm{proj}_{E^u_{<1}} X(f)Y + f\nabla_X Y = X(f)Y + f\nabla_X^{<1} Y.$$

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Here we have used that $\text{proj}_{E^u_{<1}} Y = Y$. Hence $\nabla^{<1}$ is $C^1(W^u(v))$-linear in X, and satisfies the derivation property of connections. Observe that $\nabla^{<1}$ can then be extended to a map of sections $\nabla^{<1} : \Gamma^1(W^u(v), E^u_{<1}) \times \Gamma^1(W^u(v), E^u_{<1}) \to \Gamma^1(W^u(v), E^u_{<1})$.

For $v \in \mathcal{U}$, $X, Y \in \Gamma^1(W^u(v), E^u_{<1})$ the torsion tensor $T(X, Y) = \nabla^{<1}_X Y - \nabla^{<1}_Y X - [X, Y]$ is indeed a tensor, due to the derivation property of the connection and bracket where we take the bracket of vector fields in $W^u(v)$.

Next we show that $\nabla^{<1}$ is torsion free. Since $[X, Y]$ and $\nabla^{<1}$ are invariant under Dg_t, so is $T(X, Y)$. Also, since $v \in \mathcal{P}$, the sum of any two Lyapunov exponents lies in $(1, 2]$. By Fubini, and absolute continuity of the W^u foliation, we may choose $\mathcal{Q} \subset \mathcal{U} \cap \mathcal{P} \cap \widehat{\mathcal{R}}$ to be an invariant full measure set where for each $v \in \mathcal{Q}$ almost everywhere, $w \in W^u(v)$ is forward and backward recurrent. By Lusin's theorem, for all $\epsilon > 0$, there is a compact set A of measure $>1 - \epsilon$ such that the torsion tensor T is continuous on A. In particular, there is a constant $C > 0$ such that for all tangent vectors X, Y at $a \in A$, $\|T(X, Y)\| \leq C\|X\|\|Y\|$. Also note that for almost every $x \in A$, $g^t(x) \in A$ for infinitely many $t \to \infty$, by ergodicity of g^t. Then the argument from Lemma 5.2 shows that T vanishes on A. As $\epsilon > 0$ is arbitrary, T vanishes on a set of full measure. In addition, this set is automatically g^t-invariant as desired. □

COROLLARY 6.3. *The slow unstable Lyapunov distribution $E^u_{<1}$ is integrable on every leaf $W^u(v)$ for $v \in \mathcal{Q}$.*

Proof. For $v \in \mathcal{Q}$, and $X, Y \in \Gamma^1(W^u(v), E^u_{<1})$ the vanishing of the torsion tensor implies $T(X, Y) = 0 = \nabla^{<1}_X Y - \nabla^{<1}_Y X - [X, Y]$. However, by definition $\nabla^{<1}_X Y$ and $\nabla^{<1}_Y X$ belong to $E^u_{<1}$, and therefore so does $[X, Y]$. In particular, $E^u_{<1}$ is integrable. □

This corollary gives us well-defined slow unstable foliations on almost every $W^u(v)$. For such v, we denote its leaves by $W^u_{<1}(w)$ for $w \in W^u(v)$. Note that the fast unstable distribution is also integrable with leaves we denote by $W^u_1(w)$. Next, we will show that these foliations are invariant under stable holonomy. This is substantially more difficult in the non-strict $\frac{1}{4}$-pinched case since the unstable holonomy maps *a priori* are not known to be C^1.

6.2. *Stable holonomies are C^1 almost everywhere.* We will now address the second difficulty, namely that the stable holonomy maps are not globally C^1. Essentially we will approximate the stable holonomy by C^1 approximate holonomies, and show that the latter have differentiable limits almost everywhere. This approach is inspired by work of Avila, Santamaria and Viana on cocycle holonomy maps in [**AV10**, **ASV13**]. These are easier to control than foliation holonomies. For us, the cocycle is the unstable derivative cocycle. We will show existence and estimates of such, and relate them to the stable foliation holonomy. The latter is similar to work of Burns and Wilkinson [**BW05**] and Brown [**Bro16**]. Our situation is somewhat more technical as we only have pinching of Lyapunov exponents, not pinching of uniform hyperbolic estimates.

To simplify notation, we use $Dg_{t,v}$ for the derivative of g_t at v restricted to $E^u(v)$. Since M is strictly $\frac{1}{4} + \delta$-pinched for any $\delta > 0$, Corollary 1.7 of [**Has94**] implies the following.

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LEMMA 6.4. *The foliations W^s and W^u are α-Hölder for all $\alpha < 1$.*

Now choose an $\alpha > 1 - \nu/4$. As in Kalinin–Sadovskaya [**KS13**, §2.2], we have local linear identifications $I_{v,w} : E^u(v) \to E^u(w)$, such that $I_{v,v} = \mathrm{Id}$ and $I_{v,w}^{-1} = I_{w,v}$, that vary in an α-Hölder way on a neighborhood of the diagonal in $SM \times SM$. We also have that Dg_t is an α-Hölder cocycle, since it is the restriction of the smooth Dg_t to an α-Hölder bundle. In other words, with respect to these identifications, we have

$$\|Dg_{t,v} - I_{g_t v, g_t w}^{-1} \circ Dg_{t,w} \circ I_{v,w}\| \le C(T_0) d(v, w)^\alpha \tag{6.1}$$

for any $T_0 > 0$ and all $t \le T_0$.

We will now construct local smooth maps approximating the stable holonomy whose derivatives have certain properties. This will allow us to connect the holonomy $h_{v,w}$ to linearized approximations by providing a superior choice of identifications $I_{vw} : E^u(v) \to E^u(w)$.

For $v \in S\widetilde{M}$, let $W^{cs}(v) = \bigcup_{t \in \mathbb{R}} W^s(g^t v)$, the weak unstable leaf of v, and set $W^u(v, \epsilon) = B(v, \epsilon) \cap W^u(v)$.

LEMMA 6.5. *There are constants $\epsilon_0, C > 0$ such that for any $v \in S\widetilde{M}$ and $w \in W^{cs}(v)$ with $d(v, w) < \epsilon_0$, there are C^∞ maps $\mathcal{I}_{v,w} : W^u(v, \epsilon_0) \to W^u(w)$ satisfying:*

(1) *$d(\mathcal{I}_{v,w}(v), w) < Cd(v, w)$;*

(2) *$d(D\mathcal{I}_{v,w}(\xi), \xi) < Cd(v, w)^\alpha$ for all $\xi \in S_v W^u$;*

(3) *$|\|D\mathcal{I}_{v,w}\| - 1| < Cd(v, w)^\alpha$;*

(4) *if $v' \in W^u(v)$ and $w' \in (\bigcup_{t \in \mathbb{R}} g_t W^s(v', \epsilon_0)) \cap W^u(w)$ then $\mathcal{I}_{v,w} = \mathcal{I}_{v',w'}$ on the common part of their domain; and*

(5) *for any $T_0 > 0$ and all $t \le T_0$, we have*

$$d(Dg_{t,v}(\xi), D_{g_t \mathcal{I}_{v,w}(v)} (\mathcal{I}_{g_t v, g_t w})^{-1} \circ Dg_{t, \mathcal{I}_{v,w}(v)} \circ D_v \mathcal{I}_{v,w}(\xi)) \le C(T_0) d(v, w)^\alpha \|\xi\|, \tag{6.2}$$

for every $\xi \in E^u(v)$.

Proof. We consider $S\widetilde{M}$ with the Sasaki metric. By compactness of SM the operator norm of the second fundamental forms at all points of the individual smooth leaves $W^u(v)$ are bounded above by a universal constant. Hence the focal radius of each $W^u(v)$ is uniformly bounded from below by a number i_0 independent of $v \in S\widetilde{M}$. Thus the exponential map, exp, for the Sasaki metric applied to the normal bundle ν to $E^u(v) = T_v W^u$ is injective on all of $B^\perp(W^u(v), t) = \bigcup_{y \in W^u(v)} B^\perp(y, t)$ for all $t < i_0$ and $v \in S\widetilde{M}$. (Here $B^\perp(y, t)$ indicates the ball of radius t in the vector space ν_y.)

By compactness of SM and continuity of E^u, there is a universal $r_0 > 0$ such that the leaves $W^u(w)$ will be uniformly transverse to $\exp_y B^\perp(y, i_0)$ whenever they intersect for some $y \in W^u(v)$ and whenever $d(v, w) < r_0$.

Let $\epsilon_0 = \min\{i_0/2, r_0/2\}$. The images $\exp_y B^\perp(y, i_0/2)$ for $y \in W^u(v)$ foliate a normal neighborhood of $W^u(v)$. We call this the *normal disk foliation*.

When $d(v, w) < \epsilon_0$, define $\mathcal{I}_{v,w}(v')$ to be the intersection of $\exp_{v'} B^\perp(v', \epsilon_0) \cap W^u(w)$ on the maximal domain of $W^u(v)$ where the intersection exists. By construction the intersection point will be unique if it exists. In other words, $\mathcal{I}_{v,w}$ is the holonomy for

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the normal disk foliation and thus we may extend it to every leaf $\exp_y B^\perp(y, \epsilon_0)$ of the normal disk bundle whenever this intersects $W^u(w)$. Property (4) now follows from the construction since the normal disk foliation is defined in a normal neighborhood of $W^u(v)$.

By compactness of M, the derivatives of the normal bundle to $W^u(v)$ are bounded. Hence for any $v \in S\widetilde{M}$ and $v' \in W^u(v, \epsilon_0)$, $t < \epsilon_0$, and $\xi \in T^\perp_{v'} W^u(v)$ we have $d(\exp_{v'}(t\xi), \exp_v(B^\perp(v, \epsilon_0))) < C' d(v, v')$ for some uniform constant C. Let $\beta : [0, d_{W^{cs}(v)}(v, w)] \to W^u(w)$ be the projection of the unit speed $W^{cs}(v)$-geodesic from v to w to $W^u(w)$ under $\mathcal{I}_{v,w}$. We measure

$$d(w, \mathcal{I}_{v,w}(v)) \le \ell(\beta) \le \int_0^{d_{W^{cs}(v)}(v,w)} C' \sin(\angle(t))\, dt \le C' d_{W^{cs}(v)}(v, w),$$

where $\angle(t)$ represents the angle between the $W^{cs}(v)$ geodesic and the $T \exp_y B^\perp(y, \epsilon_0)$ distribution. We note that for small distances, Theorem 4.6 of [**HeiImH**] implies that the induced distance on a horosphere of $\widetilde{M}$ is comparable to the distance in $\widetilde{M}$. It then follows from the definition of the Sasaki metric and the universal bound on the second fundamental form of the horospheres, that there is a uniform constant C'' such that $d_{\bigcup_{t\in\mathbb{R}} g_t W^s(v)}(v, w) \le C'' d(v, w)$. Statement #1 now follows with $C = C'C''$.

For statements (2) and (3), choose a smooth coordinate chart ϕ on a neighborhood $U \subset W^u(v)$ of $v \in W^u(v)$ and on each $y \in U$ use $\exp_y^{-1}$ to obtain a global smooth chart ψ from a neighborhood O of $v \in S\widetilde{M}$ to $\mathbb{R}^n \times \mathbb{R}^{n-1}$ such that $W^u(v)$ maps to $\{0\} \times \mathbb{R}^{n-1}$. Note that for each $v' \in U$, the normal disk $\exp_{v'} B^\perp(v', \epsilon_0)$ is carried into $\mathbb{R}^n \times \{\phi(v')\}$ isometrically at v' since $d_0 \exp_{v'} = \mathrm{Id}$ and, moreover, the orthogonality is preserved. In particular, the map $\mathcal{I}_{v,w}$ becomes the horizontal holonomy onto the image $\psi(W^u(w))$, and $D_v \mathcal{I}_{v,w}$ maps to the projection of $T_0\{0\} \times \mathbb{R}^{n-1} \cong \{0\} \times \mathbb{R}^{n-1}$ onto $T_{\psi(\mathcal{I}_{v,w}(v))} \psi(W^u(w))$. However, since E^u is α-Hölder , $d_{G_{n-1}}(\{0\} \times \mathbb{R}^{n-1}, T_{\psi(\eta_{v,w}(v))} \psi(W^u(w))) < C d(v, w)^\alpha$ with respect to the distance on the Grassmanian of $n-1$ subspaces of $\mathbb{R}^{2n-1}$. Statements (2) and (3) now follow from the fact that this projection limits to the identity as $w \to v$.

For the final statement(5), by (2) we have

$$d(Dg_{t,v}(\xi), Dg_{t,\mathcal{I}_{v,w}(v)} \circ D_v \mathcal{I}_{v,w}(\xi)) \le C(T_0) d(\xi, D_v \mathcal{I}_{v,w}(\xi)) \le C(T_0) C d(v, w)^\alpha \|\xi\|.$$

We also have,

$$\begin{aligned} d(Dg_{t,\mathcal{I}_{v,w}(v)} &\circ D_v \mathcal{I}_{v,w}(\xi), D_{g_t \mathcal{I}_{v,w}(v)}(\mathcal{I}_{g_t v, g_t w})^{-1} \circ Dg_{t,\mathcal{I}_{v,w}(v)} \circ D_v \mathcal{I}_{v,w}(\xi)) \\ &\le C d(g_t v, g_t w)^\alpha \| D_{g_t \mathcal{I}_{v,w}(v)}(\mathcal{I}_{g_t v, g_t w})^{-1} \circ Dg_{t,\mathcal{I}_{v,w}(v)} \circ D_v \mathcal{I}_{v,w}(\xi) \| \\ &\le C d(g_t v, g_t w)^\alpha C'(T_0) \|\xi\| \le C C'(T_0) C(T_0) d(v, w)^\alpha \|\xi\| \end{aligned}$$

where the first inequality follows from property (2) applied to the map $\mathcal{I}_{g_t v, g_t w}$ and vector $D_{g_t \mathcal{I}_{v,w}(v)}(\mathcal{I}_{g_t v, g_t w})^{-1} \circ Dg_{t,\mathcal{I}_{v,w}(v)} \circ D_v \mathcal{I}_{v,w}(\xi)$. The final statement now follows from the triangle inequality. □

Now define the approximate holonomies $h^t_{vw} = g_{-t} \circ \mathcal{I}_{g_t v, g_t w} \circ g_t$. Note that h^t_{vw} is defined in the neighborhood $W^u(v, \epsilon'_0)$ of v in $W^u(v)$ for suitable $\epsilon'_0 > 0$, for all $t \ge 0$. Indeed, $\mathcal{I}_{g_t v, g_t w}$ is well defined on $g_t W^u(v, \epsilon'_0)$ for all $t \ge 0$ by property (4) of Lemma 6.5, and since, for suitable $\epsilon'_0 > 0$, $g_t W^u(v, \epsilon'_0)$ is in the ϵ_0-neighborhood of $W^u(g_t w)$ for all $t \ge 0$.

LEMMA 6.6. *Let $v \in S\widetilde{M}$ and $w \in W^{cs}(v)$ with $d(v, w) < \epsilon_0$. The approximate holonomy maps $h^t_{v,w}$ converge to the foliation holonomy maps $h_{v,w}$ uniformly on compact sets in a neighborhood of v in unstable manifolds.*

Proof. For v' in a neighborhood of v we have,

$$\begin{aligned} d(h^t(v'), h_{v,w}(v')) &= d(g_{-t} \circ \mathcal{I}_{g_t v, g_t w} \circ g_t(v'), g_{-t} \circ h_{g_t v, g_t w} \circ g_t(v')) \\ &< e^{-t/2} d(\mathcal{I}_{g_t v, g_t w} \circ g_t(v'), h_{g_t v, g_t w} \circ g_t(v')) < Ce^{-t/2} d(g_t v', h_{g_t v, g_t w}(g_t v')) \\ &= Ce^{-t/2} d(g_t v', g_t h_{v,w}(v')) < Ce^{-t/2}(e^{-t/2} d(v', h_{v,w}(v')) + C'). \end{aligned}$$

The second inequality follows from the first property of the approximation $\mathcal{I}$ in Lemma 6.5. The last inequality holds because v' and $h_{v,w}(v')$ belong to the same weak stable leaf and C' depends on the compact set in $W^u(v) - \{\pi_v^{-1}(c_w(-\infty))\}$ containing v'. □

Since $\mathcal{I}_{vw}$ is $C^{1+\alpha}$, $h^t_{v,w}$ is differentiable. In what follows we will redefine the $I_{v,w}$ used above from [**KS13**] to be $I_{vw} := D_v \mathcal{I}_{vw}$ instead. It follows that $D_{v'} h^t_{v,w} = Dg^{-1}_{t, g_{-t}\mathcal{I}_{g_t v', g_t w'}(g_t v')} \circ I_{g_t v', g_t w'} \circ Dg_{t,v'}$, where v is in a neighborhood of v and $w' = h_{v,w}(v')$. We note that $I_{v,w}$ does not map $E^u(v)$ into $E^u(w)$ as in [**KS13**]. Nevertheless, by the same argument as in [**KS13**, Proposition 4.2], $D_{v'} h^t_{v,w}$ converges as $t \to \infty$ when $v \in \mathcal{Q}$, w belongs to a neighborhood of v in $W^s(v)$, and $v' \in W^u(v)$ has Lyapunov exponents greater than $\frac{1}{2}$.

Since $Dh^t_{v,w}$ may not converge everywhere on $W^u(v) - \{\pi_v^{-1}(c_w(-\infty))\}$, $h_{v,w}$ may not be C^1. We resolve this issue by composing the map with the projection onto the fast direction. To be more precise, we choose a common foliation chart for the slow unstable foliation and fast unstable foliation in $W^u(v)$. This is a chart for whole $W^u(v)$ so that slow unstable leaves are horizontal and fast unstable leaves are vertical. We also choose a similar chart for $W^u(w)$. Let $\mathcal{H}$ be the plaque of v. Similarly, let $\mathcal{V}$ be the plaque of the fast unstable leaf containing w. Assume first that almost every vector in $\mathcal{H}$ is forward recurrent. Let $f : \mathcal{H} \to \mathcal{V}$ be defined by choosing $f(\eta) \in \mathcal{V}$ to be the intersection of the slow unstable leaf containing $h_{v,w}(\eta)$ and $\mathcal{V}$. We want to show that f is constant by showing that f is differentiable in the distribution sense and has a zero derivative.

Let $p_{W^u_1(w)}$ be the projection of points in $W^u(w)$ along slow unstable leaves onto the fast unstable leaf passing through w. We note that $f = p_{W^u_1(w)} \circ h_{v,w}|_{W^u_{<1}(v)}$. By Lemma 6.6, we have that $f_t = p_{W^u_1(w)} \circ h^t_{v,w}|_{W^u_{<1}(v)}$ converges uniformly on compact sets to f.

Let $v, w \in \widehat{\mathcal{R}}$ be backward recurrent under g_t. It follows that $W^u(v) \subset \widehat{\mathcal{R}}$ and $W^u(w) \subset \widehat{\mathcal{R}}$. Recall that for any η in the weak unstable manifold of w, the projection to M of every vector in $E^u_1(\eta)$ has parallel translate making curvature -1 with $g_t\eta$ for all time $t \in \mathbb{R}$. Denote by $\mathrm{proj}_{E^u_1(w)}$ the projection from $E^u(w)$ onto $E^u_1(w)$ along $E^u_{<1}(w)$. We note that, since $E^u_1(w) \perp E^u_{<1}(w)$ by Lemma 4.3, this is an orthogonal projection. We define $H^{t,1}_{v'w'} = Dg^{-1}_{t, g_{-t}\mathcal{I}_{g_t v', g_t w'}(g_t v')} \circ \mathrm{proj}_{E^u_1(\mathcal{I}_{g_t v', g_t w'}(g_t v'))} \circ I_{g_t v', g_t w'} \circ Dg_{t,v'}$ (where the superscript "1" indicates the fast subspace E^u_1 as before). In the chosen foliation chart for $W^u(w)$ we identify unstable $E^u_1(w')$ and $W^u_1(f(v'))$ along the horizontal leaf containing $h_{v,w}(v')$. Then we see that $H^{t,1}_{v'w'} = D(p_{W^u_1(w)} \circ h^t_{v,w}|_{W^u_{<1}(v)})|_{v'}$.

We have the following.

LEMMA 6.7. *For every v, $w \in \mathcal{Q}$ with $w \in W^s(v)$, we have:*

(1) $H^{t,1}_{v,w}$ *converges to a limit, denoted $H^1_{v,w}$.*

(2) *If $\xi \in E^u(v)$ with forward Lyapunov exponent $\chi(v, \xi) < 1$, then $H^1_{v,w}(\xi) = 0$.*

Proof. For almost every $v \in SM$, we let $T(v) = \inf\{s > 0 : (1/t)\log\|Dg_{t,v}\zeta\| > 1/2 + \nu/2$ for all $t > s$ and for all $\zeta \in E^u(v)\}$.

For (1), consider $t_1 > T(v)$ such that the identifications $I_{g_{t_1}v, g_{t_1}w}$ are defined and have Hölder dependence. Then for all such t_1 sufficiently large and $t < T_0$, we claim $\|H^{t_1+t,1}_{v,w} - H^{t_1,1}_{v,w}\|$ is exponentially small in terms of t_1.

We follow the mode of proof of [**KS13**, Proposition 4.2]. We denote $u = \mathcal{I}_{v,w}(v)$, $u_t = \mathcal{I}_{g_t v, g_t w}(g_t v)$, and let $\xi \in E^u(v)$. For t_1 large and $t < T_0$ consider

$$\begin{aligned} &d(H^{t_1+t,1}_{v,w}(\xi), H^{t_1,1}_{v,w}(\xi)) \\ &\quad = d((Dg_{t_1, g_{-t-t_1}u_{t+t_1}})^{-1} \circ (Dg_{t, g_{-t}u_{t+t_1}})^{-1} \circ \mathrm{proj}_{E^u_1(u_{t_1+t})} \circ I_{g_{t_1+t}v, g_{t_1+t}w} \\ &\qquad \circ Dg_{t, g_{t_1}v} \circ Dg_{t_1,v}(\xi), (Dg_{t_1, g_{-t_1}u_{t_1}})^{-1} \circ \mathrm{proj}_{E^u_1(u_{t_1})} \circ I_{g_{t_1}v, g_{t_1}w} \circ Dg_{t_1,v}(\xi)). \end{aligned}$$

Since

$$(Dg_{t, g_{-t}u_{t+t_1}})^{-1} \circ \mathrm{proj}_{E^u_1(u_{t_1+t})} \circ I_{g_{t_1+t}v, g_{t_1+t}w} \circ Dg_{t, g_{t_1}v} \circ Dg_{t_1,v}(\xi) \in E^u_1(g_{-t}u_{t_1+t}),$$

and

$$\mathrm{proj}_{E^u_1(u_{t_1})} \circ I_{g_{t_1}v, g_{t_1}w} \circ Dg_{t_1,v}(\xi) \in E^u_1(u_{t_1}),$$

we have that

$$\begin{aligned} &\frac{e^{t_1}}{C} d(H^{t_1+t,1}_{v,w}(\xi), H^{t_1,1}_{v,w}(\xi)) \\ &\quad < d((Dg_{t, g_{-t}u_{t+t_1}})^{-1} \circ \mathrm{proj}_{E^u_1(u_{t_1+t})} \circ I_{g_{t_1+t}v, g_{t_1+t}w} \circ Dg_{t, g_{t_1}v} \circ Dg_{t_1,v}(\xi), \\ &\qquad \mathrm{proj}_{E^u_1(u_{t_1})} \circ I_{g_{t_1}v, g_{t_1}w} \circ Dg_{t_1,v}(\xi)). \end{aligned}$$

Since the projection to the fast unstable distribution commutes with geodesic flow, we observe that

$$\begin{aligned} \mathrm{proj}_{E^u_1(u_{t_1}t)} \circ I_{g_{t_1}v, g_{t_1}w} \circ Dg_{t_1,v} &= (Dg_{t, g_{-t}u_{t+t_1}})^{-1} \circ \mathrm{proj}_{E^u_1(u_{t+t_1})} \circ I_{g_{t_1+t}v, g_{t_1+t}w} \\ &\quad \circ I^{-1}_{g_{t_1+t}v, g_{t_1+t}w} \circ Dg_{t, g_{-t}u_{t+t_1}} \circ I_{g_{t_1}v, g_{t_1}w} \circ Dg_{t_1,v}. \end{aligned}$$

Therefore by (6.2) applied to $g_{t_1}v$ and $g_{t_1}w$ we have

$$\begin{aligned} &d((Dg_{t, g_{-t}u_{t+t_1}})^{-1} \circ \mathrm{proj}_{E^u_1(u_{t_1+t})} \circ I_{g_{t_1+t}v, g_{t_1+t}w} \circ Dg_{t, g_{t_1}v} \circ Dg_{t_1,v}(\xi), \\ &\mathrm{proj}_{E^u_1(u_{t_1})} \circ I_{g_{t_1}v, g_{t_1}w} \circ Dg_{t_1,v}(\xi)) \\ &\quad \le C(T_0)Cd(Dg_{t, g_{t_1}v} \circ Dg_{t_1,v}(\xi), I^{-1}_{g_{t_1+t}v, g_{t_1+t}w} \circ Dg_{t, g_{-t}u_{t+t_1}} \circ I_{g_{t_1}v, g_{t_1}w} \circ Dg_{t_1,v}(\xi)) \\ &\quad \le C(T_0)CC(T_0)d(g_{t_1}v, g_{t_1}w)^{\alpha}\|Dg_{t_1,v}(\xi)\| \le C(T_0)d(w, v)^{\alpha}e^{-(1/2)\alpha t_1}e^{t_1}\|\xi\|. \end{aligned}$$

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The last inequality holds because of the curvature condition and because $w \in W^s(v)$. Here we have absorbed $C_1(T_0)$ into the generic constant $C(T_0)$. Combining inequalities, we get the estimate:

$$\begin{aligned} d(H^{t_1+t,1}_{v,w}(\xi), H^{t_1,1}_{v,w}(\xi)) &\leq C(T_0)e^{-t_1} d(w, v)^\alpha e^{-(1/2)\alpha t_1} e^{t_1} \|\xi\| \\ &= C(T_0) d(w, v)^\alpha e^{-(1/2)\alpha t_1} \|\xi\|. \end{aligned}$$

This proves the convergence claim of property (1).

Next, for (2), assuming $\chi(v, \xi) = 1 - \delta$ for some $\delta > 0$, for t large we have

$$\begin{aligned} \|H^{t,1}_{v,w}(\xi)\| &= e^{-t} \cdot \| \mathrm{proj}_{E^u_1(u_t)} \circ I_{g_t v, g_t w} \circ Dg_{t,v}(\xi)\| \\ &\leq e^{-t} e^{(1-\delta/2)t} \cdot \| \mathrm{proj}_{E^u_1(u_t)} \circ I_{g_t v, g_t w}\| \cdot \|\xi\| \leq C e^{-\delta t/2} \|\xi\|, \end{aligned}$$

where C is a constant that bounds norms of identifications between close enough points. It follows that $H^1_{v,w}(\xi) = 0$. □

COROLLARY 6.8. *f has a zero derivative in the distribution sense. It follows that f is constant. In other words, the holonomy $h_{v,w}$ maps the entire leaf $W^u_{<1}(v)$ to a slow unstable leaf.*

Proof. Recall that f_t converges uniformly on compact sets in $W^u(v) - \{\pi_v^{-1}(c_w(-\infty))\}$. For $v' \in W^u(v) - \{\pi_v^{-1}(c_w(-\infty))\}$, we let $w' \in W^s(v')$ so that $h_{v,w}(v') = g_s(w')$ for some $s \in \mathbb{R}$. Since $Df_t = Dg_s \circ H^{t,1}_{v',w'}$, they converge by Lemma 6.7. Moreover, $\|H^{t,1}_{v',w'}\| < e^{-t} C d(g_t v', g_t w')^\alpha e^t < C$ for t large. It follows that Df_t is bounded on compact sets. Hence, by the dominated convergence theorem, f is differentiable in the sense of distribution, and for almost every $v' \in W^u_{<1}(v)$ we have $Df(v') = Dg_s \circ H^1_{v',w'}\big|_{E^u_{<1}(v')}$, where $w' \in W^s(v') \cap \bigcup_{t\in\mathbb{R}} g_t W^u(v)$ and $s \in \mathbb{R}$ such that $g_s w' = h_{v,w}(v')$. Furthermore, by item (2) of Lemma 6.7, this distribution derivative is zero. It follows that all higher order distribution derivatives of f are also zero. By Sobolev embedding, and since f is continuous, we have that f is a constant everywhere. Thus the holonomy $h_{v,w}$ maps the entire slow unstable leaf containing v to a slow unstable leaf in $W^u(w)$. □

LEMMA 6.9. *Let $v, w \in \mathcal{Q}$ be backward recurrent under g_t. Assume further that v and w are chosen such that almost every vector of $W^u(v)$ and $W^u(w)$ are in $\mathcal{R}$ and forward recurrent under g_t. Then the holonomy $h_{v,w}$ maps slow unstable leaves in $W^u(v)$ to slow unstable leaves in $W^u(w)$.*

Proof. The image of the C^1 slow unstable foliation in $W^u(v)$ under $h_{v,w}$ is a C^0 foliation in $W^u(w)$.

Now consider the collection of slow unstable leaves with almost every vector being recurrent and with Lyapunov exponents larger than $\frac{1}{2}$. By the same argument, such leaves map to slow unstable leaves by Corollary 6.8. Since the foliation by slow unstable leaves is C^1 in $W^u(v)$, such leaves are generic by Fubini. Hence $h_{v,w}$ maps every leaf to a leaf by continuity. □

COROLLARY 6.10. *The slow unstable distributions are trivial.*

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Proof. For any v, $w \in \mathcal{Q}$ with $w \in W^{cs}(v)$, the projections of the slow unstable foliations on $W^u(v)$ and $W^u(w)$ to $\partial\widetilde{M}$ agree on the complement of the backward endpoints of the geodesics through v and w in $\partial\widetilde{M}$ by Lemma 6.9. Hence we obtain a common C^0 foliation of $\partial\widetilde{M}$. Moreover, this foliation is invariant under $\pi_1 M$ since the $E^u_{<1}$ distributions are $\pi_1(M)$ invariant. Again by Foulon [**Fou94**, Corollaire], this foliation is trivial. □

The last step in the proof of our main result is now essentially the same as in the strict pinching case.

Proof of Theorem 1.1. (Non-strict $\frac{1}{4}$-pinching case). Since the slow unstable distribution is trivial, all unstable Jacobi fields belong to E^u_1. Hence all sectional curvatures are -1 on $\mathcal{Q}$. Since $\mathcal{Q}$ is dense in SM, it follows that all sectional curvatures are -1. □

Proof of Corollary 1.4. Since μ is ergodic and invariant, the set of vectors that are recurrent with positive frequency has full measure. Since μ has full support, this set is therefore dense. By Lemma 4.3, the unstable Lyapunov space of exponent 1 for the geodesic through v coincides with $\mathcal{E}(v)$ everywhere on this set. In particular, $\mathcal{E}(v)$ has positive dimension everywhere, and the hyperbolic rank of M is positive. □

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REFERENCES

[**AV10**] A. Avila and M. Viana. Extremal Lyapunov exponents: an invariance principle and applications. *Invent. Math.* **181** (2010), 115–178.

[**ASV13**] A. Avila, J. Santamaria and M. Viana. Holonomy invariance: rough regularity and applications to Lyapunov exponents. *Astérisque* **358** (2013), 13–74.

[**Bal85**] W. Ballmann. Nonpositively curved manifolds of higher rank. *Ann. of Math. (2)* **122**(3) (1985), 597–609.

[**Bal95**] W. Ballmann. *Lectures on Spaces of Nonpositive Curvature (DMV Seminar, 25)*. Birkhäuser, Basel, 1995, with an appendix by Misha Brin.

[**BBE85**] W. Ballmann, M. Brin and P. Eberlein. Structure of manifolds of nonpositive curvature. I. *Ann. of Math. (2)* **122**(1) (1985), 171–203.

[**BFL90**] Y. Benoist, P. Foulon and F. Labourie. Flots d'Anosov à distributions de Liapounov différentiables. I. Hyperbolic behaviour of dynamical systems (Paris, 1990). *Ann. Inst. H. Poincaré Phys. Théor.* **53**(4) (1990), 395–412.

[**BS16**] R. G. Bettiol and B. Schmidt. Three-manifolds with many flat planes. *Trans. Amer. Math. Soc.* **370**(1) (2018), 669–693.

[**Bri82**] M. Brin. Ergodic theory of frame flows. *Ergodic Theory and Dynamical Systems, II (College Park, MD, 1979/1980) (Progress in Mathematics, 21)*. Birkhäuser, Boston, MA, 1982, pp. 163–183.

[**Bro16**] A. Brown. Smoothness of stable holonomies inside center-stable manifolds and the C^2 hypothesis in Pugh–Shub and Ledrappier–Young theory. *Preprint*, 2016, arXiv:1608.05886.

[**BS87**] K. Burns and R. Spatzier. Manifolds of nonpositive curvature and their buildings. *Publ. Math. Inst. Hautes Études Sci.* **65** (1987), 35–59.

[**BW05**] K. Burns and A. Wilkinson. A note on stable holonomy between centers. *Preprint*, 2005, http://www.math.northwestern.edu/~burns/papers/bwilk2a/c1hol0611.pdf.

[**But15**] C. Butler. Rigidity of equality of Lyapunov exponents for geodesic flows. *J. Differential Geom.* **109**(1) (2018), 39–79.

[**Con02**] C. Connell. A characterization of homogeneous spaces with positive hyperbolic rank. *Geom. Dedicata* **93** (2002), 205–233.

[**Con03**] C. Connell. Minimal Lyapunov exponents, quasiconformal structures, and rigidity of non-positively curved manifolds. *Ergod. Th. & Dynam. Sys.* **23**(2) (2003), 429–446.

[**Con08**] D. Constantine. 2-frame flow dynamics and hyperbolic rank-rigidity in nonpositive curvature. *J. Mod. Dyn.* **2**(4) (2008), 719–740.

[**EH90**] P. Eberlein and J. Heber. A differential geometric characterization of symmetric spaces of higher rank. *Publ. Math. Inst. Hautes Études Sci.* **71** (1990), 33–44.

[**FK90**] R. Feres and A. Katok. Anosov flows with smooth foliations and rigidity of geodesic flows on three-dimensional manifolds of negative curvature. *Ergod. Th. & Dynam. Sys.* **10**(4) (1990), 657–670.

[**Fou94**] P. Foulon. Feuilletages des sphères et dynamiques Nord–Sud. *C. R. Acad. Sci. Paris Sér. I Math.* **318**(11) (1994), 1041–1042.

[**GHL04**] S. Gallot, D. Hulin and J. Lafontaine. *Riemannian Geometry (Universitext)*, 3rd edn. Springer, Berlin, 2004.

[**Ham91a**] U. Hamenstädt. Compact manifolds with 1/4-pinched negative curvature. *Global Differential Geometry and Global Analysis (Lecture Notes in Mathematics, 1481)*. Eds. D. Ferus, U. Pinkall, U. Simon and B. Wegner. Springer, Berlin, 1991.

[**Ham91b**] U. Hamenstädt. A geometric characterization of negatively curved locally symmetric spaces. *J. Differential Geom.* **34**(1) (1991), 193–221.

[**Has94**] B. Hasselblatt. Regularity of the Anosov splitting and of horospheric foliations. *Ergod. Th. & Dynam. Sys.* **14**(4) (1994), 645–666.

[**HeiImH**] E. Heintze and C. Im Hof. Geometry of horospheres. *J. Differential Geom.* **12**(4) (1977), 481–491.

[**HP75**] M. Hirsch and C. Pugh. Smoothness of horocycle foliations. *J. Differential Geom.* **10** (1975), 225–238.

[**Kan88**] M. Kanai. Geodesic flows of negatively curved manifolds with smooth stable and unstable foliations. *Ergod. Th. & Dynam. Sys.* **8**(2) (1988), 215–239.

[**KS13**] B. Kalinin and V. Sadovskaya. Cocycles with one exponent over partially hyperbolic systems. *Geom. Dedicata* **167** (2013), 167–188.

[**KS17**] B. Kalinin and V. Sadovskaya. Normal forms for non-uniform contractions. *J. Mod. Dyn.* **11**(3) (2017), 341–368.

[**Lan95**] S. Lang. *Differential and Riemannian manifolds (Graduate Texts in Mathematics, 160)*, 3rd edn. Springer, New York, 1995.

[**LS16**] S. Lin and B. Schmidt. Manifolds with many hyperbolic planes. *Differential Geom. Appl.* **52** (2017), 121–126.

[**Mel17**] K. Melnick. Non-stationary smooth geometric structures for contracting measurable cocycles. *Ergod. Th. & Dynam. Sys.* (2017), doi:10.1017/etds.2017.38. Published online 28 June 2017.

[**SS90**] R. J. Spatzier and M. Strake. Some examples of higher rank manifolds of nonnegative curvature. *Comment. Math. Helv.* **65**(2) (1990), 299–317.

[**SSS16**] B. Schmidt, K. Shankar and R. Spatzier. Positively curved manifolds with large spherical rank. *Comment. Math. Helv.* **91**(2) (2016), 219–251.

[**SSW05**] K. Shankar, R. Spatzier and B. Wilking. Spherical rank rigidity and Blaschke manifolds. *Duke Math. J.* **128**(1) (2005), 65–81.

[**Wat13**] J. Watkins. The higher rank rigidity theorem for manifolds with no focal points. *Geom. Dedicata* **164** (2013), 319–349.

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