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CONSISTENCY OF M-THEORY ON NON-ORIENTABLE MANIFOLDS

by DANIEL S. FREED[†]

(*Department of Mathematics, University of Texas, Austin, TX 78712, USA*)

and MICHAEL J. HOPKINS[‡]

(*Department of Mathematics, Harvard University, Cambridge, MA 02138, USA*)

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To Sir Michael

Abstract

We prove that there is no parity anomaly in M-theory in the low-energy field theory approximation. Our approach is computational. We determine the generators for the 12-dimensional bordism group of pin manifolds with a w_1-twisted integer lift of w_4; these are the manifolds on which Wick-rotated M-theory exists. The anomaly cancellation comes down to computing a specific η-invariant and cubic form on these manifolds. Of interest beyond this specific problem are our expositions of computational techniques for η-invariants, the algebraic theory of cubic forms, Adams spectral sequence techniques and anomalies for spinor fields and Rarita–Schwinger fields.

1. Introduction

Time-reversal symmetry is a topic of renewed interest, in part because of its prevalence in condensed matter models. Under Wick rotation, time reversal is connected to reflections in Euclidean space, and time-reversal symmetric theories may be formulated unoriented manifolds. The obstruction to doing so is often termed a 'parity anomaly', though 'parity' is not synonymous with 'time-reversal symmetry'. ('Parity' typically refers to a spatial reflection through a point in Minkowski spacetime, relative to a splitting into time cross space. As this is orientation preserving in even space dimensions, more relevant is reflection in a timelike hyperplane, which is always orientation reversing. A time reversal is reflection in a spacelike hyperplane.) Witten [**56**] recently showed that there is no anomaly for this symmetry on an M2-brane in M-theory. He suggested that we investigate the analogous issue in the bulk on suitable 11-manifolds. We do so here and prove that there is no time-reversal anomaly in M-theory.

We work in the low-energy field theory approximation to M-theory, which is a classical 11-dimensional supergravity with a gravitational correction term [**20**, **54**]. The theory includes a fermionic field, and so X carries a pin structure—the appropriate choice is a pin$^+$ structure, as opposed to a pin$^-$ structure—on the *tangent* bundle. The C-field in M-theory, which is odd under time-reversal symmetry, induces an additional topological structure on X: a w_1-twisted integer lift of

[†]Corresponding author. E-mail: dafr@math.utexas.edu
[‡]E-mail: mjh@math.harvard.edu

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the fourth Stiefel–Whitney class $w_4(X)$ (see [57, Section 2.3]). A pin$^+$-manifold with a w_1-twisted integer lift of w_4 is called an $\mathfrak{m}_c$-*manifold*. There are two sources of anomalies. The first is the standard fermion anomaly, though there are subtleties: the fermion field is a Rarita–Schwinger field, rather than a spinor field, and the background is a pin manifold, rather than a spin manifold. The second anomaly is non-standard, due to the cubic form for the C-field. In the spin case, Witten [57, Section 4] represents the C-field as a connection on a principal E_8-bundle, and he uses this to prove that these two anomalies cancel. In the pin case, this argument is not available, so we resort to a computational approach. Each anomaly is encoded in an invertible unitary topological 12-dimensional field theory, hence is determined by its partition function. Furthermore, the partition function is a bordism invariant, so it suffices to check that the partition functions of the two theories agree on a set of generators for the appropriate bordism group. We use the Adams spectral sequence, together with computer assistance and geometric arguments, to compute a set of generators for the relevant bordism group. We deploy a mix of topological and geometric techniques to compute the partition functions on these generators and so prove anomaly cancellation.

To define M-theory, we must not only prove that anomalies cancel but provide data which perform the anomaly cancellation. In the spin case, ignoring time-reversal symmetry, this 'setting of the quantum integrand' can be achieved using Witten's E_8-bundle technique [29]. We do not know a canonical setting in the pin case, and indeed, isomorphism classes of settings form a torsor over isomorphism classes of 11-dimensional invertible field theories on the same class of manifolds. The latter group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, at least conjecturally, as we explain in Section 7. Since this group is non-zero, the problem of setting the quantum integrand remains open.

Now we give a more detailed summary. We begin in the expository Section 2 by reviewing the Wick-rotated setting for M-theory as a theory on a certain geometric bordism category. We recall that the anomaly of an 11-dimensional theory is an invertible 12-dimensional theory α and that invertible topological theories may be represented as maps of spectra in stable homotopy theory. The aim of this paper is to prove Theorem 2.2: the tensor product $\alpha_{RS} \otimes \alpha_C$ of the anomaly theories arising from the Rarita–Schwinger and C-fields of M-theory is trivializable. In Section 3, we define α_{RS}. As with the anomaly of any fermionic field, its partition function is the exponential of an Atiyah–Patodi–Singer η-invariant. We elucidate some aspects of the general theory in Appendix A, and in Section 3.2, we give a general formula for the anomaly theory of a Rarita–Schwinger field (see (3.10)). In our situation, the anomaly partition function is independent of the Riemannian metric, so is a topological invariant. It turns out to be ± 1 on $\mathfrak{m}_c$-manifolds, though at this stage the only apparent statement is that it is a root of unity. Indeed, on a general pin$^+$ manifold, it does not necessarily have the order 2. We develop formulas to compute it, following the work of Donnelly, Stolz and Zhang. Of particular interest is a topological formula which, as far as we know, has only an analytic proof in the literature [58]; it was brought to our attention by Jonathan Campbell. The partition function of the anomaly theory α_C is an inhomogeneous cubic polynomial in the C-field. It is also topological and by definition is equal to ± 1. In Section 4, we develop an algebraic theory of the cubic form, imitating the standard algebraic theory of quadratic forms, and then define α_C. We also review Witten's proof that $\alpha_{RS} \otimes \alpha_C$ is trivializable when restricted to spin manifolds. Section 5 is a geometric interlude to review some basic spin and pin$^+$ manifolds and their topological invariants. We also introduce more complicated manifolds used as representative elements of bordism groups. Our main computational result, whose proof we sketch in Section 8, is stated as Theorem 6.1. We specify the generators of the relevant 12-dimensional bordism group, which we represent by specific 12-dimensional $\mathfrak{m}_c$-manifolds. For each of these, we compute that the partition function of $\alpha_{RS} \otimes \alpha_C$

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vanishes, which suffices to demonstrate the anomaly cancellation. We employ a potpourri of techniques to make the computations. The aforementioned ambiguity in the definition of M-theory is discussed in Section 7. Section 8 contains a computation of the low-dimensional bordism groups of $\mathfrak{m}_c$-manifolds. In particular, we provide a proof of Theorem 6.1. For the computation, we need a set of generators of 12-dimensional spin bordism (localized at 2), which we produce in Appendix B, based on the work of Anderson–Brown–Peterson. Appendix C details the mod 2 cohomology of the Thom spectrum of $\mathfrak{m}_c$-manifolds, a key input into the Adams spectral sequence computation. A more detailed computer-free version of these computations will appear in [**31**].

Aspects of this paper have interest beyond our proof that M-theory is time-reversal invariant. This includes the algebraic theory of cubic forms in Section 4; our techniques to compute η-invariants of pin manifolds; the Adams spectral sequence techniques in Section 8 and the cohomology computations in Appendix C; the discussion of spinor field anomalies in Appendix A; and the interplay between invertible unitary topological field theories and stable homotopy theory, which is developed and plays a key role in an application to condensed matter physics in [**26**].

The authors take this opportunity to express our deep sense of gratitude and indebtedness to Michael Atiyah for his mentoring, encouragement and support. Michael's enthusiasm for mathematics and for its interaction with physics has long been an inspiration. We appreciate his unfailing sense of what constitutes an enlightening and 'correct' proof, and we join him in lamenting the lack of such a proof for this anomaly cancellation.

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2. Time-reversal, anomalies and bordism

A relativistic quantum field theory on n-dimensional Minkowski spacetime $\mathbb{M}^n$ has a symmetry group $\mathscr{H}_{1,n-1}$, equipped with a homomorphism to the group of isometries of $\mathbb{M}^n$. (See [**26**, Section 2] for an account of symmetry groups in quantum field theory.) Divide by translations and Wick rotate to Euclidean signature to obtain a compact Lie group H_n of vector symmetries, equipped with a homomorphism $\rho_n\colon H_n \to O_n$ whose image is (i) SO_n in the absence of time-reversal symmetry, or (ii) O_n if the theory has time-reversal symmetry. Eleven-dimensional M-theory has both time-reversal symmetry and fermionic fields, and no additional global symmetries, so the Wick-rotated symmetry group is one of the two Pin groups. (One could regard the C-field as the background field for a higher symmetry, but as the primary objects of interest are the background fields we do not pursue this point of view.) Because time-reversal squares in Minkowski spacetime to $(-1)^F$, the appropriate group is $H_{11} = \mathrm{Pin}^+_{11}$; see [**26**, Appendix A]. We consider M-theory on curved compact 11-dimensional Riemannian manifolds X, and so we require that X have a *tangential* pin^+ structure; equivalently, the stable normal bundle of X has a pin^- structure. There is an additional topological structure, first identified in [**57**, Section 2.3]. The C-field is an abelian gauge field, thus obeys a Dirac quantization condition. The correct condition is that the de Rham cohomology class of its field strength, a closed 4-form twisted by the orientation bundle, refines to a w_1-twisted integer cohomology class $c \in H^4(X;\widetilde{\mathbb{Z}})$ whose mod 2 reduction is the fourth Stiefel–Whitney class $w_4(X)$. Here $\widetilde{\mathbb{Z}}$ is the local coefficient system induced from the orientation double cover of X. This motivates the following terminology, taken from [**56**, Section 4.3], where it is introduced by analogy with a *spin*^{c} structure.

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DEFINITION 2.1 Let M be a pin^+ manifold. An $\mathfrak{m}_c$ *structure* on M is a w_1-twisted integer lift of $w_4(M)$. We say M is an $\mathfrak{m}_c$-*manifold* if M is equipped with an $\mathfrak{m}_c$ structure.

A necessary and sufficient condition to be $\mathfrak{m}_c$ is $\tilde{\beta}w_4(M) = 0$, where $\tilde{\beta}$ is the Bockstein map into w_1-twisted integral cohomology (see 4.21). The Wick rotation of M-theory is defined on a geometric bordism category of $\mathfrak{m}_c$-manifolds.

Once an n-dimensional field theory is formulated on compact Riemannian manifolds, then there is the possibility of an anomaly: the partition function may not be well-defined as a complex number, but rather may be an element of a complex line. These complex lines depend *locally* on the Riemannian manifold, which is expressed by saying that they are the quantum state spaces of a field theory α. The theory α is called a *gravitational anomaly*. In addition to the coupling to a gravitational background, if the kernel of $\rho_n\colon H_n \to O_n$ is non-trivial, then there is also a coupling to a background gauge field, in which case we have a mixed gravitational and gauge anomaly. In most examples, the anomaly theory α extends to an $(n+1)$-dimensional theory which has a partition function on closed $(n+1)$-manifolds. That is so in this paper. The 11-dimensional M-theory is not rigorously defined, but nonetheless we do define the 12-dimensional anomaly theory that is our main focus. Anomalies are very special among field theories: they are *invertible*. Recall that field theories have a composition law of tensor product, and there is a trivial theory 1 which is an identity for this composition law. So a field theory α is invertible if there exists a theory β such that $\alpha \otimes \beta$ is isomorphic to 1. An invertible field theory has non-zero partition functions, one-dimensional state spaces, etc. We refer to [**23**] and the references therein for exposition on this point of view about anomalies.

Recall that M-theory has two bosonic fields—a metric and C-field—and a single fermionic field—the Rarita–Schwinger field ψ. To analyse anomalies we work in the effective theory after integrating out ψ; the metric and C-field are treated as background fields. One source of anomalies is the fermionic integration of ψ, which we review in Section 3. Let α_{RS} denote that 12-dimensional anomaly theory. The other source of anomalies is the 'Chern-Simons coupling' of the C-field, which is an inhomogeneous cubic form we review in Section 4. Let α_C denote that 12-dimensional anomaly theory. Our main result is the cancellation of these anomalies.

THEOREM 2.2 *The total anomaly theory $\alpha_{RS} \otimes \alpha_C$ is trivializable.*

That is, $\alpha_{RS} \otimes \alpha_C \cong 1$. This implies that M-theory should exist as an 'absolute' quantum field theory whose partition functions are complex numbers (not merely elements of an abstract complex line), whose state spaces are vector spaces (not merely well-defined as projective spaces), etc. In other words, M-theory is anomaly-free. As explained in [**56**, Section 1], this is a strong form of the vanishing of the 'parity anomaly'.

An important feature is that both α_{RS} and α_C are *topological* field theories. That means they are each independent of the metric and α_C only depends on the C-field through its topology: a C-field on X represents a class in the twisted differential cohomology group $\check{H}^4(X;\widetilde{\mathbb{Z}})$ (Section 4.5); the statement is that the anomaly theory α_C only depends on the representative of its image under $\check{H}^4(X;\widetilde{\mathbb{Z}}) \to H^4(X;\widetilde{\mathbb{Z}})$. Furthermore, as already stated these theories are *invertible*. Finally, due to their physical origins these theories are *unitary*, or equivalently in the Wick-rotated version they satisfy reflection positivity. The main theorem in [**26**] asserts that, assuming reasonable ansätze, reflection positive invertible topological field theories live in the world of stable homotopy theory: they are spectrum maps from a Thom spectrum to a universal target, the shifted Pontrjagin

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dual to the sphere spectrum. (In the absence of unitarity, an invertible topological theories has domain a Madsen–Tillmann spectrum; see [27], [47].) This result uses a strong form of locality—a *fully extended* field theory—and also a companion strong form of reflection positivity for invertible topological theories. Thus the anomaly theories are maps

$$\alpha_{RS}, \alpha_C \colon M\mathfrak{m}_c \longrightarrow \Sigma^{12} I\mathbb{C}^\times. \tag{2.1}$$

Here $M\mathfrak{m}_c$ is the Thom spectrum of $\mathfrak{m}_c$-manifolds: manifolds with a stable tangential pin^+ structure and a w_1-twisted integer lift of w_4. We construct $M\mathfrak{m}_c$ in Section 8.1. Also, $I\mathbb{C}^\times$ is the character dual to the sphere spectrum, closely related to the Brown–Comenetz dual [8]. The universal property which characterizes $I\mathbb{C}^\times$ (see [26, Section 5.3]) implies that the group of homotopy classes of maps (2.1) is isomorphic to the group $\mathrm{Hom}(\pi_{12}M\mathfrak{m}_c, \mathbb{C}^\times)$ of characters of $\pi_{12}M\mathfrak{m}_c$. In other words, the maps (2.1) are determined up to homotopy—and the corresponding topological field theories up to isomorphism—by abelian group homomorphisms

$$\hat{\alpha}_{RS}, \hat{\alpha}_C \colon \pi_{12}M\mathfrak{m}_c \longrightarrow \mathbb{C}^\times. \tag{2.2}$$

These homomorphisms encode the partition functions of the respective anomaly theories. We prove Theorem 2.2 by demonstrating that the product

$$\hat{\alpha}_{RS} \cdot \hat{\alpha}_C \colon \pi_{12}M\mathfrak{m}_c \longrightarrow \mathbb{C}^\times \tag{2.3}$$

of partition functions is identically one. Both $\hat{\alpha}_{RS}$ and $\hat{\alpha}_C$ take values in the group $\mathbb{T} \subset \mathbb{C}^\times$ of unit norm complex numbers. From its definition (4.41), the homomorphism $\hat{\alpha}_C$ takes values in $\{\pm 1\} \subset \mathbb{C}^\times$, and so the field theory α_C has order two: its square is isomorphic to the trivial theory. It emerges from our computations that α_{RS} also has order two. (We remark that α_{RS} is pulled back from an invertible theory defined on $MT\,\mathrm{Pin}^+$ which has order 2^8.)

Theorem 2.2 asserts that the total anomaly is trivializable but does not specify a trivialization. (For further discussion, see [29] where the trivialization is called a 'setting of the quantum integrand'.) Homotopy classes of trivializations form a torsor over the group of invertible 11-dimensional reflection positive topological theories. That is, given one trivialization, and so in principle one realization of M-theory, any other one differs by inserting a 'topological term' in the 11-dimensional theory. In Section 7, based on computations to appear in [31], we discuss the following.

CONJECTURE 2.3 *The group of homotopy classes of spectrum maps $M\mathfrak{m}_c \to \Sigma^{11} I\mathbb{C}^\times$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})$. The partition function of the non-trivial theory is the mod 2 index of the Dirac operator.*

3. The Rarita–Schwinger operator and η-invariants

The reader may want to refer to recent expositions of fermions and anomalies in [55, 56]. We recall the relation between fermion integrals and pfaffians in Section 3.1. In Section 3.2, we indicate the modifications engendered by a Rarita–Schwinger field, as opposed to a spinor field, and then specialize to M-theory and define the Rarita–Schwinger anomaly theory α_{RS}. This all relies on material in Appendix A. In Section 3.3, we recall and prove some properties of and formulas for η-invariants

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on pin$^+$ 12-manifolds that we use in our subsequent computations (Section 6). This exponentiated η-invariant is topological—independent of the Riemannian metric—and there is a topological formula (Theorem 3.9) for its value.

3.1. *Brief recollection of free fermionic path integrals*

The material in this section also appears in [**25**, Section 11] as part of a broader discussion of anomalies.

Suppose W is a finite dimensional complex vector space and

$$B\colon W \times W \longrightarrow \mathbb{C} \tag{3.1}$$

a skew-symmetric bilinear form. We identify B as a skew-symmetric map $W \to W^*$, and so an element $\omega_B \in \bigwedge^2 W^*$. The natural integral on the exterior algebra is the linear map

$$\int_{\Pi W} : \bigwedge\nolimits^\bullet W^* \longrightarrow \operatorname{Det} W^* \tag{3.2}$$

which projects a form of mixed degree to its highest degree component in $\operatorname{Det} W^* = \bigwedge^{\max} W^*$. The odd vector space ΠW, the parity-reversal of W, has as its ring of functions the $\mathbb{Z}/2\mathbb{Z}$-graded exterior algebra $\bigwedge^\bullet W^*$. The fermionic integration (3.2) is purely algebraic—there is no measure—and it is defined on functions rather than forms or densities. If $\dim W = 2m$ is even, then

$$\int_{\Pi W} e^{\omega_B} = \frac{\omega_B^m}{m!} = \operatorname{pfaff} B \in \operatorname{Det} W^* \tag{3.3}$$

is the *pfaffian* of B; if $\dim W$ is odd, then the integral vanishes. It is natural to regard $\operatorname{Det} W^*$ as $\mathbb{Z}/2\mathbb{Z}$-graded by the parity of $\dim W$, which is equal to the parity of the dimension of the null space $\ker B$. There is an infinite dimensional version of the pfaffian for W a Hilbert space and B a *Fredholm form*: B is Fredholm if $\ker B$ is a closed, finite dimensional subspace. Then the $\mathbb{Z}/2\mathbb{Z}$-graded line $\operatorname{Pfaff} B$ depends on B and forms a non-trivial complex line bundle over the space of Fredholm skew forms, and the pfaffian elements

$$\operatorname{pfaff} B \in \operatorname{Pfaff} B \tag{3.4}$$

form a section of the Pfaffian bundle. See [**45**], [**49**, Appendix B] for the case of the Fredholm determinant (Remark 3.2 below). (We remark that the space of Fredholm skew forms has two components distinguished by the parity of $\dim \ker B$, the mod 2 index. Over each component the Pfaffian line bundle represents a generator of $H^2(-;\mathbb{Z})$. The pfaffian section vanishes if and only if $\ker B \neq 0$.)

REMARK 3.1 There are real and quaternionic refinements; the latter applies to M-theory on spin manifolds [**29**, Section 1].

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REMARK 3.2 Suppose $T\colon U \to V$ is a linear map between complex vector spaces. Set $W = V^* \oplus U$ and

$$B\big((v_1^*, u_1), (v_2^*, u_2)\big) = \langle v_1^*, Tu_2\rangle - \langle v_2^*, Tu_1\rangle. \tag{3.5}$$

Then the Pfaffian line of B is canonically isomorphic to the determinant line $\operatorname{Det} V \otimes (\operatorname{Det} U)^*$ of T, and under that isomorphism $\operatorname{pfaff} B = \det T$ as an element of $\operatorname{Det}(U)^* \otimes \operatorname{Det}(V)$.

A spinor field in an n-dimensional relativistic field theory on Minkowski spacetime (without time-reversal symmetry) is specified by a real spinor representation $\mathbb{S}$ of $\mathrm{Spin}_{1,n-1}$ together with a symmetric non-negative $\mathrm{Spin}_{1,n-1}$-invariant bilinear form $\Gamma\colon \mathbb{S}\times\mathbb{S} \to \mathbb{R}^{1,n-1}$ (see Section A.1) for details. The complexification $\mathbb{S}_{\mathbb{C}}$ is a representation of the compact spin group Spin_n. On a closed Riemannian spin n-manifold X there is an associated complex vector bundle whose sections are spinor fields ψ. Define the complex skew-symmetric form

$$B_X(\psi_1, \psi_2) = \int_X \Gamma_{\mathbb{C}}(\psi_1, \nabla\psi_2)\,|\mathrm{d}x|, \tag{3.6}$$

where ∇ is induced from the Levi-Civita covariant derivative, $\Gamma_{\mathbb{C}}$ is the complexification of Γ and $|dx|$ is the Riemannian measure. On appropriate function spaces B_X is Fredholm. The Feynman path integral over ψ is the formal analog of (3.3), and we define the result to be (3.4), the pfaffian element of the Pfaffian line. In particular, the fermionic path integral is anomalous.

As explained in Section 2, the Pfaffian line of the Dirac form B_X is the quantum state space on X of an invertible $(n+1)$-dimensional field theory α, called the *anomaly theory*. To define its partition function, we must use the data $(\mathbb{S}, \Gamma)$ to define a Dirac operator on a Riemannian spin $(n+1)$-dimensional manifold W. That construction is carried out in Appendix A; the partition function $\alpha(W)$ for W closed is the exponentiated η-invariant (A10).

3.2. *The Rarita–Schwinger anomaly*

The Rarita–Schwinger field occurs in theories of supergravity; it is the super-partner to the metric. In n spacetime dimensions there is an associated anomaly theory, which is an $(n+1)$-dimensional invertible field theory, just as for a spinor field. Here we explain the modifications to the discussions in the previous section and Section A.1 required to specify the anomaly theory. More information may be found in [**28**], [**29**, Appendix A] and the references therein.

Suppose given data $(\mathbb{S}, \Gamma)$ for a spinor field in n-dimensional Minkowski spacetime M^n, as above. Let V be the standard n-dimensional real representation of $\mathrm{Spin}_{1,n-1}$. The Rarita–Schwinger field is a function $\chi\colon M^n \to \mathbb{S}\otimes V$. (More precisely, we should view $\mathbb{S}\otimes V$ as an *odd* super vector space.) There is a correspondence between free fields and particles, and under this correspondence the Rarita–Schwinger field gives rise to four particles, as recounted in [**28**, Section A.2]: a single spin 3/2 particle, which is the desired gravitino, as well as three spurious spin 1/2 particles. Two of the spin 1/2 particles are associated with spinor fields with values in $\mathbb{S}$ and the third to a spinor field with values in $\mathbb{S}^*$. Wick rotation of χ proceeds by complexification, and one obtains a skew form B'_X analogous to (3.6), now built on sections of a spinor bundle tensored with the tangent bundle. To eliminate the extra spin 1/2 particles, we divide the pfaffian of B'_X by the product of the pfaffians of the forms B_X associated with the three Wick rotated spin 1/2 fields [**28**, Section A.5].

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We now determine the associated anomaly theory, which is an invertible $(n+1)$-dimensional field theory of Riemannian spin manifolds. In Appendix A, we define the anomaly theory $\alpha_{\mathbb{S}}$ associated with spinor data $(\mathbb{S}, \Gamma)$. There is a variation which gives the anomaly of the pfaffian of B'_X. Motivated by (A7) define

$$\mathbb{E}' = \mathrm{Cliff}_{+2} \otimes (\mathbb{S} \oplus \mathbb{S}^*) \otimes \mathbb{R}^{n+1}. \tag{3.7}$$

(Our notation: the real Clifford algebra $\mathrm{Cliff}_{p,q}$ has p generators squaring to $+1$ and q generators squaring to -1. Set $\mathrm{Cliff}_{+p} = \mathrm{Cliff}_{p,0}$ and $\mathrm{Cliff}_{-q} = \mathrm{Cliff}_{0,q}$.) Let Spin_{n+1} act as after (A7) on $\mathbb{E} = \mathrm{Cliff}_{+2} \otimes (\mathbb{S} \oplus \mathbb{S}^*)$ and tensor with the usual vector representation on $\mathbb{R}^{n+1}$. There is a commuting Cliff_{-3} action, as after (A7), and the Dirac operator (A8) and exponentiated η-invariant (A10) are defined. Denote the resulting $(n+1)$-dimensional theory as $\alpha_{\mathbb{S}\otimes V}$. Since Spin_n acts reducibly on $\mathbb{R}^{n+1} = \mathbb{R} \oplus \mathbb{R}^n$, the specialization to a product manifold $\mathbb{R} \times X$ gives the Dirac operator coupled to the tangent bundle plus an extra copy of the Dirac operator on spinor fields. Therefore, the anomaly theory associated with the pfaffian of B'_X is

$$\alpha_{\mathbb{S}\otimes V} \otimes \alpha_{\mathbb{S}}^{\otimes(-1)}. \tag{3.8}$$

Put (3.8) together with the anomalies of the spurious spin 1/2 fields to obtain the total anomaly

$$\alpha_{\mathbb{S}\otimes V} \otimes \alpha_{\mathbb{S}}^{\otimes(-3)} \otimes \alpha_{\mathbb{S}^*}^{\otimes(-1)}. \tag{3.9}$$

Proposition A.1 implies that the product $\alpha_{\mathbb{S}} \otimes \alpha_{\mathbb{S}^*}$ is trivializable. Hence (3.9) is isomorphic to

$$\alpha_{\mathbb{S}\otimes V} \otimes \alpha_{\mathbb{S}}^{\otimes(-2)}. \tag{3.10}$$

This is a general formula for the anomaly theory of a Rarita–Schwinger field in n dimensions built from the spinor data $(\mathbb{S}, \Gamma)$.

Now we specialize to $n = 11$ and M-theory on pin^+ manifolds. The spinor data $(\mathbb{S}, \Gamma)$ are specified in Section A.2. Let W be a closed Riemannian pin^+ 12-manifold. We compute the partition function $\hat{\alpha}_{RS}(W)$ of the total anomaly theory (3.10). The (s)pinor bundle on W is a rank 32 quaternionic bundle, and the appropriate Dirac operator (A8) is a self-adjoint operator on its sections. The partition function (A10) is, in this case, a ratio of exponentiated η-invariants, which we write as

$$\hat{\alpha}_{RS}(W) = \exp\left(2\pi i\, \frac{\eta(TW-2)}{4}\right). \tag{3.11}$$

Here $\eta(TW-2)$ is the difference of the η-invariant of the Dirac operator coupled to the tangent bundle and twice the η-invariant of the pure Dirac operator.

PROPOSITION 3.3 *The Rarita–Schwinger partition function* $\hat{\alpha}_{RS}(W)$ *is* (i) *independent of the metric on W,* (ii) *a* pin^+ *bordism invariant and* (iii) *a root of unity.*

Proof. Similar assertions for even dimensional *pin*^{c} manifolds are proved in [**30**, Section 1], so we will be brief. The space of Riemannian metrics is connected (better: contractible), so it suffices to

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compute the derivative with respect to the metric. That variation formula [5] is an integral of a w_1-twisted 13-form over W; the result is a 1-form on the space of metrics. The integrand is a local invariant of the geometry by a general theory developed by Seeley [46], and here it vanishes for parity reasons (see [30, Lemma 1.5]). This proves (i). Now suppose W is the boundary of a compact pin^+ 13-manifold Z. Then [4, Theorem 3.10] computes (3.11) as the exponential of the integral of the same w_1-twisted 13-form over Z. But as above, this 13-form vanishes identically, and so (ii) holds. For (iii) we need only use that the relevant bordism group is finite, in fact [38]

$$\pi_{12}MT\mathrm{Pin}^+ \cong \mathbb{Z}/2^8\mathbb{Z} \oplus \mathbb{Z}/2^4\mathbb{Z} \oplus \mathbb{Z}/2^2\mathbb{Z}. \tag{3.12}$$

□

COROLLARY 3.4 *The Rarita–Schwinger partition function factors through a homomorphism*

$$\hat{\alpha}_{RS}\colon \pi_{12}MT\mathrm{Pin}^+ \longrightarrow \mathbb{C}^\times. \tag{3.13}$$

As reviewed in Section 2, the homomorphism (3.13) determines an invertible unitary topological field theory

$$\alpha_{RS}\colon MT\mathrm{Pin}^+ \longrightarrow \Sigma^{12}I\mathbb{C}^\times \tag{3.14}$$

up to isomorphism. (The maps (2.1), (2.2) which correspond to (3.13), (3.14) are lifted to the bordism spectrum $M\mathfrak{m}_c$ of manifolds with a pin^+ structure *and* a w_1-twisted integer lift of w_4.) We stretch notation slightly and use the notation '$\alpha_{RS}(W)$' for the partition function of a closed pin^+ 12-manifold W.

3.3. *Properties of the η-invariant*

On a *spin* manifold W, the partition function $\alpha_{RS}(W)$ has a natural logarithm defined using an integer index.

PROPOSITION 3.5 *Let W be a closed spin 12-manifold. Then $\alpha_{RS}(W) = (-1)^{RS(W)}$, where*

$$RS(W) = \frac{1}{2}\left\langle \hat{A}(W)\,\mathrm{ch}(TW-2),\,[W]\right\rangle. \tag{3.15}$$

Proof. As remarked after (A21), the pin^+ spinor data restrict to spin spinor data, so on a spin manifold we apply the discussion in Section A.1. The crucial point, which holds in general for n odd, is that the action of the volume form $\omega = \delta^1\delta^2\gamma^1\cdots\gamma^{10} \subset \mathrm{Cliff}_{+12}$ commutes with the action of $Spin_{12}$ and so anticommutes with the Dirac operator D^0. This implies that the spectrum of D^0 is invariant under $\lambda \mapsto -\lambda$. Choose a in (A9) to be negative and greater than the first negative eigenvalue of D^0. Then $\eta_a(s) = \dim\ker D^0$ for all s—the non-zero eigenvalues in the sum cancel—and so from (A10) we have

$$\alpha_{RS}(W) = \exp(2\pi i \dim_{\mathbb{C}}\ker D^0/4) = (-1)^{\dim_{\mathbb{H}}\ker D^0}. \tag{3.16}$$

The quaternionic dimension of the kernel is congruent mod 2 to the quaternionic index of the chiral Dirac operator, which maps the $+1$-eigenspace of ω to its -1-eigenspace. The Atiyah–Singer index formula completes the proof. □

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REMARK 3.6 The expansion of (3.15) in terms of Pontrjagin numbers of W is

$$RS(W) = \left\langle \frac{97p_1^3 - 788p_1p_2 + 3952p_3}{967680}, [W] \right\rangle. \tag{3.17}$$

REMARK 3.7 In Section 4.3 we encounter a shift of (3.15) by an integer, namely

$$RS'(W) = \frac{1}{2}\left\langle \hat{A}(W)\,\mathrm{ch}(TW-4), [W] \right\rangle. \tag{3.18}$$

It has the same mod 2 reduction as $RS(W)$.

For a real vector bundle $V \to W$ over a closed pin^+ 12-manifold set

$$\tau_W(V) = \exp\left(2\pi i\,\frac{\eta_W(V)}{4}\right). \tag{3.19}$$

The η-invariant (A9) depends on a parameter $a \in \mathbb{R}$, as we defined it, but the exponential (3.19) is independent of a. Comparing with (3.11) our notation is $\alpha_{RS}(W) = \tau_W(TW-2)$.

PROPOSITION 3.8 *Let $V^0, V^1 \to W$ be real vector bundles over a closed* pin^+ 12*-manifold W. Then the ratio $\tau_W(V^0)/\tau_W(V^1)$ of exponentiated η-invariants depends only on the class of the virtual bundle $[V^0] - [V^1] \in KO^0(W)$.*

In particular, it does not depend on the choices of covariant derivative.

Proof. The independence from the covariant derivative follows from the variation formula, as in the proof of Proposition 3.3. Then simply observe that $\tau_W(V)$ is multiplicative: $\tau_W(V \oplus V') = \tau_W(V)\tau_W(V')$. □

Propositions 3.3 and 3.8 suggest that there is a KO-theory formula for $\tau_W(V)$. Indeed, such a formula was recently proved by Zhang [**58**], based on an analytic theorem of Bismut–Zhang [**14**] telling the behavior of η-invariants under immersions. While the other formulas and techniques in this section suffice for most of the computations in Section 6.2, we were only able to compute α_{RS} in Section 6.2.6 using this topological formula. We express (3.20) in a different, but equivalent, form than [**58**], and we have used the pin^+ variant of his pin^- theorem (which he remarks holds in the pin^+ case).

THEOREM 3.9 (Zhang) *Let $V \to W$ be a real vector bundle over a closed* pin^+ 12*-manifold W. Let $L \to W$ be the orientation real line bundle, $H \to \mathbb{RP}^{20}$ the tautological line bundle and $\gamma\colon W \to \mathbb{RP}^{20}$ a map such that $\gamma^*H \cong L$. Then*

$$\gamma_*\big([V]\big) = 2^{11}\,\frac{\eta_W(V)}{4}\,\big(1 - [H]\big) \qquad \text{in } \widetilde{KO}^0(\mathbb{RP}^{20}). \tag{3.20}$$

In this formula $[V] \in KO^0(W)$ is the KO-class of $V \to W$; the map γ has a spin structure induced from the pin^+ structures on W and $\mathbb{RP}^{20}$ together with a choice of isomorphism $\gamma^*H \cong L$; and γ_* is the

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induced pushforward on *KO*-theory, after multiplication by the Bott class. The group $\widetilde{KO}^0(\mathbb{RP}^{20})$ is cyclic of order 2^{11} with generator $1-[H]$.

A pin$^+$ structure on a smooth manifold M has an *opposite*, obtained by tensoring with the orientation double cover.

PROPOSITION 3.10 *Let $V \to W$ be a real vector bundle over a closed* pin$^+$ 12*-manifold W, and let $L \to W$ be the real line bundle associated with the orientation double cover $\pi\colon \widehat{W} \to W$. Then*

$$\tau_W(V \otimes L) = \tau_W(V)^{-1}. \tag{3.21}$$

Proof. Let σ be the deck transformation of the double cover π. Then V-valued spinor fields on W lift to σ-invariant π^*V-valued spinor fields on $\widehat{W}$ and $V \otimes L$-valued spinor fields on W lift to σ-anti-invariant π^*V-valued spinor fields on $\widehat{W}$. Hence $\eta_W(V) + \eta_W(V \otimes L) = \eta_{\widehat{W}}(\pi^*V)$. The pullback of the *pin*$^+$ structure on W combines with the orientation of $\widehat{W}$ to produce a spin structure on $\widehat{W}$, so $\tau_{\widehat{W}}(\pi^*V)$ is computed using the mod 2 reduction of

$$\frac{1}{2}\left\langle \pi^*\left[\hat{A}(W)\operatorname{ch}(V)\right], [\widehat{W}]\right\rangle, \tag{3.22}$$

as in Proposition 3.5. Since σ is an orientation-reversing involution, it follows that the integer (3.22) equals its negative, hence vanishes. □

PROPOSITION 3.11 *Suppose $W = W' \times W''$ is the product of a* pin$^+$ 4*-manifold W' and a spin* 8*-manifold W''. Let $V' \to W'$ and $V'' \to W''$ be real vector bundles. Then*

$$\tau_W(V' \otimes V'') = \tau_{W'}(V')^{\operatorname{ind} D_{W''}(V'')}, \tag{3.23}$$

where the exponent is the index of the Dirac operator coupled to V''.

Proof. This follows directly from the topological index formula (3.20), but there is a straightforward analytic proof which we outline here. Use the setup of Appendix A. Let $\mathscr{E}'^i, \mathscr{E}''^i$, $i = 0, 1$, denote the spaces of spinor fields on W', W'', and D', D'' the Dirac operators. Let $\omega = \gamma^0\delta^1\delta^2$ denote the volume form of the commuting $Cliff_{-3}$. Then the space of spinor fields on W is $\mathscr{E}'^0 \otimes \mathscr{E}''^0 \oplus \mathscr{E}'^1 \otimes \mathscr{E}''^1$ and the Dirac operator on W is $D_W^0 = \omega D' \otimes \mathrm{id} + \omega \otimes D''$. (As we eventually compute using an orthogonal decomposition into finite dimensional eigenspaces, we do not worry about the topology in these tensor products.) Write spectral decompositions

$$\begin{aligned}
\mathscr{E}'^0 &= \bigoplus_{\lambda \in \operatorname{spec} \omega D'} \mathscr{E}'^0_\lambda \\
\mathscr{E}''^0 &= \bigoplus_{\mu \in \operatorname{spec}(D'')^2} \mathscr{E}''^0_{\mu^2} \\
\mathscr{E}''^1 &= \bigoplus_{\mu \in \operatorname{spec}(D'')^2} \mathscr{E}''^1_{\mu^2}.
\end{aligned} \tag{3.24}$$

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If $\mu^2 \neq 0$ then D^0_W acts on $\mathscr{E}'^0_\lambda \otimes \mathscr{E}''^0_{\mu^2} \oplus \mathscr{E}'^1_\lambda \otimes \mathscr{E}''^1_{\mu^2}$ with trace 0: we compute

$$D^0_W(\psi' \otimes \psi'' \pm \mu^{-1}\omega\psi' \otimes D''\psi'') = (\lambda \mp \mu)(\psi' \otimes \psi'' \mp \mu^{-1}\omega\psi' \otimes D''\psi'') \tag{3.25}$$

and let ψ', ψ'' run over orthonormal bases of $\mathscr{E}'^0_\lambda, \mathscr{E}''^0_{\mu^2}$, respectively. Choose a in (A9) to be less than zero and greater than the first negative eigenvalue of D_W. to conclude that the only contributions to the η-invariant of D^0_W come from $\mathscr{E}'^0 \otimes (\ker D'')^0$ and $\mathscr{E}'^1 \otimes (\ker D'')^1$. If $\psi' \in \mathscr{E}'^0_\lambda$ and $\psi''^i \in (\ker D'')^i$, $i = 0, 1$, then since $\omega D' = -D'\omega$ we compute

$$\begin{aligned} D^0_W(\psi' \otimes \psi''^0) &= \lambda(\psi' \otimes \psi''^0) \\ D^0_W(\omega\psi' \otimes \psi''^1) &= -\lambda(\omega\psi' \otimes \psi''^1) \end{aligned} \tag{3.26}$$

and (3.23) quickly follows. □

The next result is inspired by techniques in [5]. Suppose W is a closed pin$^+$ 12-manifold and $\pi\colon \widehat{W} \to W$ its orientation double cover. Let $\sigma\colon \widehat{W} \to \widehat{W}$ be the canonical orientation-reversing free involution. If $P \to W$ is the principal Pin^+_{12}-bundle of frames, then σ lifts canonically to an involution of $\pi^*P \to \widehat{W}$ which reverses the spin structure on $\widehat{W}$. Suppose that Z is a compact spin 13-manifold with boundary $\partial Z = \widehat{W}$ and σ' an orientation-reversing involution of Z which extends σ and is equipped with a lift to a spin-reversing involution of the Pin^+_{13}-bundle of frames. Let $F \subset Z$ denote the fixed point set of σ'. At an isolated fixed point $f \in F$ the action of σ' on $T_f Z$ is by -1, so its lift to the pin$^+$ frames acts by $\pm\omega$, where $\omega = \gamma^1\gamma^2\cdots\gamma^{13}$ is the volume form. Let $i_f = \pm 1$ denote the sign. If $V \to W$ is a real vector bundle, assume $\pi^*V \to \widehat{W}$ extends over Z and the involution σ' lifts, extending the lift of σ on the boundary. Let τ_f denote the trace of the lifted action at a fixed point $f \in F$.

Proposition 3.12 *If F consists of isolated points, then*

$$\tau_W(V) = \exp\left(2\pi i \sum_{f \in F} \frac{i_f \tau_f}{2^8}\right). \tag{3.27}$$

If V is the trivial real line bundle, then this is [50, Proposition 5.3], which is based on the general equivariant index theorem [21, Theorem 1.2] for manifolds with boundary. Donnelly's theorem identifies the contribution at a fixed point in terms of an asymptotic expansion of a heat kernel. The general cohomological expression for that contribution appears in [7, (3.9)] in the context of the general Lefschetz theorem, and it applies to fixed point manifolds of positive dimension as well as isolated fixed points. That this is the correct fixed point contribution in Donnelly's theorem is proved in [22] for the signature operator. We use it for the Dirac operator and an orientation-reversing isometry in Section 6.2.5.

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4. Cubic forms and the *C*-field

4.1. *Motivation: spin^c manifolds*

Recall that the compact Lie group Spin^c_n is a group extension

$$1 \longrightarrow \mathbb{T} \longrightarrow \mathrm{Spin}^c_n \longrightarrow SO_n \longrightarrow 1 \tag{4.1}$$

where $\mathbb{T}$ is the circle group of complex numbers of unit norm; it is defined as the quotient $(\mathrm{Spin}_n \times \mathbb{T})/\{\pm 1\}$. Let M be an n-dimensional *spin^c* manifold. A *spin^c structure* on M is a principal Spin^c_n-bundle $\mathcal{B}_{\mathrm{Spin}^c}(M) \to M$ together with an isomorphism $\mathcal{B}_{\mathrm{Spin}^c}(M)/\mathbb{T} \cong \mathcal{B}_{SO}(M)$ with the principal SO_n-bundle of oriented orthonormal frames. The $\mathbb{T}$-bundle over M associated with the homomorphism $\mathrm{Spin}^c_n \to \mathbb{T}$ is called the *characteristic bundle*, and its first Chern class $c \in H^2(M;\mathbb{Z})$ is an integer lift of the second Stiefel–Whitney class:

$$c \equiv w_2(M) \pmod 2. \tag{4.2}$$

Furthermore, any other *spin^c* structure is obtained by 'tensoring' with a circle bundle $Q \to M$ using the homomorphism $\mathrm{Spin}^c_n \times \mathbb{T} \to \mathrm{Spin}^c_n$; the characteristic class of the new *spin^c* structure is $c + 2x$, where $x = c_1(Q)$. Finally, there is an involution on *spin^c* structures which inverts the characteristic bundle and so changes the sign of c.

Suppose $n = \dim M$ is even and M is compact without boundary. The ($\mathbb{Z}/2\mathbb{Z}$-graded) complex spin representation of Spin^c_n gives rise to a Dirac operator D_M whose index is a topological invariant. It is computed by the Atiyah–Singer formula

$$\mathrm{index}\, D_M = \langle \hat{A}(M) e^{c/2},\, [M] \rangle, \tag{4.3}$$

where $\hat{A}(M) = 1 - p_1(M)/24 + \dots$ and $[M]$ is the fundamental class of M. As a function $\kappa(c)$ of the characteristic class c it is a polynomial, which for $n = 4, 6$ may be written

$$\kappa_2(c) = \frac{c^2 - \sigma(M)}{8}, \tag{4.4}$$

$$\kappa_3(c) = \frac{c^3 - p_1(M)c}{48}. \tag{4.5}$$

The subscript indicates the degree of the polynomial, $\sigma(M)$ is the signature of the 4-manifold M and we omit evaluation on $[M]$ from the notation for convenience. One may continue to $n = 8, 10, \dots$ to obtain polynomials of higher degree. These polynomials satisfy a symmetry property:

$$\kappa_2(-c) = \kappa_2(c), \qquad \kappa_3(-c) = -\kappa_3(c). \tag{4.6}$$

For a fixed characteristic element c define $q^c : H^2(M;\mathbb{Z}) \to \mathbb{Z}$ as

$$q^c(x) = \kappa(c + 2x) - \kappa(c), \qquad x \in H^2(M;\mathbb{Z}). \tag{4.7}$$

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For $n = 4, 6$ we find

$$q_2^c(x) = \frac{1}{2}(x^2 + cx) \tag{4.8}$$

$$\begin{aligned} q_3^c(x) &= \frac{1}{24}\big(p_1(M)x + 4x^3 + 6cx^2 + 3c^2x\big) \\ &= \frac{1}{6}x^3 + \dots \end{aligned} \tag{4.9}$$

Note q_2^c is a quadratic refinement of the intersection pairing on $H^2(M^4;\mathbb{Z})/\text{torsion}$, and q_3^c is a cubic refinement of the symmetric trilinear form on $H^2(M^6;\mathbb{Z})/\text{torsion}$.

The general mathematical problem suggested here is: Replace c by a cohomology class of arbitrary even degree and extend the topological invariants (4.4), (4.5). Of course, one may pose this as well for the higher degree polynomials of c deduced from the index formula (4.3) in higher dimensions. In the quadratic case, we have $n = 4k$ for some $k \in \mathbb{Z}^{>0}$ and $c \in H^{2k}(M;\mathbb{Z})$ lies in the middle degree. The associated topological invariant was investigated by Brown [12] and Browder [11]. In this instance c is an integer lift of the middle Wu class $\nu_{2k} \in H^{2k}(M;\mathbb{Z}/2\mathbb{Z})$, which may or may not exist. Corresponding geometric invariants were constructed in [33]. We take up the next interesting case—the cubic form for $n = 12$ and $\deg c = 4$—which appears in the action of the C-field in M-theory.

4.2. *Algebraic theory of cubic forms*

We begin with a review of the algebraic theory of quadratic forms. Let L be a finitely generated free abelian group and $\langle \cdot, \cdot \rangle : L \times L \to \mathbb{Z}$ a non-degenerate (that is, unimodular) symmetric bilinear form. The non-degeneracy implies the existence of a unique element $\bar{c} \in L \otimes \mathbb{Z}/2\mathbb{Z}$ such that

$$\langle \bar{x}, \bar{x} \rangle \equiv \langle \bar{c}, \bar{x} \rangle \pmod 2, \qquad \bar{x} \in L \otimes \mathbb{Z}/2\mathbb{Z}, \tag{4.10}$$

since the left-hand side is linear in $\bar{x}$. An element $c \in L$ with $c \equiv \bar{c} \pmod 2$ is called *characteristic*. The set $L_{\text{char}} \subset L$ of characteristic elements is a torsor for L: if $c \in L_{\text{char}}$ and $x \in L$ then $c + 2x \in L_{\text{char}}$. Now an easy check shows that $\langle c, c \rangle \pmod 8$ is independent of $c \in L_{\text{char}}$, so for any integer lift $\sigma \in \mathbb{Z}$ of $\langle c, c \rangle \pmod 8$,

$$\kappa_2(c) = \frac{\langle c, c \rangle - \sigma}{8} \tag{4.11}$$

is an integer. It is a standard result [48, Chapter 5] that σ may be chosen to be the signature of $\langle \cdot, \cdot \rangle$, defined by extending the form to the real vector space $L \otimes \mathbb{R}$. This is the algebraic theory which underlies (4.4). Note $\kappa_2(-c) = \kappa_2(c)$.

We develop a similar theory for the cubic (4.5). Consider the triple $(L, \langle \cdot, \cdot, \cdot \rangle, \bar{c})$ where L is a finitely generated free abelian group, $\langle \cdot, \cdot, \cdot \rangle : L \times L \times L \to \mathbb{Z}$ is a symmetric trilinear form and $\bar{c} \in$

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$L \otimes \mathbb{Z}/2\mathbb{Z}$ is assumed to satisfy

$$\langle \bar{c}, \bar{x}, \bar{y} \rangle \equiv \langle \bar{x}, \bar{x}, \bar{y} \rangle + \langle \bar{x}, \bar{y}, \bar{y} \rangle \pmod 2, \qquad \bar{x}, \bar{y} \in L \otimes \mathbb{Z}/2\mathbb{Z}. \tag{4.12}$$

(Equation (4.12) for trilinear forms appears in Postnikov's study [**44**] of the mod 2 cohomology ring of a closed 3-manifold, for example.) As we do not know a notion of non-degeneracy for trilinear forms which guarantees the existence of $\bar{c}$, we postulate its existence. Define the torsor $L_{\text{char}} \subset L$ of characteristic elements as above. Let $L^* = \text{Hom}(L, \mathbb{Z})$ and for convenience write the trilinear form as a simple product.

LEMMA 4.1 *There exists a unique $\hat{p} \in L^* \otimes \mathbb{Z}/24\mathbb{Z}$ such that*

$$\hat{p} \cdot \hat{x} \equiv 4\hat{x}^3 + 6\hat{c}\hat{x}^2 + 3\hat{c}^2\hat{x} \pmod{24} \tag{4.13}$$

for all $\hat{x} \in L \otimes \mathbb{Z}/24\mathbb{Z}$ and mod 24 reductions $\hat{c}$ of characteristic elements $c \in L_{char}$.

Proof. Use (4.12) to check that, as a function of $\hat{x}$, the right-hand side of (4.13) defines a homomorphism $L \otimes \mathbb{Z}/24\mathbb{Z} \to \mathbb{Z}/24\mathbb{Z}$. □

LEMMA 4.2 *Let $p \in L^*$ satisfy $p \equiv \hat{p} \pmod{24}$. Then*

$$\frac{c^3 - p \cdot c}{24} \pmod 2 \tag{4.14}$$

lies in $\mathbb{Z}/2\mathbb{Z}$ and is independent of $c \in L_{char}$. Furthermore, there exist lifts $p \in L^*$ of $\hat{p}$ such that this invariant vanishes, in which case

$$\kappa_3(c) = \frac{c^3 - p \cdot c}{48} \tag{4.15}$$

is an integer. Also, $\kappa_3(-c) = -\kappa_3(c)$.

Proof. To check the independence of $c \in L_{\text{char}}$, replace c in (4.14) with $c + 2x$ for $x \in L$ and use the fact that cx^2 is even, which follows from (4.12). To see that the fraction in (4.14) is an integer, use (4.13) and the fact that c^3 is even, which also follows from (4.12). To find the lift p, if $\bar{c} = 0$, then any p works since we can compute (4.14) using $c = 0$. If $\bar{c} \neq 0$, and if for a chosen lift p the invariant (4.14) is non-zero, choose $x^* \in L^*$ such that $x^* \cdot c$ is odd for any characteristic c and replace p with $p + x^*$. □

4.3. *The cubic form on spin 12-manifolds*

In (4.5), we gave an example of the cubic form (4.15) for a closed oriented 6-manifold M^6, where $L = H^2(M; \mathbb{Z})/\text{torsion}$, $\langle x, y, z \rangle = \langle x \smile y \smile z, [M] \rangle$, $\bar{c} = w_2(M)$ and $p = p_1(M)$. We now consider a

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closed *spin* 12-manifold W^{12} and set

$$\begin{aligned} L &= H^4(W;\mathbb{Z})/\text{torsion} \\ \langle x,y,z\rangle &= \langle x \smile y \smile z,\, [W]\rangle \\ \bar{c} &= w_4(W). \end{aligned} \tag{4.16}$$

REMARK 4.3 Let $T^4 \subset H^4(W;\mathbb{Z})$ denote the torsion subgroup, which fits into the exact sequence

$$0 \longrightarrow T^4 \longrightarrow H^4(W;\mathbb{Z}) \longrightarrow L \longrightarrow 0. \tag{4.17}$$

Tensoring with $\mathbb{Z}/2\mathbb{Z}$ defines a homomorphism $H^4(W;\mathbb{Z}) \longrightarrow H^4(W;\mathbb{Z}/2\mathbb{Z})$, and the precise definition of $\bar{c}$ is the image of $w_4(W)$ under the quotient map

$$H^4(W;\mathbb{Z}/2\mathbb{Z}) \longrightarrow H^4(W;\mathbb{Z}/2\mathbb{Z})/(T^4 \otimes \mathbb{Z}/2\mathbb{Z}). \tag{4.18}$$

In the classifying space *BSpin* there is a characteristic class $\lambda \in H^4(B\text{Spin};\mathbb{Z})$ such that (i) $2\lambda = p_1$ and (ii) the image of λ under $H^4(B\text{Spin};\mathbb{Z}) \to H^4(B\text{Spin};\mathbb{Z}/2\mathbb{Z})$ is w_4. A spin manifold W has a corresponding integer characteristic class $\lambda(W)$. The existence of this integer lift of $w_4(W)$ implies that the image of $w_4(W)$ under (4.18) lies in the subgroup $\left(H^4(W;\mathbb{Z})/T^4\right) \otimes \mathbb{Z}/2\mathbb{Z} = L \otimes \mathbb{Z}/2\mathbb{Z}$. The computations below are written in $H^\bullet(W;\mathbb{Z}/2\mathbb{Z})$, but the results should be interpreted in terms of this subquotient. (To do so, use the fact that torsion integer cohomology classes evaluate trivially on the fundamental class.)

LEMMA 4.4 *The Stiefel–Whitney class* $\bar{c} = w_4(W)$ *of a closed spin* 12*-manifold* W *satisfies* (4.12).

The proof uses the Cartan formula and Adem relations for Steenrod squares, as well as the Wu formula, which states that on a closed n-manifold M there is a class $\nu_i(M) \in H^i(M;\mathbb{Z}/2\mathbb{Z})$ such that squaring to the top,

$$Sq^i\colon H^{n-i}(M;\mathbb{Z}/2\mathbb{Z}) \longrightarrow H^n(M;\mathbb{Z}/2\mathbb{Z}), \tag{4.19}$$

is cup product with $\nu_i(M)$. In low degrees we have

$$\begin{aligned} \nu_1 &= w_1 \\ \nu_2 &= w_1^2 + w_2 \\ \nu_3 &= w_1 w_2 \\ \nu_4 &= w_4 + w_1 w_3 + w_2^2 + w_1^4 \end{aligned} \tag{4.20}$$

in terms of the Stiefel–Whitney classes of the *tangent* bundle. (See [**33**, Section E.1.1] for characteristic properties of Wu classes from which (4.20) may be computed.) The Bockstein β is defined as the connecting homomorphism induced from the coefficient sequence $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$:

$$\cdots \longrightarrow H^i(M;\mathbb{Z}) \xrightarrow{r} H^i(M;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\beta} H^{i+1}(M;\mathbb{Z}) \xrightarrow{2} H^{i+1}(M;\mathbb{Z}) \longrightarrow \cdots \tag{4.21}$$

Also, $Sq^1 = r \circ \beta$. It follows that if x is an integer cohomology class, Sq^1 vanishes on its mod 2 reduction $\bar{x} = r(x)$.

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Proof. If $x, y \in H^4(W;\mathbb{Z})$ and $\bar{x}, \bar{y} \in H^4(W;\mathbb{Z}/2\mathbb{Z})$ are their mod 2 reductions, then (4.20) with $w_1 = w_2 = 0$ implies

$$\begin{aligned} w_4(W)\bar{x}\bar{y} &= Sq^4(\bar{x}\bar{y}) \\ &= (Sq^4\bar{x})\bar{y} + Sq^2\bar{x}\,Sq^2\bar{y} + \bar{x}\,Sq^4\bar{y} \\ &= \bar{x}\bar{x}\bar{y} + Sq^2\bar{x}\,Sq^2\bar{y} + \bar{x}\bar{y}\bar{y}, \end{aligned} \tag{4.22}$$

since Sq^1 vanishes on reductions of integer classes. Then from (4.20) again

$$\begin{aligned} 0 &= w_2(W)\bar{x}Sq^2\bar{y} \\ &= Sq^2(\bar{x}\,Sq^2\bar{y}) \\ &= Sq^2\bar{x}\,Sq^2\bar{y} + \bar{x}Sq^2Sq^2\bar{y} \\ &= Sq^2\bar{x}\,Sq^2\bar{y} + \bar{x}Sq^3Sq^1\bar{y} \\ &= Sq^2\bar{x}\,Sq^2\bar{y}. \end{aligned} \tag{4.23}$$

□

PROPOSITION 4.5 *In $B\,\mathrm{Spin}$ there is a unique characteristic class $p \in H^8(B\,\mathrm{Spin};\mathbb{Z})$ with*

$$2p = p_2 - \lambda^2. \tag{4.24}$$

Furthermore, $p \equiv w_8 \pmod 2$.

For a smooth manifold M we obtain a characteristic class $p(M) \in H^8(M;\mathbb{Z})$, and we use the same symbol to denote its reduction modulo torsion.

Proof. $H^8(B\mathrm{Spin};\mathbb{Z})$ is torsionfree and $p_2 \equiv \lambda^2 \equiv w_4^2 \pmod 2$, which proves the existence and uniqueness of p. To compute its reduction mod 2 we restrict to the classifying space of a maximal torus of $Spin_N$ for $N \geq 8$. The computation is carried out in [**13**, Section 3], where p is the class called '$-q_2$.' Its reduction mod 2 equals the reduction of the class called 'c_4,' which is the Stiefel–Whitney class w_8. □

REMARK 4.6 Hopkins–Singer [**33**, Appendix E] define spin Wu classes in $H^\bullet(B\mathrm{Spin};\mathbb{Z})$, in terms of which we have $p = \nu_8^{\mathrm{Spin}} - 2p_2 + p_1^2$.

PROPOSITION 4.7 *On a closed spin 12-manifold W the mod 24 reduction of $p(W)$ satisfies* (4.13).

Proof. We follow Witten's argument in [**57**, Section 4]. Namely, a principal E_8-bundle over a 12-manifold is determined up to isomorphism by an element $x \in H^4(W;\mathbb{Z})$. Let $V(x)$ denote the (real) adjoint vector bundle to the principal E_8-bundle with characteristic class x, and set $c = \lambda(W) + 2x$.

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The Chern character of $V(x)\to W$ is

$$\operatorname{ch} V(x) = 248 - 60x + 6x^2 - \frac{1}{3}x^3. \tag{4.25}$$

To see this, observe that *a priori* the Chern character is a cubic polynomial in x, so we need only determine the coefficients. The restriction of the adjoint representation of E_8 to $\mathrm{Spin}_{16} \subset E_8$ is the sum of a half-spin representation and the adjoint representation of $Spin_{16}$. The restriction of its complexification to $\mathrm{Spin}_3 \subset \mathrm{Spin}_{16}$ is $78V_1 \oplus 64V_2 \oplus 14V_3$, where V_n is the n-dimensional irreducible representation of $\mathrm{Spin}_3 \cong SU_2$; the Chern character of this representation is easily computed. Finally, the generator of $H^4(BE_8;\mathbb{Z})$ restricts to minus twice the generator of $H^4(BSU_2;\mathbb{Z})$. (The generator of $H^4(BSO_{16};\mathbb{Z})$ restricts to the generator of $H^4(BSO_3;\mathbb{Z})$. The former pulls back to twice the generator of $H^4(B\mathrm{Spin}_{16};\mathbb{Z})$, whereas the latter pulls back to minus four times the generator of $H^4(BSU_2;\mathbb{Z})$.) Then a long computation verifies the following identity:

$$\left\langle \frac{c^3 - pc}{48} + \frac{1}{2}\hat{A}(W)\operatorname{ch} V(x) + \frac{1}{4}\hat{A}(W)\operatorname{ch}(TW - 4), [W] \right\rangle = 0. \tag{4.26}$$

The second term is an integer; it is the *KO*-theory direct image of the real bundle $V(x)$, defined using the spin structure, which by the Atiyah–Singer index theorem is the index of the Dirac operator coupled to $V(x)$. Similarly, the last term is a half-integer, hence so is the cubic expression. Replace the denominator in the cubic expression by 24 to obtain an integer, and now subtract the integers for arbitrary x and $x = 0$ to establish the congruence

$$\big(p_2(W) - \lambda(W)^2\big)x \equiv 8x^3 + 12\lambda(W)x^2 + 6\lambda(W)^2 x \pmod{24}, \tag{4.27}$$

where we omit evaluation on $[W]$ from the notation for convenience. If necessary, use the last argument in Lemma 4.2 to replace $p(W)$ by $p' = p(W) + 24a$ for $a \in H^8(W;\mathbb{Z})/\text{torsion}$ so that

$$\frac{c^3 - p'c}{48} \in \mathbb{Z}, \qquad c = \lambda(W) + 2x, \quad x \in H^4(W;\mathbb{Z}), \tag{4.28}$$

and so deduce the desired mod 24 congruence. □

Note that $p(W)$ is *not* necessarily a distinguished lift of $\hat{p}$ described in Lemma 4.2; rather we need to add the constant term $1/4RS'(W)$ (see (3.18)) in (4.26) is needed to obtain integrality. Define the integer-valued cubic form

$$\kappa_W(c) = \frac{c^3 - p(W)c}{48} + \frac{1}{2}RS'(W) \tag{4.29}$$

on characteristic elements; it satisfies a shifted version of the symmetry (4.6):

$$\kappa_W(-c) = RS'(W) - \kappa_W(c). \tag{4.30}$$

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4.4. *The cubic form on pin$^+$ 12-manifolds*

Any manifold M has a canonical *orientation double cover* $\widehat{M} \to M$: the fiber at $m \in M$ is the set of orientations on T_mM. There results a canonical local system $\widetilde{\mathbb{Z}} \to M$ of coefficients; we call $H^\bullet(M;\widetilde{\mathbb{Z}})$ the *w_1-twisted cohomology*. An orientation is a trivialization of $\widetilde{\mathbb{Z}} \to M$, and on an oriented manifold w_1-twisted integer cohomology reduces to untwisted integer cohomology. The fundamental class $[M]$ of a closed manifold M lives in w_1-twisted integer homology, so we can integrate w_1-twisted cohomology classes.

LEMMA 4.8 *Let M be a closed n-manifold with no orientable components and $\pi\colon \widehat{M} \to M$ the orientation double cover. Then the image of*

$$\pi^*\colon H^n(M;\widetilde{\mathbb{Z}}) \longrightarrow H^n(\widehat{M};\mathbb{Z}) \tag{4.31}$$

is $2H^n(\widehat{M};\mathbb{Z})$, and if $\bar{\omega} \in H^n(M;\widetilde{\mathbb{Z}})$, then

$$\langle \bar{\omega}, [M] \rangle = \left\langle \frac{1}{2}\pi^*\bar{\omega}, [\widehat{M}] \right\rangle. \tag{4.32}$$

As the domain and codomain of (4.31) are torsionfree, we can prove Lemma 4.8 using de Rham theory, a task we leave to the reader.

Let W be a closed pin$^+$ 12-manifold W. The existence of a pin$^+$ structure on W is equivalent to $w_2(W) = 0$, but in general $w_1(W) \neq 0$. Also, $w_3(W) = 0$ since $w_3 = Sq^1 w_2 + w_1 w_2$ (Wu formula). Note then that the Wu classes (4.20) simplify to $\nu_2(W) = w_1(W)^2$ and $\nu_4(W) = w_4(W) + w_1(W)^4$. There is a short exact sequence of coefficients $0 \to \widetilde{\mathbb{Z}} \xrightarrow{2} \widetilde{\mathbb{Z}} \to \mathbb{Z}/2\mathbb{Z} \to 0$, and the connecting homomorphism in the resulting long exact sequence—(4.21) with twisted coefficients—is the twisted Bockstein $\tilde{\beta}$. In this case $r \circ \tilde{\beta} = Sq^1 + w_1$, so that if x is a w_1-twisted integer class then

$$Sq^1 \bar{x} = w_1(M) \smile x. \tag{4.33}$$

For a closed $\mathfrak{m}_c$ 12-manifold W we modify (4.16) to

$$\begin{aligned} L &= H^4(W;\widetilde{\mathbb{Z}})/\text{torsion} \\ \langle x,y,z \rangle &= (x \smile y \smile z)[W] \\ \bar{c} &= w_4(W). \end{aligned} \tag{4.34}$$

Remark 4.3 applies if we replace integer cohomology with w_1-twisted integer cohomology and assume W is an $\mathfrak{m}_c$-manifold. The dual lattice $L^* = H^8(W;\mathbb{Z})/\text{torsion}$ is untwisted integer cohomology as in the spin case.

PROPOSITION 4.9 *In B*Pin$^+$ *there is a unique characteristic class $\bar{p} \in H^8(BPin^+;\mathbb{Z})/$torsion whose restriction to B*Spin *is the class p of Proposition 4.5.*

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Proof. Let $\{E_r^{p,q}\}$ denote the Leray–Serre spectral sequence for the fibration

$$B\mathrm{Spin} \longrightarrow B\mathrm{Pin}^+ \xrightarrow{w_1} \mathbb{RP}^\infty. \tag{4.35}$$

Then $E_2^{0,8} \cong H^8(B\mathrm{Spin};\mathbb{Z})$ and $E_\infty^{0,8} \cong H^8(B\mathrm{Pin}^+;\mathbb{Z})/\text{torsion} \cong \ker(d_2\colon E_2^{0,8} \to E_2^{2,7})$. Note that $E_2^{2,7} \cong \mathbb{Z}/2\mathbb{Z}$, since $H^7(B\mathrm{Spin};\mathbb{Z})$ is cyclic of order 2, generated by the integer Bockstein of w_6, and $H^2\big(\mathbb{RP}^\infty;\mathbb{Z}/2\mathbb{Z}\big) \cong \mathbb{Z}/2\mathbb{Z}$. The proposition follows from $d_2(p) = 0$, which in turn follows since $d_2(p)$ is detectable mod 2 and $p \pmod 2 = w_8$ survives the differentials. □

Remark 4.10 There is a (homotopy) splitting of the map w_1 in (4.35), namely the classifying map $\mathbb{RP}^\infty \to BO$ of the reduced tautological bundle $H \to \mathbb{RP}^\infty$, which lifts since H has a pin^+ structure. (Introduce an inner product on $H \to \mathbb{RP}^\infty$ and use a splitting of the homomorphism $\mathrm{Pin}_1^+ \to O_1$.) Then the product map

$$B\mathrm{Spin} \times \mathbb{RP}^\infty \to B\mathrm{Pin}^+ \tag{4.36}$$

is a homotopy equivalence, since it induces an isomorphism on homotopy groups. This yields an isomorphism $H^\bullet(B\mathrm{Pin}^+;\mathbb{Z})/\text{torsion} \to H^\bullet(B\mathrm{Spin};\mathbb{Z})/\text{torsion}$, which re-proves Proposition 4.9.

Proposition 4.11 *Let W be a closed $\mathfrak{m}_c$ 12-manifold. Then $\bar{c} = w_4(W)$ satisfies equation (4.12). Furthermore, the mod 24 reduction of $\bar{p}(W)$, viewed as a class in $L^* = H^8(W;\mathbb{Z})/\text{torsion}$, satisfies the condition in Lemma 4.1.*

Proof. We modify the proof of Lemma 4.4. So (4.22) becomes

$$\big(w_4(W) + w_1^4(W)\big)\bar{x}\bar{y} = \bar{x}\bar{x}\bar{y} + \bar{x}\bar{y}\bar{y} + w_1(W)(\bar{x}\,Sq^3\bar{y} + \bar{y}\,Sq^3\bar{x}) + Sq^2\bar{x}\,Sq^2\bar{y} \tag{4.37}$$

and (4.23) becomes

$$w_1^2(W)\bar{x}\,Sq^2\bar{y} = Sq^2\bar{x}\,Sq^2\bar{y} + w_1(W)\bar{x}Sq^3\bar{y} + \bar{x}w_1^2(W)Sq^2\bar{y} + \bar{x}w_1(W)Sq^3\bar{y}, \tag{4.38}$$

which implies $Sq^2\bar{x}\,Sq^2\bar{y} = 0$. Then using $\nu_3(W) = 0$ from (4.20), we find

$$\begin{aligned} 0 = Sq^3(w_1(W)\bar{x}\bar{y}) &= w_1^2(W)Sq^2(\bar{x}\bar{y}) \;+\; w_1(W)Sq^3(\bar{x}\bar{y}) \\ &= w_1^4(W)\bar{x}\bar{y} \;+\; w_1(W)\big(\bar{x}Sq^3\bar{y} + \bar{y}Sq^3\bar{x}\big). \end{aligned} \tag{4.39}$$

Combine these equations to complete the proof that $\bar{c}$ satisfies (4.12).

For the last statement in the proposition we observe that Proposition 4.9 implies $\bar{p}(W) = p(W)$ if W is spin, and also if $\pi\colon \widehat{W} \to W$ is the orientation double cover then $\pi^*\bar{p}(W) = p(\widehat{W})$. The last statement reduces to Proposition 4.7 on orientable components of W, and on non-orientable components we use Lemma 4.8 to reduce to Proposition 4.7 on the orientation double cover. □

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LEMMA 4.12 *Let W be a closed $\mathfrak{m}_c$ 12-manifold and $\tilde{c} \in H^4(W;\widetilde{\mathbb{Z}})$ a w_1-twisted integer lift of $w_4(W)$. Then*

$$\frac{\tilde{c}^3 - \bar{p}(W)\tilde{c}}{48} \pmod{\mathbb{Z}} \tag{4.40}$$

lies in $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$, is independent of the choice of $\tilde{c}$, and is a bordism invariant of $\mathfrak{m}_c$-manifolds. It is additive under disjoint union.

Proof. That the fraction in (4.40) is a half-integer follows from Lemma 4.2 in the algebraic theory of cubic forms. Any w_1-twisted integer lift of $w_4(M)$ has the form $\tilde{c} + 2\tilde{x}$ for some $\tilde{x} \in H^4(W;\widetilde{\mathbb{Z}})$, and an easy check from (4.13) proves that (4.40) is unchanged by the replacement. If $W = \partial Z$ is the boundary of a compact $\mathfrak{m}_c$ 13-manifold Z, then Z has a fundamental class in relative homology and the usual adjunction (integer Stokes' theorem) argument implies that (4.40) vanishes, even before reducing modulo $\mathbb{Z}$. □

Define

$$\hat{\alpha}_C(W) = \exp\left(2\pi i\, \frac{\tilde{c}^3 - \bar{p}(W)\tilde{c}}{48}\right). \tag{4.41}$$

Recall that $M\mathfrak{m}_c$ is the bordism spectrum of pin$^+$ manifolds with an $\mathfrak{m}_c$ structure.

COROLLARY 4.13 *The exponential of the cubic form factors through a homomorphism*

$$\hat{\alpha}_C\colon \pi_{12}M\mathfrak{m}_c \longrightarrow \mathbb{C}^\times \tag{4.42}$$

which takes values in $\{\pm 1\} \subset \mathbb{C}^\times$.

As discussed in Section 2, the homomorphism (3.13) determines an invertible topological field theory

$$\alpha_C\colon M\mathfrak{m}_c \longrightarrow \Sigma^{12} I\mathbb{C}^\times \tag{4.43}$$

up to isomorphism. The square $\alpha_C^{\otimes 2}$ is isomorphic to the trivial theory.

REMARK 4.14 Let $\pi\colon \widehat{W} \to W$ be the orientation double cover of an $\mathfrak{m}_c$-manifold which has no orientable components, and suppose $\tilde{c} \in H^4(W;\widetilde{\mathbb{Z}})$ is a w_1-twisted integer lift of $w_4(W)$. Set $c = \pi^*\tilde{c} \in H^4(\widehat{W};\mathbb{Z})$. As in the proof of Proposition 4.11 we have $p(\widehat{W}) = \pi^*\bar{p}(W)$. Apply Lemma 4.8 to evaluate the *integer* cubic form—twice (4.40)—on the orientation double cover:

$$\left\langle \frac{\tilde{c}^3 - \bar{p}(W)\tilde{c}}{24},\ [W] \right\rangle = \left\langle \frac{c^3 - p(W)c}{48},\ [\widehat{W}] \right\rangle. \tag{4.44}$$

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4.5. *The C-field and its anomaly; cancellation on spin manifolds*

The C-field in M-theory is an example of an abelian gauge field. Classically, all information is captured by its field strength Ω, which is a closed 4-form. In the quantum theory Dirac's quantization of charge applies: the de Rham cohomology class of Ω is constrained to lie in a full lattice in the degree 4 real cohomology. There is more information, as inspired by the Aharanov–Bohm effect in the case of ordinary electromagnetism and the resulting refinement of the electromagnetic field—a closed 2-form—to a connection on a principal $\mathbb{T}$-bundle. In higher degrees a suitable language for quantum abelian gauge fields is *differential cohomology*, which is developed in [**33**] in part to model the C-field; the focus there is on the M5 brane and so on a quadratic form. (See [**24**, Section 3] for a general exposition of abelian gauge fields as differential cocycles.) Here we work in the 'bulk' on a Wick-rotated spacetime which we take to be an 11-dimensional Riemannian pin$^+$ manifold X. Dirac's quantization of charge for the C-field, which is determined in [**57**], is encoded by positing the C-field as a geometric representative of a w_1-twisted differential cohomology class which lifts $w_4(X)$. (If X is spin, then there is a model [**18**] in terms of E_8-bundles, as in the proof of Proposition 4.7.) Locally C-fields exist but there is a global obstruction, as explained after Definition 2.1. In that spirit, a C-field $\breve{\Omega}$ is a *differential* $\mathfrak{m}_c$ *structure* on X; a precise model is established in [**33**], where it is termed a *differential integral Wu structure*. Its field strength Ω is a closed w_1-twisted 4-form whose de Rham cohomology class in $H^4(X;\widetilde{\mathbb{R}})$ is the real image of a w_1-twisted integer lift $\tilde{c} \in H^4(X;\widetilde{\mathbb{Z}})$ of $w_4(X)$. (The form Ω lifts to a closed 4-form $\widehat{\Omega}$ on the total space of the orientation double cover $\widehat{X} \to X$; then $\sigma^*\widehat{\Omega} = -\widehat{\Omega}$, where $\sigma\colon \widehat{X} \to \widehat{X}$ is the non-trivial deck transformation.)

The effective action of M-theory has a cubic term of the form

$$\exp\left(2\pi i\,\frac{\breve{\Omega}^3 - \breve{p}(X)\breve{\Omega}}{48}\right), \tag{4.45}$$

where $\breve{p}(X)$ is a lift to differential cohomology of the class $\bar{p}(X) \in H^8(X;\mathbb{Z})/\text{torsion}$. This differential cohomology version of the cubic form is analogous to a Chern–Simons invariant. We do not need its precise definition, so will not elaborate further.

Remark 4.15 The $\breve{\Omega}^3$ term in (4.45) is part of the classical 11-dimensional supergravity action [**15**]. The $\breve{p}(X)\breve{\Omega}$ term is a quantum correction, introduced in [**20**, (3.14)] in the spin case, in part inspired by [**54**, Section 3] who introduce an analogous correction in the Type IIA superstring. We do not know of any literature about this quantum correction in the *pin*$^+$ case.

Remark 4.16 We have only defined the class $\breve{p}$ in (4.45) up to an element of $H^7(B\text{Pin}^+;\mathbb{R}/\mathbb{Z})$, but we now argue that (4.45) is independent of the lift. First, $H^7(B\text{Pin}^+;\mathbb{R}/\mathbb{Z}) \cong H^8(B\text{Pin}^+;\mathbb{Z})_{\text{tor}}$, since $H^7(B\text{Pin}^+;\mathbb{R}) = 0$. ($A_{\text{tor}}$ is the torsion subgroup of the abelian group A.) Recall from (4.36) that $B\text{Pin}^+ \simeq B\text{Spin} \times \mathbb{RP}^\infty$. Then the main theorem in [**37**] implies that $2H^8(B\text{Pin}^+;\mathbb{Z})_{\text{tor}} = 0$. Use the short exact sequence

$$0 \longrightarrow \frac{1}{2}\mathbb{Z}/\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z} \xrightarrow{2} \mathbb{R}/\mathbb{Z} \longrightarrow 0 \tag{4.46}$$

to deduce that $H^7(B\text{Pin}^+;\frac{1}{2}\mathbb{Z}/\mathbb{Z}) \longrightarrow H^7(B\text{Pin}^+;\mathbb{R}/\mathbb{Z})$ is surjective. It follows that the ambiguity in (4.45) is expressed as a characteristic number of mod 2 cohomology. Our Adams spectral sequence

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computation (see Fig. 1 in Section 8) shows that there is no element of $\pi_{11}M\mathfrak{m}_c$ in Adams filtration 0, and so every mod 2 characteristic number vanishes on closed 11-dimensional $\mathfrak{m}_c$-manifolds.

Note that there is an ambiguity in the M-theory action from a topological term which is not mod 2 characteristic numbers but rather a mod 2 *KO*-characteristic number, a mod 2 index of a Dirac operator (see Section 7).

Our focus is on anomalies, and here the crucial point is that only the square of (4.45) is unambiguously defined as an element of $\mathbb{C}$. This is equivalent to the assertion that on a closed $\mathfrak{m}_c$ 12-manifold W the cubic form

$$\frac{\tilde{c}^3 - \bar{p}(W)\tilde{c}}{24} \tag{4.47}$$

is integral, but is not necessarily even. Hence the square root (4.45) is an element of a complex line $\alpha_C(X)$ whose square $\alpha_C(X)^{\otimes 2}$ is trivialized. As the notation suggests, this line is the state space of the invertible 12-dimensional field theory α_C. The field theory α_C is topological: it does not depend on the Riemannian metric or differential $\mathfrak{m}_c$ structure, only on the underlying topological $\mathfrak{m}_c$ structure.

Witten's argument [**57**, Section 4], reproduced in the proof of Proposition 4.7, proves the Anomaly Cancellation Theorem 2.2 on spin manifolds. Let $M\mathrm{Spin}\langle\beta w_4\rangle$ denote the bordism spectrum of spin manifolds with an integer lift of w_4. There is a map $M\mathrm{Spin}\langle\beta w_4\rangle \to M\mathfrak{m}_c$, where $M\mathfrak{m}_c$ is the bordism spectrum of $\mathfrak{m}_c$-manifolds.

THEOREM 4.17 (Witten) *The lift* $\alpha_{RS} \otimes \alpha_C \colon M\mathrm{Spin}\langle\beta w_4\rangle \to \Sigma^{12} I\mathbb{C}^\times$ *is trivializable.*

The E_8-model for the C-field leads to a distinguished trivialization [**29**].

Proof. Because an invertible topological field theory is determined up to isomorphism by its partition functions, to prove Theorem 4.17 we show that for any closed spin 12-manifold W with an $\mathfrak{m}_c$ structure we have

$$\hat{\alpha}_{RS}(W)\hat{\alpha}_C(W) = 1. \tag{4.48}$$

This follows immediately from the integrality of (4.29) (see Proposition 3.5 and Remark 3.7). □

5. Some spin and pin manifolds

This section is a geometric interlude to review and introduce some special manifolds and their topological invariants. We use these manifolds as building blocks for the closed pin$^+$ 12-manifolds we need in Section 6, where we also specify $\mathfrak{m}_c$ structures.

If M is a smooth manifold, then we use the notations

$$\begin{aligned} w(M) &= 1 + w_1(M) + w_2(M) + \cdots \\ p(M) &= 1 + p_1(M) + p_2(M) + \cdots \end{aligned} \tag{5.1}$$

for the total Stiefel–Whitney class and total Pontrjagin class, respectively. The former satisfies the Whitney sum formula $w(M_1 \times M_2) = w(M_1)w(M_2)$ for Cartesian products; the analogous equation

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for the total Pontrjagin class is true modulo torsion. Also, these characteristic classes are defined for arbitrary real vector bundles, not just the tangent bundle, and are *stable* in the sense that they are unchanged by adding a trivial bundle. Recall also the characteristic class λ of a spin manifold, or of a real vector bundle with a spin structure, characterized after (4.18); it satisfies $2\lambda = p_1$.

5.1. *K3 surface*

There is a moduli space of inequivalent complex K3 surfaces whose underlying real 4-manifolds are all diffeomorphic. For definiteness, then, we define $K \subset \mathbb{CP}^3$ as the zero locus of the quartic

$$(z^0)^4 + (z^1)^4 + (z^2)^4 + (z^3)^4 = 0, \tag{5.2}$$

where z^0, z^1, z^2, z^3 are the standard homogeneous coordinates on $\mathbb{CP}^3$. It is a smooth closed real 4-manifold which is simply connected, and the complex structure induces an orientation. The Chern classes can be computed from those of $\mathbb{CP}^3$ and that of the normal bundle, which is the restriction of $\mathcal{O}(4) \to \mathbb{CP}^3$ to K, and from there we derive the Stiefel–Whitney and Pontrjagin classes:

$$\begin{aligned} w(K) &= 1 \\ p(K) &= 1 - 48k, \end{aligned} \tag{5.3}$$

where $k \in H^4(K;\mathbb{Z}) \cong \mathbb{Z}$ is the positive generator. In particular, $w_2(K) = 0$ and so K admits a spin structure compatible with the orientation, which is unique up to isomorphism since K is simply connected. Also,

$$\lambda(K) = -24k. \tag{5.4}$$

5.2. *Quaternionic projective plane*

Let $\mathbb{HP}^2$ denote the space of one-dimensional quaternionic subspaces of the quaternionic vector space $\mathbb{H}^3$. For definiteness we let the division algebra $\mathbb{H}$ act on the right of $\mathbb{H}^3$. In coordinates write

$$\mathbb{HP}^2 = \left\{[q^0, q^1, q^2] : q^i \in \mathbb{H}\right\} / \sim, \qquad [q^0, q^1, q^2] \sim [q^0 h, q^1 h, q^2 h], \quad h \in \mathbb{H}^{\neq 0}. \tag{5.5}$$

$\mathbb{HP}^2$ is a simply connected 8-manifold. In fact, the filtration $* \subset \mathbb{HP}^1 \subset \mathbb{HP}^2$ provides a CW structure with a single 0-cell, 4-cell and 8-cell. The simple connectivity implies that up to isomorphism $\mathbb{HP}^2$ has a unique spin structure compatible with a given orientation.

Let $L \to \mathbb{HP}^2$ be the tautological quaternionic line bundle; its fiber at a point $\ell \in \mathbb{HP}^2$ is the quaternionic line ℓ. There is a short exact sequence

$$0 \longrightarrow L \longrightarrow \underline{\mathbb{H}^3} \longrightarrow Q \longrightarrow 1 \tag{5.6}$$

of (right) quaternionic vector bundles; in the middle is the trivial bundle with fiber $\mathbb{H}^3$ and the sequence defines the rank two quotient bundle $Q \to \mathbb{HP}^2$. Note that the dual $L^* \cong \mathrm{Hom}_{\mathbb{H}}(L, \mathbb{H})$ is

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canonically a *left* $\mathbb{H}$-module. The tangent bundle is identified as the real vector bundle $\mathrm{Hom}_{\mathbb{H}}(L, Q) \cong Q \otimes_{\mathbb{H}} L^*$, and it is the quotient in the short exact sequence of real vector bundles

$$0 \longrightarrow L \otimes_{\mathbb{H}} L^* \longrightarrow \underline{\mathbb{H}^3} \otimes_{\mathbb{H}} L^* \longrightarrow Q \otimes_{\mathbb{H}} L^* \longrightarrow 0, \tag{5.7}$$

so its total Pontrjagin class is the quotient

$$p(\mathbb{HP}^2) = \frac{p(\underline{\mathbb{H}^3} \otimes_{\mathbb{H}} L^*)}{p(L \otimes_{\mathbb{H}} L^*)}, \tag{5.8}$$

since $H^\bullet(\mathbb{HP}^2; \mathbb{Z})$ is torsionfree. The quaternionic line bundle $L^* \to \mathbb{HP}^2$ is, by restriction of scalars to $\mathbb{C} \subset \mathbb{H}$, a rank 2 complex vector bundle isomorphic to its complex conjugate, so its total Chern class has the form $1 - x$, where $x \in H^4(\mathbb{HP}^2; \mathbb{Z})$; we call x the *quaternionic first Pontrjagin class*. Restrict to $\mathbb{HP}^1 \subset \mathbb{HP}^2$ and fix a non-zero quaternionic functional $\mathbb{H}^2 \to \mathbb{H}$ to define a section of $L^* \to \mathbb{HP}^1$ which vanishes transversely at a single point. It follows that x generates $H^4(\mathbb{HP}^2; \mathbb{Z})$. Now $L \otimes_{\mathbb{H}} L^*$ splits off a trivial real line bundle, and the orthogonal rank 3 bundle is the real adjoint bundle of the complex 2-plane bundle underlying $L^* \to \mathbb{HP}^2$; the first Pontrjagin class of the real adjoint bundle is $4x$. Therefore, from (5.8)

$$p(\mathbb{HP}^2) = \frac{(1+x)^6}{(1+4x)} = 1 + 2x + 7x^2. \tag{5.9}$$

(See [9, Section 15.5] for an alternative derivation.) It follows that

$$\lambda(\mathbb{HP}^2) = x, \qquad w_4(\mathbb{HP}^2) = \bar{x}, \tag{5.10}$$

where $\bar{x} \in H^4(\mathbb{HP}^2; \mathbb{Z}/2\mathbb{Z})$ is the mod 2 reduction of x.

REMARK 5.1 As mentioned above, a quaternionic line bundle $L \to X$ has a quaternionic first Pontrjagin class $p_1^{\mathbb{H}}(L) \in H^4(X; \mathbb{Z})$ which equals minus the second Chern class after restricting scalars to $\mathbb{C} \subset \mathbb{H}$. We can also restrict scalars to $\mathbb{R} \subset \mathbb{H}$ to obtain a rank 4 real vector bundle $L_{\mathbb{R}} \to X$, whose first Pontrjagin class satisfies $p_1(L_{\mathbb{R}}) = 2p_1^{\mathbb{H}}(L)$. The following general formula is useful, and can be used in the derivation of (5.9). Suppose $R, L \to X$ are right and left quaternionic line bundles with quaternionic first Pontrjagin classes $r, \ell \in H^4(X; \mathbb{Z})$. Then $R \otimes_{\mathbb{H}} L \to X$ is a real 4-plane bundle with total Pontrjagin class $1 + 2(r + \ell) + (r - \ell)^2$.

Use $x^2 \in H^8(\mathbb{HP}^2; \mathbb{Z}) \cong \mathbb{Z}$ to orient $\mathbb{HP}^2$: choose the fundamental class such that $\langle x^2, [\mathbb{HP}^2] \rangle = 1$.

5.3. *Bott manifold*

The bordism group $\pi_8 M\,\mathrm{Spin}$ is free abelian of rank two: there is an isomorphism

$$\begin{aligned} \pi_8 M\,\mathrm{Spin} &\longrightarrow \mathbb{Z} \oplus \mathbb{Z} \\ [M] &\longmapsto \big(\hat{A}(M), \sigma(M)\big) \end{aligned} \tag{5.11}$$

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which maps a closed spin 8-manifold to its $\hat{A}$-genus and its signature. The quaternionic projective plane has $\hat{A}(\mathbb{HP}^2) = 0$, $\sigma(\mathbb{HP}^2) = 1$. A closed spin manifold B with $\hat{A}(B) = 1$ is called a *Bott manifold*. We need not insist on vanishing signature, as that can always be achieved by connected sum with copies of $\mathbb{HP}^2$ or its orientation-reversal, and indeed the Bott manifold we use has signature -224.

We do not know of an elementary construction of a Bott manifold. One possibility is a Riemannian manifold B of special holonomy $\mathrm{Spin}_7 \subset \mathrm{Spin}_8$, which necessarily satisfies $\hat{A}(B) = 1$ and is simply connected (see [**34**, Section 10.6]). Closed 8-manifolds with Spin_7 holonomy were first produced by Joyce. A more topological approach leans on the work of Kervaire and Milnor [**42**], [**36**]. The Bott manifold B so constructed is also simply connected, so admits a unique spin structure. Briefly, plumb together eight copies of the disk bundle of the tangent bundle to S^4 according to the E_8 Dynkin diagram. The resulting compact 8-manifold N has a boundary which is an exotic 7-sphere. The Kervaire–Milnor results imply that a connect sum of 28 copies of the exotic sphere bounds a ball, hence we define B as the boundary connect sum of 28 copies of N and cap off with a standard ball (see [**32**, Section 6.5]) for details. The manifold B is *almost parallelizable*, that is, admits a trivialization of the tangent bundle away from a point. This implies that $p_1(B) = 0$, and from a computation with the signature we deduce the total Pontrjagin class

$$p(B) = 1 - 1440b, \tag{5.12}$$

where $b \in H^8(B;\mathbb{Z}) \cong \mathbb{Z}$ is the positive generator. Note $\lambda(B) = 0$ and $w_4(B) = 0$. Then

$$\hat{A}(B) = \left\langle \frac{7p_1^2 - 4p_2}{5760}, [B] \right\rangle \tag{5.13}$$

implies $\hat{A}(B) = 1$. We use this Bott manifold in the sequel.

5.4. *Real projective spaces*

Let $L \to \mathbb{RP}^n$ be the tautological real line bundle. Arguing as in the second paragraph of Section 5.2 we deduce that the tangent bundle to $\mathbb{RP}^n$ is stably equivalent to

$$(n+1)L - 1. \tag{5.14}$$

Then if $\alpha \in H^1(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ is the generator, we conclude

$$w(\mathbb{RP}^n) = (1+\alpha)^{n+1}. \tag{5.15}$$

Projective 4-space $\mathbb{RP}^4$ has $w_2 = 0$, so admits a pin$^+$ structure, in fact two distinct ones which are opposite in the sense of Proposition 3.10. Of course, $w_1(\mathbb{RP}^4) = \alpha$ so that $\mathbb{RP}^4$ is not orientable, so not spin either. Also, $w_4(\mathbb{RP}^4) = \alpha^4$ is non-zero, and we fix a w_1-twisted lift $\tilde{c}_{\mathbb{RP}^4} \in H^4(\mathbb{RP}^4;\widetilde{\mathbb{Z}}) \cong \mathbb{Z}$ which is a generator.

For $n = 12$ we compute from (5.15) that $\mathbb{RP}^{12}$ is not orientable; is pin$^+$ with two opposite pin$^+$ structures; that $w_4(\mathbb{RP}^{12}) = \alpha^4$; and since $H^4(\mathbb{RP}^{12};\widetilde{\mathbb{Z}}) = 0$ it does not admit an $\mathfrak{m}_c$ structure.

The η-invariants of $\mathbb{RP}^4$ and $\mathbb{RP}^{12}$ are computed in [**50**, Corollary 5.4]. The results are reciprocal for the two opposite pin$^+$ structures (Proposition 3.10), and we use the η-invariant to pin down a

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choice. Stolz's result follows from Proposition 3.12 (see (3.19) for notation):

$$\tau_{\mathbb{RP}^4} = \exp\left(\frac{2\pi i}{2^4}\right) \tag{5.16}$$

$$\tau_{\mathbb{RP}^{12}} = \exp\left(\frac{2\pi i}{2^8}\right). \tag{5.17}$$

For later use we quote from [**38**] the position of these real projective spaces in pin^+ bordism. In dimension 4 we have

$$\pi_4 MT\mathrm{Pin}^+ \cong \mathbb{Z}/2^4\mathbb{Z} \tag{5.18}$$

and $\mathbb{RP}^4$ represents a generator. In dimension 12 we have, as already quoted in (3.12),

$$\pi_{12} MT\mathrm{Pin}^+ \cong \mathbb{Z}/2^8\mathbb{Z} \oplus \mathbb{Z}/2^4\mathbb{Z} \oplus \mathbb{Z}/2^2\mathbb{Z} \tag{5.19}$$

and $\mathbb{RP}^{12}$ represents a generator of the first factor. Proposition 8.11 below proves that

$$\pi_4 M\mathfrak{m}_c \cong \mathbb{Z} \oplus \mathbb{Z}/2^3\mathbb{Z}. \tag{5.20}$$

The pair $(\mathbb{RP}^4, \tilde{c}_{\mathbb{RP}^4})$ represents a generator of the infinite cyclic summand. The connected sum $\mathbb{RP}^4 \# \mathbb{RP}^4$ has order 8 in (5.18) and has vanishing w_4 (since its value on the fundamental class is the mod 2 Euler number). The pair $(\mathbb{RP}^4 \# \mathbb{RP}^4, 0)$ represents a generator of the second summand in (5.20).

Remark 5.2 Stolz uses Proposition 3.12 to compute (5.16) and (5.17). These results also follow from the topological formula (3.20), stated for pin^+ 12-manifolds but with an extension to pin^+ $(8k+4)$-manifolds, $k \in \mathbb{Z}^{\geq 0}$. (Zhang uses the η-invariant of real projective spaces to prove Theorem 3.9, so logically we are only illustrating the theorem here, not using it to derive (5.16) and (5.17).) For example, let $\gamma\colon \mathbb{RP}^{12} \to \mathbb{RP}^{20}$ be a linear embedding, and $H \to \mathbb{RP}^{20}$ the tautological real line bundle. The normal bundle to γ is the restriction of $H^{\oplus 8} \to \mathbb{RP}^{20}$ to $\mathbb{RP}^{12}$, and $\mathbb{RP}^{12}$ is the 0-set of a section of $H^{\oplus 8} \to \mathbb{RP}^{20}$ (8 linearly independent linear functions). It follows that $\gamma_*(1)$ is the KO Euler class of $H^{\oplus 8} \to \mathbb{HP}^{20}$, which we compute to be $8\big(1-[H]\big)$ after multiplication by the Bott class. (The Euler class is associated with the difference of the half spin representations of $Spin_8$, which restricted to the diagonal $\mathbb{Z}/2\mathbb{Z} \subset \mathrm{Spin}_8$ is $8(1-\epsilon)$, where ϵ is the sign representation.) It remains to observe that $8\big(1-[H]\big)$ has order 2^8 in $\widetilde{KO}(\mathbb{RP}^{20})$.

5.5. *Three special manifolds*

We define three 12-dimensional manifolds W_0', W_0'', W_1 which appear in Theorem 6.1 below. Each of W_0', W_0'' is presented as the quotient of its orientation double cover by a free involution.

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W_0'. Set

$$\widehat{W_0'} = S^4 \times (\mathbb{HP}^2 \# \mathbb{HP}^2), \tag{5.21}$$

the Cartesian product of the 4-sphere and the connected sum of two quaternionic projective planes. As an explicit model of the connected sum, fix a line through the origin in real affine space $\mathbb{A}^9$, remove two small antipodal balls from $S^8 \subset \mathbb{A}^9$ which are exchanged by the half-turn about that line, and glue in two identical copies of $\mathbb{HP}^2 \setminus B^8$. Then (5.21) has a free orientation-reversing involution which is the Cartesian product of the antipodal involution of S^4 and the half-turn of $\mathbb{HP}^2 \# \mathbb{HP}^2$ with its two fixed points. The quotient is the manifold W_0'. Since $\widehat{W_0'}$ is simply connected, we have $\pi_1 W_0' \cong \mathbb{Z}/2\mathbb{Z}$ and hence $H^1(W_0'; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Since the involution is free on S^4, the manifold W_0' fits into a fiber bundle

$$\mathbb{HP}^2 \# \mathbb{HP}^2 \longrightarrow W_0' \longrightarrow \mathbb{RP}^4. \tag{5.22}$$

The simply connected manifold $\mathbb{HP}^2 \# \mathbb{HP}^2$ has a unique spin structure, and so the half-turn lifts to a spin automorphism. Its square is either the identity or the spin flip; we show it is the identity by computing at a fixed point on S^8. The differential of the half-turn is the linear map -1 on the eight-dimensional tangent space. The linear map -1 lifts to the volume form in Spin_8, which squares to $+1$. Therefore, the vertical tangent bundle of (5.22) is spin, and so $w_i(W_0')$, $i = 1, 2$, are pulled back from $\mathbb{RP}^4$. Using (5.15) we see that W_0' is pin$^+$: it admits two opposite pin$^+$ structures.

The cohomology ring of the connected sum is

$$H^\bullet(\mathbb{HP}^2 \# \mathbb{HP}^2; \mathbb{Z}) \cong \mathbb{Z}[x_1, x_2]/(x_1 x_2, x_1^2 - x_2^2), \qquad \deg x_1 = \deg x_2 = 4. \tag{5.23}$$

Let $t \in H^4(S^4; \mathbb{Z})$ denote a positive generator. Then under the antipodal involution the class $2t$ descends to the generator $\tilde{c} \in H^4(\mathbb{RP}^4; \widetilde{\mathbb{Z}})$ (see Lemma 4.8). Recalling (5.10) and the fact that $w_4(\mathbb{RP}^4) = \alpha^4$, as mentioned following (5.15), we deduce that $w_4(W_0') = \alpha^4 + \bar{x}_1 + \bar{x}_2$, where $\bar{x}_i$ is the mod 2 reduction of x_i. The class $\bar{x}_1 + \bar{x}_2$ is invariant under the involution of $\mathbb{HP}^2 \# \mathbb{HP}^2$ and descends to w_4 of the vertical tangent bundle in (5.22). Define the w_1-twisted integer lift $\tilde{c}_0' \in H^4(W_0'; \widetilde{\mathbb{Z}})$ of $w_4(W_0')$ by

$$\pi^* \tilde{c}_0' = 2t + x_1 - x_2, \tag{5.24}$$

where $\pi\colon \widehat{W}_0' \to W_0'$ is the orientation double cover.

W_0''. Let $K_{\mathbb{R}} \to S^4 = \mathbb{HP}^1$ be the underlying real 4-plane bundle of the quaternionic line bundle $K \to S^4$ with $p_1^{\mathbb{H}}(K) \in H^4(S^4; \mathbb{Z})$ a positive generator. Define $W_0'' = \mathbb{P}(K_{\mathbb{R}}^{\oplus 2} \oplus \underline{\mathbb{R}})$ as the total space of the real projective bundle

$$\mathbb{RP}^8 \longrightarrow \mathbb{P}(K_{\mathbb{R}}^{\oplus 2} \oplus \underline{\mathbb{R}}) \xrightarrow{\rho} S^4. \tag{5.25}$$

Let $L \to W_0''$ be the tautological real line bundle. Since the stable tangent bundle to S^4 is trivial, the stable tangent bundle to W_0'' is the stable tangent bundle along the fibers, which is

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$$(L - \underline{\mathbb{R}}) + (\rho^* K_{\mathbb{R}}^{\oplus 2} \otimes L). \tag{5.26}$$

This comes from the short exact sequence $0 \to L \to \rho^*(K_{\mathbb{R}}^{\oplus 2} \oplus \underline{\mathbb{R}}) \to Q \to 0$ of real vector bundles over W_0'' (compare (5.6)). Using (5.26) we compute $w_1(W_0'') = w_1(L)$ and $w_2(W_0'') = 0$: it suffices to restrict to a fiber of (5.25) since that restriction induces an isomorphism on $H^i(-; \mathbb{Z}/2\mathbb{Z})$, $i = 1, 2$. The orientation double cover is an S^8-bundle over S^4, which is simply connected. Hence W_0'' admits two opposite pin$^+$ structures.

The bundle $\rho^* K_{\mathbb{R}} \otimes L$ has total Stiefel–Whitney class of the form $1 + w_4$, and it follows easily from the Whitney formula applied to (5.26) that $w_4(W_0'') = 0$.

W_1. The projective group $\mathbb{P}Sp_1 \cong SO_3$ acts on $\mathbb{HP}^2$ via (see (5.5) for notation)

$$\lambda \cdot [q^0, q^1, q^2] = [\lambda q^0, \lambda q^1 \lambda q^2], \qquad \lambda \in Sp_1. \tag{5.27}$$

So a principal SO_3-bundle has an associated fiber bundle with fiber $\mathbb{HP}^2$. The action (5.27) lifts to the spin bundle of frames of $\mathbb{HP}^2$. To see this, choose a basepoint $[1, 0, 0]$ and write $\mathbb{HP}^2 = Sp_3/Sp_1 \times Sp_2$. The principal Spin_8-bundle of frames is associated with the principal $(Sp_1 \times Sp_2)$-bundle $Sp_3 \to \mathbb{HP}^2$ via the representation

$$Sp_1 \times Sp_2 \cong \mathrm{Spin}_3 \times \mathrm{Spin}_5 \longrightarrow \mathrm{Spin}_8. \tag{5.28}$$

The $\mathbb{P}Sp_1$ action fixes the basepoint, and the 'diagonal' map $Sp_1 \to Sp_1 \times Sp_2 \to \mathrm{Spin}_8$ descends to $\mathbb{P}Sp_1$. The induced map $\mathbb{P}Sp_1 \to \mathrm{Spin}_8$ gives the desired lift. Define W_1 as the fiber bundle

$$\mathbb{HP}^2 \longrightarrow W_1 \longrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1 \tag{5.29}$$

obtained from the principal SO_3-bundle of oriented orthonormal frames of the real 3-plane bundle $\mathcal{O}(1,1)_{\mathbb{R}} \oplus \underline{\mathbb{R}} \to \mathbb{CP}^1 \times \mathbb{CP}^1$, where $\mathcal{O}(1,1) \to \mathbb{CP}^1 \times \mathbb{CP}^1$ is the tensor product of the hyperplane line bundles on the factors. The manifold W_1 is simply connected, hence orientable. The stable tangent bundle to $\mathbb{CP}^1 \times \mathbb{CP}^1$ is trivial, and the vertical tangent bundle is spin, hence W_1 is spin with a unique spin structure refining each orientation.

To compute the Pontrjagin classes of the vertical tangent bundle of (5.29), we use the $\mathbb{P}Sp_1$ action to construct a fiber bundle

$$\mathbb{HP}^2 \longrightarrow E \longrightarrow \mathbb{CP}^\infty \tag{5.30}$$

from the rank three real vector bundle $\mathcal{O}(1)_{\mathbb{R}} \oplus \underline{\mathbb{R}} \to \mathbb{CP}^\infty$. The squaring map $\mathbb{T} \to \mathbb{T}$ induces a degree two map f on $B\mathbb{T} = \mathbb{CP}^\infty$; the pullback $f^*\left(\mathcal{O}(1)_{\mathbb{R}} \oplus \underline{\mathbb{R}}\right) \cong \mathcal{O}(2)_{\mathbb{R}} \oplus \underline{\mathbb{R}}$ is the adjoint bundle of a principal $Sp_1 \cong SU_2$-bundle we write as the quaternionic line bundle $K \to \mathbb{CP}^\infty$. Then the pullback of (5.30) under f is the projectivization of the rank three quaternionic vector bundle $K^{\oplus 3} \to \mathbb{CP}^\infty$. Let $a \in H^2(\mathbb{CP}^\infty; \mathbb{Z})$ be a generator; then the quaternionic Pontrjagin class of $K \to \mathbb{CP}^\infty$ is $p_1^{\mathbb{H}}(K) = a^2$. Let $L \to \mathbb{P}(K^{\oplus 3})$ be the tautological quaternionic line bundle. Using the technique in Section 5.2, including Remark 5.1, we compute the total Pontrjagin class of the vertical tangent bundle to

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$E' = \mathbb{P}(K^{\oplus 3}) \to \mathbb{CP}^\infty$ as

$$\begin{aligned} p(E'/\mathbb{CP}^\infty) &= \frac{\left[1 + 2(x+a^2) + (x-a^2)^2\right]}{1+4x} \\ &= 1 + (2x + 6a^2) + (7x^2 - 6a^2x) + (12a^2x^2) + \cdots \end{aligned} \tag{5.31}$$

where $x = p_1^{\mathbb{H}}(L) \in H^4(E';\mathbb{Z})$. Grothendieck's formula for projective bundles implies

$$x^3 = 3a^2x^2 - 3a^4x + a^6. \tag{5.32}$$

(A quick explanation: Let $V \to X$ be a vector bundle of rank $r > 0$ (over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$), $\mathbb{P}(V) \xrightarrow{p} X$ its projectivization, and $L \to \mathbb{P}(V)$ the tautological line bundle. Over $\mathbb{P}(V)$ there is a short exact sequence $0 \to L \to p^*V \to Q \to 0$ of vector bundles, where $Q \to \mathbb{P}(V)$ has rank $r-1$. Grothendieck's formula expresses the vanishing of its rth Chern or Pontrjagin class.) Use the pullback diagram

$$\begin{array}{ccc} E' & \xrightarrow{\tilde f} & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbb{CP}^\infty & \xrightarrow{f} & \mathbb{CP}^\infty \end{array} \tag{5.33}$$

of fiber bundles to compute π_* of the degree 12 Pontrjagin classes p_3, p_1p_2, p_1^3 of the vertical tangent bundle to $E/\mathbb{CP}^\infty$; they pull back under $\tilde f$ to the corresponding Pontrjagin classes of $E'/\mathbb{CP}^\infty$. Note $f^*a = 2a$. Thus

$$f^*\pi_* p_3 = \pi'_* \tilde f^* p_3 = \pi'_*(12a^2x^2) = 12a^2 \tag{5.34}$$

from which

$$\pi_* p_3 = 3a^2. \tag{5.35}$$

Similarly,

$$\pi_* p_1 p_2 = 18a^2 \tag{5.36}$$

$$\pi_* p_1^3 = 24a^2 \tag{5.37}$$

and hence

$$\pi_* \lambda^3 = 3a^2. \tag{5.38}$$

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Finally, we pull back by the degree $(1,1)$ map $\mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^\infty$ to compute the corresponding quantities on W_1. After evaluating on the fundamental class $[\mathbb{CP}^1 \times \mathbb{CP}^1]$ we find

$$\begin{aligned} \langle p_3(W_1), [W_1]\rangle &= 6 \\ \langle p_1 p_2(W_1), [W_1]\rangle &= 36 \\ \langle p_1^3(W_1), [W_1]\rangle &= 48 \\ \langle \lambda^3(W_1), [W_1]\rangle &= 6. \end{aligned} \tag{5.39}$$

6. The anomaly cancellation

In this section we state the main computational result, Theorem 6.1, which provides generators for the dimension 12 bordism of manifolds which occur in M-theory. We give the proof in Section 8. Here, in six subsections, we use this bordism computation to prove the Anomaly Cancellation Theorem 2.2 by computing the invariants α_{RS}, α_C for a generator of each factor. We organize the presentation by Adams filtration (see Section 8.4). Two of the six generators are represented by spin manifolds, so in these cases the anomaly cancellation is already proved in Theorem 4.17. Nonetheless, we check directly by computing the invariants.

Since our invariants take values in a finite abelian 2-group (see Corollaries 3.4 and 4.13), it suffices to compute after completing at the prime 2. (The structure of the bordism group at odd primes is much simpler, but we do not treat it here.) Let $\mathbb{Z}_2$ denote the 2-adic numbers.

6.1. *The bordism group*

Recall that $M\mathfrak{m}_c$ denotes the bordism spectrum of $\mathfrak{m}_c$-manifolds. We use the manifolds and cohomology classes defined in Section 5. The cohomology class λ of a spin manifold is the canonical integer lift of w_4.

THEOREM 6.1 *The following six $\mathfrak{m}_c$-manifolds generate the group $\pi_{12} M\mathfrak{m}_c \otimes \mathbb{Z}_2$:*

$$\begin{gathered} (W'_0, \tilde{c}'_0), \quad (W''_0, 0), \quad (W_1, \lambda) \\ (K \times \mathbb{HP}^2, \lambda), \quad (\mathbb{RP}^4, \tilde{c}'_{\mathbb{RP}^4}) \times B, \quad (\mathbb{RP}^4 \# \mathbb{RP}^4, 0) \times B. \end{gathered} \tag{6.1}$$

Note that W_1 and $K \times \mathbb{HP}^2$ are spin manifolds.

6.2. *Computations*

As explained in (2.3), the Anomaly Cancellation Theorem 2.2 is a consequence of Theorem 6.1 and the following.

THEOREM 6.2 *For each of the pairs $(W, \tilde{c})$ listed in (6.1) the anomaly cancellation condition $\alpha_{RS}(W)\alpha_C(W) = 1$ holds.*

The proof of Theorem 6.2 is divided into six parts, one for each generator. It occupies the remainder of Section 6. Recall that α_{RS} is defined in (3.11) and α_C in (4.41). Strictly speaking, we do not rely

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on the particular w_1-twisted integer lift $\tilde{c}$ of w_4 since the mod 2 cubic invariant α_C is independent of the choice (Lemma 4.12). For convenience, we summarize the computations in the following chart:

$(W, \tilde{c})$	$\alpha_{RS}(W)$	$\alpha_C(W)$
$(W_0', \tilde{c}_0')$	$+1$	$+1$
$(W_0'', 0)$	$+1$	$+1$
(W_1, λ)	$+1$	$+1$
$(K \times \mathbb{HP}^2, \lambda)$	-1	-1
$(\mathbb{RP}^4, \tilde{c}'_{\mathbb{RP}^4})$	$+1$	$+1$
$(\mathbb{RP}^4 \# \mathbb{RP}^4, 0)$	$+1$	$+1$

(6.2)

Adams filtration 4

To compute the Rarita–Schwinger anomaly partition function of $\mathbb{RP}^4 \times B$, we apply Propositions 3.8 and 3.11:

$$\alpha_{RS}(\mathbb{RP}^4 \times B) = \tau_{\mathbb{RP}^4 \times B}(T\mathbb{RP}^4 + TB - 2) = \frac{\tau_{\mathbb{RP}^4}(T\mathbb{RP}^4)^{\operatorname{ind} D_B}\, \tau_{\mathbb{RP}^4}^{\operatorname{ind} D_B(TB)}}{\tau_{\mathbb{RP}^4}^{2\operatorname{ind} D_B}}. \tag{6.3}$$

From (5.14) the stable tangent bundle to $\mathbb{RP}^4$ is $5L - 1$. Proposition 3.10 implies that as far as η-invariants are concerned, tensoring with L induces a change of sign. Hence using Proposition 3.8 and (5.17)

$$\tau_{\mathbb{RP}^4}(T\mathbb{RP}^4) = \tau_{\mathbb{RP}^4}(5L - 1) = \tau_{\mathbb{RP}^4}(-6) = \exp\left(-\frac{3\pi i}{4}\right). \tag{6.4}$$

An alternative computation uses the four-dimensional analog of Proposition 3.12 in which the denominator of 2^8 in (3.27) is replaced by 2^4. Bound the orientation double cover S^4 by the closed 5-ball D^5 with its antipodal involution. The pullback of $T\mathbb{RP}^4$ extends over D^5 as $TD^5 - 1$. The trace τ_f at the unique fixed point is $-5 - 1 = -6$, and we recover (6.4) from (3.27). Still another computation uses a variant of Theorem 3.9 for pin^+ 4-manifolds: replace $\mathbb{RP}^{20}$ with $\mathbb{RP}^{12}$ and 2^{11} in (3.20) with 2^7. Following the notation in Remark 5.2 with these replacements we compute

$$\begin{aligned} \gamma_*(T\mathbb{RP}^4) &= \gamma_*\big(\gamma^*(5[H] - 1)\big) \\ &= (5[H] - 1)\,\gamma_*(1) \\ &= (5[H] - 1)\,8(1 - [H]) \\ &= -48(1 - [H]), \end{aligned} \tag{6.5}$$

and now (6.4) follows from the adapted (3.20). For the index computation on the Bott manifold we use the Atiyah–Singer index theorem and (5.12):

$$\operatorname{ind} D_B(TB) = \hat{A}(B)\operatorname{ch}(TB)[B] = \left(1 - \frac{p_2}{1440}\right)\left(8 - \frac{p_2}{6}\right)[B] = 248. \tag{6.6}$$

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Combining (5.16), (6.3), (6.4), (6.6) and $\operatorname{ind} D_B = 1$ (see (5.13)) we conclude

$$\alpha_{RS}(\mathbb{RP}^4 \times B) = \frac{e^{-3\pi i/4}\, e^{\pi i}}{e^{\pi i/4}} = 1. \tag{6.7}$$

The C-field anomaly partition function is the exponential of the mod 2 reduction of the cubic form (4.47). As in Section 6.2 the class $\bar{p}(\mathbb{RP}^4 \times B) = p(B) = -720b$. Evaluating on the generator $\tilde{c}_{\mathbb{RP}^4} \in H^4(\mathbb{RP}^4, \widetilde{\mathbb{Z}})$ we find

$$\left(\frac{\tilde{c}^3_{\mathbb{RP}^4} - \bar{p}\tilde{c}_{\mathbb{RP}^4}}{24}\right)[\mathbb{RP}^4 \times B] = 30. \tag{6.8}$$

Since this is even, $\alpha_C(\mathbb{RP}^4 \times B) = 1$.

Adams filtration 5
The Rarita–Schwinger partition function only depends on the image of $(\mathbb{RP}^4 \# \mathbb{RP}^4) \times B$ in pin^+ bordism. The connected sum $\mathbb{RP}^4 \# \mathbb{RP}^4$ represents twice $\mathbb{RP}^4$ in pin^+ bordism, and the same is true after crossing with the Bott manifold. (See Section 5.4.) Thus we deduce from (6.7) that $\alpha_{RS}\big((\mathbb{RP}^4 \# \mathbb{RP}^4) \times B\big) = \alpha_{RS}(\mathbb{RP}^4 \times B)^2 = 1$. Since $w_4\big((\mathbb{RP}^4 \# \mathbb{RP}^4) \times B\big) = 0$ we can take the w_1-twisted integer lift to vanish, and hence $\alpha_C\big((\mathbb{RP}^4 \# \mathbb{RP}^4) \times B\big) = 1$.

Adams filtration 3
The manifold $W_3 = K \times \mathbb{HP}^2$ is spin, so by Proposition 3.5 the Rarita–Schwinger anomaly partition function is the mod 2 reduction of an integer $RS(W_3)$ defined in (3.15). Using (5.3) and (5.9) we compute

$$\begin{aligned}\frac{1}{2}\hat{A}(W_3)\mathrm{ch}(TW_3 - 2) &= \frac{1}{2}\hat{A}(K)\hat{A}(\mathbb{HP}^2)\big(\mathrm{ch}(TK) + \mathrm{ch}(T\mathbb{HP}^2) - 2\big) \\ &= \frac{1}{2}(1+2k)\left(1 - \frac{x}{12}\right)\left((4-48k) + \left(8 + 2x - \frac{5}{6}x^2\right) - 2\right) \\ &= -kx^2 + \cdots\end{aligned} \tag{6.9}$$

Therefore, $RS(W_3) = -1$ and $\alpha_{RS}(W_3) = -1$.

The C-field anomaly partition function is computed from the cubic form

$$\frac{\lambda^3 - p\lambda}{24} = \frac{(x-24k)^3 - (3x^2 - 24xk)(x-24k)}{24} = kx^2 + \cdots \tag{6.10}$$

Evaluating on the fundamental class and exponentiating we deduce $\alpha_C(W_3) = -1$.

Adams filtration 1
The spin manifold W_1 is defined in Section 5.5.3; it is an $\mathbb{HP}^2$-bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1$. We plug (5.39) into (3.17) to compute $\alpha_{RS}(W_1) = 1$ and into (4.47) with $\tilde{c} = \lambda$ to compute $\alpha_C(W_1) = 1$.

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Adams filtration 0*, part* 1

The manifold W_0' is an $(\mathbb{HP}^2\#\mathbb{HP}^2)$-bundle over $\mathbb{RP}^4$ (see (5.22)). We claim that its Rarita–Schwinger anomaly partition function is trivial: $\alpha_{RS}(W_0') = 1$. To prove this we apply Proposition 3.12. The total space (5.21) of the orientation double cover bounds $D^5 \times (\mathbb{HP}^2\#\mathbb{HP}^2)$ with the antipodal involution on D^5 times the half-turn about an axis through $S^8 \subset \mathbb{A}^9$ acting on the connected sum. There are two fixed points: the center of D^5 times antipodal points $p, p' \in S^8$. In (3.27) the traces $\tau_p = \tau_{p'}$, and we claim $i_{p'} = -i_p$. To prove this choose the center of S^8 as the origin of $\mathbb{A}^9$, so identify the affine space $\mathbb{A}^9$ with the vector space $\mathbb{R}^9$. The half-turn is implemented on spinors by the element $\omega = \gamma^1\gamma^2\cdots\gamma^8$ in Cliff_{+9}, where we choose the axis to be the last coordinate. But the standard basis vectors $e_1, e_2, \dots, e_8$ form an *oriented* basis of exactly one of T_pS^8, $T_{p'}S^8$; it is negatively oriented at the other point. So the action of frames at the other point is by the conjugate $e_1\omega e_1^{-1} = -\omega$. Multiply by the volume element of Cliff_{+5}, which gives the action of the involution on pin^+ frames at the center of D^5.

For the C-field anomaly partition function we compute the cubic form on the orientation double cover using (4.44). Use the w_1-twisted integer lift $\tilde{c}$ specified in (5.24) via its lift c to $\widehat{W}_0'$. Use (5.9) to compute that $p(\widehat{W}_0') = 3(x_1^2 + x_2^2)$. Thus

$$\frac{c^3 - pc}{48} = \frac{12tx^2 - 12tx^2}{48} = 0, \tag{6.11}$$

since $x^2 = x_1^2 = x_2^2$ in $H^8(\mathbb{HP}^2\#\mathbb{HP}^2)$ (see (5.23)). Therefore, $\alpha_C(W_0') = 1$.

Adams filtration 0*, part* 2

The manifold $W_0'' = \mathbb{P}(2K_{\mathbb{R}} \oplus \underline{\mathbb{R}})$, defined in (5.25) as an $\mathbb{RP}^8$-bundle over S^4. Recall that $K \to S^4$ is the quaternionic line bundle with $p_1^{\mathbb{H}}(K) \in H^4(S^4;\mathbb{Z})$ the positive generator. We use Theorem 3.9 to compute $\alpha_{RS}(W_0'') = 1$.

As a preliminary define $\mu \in \widetilde{KO}^4(S^4)$ as $\iota_*(1)$ for $\iota\colon \mathrm{pt} \to S^4$, and $\lambda \in \widetilde{KO}^0(S^4)$ as the KO-class $[K_{\mathbb{R}}] - 4$, where $K_{\mathbb{R}} \to S^4$ is the real 4-plane bundle underlying $K \to S^4$. Identify $\widetilde{KO}^4(S^4)$ as the Grothendieck group of quaternionic vector bundles over S^4 of virtual rank zero; then $\mu = [K] - [\underline{\mathbb{H}}]$. Let $\pi\colon S^4 \to \mathrm{pt}$ be the unique map. Then we claim

$$\lambda\mu = 0 \tag{6.12}$$

$$\lambda[\underline{\mathbb{H}}] = 4\mu \tag{6.13}$$

$$\pi_*\mu = 1 \tag{6.14}$$

$$\pi_*[\underline{\mathbb{H}}] = 0. \tag{6.15}$$

Since we can trivialize $K_{\mathbb{R}} \to S^4$ away from a point, we can arrange representatives of λ, μ with disjoint support, from which (6.12) follows. For (6.13) we observe that if M is any quaternionic line, then there is a natural isomorphism

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$$\begin{aligned} M^{\oplus 4} &\longrightarrow M_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{H} \\ (\xi_1, \xi_2, \xi_3, \xi_4) &\longmapsto \sum_{s=1}^{4} -\xi_s \otimes 1 + \xi_s i \otimes i + \xi_s j \otimes j + \xi_s k \otimes k \end{aligned} \tag{6.16}$$

of quaternionic vector spaces. Equation (6.14) is immediate: $\pi_*\mu = (\pi \circ \iota)_*(1) = 1$. Finally, $\underline{\mathbb{H}} \to S^4$ is pulled back from $\underline{\mathbb{H}} \to \mathrm{pt}$, so $\pi_*[\underline{\mathbb{H}}] = [\underline{\mathbb{H}}]\pi_*(1) = 0$ since $1 \in KO^0(S^4)$ extends over the 5-ball.

Another preliminary: If $M_{\mathbb{R}} \to Y$ is the real 4-plane bundle underlying a quaternionic line bundle $M \to Y$, then its KO-theory Euler class is

$$[\underline{\mathbb{H}}] - [M] \in \widetilde{KO}^4(Y). \tag{6.17}$$

Proof: $M \to Y$ is associated with a principal Sp_1-bundle via (i) the embedding $Sp_1 \hookrightarrow \mathrm{Spin}_4 \cong Sp_1 \times Sp_1$ onto the second factor and (ii) the action of $Sp_1 \times Sp_1$ on $\mathbb{H}$ in which the first factor acts trivially and the second by right multiplication. Then the KO-Euler class is pulled back from the vector bundle associated with the difference of the quaternionic half-spin representations.

Let $J \to S^4$ be the quaternionic line bundle with $p_1^{\mathbb{H}}(J) = -2p_1^{\mathbb{H}}(K)$. Then $K_{\mathbb{R}}^{\oplus 2} \oplus J_{\mathbb{R}} \to S^4$ is trivializable. Define

$$\gamma\colon W_0'' = \mathbb{P}(K_{\mathbb{R}}^{\oplus 2} \oplus \underline{\mathbb{R}}) \xrightarrow{i} \mathbb{P}(K_{\mathbb{R}}^{\oplus 2} \oplus \underline{\mathbb{R}} \oplus J_{\mathbb{R}}) \cong S^4 \times \mathbb{RP}^{12} \xrightarrow{\pi} \mathbb{RP}^{12} \xrightarrow{j} \mathbb{RP}^{20}, \tag{6.18}$$

where π is the projection onto the second factor and j is a linear embedding as in Remark 5.2. Let $L \to \mathbb{P}(K_{\mathbb{R}}^{\oplus 2} \oplus \underline{\mathbb{R}} \oplus J_{\mathbb{R}})$ be the tautological real line bundle. Then $L \cong \pi^*H = \pi^*j^*H$ for $H \to \mathbb{RP}^{20}$ the tautological line bundle, and i^*L is also isomorphic to the tautological line bundle. We identify $L^* \cong L$. The normal bundle to i is the quotient of tangent bundles (see (5.26)):

$$\begin{aligned} \left[(i^*L - \underline{\mathbb{R}}) \,\oplus\, K_{\mathbb{R}}^{\oplus 2} \otimes i^*L \,\oplus\, J_{\mathbb{R}} \otimes i^*L\right] \,/\, \left[(i^*L - \underline{\mathbb{R}}) \,\oplus\, K_{\mathbb{R}}^{\oplus 2} \otimes i^*L\right] &\cong J_{\mathbb{R}} \otimes i^*L \\ &\cong i^*(J_{\mathbb{R}} \otimes L). \end{aligned} \tag{6.19}$$

There is a canonical section of

$$J_{\mathbb{R}} \otimes L \cong \mathrm{Hom}(L, J_{\mathbb{R}}) \longrightarrow \mathbb{P}(K_{\mathbb{R}}^{\oplus 2} \oplus \underline{\mathbb{R}} \oplus J_{\mathbb{R}}) \tag{6.20}$$

given by projection $K_{\mathbb{R}}^{\oplus 2} \oplus \underline{\mathbb{R}} \oplus J_{\mathbb{R}} \to J_{\mathbb{R}}$, and its zero set is the image of i. It follows that $i_*(1)$ is the KO-Euler class of (6.20), which we compute using (6.17):

$$\begin{aligned} i_*(1) = [\underline{\mathbb{H}}] - [J \otimes L] &= (1 - [L])[\underline{\mathbb{H}}] + 2[L]\mu \\ &= \pi^*\big\{(1 - [H])[\underline{\mathbb{H}}] + 2[H]\mu\big\}. \end{aligned} \tag{6.21}$$

Using (5.26) we find

$$[TW_0''] - 2 = i^*\big\{2[H]\lambda + 9[H] + 1\big\}. \tag{6.22}$$

Combining (6.21) and (6.22) with (6.12)–(6.15) we calculate

$$\begin{aligned} \pi_* i_*([TW_0''] - 2) &= \pi_*\Big\{\big(2[H]\lambda + 9[H] + 1\big)\big((1 - [H])[\underline{\mathbb{H}}] + 2[H]\mu\big)\Big\} \\ &= 10[H] + 10. \end{aligned} \tag{6.23}$$

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Now $j_*(1) = 8(1 - [H])$ is computed in Remark 5.2, and so

$$j_*\big([H]\big) = [H]j_*(1) = 8[H](1 - [H]) = -j_*(1), \tag{6.24}$$

from which

$$\gamma_*([TW_0''] - 2) = j_*\pi_*i_*([TW_0''] - 2) = 0. \tag{6.25}$$

Then $\alpha_{RS}(W_0'') = 1$ follows immediately from (3.20).

The C-field anomaly is also trivial—$\alpha_C(W_0'') = 1$—since $w_4(W_0'') = 0$ and we can choose the w_1-twisted integer lift $\tilde{c}$ in (4.41) to be zero.

7. Ambiguities in the M-theory action

As mentioned in the introduction, to define an M-theory action it is not sufficient to demonstrate the cancellation of anomalies; we must also give a trivialization of the product $\alpha_{RS} \otimes \alpha_C$, a so-called *setting of the quantum integrand*. The ratio β of two trivializations is an invertible 11-dimensional field theory. Unitarity of M-theory requires that β be reflection positive. If β were to depend on the metric or the field strength of the C-field, then it would be detected locally. Since the local physics is fixed by considerations other than anomaly cancellation, we restrict β to be a *topological* field theory. As explained in Section 2 a reflection positive invertible 11-dimensional topological field theory of $\mathfrak{m}_c$-manifolds is determined by a homomorphism $\pi_{11}M\mathfrak{m}_c \to \mathbb{C}/\mathbb{Z}$. (Reflection positivity imposes a restriction, which is satisfied here by all such homomorphisms, since they take values in $\{\pm 1\} \subset \mathbb{C}/\mathbb{Z}$.) The following conjecture describes the group of these theories.

Let Σ be the Klein bottle. It has four pin$^+$ structures of which two are non-bounding [**39**, Proposition 3.9]; fix one of those. Also, let S^1 denote the circle with its non-bounding string structure (see Remark 8.1). Define the 11-manifold

$$N = S^1 \times \Sigma \times B, \tag{7.1}$$

where B is the Bott manifold (Section 5.3). The following is based on computations to appear in [**31**], and out of an abundance of caution we state it here as a conjecture, a more precise version of Conjecture 2.3.

CONJECTURE 7.1 *The group $\pi_{11}M\mathfrak{m}_c$ is cyclic of order 2. The bordism class of the pair $(N, 0)$ represents the generator. The mod 2 index of the* pin$^+$ *Dirac operator is an isomorphism $\pi_{11}M\mathfrak{m}_c \to \mathbb{Z}/2\mathbb{Z}$.*

See Section 8.5.4 for a justification of Conjecture 7.1 using the Adams spectral sequence.

REMARK 7.2 Index invariants of *pin*$^+$ n-manifolds correspond to index invariants of spin $(n-1)$-manifolds (see [**26**, Section 9.2.3]). Hence the mod 2 indices of spin manifolds in dimensions 9, 10 correspond to mod 2 indices of *pin*$^+$ manifolds in dimensions 10, 11. Let P be a *pin*$^+$ 10-manifold. Then the mod 2 index of the product $S^1 \times P^{10}$ equals the mod 2 index of P. We use product formulas analogous to Proposition 3.11 to compute that the mod 2 index of $\Sigma \times B$ is non-zero.

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8. The bordism computation

In this section, we present the computations which prove Theorem 6.1 and justify Conjecture 7.1. We begin in Section 8.1 by constructing the Thom spectrum $M\mathfrak{m}_c$. In Section 8.2, we discuss some characteristic classes of $\mathfrak{m}_c$-manifolds and their behavior under transfer maps from the orientation double cover. We compute the values of some $\mathfrak{m}_c$-characteristic classes on two special manifolds in Section 8.3. The Adams spectral sequence is introduced in Section 8.4. A crucial input is the structure of the mod 2 cohomology of $B\mathfrak{m}_c$, which is discussed in Appendix C. The main work in this section occurs in Section 8.5. We present arguments to determine the facts we need about $\mathfrak{m}_c$-bordism groups in dimensions 11 and 12, and along the way compute low dimensional $\mathfrak{m}_c$-bordism groups.

8.1. *The Thom spectrum*

Our aim in this section is to justify the claim that the manifolds listed in Theorem 6.1 generate the bordism group of $\mathfrak{m}_c$-manifolds (Definition 2.1). We begin by identifying the relevant Thom spectrum.

Suppose that (X,ζ) is a space X equipped with a stable vector bundle ζ of virtual dimension 0, which one may think of as a map $\zeta : X \to BO$ from X to the classifying space of the infinite orthogonal group. Write $\mathrm{Thom}(X,\zeta)$ for the Thom spectrum of ζ. The homotopy group $\pi_m\mathrm{Thom}(X,\zeta)$ is the bordism group of triples (M,f,ϕ) consisting of an m-manifold M equipped with a map $f : M \to X$, and an isomorphism

$$\pi : f^*\zeta \approx \mathbb{R}^m - TM$$

of virtual vector bundles. Put more colloquially it is the bordism group of manifolds whose stable *normal* bundle has an (X,ζ)-structure. The bordism group of manifolds whose stable *tangent* bundle has an (X,ζ)-structure is the homotopy group $\pi_n\ \mathrm{Thom}(X,-\zeta)$.

We are interested in manifolds M whose whose stable *tangent* bundle has a pin^+ structure and which are equipped with a w_1-twisted integer lift of w_4. We therefore consider the space $B\mathfrak{m}_c$ defined by the homotopy pullback square

$$\begin{array}{ccc} B\mathfrak{m}_c & \longrightarrow & E\mathbb{Z}/2 \underset{\mathbb{Z}/2}{\times} K(\mathbb{Z},4) \\ \downarrow & & \downarrow \\ B\mathrm{Pin}^+ & \xrightarrow[(w_1,w_4)]{} & B\mathbb{Z}/2 \times K(\mathbb{Z}/2,4), \end{array} \tag{8.1}$$

let ζ be the virtual vector bundle classified by the pullback

$$B\mathfrak{m}_c \to B\mathrm{Pin}^+ \to BO,$$

and write $M\mathfrak{m}_c = \mathrm{Thom}(B\mathfrak{m}_c, -\zeta)$. The homotopy groups $\pi_* M\mathfrak{m}_c$ are then the bordism groups of manifolds equipped with a tangential pin^+-structure and a w_1-twisted integer lift of w_4.

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As in Definition 2.1, an $\mathfrak{m}_c$-*manifold* is a pair (M, c) in which M is a pin$^+$-manifold and c is a w_1-twisted integer lift of tangential w_4. A Spin-manifold M gives rise to an $\mathfrak{m}_c$-manifold by taking c to be the tangential characteristic class λ.

REMARK 8.1 Given two $\mathfrak{m}_c$-manifolds (M_1, c_1) and (M_1, c_2) it is tempting to imagine that the product $(M_1 \times M_2, c_1 + c_1)$ is an $\mathfrak{m}_c$-manifold. While it is true that $w_4(M_1 \times M_2) = w_4(M_1) + w_4(M_2)$, the sum $c_1 + c_2$ does not really make sense as the two summands lie in different twisted cohomology groups. The expression does make sense if $w_1(M_2) = 0$ and $c_2 = 0$, and in particular, if M_2 is a *Spin*-manifold equipped with a trivialization of λ (a *String*-manifold). The 'Bott manifold' B of Section 5.3 is a *String*-manifold so if (N, c) is an $\mathfrak{m}_c$-manifold, then $(N \times B, \pi_1^* c)$ is also an $\mathfrak{m}_c$-manifold.

8.2. *Characteristic classes*

To describe bordism invariants of $\mathfrak{m}_c$-manifolds we will require some cohomology classes in $B\mathfrak{m}_c$. First note that under the equivalence

$$B\mathrm{O}_1 \times B\,\mathrm{SO} \xrightarrow{\oplus} B\mathrm{O}$$

the characteristic class w_2 pulls back to $(0, w_2)$. Passing to the homotopy fiber of the classifying map to $K(\mathbb{Z}/2, 2)$ gives an equivalence

$$B\mathrm{O}_1 \times B\mathrm{Spin} \to B\mathrm{Pin}^+.$$

From this one sees that a pin$^+$-structure on a vector bundle T may be identified with an equivalence $T \approx L \oplus V$ of stable vector bundles, in which L is a line bundle and V is a Spin-bundle.

Suppose M is a pin$^+$ manifold and, using the above, regard the pin$^+$-structure as giving a stable isomorphism $TM \approx L \oplus V$ with L a real line bundle and V a Spin-bundle. Set

$$\begin{aligned} \alpha &= w_1(TM) = w_1(L), \\ w_i &= w_i(TM) = w_i(V), \qquad 1 < i \leq 4, \end{aligned}$$

and, as in Section 4.3, write

$$\lambda = \lambda(V)$$

for the characteristic class of Spin-bundles, twice which is p_1. The mod 2 reduction of λ is w_4, so every pin$^+$-manifold has an untwisted integer lift of w_4.

Now suppose that (M, c) is an $\mathfrak{m}_c$-manifold. The total space of the orientation double cover $\pi : \widehat{M} \to M$ is a Spin-manifold, and in fact $T\widehat{M}$ is equipped with a stable isomorphism $T\widehat{M} \approx \pi^* V$. The

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class $\hat{c} = \pi^* c$ is an untwisted integer lift of $w_4(T\widehat{M})$. This specifies a class

$$\iota \in H^4(\widehat{M}; \mathbb{Z})$$

satisfying

$$2\iota = \lambda - \hat{c} \tag{8.2}$$

REMARK 8.2 The fact that ι is specified uniquely and not just up to elements of order two follows from the fact that both λ and $\hat{c}$ are *integer lifts* of w_4. The integer lifts of a fixed mod 2 cohomology class of dimension k form a torsor for integer cohomology in dimension k, under the action in which an integer cohomology class ι changes an integer lift c to $c + 2\iota$. See [**33**, Section 2.5] for a more systematic discussion of this from the point of view of cocycles. It is also not difficult to show (for example using (8.1)) that the classifying space $B\mathrm{Spin}\langle \beta w_4 \rangle$ for *Spin* bundles with an integer lift of w_4 is homotopy equivalent to $B\mathrm{Spin} \times K(\mathbb{Z}, 4)$ and in particular has torsion free H^4. So in fact equation (8.2) specifies ι uniquely as a cohomology class in $B\mathrm{Spin}\langle \beta w_4 \rangle$.

Transfer

We will make use of the additive and multiplicative transfers

$$\mathrm{tr} : H^k(\widehat{M}; \mathbb{Z}/2) \to H^k(M; \mathbb{Z}/2)$$
$$P : H^k(\widehat{M}; \mathbb{Z}/2) \to H^{2k}(M; \mathbb{Z}/2).$$

Most computations of the additive transfer can be made in terms of $\widehat{M}$: for $y \in H^* M$ one has

$$\int_M \mathrm{tr}(x)\, y = \int_{\widehat{M}} (x \pi^* y).$$

Computing the map P can be a little tricky, however there are some useful methods in special cases. For one thing, the map P is quadratic:

$$P(x+y) = P(x) + P(y) + \mathrm{tr}(x\tau(y)),$$

where τ is the cohomology homomorphism induced by the involution. One can also compute $P(x)$ in terms of characteristic classes, when x itself is a characteristic class. The following, which is all we need, is a special case of [**35**, Theorem 1.1].

LEMMA 8.3 *Suppose that $p : \widehat{X} \to X$ is a double cover and W is a* Spin*-vector bundle on $\widehat{X}$ of dimension d. If $x = w_4(W) \in H^4(\widehat{X}; \mathbb{Z}/2)$ then*

$$P(x) = w_8(p_* W - p_* \mathbb{R}^d) \in H^8(X; \mathbb{Z}/2)$$

where, for a vector bundle V, $p_* V = \widehat{X} \underset{\mathbb{Z}/2}{\times} (V \oplus V)$. □

REMARK 8.4 In the situation of Lemma 8.3, if $W = p^* U$ for some vector bundle U on X then $p_* W = U \oplus (U \otimes L)$ where $L = \widehat{X} \underset{\mathbb{Z}/2}{\times} \mathbb{R}$, with $\mathbb{Z}/2$ is acting on $\mathbb{R}$ by the sign representation. In that case $P(x) = w_8(U + U \otimes L - L^{\oplus d})$.

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8.3. *Two examples*

Two characteristic classes play an important role in our computation of $\pi_{12}M\mathfrak{m}_c$. They are $\mathrm{tr}(\iota^3 + \iota^2 w_4)$ and $\alpha^4 P(\iota)$.

EXAMPLE 8.5 Recall from Section 5.5.1 the pair $(W_0', \tilde{c}_0')$ in which

$$W_0' = S^4 \underset{\mathbb{Z}/2}{\times} \mathbb{HP}^2 \# \mathbb{HP}^2.$$

The orientation double cover is

$$\widehat{W}_0' = S^4 \times \mathbb{HP}^2 \# \mathbb{HP}^2 \xrightarrow{\pi} S^4 \underset{\mathbb{Z}/2}{\times} \mathbb{HP}^2 \# \mathbb{HP}^2$$

and the involution of $\mathbb{HP}^2 \# \mathbb{HP}^2$ exchanges the two summands, is orientation preserving, and has two fixed points. The w_1-twisted cohomology class $\tilde{c}_0'$ satisfies

$$\pi^* \tilde{c}_0' = 2t + x_1 - x_2.$$

Since $\lambda = x_1 + x_2$ (see (5.10)) we have

$$\begin{aligned} \iota &= \frac{1}{2}(\lambda - \tilde{\pi}^* c_0') \\ &= x_2 - t \\ \iota^3 + \iota^2 w_4 &= x_2^2 t \end{aligned}$$

and so

$$\int_{W_0'} \mathrm{tr}(\iota^3 + \iota^2 w_4) = \int_{\widehat{W}_0'} (\iota^3 + \iota^2 w_4) = \int_{\widehat{W}_0'} (x_2^2 t) = 1.$$

REMARK 8.6 One can check that $\int_{W_0'} \alpha^4 P(\iota) = 0$, though we will not make use of this fact.

EXAMPLE 8.7 Consider the manifold $W_0'' = P(K_{\mathbb{R}}^{\oplus 2} \oplus \mathbb{R})$ described in Section 5.5.2. Since the Stiefel–Whitney classes of $K_{\mathbb{R}}^{\oplus 2}$ vanish on S^4, the projective bundle formula gives

$$H^*(W_0''; \mathbb{Z}/2) = \mathbb{Z}/2[t, \alpha]/(t^2, \alpha^9)$$

where t is the generator of $H^4(S^4)$. The orientation double cover is $S(K_{\mathbb{R}}^{\oplus 2} \oplus \mathbb{R})$ and the mod 2 reduction of

$$\iota = \frac{1}{2}(\lambda - c)$$

is $t = w_4(K_{\mathbb{R}})$. Since the map

$$H^4(S^4) \to H^4(S(K_{\mathbb{R}}^{\oplus 2} \oplus \mathbb{R}))$$

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is an isomorphism, we have

$$\iota^3 + \iota^2 w_4 = 0,$$

and so

$$\int_{W_0''} \mathrm{tr}(\iota^3 + \iota^2 w_4) = 0.$$

For the characteristic number $\alpha^4 P(\iota)$ we first appeal to Lemma 8.3 and compute

$$P(\iota) = w_8(p_* K_{\mathbb{R}} - p_* \mathbb{R}^4).$$

Since $K_{\mathbb{R}}$ is pulled back from W_0'' we are in the situation of Remark 8.4. Writing L_α for the real line bundle with $w_1(L_\alpha) = \alpha$ we are led to the total Stiefel–Whitney class

$$\begin{aligned} w(K_{\mathbb{R}} + K_{\mathbb{R}} \otimes L_\alpha - 4L_\alpha) &= (1+t)(1+t+\alpha^4)(1+\alpha)^{-4} \\ &= 1 + t^2 + \alpha^4 t + O[9] = 1 + \alpha^4 t + O[9]. \end{aligned}$$

Thus

$$P(\iota) = \alpha^4 t$$

and

$$\int_{W_0''} \alpha^8 P(\iota) = 1.$$

8.4. *The Adams spectral sequence*

Our aim is to identify generators for $\pi_{12} M\mathfrak{m}_c$. Since our main concern is the comparison of two different homomorphisms from $\pi_{12} M\mathfrak{m}_c$ to a finite abelian 2-group, it suffices to do so after completing at 2. For this we can appeal to the Adams spectral sequence, and this can be done by computer calculation. For the purposes of this paper the authors used *Mathematica* to determine the mod 2 cohomology of $B\mathfrak{m}_c$ as a module over the mod 2 Steenrod algebra, and Rob Bruner's program [**10**] for computing the E_2-term of the Adams spectral sequence, as well as the map of Adams spectral sequences induced by the map from M Spin to $M\mathfrak{m}_c$. The results are displayed in Figs 1 and 2. Appendix C contains much more information about the cohomology of $B\mathfrak{m}_c$. In [**31**], a more detailed version of this computation is described, as well as means of doing it by hand.

The Adams spectral sequence begins with

$$E_2^{s,t} = \mathrm{Ext}_{\mathbf{A}}^{s,t}(H^* M\mathfrak{m}_c, \mathbb{Z}/2) = \mathrm{Ext}_{\mathbf{A}}^{s}(H^{*+t} M\mathfrak{m}_c, \mathbb{Z}/2),$$

with $\mathbf{A}$ the mod 2 Steenrod algebra, and converges to the homotopy groups of the 2-adic completion $\widehat{M\mathfrak{m}_c}$ of $M\mathfrak{m}_c$. Since the homology groups of $B\mathfrak{m}_c$ are each finitely generated, the homotopy groups of $M\mathfrak{m}_c$ are each finitely generated and so $\pi_{t-s}\widehat{M\mathfrak{m}_c}$ is just the 2-adic completion of $\pi_{t-s} M\mathfrak{m}_c$.

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REMARK 8.8 For the remainder of this section all bordism groups will be 2-adically completed. Except for the appearance of the symbol $\mathbb{Z}_2$ for the 2-adic numbers, this will not be indicated in the notation.

The differential d_r of the Adams spectral sequence goes from bidegree (s,t) to bidegree $(s+r, t+r-1)$. It is customary to display the Adams spectral sequence with the horizontal axis numbered by $(t-s)$ and the vertical axis s. With this convention the differential d_r goes one square to the left and r-squares upward. The groups contributing to a given homotopy group lie in a column.

The 's' in the Adams spectral sequence direction corresponds to a decreasing filtration of stable homotopy groups known as the *Adams filtration*.

DEFINITION 8.9 A map $f : X \to Y$ of spectra has (mod 2) *Adams filtration greater than or equal to n* if there is a factorization

$$X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \to X_n = Y$$

in which each f_i induces the zero map in mod 2 cohomology.

The Adams filtration is natural in both variables, in the sense that composition with a map $X' \to X$ or $Y \to Y'$ sends maps of Adams filtration greater than or equal to n to maps of Adams filtration greater than or equal to n.

DEFINITION 8.10 A map has *Adams filtration n* if it has Adams filtration greater than or equal to n but not greater than or equal to $(n+1)$.

The maps in Adams filtration greater than or equal to n appear in the Adams charts with s-coordinate greater than or equal to n.

The Adams spectral sequence for $\pi_* X$ is a module over the Adams spectral sequence for the stable homotopy groups $\pi_* S^0$ of spheres. The element $2 \in \pi_0 S^0$ has Adams filtration 1 and is represented by a class traditionally denoted

$$h_0 \in \mathrm{Ext}_{\mathbf{A}}^{1,1}(\mathbb{Z}/2, \mathbb{Z}/2).$$

Multiplication by h_0 in any Adams spectral sequence is indicated in the chart by a vertical line. Similarly, the Hopf map $\eta \in \pi_1 S^0$ is represented by the element

$$h_1 \in \mathrm{Ext}_{\mathbf{A}}^{2,1}(\mathbb{Z}/2, \mathbb{Z}/2)$$

and multiplication by h_1 is indicated by a $(1,1)$ diagonal line. A little care must be used in drawing conclusions from these notations. For example, Fig. 1 shows the Adams spectral sequence for $\pi_* M\mathfrak{m}_c$. From the chart it appears that $\pi_4 M\mathfrak{m}_c = \mathbb{Z}/8 \oplus \mathbb{Z}_2$, with the generator of the $\mathbb{Z}/8$ appearing in Adams filtration 1. However all that the chart implies is that 8 times the apparent generator in filtration 1 has Adams filtration greater than 4. Some additional argument is needed to conclude that there is an element of order 8 in filtration 1. One can conclude from the chart that π_4 is generated by 2 elements and has rank 1. The computation of $\pi_4 M\mathfrak{m}_c$ will be given in detail in Section 8.5.2.

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The $s=0$ line of the Adams spectral sequence consists of the groups

$$\hom_{\mathbf{A}}(H^tX, H^0S^0) \subset H_tX.$$

The kernel of the higher differentials pick out the image of the Hurewicz homomorphism in H_tX. When $X = \mathrm{Thom}(B,V)$ is a Thom spectrum it is often useful to label an element $x \in E_2^{0,t} = \hom_{\mathbf{A}}(H^tX, H^0S^0)$ with a cohomology class $\beta \in H^t(B)$ whose image under

$$H^t(B) \overset{\text{Thom iso}}{\longrightarrow} H^t(X) \overset{x^*}{\longrightarrow} H^t(S^t) = \mathbb{Z}/2 \tag{8.3}$$

is non-zero. This can be a little perilous as there can be many cohomology classes having a non-zero image under given class, and some care must be taken to ensure that the labeled cohomology classes are linearly independent on the image of the Hurewicz homomorphism. In the end it provides useful information. If x survives the Adams spectral sequence and is represented by a manifold M, the image of β under (8.3) is

$$\int_M \beta.$$

Such labels therefore provide a means of identifying specific manifolds as representing a basis of the image of the Hurewicz homomorphism. The class β is a characteristic class of some kind.

8.5. *Computations*

Armed with these spectral sequences, we first turn to the computation of $\pi_* M\mathfrak{m}_c$ in low degrees and then proceed to $\pi_{12} M\mathfrak{m}_c$. We remind the reader of Remark 8.8, that all homotopy groups have been 2-adically completed.

Dimension less than 4
The homotopy fiber of the map $B\mathfrak{m}_c \to B\mathrm{Pin}^+$ is the Eilenberg–MacLane space $K(\mathbb{Z},4)$. It follows easily from this that the map

$$\pi_* M\mathfrak{m}_c \to \pi_* MT\mathrm{Pin}^+$$

is an isomorphism for $*<4$ and an epimorphism when $*=4$. From [**38**] one concludes that

$$\begin{aligned} \pi_0 M\mathfrak{m}_c &= \mathbb{Z}/2 \qquad \text{generated by a point} \\ \pi_1 M\mathfrak{m}_c &= 0 \\ \pi_2 M\mathfrak{m}_c &= \mathbb{Z}/2 \qquad \text{generated by } (\Sigma, 0) \end{aligned}$$

where Σ is a Klein bottle in a non-bounding pin$^+$-structure, and

$$\pi_3 M\mathfrak{m}_c = \mathbb{Z}/2 \qquad \text{generated by } S^1 \times \Sigma,$$

where S^1 is given the non-bounding String-structure (on S^1 a String structure is equivalent to a Spin-structure).

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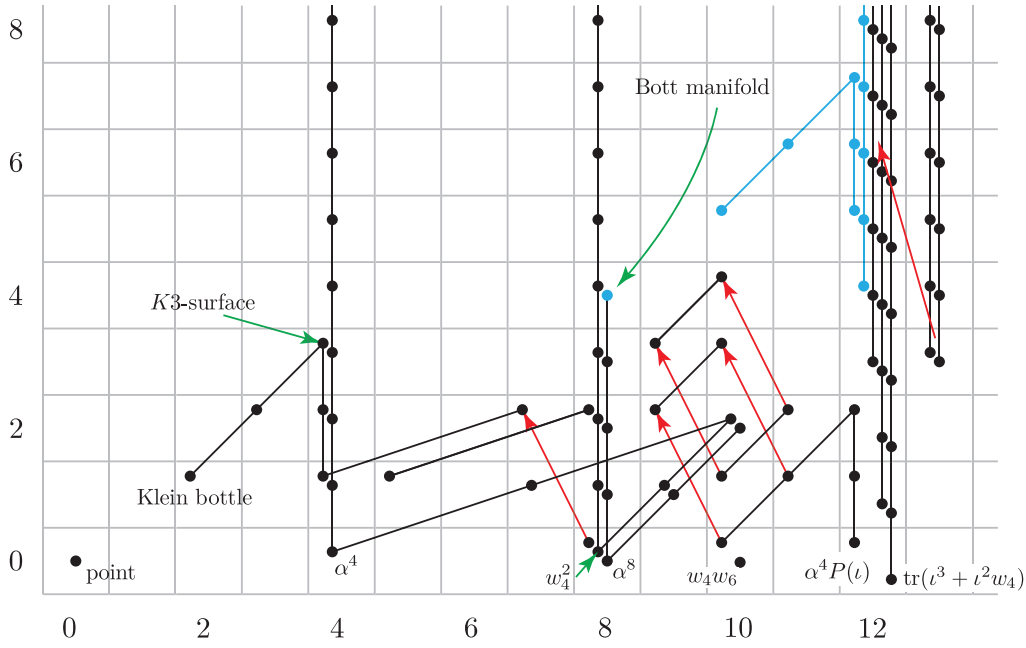

Figure 1. The Adams spectral sequence for $\pi_* M\mathfrak{m}_c$

Dimension 4

We define a homomorphism $e : \pi_4 M\mathfrak{m}_c \to \mathbb{Z}_2$ by

$$e(M,c) = \int_M c.$$

Forgetting the twisted lift of w_4 gives a map

$$u : \pi_* M\mathfrak{m}_c \to \pi_* MT\mathrm{Pin}^+.$$

By [38], the group $\pi_4 MT\mathrm{Pin}^+$ is cyclic of order 16, with generator $\mathbb{RP}^4$. Combined, these two homomorphisms give a map

$$\pi_4 M\mathfrak{m}_c \xrightarrow{(e,u)} \mathbb{Z}_2 \oplus \mathbb{Z}/16. \tag{8.4}$$

PROPOSITION 8.11 *The map above gives an isomorphism of $\pi_4 M\mathfrak{m}_c$ with the set of elements $(a,b) \in \mathbb{Z}_2 \oplus \mathbb{Z}/16$ with $a \equiv b \mod 2$. The group $\pi_4 M\mathfrak{m}_c$ is generated by $(\mathbb{RP}^4, \tilde{c}_{\mathbb{RP}^4})$ and $(\mathbb{RP}^4 \# \mathbb{RP}^4, 0)$.*

Proof. By definition, the map

$$\pi_4 M\mathfrak{m}_c \xrightarrow{e} \mathbb{Z}_2 \to \mathbb{Z}/2$$

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is given by $\int_M w_4$, and the map

$$\pi_4 M\mathfrak{m}_c \to \pi_4 MT\mathrm{Pin}^+ \to \pi_4 MT\mathrm{Pin}^+ \otimes \mathbb{Z}/2 = \mathbb{Z}/2$$

is given by $\int_M w_1^4$. Since $Sq^4\colon H^0(M;\mathbb{Z}/2) \to H^4(M;\mathbb{Z}/2)$ is zero, the fourth Wu class (4.20) vanishes. Therefore, $\int_M (w_4 + w_1 w_3 + w_2^2 + w_1^4) = \int_M (w_4 + w_1^4) = 0$. This implies

$$\int_M w_1^4 = \int_M w_4,$$

so the image of (8.4) is contained in the subgroup of elements (a, b) with $a \equiv b \mod 2$. On the other hand, the Adams spectral sequence shows that the kernel of e has order at most 8. The $M\mathfrak{m}_c$-manifold $(\mathbb{RP}^4 \# \mathbb{RP}^4, 0)$ is in the kernel of e. Its image in $\pi_4 MT\mathrm{Pin}^+ = \mathbb{Z}/16$ is $2[\mathbb{RP}^4]$. Since $[\mathbb{RP}^4]$ generates $\pi_4 MT\mathrm{Pin}^+$, the image of $(\mathbb{RP}^4 \# \mathbb{RP}^4, 0)$ actually has order 8. The assertion about generators follows from the computation

$$(\mathbb{RP}^4, \tilde{c}_1) \mapsto (1, 1)$$
$$(\mathbb{RP}^4 \# \mathbb{RP}^4, 0) \mapsto (0, 2).$$

This completes the proof. □

EXAMPLE 8.12 If M is a *Spin*-manifold of dimension 4 then under (8.4) one has

$$(M, \lambda) \to (\lambda(M), \lambda(M)).$$

It follows that $(M, \lambda) \equiv \lambda(M)(\mathbb{RP}^4, \tilde{c}_1)$. In particular

$$[K, \lambda] = -24[\mathbb{RP}^4, \tilde{c}_1]$$

when K is a Kummer surface.

Dimension 12
Our main result in dimension 12 is the following restatement of Theorem 6.1.

PROPOSITION 8.13 *The group $\pi_{12} M\mathfrak{m}_c$ is generated (over $\mathbb{Z}_2$) by the six manifolds*

$$(W_0', \tilde{c}_0'), \quad (W_0'', 0), \quad (W_1, \lambda)$$
$$(K \times \mathbb{HP}^2, \lambda), \quad (\mathbb{RP}^4, \tilde{c}'_{\mathbb{RP}^4}) \times B, \quad (\mathbb{RP}^4 \# \mathbb{RP}^4, 0) \times B.$$

□

The proof makes use of the following fact about Spin-bordism, which we prove in Appendix B.

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PROPOSITION 8.14 *The group* $\pi_{12}M\,\mathrm{Spin}$ *is free of rank* 3*, and generated by* $K \times B$*,* $K \times \mathbb{HP}^2$*, and the manifold* W_1 *described in Section* 5.5.3*, sitting in the fibration sequence*

$$\mathbb{HP}^2 \to W_1 \to \mathbb{CP}^1 \times \mathbb{CP}^1.$$

□

Proof. Proof of Proposition 8.13 We begin by extracting some facts from the Adams spectral sequence. First of all, the map

$$\pi_{12}M\,\mathfrak{m}_c \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \tag{8.5}$$

with components

$$\int_M \alpha^4 P(\iota) \quad \text{and} \quad \int_M \mathrm{tr}(\iota^3 + \iota^2 w_4)$$

gives an isomorphism of the quotient of $\pi_{12}M\,\mathfrak{m}_c$ by the elements of positive Adams filtration with a subgroup of $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. The computations of examples 8.5 and 8.7 show that this subgroup is in fact all of $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. The kernel of this map contains the image of $\pi_{12}M\,Spin$ and the image of multiplication by B. This follows from a consideration of Adams filtrations, but it is easily checked directly. Indeed if M is a *Spin*-manifold, then M is oriented, $\alpha = 0$, and $\int_M \alpha^4 P(\iota) = 0$. Also, since the orientation double cover of M is $M \amalg M$, one has

$$\begin{aligned}\int_M \mathrm{tr}(\iota^3 + \iota^2 w_4) &= \int_{M \amalg M} (\iota^3 + \iota^2 w_4) \\ &= 2 \int_M (\iota^3 + \iota^2 w_4) = 0.\end{aligned}$$

In the case $(M, c) = (N_4, c') \times B$ all of the characteristic classes to be integrated are pulled back from $H^8(N_4) = 0$.

Let $J' \subset \pi_{12}M\,\mathfrak{m}_c$ be the subgroup generated by the image of $\pi_{12}M\,\mathrm{Spin}$ and the image of multiplication by B, and let

$$C = \pi_{12}M\,\mathfrak{m}_c / J'.$$

A portion of the Adams spectral sequence for computing the map $\pi_{12}M\,\mathrm{Spin} \to \pi_{12}M\,\mathfrak{m}_c$ is shown in Fig. 2. The map, which is part of the (machine) computation of *Ext*, can also be determined by composing with the map $\pi_{12}M\,\mathfrak{m}_c \to \pi_{12}MT\mathrm{Pin}^+$. From it one can read off that the map (8.5) gives an isomorphism

$$C \otimes \mathbb{Z}/2 \approx \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

Since C is finitely generated, Nakayama's Lemma and the computations of Examples 8.5 and 8.7 show that C is generated by $(W_0', \tilde{c}_0')$ and $(W_0'', 0)$.

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Let $J \subset \pi_{12} M\mathfrak{m}_c$ be the subgroup generated by

$$(W_1, \lambda), \quad (K \times \mathbb{HP}^2, \lambda), \quad (\mathbb{RP}^4, \tilde{c}'_{\mathbb{RP}^4}) \times B, \quad \text{and} \quad (\mathbb{RP}^4 \# \mathbb{RP}^4, 0) \times B.$$

We are done if we can show that the manifolds $(W'_0, \tilde{c}'_0)$ and $(W''_0, 0)$ generate $\pi_{12} M\mathfrak{m}_c/J$. Note that by Proposition 8.11, the subgroup J contains J': it contains image of multiplication by B and by Proposition 8.14 it contains the image of $\pi_{12} M\,Spin$. By the above discussion, the manifolds $(W'_0, \tilde{c}'_0)$ and $(W''_0, 0)$ generate $\pi_{12} M\mathfrak{m}_c/J'$, so they certainly generate $\pi_{12} M\mathfrak{m}_c/J$. This completes the proof. □

Dimension 11

The ambiguity of the M theory action has to do with the group $\pi_{11} M\mathfrak{m}_c$. In this section, we offer a tentative evaluation of this group. Since the ambiguity involves the entire group, we drop the convention that groups have been completed at 2.

PROPOSITION 8.15 *The group $\pi_{11} M\mathfrak{m}_c$ is a finite abelian 2-group.*

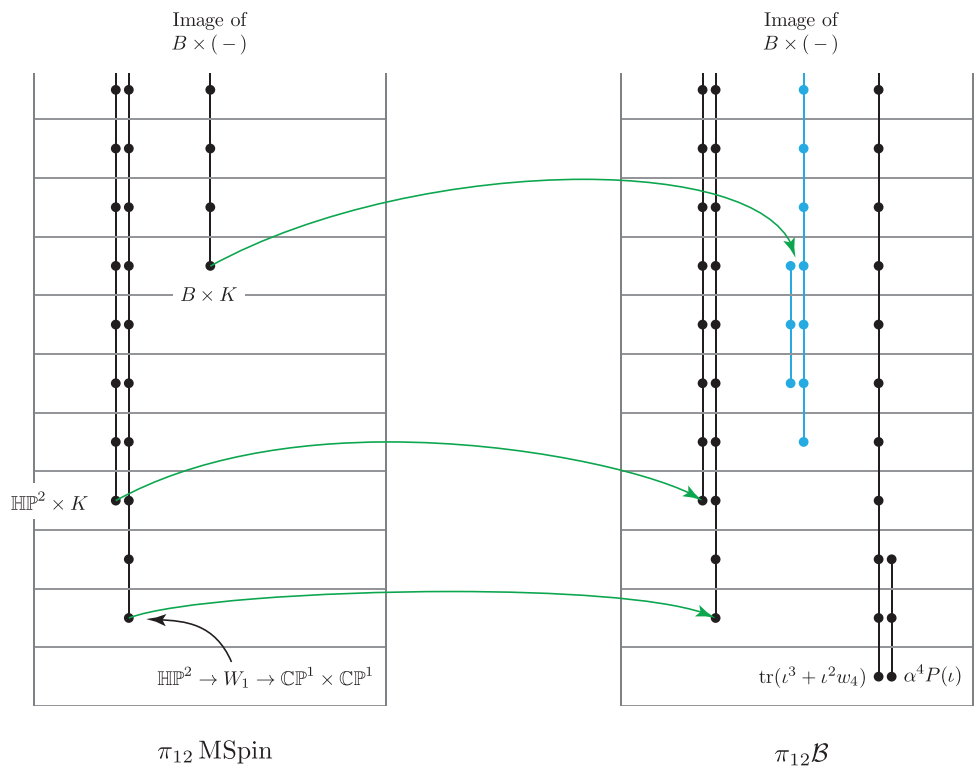

Figure 2. The map from *Spin* bordism to $M\mathfrak{m}_c$-bordism in dimension 12

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Proof. From the theory of Serre classes, one knows that $\pi_{11}M\mathfrak{m}_c$ is finitely generated. It therefore suffices to show that

$$\pi_{11}M\mathfrak{m}_c \otimes \mathbb{Z}[1/2] = 0.$$

The homotopy fiber of the map $B\mathfrak{m}_c \xrightarrow{w_1} K(\mathbb{Z}/2,1)$ is $B\mathrm{Spin} \times K(\mathbb{Z},4)$, representing $B\mathfrak{m}_c$ as the homotopy quotient of an action of $\mathbb{Z}/2$ on $B\mathrm{Spin} \times K(\mathbb{Z},4)$. A map $T \to B\mathrm{Spin} \times K(\mathbb{Z},4)$ classifies a pair consisting (V,ι) consisting of a *Spin* bundle $V \to T$ and an element $\iota \in H^4(T;\mathbb{Z})$. The pullback of the universal w_1-twisted integer lift c of w_4 is

$$c = \lambda(V) - 2\iota$$

(see (8.2)), and the generator of the $\mathbb{Z}/2$-action sends (V,ι) to

$$(V, \lambda(V) - \iota).$$

By passing to Thom spectra, this depicts $M\mathfrak{m}_c$ as the homotopy quotient of the $\mathbb{Z}/2$ spectrum $M\mathrm{Spin} \wedge K(\mathbb{Z},4)_+ \wedge S^{1-\sigma}$, where σ is the sign representation of $\mathbb{Z}/2$. This means that after inverting 2, the map

$$\pi_* M\mathrm{Spin} \wedge K(\mathbb{Z},4)_+ \to \pi_* M\mathfrak{m}_c$$

is projection to a summand. The claim now follows from Stong's Theorem [**51**], which states that $\pi_{11}M\mathrm{Spin} \wedge K(\mathbb{Z},4)_+ = 0$. □

From the discussion in the above proof, it is an easy matter to compute

$$\pi_* M\mathfrak{m}_c \otimes \mathbb{Q}.$$

Let $J = (j_1, \dots,)$ run through the sequences of non-negative integers, almost all of which are 0, and write

$$|J| = j_1 + 2j_2 + \cdots + nj_n + \cdots$$
$$p^J = p_1^{j_1} p_2^{j_2} \cdots.$$

PROPOSITION 8.16 *The group $\pi_m M\mathfrak{m}_c \otimes \mathbb{Q}$ is zero if m is not divisible by* 4. *The map*

$$(M,c) \longmapsto \int_M c^{2k+1} p^J$$

gives an isomorphism

$$\pi_{4n} M\mathfrak{m}_c \otimes \mathbb{Q} \longrightarrow \prod_{2k+1+|J|=n} \mathbb{Q}.$$

For example, this implies that the group $\pi_{12}M\mathfrak{m}_c$ has rank 3, corresponding to the indices $(k,J) = (0,(2,0)), (0,(0,1)), (1,(0))$. This implies that there must be non-trivial differentials in the chart (Fig. 1) from dimension 13 to dimension 12.

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REMARK 8.17 In [51] (note after Item 6), Stong also shows that for $* < 12$ the groups

$$\pi_* M\mathrm{Spin} \wedge K(\mathbb{Z}, 4)_+ \otimes \mathbb{Z}[1/2]$$

are torsion free. In fact his argument for dimension 8 can also be adapted to dimension 12 to establish the same conclusion for $* = 12$. So the above also provides an evaluation of the groups

$$\pi_* M\mathfrak{m}_c \otimes \mathbb{Z}[1/2], \qquad * \leq 12.$$

Because of Proposition 8.15, the group $\pi_{11} M\mathfrak{m}_c$ can be determined from the Adams spectral sequence, which is displayed in Fig. 1. The E_2-term provides an upper bound and shows that the group has order at most 8. In the table, there are two d_3-differentials indicated, originating in Adams filtrations 1 and 2. These should be regarded as tentative at the moment and will appear in [31]. Assuming them, the Adams spectral sequence shows that after 2-completion, the group $\pi_{11} M\mathfrak{m}_c$ is cyclic of order 2 and that an isomorphism is given by the mod 2 index of the pin^+-Dirac operator. We state the outcome of this argument as a restatement of Conjecture 7.1.

CONJECTURE 8.18 *The map*

$$\pi_{11} M\mathfrak{m}_c \to \mathbb{Z}/2$$

given by the mod 2 index of the pin^+ *Dirac operator is an isomorphism.*

REMARK 8.19 Let M be the product of the Bott manifold, S^1 in its non-bounding *String*-structure and $(\Sigma, 0)$ where Σ is the Klein bottle in a non-bounding *pin*$^+$-structure (see Section 8.5.1). The mod 2 index of the *pin*$^+$ Dirac operator on M is 1 and so the above conjecture implies that $\pi_{11} M\mathfrak{m}_c$ is generated by M.

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Appendix A On the anomaly theory of a spinor field

In an n-dimensional field theory F, the partition function of a spinor field on a closed n-dimensional Riemannian manifold is the pfaffian of a Dirac operator, which is an element of a Pfaffian line, as reviewed in Section 3. The Pfaffian line is the quantum state space of the associated *anomaly theory*, which is an invertible $(n+1)$-dimensional theory α, but initially truncated to manifolds of dimension $\leq n$, since such manifolds form the domain of F. To extend α to an $(n+1)$-dimensional theory, we must define the partition function of a closed $(n+1)$-manifold, as well as an element in the state space of the boundary of a compact $(n+1)$-manifold with boundary, and these elements must satisfy a gluing law. The results in [17] imply that an exponentiated η-invariant works as the partition function: on a compact $(n+1)$-manifold with boundary, it takes values in the Pfaffian line. But to define it, we must construct a Riemannian Dirac operator in $(n+1)$-dimensions

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from the n-dimensional Lorentzian data which define the spinor field. The construction was given in [**26**, Section 9.2.5], but only in passing; in this appendix, we give more detail. We discuss the base case of spin manifolds (no time-reversal symmetry) in Section A.1. In Section A.2, we specialize to 11 dimensions and the pin module relevant to M-theory.

A.1. The spin case in general dimensions

A spinor field in a relativistic quantum field theory is specified by [**16**, Section 6], a real spin representation $\mathbb{S}$ of the Lorentz spin group $\mathrm{Spin}_{1,n-1}$ together with a symmetric positive $\mathrm{Spin}_{1,n-1}$-invariant map

$$\Gamma\colon \mathbb{S}\otimes\mathbb{S}\longrightarrow \mathbb{R}^{1,n-1}. \tag{A1}$$

the positivity condition is that $\Gamma(s,s)$ lie in the closure of the forward timelike vectors in $\mathbb{R}^{1,n-1}$. Thus, $\mathbb{S}$ is an ungraded module over the even subalgebra $\mathrm{Cliff}^0_{n-1,1}$ of the Clifford algebra, with $n-1$ generators squaring to $+1$ and a single generator squaring to -1. The pair $(\mathbb{S},\Gamma)$ Wick rotates to define a Dirac operator on a Riemannian spin n-manifold X as follows. First, the complexification $\mathbb{S}_{\mathbb{C}}$ is a module over the complex algebra $\mathrm{Cliff}^0_n(\mathbb{C})$, so restricts to a representation of the compact spin group Spin_n. Also, Γ complexifies to a Spin_n-equivariant morphism

$$\Gamma_{\mathbb{C}}\colon (\mathbb{C}^n)^*\otimes\mathbb{S}_{\mathbb{C}}\longrightarrow \mathbb{S}^*_{\mathbb{C}}. \tag{A2}$$

Let $S\to X$ be the complex vector bundle on X associated with the Spin_n representation $\mathbb{S}_{\mathbb{C}}$. Then as usual the Dirac operator on X is the composition

$$D_X=\Gamma_{\mathbb{C}}\circ\nabla\colon C^\infty(X;S)\longrightarrow C^\infty(X;S^*), \tag{A3}$$

where ∇ is the covariant derivative on sections of $S\to X$. The operator D_X is complex skew-adjoint. (The metric on $S\to X$ is constructed in the next paragraph.) For X closed this operator appears in the Dirac form (3.6), and its pfaffian is the fermionic path integral (3.4).

The construction of a Dirac operator on a Riemannian spin $(n+1)$-manifold W from the data $(\mathbb{S},\Gamma)$ uses the Clifford linear Dirac operator [**40**, Section II.7] and Morita equivalence of Clifford algebras. By [**16**, Corollary 6.2], the data $(\mathbb{S},\Gamma)$ determine a unique $\mathbb{Z}/2\mathbb{Z}$-graded $\mathrm{Cliff}_{n-1,1}$-module structure on $\mathbb{S}\oplus\mathbb{S}^*$. Let γ^0 denote the Clifford generator with $(\gamma^0)^2=-1$, acting as an odd endomorphism of $\mathbb{S}\oplus\mathbb{S}^*$, and $\gamma^1,\dots,\gamma^{n-1}$ the Clifford generators with $(\gamma^i)^2=+1$. Fix an inner product on $\mathbb{S}\oplus\mathbb{S}^*$ such that the finite group consisting of products of the γ^μ, $\mu=0,1,\dots,n-1$ acts orthogonally. (It follows that Spin_n acts unitarily on $\mathbb{S}_{\mathbb{C}}$, which induces the hermitian metric on $S\to X$ used in (A3).) Now

$$\mathrm{Cliff}_{+(n+1)}\otimes(\mathbb{S}\oplus\mathbb{S}^*) \tag{A4}$$

is a real super vector space which carries a left action of Spin_{n+1}—by left multiplication on

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$\mathrm{Cliff}_{+(n+1)}$ tensored the identity on the second factor—and a commuting left action of the real super algebra

$$A = \mathrm{Cliff}_{-(n+1)} \otimes \mathrm{Cliff}_{n-1,1}. \tag{A5}$$

Elements of $\mathrm{Cliff}_{+(n+1)}$ act by right multiplication on the first factor of (A4), tensored with the identity on the second factor, which is equivalent to a left action of $\mathrm{Cliff}_{-(n+1)}$, the opposite super algebra. Elements of $\mathrm{Cliff}_{n-1,1}$ act by the identity on the first factor of (A4) tensored with the action above on the second factor. Now the left Spin_{n+1} action on (A4) defines a bundle of real A-modules over W as well as a Dirac operator on its sections which commutes with the action of A. We claim this solves the problem of defining an $(n+1)$-dimensional Riemannian Dirac operator from $(\mathbb{S}, \Gamma)$ which can be used in the anomaly theory α. To verify that claim we must (i) define the exponentiated η-invariant of this operator and (ii) identify the induced operator in n dimensions with (A3).

For (i) we use a (super) Morita equivalence of A with Cliff_{-3}, 'canceling' the last $n-1$ generators of $\mathrm{Cliff}_{-(n+1)}$ with the $n-1$ positive generators of $\mathrm{Cliff}_{n-1,1}$. (The cancellation identifies $\mathrm{Cliff}_{-(n-1)} \otimes \mathrm{Cliff}_{+(n-1)}$ with the super algebra of endomorphisms of the vector space $\mathrm{Cliff}_{+(n-1)}$, which is Morita trivial.) Under the Morita isomorphism, left A-modules are identified with left Cliff_{-3}-modules, and so the A-module (A4), rewritten as

$$\mathrm{Cliff}_{+(n+1)} \otimes (\mathbb{S} \oplus \mathbb{S}^*) \cong \mathrm{Cliff}_{+(n-1)} \otimes \mathrm{Cliff}_{+2} \otimes (\mathbb{S} \oplus \mathbb{S}^*), \tag{A6}$$

is identified with

$$\begin{aligned} \mathbb{E} &= \mathrm{Hom}_{\mathrm{Cliff}_{+(n-1)} \otimes \mathrm{Cliff}_{-(n-1)}} \left(\mathrm{Cliff}_{+(n-1)}\,,\ \mathrm{Cliff}_{+(n-1)} \otimes \mathrm{Cliff}_{+2} \otimes (\mathbb{S} \oplus \mathbb{S}^*) \right) \\ &\cong \mathrm{Cliff}_{+2} \otimes (\mathbb{S} \oplus \mathbb{S}^*). \end{aligned} \tag{A7}$$

Let δ^1, δ^2 be the generators of Cliff_{+2}, and γ^0 as above the (negative) generator of the $\mathrm{Cliff}_{n-1,1}$-action on $\mathbb{S} \oplus \mathbb{S}^*$. Then the Cliff_{-3} which acts on $\mathbb{E}$ is generated by $\gamma^0, \delta^1, \delta^2$, with δ^1, δ^2 acting by *right* multiplication on Cliff_{+2}, tensored with the identity on $\mathbb{S} \oplus \mathbb{S}^*$. Furthermore, $\mathrm{Spin}_{n+1} \subset \mathrm{Cliff}_{+(n+1)}$ acts on $\mathbb{E}$ using left multiplication by δ^1, δ^2 on Cliff_{+2}, tensored with the identity on $\mathbb{S} \oplus \mathbb{S}^*$, and by $\gamma^1, \ldots, \gamma^{n-1}$ acting on $\mathbb{S} \oplus \mathbb{S}^*$, tensored with the identity on Cliff_{+2}. The latter actions determine an odd skew-adjoint Dirac operator D on sections of the real vector bundle $E = E^0 \oplus E^1 \to W$ associated with the representation $\mathbb{E}$ of Spin_{n+1}. The operator D commutes with the left Cliff_{-3}-action on E. Now $\delta^1\delta^2 \in \mathrm{Cliff}_{-3}$ acts as a complex structure on E and $\gamma^0\delta^2 \in \mathrm{Cliff}_{-3}$ acts as a complex antilinear operator which squares to $-\mathrm{id}_E$. Thus E has a quaternionic structure. (More

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simply, the ungraded algebra Cliff^0_{-3} is isomorphic to the quaternion algebra.) The even self-adjoint operator

$$D^0 := \gamma^0\delta^1\delta^2 D\colon C^\infty(W;E^0) \to C^\infty(W;E^0) \tag{A8}$$

commutes with this quaternionic structure. Assume W is compact without boundary. Then D^0 is elliptic, so has a discrete spectrum and the eigenspaces E^0_λ are finite dimensional quaternionic vector spaces. Let $a \in \mathbb{R}$ be in the complement of the spectrum. Define

$$\eta_a(s) = \sum_{\lambda\neq 0} \mathrm{sign}(\lambda - a)\ (\dim_{\mathbb{C}} E^0_\lambda)\ |\lambda|^{-s} \ -\ \mathrm{sign}(a)\dim_{\mathbb{C}}\ker D^0, \qquad \mathrm{Re}(s) >> 0, \tag{A9}$$

where the sum is over the non-zero eigenvalues of D^0 and s is a complex number. According to the results of [4–6] the sum converges to a holomorphic function of s if the real part of s is sufficiently large, it has a meromorphic continuation to $\mathbb{C}$, and $s = 0$ is a regular point. Then

$$\alpha(W) = \exp(2\pi i\eta_a(0)/4) \tag{A10}$$

is independent of a. It is the partition function of the anomaly theory.

We now verify (ii) above, namely that the Dirac operator in n dimensions induced from (A8) (see [4, (3.1)]) can be identified with (A3). Let X be an n-dimensional Riemannian spin manifold and consider $W = \mathbb{R}\times X$ with the product Riemannian metric and spin structure. Then W has a Dirac operator (A8), which we view as complex since it commutes with the complex structure $\delta^1\delta^2$. Choose a local orthonormal framing on X, with basis numbered $0, 1, \dots, n-1$, and a global coordinate t on $\mathbb{R}$. Order the Clifford generators of $\mathrm{Cliff}_{+(n+1)}$ as $\delta^1, \delta^2, \gamma^1, \dots, \gamma^{n-1}$, as in the previous paragraph. Then the Dirac operator D^0 on W can be written locally as

$$D^0 = \gamma^0\delta^1\delta^2\left(\delta^1\frac{\partial}{\partial t} + \delta^2\nabla_0 + \gamma^i\nabla_i\right), \tag{A11}$$

where the sum over i runs from 1 to $n-1$. The symbol of D, evaluated on dt, is induced from the algebraic operator

$$J(x\otimes\chi) = -(-1)^{|x|}\,\delta^1 x\delta^1\delta^2\otimes\gamma^0\chi, \qquad x\in\mathrm{Cliff}_{+2}, \quad \chi\in\mathbb{S}\oplus\mathbb{S}^*, \tag{A12}$$

where $|x| \in \mathbb{Z}/2\mathbb{Z}$ is the parity of the homogeneous element x. Then J commutes with the complex structure $\delta^1\delta^2$, anticommutes with D^0, and $J^2 = -\mathrm{id}$. According to [17, Section 1] the induced Dirac operator D^0_X on X is the operator (A11) restricted to functions on $\mathbb{R}\times X$ which are invariant under translation in t, mapping the $+\sqrt{-1}$-eigenspace of J to the $-\sqrt{-1}$-eigenspace of J. Thus, if now

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$x \otimes \chi$ is a section of $E^0 \to X$, we compute

$$D_X^0(x \otimes \chi) = -(-1)^{|x|} (\delta^2 \nabla_0 + \gamma^i \nabla_i)(x \delta^1 \delta^2 \otimes \gamma^0 \chi). \tag{A13}$$

Write the $\pm\sqrt{-1}$ eigenbundles of J as $E^0_\pm \to X$, and recall the complex vector bundle $S \to X$ associated with $\mathbb{S}_\mathbb{C}$. Then there are isomorphisms

$$\begin{aligned} S &\longrightarrow E^0_+ \\ \psi + \sqrt{-1}\psi' &\longmapsto 1 \otimes \psi - \delta^1 \otimes \gamma^0 \psi - \delta^2 \otimes \gamma^0 \psi + \delta^1 \delta^2 \otimes \psi' \end{aligned} \tag{A14}$$

and

$$\begin{aligned} S^* &\longrightarrow E^0_- \\ \lambda + \sqrt{-1}\lambda' &\longmapsto -1 \otimes \gamma^0 \lambda' + \delta^1 \otimes \lambda' - \delta^2 \otimes \lambda + \delta^1 \delta^2 \otimes \gamma^0 \lambda \end{aligned} \tag{A15}$$

A straightforward computation demonstrates that these isomorphisms intertwine the operators D_X^0 in (A13) and D_X in (A3), where the latter is

$$D_X = \sqrt{-1}\gamma^0 \nabla_0 + \gamma^i \nabla_i \tag{A16}$$

in the local moving frame on X. The factor $\sqrt{-1}$ comes from Wick rotation when passing from (A1) to (A2).

As a companion to (A8) we have the operator

$$D^1 := \gamma^0 \delta^1 \delta^2 D \colon C^\infty(W; E^1) \to C^\infty(W; E^1) \tag{A17}$$

of the odd subspace of $E = D^0 \oplus E^1$. Note that swapping $\mathbb{S}$ and $\mathbb{S}^*$ swaps the even and odd parts of E (see (A7)).

PROPOSITION A.1 *The exponentiated η-invariant formed with D^1 is the reciprocal of the exponentiated η-invariant* (A10) *formed with D^0.*

Proof. From the definition of D following (A7), since $Cliff_{-3}$ graded commutes with $Cliff_{+(n+1)}$ it follows that $\omega = \gamma^0 \delta^1 \delta^2$ satisfies $\omega D = -D\omega$, and then since $D^i = \omega D$, $i = 0, 1$, we deduce $\omega D^0 = -D^1 \omega$. Therefore, the spectrum of D^1 is the negative of the spectrum of D^0. Then, distinguishing the η-functions (A9) for D^0, D^1, we have $\eta^0_a(s) = -\eta^1_{-a}(s)$ for all s. The desired conclusion follows by analytic continuation. □

The exponentiated η-invariants are the partition functions of invertible field theories α^0, α^1, and the stronger version of Proposition A.1 is that α^0 and α^1 are inverse theories. If both are topological, which is the case for the application to M-theory, then the stronger assertion follows from Proposition A.1 since the partition function determines the isomorphism class of the theory. Here we will not attempt to justify the stronger assertion in the non-topological case, nor the conjecture that $\alpha^0 \otimes \alpha^1$ admits a *canonical* trivialization.

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A.2. The pin case in dimension 11

We describe the relevant pin representation and check that (A11) produces the Dirac operator in 12 dimensions which appears in [**50**].

We follow [**26**, Section 9.2] in which the Pin^+ case is described by a parameter $s = -1$. The point is to use the embeddings [**26**, Lemma 9.25] and [**26**, (9.44)], which specialize to

$$\begin{aligned} \mathrm{Pin}^+_{12} &\longrightarrow \mathrm{Cliff}^0_{12,1} \\ \gamma^i &\longmapsto \gamma^i \otimes \gamma^- \end{aligned} \tag{A18}$$

and

$$\begin{aligned} \mathrm{Pin}_{10,1} &\longrightarrow \mathrm{Cliff}^0_{10,2} \\ \gamma^i &\longrightarrow \gamma^i \otimes \gamma^-, \end{aligned} \tag{A19}$$

where $(\gamma^-)^2 = -1$ and $i = 1, 2, \dots, 12$. These give embeddings of groups $\mathrm{Pin}^+_{12} \hookrightarrow \mathrm{Spin}_{12,1}$ and $\mathrm{Pin}_{10,1} \hookrightarrow \mathrm{Spin}_{10,2}$. The starting data are a real representation of $\mathrm{Pin}_{10,1}$ obtained by restriction from an ungraded real $\mathrm{Cliff}^0_{10,2}$-module. There are isomorphisms

$$\mathrm{Cliff}_{10,2} \cong \mathrm{Cliff}_{+8} \otimes \mathrm{Cliff}_{2,2} \cong \mathrm{End}(\mathbb{M}^0 \oplus \mathbb{M}^1) \otimes \mathrm{End}(\mathrm{Cliff}_{+2}) \tag{A20}$$

where $\mathbb{M}^i$ is a real vector space of dimension 8. A minimal real $\mathrm{Cliff}^0_{10,2}$-module is the even subspace

$$\mathbb{S} := \mathbb{M}^0 \otimes \mathrm{Cliff}^0_{+2} \;\oplus\; \mathbb{M}^1 \otimes \mathrm{Cliff}^1_{+2} \tag{A21}$$

of $(\mathbb{M}^0 \oplus \mathbb{M}^1) \otimes \mathrm{Cliff}_{+2}$, which has real dimension 32. (We could as well take the odd subspace (see Proposition A.1.)) The restriction of $\mathbb{S}$ to $\mathrm{Cliff}^0_{10,1} \subset \mathrm{Cliff}^0_{10,2}$, or equivalently to $\mathrm{Spin}_{10,1} \subset \mathrm{Spin}_{10,2}$, is irreducible. (The Cliff_{+8} in (A20) splits off and one simply checks for $\mathrm{Cliff}_{2,1} \subset \mathrm{Cliff}_{2,2}$.) By [**16**, Theorem 6.1] there is a $\mathrm{Spin}_{10,1}$-invariant pairing (A1), unique up to a positive scalar, and it is then automatically $\mathrm{Pin}_{10,1}$-invariant. This defines the starting data $(\mathbb{S}, \Gamma)$.

The Wick rotation on 12-manifolds, carried out in the second paragraph of Section A.1, is modified in the first instance by tensoring (A4) with Cliff_{-1} and using the embedding (A18), of course setting $n = 11$. Then the commuting super algebra (A5) is $\mathrm{Cliff}_{1,12} \otimes \mathrm{Cliff}_{10,2}$, which as before is Morita equivalent to Cliff_{-3}. Then $\mathbb{E} = \mathrm{Cliff}_{+2} \otimes (\mathbb{S} \oplus \mathbb{S}^*)$ is as in (A7), but is a left $\mathrm{Cliff}_{12,1} \otimes \mathrm{Cliff}_{-3}$-module: the last Clifford generator in $\mathrm{Cliff}_{12,1}$ acts via the action of the last Clifford generator on the $\mathrm{Cliff}_{10,1}$-module $\mathbb{S} \oplus \mathbb{S}^*$. The even subspace $\mathbb{E}^0 \subset \mathbb{E}$ has real dimension 128 and carries a quaternionic structure, so is a 32-dimensional quaternionic vector space. The resulting representation of Pin^+_{12} agrees with the one described at the end of [**50**, Section 3]. (Stolz distinguishes between two representations of Pin^+_{13}, but they are isomorphic when restricted to Pin^+_{12}.)

Appendix B Spin bordism in dimension 12

The purpose of this appendix is to give a proof of Proposition 8.14, which asserts that after 2-adic completion the group $M\mathrm{Spin}_4 = \pi_{12} M\mathrm{Spin}$ is freely generated by the bordism class of the manifolds

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$K \times B$, $K \times \mathbb{HP}^2$ and W_1. Here B is a Bott-manifold, K is a $K3$-surface, $\mathbb{HP}^2$ is quaternionic projective space and W_1 is the manifold described in Section 5.5.3. This is done by direct application of the computation of Anderson, Brown and Peterson [**3**]. The material in the section owes much to conversations with Meng Guo.

Following Anderson, Brown and Peterson [**2**, Section 4], associated with an oriented vector bundle V over a space X is the *total KO-Pontrjagin class*

$$\pi_t(V) = \sum_{n=0}^{\infty} \pi^n(V)\, t^n \in KO^0(X)[[t]].$$

It is uniquely determined by the following properties:

(1) (Naturality) If $f : Y \to X$ is a continuous map then $\pi_t(f^*V) = f^*\pi_t(V)$.
(2) For vector bundles V and W one has $\pi_t(V \oplus W) = \pi_t(V)\pi_t(W)$.
(3) If V is an oriented 2-plane bundle then $\pi_t(V) = 1 + t(V - 2)$.

Because of property (2) the total KO-Pontrjagin class $\pi_t(V)$ can be defined for virtual oriented vector bundles V.

In practice one often computes $\pi_t(V)$ by following the splitting principle and finding a map $f : Y \to X$ for which KO^*f is a monomorphism and f^*V is isomorphic to the oriented real vector bundle underlying a sum of complex line bundles L_i. In this case one has

$$\pi_t(f^*V) = \prod (1 + t(L_i^u - 2)) \in KO^0(Y) \quad \text{and}$$
$$\pi_t(f^*V) \otimes \mathbb{C} = \prod (1 + t(L_i + L_i^{-1} - 2)) \in K^0(Y),$$

in which the notation W^u is being used for the real vector bundle underlying a complex vector bundle W.

For a Spin-manifold M of dimension d, and a sequence $J = (j_1, j_2, \dots,)$, of non-negative integers j_i, with $j_i = 0$ for $i \gg 0$, one defines the KO-Pontrjagin number $\pi^J(M) \in KO_d$ to be the index of the Clifford linear Dirac operator on M coupled to the virtual bundle

$$\left(\pi^1(TM)\right)^{j_1} \left(\pi^2(TM)\right)^{j_2} \cdots.$$

Anderson, Brown and Peterson showed that the map

$$M\mathrm{Spin}_d \xrightarrow{(\pi^J(M),\, w^N(M))} \prod_J KO_d \times \prod_N \mathbb{Z}/2 \tag{B1}$$

is an isomorphism after completion at 2. Here J runs over the sequences $(j_1, \dots)$ with $j_1 = 0$ and

$$n(J) = (j_1 + 2j_2 + 3j_3 + \cdots) \le \begin{cases} d/4 & n(J) \text{ even} \\ (d+2)/4 & n(J) \text{ odd.} \end{cases}$$

The invariants $w^N(M)$ are certain Stiefel–Whitney numbers of M, and do not occur in dimension less than 20. Both sides of (B1) are finitely generated abelian groups, so that the property of being

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an isomorphism after 2-adic completion is equivalent to being an isomorphism after localization at 2 and also to being an isomorphism after reducing mod 2. For further information see [1, Corollary 1.4], [3, Theorem 2.2] or the Manifold Atlas [41]. We alert the reader consulting these sources that our convention associating Pontrjagin numbers to sequences J differs from the one used in these references.

For the purpose of writing down KO-Pontrjagin numbers it is helpful to choose a basis of KO_{4k} that is compatible with multiplication. We do this by identifying KO_{4k} with its image in K_{4k}. Writing $v_1 \in K_2$ for the Bott periodicity class, a basis for the image of KO_* consists of the elements

$$\{v_1^{8k}, 2v_1^{8k+4}\}.$$

For convenience we will use the same names for basis elements of KO_{4k}, with the reminder that the element $2v_1^{8k+4} \in KO_{8k+4}$ is not divisible by 2.

We are interested in dimensions $d = 4, 8, 12$ where the KO-Pontrjagin numbers one encounters are those for which J is the zero sequence, in which case $\pi^J(M)$ (which we denote $\pi^0(M)$) is the index of the Clifford linear Dirac operator on M, or the sequence J_i whose only non-zero entry is a 1 in the ith spot. In this latter case $\pi^{J_i}(M)$ is the index of the Clifford linear Dirac operator on M with coefficients in $\pi^i(TM)$. We write

$$\pi^i(M) = \pi^{J_i}(M),$$

and note that sending M to the total class

$$\pi_t(M) = \sum_{i \geq 0} \pi^i(M)\, t^i$$

defines a ring homomorphism

$$M\mathrm{Spin}_* \to KO_*[\![t]\!].$$

Remark B.1 Hopefully our notation will not be confusing to the reader. The symbol $\pi_t(-)$ has a meaning that depends on the mathematical type of the argument. For a vector bundle V over a space X, $\pi_t(V)$ is an element of $KO^0(X)[\![t]\!]$. If M is a *Spin*-manifold of dimension d then $\pi_t(M)$ is an element of $KO_d[\![t]\!]$. One has

$$\pi_t(M) = f_* \pi_t(TM)$$

where f is the unique map from M to a point, and f_* is the pushforward in KO-theory.

By [2, Theorem 4.6], when M has dimension $d = 4i$ the KO-Pontrjagin number $\pi^i(M)$ is the same as the ordinary Pontrjagin number $p_i(M)$

$$\pi^i(M) = p_i(M) v_1^{2i}. \tag{B2}$$

Note that in terms of our chosen basis for KO_{4i}, when i is odd, (B2) means that

$$\pi^i(M) = (2v_1^{2i}) \frac{p_i(M)}{2}.$$

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We now turn to our specific manifolds. In dimension 4 we have (by (B2))

$$\pi_t(M) = (2v_1^2)\left(-\frac{p_1}{48} + \frac{p_1}{2}\,t\right).$$

For a $K3$-surface one has $p_1 = -48$ (see (5.3)), and so

$$\pi_t(K) = (2v_1^2)(1 - 24t).$$

In the Anderson, Brown, Peterson isomorphism for $d = 4$, only the zero sequence J occurs, and the map

$$\pi^0(M) : M\,\mathrm{Spin}_4 \to KO_4$$

is an isomorphism after completing at 2. Since $\pi^0(K) = (2v_1^2)$ the composition

$$\mathbb{Z} \xrightarrow{K} M\,\mathrm{Spin}_4 \xrightarrow{\pi^0} KO_4$$

is an isomorphism, and so

$$\mathbb{Z} \xrightarrow{K} M\,\mathrm{Spin}_4$$

is an isomorphism after completing at 2. Less formally, the 2-adic completion of $M\,\mathrm{Spin}_4$ is freely generated (2-adically) by K.

In dimension 8, a little calculation gives

$$\pi_t(M) = v_1^4\left(\frac{7p_1^2 - 4p_2}{5760} + \frac{p_1^2 - 4p_2}{24}\,t + p_2\,t^2\right)[M]. \tag{B3}$$

For the Bott manifold, $p_1^2 = 0$ and $p_2 = -1440$ (see (5.12)), and so

$$\pi_t(B) = v_1^4(1 + 240t - 1440t^2).$$

For $\mathbb{HP}^2$ one can plug the values of the Pontrjagin classes from (5.9) into (B3) to get

$$\pi_t(\mathbb{HP}^2) = v_1^4(-t + 7t^2),$$

though it is a little more instructive to work through the splitting principle approach (after all, that

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was what was used to determine the Pontrjagin classes in the first place). Let $f\colon \mathbb{CP}^5 \to \mathbb{HP}^2$ be the map

$$S^{11}/U(1) \to S^{11}/SU(2).$$

As described in Section 5.2, the KO class of $f^*T\mathbb{HP}^2$ is the KO class underlying the virtual complex bundle

$$3(\mathcal{O}(1)+\mathcal{O}(-1)) - (\mathcal{O}(2)+1).$$

This means that

$$\begin{aligned} f^*\pi_t(T\mathbb{HP}^2) &= \frac{(1+t(\mathcal{O}(1)_u-2))^3(1+t(\mathcal{O}(-1)_u-2)))^3}{(1+t(\mathcal{O}(2)_u-2))} \\ &= \frac{(1+t(\mathcal{O}(1)_u-2))^6}{(1+t(\mathcal{O}(2)_u-2))} \end{aligned}$$

since $\mathcal{O}(n)_u = \mathcal{O}(-n)_u$.

We need to complexify the above and compute the Chern character. Complexifying gives

$$f^*\pi_t(T\mathbb{HP}^2)\otimes\mathbb{C} = \frac{(1+t(\mathcal{O}(1)+\mathcal{O}(-1)-2))^6}{(1+t(\mathcal{O}(2)+\mathcal{O}(-2)-2))} \in K^0(\mathbb{CP}^5)$$

and so

$$\begin{aligned} f^*\hat{A}(T\mathbb{HP}^2) &= (y/(e^{y/2}-e^{-y/2}))^6(2y/(e^y-e^{-y}))^{-1} \\ \mathrm{ch}(f^*\pi_t(T\mathbb{HP}^2)) &= (1+t(e^y+e^{-y}-2))^6/(1+t(e^{2y}+e^{-2y}-2)) \\ f^*\hat{A}(T\mathbb{HP}^2)\mathrm{ch}(\pi_t(T\mathbb{HP}^2)) &= 1+\frac{1}{12}(-1+24t)x+\left(-t+7t^2\right)x^2 \end{aligned}$$

in which $y = c_1(\mathcal{O}(1))$ is the generator of $H^2(\mathbb{CP}^5)$ and $x = y^2$. From the coefficient of x^2 we recover the formula

$$\pi_t(\mathbb{HP}^2) = v_1^4(-t+7t^2). \tag{B4}$$

In dimension 8 the Anderson, Brown, Peterson map is

$$(\pi^0,\pi^2) : M\mathrm{Spin}_8 \to (KO_8)^2.$$

The matrix of the composite

$$\mathbb{Z}^2 \xrightarrow{\begin{bmatrix} B & \mathbb{HP}^2 \end{bmatrix}} M\mathrm{Spin}_4 \xrightarrow{\begin{bmatrix} \pi^0 \\ \pi^2 \end{bmatrix}} (KO_8)^2$$

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is

$$\begin{pmatrix} 1 & 0 \\ -1440 & 7 \end{pmatrix}$$

which is the identity modulo 2. This implies that after 2-adic completion, (or even just localization at 2), $M\,\mathrm{Spin}_8$ is freely generated by $\{B, \mathbb{HP}^2\}$.

Now for dimension 12. The invariants are π^0, π^2 and π^3, and we need to calculate them on $K \times B$, $K \times \mathbb{HP}^2$ and W_1. Using the fact that $\pi_t(M)$ is multiplicative in the Spin-manifold M we find

$$\begin{aligned} \pi_t(K \times B) &= (2\nu_1^6)(1-24t)(1+240t-1440t^2) \\ &= (2\nu_1^6)(1+216t+-7200t^2+34560t^3) \end{aligned}$$

and

$$\begin{aligned} \pi_t(K \times \mathbb{HP}^2) &= (2\nu_1^6)(1-24t)(-t+7t^2) \\ &= (2\nu_1^6)(-t+31t^2-168t^3). \end{aligned}$$

We also know from (5.39) that $p_3(W_1) = 6$ so that

$$\pi_3(W_1) = 6\nu_1^6 = (2\nu_1^6)(3).$$

The matrix of the composition

$$\mathbb{Z}^3 \overset{\begin{bmatrix} K\times B & K\times\mathbb{HP}^2 & W_1 \end{bmatrix}}{\longrightarrow} M\mathrm{Spin}_{12} \overset{\begin{bmatrix} \pi^0 \\ \pi^2 \\ \pi^3 \end{bmatrix}}{\longrightarrow} (KO_{12})^3$$

is therefore

$$\begin{pmatrix} 1 & 0 & * \\ -7200 & 31 & * \\ 34560 & -168 & 3 \end{pmatrix}$$

which reduces modulo 2 to

$$\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

which is clearly invertible. This shows that $\{K \times B, K \times \mathbb{HP}^2, W_1\}$ freely generate the 2-adic completion of $M\,\mathrm{Spin}_{12}$, which is the assertion of Proposition 8.14.

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Appendix C Cohomology of $M\mathfrak{m}_c$

As explained earlier, the Adams spectral sequence chart in Section 9 was determined by specifying a basis for $H^*(M\mathfrak{m}_c)$ and then using a Mathematica program to encode the action of the Steenrod operations in this basis and generate the input for Bruner's program [10] for computing the cohomology of the Steenrod algebra with coefficients in a given module. In this appendix, we describe the cohomology ring $H^*(B\mathfrak{m}_c)$, the module $H^*(M\mathfrak{m}_c)$ and the action of the Steenrod algebra. Using this, the interested reader should be able to reproduce the computation of the E_2-term of the Adams spectral sequence.

Throughout this appendix all cohomology will be with coefficients in $\mathbb{Z}/2$.

We begin with the cohomology of BSpin. For a number $n = \sum \epsilon_i 2^i$, $\epsilon \in \{0,1\}$ write $\alpha(n) = \sum \epsilon_i$.

PROPOSITION C.1 (Thomas [53]) *The map BSpin$\to BSO$ induces an isomorphism*

$$\mathbb{Z}/2[w_i \mid \alpha(i-1) > 1] = \mathbb{Z}/2[w_4, w_6, \dots] \approx H^*(B\text{Spin}). \tag{C1}$$

□

REMARK C.2 The kernel of the map

$$H^*(BSO) = \mathbb{Z}/2[w_2, w_3, \dots] \to H^*(B\text{Spin})$$

is the regular ideal with generators

$$\text{Sq}^{2^n}\text{Sq}^{2^{n-1}}\cdots\text{Sq}^1 w_2. \tag{C2}$$

It is *not* generated by the classes w_i with $\alpha(i-1) = 1$. The first place these ideals differ is in dimension 17. In $H^*(B\text{Spin})$ one has

$$w_{17} = w_7 w_{10} + w_6 w_{11} + w_4 w_{13}.$$

Consider the fibration sequence

$$\text{BString} \to B\text{Spin} \xrightarrow{\lambda} K(\mathbb{Z}, 4). \tag{C3}$$

PROPOSITION C.3 (Stong [52]) *The map* BString$\to BO$ *induces an isomorphism*

$$\mathbb{Z}/2[w_i \mid \alpha(i-1) > 2] = \mathbb{Z}/2[w_8, w_{12}, \dots] \approx H^*(\text{BString}).$$

REMARK C.4 The kernel of $H^*(B\text{Spin}) \to H^*(\text{BString})$ is the regular ideal generated by the elements $\text{Sq}^I w_4$ in which $I = (i_1, i_2, \dots, i_k)$ is a sequence of non-negative integers satisfying

$$\begin{gathered} i_\ell \geq 2i_\ell \qquad 1 \leq \ell \leq k \\ (i_1 - 2i_2) + \cdots + (i_{k-1} - 2i_k) + i_k < 4 \\ i_k > 1. \end{gathered}$$

The first condition is called *admissibility* and the quantity on the left side of the second inequality is the *excess* of the sequence. The kernel is *not* the ideal generated by the w_i with $\alpha(i-1) = 2$.

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The Leray–Hirsch theorem applied to (C3) implies

PROPOSITION C.5 *The map BSpin* $\to K(\mathbb{Z},4)$ *induces an isomorphism*

$$H^*(K(\mathbb{Z},4))[w_i \mid \alpha(i-1) > 2] = H^*(K(\mathbb{Z},4))[w_8, w_{12}, \dots] \xrightarrow{\approx} H^*(BSpin). \tag{C4}$$

□

REMARK C.6 To work out the action of the Steenrod operations under the above isomorphism, one must use the relations setting the terms (C2) equal to zero, and the Wu formulae. This can get complicated. For example one has

$$\begin{aligned} \mathrm{Sq}^1 w_{16} &= w_{17} \\ &= w_7 w_{10} + w_6 w_{11} + w_4 w_{13} \\ &= (\mathrm{Sq}^4\mathrm{Sq}^2 w_4)(\mathrm{Sq}^3 w_4) + (\mathrm{Sq}^5\mathrm{Sq}^2 w_4)(\mathrm{Sq}^2 w_4) \\ &\quad + (\mathrm{Sq}^6\mathrm{Sq}^3 w_4)(w_4) + (\mathrm{Sq}^2 w_4)(\mathrm{Sq}^3 w_4)\, w_4. \end{aligned}$$

From (C3) one constructs a fibration sequence

$$\mathrm{BString} \to B\mathrm{Spin} \times K(\mathbb{Z},4) \xrightarrow{(\iota, -\lambda - \iota)} K(\mathbb{Z},4) \times K(\mathbb{Z},4). \tag{C5}$$

in which the rightmost map is equivariant when $K(\mathbb{Z},4) \times K(\mathbb{Z},4)$ is given permutation $\mathbb{Z}/2$-action.

For a space X write

$$D_2(X) = E\mathbb{Z}/2 \underset{\mathbb{Z}/2}{\times} (X^2)$$

in which $X^2 = X \times X$ is equipped with the permutation action. Passing to homotopy orbits from (C5) gives the fibration sequence

$$\mathrm{BString} \to B\mathfrak{m}_c \to D_2(K(\mathbb{Z},4)).$$

As before the Leray–Hirsch theorem leads to an isomorphism

$$H^*(D_2(K(\mathbb{Z},4))[w_8, w_{12}, \dots] \xrightarrow{\approx} H^*(B\mathfrak{m}_c).$$

To go further we must describe the cohomology of $D_2(X)$. The computation of the cohomology ring of $D_2(X)$ is due to Dyer and Lashof [**19**], and the action of the Steenrod operations was determined by Nishida [**43**]. To describe the computation we first recall the additive and multiplication

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transfers described in Section 8.2.1. Suppose that

$$\pi : \widehat{M} \to M$$

is a double cover, classified by

$$\alpha \in H^1(M;\mathbb{Z}/2),$$

and write

$$\tau : H^*(\widehat{M}) \to H^*(\widehat{M})$$

for the map induced by the deck transformation. There are additive and multiplicative transfers

$$\mathrm{tr} : H^k(\widehat{M};\mathbb{Z}/2) \to H^k(M;\mathbb{Z}/2)$$
$$P = P_\alpha : H^k(\widehat{M};\mathbb{Z}/2) \to H^{2k}(M;\mathbb{Z}/2)$$

which are natural in the sense that they commute with base change in M. They satisfy the following properties (for $x, y \in H^*(\widehat{M})$, $z \in H^*(M)$)

(1) $\mathrm{tr}(x+y) = \mathrm{tr}(x) + \mathrm{tr}(y)$
(2) $\mathrm{tr}(x)z = \mathrm{tr}(x\pi^*(z))$
(3) $\pi^*\mathrm{tr}(x) = x + \tau(x)$
(4) $P(xy) = P(x)P(y)$
(5) $P(x+y) = P(x) + P(y) + \mathrm{tr}(x\tau(y))$
(6) $\pi^*P(x) = x\tau(x)$.

Note that property (2) implies that $\mathrm{tr}(x)\alpha = 0$.

We will refer to the additive transfer tr simply as the *transfer* and the multiplicative transfer P as the *norm*.

The transfer map commutes with Steenrod operations

$$\mathrm{Sq}^k\mathrm{tr}(x) = \mathrm{tr}(\mathrm{Sq}^k x).$$

Suitably interpreted, the norm also commutes with the Steenrod operations. Let $\beta \in H^1(B\mathbb{Z}/2)$ be the non-zero element and for $x \in H^n(X)$ write

$$\mathrm{Sq}^\beta(x) = \sum \mathrm{Sq}^{n-i}(x)\beta^i = x^2 + \beta\mathrm{Sq}^{n-1}(x) + \cdots + \beta^n x \in H^{2n}(B\mathbb{Z}/2 \times X).$$

As Nishida [43] observed, the values of the Steenrod operations on the norm $P(x)$ are determined by the formula

$$\mathrm{Sq}^\beta(P(x)) = P(\mathrm{Sq}^\beta(x)). \tag{C6}$$

Proposition C.7 ([19], Proposition 2.2) *Suppose X is a space and $\{e_i\}$ is a basis of $H^*(X;\mathbb{Z}/2)$. The vector space $H^*(D_2(X))$ has basis*

$$\{\mathrm{tr}(e_i \otimes e_j) \mid i < j\} \cup \{\alpha^j P(e_i \otimes 1) \mid j \geq 0\}.$$

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To relate the elements $P(x)$ and $\mathrm{tr}(x)$ to other naturally occurring elements, it is useful to exploit both the covering map

$$\pi : E\mathbb{Z}/2 \times X \times X \to D_2(X))$$

and the diagonal map

$$\Delta : B\mathbb{Z}/2 \times X \to D_2(X),$$

the latter obtained by passing to homotopy orbits from the diagonal inclusion (denoted by the same symbol)

$$\Delta : X \to X \times X.$$

From the definition of Steenrod operations, one has

$$\Delta^*(P(x)) = \mathrm{Sq}^{\alpha}(x)$$

where $\alpha \in H^1(B\mathbb{Z}/2)$ is the non-zero element. From naturality and the pullback square,

$$\begin{array}{ccc} E\mathbb{Z}/2 \times X & \longrightarrow & E\mathbb{Z}/2 \times X \times X \\ \downarrow & & \downarrow \\ B\mathbb{Z}/2 \times X & \longrightarrow & D_2(X), \end{array}$$

one also sees that

$$\Delta^*(\mathrm{tr}(x)) = \mathrm{tr}(\Delta^*(x)) = 0.$$

The next result also follows from [19, Proposition 2.2]

PROPOSITION C.8 *If* $\pi^*(x) = 0$ *and* $\Delta^*(x) = 0$ *then* $x = 0$.

In practice, it is easier not to separate out the factor of $D_2K(\mathbb{Z}, 4)$ but rather to work directly with $B\mathrm{Spin} \times K(\mathbb{Z}, 4)$. In these terms, the diagonal inclusion corresponds to the map

$$B\mathrm{Spin}\langle w_4\rangle \overset{(\mathrm{id}, -\lambda/2)}{\longrightarrow} B\mathrm{Spin} \times K(\mathbb{Z}, 4) \tag{C7}$$

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in which $B\mathrm{Spin}\langle w_4\rangle$ is defined by left square in the diagram of homotopy pullback squares

$$\begin{array}{ccccc} B\mathrm{Spin}\langle w_4\rangle & \xrightarrow{\lambda/2} & K(\mathbb{Z},4) & \longrightarrow & * \\ \downarrow & & \downarrow{\scriptstyle 2} & & \downarrow \\ B\mathrm{Spin} & \xrightarrow[\lambda]{} & K(\mathbb{Z},4) & \longrightarrow & K(\mathbb{Z}/2,4)\ . \end{array} \tag{C8}$$

Since the right square in (C8) is a homotopy pullback square, the space $B\mathrm{Spin}\langle w_4\rangle$ can is canonically equivalent to homotopy fiber of the map

$$B\mathrm{Spin} \xrightarrow{w_4} K(\mathbb{Z}/2,4).$$

The left square in (C8) gives a cohomology class $\lambda/2 \in H^4(B\mathrm{Spin}\langle w_4\rangle)$ with the property that $2(\lambda/2)$ is the restriction of λ.

One arrives at (C7) by considering first

$$\begin{array}{ccccc} & & B\mathrm{Spin}\times K(\mathbb{Z},4) & \xrightarrow{2\iota+\lambda} & K(\mathbb{Z},4) \\ & & \downarrow{\scriptstyle (\iota,-\lambda-\iota)} & & \| \\ K(\mathbb{Z},4) & \xrightarrow[\mathrm{diag}]{} & K(\mathbb{Z},4)\times K(\mathbb{Z},4) & \xrightarrow[[1,-1]]{} & K(\mathbb{Z},4) \end{array}$$

to identify the homotopy pullback of the left horizontal and vertical maps with the homotopy fiber of the map $(2\iota+\lambda)$, and the homotopy pullback square

$$\begin{array}{ccc} B\mathrm{Spin}\times K(\mathbb{Z},4) & \xrightarrow{2\iota+\lambda} & K(\mathbb{Z},4) \\ \downarrow & & \downarrow \\ B\mathrm{Spin} & \xrightarrow[w_4]{} & K(\mathbb{Z}/2,4), \end{array}$$

to identify the fiber of $(2\iota+\lambda)$ with $B\mathrm{Spin}\langle w_4\rangle$.

Passing to homotopy orbits from

$$B\mathrm{Spin}\langle w_4\rangle \to B\mathrm{Spin}\times K(\mathbb{Z},4) \to B\mathrm{Spin}$$

one sees that the composition

$$B\mathbb{Z}/2\times B\mathrm{Spin}\langle w_4\rangle \to B\mathfrak{m}_c \to B\mathrm{Pin}^+ \approx B\mathbb{Z}/2\times B\mathrm{Spin}$$

is the product of the identity map and the map $B\mathrm{Spin}\langle w_4\rangle \to B\mathrm{Spin}$. It follows that the virtual vector bundle classified by

$$B\mathbb{Z}/2\times B\mathrm{Spin}\langle w_4\rangle \to B\mathfrak{m}_c \to BO$$

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is the sum of the pullback of the tautological line bundle L on $B\mathbb{Z}/2$ and the pullback of the virtual vector bundle V classified by

$$B\mathrm{Spin}\langle w_4\rangle \to B\mathrm{Spin} \to BO.$$

See Section 8.2. With this in hand, one can appeal to the pullback square

$$\begin{array}{ccc} B\mathrm{Spin}\langle w_4\rangle & \longrightarrow & B\mathrm{Spin}\times K(\mathbb{Z},4) \\ \downarrow & & \downarrow \\ B\mathbb{Z}/2\times B\mathrm{Spin}\langle w_4\rangle & \longrightarrow & B\mathfrak{m}_c \end{array}$$

and Proposition C.8 to conclude that the map from $H^*(B\mathfrak{m}_c)$ to the pullback of

$$\begin{array}{ccc} & & H^*(B\mathrm{Spin}\times K(\mathbb{Z},4)) \\ & & \downarrow \\ H^*(B\mathbb{Z}/2)\times B\mathrm{Spin}\langle w_4\rangle & \longrightarrow & H^*(B\mathrm{Spin}\langle w_4\rangle) \end{array}$$

is a monomorphism and identify classes like $\mathrm{tr}(e)$ and $P(e)$ in terms of Stiefel–Whitney classes when possible.

EXAMPLE C.9 Consider the class $tr(\iota)$. The restriction of $tr(\iota)$ to $H^*(B\mathrm{Spin}\times K(\mathbb{Z},4))$ is

$$\iota+\tau(\iota)=\iota+(\iota+w_4)=w_4.$$

This suggests that $tr(\iota)=w_4$. To verify this, one needs to only check that the restriction of w_4 to $B\mathbb{Z}/2\times B\mathrm{Spin}\langle w_4\rangle$ is zero. For this, one computes

$$w_4(L\oplus V)=w_4(V)+w_1(L)w_3(V)=0.$$

It follows that $\mathrm{tr}(\mathrm{Sq}^I\iota)=\mathrm{Sq}^I(w_4)$.

REMARK C.10 For the interested reader, the map

$$H^*(B\mathrm{Spin})\to H^*(B\mathrm{Spin}\langle w_4\rangle)$$

factors uniquely as

$$H^*(B\mathrm{Spin})\to H^*(\mathrm{BString})\to H^*(B\mathrm{Spin}\langle w_4\rangle).$$

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Using this, the pullback square

$$\begin{array}{ccc} B\mathrm{Spin}\langle w_4\rangle & \longrightarrow & K(\mathbb{Z},4) \\ \downarrow & & \downarrow 2 \\ B\mathrm{Spin} & \longrightarrow & K(\mathbb{Z},4) \end{array}$$

and the Eilenberg–Moore spectral sequence imply that the map

$$H^*(\mathrm{BString}) \otimes H^*(K(\mathbb{Z},4)) \to H^*(B\mathrm{Spin}\langle w_4\rangle) \tag{C9}$$

is an isomorphism of algebras over the Steenrod algebra.

To put this all together, let $\{e_i\}$ be the basis of $H^*(K(\mathbb{Z},4))$ consisting of monomials in the admissible Steenrod operations on ι and $\{w_I\}$ be the basis of monomials in the Stiefel–Whitney classes

$$\{w_i \mid \alpha(i-1) > 2\}.$$

Then, a basis of $H^*(B\mathfrak{m}_c)$ is given by

$$\{w_I \mathrm{tr}(e_i\tau(e_j)) \mid i<j\} \cup \{w_I \alpha^j P(e_i)\}. \tag{C10}$$

The Steenrod operations, products and the relation with the Stiefel–Whitney classes are as described above.

The module $H^*(M\mathfrak{m}_c)$ is a free module over $H^*(B\mathfrak{m}_c)$ on the Thom class U of the vector bundle classified by the *negative* of the map

$$B\mathfrak{m}_c \to B\mathrm{Pin}^+ \to BO$$

As described in Section 8.2, we use the equivalence

$$B\mathbb{Z}/2 \times B\mathrm{Spin} \approx B\mathrm{Pin}^+$$

obtained by writing a stable Pin^+ bundle in the form $L \oplus V$ and write $\alpha = w_1(L)$, $w_i = w_i(V)$. With these conventions, the Thom formula for the total squaring operation on U gives

$$\mathrm{Sq}_t(U) = (1+t\alpha)^{-1}(1+t^4 w_4 + \cdots)^{-1} U. \tag{C11}$$

For the computations reported in this paper, the authors restricted to dimensions less than 16 and used the basis consisting of the product of the basis elements in (C10) with the Thom class U.

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