

Affine maps between CAT(0) spaces

Hanna Bennett¹ · Christopher Mooney² ·
Ralf Spatzier¹

Received: 20 September 2014 / Accepted: 22 May 2015 / Published online: 7 June 2015
© Springer Science+Business Media Dordrecht 2015

Abstract We study affine maps between CAT(0) spaces with cocompact group actions, and show that they essentially split as products of dilations and linear maps (on the Euclidean factor). This extends known results from the Riemannian case. Furthermore, we prove a splitting lemma for the Tits boundary of a CAT(0) space with cocompact group action, a variant of a splitting lemma for geodesically complete CAT(1) spaces by Lytchak.

Keywords CAT(0) spaces · Geometric actions · Affine maps

1 Introduction

Let X and Y be geodesic spaces, and $f: X \mapsto Y$ a map. Recall that a *geodesic space* is a metric space in which every pair of points is joined by a path of shortest length, called a *geodesic*. We will always parametrize geodesics by arc length. We call f *affine* if f maps geodesics γ in X to geodesics in Y , and f rescales γ with constant speed $\rho(\gamma)$, which a priori depends on the geodesic γ . We call ρ the *rescaling function*, and say that an affine map is a *dilation* if the rescaling function is constant. In this paper we classify affine maps between CAT(0) spaces. We will not touch on the much more difficult question of the extent to which the set of geodesics determines the metric. Matveev has obtained strong positive results, especially for closed Riemannian manifolds with negative curvature [21].

If X and Y are Riemannian, the answer to our problem is classical [16]: any self-affine map of an irreducible, non-Euclidean Riemannian manifold is a dilation. Remarkably, Lytchak

H. Bennett is Supported in part by an EAF Grant for the IBL Center at the University of Michigan
C. Mooney is Supported in part by the NSF EMSW21 Grant RTG-0602191 and a Caterpillar Fellowship
R. Spatzier is Supported in part by the NSF Grants DMS-0906085, DMS-1307164 and DMS-0932078 000,
the latter while this author was in residence at MSRI in Berkeley during the Spring 2015 semester.

✉ Ralf Spatzier
spatzier@umich.edu

¹ University of Michigan, Ann Arbor, MI, USA

² Epic, Verona, WI 53593, USA

[19], following work by Ohta [23], classified affine maps from a Riemannian manifold X to any metric space Y as dilations, as long as X is not a product or a higher rank symmetric space. In the latter cases, they produce counterexamples by endowing these spaces with suitable Finsler metrics. Lytchak and Schröder [20], and later Hitzelberger and Lytchak [13] further investigated the case of real-valued functions on a $CAT(\kappa)$ metric space, and obtained severe restrictions.

Understanding affine maps has been important in several applications, and has connections to superrigidity problems. (cf. [23]). When the source is a higher rank non-positively curved symmetric space or quaternionic hyperbolic space and the target is a Riemannian manifold or Euclidean building, then Gromov and Schoen proved that harmonic maps are affine [11]. This was generalized to hyperbolic buildings as targets by Daskalopoulos, Mese and Vdovina [9]. One hopes for similar results when the source is a singular space. There have been various attempts in this direction, e.g., by Daskalopoulos and Mese [7, 8] or Izeki and Nayatani [14]. We refer the reader to Ohta [23] for further discussion.

To state our main result, we recall a fundamental result about splittings of metric spaces. Foertsch and Lytchak [10, Theorem 1.1] prove that such splittings exist and are essentially unique when X is a geodesic metric space of finite topological dimension, i.e., under those conditions $X = X_1 \times \cdots \times X_n \times \mathbb{E}^d$ such that each $X_i \neq \mathbb{R}$ and is irreducible. In particular their result applies to geodesically complete $CAT(0)$ spaces (i.e., every geodesic segment extends to a geodesic line defined on $(-\infty, \infty)$) with a cocompact group action (i.e., there is a group of isometries with compact quotient) [15, Theorem C]. This case was proved earlier by Caprace and Monod [5].

Recall that a metric space is called *proper* if all closed balls are compact.

Theorem 1.1 (Main Theorem) *Let X be a geodesically complete proper $CAT(0)$ space admitting a cocompact group action, and $X = X_1 \times \cdots \times X_n \times \mathbb{E}^d$ be a factorization into irreducible factors such that no $X_i = \mathbb{R}$. Let Y also be a $CAT(0)$ space, and $f: X \rightarrow Y$ be a continuous affine map. Then the image of f is convex and hence $CAT(0)$ and splits as $f(X_1) \times \cdots \times f(X_n) \times f(\mathbb{E}^d)$, where $f(\mathbb{E}^d)$ is the Euclidean factor of the image. The restriction of f to every factor X_i is a dilation (possibly with rescaling constant zero) and the restriction to \mathbb{E}^d is a standard affine map between Euclidean spaces.*

Combining the Main Theorem with work of Bosché [3], we get two applications to self maps of $CAT(0)$ spaces. We only formulate this result for $CAT(0)$ spaces with geometric actions as this is the generality Bosché works in. Thus we assume that the action is properly discontinuous by isometries with compact quotient.

Corollary 1.2 (Self-Affine Maps) *Let X be a proper geodesically complete $CAT(0)$ space with geometric action and $f: X \rightarrow X$ be a strictly contracting continuous affine map. Then X is flat.*

Corollary 1.3 (Self-Affine Homeomorphisms) *Let X be a proper geodesically complete $CAT(0)$ space admitting a geometric group action and $f: X \rightarrow X$ be an affine homeomorphism. Assume further that X has no Euclidean factor. Then f preserves the de Rham decomposition of X and induces a permutation of the pairwise homothetic irreducible factors. Moreover, we have:*

- (1) *If f preserves the factors of X , then f is an isometry.*
- (2) *If X is irreducible, then f is an isometry.*

- (3) *Some power of f is an isometry.*
- (4) *The group of isometries has finite index in the group of affine homeomorphisms.*

One of the key tools in the proof of the Main Theorem is the following Splitting Lemma, which is an analogue to Lytchak's result [18, Proposition 4.2] for splittings of geodesically complete CAT(1) spaces. We will work with boundaries of CAT(0) spaces that admit cocompact actions. Endowed with the Tits metric, such boundaries are always CAT(1) spaces but often not geodesically complete. See, for instance, the Croke–Kleiner examples [6]. A subset P of a CAT(1) space Y is called π -convex if whenever $x, y \in P$ such that $d(x, y) < \pi$, then the unique geodesic joining them is also in P . Given $x, y \in Y$, x and y are called *antipodal* or *antipodes* if $d(x, y) \geq \pi$. A nonempty subset $P \subset Y$ is *involutive* if it contains all of its antipodes.

Theorem 1.4 (Splitting Lemma) *Let X be a proper CAT(0) space admitting a cocompact group action. Suppose ∂X contains a proper subset P that is π -convex and involutive and closed in the cone topology. Then ∂X splits as the spherical join $\partial X = P * P^\perp$ where P^\perp is the set of points that have Tits distance exactly $\pi/2$ from all points in P .*

We note that our definition of the perpendicular set P^\perp is different from Lytchak's $\text{Pol}(P)$, which is the set of points that are at least $\pi/2$ from all points in P .

To prove the Splitting Lemma, we critically use the π -convergence theorem of Papasoglu–Swenson [24] and the theorem of Kleiner [15] that the boundary ∂X of X contains isometrically embedded round spheres of dimension equal to the geometric dimension of the Tits boundary (cf. Sect. 2.1).

We then apply the Splitting Lemma to prove the Main Theorem. We first show that asymptotic geodesics are rescaled by the same constant. Thus the rescaling function extends to the boundary ∂X . If the rescaling function ρ is not constant on ∂X , then let P be the set of points on which ρ attains its maximum. We show that P is closed, π -convex, and involutive. Therefore the Tits boundary splits off a factor. We can then apply a theorem of Bridson–Haefliger [4, Theorem II.9.24] to get that the underlying CAT(0) space splits as a product.

We do not know if our results extend to affine maps between CAT(κ) spaces, or at least from CAT(0) to CAT(κ) spaces. As we mentioned above, affine maps between irreducible metric spaces are not always dilations. We also do not know if affine maps are always continuous. This is certainly the case in many situations.

2 CAT(0) spaces

2.1 Tools from CAT(0) spaces

In this section we review some definitions and techniques in CAT(0) spaces. For a more thorough treatment the reader should review [4]. Let X be a geodesic metric space. Given three distinct points $x, y, z \in X$, choose points \bar{x}, \bar{y} , and \bar{z} in the Euclidean plane \mathbb{E}^2 such that $d(\bar{x}, \bar{y}) = d(x, y)$, $d(\bar{x}, \bar{z}) = d(x, z)$, and $d(\bar{y}, \bar{z}) = d(y, z)$. We denote the resulting triangle in \mathbb{E}^2 by $\bar{\Delta}xyz$ and call it a *comparison triangle* for Δxyz in X . Choose any p in the geodesic $[x, y]$ and q in $[x, z]$ and get corresponding points $\bar{p} \in [\bar{x}, \bar{y}]$ and $\bar{q} \in [\bar{x}, \bar{z}]$. If for every choice of p and q , we have $d(p, q) \leq d(\bar{p}, \bar{q})$, then Δxyz is said to be *no fatter* than $\bar{\Delta}xyz$. If every triangle in X is no fatter than its comparison triangle in \mathbb{E}^2 , then X is called *CAT(0)*. We will henceforth assume that our CAT(0) spaces are proper (i.e., that closed balls are compact).

Next we recall the definition of Alexandrov angles in CAT(0) spaces. If $x, y, z \in X$, then we denote by $\bar{\angle}_x(y, z)$ the corresponding angle in the comparison triangle $\bar{\triangle}xyz$. If α and β are the geodesics $[x, y]$ and $[x, z]$, then the CAT(0) condition implies that

$$t \mapsto \bar{\angle}_x(\alpha(t), \beta(t)) \text{ is a nondecreasing function.} \quad (1)$$

Its limit as $t \rightarrow 0$ is called the *Alexandrov angle* between α and β , denoted by $\angle_x(y, z)$, or $\angle_x(\alpha, \beta)$. The condition that $\angle_x(y, z) \leq \bar{\angle}_x(y, z)$ for every choice of $x, y, z \in X$ is equivalent to the CAT(0) property for X [4, Proposition II.1.7(4)].

Many geometric properties of nonpositively curved manifolds carry over to the CAT(0) setting. For instance, angle sums of triangles are bounded above by π . Furthermore we can define the *visual boundary* of X , denoted ∂X , as the set of equivalence classes of geodesic rays in X . Two geodesic rays α and β are equivalent, or asymptotic, if one lies in a tubular neighborhood of the other. Equivalently, if a basepoint $x_0 \in X$ is fixed, then ∂X may be defined as the set of geodesic rays emanating from x_0 [4, § II.8]. We think of ∂X as “attached to X at infinity” and it captures the notion of infinity of X .

We endow $\bar{X} = X \cup \partial X$ with a topology by identifying points in X with geodesics emanating from a common basepoint x_0 and points in ∂X with geodesic rays emanating from the same point. Then a sequence of points $(x_n) \subset \bar{X}$ converges to a point $y \in X$ if the corresponding geodesics converge uniformly on compact sets. The subspace topology on ∂X is called the *cone topology*. When X is proper, \bar{X} is a compactification for X and ∂X is compact. In this topology geodesic rays are close if they track a long time before diverging.

A second topology on ∂X comes from a metric. Given a pair of points $\zeta, \eta \in \partial X$, the *Tits angle* between them, denoted $\angle_{Tits}(\zeta, \eta)$, is defined as the supremum of Alexandrov angles $\angle_x(\zeta, \eta)$ between the geodesic rays α and β emanating from x going out to ζ and η as x ranges over X . If x is fixed, we know from [4, Proposition II.9.8(1)] that

$$\lim_{t \rightarrow \infty} \bar{\angle}_x(\alpha(t), \beta(t)) = \angle_{Tits}(\zeta, \eta) \quad (2)$$

where $\bar{\angle}_x(\alpha(t), \beta(t))$ is nondecreasing by (1).

The *Tits metric* d_{Tits} is the corresponding length metric (possibly taking the value infinity). This metric induces the *Tits topology*, which is finer than the cone topology. The boundary with the Tits topology is called the *Tits boundary* and denote it by $\partial_{Tits} X$. It is well-known that d_{Tits} is lower semicontinuous in the cone topology [4, Proposition II.9.5(2)]. Also, when $\angle_{Tits}(\zeta, \eta) < \pi$, then $d_{Tits}(\zeta, \eta) = \angle_{Tits}(\zeta, \eta)$ [4, Remark II.9.19(2)] and $d_{Tits}(\zeta, \eta) \geq \pi$ iff $\angle_{Tits}(\zeta, \eta) = \pi$.

Amazingly, the Tits boundary has an elegant geometry. Given any real κ , a geodesic space is called CAT(κ) if triangles are no fatter than comparison triangles in a simply connected Riemannian manifold with constant curvature κ . The following theorem is due to Gromov in the manifold setting [1], and to Bridson and Haefliger in full generality.

Theorem 2.1 [4, Theorem II.9.13] *Let X be a complete CAT(0) space. Then its Tits boundary is a complete CAT(1) space. In particular, any two points of finite Tits distance are connected by a Tits geodesic.*

Geodesics in $\partial_{Tits} X$ reflect flatness in X . For instance, if $x \in X$ and $\zeta, \eta \in \partial X$ such that $\angle_x(\zeta, \eta) = \angle_{Tits}(\zeta, \eta)$, then the convex hull of the union of the two rays emanating from x going out to ζ and η is isometric to a sector in \mathbb{E}^2 [4, Corollary II.9.9].

Another important example of a CAT(1) space is the space of directions based at a point. Given a point y in a CAT(κ) space Y , deem two geodesic segments emanating from y to be equivalent if the angle between them is zero. Equivalence classes are called *geodesic*

germs. The completion of this space with the metric induced by angles is called the *link of* y . Elements in the link are thought of as directions. A direction is *genuine* if it has a geodesic representative. Links in a CAT(1) space are always CAT(1) [22].

Following Kleiner in [15], we define the *geometric dimension* of a CAT(κ) space Y to be the smallest function \dim on the class of such spaces (taking on non-negative integer values and infinity) such that

- $\dim Y = 0$ if Y is discrete and
- $\dim Y$ is strictly greater than the dimension of every link in Y .

This is equal to the maximal topological dimension of compact subspaces of Y [15, Theorem A]. Moreover, if Y is a CAT(0) space with cocompact action, then the topological and geometric dimensions are equal and finite [15, Theorem C].

2.2 Density of round spheres

A *round sphere* in a CAT(1) space Y of geometric dimension $d > 0$ is an isometrically embedded d -sphere Σ of curvature 1. In a zero-dimensional CAT(1) space, a *round sphere* is just a pair of points with distance ∞ . For boundaries of certain CAT(0) spaces, Kleiner proved the existence of round spheres.

Theorem 2.2 [15, Theorem C] *Let X be a CAT(0) space admitting a cocompact group action. Let d be the geometric dimension of its Tits boundary. Then there is a round sphere of dimension d in $\partial_{\text{Tits}} X$. In addition, this round sphere is the boundary of an isometrically embedded $(d+1)$ -flat in X .*

We remark that Leeb [17, Proposition 2.1] proved that any top-dimensional round sphere in the boundary of X is the boundary of a flat.

We will also need the π -convergence technique of Papasoglu–Swenson [24]. We use the following more general version by Swenson.

Theorem 2.3 [25, Theorem 2] *Let X be a geodesically complete CAT(0) space. Suppose (g_n) is a sequence of isometries with the property that for any $x \in X$, $g_i(x) \rightarrow p \in \partial X$ and $g_i^{-1}(x) \rightarrow n \in \partial X$. Then for any $0 \leq \theta \leq \pi$ and every compact $K \subset \partial X \setminus \overline{B_{\text{Tits}}}(n, \theta)$ and neighborhood U of $\overline{B_{\text{Tits}}}(p, \pi - \theta)$, we have $g_k(K) \subset U$ for large enough k .*

The existence of antipodes was proven by Balser–Lytchak.

Lemma 2.4 [2, Lemma 3.1] *Let X be a CAT(1) space with geometric dimension $d < \infty$ and $K \subset X$ be a round sphere. Then every point of X has an antipode in K .*

Corollary 2.5 (Density of Round Spheres) *Let G be a group acting cocompactly on a CAT(0) space X . Then the union of round spheres in ∂X is dense (in the the cone topology). In addition, we may pick these round spheres to consist of boundaries of flats in X .*

Proof Fix $x_0 \in X$. Choose any point $p \in \partial X$. Since the action of G is cocompact, there is a sequence of group elements $(g_n) \subset G$ such that $(g_n x_0)$ converges to p . After passing to a subsequence (if necessary), we may assume that $(g_n^{-1} x_0)$ also converges to some point $n \in \partial X$. By [15, Theorem C] we know that there exists a round sphere $K \subset \partial_{\text{Tits}} X$. By Lemma 2.4, we can get a $q \in K$ for which $d_{\text{Tits}}(n, q) \geq \pi$. Apply π -convergence now to get that $g_n q \rightarrow p$. Since G -translates of round spheres are round spheres, we see that the family of round spheres GK is dense. \square

Suppose X is a CAT(0) space with cocompact group action. To prove the Main Theorem, we will need to know that any factor of X also has a dense set of round spheres at infinity. We conclude this section by proving this. Recall that a flat in a CAT(0) space X is called *maximal* if it is not contained as a proper subspace of any other flat in X .

Lemma 2.6 *Let X be a CAT(0) space splitting as $X = X_1 \times X_2$. Then $F \subset X$ is a maximal flat iff it can be written as $F = F_1 \times F_2$ where F_1 and F_2 are maximal flats in X_1 and X_2 respectively.*

Proof Let $\pi_i : X \rightarrow X_i$ denote coordinate projection. Let F be a maximal flat in X and denote $F_i = \pi_i(F)$. Then $F_1 = \pi_1(F)$ is a flat in X_1 because it is isometric to a totally geodesic subspace of F , namely $F_1 \times \{x_2\}$ where $x_2 \in F_2$. Similarly, F_2 is a flat in X_2 . Since $F_1 \times F_2$ is a flat in X containing F and F is maximal, we must have $F = F_1 \times F_2$. Finally, if F_1 were not maximal in X_1 , then there would be a flat $F'_1 \subset X_1$ containing F_1 as a proper subspace and $F'_1 \times F_2$ would contain F as a proper subspace. So F_1 and F_2 must both be maximal. This proves the forward implication.

Now suppose we have been given maximal flats F_1, F_2 in X_1, X_2 , and suppose $F \subset X$ is a flat containing $F_1 \times F_2$. Then $\pi_1(F)$ is a flat in X_1 containing F_1 and $\pi_2(F)$ is a flat in X_2 containing F_2 . Therefore $\pi_1(F) = F_1$ and $\pi_2(F) = F_2$, which means that $F \subset F_1 \times F_2$. Therefore $F = F_1 \times F_2$, showing that $F_1 \times F_2$ is maximal. \square

As an immediate consequence, we get

Corollary 2.7 *Let X be a CAT(0) space splitting as $X = X_1 \times X_2$. A subspace of $\partial_{\text{Tits}} X$ is a round sphere iff it is a spherical join of round spheres in $\partial_{\text{Tits}} X_1$ and $\partial_{\text{Tits}} X_2$.*

Lemma 2.8 *Let X be a CAT(0) space splitting as $X = X_1 \times X_2$ such that ∂X has a dense family of round spheres. Then the factors ∂X_1 and ∂X_2 also have dense families of round spheres.*

Proof Let $q : \partial X_1 \times \partial X_2 \times [0, \pi/2] \rightarrow \partial X_1 * \partial X_2$ be a quotient map with the conventions

- The restriction to $\partial X_1 \times \partial X_2 \times \{0\}$ is projection to the ∂X_1 -coordinate.
- The restriction to $\partial X_1 \times \partial X_2 \times \{\pi/2\}$ is projection to the ∂X_2 -coordinate.
- The restriction to $\partial X_1 \cup \partial X_2 \times (0, \pi/2)$ is a homeomorphic embedding.

Choose any $\zeta_1 \in \partial X_1$ and let $U \subset \partial X_1$ be an open neighborhood of ζ_1 (in the cone topology). Then $U' = q(U \times \partial X_2 \times [0, \pi/4])$ is a nonempty open set whose intersection with ∂X_1 is U . By hypothesis, there is a round sphere K which intersects U' at a point, say $q(\eta_1, \eta_2, t)$. By the previous corollary, we know that $K = K_1 * K_2$ where K_1 is a round sphere in ∂X_1 and K_2 is a round sphere in ∂X_2 . Therefore $U \cap K_1$ contains the point η_1 . \square

2.3 A splitting lemma

In this section we prove a strengthened version of Theorem 1.4 from the Introduction. In particular, by Corollary 2.5 and Lemma 2.8, this theorem applies to irreducible factors of a CAT(0) space with a cocompact group action, which is precisely what we need for the proof of Theorem 1.1.

Theorem 2.9 *Let X be a proper CAT(0) space, where ∂X has a dense family of round spheres. Suppose ∂X contains a proper subset P that is π -convex and involutive and closed in the cone topology. Then ∂X splits as the spherical join $\partial X = P * P^\perp$ where P^\perp is the set of points that have Tits distance exactly $\pi/2$ from all points in P .*

Recall that for a subset P of a metric space X ,

$$P^\perp = \left\{ x \in \partial X \mid d_{Tits}(x, y) = \frac{\pi}{2} \text{ for all } y \in P \right\}.$$

To prove Theorem 2.9 we first need to establish some lemmas.

Lemma 2.10 *Let P be an involutive subset of ∂X and K be a round sphere in ∂X . Define*

$$P_K^\perp = \left\{ x \in K \mid d_{Tits}(x, y) = \frac{\pi}{2} \text{ for all } y \in P \cap K \right\}.$$

Then $P \cap K$ is nonempty and $P_K^\perp = P^\perp \cap K$.

Proof By Lemma 2.4, every point of P has an antipode in K . Thus $P \cap K$ is nonempty. It is clear that $P^\perp \cap K \subset P_K^\perp$. To prove the converse, let $x \in P_K^\perp$ and $y \in P$. Draw the geodesic $[y, x]$. As in the proof of [2, Lemma 3.1] we can extend $[y, x]$ inside K to get a geodesic $[y, z]$ of length π that passes through x . Since P is involutive, $z \in P$. Thus $d(x, z) = \pi/2$ by definition of P_K^\perp and so $d(x, y) = \pi/2$ as well. \square

Lemma 2.11 *If $P \subset \partial X$ is an involutive set, then P^\perp is closed in the cone topology.*

Proof Let $\{\zeta_n\} \subset P^\perp$ be a sequence of points converging to a point $\zeta \in \partial X$. Then for every $\eta \in P$, $d(\zeta, \eta) \leq \pi/2$ since the Tits metric is lower semicontinuous in the cone topology. But since X is geodesically complete, η has an antipode η' . Since P is involutive, $\eta' \in P$, so $d(\zeta, \eta') \leq \pi/2$, and therefore we must have $d(\zeta, \eta) = \pi/2$. \square

Proof of Theorem 2.9 Our goal is to prove that every $x \in \partial X$ lies between points $y \in P$ and $z \in P^\perp$, by [18, Lemma 4.1], for example. Let $x \in \partial X$ be given. By assumption we may choose round spheres K_n and $x_n \in K_n$ such that $x_n \rightarrow x$. If for large enough n , $P \supset K_n$, then $x \in P$ and we are done. On the other hand, every $P_n = P \cap K_n$ is nonempty by the lemma above. So P_n is a proper closed involutive π -convex subset of K_n , hence a subsphere, hence $K_n = P_n * P_n^\perp$ where $P_n^\perp = P^\perp \cap K_n$. Choose $y_n \in P_n$ and $z_n \in P_n^\perp$ such that $x_n \in [y_n, z_n] \subset K_n$. Since ∂X is compact, we may pass to subsequences so that $y_n \rightarrow y \in \partial X$ and $z_n \rightarrow z \in \partial X$. Since P and P^\perp are both closed, $y \in P$ and $z \in P^\perp$.

It remains to verify that $d(y, x) + d(x, z) = \pi/2$. Since the Tits metric is lower semicontinuous in the cone topology,

$$\pi/2 = d(y, z) \leq d(y, x) + d(x, z) \leq \liminf d(y_n, x_n) + d(x_n, z_n) = \pi/2. \quad \square$$

Proof of Theorem 1.4 By Corollary 2.5 there is a dense family of round spheres in ∂X . Hence the claim follows from Theorem 2.9. \square

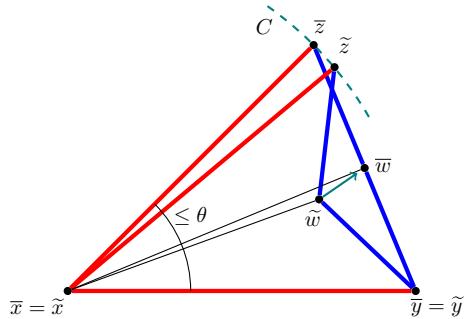
2.4 Almost-flat triangles

Recall that the boundary of a flat sector in X is a Tits geodesic in $\partial_{Tits} X$ by [4, Corollary II.9.9]. While the converse is not true in general, it is true approximately, as we will see in this section. We will need two lemmas in Euclidean geometry: the well-known Alexandrov Lemma and a controlled version.

Lemma 2.12 [4, Lemma I.2.16] *Let $\bar{x}, \bar{y}, \bar{z}, \bar{w}, \tilde{x}, \tilde{y}, \tilde{z}$, and \tilde{w} be points in \mathbb{E}^2 such that*

- (1) \bar{w} is between \bar{y} and \bar{z} ,

Fig. 1 The proof of the controlled alexandrov Lemma



- (2) \tilde{z} and \tilde{y} are on opposite sides of the line passing through \tilde{x} and \tilde{w} ,
- (3) $d(\tilde{y}, \tilde{x}) + d(\tilde{x}, \tilde{z}) \geq d(\tilde{y}, \tilde{w}) + d(\tilde{w}, \tilde{z})$,
- (4) $d(\bar{x}, \bar{y}) = d(\tilde{x}, \tilde{y})$, $d(\bar{x}, \bar{z}) = d(\tilde{x}, \tilde{z})$, $d(\bar{y}, \bar{w}) = d(\tilde{y}, \tilde{w})$, $d(\bar{w}, \bar{z}) = d(\tilde{w}, \tilde{z})$, and
- (5) $\pi \leq \angle_{\tilde{w}}(\tilde{x}, \tilde{y}) + \angle_{\tilde{w}}(\tilde{x}, \tilde{z})$. Then
- (6) $\angle_{\bar{x}}(\bar{y}, \bar{z}) \geq \angle_{\tilde{x}}(\tilde{y}, \tilde{w}) + \angle_{\tilde{x}}(\tilde{w}, \tilde{z})$,
- (7) $\angle_{\bar{y}}(\bar{x}, \bar{w}) \geq \angle_{\tilde{y}}(\tilde{x}, \tilde{w})$,
- (8) $\angle_{\bar{z}}(\bar{x}, \bar{w}) \geq \angle_{\tilde{z}}(\tilde{x}, \tilde{w})$, and
- (9) $d(\bar{x}, \bar{w}) \geq d(\tilde{x}, \tilde{w})$.

The next lemma is a modified version of the Alexandrov Lemma, which gives a lower bound on $d(\tilde{x}, \tilde{w})$ under additional hypotheses.

Lemma 2.13 (Controlled Alexandrov Lemma) *Let $0 < \theta < \pi$ be fixed. Given $\epsilon > 0$, there is a $\delta > 0$ such that whenever $\bar{x}, \bar{y}, \bar{z}, \bar{w}, \tilde{x}, \tilde{y}, \tilde{z}$, and $\tilde{w} \in \mathbb{E}^2$ satisfy the conditions of the Alexandrov lemma and in addition:*

- $d(\bar{x}, \bar{y}) = d(\bar{x}, \bar{z}) = 1$,
- \bar{w} is the midpoint of the segment $[\bar{z}, \bar{y}]$,
- $\theta - \delta \leq \angle_{\tilde{x}}(\tilde{y}, \tilde{z}) \leq \angle_{\bar{x}}(\bar{y}, \bar{z}) \leq \theta$

then $d(\bar{x}, \bar{w}) - \epsilon \leq d(\tilde{x}, \tilde{w})$.

Proof Let $0 < \theta < \pi$ be given. Suppose \bar{x} , \bar{y} , and \bar{z} satisfy the hypotheses. Without loss of generality, choose $\tilde{x} = \bar{x}$ and $\tilde{y} = \bar{y}$. Let $C_{\bar{z}}$ be the subarc of the circle of radius 1 centered at \bar{x} joining \bar{z} to \bar{y} which has length $< \pi$. Given \tilde{z} , the circles centered at \tilde{z} and \tilde{y} of radius $d(\tilde{z}, \bar{w})$ must intersect in one or two points because of (6) in the previous lemma. Condition (5) guarantees that \tilde{w} is the point closer to \tilde{x} . This shows that $d(\tilde{w}, \bar{w})$ is a continuous function of the pair (\tilde{z}, \bar{z}) whose domain is a compact set (see Fig. 1). Therefore this function is uniformly continuous and since it attains zero whenever $\tilde{z} = \bar{z}$, the conclusion follows. \square

We now return our attention to studying triangles in CAT(0) spaces. The proof of the next lemma echoes that of the flat triangle Lemma in [4, Proposition II.2.9].

Lemma 2.14 (Approximately Flat Triangle Lemma) *Let (X, d) be a CAT(0) space and let $\theta \in (0, \pi)$ be fixed. Given $\epsilon > 0$, there is a $\delta > 0$ such that for all $t > 0$, whenever $x, y, z \in X$ with $d(x, y) = d(x, z) = t$ and*

$$\theta - \delta \leq \angle_x(y, z) \leq \overline{\angle_x}(y, z) \leq \theta.$$

then for the midpoint w in the geodesic $[y, z]$,

$$d(w, x) > d_{\mathbb{E}^2}(\overline{w}, \overline{x}) - \epsilon t$$

where $\Delta(\bar{x}, \bar{y}, \bar{z})$ is a comparison triangle in Euclidean space with \bar{w} the point corresponding to w (i.e., \bar{w} is the midpoint of $[\bar{y}, \bar{z}]$).

Proof Choose $\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w}$ in \mathbb{E}^2 so that $\Delta(\tilde{x}, \tilde{y}, \tilde{w})$ and $\Delta(\tilde{x}, \tilde{z}, \tilde{w})$ are comparison triangles for $\Delta(x, y, w)$ and $\Delta(x, z, w)$ respectively, arranged so that \tilde{z} and \tilde{y} are on opposite sides of the line passing through \tilde{x} and \tilde{w} . The reader may check that the hypothesis of Alexandrov Lemma (Lemma 2.12) are satisfied. Then

$$\begin{aligned}
 \theta - \delta &\leq \angle_x(y, z) && \text{(by hypothesis)} \\
 &\leq \angle_x(y, w) + \angle_x(w, z) && \text{(by triangle inequality)} \\
 &\leq \bar{\angle}_x(y, w) + \bar{\angle}_x(w, z) && \text{(by definition of comparison angle)} \\
 &= \angle_{\tilde{x}}(\tilde{y}, \tilde{w}) + \angle_{\tilde{x}}(\tilde{w}, \tilde{z}) = \angle_{\tilde{x}}(\tilde{y}, \tilde{z}) && \text{(by euclidean geometry)} \\
 &\leq \bar{\angle}_x(y, z) && \text{(by (6) of Lemma 2.12)} \\
 &\leq \theta && \text{(by hypothesis.)}
 \end{aligned}$$

This verifies the last hypothesis of Lemma 2.13. Therefore the claim follows after rescaling. \square

Corollary 2.15 (Approximately Flat Sectors) *Let $\zeta, v \in \partial X$ such that $\theta = d_{Tits}(\zeta, v) < \pi$ and $\epsilon > 0$ be given. Then $p \in X$ may be chosen so that the following statement holds: if α and β are the unit speed geodesics emanating from p going out to ζ and v respectively and z_t is the midpoint of the geodesic $[\alpha(t), \beta(t)]$, then $d(p, z_t) \geq t \cos(\theta/2) - \epsilon t$.*

Proof By definition of $\angle_{Tits}(\zeta, v)$, $p \in X$ may be chosen so that $\theta - \angle_p(\zeta, v)$ is smaller than the $\delta = \delta(\theta, \epsilon)$ provided in Lemma 2.14. For any fixed t , the comparison triangle $\Delta(p, \alpha(t), \beta(t))$ in Euclidean space is an isosceles triangle with two sides of length t and apex angle of measure $\theta_t = \bar{\angle}_p(\alpha(t), \beta(t))$. Since θ_t is nondecreasing in t (1), θ_t lies between $\theta - \delta$ and θ . Therefore, by Lemma 2.14,

$$d(p, z_t) \geq t \cos(\theta_t/2) - \epsilon t \geq t \cos(\theta/2) - \epsilon t.$$

\square

3 Affine maps

3.1 Properties of affine maps between CAT(0) spaces

Let $f: X \rightarrow Y$ be a continuous affine map between proper CAT(0) spaces. We first establish Lemma 3.1 below, which allows us to assume that f is surjective. To this end, recall that CAT(0) metrics are convex, meaning that the distance between a pair of points on geodesics is bounded above by a convex combination of the distances between their endpoints. This implies that geodesic segments are uniquely determined by their endpoints. Moreover, after reparameterizing as constant speed maps over $[0, 1]$, they depend continuously on their endpoints (in the uniform topology on maps). We remind the reader that we will denote geodesic rays, lines, and their parameterizations by the same letters.

Lemma 3.1 *Let $f: X \rightarrow Y$ be a continuous affine map between proper CAT(0) spaces. The image Y' of f is a closed, convex subspace of Y . If X is geodesically complete, then so is Y' .*

Proof Any two points $p' \neq q' \in Y'$ are images of points $p \neq q \in X$. Hence p' and q' belong to $f([p, q])$. Since geodesics are unique, it follows that Y' is convex.

To see that Y' is closed, choose any sequence $(y_n) \subset Y'$ converging to a point $y \in Y$. Choose preimages $(x_n) \subset X$. By passing to a subsequence, we may assume that $x_n \rightarrow x \in X \cup \partial X$. If $x \in X$, then $y = f(x) \in Y'$ by continuity and we are done. Otherwise the sequence of geodesics $[x_0, x_n]$ converges to a ray α . By continuity of geodesics in their endpoints, $[y_0, y_n] \rightarrow f(\alpha)$, which is either a ray or a point. If it is a ray, then $\{y_n\}$ is unbounded, giving us a contradiction. So $f(\alpha)$ is a point and $y = y_0 \in Y'$. \square

Remark 3.2 The continuous affine image of a CAT(0) space need not be CAT(0). For instance, the identity map $\mathbb{E}^2 \rightarrow (\mathbb{R}^2, l_1)$ is a continuous affine map. Similar examples can be obtained by replacing l_1 with a norm determined by a suitable centrally symmetric convex body. Indeed, this idea gives rise to the Finsler norms explored in [19].

Recall that for a geodesic α in X , $\rho(\alpha)$ denotes the constant by which α is rescaled by f .

Lemma 3.3 *Let $f: X \rightarrow Y$ be a continuous affine map between proper CAT(0) spaces with rescaling function ρ . Then:*

- (1) *ρ determines a function $\partial X \rightarrow [0, \infty)$, which we still call ρ . In other words, $\rho(\alpha) = \rho(\beta)$ if α and β are asymptotic geodesic rays in X .*
- (2) *If $\rho(\alpha) > 0$ and β is a geodesic ray asymptotic to α , then $f(\alpha)$ and $f(\beta)$ are also asymptotic geodesic rays.*
- (3) *ρ is a continuous function on ∂X in the cone topology.*
- (4) *If f is surjective, X is geodesically complete, and there is no $\zeta \in \partial X$ such that $\rho(\zeta) = 0$, then f extends to a homeomorphism $X \cup \partial X \rightarrow Y \cup \partial Y$.*

Proof Assume all geodesics in X are parameterized to have unit speed.

Proof of (1) and (2). Suppose α and β are a pair of asymptotic geodesics in X . Let γ_n be the geodesic joining $\beta(0)$ to $\alpha(n)$. We will break the proof into two cases.

Case 1: Assume $\rho(\alpha) > 0$. Then the image of α is also a geodesic ray (with new speed). Then, by the triangle inequality,

$$\lim_{n \rightarrow \infty} \frac{d(\beta(0), \alpha(n))}{d(\alpha(0), \alpha(n))} = \lim_{n \rightarrow \infty} \frac{d(f\beta(0), f\alpha(n))}{d(f\alpha(0), f\alpha(n))} = 1.$$

This implies that $\rho(\gamma_n) \rightarrow \rho(\alpha)$.

By convexity of the metric, $\gamma_n \rightarrow \beta$ uniformly on compact sets. In Y , $f(\gamma_n)$ is the geodesic joining $f(\beta(0))$ to $f(\alpha(n))$ and $f(\gamma_n)$ converges to the unique geodesic ray $\widehat{\beta}$ emanating from $f(\beta(0))$ which is asymptotic to $f(\alpha)$. On the other hand, $f(\gamma_n)$ converges to $f(\beta)$. This implies that $f(\beta) = \widehat{\beta}$ and hence $\rho(\alpha) = \rho(\beta)$. This establishes (1) and (2).

Case 2: Assume $\rho(\alpha) = 0$. Then the image of α is a single point, and all γ_n have the same image – a finite geodesic segment $\widehat{\gamma}$ emanating from $f\beta(0)$. Since the lengths of the γ_n go to infinity, $\rho(\gamma_n)$ converges to zero. Again, γ_n converges to β , and so by continuity of f , $\rho(\beta) = 0$.

Proof of (3). Observe that whenever γ_n is a sequence of geodesics (either segments or rays) converging to a geodesic ray β , then $f(\gamma_n)$ converges to $f(\beta)$ and $\rho(\gamma_n)$ converges to $\rho(\beta)$.

Proof of (4). First note that f is injective; if not, there are two points p and q such that $f(p) = f(q)$. Then the geodesic containing p and q is mapped to a single point. Since X is geodesically complete, we can extend this geodesic segment to the boundary, giving us a point ζ on the boundary such that $\rho(\zeta) = 0$. We observe that since ρ is continuous on the compact set ∂X , it attains upper and lower bounds. Therefore f is bi-Lipschitz on X . Hence

f^{-1} is continuous on Y , extends to ∂Y by (2). Clearly this extension is the inverse of the extension of f to $X \cup \partial X$. \square

Next we reduce to the case where f is injective.

Lemma 3.4 *Let X and Y be $CAT(0)$ spaces, X geodesically complete and proper. Let $f: X \rightarrow Y$ be a continuous affine map. If ρ is 0 anywhere on ∂X , then X splits as a product $X = X_0 \times X_1$ such that $\partial(X_0) = \rho^{-1}(0)$. More precisely,*

(1) *for every $x \in X_0$, f is injective on the subspace $\{x\} \times X_1$ and*

(2) *for every $y \in X_1$, $f(X_0 \times \{y\})$ is a single point.*

Proof Our proof resembles the proof of [4, Proposition II.6.23]. Fix x and y in Y , and let Z_x and Z_y be their preimages under f . Observe first of all that each Z_x is totally geodesic. For, given $p, q \in Z_x$, $\rho([p, q]) = 0$. Thus every ray α which extends $[p, q]$ stays inside Z_x .

Now define $\phi: Z_x \rightarrow [0, \infty)$ by letting $\phi(p)$ be the distance from $p \in Z_x$ to the closest point on Z_y . This function is convex. We will show that ϕ is constant by proving that it is bounded. Suppose $\phi(p) > \phi(q)$. Extend $[q, p]$ to get a geodesic ray α . Since Z_x is totally geodesic, α is contained in Z_x . The fact that ϕ is convex and increasing on α implies that ϕ is unbounded here.

Now, let $q' \in Z_y$ be the point closest to q and α' be the geodesic ray emanating from q' asymptotic to α . By the previous proposition, $\rho(\alpha') = \rho(\alpha) = 0$ and therefore $f(\alpha') = f(q')$. This shows that $\alpha' \subset Z_y$. But this means that ϕ is bounded on α , giving us a contradiction.

Hence, Z_x and Z_y are equidistant totally geodesic subspaces. As in the proof of [4, Proposition II.6.23], we may use the Sandwich Lemma [4, Exercise II.2.12(2)] to get an isometry $X \rightarrow X_0 \times X_1$ where X_0 is the set $\{Z_x\}_{x \in X}$ and $X_1 = Z_{x_1}$ for some $x_1 \in X$. \square

From this point forward, we will assume that ρ is bounded away from zero, which is equivalent to the assumption that f is injective.

We will need to know that ρ is constant on antipodes. If an antipodal pair $\zeta, \eta \in \partial X$ is joined by a geodesic line in X , then this statement is obvious. This is guaranteed when $d_{Tits}(\zeta, \eta) > \pi$, for instance, by [4, Proposition II.9.21(1)]. However, one can construct $CAT(0)$ 2-complexes in which there is a pair of points ζ, η in the boundary where $d_{Tits}(\zeta, \eta) = \pi$ but there is no geodesic in the space joining them [4, Example II.9.23(2)]. Therefore a more robust argument is needed.

First, a technical lemma about Euclidean space.

Lemma 3.5 *Let $0 < m \leq M$ be given. Then there is a continuous function $\sigma: [0, \pi) \rightarrow (0, \infty)$ such that $\sigma(0) = 0$ and whenever $\triangle xyz$ is an isosceles triangle in Euclidean space, with $d(x, y) = d(x, z)$ and $\triangle x'y'z'$ is another triangle satisfying*

$$\frac{d(x, y)}{d(x', y')}, \frac{d(x, z)}{d(x', z')}, \text{ and } \frac{d(y, z)}{d(y', z')}$$

lying between m and M , then

$$\angle_{x'}(y', z') \leq \sigma(\angle_x(y, z)).$$

Proof Begin by defining $D(\theta)$ to be the length of the third side of an isosceles triangle with legs 1 and apex angle of measure θ . Define $\sigma(\theta) = \arccos\left(1 - \frac{M^2}{2m^2} D(\theta)^2\right)$. This is continuous

in θ with $\sigma(0) = 0$. Let Δxyz and $\Delta x'y'z'$ be as in the statement of the lemma. Denote $t = d(x, y) = d(x, z)$ and $\theta = \angle_x(y, z)$. Then $d(y, z) = D(\theta)t$. Set also $A = d(x', y')$, $B = d(x', z')$, $C = d(y', z')$, and $\theta' = \angle_{x'}(y', z')$. From the Law of Cosines and since $A^2 + B^2 \geq 2AB$ for all real numbers A and B , we obtain

$$\begin{aligned}\cos \theta' &= \frac{A^2 + B^2}{2AB} - \frac{C^2}{2AB} \\ &\geq 1 - \frac{M^2}{2m^2} D(\theta)^2 \\ &= \cos(\sigma(\theta)).\end{aligned}$$

Since $\cos(\theta)$ is decreasing in θ , we have $\theta' \leq \sigma(\theta)$ as desired. \square

Lemma 3.6 (Small Angles Lemma) *Let X and Y be CAT(0) spaces and $f : X \rightarrow Y$ be a continuous affine map. For every $\epsilon > 0$, there is a $\delta > 0$ such that whenever $x, y, z \in X$ are distinct and $\angle_x(y, z) < \delta$, then $\angle_{f(x)}(f(y), f(z)) < \epsilon$. Furthermore, for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever $x, y, z \in X$ are distinct and $\angle_x(y, z) < \delta$, then $|\rho([x, y]) - \rho([x, z])| < \epsilon$.*

Proof Let $x, y, z \in X$ be distinct, and let σ be the function constructed in the previous lemma where m and M are the minimum and maximum of ρ . Let α and β denote the unit speed geodesics $[x, y]$ and $[x, z]$ with $\alpha(0) = \beta(0) = x$ and choose any $0 < t < \min\{d_X(x, y), d_X(x, z)\}$. Recall that $\angle_x(y, z)$ denotes the angle of a Euclidean comparison triangle at the vertex corresponding to x . Observe that the ratios

$$\frac{d(f\alpha(t), f\beta(t))}{d(\alpha(t), \beta(t))}, \frac{d(f(x), f\alpha(t))}{d(x, \alpha(t))}, \text{ and } \frac{d(f(x), f\beta(t))}{d(x, \beta(t))}$$

all lie between m and M . Therefore

$$\begin{aligned}\angle_{f(x)}(f(y), f(z)) &\leq \overline{\angle}_{f(x)}(f\alpha(t), f\beta(t)) \\ &\leq \sigma(\overline{\angle}_x(\alpha(t), \beta(t))) \\ &\rightarrow \sigma(\angle_x(y, z))\end{aligned}$$

as $t \rightarrow 0$. Since σ is continuous, this establishes the first part of the lemma.

Now consider the second part of the lemma. Denote $A = \rho([x, y])$ and $B = \rho([x, z])$. Applying the triangle inequality to the triple $(f\alpha(t), f\beta(t), f(x))$ for small $t > 0$, we get

$$\begin{aligned}|At - Bt| &\leq d(f\alpha(t), f\beta(t)) \\ &\leq M d(\alpha(t), \beta(t)) \\ &= Mt D(\overline{\angle}_x(\alpha(t), \beta(t)))\end{aligned}$$

where $D(\theta)$ is the continuous function defined in the proof of the previous lemma. Divide both sides by t and let $t \rightarrow 0$ to get

$$|A - B| \leq MD(\angle_x(y, z)).$$

\square

Lemma 3.7 *Assume $\angle_{Tits}(\zeta, \zeta') = \pi$, $x_0 \in X$, and α a geodesic ray from x_0 to ζ . Then*

$$\lim_{t \rightarrow \infty} \angle_{\alpha(t)}(x_0, \zeta') = 0.$$

Proof Consider the ideal triangle with the vertices ζ' , $\alpha(t)$, and $\alpha(t/2)$. We know that $\angle_{\alpha(t)}(\alpha(t/2), \zeta') + \angle_{\alpha(t/2)}(\alpha(t), \zeta') \leq \pi$. By [4, Proposition II.9.8(2)], $\angle_{\alpha(t/2)}(\alpha(t), \zeta') \rightarrow \pi$ as $t \rightarrow \infty$; thus

$$\lim_{t \rightarrow \infty} \angle_{\alpha(t)}(\alpha(t/2), \zeta') = \lim_{t \rightarrow \infty} \angle_{\alpha(t)}(x_0, \zeta') = 0. \quad \square$$

Lemma 3.8 (Involutive Invariance) *The rescaling function ρ is constant on pairs of antipodes. Specifically, whenever $d_{Tits}(\zeta, \zeta') \geq \pi$, then $\rho(\zeta) = \rho(\zeta')$.*

Proof Let α be a geodesic ray going out to ζ . For each $t > 0$, let β_n be the geodesic ray based at $\alpha(n)$ going out to ζ' . By Lemma 3.7, $\angle_{\alpha(n)}(\alpha(0), \beta_n(1)) \rightarrow 0$, so by Lemma 3.6, $\rho(\beta_n)$ converges to $\rho(\alpha)$ as $n \rightarrow \infty$. Thus we establish that $\rho(\zeta') = \rho(\zeta)$. \square

3.2 Proof of Theorem 1.1

Finally we address the problem of how to split affine maps. Recall that a geodesically complete CAT(0) space X is called *irreducible* if it does not split as a product. By [4, Theorem II.9.24], this is equivalent to saying that the boundary does not split as a spherical join in the Tits topology (if X is geodesically complete).

Lemma 3.9 *Let X and Y be CAT(0) spaces, where X is irreducible and non-Euclidean. Let $f: X \times L \rightarrow Y$ be an injective affine map, where $L = \mathbb{R}$. Then f splits. More precisely,*

- (1) *the image of f splits as $Y' \times L'$, where L' is a line and Y' is a convex subset of Y ,*
- (2) *for any $p \in L$, $f(X \times \{p\}) = Y' \times f(p)$, and*
- (3) *for any $x \in X$, $f(\{x\} \times L) = f(x) \times L'$.*

Proof Note that for all $x \in X$, $f(\{x\} \times L)$ is a geodesic, and for any two such $x_1, x_2 \in X$, these geodesics are parallel. From Lemma 3.3 $f(\{x_1\} \times L)$ and $f(\{x_2\} \times L)$ are also parallel. By [4, Proposition II.2.14(2)] the image $f(X \times L)$ splits, where the lines of the new splitting correspond to images of lines $f(\{x\} \times L)$.

Let $\pi_{L'}$ be the projection from the image $f(X \times L)$ to L' . To show (2), we need to show that for any $p \in L$, the composition $\pi_{L'} \circ f$ is constant on $X \times \{p\}$. Note that by [4, Proposition I.5.3(3)], $\pi_{L'}$ is itself an affine map. Then $\pi_{L'} \circ f$ restricted to X is an affine map from an irreducible non-Euclidean CAT(0) space to \mathbb{R} . By [20, Lemma 4.1], since X does not split, this function must be constant. \square

Let X and Y be CAT(0) spaces such that X is proper, geodesically complete, and admits a cocompact group action. In addition, let $f: X \rightarrow Y$ be a continuous affine map. By Lemma 3.1, we may assume f is surjective. Recall that by Lemma 3.4 we may assume that f is injective, since otherwise, f will be a point map on a factor of X . Suppose X admits a splitting, and let X_1 and X_2 be two factors of the splitting, where X_2 is non-Euclidean. Choosing any line L in X_1 , apply Lemma 3.9 to find that $f(L)$ is perpendicular to all of the image of X_2 . If one of the factors $f(X_1)$ splits, then applying the same lemma to f^{-1} tells us that X_1 itself must split. Thus the image of an irreducible factor of X is an irreducible factor of Y , and hence the image of the Euclidean factor of X must be the Euclidean factor of Y .

Define $\mathbf{Max} \subset \partial X$ to be the subset on which ρ attains its maximum. It remains to show that, when restricted to a non-Euclidean irreducible factor, f must be a dilation. Indeed, suppose f is not a dilation. Using Theorem 2.9 and the set \mathbf{Max} we will then show that X admits a splitting with two nontrivial factors. To use this theorem, we need the following lemma.

Lemma 3.10 (**Max** is π -Convex) *Assume $\zeta, \eta \in \mathbf{Max}$ such that $d(\zeta, \eta) < \pi$, and let v be the midpoint of the geodesic $[\zeta, \eta] \subset \partial_{Tits} X$. Then $v \in \mathbf{Max}$ as well.*

Proof Since f is bijective, ρ cannot attain zero. So by rescaling the metric on Y (if necessary), we may assume that the maximum attained by ρ is 1. Let ζ' and η' be the images of ζ and η and fix $\epsilon > 0$. Set $\theta = d_{Tits}(\zeta, \eta)/2$, and $\theta' = d_{Tits}(\zeta', \eta')/2$. By Corollary 2.15 we may choose p' such that the rays α' from p' to ζ' and β' from p' to η' satisfy the following: if z'_t is the midpoint of $[\alpha'(t), \beta'(t)]$, then $d(p', z'_t) \geq t \cos(\theta') - t\epsilon$.

Now let p be the preimage of p' and α and β be the preimages of α' and β' . Note that α and β determine the points ζ and η at infinity. Denote by z_t the midpoint of $[\alpha(t), \beta(t)]$. By construction, $f(z_t) = z'_t$. By the Law of Cosines and comparison geometry and Equation (2), we may choose t large enough so that

$$d(p, z_t) \leq t \cos(\angle_p(\alpha(t), \beta(t))/2) \leq t \cos(\theta) + t\epsilon.$$

Thus

$$\frac{d(p', z'_t)}{d(p, z_t)} \geq \frac{\cos(\theta') - \epsilon}{\cos(\theta) + \epsilon}.$$

As shown in the proof of [4, Lemma II.9.14], z_t converges to the midpoint v of the Tits geodesic $[\zeta, \eta]$ as $t \rightarrow \infty$. Letting $\epsilon \rightarrow 0$, we get $\rho(v) \geq \cos(\theta')/\cos(\theta)$. Since we assumed $\rho \leq 1$, we have $d(\alpha'(t), \beta'(t)) \leq d(\alpha(t), \beta(t))$. So $\theta' \leq \theta$, and $\cos \theta' \geq \cos \theta$. Thus $\rho(v) = 1$. \square

Suppose that f is not a dilation when restricted to an irreducible, non-Euclidean factor of X . Then **Max** is a proper subset of the boundary of this factor. Since ρ is continuous, **Max** is closed in the cone topology. It is involutive by Lemma 3.8. We have now shown that it is also π -convex. Thus we can apply Theorem 2.9 to find that this irreducible factor splits, which is a contradiction. Therefore, f must be a dilation on each irreducible factor of X .

3.3 Self-affine maps

Here we consider a self-affine map f of a proper CAT(0) space X admitting a geometric group action. We will prove Corollary 1.2 that X is flat if f is a strict contraction. We first need to establish a technical lemma.

Let $R \geq 0$ and $\lambda > 0$. We say that a subset $\Sigma \subset \partial X$ is (R, λ) -wide if there is an $x \in X$ such that for every pair $\zeta, \eta \in \Sigma$, there is a $y \in B_R(x)$ for which $\angle_y(\zeta, \eta) \geq \lambda$. We will refer to x as an (R, λ) -center for Σ . There is a bound on the cardinality of wide sets:

Lemma 3.11 *Let X be a cocompact proper CAT(0) space, $R \geq 0$, and $\lambda > 0$. Then there is a bound on the cardinality of (R, λ) -wide subsets of ∂X .*

Proof Let $\Sigma \subset \partial X$ be an (R, λ) -wide subset with $x \in X$ a center. Let α, β be a pair of geodesic rays emanating from x going out to a pair of points in Σ and set

$$K = \frac{2R + 1}{\sqrt{2 - 2 \cos \lambda}}.$$

Let $y \in B_R(x)$ and α' and β' be geodesic rays starting at y asymptotic to α and β such that $\angle_y(\alpha', \beta') \geq \lambda$. By [4, Proposition II.1.7(5)], $d(\alpha'(K), \beta'(K))$ is bounded below by the length B of the base of an isosceles triangle with legs of length K and apex angle λ . Using the Law of Cosines, $B = 2R + 1$. By convexity of metric, $\alpha'(K)$ and $\beta'(K)$ are a distance

of at most R away from $\alpha(K)$ and $\beta(K)$. Thus $d(\alpha(K), \beta(K)) \geq 1$. Therefore there is a subset $\widehat{\Sigma} \subset S_K(x)$ with the same cardinality as Σ such that the distance between every pair of points is at least 1.

Suppose now that there is a sequence of (R, λ) -wide subsets $\Sigma_n \subset \partial X$ such that the cardinality of Σ_n is at least n with corresponding center $x_n \in X$. Using cocompactness, and replacing $\widehat{\Sigma}_n$ by translates we may assume that $x_n \rightarrow x$. Construct for each Σ_n the corresponding set $\widehat{\Sigma}_n \subset S_K(x_n)$ as in the previous paragraph. Choose $y_n^1 \in \widehat{\Sigma}_n$, and let $\widehat{\Sigma}_n^1$ be the remaining $n - 1$ points. By properness of X , we may pass to a subsequence so that $y_n^1 \rightarrow y^1 \in S_K(x)$. Next choose $y_n^2 \in \widehat{\Sigma}_n^1$ for $n \geq 2$ and let $\widehat{\Sigma}_n^2$ be the remaining points. Again, pass to a subsequence so that $y_n^2 \rightarrow y^2 \in S_K(x)$. Note that $d(y^1, y^2) \geq 1$. Continuing in this manner, for all m we can find a $y_m \in S_K(x)$ such that $d(y^m, y^n) \geq 1$ for every $n \neq m$. Thus we have found an infinite discrete subset of $S_K(x)$, contradicting the assumption that X is proper. \square

Our strategy for the proof of Corollary 1.2 will be to show that $\partial_{Tits} X$ is compact, and then apply the following theorem of Bosché.

Theorem 3.12 [3, Propositions 3 and 7] *Let X be a geodesically complete proper $CAT(0)$ space admitting a geometric group action. If the Tits boundary $\partial_{Tits} X$ is compact, then X is flat.*

Proof of Corollary 1.2 Suppose $\partial_{Tits} X$ is not compact. By Theorem 1.1, we may pass to a factor of X , if necessary, so that f is a dilation. Recall that a complete metric space is compact if and only if it is totally bounded. Hence there is a sequence of points (x_i) in ∂X and a $\lambda > 0$ such that the Tits distance between any two points in the sequence is at least λ . Let M be the maximal size of $(1, \lambda)$ -wide sets of $\partial_{Tits} X$. Let Σ be the first $M + 1$ elements of (x_i) . Note that we may also require $\lambda < \pi$. In consequence, for any distinct points $\zeta, \nu \in \Sigma$, we have $\angle_{Tits}(\zeta, \nu) \geq \lambda$.

Form a finite subset Σ' of X by taking for every pair of distinct $\zeta, \nu \in \Sigma$ a point $y = y(\zeta, \nu) \in X$ such that $\angle_y(\zeta, \nu) \geq \lambda$.

Since f is a dilation, it induces an isometry on $\partial_{Tits} X$. In particular, it preserves the distances between the points in Σ while shrinking the diameter of Σ' . Thus for large enough k , $f^k(\Sigma)$ is $(1, \lambda)$ -wide, contradicting the assumption that M was the maximal cardinality of such sets. Therefore $\partial_{Tits} X$ is compact and we may apply Theorem 3.12 to get the desired result. \square

Proof of Corollary 1.3 By the Main Theorem, X splits as a product $X = X_1 \times \dots \times X_n$ of irreducible factors $X_i \neq \mathbb{R}$ such that the restrictions $f|_{X_i}: X_i \mapsto X$ are dilations. Since f is a homeomorphism, no rescaling constant can be zero and hence no $f(X_i)$ is a point. By uniqueness of the splitting and the Main Theorem, f must interchange the factors. If f preserves the factors of X , then by Corollary 1.2, it must be an isometry. This proves (1). Statements (2) and (3) follow immediately from (1). For (4), let j be the induced homomorphism from the group of affine homeomorphisms to the symmetric group of rank n . Then $\ker j$ is a subgroup of isometries by (1).

Remark 3.13 Observe that surjectivity is a necessary assumption since a tree may be dilated by constant $\alpha > 1$ to a subtree.

Acknowledgments We thank Dan Guralnik and Eric Swenson for their inspiring paper [12], and Russell Ricks and Ben Schmidt for discussions of this paper. Ricks in particular suggested the simple proof of Lemma 3.7 and the reference for Theorem 2.3. We also thank the referee for a thorough report. The first two authors held postdoctoral fellowships at the University of Michigan, and are thankful for the excellent working environment provided.

References

1. Ballmann, W., Gromov, M., Schroeder, V.: Manifolds of Nonpositive Curvature. Vol. 61 of Progress in Mathematics. Birkhäuser Boston Inc., Boston, MA (1985)
2. Balser, A., Lytchak, A.: Centers of convex subsets of buildings. *Ann. Global Anal. Geom.* **28**, 201–209 (2005)
3. Bosché, A.: Tits compact CAT(0) spaces. Preprint
4. Bridson, M.R., Haefliger, A.: Metric Spaces of Non-Positive Curvature. Vol. 319 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Berlin (1999)
5. Caprace, P.-E., Monod, N.: Isometry groups of non-positively curved spaces: structure theory. *J. Topol.* **2**, 661–700 (2009)
6. Croke, C.B., Kleiner, B.: The geodesic flow of a nonpositively curved graph manifold. *Geom. Funct. Anal.* **12**, 479–545 (2002)
7. Daskalopoulos, G., Mese, C.: Harmonic maps from a simplicial complex and geometric rigidity. *J. Differ. Geom.* **78**, 269–293 (2008)
8. Daskalopoulos, G., Mese, C.: Fixed point and rigidity theorems for harmonic maps into NPC spaces. *Geom. Dedicata* **141**, 33–57 (2009)
9. Daskalopoulos, G., Mese, C., Vdovina, A.: Superrigidity of hyperbolic buildings. *Geom. Funct. Anal.* **21**, 905–919 (2011)
10. Foertsch, T., Lytchak, A.: The de Rham decomposition theorem for metric spaces. *Geom. Funct. Anal.* **18**, 120–143 (2008)
11. Gromov, M., Schoen, R.: Harmonic maps into singular spaces and p -adic superrigidity for lattices in groups of rank one. *Inst. Hautes Études Sci. Publ. Math.* **76**, 165–246 (1992)
12. Guralnik, D., Swenson, E.: A ‘transversal’ for minimal invariant sets in the boundary of a CAT(0) group. *Trans. Amer. Math. Soc.* (2012)
13. Hitzelberger, P., Lytchak, A.: Spaces with many affine functions. *Proc. Am. Math. Soc.* **135**, 2263–2271 (2007)
14. Izeki, H., Nayatani, S.: Combinatorial harmonic maps and discrete-group actions on Hadamard spaces. *Geom. Dedicata* **114**, 147–188 (2005)
15. Kleiner, B.: The local structure of length spaces with curvature bounded above. *Math. Z.* **231**, 409–456 (1999)
16. Kobayashi, S., Nomizu, K.: Foundations of differential geometry, Vol. I. Wiley Classics Library. Wiley, New York, 1996. Reprint of the 1963 original, A Wiley-Interscience Publication
17. Leeb, B.: A characterization of irreducible symmetric spaces and Euclidean buildings of higher rank by their asymptotic geometry. *Bonner Mathematische Schriften* [Bonn Mathematical Publications], vol. 326, ii+42 pp. Universität Bonn, Mathematisches Institut, Bonn (2000)
18. Lytchak, A.: Rigidity of spherical buildings and joins. *Geom. Funct. Anal.* **15**, 720–752 (2005)
19. Lytchak, A.: Affine images of Riemannian manifolds. *Math. Z.* **270**, 809–817 (2012)
20. Lytchak, A., Schroeder, V.: Affine functions on CAT(κ)-spaces. *Math. Z.* **255**, 231–244 (2007)
21. Matveev, V.S.: Hyperbolic manifolds are geodesically rigid. *Invent. Math.* **151**, 579–609 (2003)
22. Nikolaev, I.: The tangent cone of an Aleksandrov space of curvature $\leq K$. *Manuscr. Math.* **86**, 137–147 (1995)
23. Ohta, S.-I.: Totally geodesic maps into metric spaces. *Math. Z.* **244**, 47–65 (2003)
24. Papasoglu, P., Swenson, E.: Boundaries and JSJ decompositions of CAT(0)-groups. *Geom. Funct. Anal.* **19**, 559–590 (2009)
25. Swenson, E.: On cyclic CAT(0) domains of discontinuity. *Groups Geom. Dyn.* **7**, 737–750 (2013)