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On the algebras over equivariant little disks

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ABSTRACT

We describe the structure present in algebras over the little disks operads for various representations of a finite group G, including those that are not necessarily universe or that do not contain trivial summands. We then spell out in more detail what happens for $G = C_2$, describing the structure on algebras over the little disks operad for the sign representation. Here we can also describe the resulting structure in Bredon homology. Finally, we produce a stable splitting of coinduced spaces analogous to the stable splitting of the product, and we use this to determine the homology of the signed James construction.

1. Twisted equivariant "monoids"

We begin with a purely algebraic observation. Consider the free associative monoid on the set $C_2 = \{x, \bar{x}\}$. The fixed points of this are the one point set consisting of the identity element, and elements which we expect from the commutative case, like the norm classes $x\bar{x}$, are not fixed here. If we enforce commutativity, then these become fixed, but as homotopy theorists, we should instead attach cells which connect $x\bar{x}$ to $\bar{x}x$, building a C_2-CW complex. Unfortunately, since $x\bar{x}$ and $\bar{x}x$ form a free orbit, these new cells will also be C_2-free. Continuing to make the multiplication more highly commutative will never add fixed cells, so we never produce new fixed points. This is reflecting a classical observation.

Proposition 1.1. *If $\mathcal{O}^{tr}$ is an E_∞ operad with trivial G-action, and X is a free G-space, then the fixed points of the free algebra are trivial:*

$$\left(\mathbb{P}_{\mathcal{O}^{tr}}(X)\right)^G \simeq *.$$

More generally, if V is a countable dimensional vector space with trivial G-action, and if $\mathcal{D}(V)$ is the little disks operad for V, then the free algebra on X also has trivial fixed points:

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$$\big(\mathbb{P}_{\mathcal{D}(V)}(X)\big)^G \simeq *.$$

This is in stark contrast with the "genuine" commutative case, where the tom Dieck splitting shows that the putative norm classes do give rise to new fixed points.

Proposition 1.2. *If $\mathcal{O}$ is a G-E_∞ operad, and if X is a free G-space, then we have*

$$\big(\mathbb{P}_{\mathcal{O}}(X)\big)^G \simeq \mathbb{P}_{\mathcal{O}^{tr}}(X/G).$$

The obvious map

$$i_e^*\mathbb{P}_{\mathcal{O}}(X) \simeq \mathbb{P}_{\mathcal{O}^{tr}}(i_e^*X) \to \mathbb{P}_{\mathcal{O}^{tr}}(X/G) \hookrightarrow \big(\mathbb{P}_{\mathcal{O}}(X)\big)^G$$

is the operadic norm from the underlying space to the fixed points. Adams' original discussion of the construction of the transfer in the G-equivariant Spanier-Whitehead category shows that we have a similar norm map for free algebras over the little disks algebra for any orthogonal G-representation in which G equivariantly embeds [1].

The primary goal of this paper is to study these exact phenomena by carefully unpacking some of the structure that we see in these algebras over little disks operads for (possibly finite dimensional) orthogonal G-representations. Since a trivial E_∞ operad and a G-E_∞ operad are both little disks operads, we will recover our classical understanding. More generally, a little disks operad for a G-universe U is the prototype of an N_∞ operad, the definition of which and the basic properties thereof were described by Blumberg and the author in [3]. In particular, the N_∞ operads behave very much like ordinary E_∞ operads plus operations which coherently encode norm maps. A natural question then, is what operads deserve to be called $\mathcal{N}_k$ operads for finite k. Exploring the examples for little disks gives desiderata for defining the $\mathcal{N}_k$, but we leave this for later work.

Work of Rourke and Sanderson [17], Segal [19], and Hauschild [10], equivariantizing foundational work of May [15], connects this general analysis to the study of loop spaces. They show that if $\mathcal{O}$ is the little disks operad for a representation V, then the natural map

$$\mathbb{P}_{\mathcal{O}}(X) \to \Omega^V\Sigma^V X$$

is a group completion. Thus our operadic analysis describes geometric content for these generalized loop spaces, unpacking the encoded structure. In particular, it provides a basic step for describing the operations on V-fold loop spaces, in the vein of Cohen [5].

The group C_2 has a non-trivial one-dimensional representation: the sign representation σ, and the corresponding operad E_σ and its algebras have become a central object of study in algebraic topology recently. Here are several recent example:

(1) Work of Dotto–Moi–Patchkoria–Reeh has described the Real topological Hochschild homology of E_σ algebras. Their computations of $\underline{\pi}_0$ closely parallel the results below [6].
(2) Hahn–Shi have used the E_σ-algebra structure on various quotients of $MU_{\mathbb{R}}$ by regular sequences to show that Lubin–Tate spectra are all Real oriented [8] and the Real orientations are maps of E_σ-algebras.
(3) Behrens–Wilson have lifted Mahowald's description of $H\mathbb{F}_2$ as a Thom spectrum to a C_2-equivariant spectrum, showing that $H\underline{\mathbb{F}}_2$ is a Thom spectrum over $\Omega^{1+\sigma}S^{2+\sigma}$ [2], and this has been generalized to arbitrary cyclic p-groups by Hahn–Wilson [9] and by Levy [13].
(4) Ongoing work of Hahn–Wilson in their program to build an odd-primary analogue of $MU_{\mathbb{R}}$ generalizes E_σ algebras to odd primes, building what they call "spoke algebras".

In the second part of the paper, we study the basic properties of signed loop spaces. We then turn to cohomology, looking at the cohomology of a coinduced space and the resulting structure on the homology of a signed loop space. We next study the signed version of the James construction on a C_2-space, showing that after suspension by the regular representation it splits.

As an appendix, we include a result of independent interest: stable splittings of coinduction. For C_2, coinduction splits after a signed suspension, while for C_p, coinduction splits after suspending by an irreducible 2-dimensional representation.

Notation and conventions In all that follows, G will denote a finite group, and H will usually denote a subgroup thereof. Letters around X in the alphabet will denote G-spaces; letters around T in the alphabet will denote G-sets; and letters around V in the alphabet will denote G-representations.

If X is a G-space and $x \in X$ is any point, then $\mathrm{Stab}(x)$ will denote the stabilizer subgroup of x in G. If $H \subset G$, then $W_G(H)$ will denote the Weyl group of H in G.

In the second half of the paper, we will begin doing equivariant algebra, and we will work often with Mackey functors for the cyclic group of order 2, C_2. The Mackey functor versions of standard classical constructions like homotopy or homology groups will be denoted with an underline to the classical symbol. For Mackey functors, we will follow Lewis' notation, stacking the values at the two orbits and indicating the structure maps:

$$\begin{array}{c} \underline{M}(C_2/C_2) \\ res \Big\downarrow \quad \Big\uparrow tr \\ \underline{M}(C_2/e) \end{array}$$

Finally, for gradings, we follow the increasingly standard wild-card notation: $*$ will be reserved for an arbitary element of $\mathbb{Z}$, while $\star$ will denote an element of $RO(G)$.

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2. Algebras over equivariant little disks operads

2.1. The general case

We take as our model the analysis of algebras over the little disks for a G-universe from [3, Theorem 4.19]. The key idea there was to understand which finite H-sets embed in our universe. Essentially everything goes through without change: the only difference is that the spaces parameterizing our generalized multiplications are no longer necessarily contractible. Much of the analysis of the spaces is closely related to results of Rourke and Sanderson [17]; we cast things in a way that most transparently reflects how various norms arise in the algebras.

2.1.1. Basic definition

Definition 2.1. Let V be an orthogonal representation of G (not necessarily finite dimensional, and not necessarily a universe for G). Let W be a finite dimensional orthogonal subrepresentation of V. A **little disk** in W is a (not necessarily equivariant) affine map $D(W) \to D(W)$.

Define the space $\mathcal{D}_W(V)_n$ to be the space of n-tuples of nonoverlapping little disks. This is a $G\times\Sigma_n$-space, where G acts by conjugation and Σ_n by permuting the coordinates. The operadic structure is by composition.

If $W\subset W'$, then the affine maps defining any of the little disks in $\mathcal{D}_W(V)_n$ extend to give little disks in $\mathcal{D}_{W'}(V)_n$, so we define

$$\mathcal{D}(V)_n = \operatorname*{colim}_W \mathcal{D}_W(V)_n.$$

Of course, if V is finite dimensional, then $\{V\}$ is cofinal in the set of finite dimensional subspaces of V, so we can ignore any colimits.

The operadic multiplications and transfers are produced by the fixed points for various subgroups of $G\times\Sigma_n$.

2.1.2. Structure of graph fixed points

We begin with a basic observation which follows immediately from the requirement that the little n-disks not overlap.

Proposition 2.2. *For any orthogonal representation V, Σ_n acts freely on $\mathcal{D}(V)_n$.*

Thus the only subgroups which could have fixed points are the "graph subgroups" considered in [3].

Definition 2.3. A **graph subgroup** of $G\times\Sigma_n$ is a subgroup which intersects Σ_n trivially.

The name comes from the fact that if Γ is a graph subgroup of $G\times\Sigma_n$, then there is a subgroup $H\subset G$ and a homomorphism $\phi\colon H\to\Sigma_n$ such that Γ is the graph of ϕ. In particular, to each graph subgroup, associate a subgroup H of G and an H-set structure on $\{1,\ldots,n\}$. If H is a proper subgroup of G, then the analysis of the Γ fixed points of $\mathcal{D}(V)$ is most naturally an H-equivariant one, and hence is covered by an induction argument on the subgroups. We can therefore restrict attention to $H=G$.

Lemma 2.4. *Let Γ be a graph subgroup of $G\times\Sigma_n$ corresponding to a G-set T. Then the map taking a disk to its center gives a weak equivalence*

$$\mathcal{D}(V)_n^\Gamma \simeq \mathrm{Emb}^G(T,V),$$

where $\mathrm{Emb}^G(T,V)$ is the space of G-equivariant embeddings of T into V. Moreover, this equivalence is

$$W_{G\times\Sigma_n}(\Gamma)\cong \mathrm{Aut}^G(T)$$

equivariant.

Proof. The map sending a little disk to its center establishes a $G\times\Sigma_n$-weak equivalence

$$\mathcal{D}(V)_n \simeq \mathrm{Emb}\big(\{1,\ldots,n\},V\big),$$

where $G\times\Sigma_n$ acts on the latter via the Σ_n-action on the source and the G-action on the target. When restricted to Γ, this becomes the conjugation action on $\mathrm{Emb}(T,V)$. □

Since we are considering only finite G-sets, these spaces of embeddings are a kind of equivariant configuration space.

Definition 2.5. Let $n \geq 1$, and let X be a G-space. Define the **configuration space of** nG/H in X to be

$$\mathrm{Conf}_{nG/H}(X) := \big\{(x_1, \dots, x_n) \mid \forall i, \mathrm{Stab}(x_i) = H \text{ and } \forall i \neq j, \forall g \in G, x_i \neq g x_j\big\},$$

topologized as a subspace of X^n.

Note that by construction

$$\mathrm{Conf}_{nG/H}(X) \subset (X^H)^n.$$

The latter has a canonical action of $W_G(H) \wr \Sigma_n$ extending the coordinate actions of $W_G(H)$. By construction, $\mathrm{Conf}_{nG/H}(X)$ is an equivariant subspace. With this observation, the following is immediate.

Proposition 2.6. *If X is any space, then we have a homeomorphism*

$$\mathrm{Emb}^G(nG/H, X) \cong \mathrm{Conf}_{nG/H}(X)$$

given by choosing an ordering of the summands and evaluating at the cosets of the identity for each summand. This is equivariant for

$$\mathrm{Aut}^G(nG/H) \cong W_G(H) \wr \Sigma_n.$$

When X is a representation, then these are simpler still: they are hyperplane complements.

Definition 2.7. Let $H \subset G$ be a subgroup and V a representation of G.

Let

$$V^{\partial H} := \bigcup_{H \subset K} V^K$$

be the H-relative singular set of V, and let

$$\mathring{V}^H := V^H - V^{\partial H}.$$

Proposition 2.8. *The configuration space of nG/H in a representation V is a hyperplane complement:*

$$\begin{aligned}\mathrm{Conf}_{nG/H}(V) &= \big\{(v_1, \dots, v_n) \in \mathring{V}^H \mid \forall i \neq j, \forall gH \in W_G(H), v_j \neq g v_j\big\} \\ &= (V^H - V^{\partial H})^n - \bigcup_{i \neq j, gH \in W_G(H)} W_{i,j,gH},\end{aligned}$$

where

$$W_{i,j,gH} = \big\{(v_1, \dots, v_n) \mid v_i = g v_j\big\} \subset (V^H)^n.$$

Proof. The subspace $\mathring{V}^H$ is the subspace of V consisting of those points with stabilizer H. It itself is a hyperplane complement, so its n-fold Cartesian power is. The result follows immediately from the definition. □

Putting this all together, we deduce the graph fixed points for an arbitrary graph subgroup.

Theorem 2.9. *Let $T = n_1G/H_1 \amalg \cdots \amalg n_kG/H_k$, where if $i \neq j$, then H_i and H_j are not conjugate. Then we have an $\mathrm{Aut}^G(T)$-equivariant homeomorphism*

$$\mathrm{Emb}^G(T,V) \cong \prod_{i=1}^{k} \mathrm{Conf}_{n_iG/H_i}(V).$$

In particular, this is a hyperplane complement.

Proof. It only remains to prove that the embeddings for non-conjugate stabilizers can never coincide (hence we get a product decomposition). However, this is immediate, since an embedding preserves stabilizers. □

Corollary 2.10. *If T is any finite G-set such that $T^G = \emptyset$, then*

$$\mathrm{Emb}^G(T,V) \simeq \mathrm{Emb}^G(T \amalg *, V).$$

Proof. The origin is always fixed in V, thus $\mathrm{Conf}_{G/G}(V)$ is always non-empty. □

Remark 2.11. If $|T| = n$, then choose an ordering of the points of T and hence a non-equivariant identification of T and $\{1,\dots n\}$. Including $*$ as the $(n+1)$st point gives a non-equivariant identification of $T \amalg *$ with $\{1,\dots,n+1\}$. In this case, the homotopy equivalence in Corollary 2.10 can be realized by the inclusion of the unit in the $(n+1)$st coordinate.

2.1.3. Algebras over little disks

Lemma 2.4 is the key result that underpins our analysis.

Theorem 2.12 *(Compare [3, Lemma 6.6]). Let X be a $\mathcal{D}(V)$-algebra in spaces, and let T be a finite H-set. Then the space $\mathrm{Emb}^H(T,V)$ encodes H-equivariant multiplications*

$$\mathrm{Map}(T,X) \to X.$$

Proof. Let Γ be the graph subgroup associated to an ordering of T. Lemma 2.4 shows that we have an equivalence

$$\mathrm{Map}^{G\times\Sigma_n}\left(G\times\Sigma_n/\Gamma_T, \mathcal{D}(V)_n\right) \simeq \mathrm{Emb}^H(T,V),$$

and that this equivalence is compatible with the (H-)Weyl action on the source and the $\mathrm{Aut}^H(T)$-action on the target. If $\mathrm{Emb}^H(T,V)$ is empty, then we have nothing to prove, so assume that this is non-empty. In this case, each point in $\mathrm{Emb}^H(T,V)$ gives a G-equivariant map

$$G \times_H \mathrm{Map}(T,X) \cong G \times_H \left((H\times\Sigma_n/\Gamma_T)\times_{\Sigma_n} X^{\times n}\right) \to \mathcal{D}(V)_n \times_{\Sigma_n} X^n \to X.$$

The maps in the theorem are the adjoint. □

Using Corollary 2.10, we get a kind of module structure on a $\mathcal{D}(V)$-algebra.

Corollary 2.13. *If X is a $\mathcal{D}(V)$-module and T is a finite G-set such that $T^G = \emptyset$, then $\mathrm{Emb}^G(T,V)$ also parameterizes maps*

$$\mathrm{Map}(T,X) \times X \to X.$$

There are coherence questions relating to how the pieces for various subgroups fit together which arise here, and the answer is encoded in Elmendorf's Theorem allowing us to rebuild the homotopy type of a G-space out of the diagram of fixed points [7]. This gives us a way to understand the homotopy type of $\mathcal{D}(V)_n \times_{\Sigma_n} X^{\times n}$ out of the data of Lemma 2.4.

Definition 2.14. Let $\mathcal{O}rb^G$ denote the orbit category of G. Let

$$J\colon \mathcal{O}rb^G \to \mathcal{T}op^G$$

denote the natural embedding which sends an orbit G/H to G/H viewed as a G-space. If X is a G-space, let

$$X^{(-)}\colon (\mathcal{O}rb^G)^{op} \to \mathcal{T}op$$

denote the functor which sends G/H to $\mathrm{Map}^G(G/H, X) \cong X^H$.

We will also take advantage of a simplification from the freeness of the Σ_n-action on $\mathcal{D}(V)_n$. This gives a natural family of subgroups and hence subcategory of the orbit category.

Definition 2.15. If V is an orthogonal representation of G, let $\mathcal{F}_V^n$ denote the family of subgroups $\Gamma \subset G \times \Sigma_n$ such that $\mathcal{D}(V)_n^\Gamma$ is non-empty. Let $\mathcal{F}_V^n$ also denote the corresponding sieve in the orbit category.

Theorem 2.16. *If X is a G-CW complex, then we have a natural weak equivalence*

$$\left|B_\bullet\big(\mathcal{D}(V)_n^{(-)}, \mathcal{F}_V^n, J|_{\mathcal{F}_V^n} \times_{\Sigma_n} X^n\big)\right| \simeq \mathcal{D}(V)_n \times_{\Sigma_n} X^n,$$

where the left-hand side is the geometric realization of the 2-sided bar construction.

Proof. Elmendorf's Theorem [7] shows that we have a natural $G \times \Sigma_n$-weak equivalence

$$\left|B_\bullet\big(\mathcal{D}(V)_n^{(-)}, \mathcal{O}rb^{G\times\Sigma_n}, J\big)\right| \simeq \mathcal{D}(V)_n.$$

Since $\mathcal{D}(V)_n^\Gamma$ is by definition non-empty only for those $\Gamma \in \mathcal{F}_V^n$, and since $\mathcal{F}_V^n$ is a sieve, we have a simplicial homeomorphism of simplicial spaces

$$B_\bullet\big(\mathcal{D}(V)_n^{(-)}, \mathcal{O}rb^{G\times\Sigma_n}, J\big) \cong B_\bullet\big(\mathcal{D}(V)_n^{(-)}, \mathcal{F}_V^n, J\big).$$

The result follows from passing the product with X past the geometric realization and commuting the Σ_n-orbits passed the geometric realization. □

Remark 2.17. The pieces in the two-sided bar construction are ones already encountered in Theorem 2.12: for some $\Gamma_T \in \mathcal{F}_V^n$, Lemma 2.4 identifies $\mathcal{D}(V)_n^{\Gamma_T}$ with $\mathrm{Emb}^G(T, V)$, and we have

$$\big(G \times \Sigma_n/\Gamma_T\big) \times_{\Sigma_n} X^n \simeq \mathrm{Map}(T, X).$$

Theorem 2.16 shows that the free $\mathcal{D}(V)$-algebra on X is homotopically built out of exactly this data.

Corollary 2.18. *Let $K \subset H \subset G$, and let T be a finite H-set and T' a finite K-set such that $i_K^* T \cong T'$. Then the inclusion*

$$\mathrm{Map}^H(T, V) \hookrightarrow \mathrm{Map}^K(T', V)$$

describes the restriction map on the operadic multiplications.

2.1.4. Basic consequences

We first make a simple observation which olds for all algebras over all little disks operads (including for the absurd case of $V = 0$).

Proposition 2.19. *For any V and for any $\mathcal{D}(V)$-algebra X, we have a canonical G-fixed basepoint $e \in X$.*

Proof. This is the zeroth structure map corresponding to $\mathcal{D}(V)_0 = *$. □

This connects with traditional constructions in the expected way: trivial subspaces give increasingly coherently commutative multiplications.

Proposition 2.20. *For any V and for any $\mathcal{D}(V)$-algebra X, if $H \subset G$ has $\dim V^H \geq 1$, then $i_H^* X$ has an $E_{\dim V^H}$-structure. The basepoint $e \in i_H^* X$ is the unit.*

Corollary 2.21. *If V is any orthongal representation which contains infinitely many copies of the trivial representation, then any $\mathcal{D}(V)$-algebra has an underlying E_∞ multiplication.*

The analysis of the embedding spaces automatically gives us additional structure maps.

Proposition 2.22. *If V has $\mathring{V}^H \neq \emptyset$, then we have norm maps*

$$\mathrm{Map}^H(G, i_H^* X) \cong \mathrm{Map}(G/H, X) \to X,$$

and if $\pi_0 \mathring{V}^H = *$, then this is unique up to homotopy.

Remark 2.23. If V is a G-universe (so $\mathcal{D}(V)$ is an N_∞ operad), then coïnduction is both the right and the left adjoint to the forgetful functor. The norm maps realize the counit of the adjunction (with coïnduction as the left adjoint).

Proposition 2.24. *If V has $\mathring{V}^H \neq \emptyset$, then we have an action map*

$$\mu_{G/H} \colon \mathrm{Map}(G/H, X) \times X \to X$$

making X into a module over the $E_{\dim V^H}$-space $\mathrm{Map}(G/H, X)$. If $\pi_0 \mathring{V}^H = *$, then this is homotopically unique.

Proof. Proposition 2.20 shows that $i_H^* X$ is an $E_{\dim V^H}$-space, and since coinduction is a strong symmetric monoidal functor from H-spaces to G-spaces, we deduce that

$$\mathrm{Map}^H(G, i_H^* X) \cong \mathrm{Map}(G/H, X)$$

is an $E_{\dim V^H}$-space. The action map is given by any point in the space $\mathcal{D}(V)_{|G/H|+1}^{\Gamma_{G/H}}$, which Corollary 2.10 shows is non-empty.

The only part of the proposition which requires proof is the assertion that this action map makes X a module. For this, note that $\mu_{G/H}$ determines a disk d_{eH} inside of $D(V)$, namely the one corresponding to the coset eH. This in turn gives a map

$$\mathcal{D}(V)_m \xrightarrow{\Delta^G} \operatorname{Map}\big(G/H, \mathcal{D}(V)_m\big) \xrightarrow{\mu_{G/H}} \mathcal{D}(V)_{m|G/H|}$$

which is $G \times \Sigma_m$-equivariant. Here Δ^G is the twisted diagonal adjoint to the identity map on $i_H^*\mathcal{D}(V)_m$, and Σ_m acts diagonally in the coinduced space and via the diagonal copy of $\Sigma_m^{|G/H|}$ in $\Sigma_{m|G/H|}$. Moreover, we here identify the element $\mu_{G/H}$ with the equivariant map

$$G \times \Sigma_{|G/H|}/\Gamma_{G/H} \to \mathcal{D}(V)_{|G/H|}.$$

This map views $\mathcal{D}(V)_m$ as being an arrangement of m little disks in d_{eH}, forming the $|G/H|$ collections of little m disks by conjugating by g, and then using the natural embedding of $G \times_H d_{eH}$ into V given by $\mu_{G/H}$. This map is visibly compatible with compositions in the source, which gives the module structure.

The homotopical uniqueness follows from the homotopical uniqueness of $\mu_{G/H}$ in the case the space $\mathring{V}^H$ is path connected. □

Remark 2.25. We have not used the full-strength of the operadic action here. The operad $\mathcal{D}(i_H^*V)$ acts on i_H^*X and hence on $\operatorname{Map}(G/H, X)$. This also has norm maps, by this procedure, and that endows X also with a compatible module structure over this via

$$\operatorname{Map}^H\big(G, \operatorname{Map}^K(H, i_K^*X)\big) \times \operatorname{Map}(G/H, X) \times X \to \operatorname{Map}(G/H, X) \times X \to X.$$

2.2. The case of $\mathcal{D}(\sigma)$

Now let X be an algebra in spaces over $\mathcal{D}(\sigma)$. Here, however, we run into the issue that σ is one dimensional. In particular, $i_e^*\mathcal{D}(\sigma)$ is an E_1-operad, and hence we have no reason to believe that it is homotopy commutative. One of the surprising features of a $\mathcal{D}(\sigma)$-algebra is that it comes equipped with a canonical isomorphism

$$i_e^*X \xrightarrow{\cong} i_e^*X^{op}.$$

This is one of the defining features of a $\mathcal{D}(\sigma)$-algebra, and it was independently observed by Hahn-Shi in their study of the Real orientation of the Lubin-Tate spectra [8].

2.2.1. Underlying E_1-space

Since σ is one dimensional, the space i_e^*X is an E_1-space. The following is classical; we include it to emphasize the equivariance.

Proposition 2.26. *Pulling the images of points in an embedding of $\{1, \ldots, 2k\}$ to the points $-k, \ldots, -1, 1, \ldots, k$ in σ gives a Σ_{2k}-homotopy equivalence*

$$\mathcal{D}(i_e^*\sigma)_{2k} \simeq \Sigma_{2k}.$$

Pulling the images of points in an embedding of $\{1, \ldots, 2k+1\}$ to the points $-k, \ldots, k$ in σ gives a Σ_{2k+1}-homotopy equivalence

$$\mathcal{D}(i_e^*\sigma)_{2k+1} \simeq \Sigma_{2k+1}.$$

We write these symmetrically because then the underlying action of C_2 on σ preserves the preferred images: the sets $\{\pm 1, \ldots, \pm k\}$ and $\{\pm 1, \ldots, \pm k, 0\}$ are C_2-equivariant subsets of σ. Of course, the underlying story does not a priori care about what collection of points we choose, and our analysis of the underlying structure does not care what collection of points in σ we choose. By having a C_2-invariant subset, however, we underscore the connection with the fixed points.

Under any choice of points, we see that the underlying C_2 sign action turns an ordered collection of points into one with the opposite order. Since C_2 is abelian, this is also a C_2-equivariant map from $\mathcal{D}(\sigma)$ to itself. This shows the following.

Proposition 2.27. *If X is a $\mathcal{D}(\sigma)$-algebra, then the non-trivial element of C_2 acts on X as an anti-automorphism of E_1-algebras.*

This is exactly what makes the fixed points of a $\mathcal{D}(\sigma)$-algebra not have a product: the action and multiplication do not commute.

Remark 2.28. If $X = \Omega^\sigma Y$, then the effect of the anti-automorphism from Proposition 2.27 in homotopy is readily understood: it is inversion.

2.2.2. Norms and actions

The analysis in the previous section shows that X comes equipped with a collection of structure maps

$$\mathrm{Map}(kC_2 + \epsilon C_2/C_2, X) \cong \mathrm{Map}(C_2, X)^k \times X^\epsilon \to X,$$

as k ranges over the natural numbers and ϵ is 0 or 1. Since $\dim i_e^*\sigma = 1$, this is homotopically quite simple.

Proposition 2.29. *For all k and ϵ, the space of structure maps is a homotopy discrete $\mathrm{Aut}^{C_2}(kC_2)$-torsor.*

$$\mathrm{Emb}(kC_2, \sigma) \simeq \mathrm{Aut}^{C_2}(kC_2).$$

Proof. The space $(\mathring{\sigma})^e$ is $\mathbb{R}^\times \cong C_2 \times \mathbb{R}$. Given any embedding, we can pull the images like beads on a string so that they become $\{\pm 1, \dots, \pm k\}$. This is obviously $\mathrm{Aut}^{C_2}(kC_2)$-equivariant. □

Thus up to homotopy, there is a unique such map for each automorphism of kC_2, and moreover, the automorphisms generated by 2 kinds:

(1) Ordinary permutations of the C_2-sets and
(2) the action of $C_2 = \mathrm{Aut}^{C_2}(C_2)$ on each summand.

Let γ be the non-trivial element of C_2. Here, we have 2 norm maps

$$n_e^{C_2}\colon \mathrm{Map}(C_2, X) \to X \text{ and } n_e^{C_2} \circ \gamma\colon \mathrm{Map}(C_2, X) \to X,$$

and 2 action maps

$$\mu_{C_2}\colon \mathrm{Map}(C_2, X) \times X \to X \text{ and } \mu_{C_2} \circ \gamma\colon \mathrm{Map}(C_2, X) \times X \to X.$$

The identification of the embedding space from Proposition 2.29 is compatible with restriction, giving us the chosen points in Proposition 2.26. Observing that the ordering of the negative integers is again opposite those of the positive shows the following.

Proposition 2.30. *As E_1-space, we have*

$$i_e^* \mathrm{Map}(C_2, X) \cong i_e^* X \times i_e^* X^{op},$$

the enveloping algebra of $i_e^* X$.

Of course, Proposition 2.27 shows that as E_1-spaces, this is isomorphic to the Cartesian square of $i_e^* X$. Using this hides the more natural ordering of the coordinates.

Corollary 2.31. *The restriction of the norm map is the multiplication map*

$$(i_e^* X \times i_e^* X^{op}) \to i_e^* X.$$

The restriction of the action map is the bimodule action map

$$(i_e^* X \times i_e^* X^{op}) \times i_e^* X \to i_e^* X.$$

The $\mathrm{Map}(C_2, X)$*-module map condition restricts to the observation that these are maps of* $i_e^* X$*-bimodules.*

3. Cohomology of $\mathcal{D}(\sigma)$-algebras

3.1. $RO(C_2)$-graded algebra

To describe the structure seen in the homology of a $\mathcal{D}(\sigma)$-algebra, we need to work in the category of $RO(C_2)$-graded Mackey functors. A wonderful introduction is provided by Lewis and Mandell in their study of equivariant Künneth and Universal Coefficients Spectral Sequences [14], especially Sections 2 and 3 therein. We include here only what we need.

Definition 3.1 *([14, Definition 2.2]).* An $RO(C_2)$-graded Mackey functor is a collection of Mackey functors $\underline{M}_\alpha$, where α ranges over the virtual representations of C_2.

A map of $RO(C_2)$-graded Mackey functors $f\colon \underline{M}_\star \to \underline{N}_\star$ is a collection of maps of Mackey functor $f_\alpha\colon \underline{M}_\alpha \to \underline{N}_\alpha$, where α ranges over the virtual representations of C_2.

There are natural suspension functors which change the gradings in the expected ways.

Definition 3.2. If $\underline{M}_\star$ is an $RO(C_2)$-graded Mackey functor and if α is a virtual representation of C_2, then let $\Sigma^\alpha \underline{M}_\star$ be the $RO(C_2)$-graded Mackey with

$$(\Sigma^\alpha \underline{M})_\tau = \underline{M}_{\tau-\alpha}.$$

Just as with ordinary abelian groups, the category of $RO(C_2)$-graded Mackey functors inherits a closed symmetric monoidal structure from Mackey functors.

Proposition 3.3 *([14, Proposition 2.5]). There is a close symmetric monoidal category structure on the category of* $RO(C_2)$*-graded Mackey functors extending the box product on Mackey functors.*

Definition 3.4. An $RO(C_2)$**-graded Green functor** is an associative monoid for the box product in $RO(C_2)$-graded Mackey functors.

Proposition 3.5 *([14, Proposition 3.10(a)]). If* $\underline{R}_\star$ *is a commutative* $RO(C_2)$*-graded Green functor, then there is closed, symmetric monoidal category of* $\underline{R}_\star$*-modules.*

Remark 3.6. There is a subtlety as to what graded commutativity means in the $RO(C_2)$-graded setting, since commuting the sign representation past itself introduces the unit

$$\epsilon = 1 - [C_2]$$

in the Burnside ring. In the applications below, this unit of the Burnside ring maps to -1, and hence this is ordinary commutativity.

We can further unpack the internal Hom to understand maps in this category a little better. We introduce some useful notation.

Definition 3.7. Let T be a finite G-set, and let $\underline{A}^T$ denote the Mackey functor:

$$\underline{A}^T(T') := \underline{A}(T \times T').$$

Viewing this as an $RO(C_2)$-graded Mackey functor in degree zero, if $\underline{R}_\star$ is a commutative $RO(C_2)$-graded Green functor, let

$$\underline{R}_\star^T := \underline{R}_\star \square \underline{A}^T.$$

Finally, if $\underline{M}_\star$ is an $\underline{R}_\star$-module, then let

$$\underline{M}_\star^T := \underline{M}_\star \square \underline{R}_\star^T.$$

Proposition 3.8 *([14, Proposition 4.2]). Let $\underline{M}_\star$ be an $\underline{R}_\star$-module. Then we have a natural isomorphism*

$$\underline{\mathrm{Hom}}^0_{\underline{R}_\star}(\Sigma^\alpha \underline{R}_\star^T, \underline{M}_\star)(C_2/C_2) \cong \underline{M}_\alpha(T).$$

In other words, maps out of the $\underline{R}$-modules $\Sigma^\alpha \underline{R}_\star^T$ recover the value at T of the αth Mackey functor of an $\underline{R}_\star$-module, and hence $\Sigma^\alpha \underline{R}_\star$ is a projective $\underline{R}_\star$-module.

The shifts $\underline{R}_\star^T$ have a second nice interaction with the symmetric monoidal structure.

Proposition 3.9. *For any finite G-sets T and T' and for any $\underline{R}_\star$-module $\underline{M}_\star$, we have natural isomorphisms*

$$\underline{R}_\star^T \underset{\underline{R}_\star}{\square} \underline{M}_\star^{T'} \cong \underline{M}_\star^{T \times T'}.$$

In our cases of interest below, we will be considering only special modules.

Definition 3.10. Let $\underline{R}_\star$ be an $RO(C_2)$-graded Green functor, and let $\underline{M}_\star$ be an $\underline{R}_\star$-module. Then we say that $\underline{M}_\star$ is free if there is an isomorphism of $\underline{R}_\star$-modules

$$\underline{M}_\star \cong \left(\bigoplus_{s \in S_\star} \Sigma^{\alpha_s} \underline{R}_\star \right) \oplus \left(\bigoplus_{t \in S_{C_2}} \Sigma^{\beta_t} \underline{R}_\star^{C_2} \right).$$

3.2. Coinduction and the norm

Any algebra X over $D(\sigma)$ in spaces is endowed with operadic transfer maps

$$\mathrm{Map}^e(C_2, X) \to X.$$

By naturality of homology, this gives us a map

$$\underline{H}_\star\big(\mathrm{Map}^e(C_2, X); \underline{M}\big) \to \underline{H}_\star(X; \underline{M})$$

for any Mackey functor $\underline{M}$, which is a kind of twisted analogue of the Pontryagin product on the homology of an associative algebra in spaces. We make this precise here.

Definition 3.11. A **graded set** is a set S together with a map $S \xrightarrow{deg} \mathbb{Z}$.

This allows us to describe what we see in algebra for the Bredon homology of coinduction. We remark that there is almost the expected universal property: the difficulty is that maps out of $\underline{R}$ itself in degree 0 is a priori non-zero for a great many $RO(C_2)$-graded suspensions of $\underline{R}$.

Definition 3.12. Let S be a graded set. We define a kind of $\underline{RO}(C_2)$-grading on the C_2-set $\mathrm{Map}^e(C_2, S)$ as follows. If $f \in \mathrm{Map}^e(C_2, S)$, then define

$$\underline{\deg}(f) := \begin{cases} \deg\big(f(e)\big) + \deg\big(f(g)\big) & f(e) \neq f(g) \\ \deg\big(f(e)\big)\rho_2 & f(e) = f(g). \end{cases}$$

In particular, for each point $f \in \mathrm{Map}^e(C_2, S)$, we have assigned a virtual representation for the subgroup $\mathrm{Stab}(f)$. Since C_2 is abelian, this is all we need to form wedges of the form

$$\bigvee_{f \in \mathrm{Map}^e(C_2,S)} S^{\underline{\deg}(f)} \in \mathcal{S}p^{C_2}.$$

We pause here to clarify a small notational point. The set over which we take our wedge in the proof of Theorem 3.13 is a C_2-set. In particular, the action on the indexing set is combined with the action on the individual factors. Thus if $f \in \mathrm{Map}^e(C_2, S)$ has $f(e) \neq f(g)$, then the summands corresponding to f and to $g \cdot f$ are switched by the group action. Thus we could rewrite the sum as

$$\bigvee_{f \in \mathrm{Map}^e(C_2,S)/C_2} C_{2+} \underset{Stab(f)}{\wedge} S^{\underline{\deg}(f)}.$$

Theorem 3.13. *Let R be a ring and let X be a space such that the homology of X with coefficients in R is free on a graded set S. Then the Bredon homology of $\mathrm{Map}^e(C_2, X)$ with coefficients in $N_e^{C_2} R$ is free on the $\underline{RO}(C_2)$-graded set $\mathrm{Map}^e(C_2, S)$.*

Proof. The graded set S gives a weak equivalence of HR-module spectra

$$\bigvee_{s \in S} S^{\deg(s)} \wedge HR \xrightarrow{\simeq} \Sigma^\infty_+ X \wedge HR.$$

If we apply the norm $N_e^{C_2}$ to both sides, then we deduce an equivalence of $N_e^{C_2} HR$-module spectra

$$N_e^{C_2}\left(\bigvee_{s \in S} S^{\deg(s)}\right) \wedge N_e^{C_2} HR \xrightarrow{\simeq} N_e^{C_2}(\Sigma^\infty_+ X) \wedge N_e^{C_2} HR.$$

The distributive law applied to the source gives an isomorphism of $N_e^{C_2} HR$-modules

$$N_e^{C_2}\left(\bigvee_{s \in S} S^{\deg(s)}\right) \wedge N_e^{C_2} HR \simeq \bigvee_{f \in \mathrm{Map}^e(C_2,S)} S^{\underline{\deg}(f)} \wedge N_e^{C_2} HR.$$

Similarly, since the infinite suspension is strong G-symmetric monoidal, we have a natural equivalence

$$N_e^{C_2} \Sigma^\infty_+ X \simeq \Sigma^\infty \mathrm{Map}^e(C_2, X).$$

Finally, since HR can be modeled by a commutative ring spectrum, $N_e^{C_2}HR$ is a C_2-equivariant commutative ring spectrum, so the category of modules thereover is symmetric monoidal. The map from a (-1)-connected C_2-equivariant commutative ring spectrum to its zeroth Postnikov section preserves C_2-equivariant commutative ring spectra, and

$$\underline{\pi}_0 N_e^{C_2} HR \cong N_e^{C_2} R,$$

where the righthand norm is the Mazur-Hoyer norm on Mackey functors [11], [12]. Base-changing along the zeroth Postnikov map

$$N_e^{C_2} HR \to HN_e^{C_2} R$$

gives the desired result. □

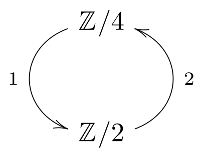
Remark 3.14. There is also a coordinate-free version of this result. We have a canonical isomorphism of simplicial G-sets

$$\mathrm{Map}^e\big(C_2, \mathrm{Sing}_\bullet(X)\big) \cong \mathrm{Sing}_\bullet\big(\mathrm{Map}^e(C_2, X)\big).$$

In particular, this produces a canonical isomorphism of simplicial Mackey functors upon applying the Burnside Mackey functor (or more generally, any other coefficients). The left-hand side is essentially the definition of the Mazur-Hoyer norm is simplicial Mackey functors. We will return to this more generally in a subsequent paper.

Since cohomology with coefficients in a field is always free, this gives us a nice family of space for which we know Bredon homology groups.

Definition 3.15. Let $\underline{B} = N_e^{C_2}\mathbb{F}_2$. This is the Green functor



Definition 3.16. Let

$$\underline{B}_\star := \underline{\pi}_\star H\underline{B}.$$

Proposition 3.17. *The $RO(C_2)$-graded Mackey functor $\underline{B}$ is naturally a commutative $RO(C_2)$-graded Green functor.*

Proof. The Eilenberg-Mac Lane spectrum $H\underline{B}$ is a commutative ring spectrum by work of Ullman. Since $\underline{\pi}_\star$ is a lax monoidal functor (see, for a complete proof [14, Appendix A]), the result follows. □

Since $\underline{B}$ is a quotient of the constant Mackey functor $\underline{\mathbb{Z}}$, the sign rule here is the ordinary one, using only the underlying dimension of the representations.

Corollary 3.18. *If X is any space, then the Bredon homology of* $\mathrm{Map}^e(C_2, X)$ *with coefficients in $\underline{B}$ is a free $\underline{B}_\star$-module.*

This can also be identified with a purely algebraic functor: the norm in $RO(C_2)$-graded Mackey functors. This is the obvious extension of the Mazur-Hoyer norm, taking into consideration the canonical isomorphism

$$N_e^{C_2}\Sigma^k M \cong \Sigma^{k\rho} N_e^{C_2} M.$$

This, plus the distributive law, immediately gives the following.

Corollary 3.19. *If X is any space, then the Bredon homology of $\mathrm{Map}^e(C_2, X)$ with coefficients in $\underline{B}$ is isomorphic to*

$$N_e^{C_2} H_*(i_e^* X; \mathbb{F}_2).$$

More generally, we get a freeness result for any module over $\underline{B}$. Since the Bredon homology with coefficients in $\underline{B}$ is a free $\underline{RO}(C_2)$-graded module, it is flat over $\underline{B}_\star$. In particular, the universal coefficients spectral sequence collapses.

Corollary 3.20. *If $\underline{M}$ is any $\underline{B}$-module, then*

$$\underline{H}_\star\big(\mathrm{Map}^e(C_2, X); \underline{M}\big) \cong N_e^{C_2}\big(H_*(X; \mathbb{F}_2)\big) \underset{\underline{B}}{\square} \underline{M},$$

which splits as an $\underline{RO}(C_2)$-graded sum of copies of $\underline{M}$.

In particular, this implies a fairly strong form of a Künneth isomorphism.

Proposition 3.21. *If X is a space and Y is a C_2-space, then we have a natural isomorphism*

$$\underline{H}_\star\big(\mathrm{Map}^e(C_2, X) \times Y; \underline{B}\big) \cong \underline{H}_\star\big(\mathrm{Map}^e(C_2, X); \underline{B}\big) \underset{\underline{B}_\star}{\square} \underline{H}_\star(Y; \underline{B}).$$

Proof. The Lewis-Mandell $RO(C_2)$-graded Künneth spectral sequence collapses, since $\underline{H}_\star\big(\mathrm{Map}^e(C_2, X); \underline{B}\big)$ is a free module. □

Finally, there is a pointed version of all of these results; the proofs are essentially unchanged.

Proposition 3.22. *If X is a pointed space, then*

$$\underline{\tilde{H}}_\star\big(N_e^{C_2} X; \underline{B}\big) \cong N_e^{C_2}\big(\tilde{H}_*(X; \mathbb{F}_2)\big).$$

3.3. Cohomology of $\mathcal{D}(\sigma)$-spaces

We can now combine the structural results from § 2.2 to determine the structure present in homology of a signed loop space. We begin with a small definition.

Definition 3.23. Let R be an associative ring. A **pointed** R-module is an R-module M together with an element $m \in M$.

By the free-forget adjunction, this is of course the same data as an R-module homomorphism $R \to M$.

Theorem 3.24. *Let X be an algebra over $\mathcal{D}(\sigma)$. Let*

$$R_* = H_*(i_e^* X; \mathbb{F}_2)$$

and let

$$\underline{R}_\star = N_e^{C_2} R_* \cong \underline{H}_\star\big(\mathrm{Map}(C_2, X); \underline{B}\big).$$

Then

(1) R_* is a graded, associative ring equipped with an isomorphism

$$\overline{(\text{-})}\colon R_* \to R_*^{op}$$

(2) $\underline{H}_\star(X; \underline{B})$ is naturally a pointed $\underline{R}_\star$-module, and the restriction of distinguished point is the multiplicative unit in R_*.

Remark 3.25. The pointing in Theorem 3.24 gives us a kind of norm map on the homology. This has been used most recently by Behrens-Wilson in their construction of $H\underline{\mathbb{F}}_2$ as a Thom spectrum [2].

3.4. Aside: even more structure

Just as the homology of any space with coefficients in a field is a co-commutative co-algebra, the homology of a C_2-space with coefficients in $\underline{B}$ has kinds of comultiplications. These arise from the two [twisted] diagonals:

$$X \xrightarrow{\Delta} X \times X \text{ and } X \xrightarrow{\Delta_g} \mathrm{Map}(C_2, X).$$

Applying homology with coefficients in $\underline{B}$ then gives structure maps.

Proposition 3.26. *If X is any C_2-space, then the homology of X with coefficients in $\underline{B}$ has a natural co-norm map*

$$\underline{H}_\star(X; \underline{B}) \to N_e^{C_2} H_*(i_e^* X; \mathbb{F}_2).$$

Proof. Corollary 3.19 identifies the homology of the target of the map $\underline{H}_\star(\Delta_g)$. $\square$

In the case we have flatness, then we can deduce also a comultiplication.

Proposition 3.27. *If X is a C_2-space such that $\underline{H}_\star(X; \underline{B})$ is a flat $\underline{B}_\star$-module, then we have a natural comultiplication*

$$\underline{H}_\star(X; \underline{B}) \to \underline{H}_\star(X; \underline{B}) \underset{\underline{B}_\star}{\square} \underline{H}_\star(X; \underline{B}).$$

In this case, $\underline{H}_\star(X; \underline{B})$ is naturally a co-Tambara functor.

Since these are induced by natural maps of spaces, we deduce that all of the structure maps described before are actually maps of co-Tambara functors.

4. The signed James construction

4.1. Construction and interpretation

Just as classically there is a combinatorial model for the loop space of the suspension of a space, there is an elegant combinatorial model due to Rybicki for the signed loops on the signed suspension of a C_2-space.

Definition 4.1 *([18, Section 2]).* If X is a pointed C_2 space, then let

$$J^{\sigma}_n(X) := \coprod_{k=0}^{n} X^{\times k}/\sim,$$

where C_2 acts on $X^{\times k}$ via

$$(x_1, \dots, x_k) \mapsto (\bar{x}_k, \dots, \bar{x}_1),$$

and where $\sim$ is the equivalence relation which simply omits any coordinate which is the basepoint.

Let $J^{\sigma}(X)$ be the colimit of $J^{\sigma}_n(X)$ as n varies.

The intertwining of the C_2-action on the space and on the Cartesian factors is what makes this definition viable. For us, we will need an inductive pushout description of the finite $J^{\sigma}_k(X)$.

Lemma 4.2. *Let X be a pointed C_2-space. For a finite C_2-set T, let $F_T(X)$ denote the "fat wedge" for $\mathrm{Map}(T,X)$, that is, the collection of points in $\mathrm{Map}(T,X)$ for which one of the coordinates is the basepoint.*

We have pushout squares of C_2-equivariant spaces:

$$\begin{array}{ccc} F_{nC_2}(X) & \longrightarrow & \mathrm{Map}(C_2,X)^n \\ \downarrow & & \downarrow \\ J^{\sigma}_{2n-1}(X) & \longrightarrow & J^{\sigma}_{2n}(X) \end{array} \quad \text{and} \quad \begin{array}{ccc} F_{nC_2\amalg *}(X) & \longrightarrow & \mathrm{Map}(C_2,X)^n \times X \\ \downarrow & & \downarrow \\ J^{\sigma}_{2n}(X) & \longrightarrow & J^{\sigma}_{2n+1}(X). \end{array}$$

4.2. Splitting the signed James construction

Classically, the James construction splits after a single suspension into a wedge of a suspension of smash powers of the original space. Here, the presence of two kinds of products requires two suspensions.

Theorem 4.3. *For any pointed C_2-space of the homotopy type of a C_2-CW complex, we have a natural weak equivalence*

$$\Sigma^{\rho}_{+} J^{\sigma}(X) \simeq \bigvee_{k\geq 0} \Sigma^{\rho}(N_e^{C_2} i_e^{*} X^{\wedge k}) \wedge (S^0 \vee X).$$

Proof. This is immediate from Lemma 4.2, once we remember that we have equivariant homeomorphisms

$$\mathrm{Map}(C_2, X^{\times k})/F_{kC_2}(X) \cong (N_e^{C_2} i_e^{*} X)^{\wedge k},$$

and

$$\big(\mathrm{Map}(C_2, X^{\times k}) \times X\big)/F_{kC_2\amalg *}(X) \cong \big(N_e^{C_2} i_e^{*} X^{\wedge k}\big) \wedge X.$$

Theorem A.3 below shows that upon a single sign or regular suspension, the top row in both pushout squares from Lemma 4.2 splits. In particular we deduce two splittings

$$\Sigma^{\rho} J^{\sigma}_{2n}(X) \simeq \Sigma^{\rho} J^{\sigma}_{2n-1}(X) \wedge N_e^{C_2}(X)^{\wedge n}$$

and

$$\Sigma^\rho J^\sigma_{2n+1}(X) \simeq \Sigma^\rho J^\sigma_{2n}(X) \wedge \left(N_e^{C_2}(X)^{\wedge n}\right) \wedge X.$$

Splicing these together gives the desired result. □

Corollary 4.4. *For any k and for any C_2-space X, we have James-Hopf maps of the form*

$$h_{kC_2}\colon \Omega^\sigma\Sigma^\sigma X \to \Omega^\rho\Sigma^\rho N_e^{C_2} i_e^* X^{\wedge k}$$

and

$$h_{kC_2+1}\colon \Omega^\sigma\Sigma^\sigma X \to \Omega^\rho\Sigma^\rho (X \wedge N_e^{C_2} i_e^* X^{\wedge k}).$$

Proof. These are adjoint to the projections onto the summands of the splitting of

$$\Sigma^\rho J^\sigma(X) \simeq \Sigma^\rho\Omega^\sigma\Sigma^\sigma X. \quad \square$$

Specializing to the case of X a sphere, we have a sequence of maps which are equivariant refinements of the James-Hopf map.

Corollary 4.5. *For any natural numbers j, k, we have a James-Hopf map*

$$h_{C_2}\colon \Omega^\sigma S^{j+(k+1)\sigma} \to \Omega^\rho S^{(j+k+1)\rho}.$$

There are two somewhat surprising features (and computationally vexing) aspects of this:

(1) We see an extra loops appearing (here in the form of $\Omega^\rho = \Omega\Omega^\sigma$), making an analysis of the fiber trickier than in the classical case, and
(2) the target of the map depends only on the underlying, non-equivariant sphere, and not on the particular equivariant sphere used.

4.3. The homology of the James construction

As an immediate consequence of the stable splitting of the James construction, we can compute the Bredon homology of the signed James construction with coefficients in $\underline{B}$.

Theorem 4.6. *For any C_2-space X, we have an isomorphism of $RO(C_2)$-graded Mackey functors*

$$\underline{H}_\star(J^\sigma X; \underline{B}) \cong \left(\bigoplus_{i=0}^{\infty} N_e^{C_2}\big(\tilde{H}_*(i_e^* X; \mathbb{F}_2)^{\otimes i}\big)\right) \square \big(\underline{B}_\star \oplus \underline{\tilde{H}}_\star(X; \underline{B})\big).$$

Proof. The stable splitting of the signed James construction yields

$$\Sigma^{\infty\rho}_+ J^\sigma X \simeq \left(\bigvee_{i=0}^{\infty} N_e^{C_2}(i_e^* X)^{\wedge i}\right) \wedge (S^0 \vee X).$$

Proposition 3.22 together with the classical Künneth theorem shows that

$$\underline{H}_\star\left(\bigvee_{i=0}^{\infty} N_e^{C_2}(i_e^* X)^{\wedge i}; \underline{B}\right) \cong \left(\bigoplus_{i=0}^{\infty} N_e^{C_2}\big(\tilde{H}_*(i_e^* X; \mathbb{F}_2)^{\otimes i}\big)\right).$$

The pointed version of Proposition 3.21 then gives the rest. □

Remark 4.7. There is a version of the Bott-Samelson theorem for the signed loops, which allows us to describe the homology as a particular algebraic functor [4]. For this case, it is possible to work out directly, though the analysis is a bit unenlightening. In short, it is the free $\mathcal{D}(\sigma)$-algebra in Mackey functors on $\underline{\tilde{H}}_\star(X;\underline{B})$. The difficulty here is describing the twisted powers, just as with the norm in spaces above. In joint work with Tyler Lawson, we will return to this point.

Appendix A. Splitting C_2-coinduction

Classically, the Cartesian product splits into the wedge and smash products after a single suspension. Equivariantly, we have instead our G-Cartesian monoidal structure on G-spaces, and this necessitates a more general splitting result. We begin with a non-example.

Proposition A.1. *All suspensions by trivial representations of the C_2-equivariant map $S^\sigma \to C_{2+} \wedge S^1$ with cofiber $\mathrm{Map}^e(C_2, S^1)$ are essential.*

Proof. The standard picture showing the 2-torus as a quotient of the square can be visibly made equivariant. In this case, all that follows is simply reinterpreting that picture.

First observe that the standard C_2-equivariant cell-structure of S^σ (and hence of $S^{k+\sigma}$) gives us an exact sequence (of pointed sets, if $k = 0$):

$$\dots \longrightarrow [S^{1+k}, C_{2+} \wedge S^{1+k}]^{C_2} \longrightarrow [C_{2+} \wedge S^{1+k}, C_{2+} \wedge S^{1+k}]^{C_2} \longrightarrow [S^{\sigma+k}, C_{2+} \wedge S^{1+k}]^{C_2} \longrightarrow [S^k, C_{2+} \wedge S^{1+k}]^{C_2} \longrightarrow \dots$$

Since the C_2-action on S^k and S^{1+k} is trivial, and since the C_2-fixed points of any space of the form $C_{2+} \wedge X$ are a point, we conclude that the two outer terms both vanish. Thus for all k, the map

$$[S^{1+k}, i_e^* C_{2+} \wedge S^{1+k}] \cong [C_{2+} \wedge S^{1+k}, C_{2+} \wedge S^{1+k}]^{C_2} \xrightarrow{q^*} [S^{\sigma+k}, C_{2+} \wedge S^{1+k}]^{C_2}$$

induced by the collapse map $S^\sigma \to C_{2+} \wedge S^1$ is an isomorphism. In particular, the suspension map is an isomorphism as well for all $k \geq 1$, by the classical result, and the suspension for $k = 0$ is abelianization.

We pause here to point out a simple geometric fact: the inverse to q^* is the function that assigns to a C_2-equivariant map

$$f\colon S^\sigma \to C_{2+} \wedge S^1$$

the non-equivariant map $f|_{Im(z)\geq 0}$, where we are viewing S^σ as the unit circle in $\mathbb{C}$. Unpacking the standard description of the attaching map for $S^1 \times S^1$ shows that the map $S^\sigma \to C_{2+} \wedge S^1$ here corresponds to the standard crush map $S^1 \to S^1 \vee S^1$, which survives abelianization. □

Remark A.2. The curious fact we used here is that while the underlying non-equivariant map is the commutator of the two canonical inclusions $S^1 \hookrightarrow S^1 \vee S^1$, as a C_2-equivariant map, the upper and lower semi-circles are oriented the same way. In particular, we never trace out the inverses, equivariantly.

In fact, for C_p in general, coinduction splits after any suspension by a representation sphere.

Theorem A.3. *For a based C_p-space X, we have equivalences*

$$\Sigma^\lambda \mathrm{Map}(C_p, X) \simeq \Sigma^\lambda N_e^{C_p}(X) \vee \Sigma^\lambda \left(C_{p+} \wedge \bigvee_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} X^{\wedge i} \right),$$

where λ is an irreducible 2-dimensional representation of C_p.

If $p = 2$, we have the same splitting but with the σ-fold suspension:

$$\Sigma^\sigma \operatorname{Map}(C_2, X) \simeq \Sigma^\sigma N_e^{C_2} X \vee \Sigma^\sigma (C_{2+} \wedge X).$$

Proof. A choice of equivariant embedding

$$* \amalg C_p \hookrightarrow \lambda$$

gives a Thom collapse

$$S^\lambda \to S^\lambda \vee C_{p+} \wedge S^\lambda.$$

Smashing this with $\operatorname{Map}(C_p, X)$ gives us then gives us a map

$$\Sigma^\lambda \operatorname{Map}(C_p, X) \to \Sigma^\lambda \operatorname{Map}(C_p, X) \vee C_{p+} \wedge \Sigma^\lambda \operatorname{Map}(C_p, X).$$

On the first summand, we can further map out via the crush the fat-wedge map:

$$\Sigma^\lambda \operatorname{Map}(C_p, X) \to \Sigma^\lambda N_e^{C_p} X.$$

For the second summand, we have an isomorphism:

$$C_{p+} \wedge \Sigma^\lambda \operatorname{Map}(C_p, X) \cong C_{p+} \underset{\{e\}}{\wedge} (\Sigma^2 X^{\times p}),$$

and we now can use the classical splitting

$$C_{p+} \underset{\{e\}}{\wedge} (\Sigma^2 X^{\times p}) \simeq C_{p+} \underset{\{e\}}{\wedge} \left(S^2 \wedge \bigvee_{i=1}^{p} \binom{p}{i} X^{\wedge p} \right).$$

The summand for $i = p$ is actually already handled by the norm summand, so we project away from that. The summands $\binom{p}{i} X^{\wedge i}$ break into orbits for the C_p action that permuted the coordinates, so choosing one for each orbit gives us an equivariant map

$$C_{p+} \wedge \Sigma^\lambda \operatorname{Map}(C_p, X) \to C_{p+} \wedge \left(S^2 \wedge \bigvee_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} X^{\wedge i} \right).$$

All told, we get a (natural in X) map

$$\Sigma^\lambda \operatorname{Map}(C_p, X) \to \Sigma^\lambda N_e^{C_p} X \vee \Sigma^\lambda \left(C_{p+} \underset{\{e\}}{\wedge} \bigvee_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} X^{\wedge i} \right).$$

Since we have chosen one summand from each orbit, the map is an underlying equivalence by the classical splitting. There are no fixed points for the second summand, and we note that the Euler class and diagonal give a map

$$q^* i_e^* X \xrightarrow{a_\lambda} \Sigma^\lambda q^* i_e^* X \xrightarrow{\Sigma^\lambda \Delta} \Sigma^\lambda \operatorname{Map}(C_p, X),$$

where q^* takes a space to a C_p-space with the trival action, and this map is an equivalence on fixed points. The projection

$$\Sigma^\lambda \operatorname{Map}(C_p, X) \to \Sigma^\lambda N_e^{C_p} X$$

is therefore also an equivalence.

For $p = 2$, we can use σ instead of λ; the proof goes through without change. □

Remark A.4. For C_2, there is also an argument using a twisted version of the join, and this appeared in the first version of this paper. The current version is heavily informed by the co-H-space approach to the splitting as explained in [16].

Remark A.5. The same kind of argument will give more general splittings for coinduction; the question is the isotropy types of the action of G on the expected stable splitting.

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