Nonlinearity

PAPER

Computability at zero temperature*

To cite this article: Michael Burr and Christian Wolf 2020 *Nonlinearity* **33** 6157

IOP Publishing | London Mathematical Society Nonlinearity

Nonlinearity **33** (2020) 6157–6175 https://doi.org/10.1088/1361-6544/ab9c71

Computability at zero temperature*

Michael Burr[1] **and Christian Wolf**[2]

[1] School of Mathematical and Statistical Sciences, Clemson University, Clemson SC, 29634, United States of America
[2] Department of Mathematics, The City College of New York, New York, NY, 10031, United States of America

E-mail: burr2@clemson.edu and cwolf@ccny.cuny.edu

Received 23 October 2019, revised 4 June 2020
Accepted for publication 12 June 2020
Published 6 October 2020

Abstract

We investigate the computability of thermodynamic invariants at zero temperature for one-dimensional subshifts of finite type. In particular, we prove that the residual entropy (i.e., the joint ground state entropy) is an upper semi-computable function on the space of continuous potentials, but it is not computable. Next, we consider locally constant potentials for which the zero-temperature measure is known to exist. We characterize the computability of the zero-temperature measure and its entropy for potentials that are constant on cylinders of a given length k. In particular, we show the existence of an open and dense set of locally constant potentials for which the zero-temperature measure can be computationally identified as an elementary periodic point measure. Finally, we show that our methods do not generalize to treat the case when k is not given.

Keywords: zero-temperature measures, residual entropy, ground states, entropy, thermodynamic formalism, computability

Mathematics Subject Classification numbers: Primary 37D35, 37E45, 03D15, Secondary 37B10, 37L40, 03D80.

1. Introduction

1.1. Motivation

It is a natural and important question to understand which mathematical invariants can (in principle) be derived by computer experiments. In particular, since computer-based approximations are often used to gain insight into theoretical questions, estimates on the quality and accuracy of computational results may be needed to have confidence in any conjectures drawn from such experiments. The answers to these questions (and the corresponding estimates) are naturally linked to questions about mathematical proofs and models. In fact, these answers lie at the boundary of mathematics, computer science, and their applications.

We provide some answers concerning the computability of basic thermodynamic invariants at zero temperature. In particular, we study the computability of the residual entropy (which coincides with the entropy of the ground states of the system) on the space of continuous potentials for subshifts of finite type (SFTs). We show that the residual entropy is an upper semi-computable function of the potentials, but it is not computable. One complication that arises is that continuous potentials may have phase transitions, which do not occur in the Hölder continuous case. Since, in general, phase transitions cannot be detected algorithmically, see, e.g., [34, 39], we are required to develop a new approach which is based on techniques from convex analysis and the thermodynamic formalism.

We also consider the computability of the zero-temperature measure for locally constant potentials. The existence of this measure was originally established by Brémont [5] by using methods from analytic geometry (for existence proofs using methods from dynamical systems, see [7, 23]). For potentials that are constant on cylinders of a given length k, we provide explicit characterizations of the sets of potentials for which the zero-temperature measure or its entropy are computable. We explicitly describe an open and dense subset $\mathcal{O}_k$ of locally constant potentials for which the zero-temperature measure is a computable periodic point measure. As a counterpart to these results, we show that once we consider the space of all locally constant potentials (i.e., without fixing the cylinder length k), the set $\mathcal{O} = \bigcup_k \mathcal{O}_k$ has empty interior. In particular, this shows that our results do not directly generalize to the case where k is not given.

There are several recent papers that study invariant sets, topological entropy, and other invariants from the computable analysis point of view. These papers include results about the computability of certain specific measures (e.g., maximal entropy and physical measures), see [1, 13, 14] and the references therein. Furthermore, there are papers proving results on the numerical computation of invariant sets, entropy, and dimension, see, e.g., [9, 19, 20] and the references therein. There are also studies concerning the computation of the topological entropy or pressure for one and multi-dimensional shift maps, see, e.g., [15, 16, 26, 27, 33, 34]. In our recent paper with Schmoll [6], we derive results about the computability of generalized rotation sets and localized entropies. In particular, our results hold for SFTs. We note that the results in [6] only consider the case of positive temperature, while the more delicate case of zero temperature is considered in this paper. To the best of our knowledge, this paper is the first attempt to study the computability of thermodynamic invariants at zero-temperature.

1.2. Statement of results

Let $f: X \to X$ be a subshift of finite type (SFT) over an alphabet with d elements and let $\mathcal{M}$ be the set of f-invariant Borel probability measures on X endowed with the weak * topology. With this topology, $\mathcal{M}$ is a compact, convex, and metrizable topological space. We use the standing assumption that f is transitive and has positive topological entropy. We consider the Banach space $(C(X,\mathbb{R}), \|\cdot\|_\infty)$, where $\|\cdot\|_\infty$ denotes the supremum norm. For $\phi \in C(X,\mathbb{R})$ and $\mu \in \mathcal{M}$, we write $\mu(\phi) = \int \phi \mathrm{d}\mu$ and define

$$I(\phi) = \{\mu(\phi) : \mu \in \mathcal{M}\}.$$

It follows, from the compactness and convexity of $\mathcal{M}$, that $I(\phi)$ is a compact interval $[a_\phi, b_\phi]$. We define $\mathcal{M}_{\max}(\phi) = \{\mu \in \mathcal{M} : \mu(\phi) = b_\phi\}$. If $\mu \in \mathcal{M}_{\max}(\phi)$, then we say μ is a *maximizing* *measure* for ϕ. Moreover, we say $\phi \in C(X, \mathbb{R})$ is *uniquely maximizing* if $\mathcal{M}_{\max}(\phi)$ is a singleton. We note that the study of maximizing measures is one of the central objectives in the area of ergodic optimization. We refer the reader to the survey article [18] for a state-of-the-art presentation of this subject. We call

$$h_{\infty,\phi} = \sup\{h_\mu(f) : \mu \in \mathcal{M}_{\max}(\phi)\}$$

the *residual entropy* of the potential ϕ. The residual entropy coincides with the entropy of the ground states of the potential ϕ (see section 2.3 for details). In particular, if the zero-temperature measure $\mu_{\infty,\phi}$ of ϕ exists (see below and section 2.2 for the definition of zero-temperature measures), then $h_{\infty,\phi}$ coincides with the entropy of $\mu_{\infty,\phi}$.

There are several recent theoretical results about the residual entropy and uniquely maximizing periodic point measures for an open and dense set of potentials in the Hölder and Lipschitz topologies [10, 11, 24, 28]. We observe, however, that these topologies are not compatible with the supremum topology since open balls in the supremum topology are not bounded in the Hölder and Lipschitz topologies. Therefore, it does not appear possible to study these genericity results from the computable analysis point of view. Consequently, the work in this paper uses the supremum norm.

Our first goal is to characterize the computability of the function $\phi \mapsto h_{\infty,\phi}$. To do this, we use two notions of computability for functions: computable functions and upper semi-computable functions (also called right recursively enumerable or right computable functions). We say that a function $g : C(X, \mathbb{R}) \to \mathbb{R}$ is computable if, for any input function ϕ, the real number $g(\phi)$ can be calculated to any prescribed accuracy. Upper semi-computability is a weaker notion of computability, where, instead, there is an algorithm to compute a sequence q_n converging to $g(\phi)$ from above. In particular, for upper semi-computability, the bounds on the convergence rate for $q_n \to g(\phi)$ are not included. We refer the reader to section 2.5 and [4, 13] for details.

The first main theorem we prove in this paper shows that the residual entropy is semi-computable, but not computable.

Theorem A. *The function $\phi \mapsto h_{\infty,\phi}$ is upper semi-computable, but not computable on $C(X, \mathbb{R})$. Moreover, the map $\phi \mapsto h_{\infty,\phi}$ is continuous at ϕ_0 if and only if $h_{\infty,\phi_0} = 0$.*

In section 2.5, we introduce the definition for the function $\phi \mapsto h_{\infty,\phi}$ to be computable at a point ϕ_0. This definition provides a computable version of being continuous at a point. Moreover, if the map $\phi \mapsto h_{\infty,\phi}$ restricted to a set S is computable and ϕ_0 is in the interior of S, then $\phi \mapsto h_{\infty,\phi}$ is computable at ϕ_0. With this definition in hand, a direct corollary of theorem A is:

Corollary 1.1. *The function $\phi \mapsto h_{\infty,\phi}$ is computable at ϕ_0 if and only if $h_{\infty,\phi_0} = 0$.*

The second goal of this paper is to study the computability of the zero-temperature measure and its entropy for locally constant potentials. We recall that $\mu \in \mathcal{M}$ is an equilibrium state of $\phi \in C(X, \mathbb{R})$ if μ maximizes $h_\nu(f) + \nu(\phi)$ among all $\nu \in \mathcal{M}$. If ϕ is Hölder continuous (and, in particular, if ϕ is locally constant), then the equilibrium state is unique and we denote it by μ_ϕ. We say $\mu_{\infty,\phi}$ is the zero-temperature measure of ϕ if $\mu_{\infty,\phi} = \lim_{\beta\to\infty} \mu_{\beta\phi}$, where the limit is taken in the weak * topology[3]. We recall that, for locally constant potentials,

[3] We point out that, in the mathematical theory of the thermodynamic formalism, it is customary to consider the inverse temperature $\beta = 1/T$ (with T being the temperature of the system) and to take the limit $\beta \to \infty$. The notation that is used for the inverse temperature in physics is $\beta = \frac{1}{k_B T}$, where k_B is Boltzmann's constant, which can be taken to be equal to one in an appropriate system of units.

the zero-temperature measure exists [5]. Let $\mathrm{LC}(X,\mathbb{R}) = \bigcup_{k\in\mathbb{N}} \mathrm{LC}_k(X,\mathbb{R})$ denote the space of locally constant potentials, where $\mathrm{LC}_k(X,\mathbb{R})$ denotes the space of potentials that are constant on cylinders of length k. Let $m_{\mathrm{c}}(k)$ denote the cardinality of the set of cylinders of X of length k (note that $m_{\mathrm{c}}(k) \leqslant d^k$). Then we can identify $\mathrm{LC}_k(X,\mathbb{R})$ with $\mathbb{R}^{m_{\mathrm{c}}(k)}$, so that $\mathrm{LC}_k(X,\mathbb{R})$ is a Banach space when endowed with the standard norm.

We note that, for the purpose of studying zero-temperature measures and their associated entropies, it suffices to consider the space $\mathrm{LC}_k(X,\mathbb{R}) \cap \overline{B}(0,1)$, where $\overline{B}(0,1)$ is the closed unit ball in $\mathbb{R}^{m_{\mathrm{c}}(k)}$. This reduction follows since $\mu_{\infty,\phi} = \mu_{\infty,\alpha\phi}$ for all $\alpha > 0$. To illustrate some of the difficulties when dealing with the computability of $h_{\infty,\phi}$ and $\mu_{\infty,\phi}$, we consider the following basic example, see [5, 38]:

Example 1.2. Let X be the full shift on two symbols, i.e., $X = \{0,1\}^{\mathbb{N}}$, and let $f\colon X \to X$ be the shift map. Let $0 < \alpha_1, \alpha_2$ be computable real numbers. Let $\phi \in \mathrm{LC}_2(X,\mathbb{R})$ be given by the matrix $\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & 0 \end{pmatrix}$, where $\phi_{i,j}$ denotes the value of ϕ on the cylinder $\mathcal{C}_2(ij) \stackrel{\mathrm{def}}{=} \{x : x_1 = i,\ x_2 = j\}$. We argue that neither $h_{\infty,\phi}$ nor $\mu_{\infty,\phi}$ are computable since they are not continuous functions: if $\alpha_1 \neq \alpha_2$, then $\mu_{\infty,\phi}$ is a periodic point measure, and, in particular, $h_{\infty,\phi} = 0$. On the other hand, if $\alpha_1 = \alpha_2$, then $\mu_{\infty,\phi}$ is the unique measure of maximal entropy (i.e., the Parry measure) of the golden mean shift, i.e., the SFT with transition matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Furthermore, $h_{\mu_{\infty,\phi}}(f) = \log \frac{1+\sqrt{5}}{2}$.

To overcome these difficulties, we partition the space of potentials $\mathrm{LC}_k(X,\mathbb{R}) \cap \overline{B}(0,1)$ into three sets with distinct computability properties, namely,

$$\mathrm{LC}_k(X,\mathbb{R}) \cap \overline{B}(0,1) = \mathcal{O}_k \,\dot{\cup}\, \mathcal{U}_k \,\dot{\cup}\, \mathcal{V}_k. \tag{1}$$

We explicitly define the three sets and identify their properties:

(a) $\mathcal{O}_k$ is the set of uniquely maximizing potentials $\phi \in \mathrm{LC}_k(X,\mathbb{R}) \cap \overline{B}(0,1)$. Moreover, the unique maximizing measure of ϕ is a k-elementary periodic point measure. Additionally, $\mathcal{O}_k$ is open and dense in $\mathrm{LC}_k(X,\mathbb{R}) \cap \overline{B}(0,1)$.

(b) $\mathcal{O}_k \,\dot{\cup}\, \mathcal{U}_k$ is the set of potentials $\phi \in \mathrm{LC}_k(X,\mathbb{R}) \cap \overline{B}(0,1)$ with $h_{\infty,\phi} = 0$. Therefore, $\mathcal{U}_k$ is the set of potentials with more than one ergodic maximizing measure, all of which are k-elementary periodic point measures[4]. Furthermore, for $\phi \in \mathcal{U}_k$, the measure $\mu_{\infty,\phi}$ is a convex combination of these k-elementary periodic point measures. It follows that $\mathcal{O}_k \,\dot{\cup}\, \mathcal{U}_k$ is an open set in $\mathrm{LC}_k(X,\mathbb{R}) \cap \overline{B}(0,1)$.

(c) $\mathcal{V}_k$ is the set of potentials $\phi \in \mathrm{LC}_k(X,\mathbb{R}) \cap \overline{B}(0,1)$ with $h_{\infty,\phi} > 0$. It follows that $\mathcal{V}_k$ is a closed set in $\mathrm{LC}_k(X,\mathbb{R}) \cap \overline{B}(0,1)$.

The properties of the sets described in this partition follow from results in [38], where a similar topological partition is considered. We note that the statement that $\mathcal{O}_k \,\dot{\cup}\, \mathcal{U}_k$ is open is not explicitly proven in [38], but its proof is analogous to the proof that $\mathcal{O}_k$ is open.

To be able to make statements about the computability of the sets $\mathcal{O}_k$ and $\mathcal{O}_k \,\dot{\cup}\, \mathcal{U}_k$, we briefly recall the notion of recursively open sets. Namely, we say an open set S is recursively open if there exists a Turing machine which for each $n \in \mathbb{N}$ produces a ball B_n such that $S = \bigcup_n B_n$ (see section 2.5 for details). We prove the following result:

[4] We note that this condition implies $h_{\infty,\phi} = 0$ for all $\phi \in \mathcal{U}_k$, see [38].

Theorem B. *Let $k \in \mathbb{N}$ be given. The following hold:*

(*a*) *The maps $\phi \mapsto \mu_{\infty,\phi}$ and $\phi \mapsto h_{\infty,\phi}$ are computable functions on $\mathcal{O}_k \subset \mathrm{LC}_k(X,\mathbb{R})$. Furthermore, the set $\mathcal{O}_k$ is a recursively open set;*

(*b*) *The map $\phi \mapsto h_{\infty,\phi}$ is a computable function on $\mathcal{O}_k \dot{\cup} \mathcal{U}_k \subset \mathrm{LC}_k(X,\mathbb{R})$. For any $\phi_0 \in \mathcal{U}_k$, the map $\phi \mapsto \mu_{\infty,\phi}$ is not continuous (and hence not computable) at ϕ_0 in $\mathcal{O}_k \dot{\cup} \mathcal{U}_k$. Furthermore, the set $\mathcal{O}_k \dot{\cup} \mathcal{U}_k$ is a recursively open set; and*

(*c*) *For any $\phi_0 \in \mathcal{V}_k$, neither the map $\phi \mapsto h_{\infty,\phi}$ nor the map $\phi \mapsto \mu_{\infty,\phi}$ are continuous (and hence not computable) at ϕ_0 in $\mathrm{LC}_k(X,\mathbb{R})$.*

We point out that in the statement of theorem B, the number k (i.e., the cylinder length on which the potentials are constant) is given, and, in particular, is not determined by the Turing machine that queries an oracle of the potential. One might suspect that either k can be calculated or that some of the results in theorem B generalize to $\mathrm{LC}(X,\mathbb{R})$ without specifying k. In this direction, we recall that $\mathcal{O} = \bigcup_k \mathcal{O}_k$ denotes the set of locally constant potentials that are uniquely maximizing. One might hope that $\mathcal{O}$ is a recursively open set, i.e., that membership in $\mathcal{O}$ is semi-decidable. A first indication that this could not be true is given example 5.1 where it is shown that $\mathcal{O}$ is not open in $\mathrm{LC}(X,\mathbb{R})$. In fact, we have the following even stronger result from proposition 5.2:

Theorem C. *Let $f: X \to X$ be a transitive SFT with positive topological entropy. Then the set $\mathcal{O}$ has no interior points in $\mathrm{LC}(X,\mathbb{R})$.*

As noted above, theorem C indicates that, from the point of view of computable analysis, there are significant differences between the cases of a given and of an arbitrary cylinder length. On the other hand, theorem C is also of theoretical interest in ergodic optimization. This is, in part, as it displays a sharp contrast between the locally constant case (in the supremum topology) and the Lipschitz case (in the Lipschitz topology). In particular, for the latter case, the set of potentials with a uniquely maximizing periodic point measure is open and dense in the space of all Lipschitz potentials, see Contreras' theorem [10].

1.3. Outline of the paper and summary of proofs

In section 2, we review some concepts from symbolic dynamics, the thermodynamic formalism, and computational analysis. In section 3, we present the proof of theorem A. We construct a sequence $(\beta_n)_n$ with associated equilibrium measures μ_n of $\beta_n\phi$ such that $(h_{\mu_n}(f))_n$ is a computable sequence that converges from above to $h_{\infty,\phi}$. This construction is fairly straight-forward in the case when the pressure function $\beta \mapsto P_{\mathrm{top}}(\beta\phi)$ is differentiable. The situation is more complicated, however, when ϕ is merely continuous due to the possibility of phase transitions (which cannot be detected algorithmically). To overcome this difficulty, we apply techniques from convex analysis to compute a sequence $h_{\mu_n}(f)$ that approximates $h_{\infty,\phi}$. The claim that $\phi \mapsto h_{\infty,\phi}$ is continuous at ϕ_0 if and only if $h_{\infty,\phi_0} = 0$ follows from basic properties of the topological pressure and the fact that the set $\mathcal{O}$ of uniquely maximizing locally constant potentials is dense in $C(X,\mathbb{R})$.

In section 4, we prove theorem B. To show that the sets of potentials $\mathcal{O}_k$ and $\mathcal{O}_k \dot{\cup} \mathcal{U}_k$ can be algorithmically detected, we approximate $\mu_x(\phi)$ to high accuracy for all k-elementary periodic points x. In particular, we compute $\mu_x(\phi)$ to identify those x's that may maximize $\mu_x(\phi)$, see propositions 4.1 and 4.2. Once the corresponding inequalities are established, $h_{\infty,\phi}$ and $\mu_{\infty,\phi}$ can be identified from the general theory presented in section 2. The noncomputability results of theorems B and C appear in sections 4 and 5. These results are based on similar approaches where we construct small, explicit perturbations of ϕ and show that such perturbations remove the potential from the appropriate set, see propositions 4.3 and 5.2.

2. Settings and generalities

We introduce the relevant background material and obtain preliminary results. In particular, we provide overviews of the pertinent results and definitions from shift spaces, zero-temperature measures, ground states, locally constant potentials, and computability theory.

2.1. Shift maps

Let $\mathcal{A} = \{0, \ldots, d-1\}$ be a finite set called the alphabet. We endow $\mathcal{A}^{\mathbb{N}} = \{x = (x_n)_{n=1}^{\infty} : x_n \in \mathcal{A}\}$ with the product topology so that $\mathcal{A}^{\mathbb{N}}$ is a compact metrizable space. For example, given $\theta \in (0, 1)$,

$$d(x, y) = d_\theta(x, y) \stackrel{\text{def}}{=} \theta^{\inf\{n\in\mathbb{N}\,:\,x_n\neq y_n\}} \quad \text{and} \quad d(x, x) = 0 \tag{2}$$

defines a metric that induces the product topology. The shift map $f : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ defined by $f(x)_n = x_{n+1}$ is a continuous d to 1 map. Given a $d \times d$ matrix A with values in $\{0, 1\}$ (called the transition matrix), we define $X = X_A = \{x \in \mathcal{A}^{\mathbb{N}} : A_{x_n,x_{n+1}} = 1\}$, which is a closed f-invariant subset of $\mathcal{A}^{\mathbb{N}}$. We say $f|_{X_A}$ is a subshift of finite type (SFT). For the remainder of this paper, we assume that $f : X \to X$ is a transitive SFT, see, e.g., [21] for details.

For $x \in X$, we write $\pi_k(x) = x_1 \ldots x_k$ and call $\tau = \tau_1 \ldots \tau_k \in \mathcal{A}^k$ a segment of length k. Moreover, we denote the *cylinder* generated by τ by $\mathcal{C}_k(\tau) = \{x \in X : x_1 = \tau_1, \ldots, x_n = \tau_k\}$, which may be empty. Given $x \in X$ and $k \in \mathbb{N}$, we call $\mathcal{C}_k(x) = \mathcal{C}_k(\pi_k(x))$ the cylinder of length k generated by x. Given $\tau \in \mathcal{A}^k$, we denote the *periodic point* generated by τ by $\mathrm{Or}(\tau) = \tau_1 \ldots \tau_k\tau_1 \ldots \tau_k\tau_1 \ldots \tau_k \ldots$, provided $\mathrm{Or}(\tau) \in X$. We denote the set of all periodic points of f with (prime) period n by $\mathrm{Per}_n(f)$. Moreover, $\mathrm{Per}(f) = \bigcup_{n\geqslant 1} \mathrm{Per}_n(f)$ denotes the set of all periodic points of f. For $x \in \mathrm{Per}_n(f)$, we call $\tau_x = x_1 \ldots x_n$ the *generating segment* of x, i.e., τ_x is the minimal length segment such that $x = \mathrm{Or}(\tau_x)$. Let $k \in \mathbb{N}$ be fixed. We define the k-cylinder support of $x \in \mathrm{Per}_n(f)$ by

$$\mathcal{S}_k(x) = \{\mathcal{C}_k(f^i(x)) : i \in \mathbb{N} \cup \{0\}\} = \{\mathcal{C}_k(f^i(x)) : \quad i = 0, \ldots, n-1\}. \tag{3}$$

Moreover, we say that $x \in \mathrm{Per}_n(f)$ is a k-elementary periodic point if $\mathcal{C}_k(f^i(x)) \neq \mathcal{C}_k(f^j(x))$ for all $i, j = 0, \ldots, n-1$ with $i \neq j$. When $k = 1$, we simply say that x is an elementary periodic point. We denote the set of all k-elementary periodic points by $\mathrm{EPer}^k(f)$. We note that the k-elementary periodic points were previously used (with other names) by Chazottes *et al* [7], Jenkinson [17] and Ziemian [40]. We recall that $m_c(k)$ denotes the cardinality of the set of cylinders of length k in X. It follows that the period of any k-elementary periodic point is at most $m_c(k)$, and, thus, $\mathrm{EPer}^k(f)$ is finite. For $x \in \mathrm{Per}_n(f)$, we denote the unique invariant measure supported on the orbit of x by μ_x, that is $\mu_x = \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i(x)}$. For $\phi \in C(X, \mathbb{R})$, we obtain the formula

$$\mu_x(\phi) = \frac{1}{n}\sum_{i=0}^{n-1} \phi(f^i(x)). \tag{4}$$

2.2. Topological pressure, ground states and zero-temperature measures

We briefly recall the relevant facts about the topological pressure, see, e.g., [35]. Let $f : X \to X$ be a transitive SFT. The topological pressure of $\phi \in C(X, \mathbb{R})$ is defined as

$$P_{\mathrm{top}}(\phi) = \sup_{\mu\in\mathcal{M}} \left(h_\mu(f) + \mu(\phi)\right), \tag{5}$$

where $h_\mu(f)$ denotes the measure-theoretic entropy of μ. Moreover, $h_{\rm top}(f) = P_{\rm top}(0)$ denotes the topological entropy of f. We recall that if $\nu \in \mathcal{M}$ satisfies $P_{\rm top}(\phi) = h_\nu(f) + \nu(\phi)$, then ν is an *equilibrium state* of ϕ. We denote the set of equilibrium states of ϕ by $\mathrm{ES}(\phi)$. Since the entropy map $\nu \mapsto h_\nu(f)$ is upper semi-continuous, $\mathrm{ES}(\phi)$ is nonempty. Furthermore, $\mathrm{ES}(\phi)$ is a compact and convex set whose extreme points are the ergodic equilibrium states.

We say $\mu \in \mathcal{M}$ is a *ground state* of the potential ϕ if there exists a sequence $\beta_n \to \infty$ and equilibrium states $\mu_n \in \mathrm{ES}(\beta_n\phi)$ such that $\mu = \lim_{n\to\infty} \mu_n$. Here, we think of β as the inverse temperature of the system (see the discussion in section 1.2). Thus, ground states are accumulation points of equilibrium states as the temperature approaches zero. We denote the set of ground states of ϕ by $\mathrm{GS}(\phi)$. By compactness, $\mathrm{GS}(\phi)$ is nonempty.

Next, we consider the case where $\beta\phi$ has a unique equilibrium state $\mu_\beta = \mu_{\beta\phi}$ for all $\beta \geqslant 0$. This case occurs, for example, when ϕ is Hölder continuous. We say $\mu_{\infty,\phi} \in \mathcal{M}$ is the zero-temperature measure of ϕ if $\mu_{\infty,\phi} = \lim_{\beta\to\infty} \mu_\beta$. We note that, in general, the uniqueness of the equilibrium states of $\beta\phi$ does not guarantee the existence of the zero-temperature measure, see, e.g., [2, 8, 12].

2.3. Entropy of ground states

We continue to use the definitions from section 1.2. Let $\phi \in C(X, \mathbb{R})$. For $w \in I(\phi)$, we define

$$\mathcal{H}(w) = \mathcal{H}_\phi(w) = \sup\{h_\mu(f) : \mu(\phi) = w\}$$

to be the localized entropy at w, see, e.g., [17, 22]. Since $\nu \mapsto h_\nu(f)$ is affine and upper semi-continuous on $\mathcal{M}$, we conclude that $\mathcal{H}$ is concave and upper semi-continuous and, therefore, continuous. We recall that $\mathcal{H}(b_\phi)$ coincides with the residual entropy $h_{\infty,\phi}$ of the potential ϕ. We make use of the following two lemmas to understand the behavior of the entropy as $\beta \to \infty$:

Lemma 2.1. *Let $(\beta_n)_n$ with $\beta_n \in \mathbb{R}^+$ be a strictly increasing sequence converging to ∞. Then, for any sequence of measures $(\mu_n)_n$, where $\mu_n \in \mathrm{ES}(\beta_n\phi)$, we have:*

(a) $b_\phi - \mu_n(\phi) \leqslant h_{\rm top}(f)/\beta_n$. *Moreover, the sequence $(\mu_n(\phi))_n$ is increasing with* $\lim_{n\to\infty} \mu_n(\phi) = b_\phi$;

(b) $(h_{\mu_n}(f))_n$ *is decreasing with* $\lim_{n\to\infty} h_{\mu_n}(f) = h_{\infty,\phi}$; *and*

(c) *If* $\mu \in \mathrm{GS}(\phi)$, *then* $h_\mu(f) = h_{\infty,\phi}$.

Proof. Let $\mu \in \mathcal{M}$ such that $\mu(\phi) = b_\phi$ and $h_\mu(f) = \mathcal{H}(b_\phi)$. Since μ_n is an equilibrium state of $\beta_n\phi$, it follows that $h_{\mu_n}(f) + \beta_n\mu_n(\phi) \geqslant h_\mu(f) + \beta_n b_\phi$. Therefore, $\beta_n(b_\phi - \mu_n(\phi)) \leqslant h_{\mu_n}(f) \leqslant h_{\rm top}(f)$, and the first and last parts of statement (a) follow.

For the remaining part of statement (a), we observe that since μ_n and μ_{n+1} are equilibrium states for $\beta_n\phi$ and $\beta_{n+1}\phi$, respectively, it follows that $h_{\mu_n}(f) + \beta_n\mu_n(\phi) \geqslant h_{\mu_{n+1}}(f) + \beta_n\mu_{n+1}(\phi)$ and $h_{\mu_{n+1}}(f) + \beta_{n+1}\mu_{n+1}(\phi) \geqslant h_{\mu_n}(f) + \beta_{n+1}\mu_n(\phi)$. Eliminating the entropies from these inequalities leads to $(\beta_{n+1} - \beta_n)(\mu_{n+1}(\phi) - \mu_n(\phi)) \geqslant 0$. Since the β_n's are strictly increasing, the final part of statement (a) follows.

The first part of statement (b) follows directly from statement (a) and the inequality $h_{\mu_n}(f) + \beta_n\mu_n(\phi) \geqslant h_{\mu_{n+1}}(f) + \beta_n\mu_{n+1}(\phi)$. For the second part of statement (b), since μ_n is an equilibrium state of $\beta_n\phi$, it follows that $h_{\mu_n}(f) = \mathcal{H}(\mu_n(\phi))$. We recall that $\mathcal{H}(b_\phi) = h_{\infty,\phi}$. Then, by statement (a) and the continuity of $\mathcal{H}$, the second part of statement (b) follows.

Finally, statement (c) follows from statement (b) and the upper semi-continuity of the entropy map. □

Lemma 2.2. *Let $\phi_0 \in C(X, \mathbb{R})$ with $h_{\infty,\phi_0} = 0$. Then, $\phi \mapsto h_{\infty,\phi}$ is continuous at ϕ_0.*

Proof. We recall that for all $\phi \in C(X, \mathbb{R})$, $\mathcal{H}(b_\phi) = h_{\infty,\phi}$. Fix $\varepsilon > 0$; by the continuity of $\mathcal{H}$, when $\mu(\phi_0)$ is sufficiently close to b_ϕ, then $h_\mu(f) \leqslant \mathcal{H}(\mu(\phi_0)) < \varepsilon$. Moreover, if $\|\phi - \phi_0\|_\infty < \delta$, then, for all $\mu \in \mathcal{M}$, $|\mu(\phi) - \mu(\phi_0)| < \delta$, and, in particular, $|b_\phi - b_{\phi_0}| < \delta$. Hence, if $\mu(\phi) = b_\phi$, then $|\mu(\phi_0) - b_{\phi_0}| < 2\delta$. Therefore, if δ is sufficiently small, then $h_\mu(f) < \varepsilon$. Then, by the definition of $\mathcal{H}$, it follows that $h_{\infty,\phi} < \varepsilon$, and the result follows. □

We show in proposition 3.5 that the converse to lemma 2.2 holds, i.e., that $\phi \mapsto h_{\infty,\phi}$ is continuous at ϕ_0 if and only if $h_{\infty,\phi_0} = 0$.

2.4. Locally constant potentials

For $\phi \in C(X, \mathbb{R})$ and $k \in \mathbb{N}$, we define $\mathrm{var}_k(\phi) = \sup\{|\phi(x) - \phi(y)| : x_1 = y_1, \ldots, x_k = y_k\}$. We denote the set of potentials that are constant on cylinders of length k by $\mathrm{LC}_k(X, \mathbb{R}) = \{\phi \in C(X, \mathbb{R}) : \mathrm{var}_k(\phi) = 0\}$. Moreover, $\mathrm{LC}(X, \mathbb{R}) = \bigcup_k \mathrm{LC}_k(X, \mathbb{R})$ denotes the set of locally constant potentials. Let $\phi \in \mathrm{LC}_k(X, \mathbb{R})$. By using a block code argument, see, e.g., [6, proposition 5.5], we may reduce the case $k > 1$ to the case $k = 1$.

We observe that since the periodic point measures are dense in $\mathcal{M}$, see, e.g., [25], $I(\phi)$ can be written in terms of the periodic points, i.e.,

$$I(\phi) = \overline{\mathrm{conv}\, \{\mu_x(\phi) : x \in \mathrm{Per}(f)\}}. \tag{6}$$

Furthermore, by decomposing the generating segment of each periodic point into the generating segments of finitely many k-elementary periodic points of X (see, e.g., [17, lemma 5]), it follows from equation (6) that

$$I(\phi) = \mathrm{conv}\, \left\{\mu_x(\phi) : x \in \mathrm{EPer}^k(f)\right\}. \tag{7}$$

We note that since $\mathrm{EPer}^k(f)$ is finite, the closure is not needed in equation (7). Therefore, the closure can also be omitted in equation (6).

Next, we characterize the decomposition of $\mathrm{LC}_k(X, \mathbb{R}) = \mathcal{O}_k \dot\cup \mathcal{U}_k \dot\cup \mathcal{V}_k$ from equation (1) in terms of the number and behavior of the elementary periodic points which achieve the maximum value in $I(\phi)$. We define

$$\mathrm{EPer}^k_{\max}(\phi) = \left\{x \in \mathrm{EPer}^k(f) : \mu_x(\phi) = b_\phi\right\}.$$

Definition 2.3. Let $f: X \to X$ be a transitive SFT.

(a) $\phi \in \mathcal{O}_k$ if $\mathrm{EPer}^k_{\max}(\phi)$ contains a single k-elementary periodic orbit.

(b) Furthermore, $\phi \in \mathcal{U}_k$ if $\mathrm{EPer}^k_{\max}(\phi) = \{x^1, \ldots, x^\ell\}$ contains more than one k-elementary periodic orbit and the k-cylinder support of different k-elementary periodic orbits are distinct. In other words, $\mathcal{C}_k(x^i) \neq \mathcal{C}_k(x^j)$ for all $x^i, x^j \in \mathrm{EPer}^k_{\max}(\phi)$ with $i \neq j$.

(c) Finally, $\phi \in \mathcal{V}_k$ if $\mathrm{EPer}^k_{\max}(\phi) = \{x^1, \ldots, x^\ell\}$ contains more than one k-elementary periodic orbit and the k-cylinder support of different k-elementary periodic orbits are not distinct. More precisely, $\mathcal{C}_k(x^i) = \mathcal{C}_k(x^j)$ for some $i, j \in \{1, \ldots, \ell\}$ with $i \neq j$.

As mentioned in section 1.2, the properties of the partition $\mathrm{LC}_k(X, \mathbb{R}) = \mathcal{O}_k \dot\cup \mathcal{U}_k \dot\cup \mathcal{V}_k$ follow from [38]. Furthermore, [38] implies the following:

Proposition 2.4. *Let $f: X \to X$ be a transitive SFT. Then*

(a) If $\phi \in \mathcal{O}_k$ and $x \in \mathrm{EPer}^k_{\max}(\phi)$, then $\mu_{\infty,\phi} = \mu_x$ and $h_{\infty,\phi} = 0$;

(*b*) *If* $\phi \in \mathcal{U}_k$, *then* $h_{\infty,\phi} = 0$ *and* $\mu_{\infty,\phi}$ *is a convex combination of the periodic point measures corresponding to periodic orbits in* $\mathrm{EPer}^k_{\max}(\phi)$*; and*

(*c*) *If* $\phi \in \mathcal{V}_k$, *then* $\mu_{\infty,\phi}$ *is a measure of maximal entropy of a non-discrete* (*not necessarily transitive*) *SFT* $X_{\max} = X_{\max}(\phi) \subset X$ *and* $h_{\infty,\phi} > 0$.

We recall that the SFT $X_{\max}$ in proposition 2.4(c) is given by the closure of the periodic points $x \in \mathrm{Per}(f)$ that maximize $\mu_x(\phi)$. For more details, see [17, 38].

2.5. Computability theory for SFTs

Computability theory provides information about the feasibility and accuracy of computational experiments when using approximate data. Without an accuracy guarantee, a computer experiment might miss or misinterpret interesting behaviors. We provide an overview of the key definitions and theorems. For a more thorough introduction to computability theory, see, e.g., [1, 3, 4, 6, 13, 29, 31, 37] and the references therein. We use different, but closely related, definitions to those in [4, 13], see also [6]. Throughout this discussion, we use a bit-based computation model, such as a Turing machine, as opposed to a real RAM (random access machine) model [32] where these questions are trivial. One can think of the set of Turing machines as a particular countable set of functions. We denote the output of a Turing machine ψ on input x by $\psi(x)$. A point is computable if it can be algorithmically approximated to any desired precision.

Definition 2.5 (cf [4, definition 1.2.1]). Let $x \in \mathbb{R}^m$. An oracle approximating x is a function ψ such that on input $n \in \mathbb{N}$, $\psi(n) \in \mathbb{Q}^m$ with $\|\psi(n) - x\| < 2^{-n}$. Moreover, x is *computable* if there is a Turing machine ψ which is an oracle for x.

We note that solely using computability, equality is not decidable. More precisely, we observe that when x is computable, we can only calculate x up to some error. Therefore, we may conclude that x is in a small interval, but we cannot conclude which point in the interval equals x. For more details, see [34, 39]. We say that a function is computable if its value at any point can be computed to any requested precision.

Definition 2.6 (cf [4, definition 1.2.5]). Let $S \subset \mathbb{R}^m$. A function $g : S \to \mathbb{R}$ is *computable* if there is a Turing machine χ so that for any $x \in S$ and any oracle ψ for x, $\chi(\psi, n)$ is a rational number so that $|\chi(\psi, n) - g(x)| < 2^{-n}$.

We also note that the definition of a computable function uses any oracle for x and applies even when x is not computable, i.e., the oracle ψ does not need to be a Turing machine. Moreover, since the accuracy of g is only dependent on an approximation for x, we conclude that computable functions are continuous, see, e.g., [4, theorem 1.5].

We also study functions which have a weaker notion of computability called upper semi-computability. In this case, the approximations converge from above, but without an accuracy guarantee.

Definition 2.7 (cf [13, definition 2.7]). Let $S \subset \mathbb{R}^m$. A function $g : S \to \mathbb{R}$ is *upper semi-computable* (also called right recursively enumerable or right computable) if there is a Turing machine χ so that for any $n \in \mathbb{N}$ and $x \in S$, and any oracle ψ for x, $\chi(\psi, n)$ is a rational number such that the sequence $(\chi(\psi, n))_n$ is nonincreasing and $\lim_{n\to\infty} \chi(\psi, n) = g(x)$.

Similarly, we may define lower semi-computable functions. We observe that a function is computable if and only if it is both upper and lower semi-computable, see, e.g., [4]. As an alternate weaker notion of computability, we introduce the definition of a function which is computable at a given point, which is a computable version of pointwise continuity.

Definition 2.8. Let $S \subset \mathbb{R}^m$ be an open set and let $x \in S$. Suppose that $g : S \to \mathbb{R}$ is a function and χ is a Turing machine such that for any oracle ψ for x, $\chi(\psi, n)$ is a rational number. We define ℓ_n to be the maximum ℓ to which χ queries $\psi(\ell)$ when evaluating $\chi(\psi, n)$.

The function $g : S \to \mathbb{R}$ is *computable at x* if there exists a Turing machine χ so that for any oracle ψ for x, $\chi(\psi, n)$ is a rational number with the following property: for all $y \in S$ such that there exists an oracle ψ' for y that agrees with ψ up to precision ℓ_n, i.e., $\psi(\ell) = \psi'(\ell)$ for all $\ell \leqslant \ell_n$, then $\chi(\psi, n) = \chi(\psi', n)$ and $|\chi(\psi, n) - g(y)| < 2^{-n}$.

We observe that this definition is existential and, in particular, does not include a decidability statement. In other words, we do not assume that there exists a Turing machine which decides if ψ is computable at $x \in S$.

Since one of our main theorems involves recursively open sets, we include its definition.

Definition 2.9 (cf [13, definition 2.4]). An open set $S \subset \mathbb{R}^m$ is a *recursively open set* (also called a semi-decidable set or a lower-computable set) if there exists a Turing machine ψ such that ψ produces a (possibly infinite) sequence of pairs (z_i, n_i) so that $z_i \in \mathbb{Q}^m$ is a rational vector and $n_i \in \mathbb{Z}$ so that

$$S = \bigcup_i B\left(z_i, 2^{-n_i}\right).$$

In the remainder of this section, we assume that f is a transitive SFT and describe its computability properties, see, e.g., [6] for further details.

Definition 2.10. Let $x \in X$. An *oracle* for x is a function ψ such that for any natural number n, $\psi(n) = x_n$. Moreover, x is *computable* if there is a Turing machine ψ which is an oracle for x.

We note that the periodic points of f are both computable and dense. Moreover, the cylinders of length k and a periodic point in each cylinder can be computed algorithmically. We observe that the locally constant potentials with rational values (denoted by $\mathrm{LC}(X, \mathbb{Q})$) are dense in $C(X, \mathbb{R})$ with respect to the supremum norm. Since $\mathrm{LC}_k(X, \mathbb{Q})$ is in bijective correspondence with $\mathbb{Q}^{m_{\mathrm{c}}(k)}$, it follows that each potential in $\mathrm{LC}(X, \mathbb{Q})$ can be represented by a pair (k, q) where $q \in \mathbb{Q}^{m_{\mathrm{c}}(k)}$.

Definition 2.11. Suppose that $\phi \in C(X, \mathbb{R})$. An oracle for ϕ is a function χ such that on input n, $\chi(n)$ is a locally constant potential in $\mathrm{LC}(X, \mathbb{Q})$ such that $\|\chi(n) - \phi\|_\infty < 2^{-n}$. Moreover, ϕ is computable if there is a Turing machine χ which is an oracle for ϕ.

We observe that if θ from the product topology (see equation (2)) is a computable real number, then the function for the distance between two points of X is a computable function and X is a computable metric space, cf [13, definition 2.2]. In addition, $C(X, \mathbb{R})$ also forms a computable metric space, see, e.g., [6, 13].

In the definition for an oracle χ of $\phi \in C(X, \mathbb{R})$, since $\chi(n)$ is a locally constant function, there is some k_n so that $\chi(n) \in \mathrm{LC}_{k_n}(X, \mathbb{Q}) \backslash \mathrm{LC}_{k_n - 1}(X, \mathbb{Q})$. We observe, however, that there is no bound on the size of k_n in the definition of such an oracle. In fact, even if $\phi \in \mathrm{LC}(X, \mathbb{R})$, it is possible that the sequence $(k_n)_{n \in \mathbb{N}}$ diverges to infinity as n increases. We now show, however, that when ϕ is locally constant, we can compute (from χ) a sequence of potentials in $\mathrm{LC}_{\ell_n}(X, \mathbb{Q})$ such that ℓ_n does not grow without bound.

Lemma 2.12. *There exists a Turing machine, which, given input $n \in \mathbb{N}$ and an oracle χ of $\phi \in \mathrm{LC}(X, \mathbb{R})$ produces $\ell_n \in \mathbb{N}$ and $\widetilde{\phi}_n \in \mathrm{LC}_{\ell_n}(X, \mathbb{Q})$ with the following properties:*

(*a*) $\|\phi - \widetilde{\phi}_n\|_\infty < 2^{-n}$ *and*

(*b*) *If* $\phi \in \mathrm{LC}_k(X, \mathbb{R})$*, then* $\ell_n \leqslant k$.

Proof. For a fixed n, let ℓ_n be the smallest cylinder length so that $\mathrm{var}_{\ell_n}(\chi(n+1)) < 2^{-n}$. We observe that $\ell_n \leqslant k$ via the triangle inequality. We define $\widetilde{\phi}_n$ to be a locally constant function in $\mathrm{LC}_{\ell_n}(X, \mathbb{Q})$ where the value in each cylinder is chosen to differ by at most 2^{-n-1} from the values of $\chi(n+1)$ in that cylinder. □

Since $C(X, \mathbb{R})$ is a computable metric space, we consider computable functions whose domains are subsets of $C(X, \mathbb{R})$.

Definition 2.13. Let $S \subset C(X, \mathbb{R})$. A function $g : S \to \mathbb{R}$ is computable if there is a Turing machine η so that for any function $\phi \in S$ and oracle χ for ϕ, $\eta(\chi, n)$ is a rational number with $|\eta(\chi, n) - g(\phi)| < 2^{-n}$.

We note that the notions of an upper semi-computable function and computability at a point carry over to the case of a function $g : S \to \mathbb{R}$ for $S \subset C(X, \mathbb{R})$. We leave the details to the reader. We also observe that the computability of the function g in definition 2.13 is defined in terms of the supremum norm. Since the supremum norm does not generate the same topology as the Hölder or Lipschitz norms, previous results on the Hölder and Lipschitz norms cannot be applied in this paper, see, e.g., [10, 11, 24, 28].

We recall that computability can also be extended to the space $\mathcal{M}$ of invariant measures on X via the Wasserstein–Kantorovich metric, see [13] for more details. All of the invariant measures considered in this paper, however, are periodic point measures. For these measures, the full theory of the computability of $\mathcal{M}$ is not needed and it is sufficient to use the computability properties of the periodic points.

3. Upper semi-computability of the residual entropy

The goal of this section is to prove theorem A. We start with a discussion of the pressure function for continuous potentials. These results are fairly standard in the Hölder continuous case, but they are more challenging for potentials which are only continuous. The difficulties arise from the lack of uniqueness results for equilibrium states and, in particular, the possibility of phase transitions. To overcome these challenges, we make use of several tools, including methods from convex analysis, see, e.g., [30].

Let $\phi : X \to \mathbb{R}$ be a fixed continuous potential. We note that we do not assume the uniqueness of the equilibrium states. We call $\beta \mapsto P(\beta) \stackrel{\text{def}}{=} P_{\text{top}}(\beta\phi)$ the pressure function of ϕ. The pressure function is convex, see, e.g., [35], and, thus, it has left and right derivatives

$$\partial_{\pm} P(\beta) = \lim_{\delta \to 0^{\pm}} \frac{P(\beta + \delta) - P(\beta)}{\delta}.$$

Moreover, since $\mu \mapsto h_\mu(f)$ is upper semi-continuous, it follows from [17, proposition 1] and [36, lemma 1] that

$$\partial_{-} P(\beta) = \min_{\mu \in \mathrm{ES}(\beta\phi)} \mu(\phi) \quad \text{and} \quad \partial_{+} P(\beta) = \max_{\mu \in \mathrm{ES}(\beta\phi)} \mu(\phi). \tag{8}$$

Furthermore, since $\mathrm{ES}(\beta\phi)$ is a compact and convex subset of $\mathcal{M}$, for all $\partial_{-} P(\beta) \leqslant \alpha \leqslant \partial_{+} P(\beta)$, there exists $\mu_\alpha \in \mathrm{ES}(\beta\phi)$ with $\mu_\alpha(\phi) = \alpha$. In particular, the minimum and maximum in equation (8) are achieved. We observe that $\beta \mapsto P(\beta)$ is differentiable at β if and only if $I_\beta \stackrel{\text{def}}{=} \{\mu(\phi) : \mu \in \mathrm{ES}(\beta\phi)\}$ is a singleton[5]. Moreover,

$$\operatorname{int} I(\phi) = (a_\phi, b_\phi) \subset \bigcup_{\beta\in\mathbb{R}} I_\beta, \tag{9}$$

see [17, corollary 2]. Since $\beta \mapsto P(\beta)$ is convex, it is differentiable on $\mathbb{R}$ with the exception of at most countably many points $\beta \in \mathbb{R}$. We define

$$h_{\max}(\beta) = \max_{\mu\in\mathrm{ES}(\beta\phi)} h_\mu(f) \quad \text{and} \quad h_{\min}(\beta) = \min_{\mu\in\mathrm{ES}(\beta\phi)} h_\mu(f).$$

Thus, equation (5) yields

$$P(\beta) = h_{\max}(\beta) + \beta\partial_- P(\beta) = h_{\min}(\beta) + \beta\partial_+ P(\beta). \tag{10}$$

Moreover, the convexity of the pressure function implies

$$\partial_+ P(\beta_1) \leqslant \partial_- P(\beta_2) \quad \text{and} \quad h_{\min}(\beta_1) \geqslant h_{\max}(\beta_2) \tag{11}$$

whenever $0 \leqslant \beta_1 < \beta_2$. We briefly discuss the case when P is differentiable at β. In this case equation (10) becomes $P(\beta) = h(\beta) + \beta\partial P(\beta)$, where $h(\beta) \stackrel{\text{def}}{=} h_{\max}(\beta) = h_{\min}(\beta)$ and $\partial P(\beta) \stackrel{\text{def}}{=} \partial_- P(\beta_1) = \partial_+ P(\beta_1)$. Since the topological pressure is convex and computable, see [6, 34], it is straight-forward to compute approximations to $\beta \mapsto \partial P(\beta)$ and $\beta \mapsto h(\beta)$ on the set of points where $\beta \mapsto P(\beta)$ is differentiable. In particular, we conclude that if ϕ is a Hölder continuous potential given by an oracle, then the functions $\beta \mapsto h(\beta)$ and $\beta \mapsto \partial P(\beta)$ are computable. Combining these observations with lemma 2.1, we conclude that theorem A holds for Hölder continuous potentials. To prove the general case, we make use of the following result to include the possibility of phase transitions:

Proposition 3.1. *Suppose $\phi \geqslant 0$ and let $0 \leqslant \beta_1 < \beta_2$. We define $\alpha = \alpha(\beta_1, \beta_2) = (P(\beta_2) - P(\beta_1))/(\beta_2 - \beta_1)$. Then, there exist $\beta_1 < \beta < \beta_2$ and $\mu \in \mathrm{ES}(\beta\phi)$ such that*

$$P(\beta_1) - \beta_2\alpha \leqslant h_\mu(f) \leqslant P(\beta_2) - \beta_1\alpha. \tag{12}$$

Proof. First, we observe that since $\phi \geqslant 0$, the map $\beta \mapsto P(\beta)$ is increasing. If $\partial_+ P(\beta_1) = \partial_- P(\beta_2)$, then $h(\beta)$ and $\partial P(\beta)$ are constant for $\beta_1 < \beta < \beta_2$, so $P|_{(\beta_1,\beta_2)}$ is an affine function of β. Moreover, for all $\beta_1 < \beta < \beta_2$, $\partial P(\beta) = \alpha$. Finally, combining this with equation (10), it follows that $P(\beta_2) - \beta_1\alpha = h(\beta) + (\beta_2 - \beta_1)\partial P(\beta) \geqslant h(\beta)$ and $P(\beta_1) - \beta_2\alpha = h(\beta) - (\beta_2 - \beta_1)\partial P(\beta) \leqslant h(\beta)$. Therefore, inequality (12) holds for all $\beta_1 < \beta < \beta_2$ and all $\mu \in \mathrm{ES}(\beta\phi)$.

It remains to consider the case where $\partial_+ P(\beta_1) < \partial_- P(\beta_2)$. Since α is the slope of the line segment joining $(\beta_1, P(\beta_1\phi))$ and $(\beta_2, P(\beta_2\phi))$, the convexity of the pressure function implies that $\partial_+ P(\beta_1) < \alpha < \partial_- P(\beta_2)$. Thus, by equation (8), $\alpha \in \operatorname{int} I(\phi)$. It now follows from equation (9) that there exists $\beta \in \mathbb{R}$ and $\mu \in \mathrm{ES}(\beta\phi)$ such that $\mu(\phi) = \alpha$. Moreover, by equation (11), we may restrict β to $\beta_1 < \beta < \beta_2$. Applying equation (5) yields

$$h_\mu(f) = P(\beta) - \beta\mu(\phi) = P(\beta) - \beta\alpha.$$

Finally, equation (12) follows since the pressure function is increasing. □

[5] We note that the nondifferentiability points of the pressure function are phase transitions, i.e., points of coexistence of multiple equilibrium states where each ergodic equilibrium state represents a phase.

The following auxiliary lemma is used in the proofs of both theorems A and B. In the lemma, we show that the endpoints of $I(\phi) = [a_\phi, b_\phi]$ are computable points.

Lemma 3.2. *The functions $\phi \mapsto a_\phi$ and $\phi \mapsto b_\phi$ are computable on $C(X, \mathbb{R})$.*

Proof. We first note that the functions $\phi \mapsto a_\phi$ and $\phi \mapsto b_\phi$ are Lipschitz continuous with Lipschitz constant 1 on $C(X, \mathbb{R})$. Let $\phi \in C(X, \mathbb{R})$ and let χ be an oracle for ϕ. We conclude that $|a_\phi - a_{\chi(n)}| < 2^{-n}$ and $|b_\phi - b_{\chi(n)}| < 2^{-n}$. Therefore, it is enough to prove the statement for locally constant potentials. Suppose that $\chi(n) \in \mathrm{LC}_{k_n}(X, \mathbb{R})$. Then, by equation (7), it is enough to approximate $\mu_x(\phi)$ for all k_n-elementary periodic points $x \in X$. We use formula (4) to approximate $\mu_x(\phi)$ to any desired precision. Since there are only finitely many k_n-elementary periodic points, we can approximate a_ϕ and b_ϕ to any desired precision. □

We are now ready to present the proof of theorem A using the computability of the topological pressure, lemma 2.1, and proposition 3.1. We begin with a technical lemma that forms the central argument of the main theorem.

Lemma 3.3. *Let $\phi \in C(X, \mathbb{R})$ with $\phi \geqslant 0$ be given by an oracle ψ. Suppose that rational numbers $0 \leqslant \beta_1 < \beta_2$ are given. There exists a Turing machine χ so that $\chi(n, \psi)$ is a rational number such that there exists[6] a β with $\beta_1 < \beta < \beta_2$ and $\mu \in \mathrm{ES}(\beta\phi)$ such that $|\chi(n, \psi) - h_\mu(f)| < 2^{-n}$.*

Proof. We observe that since the pressure function $\beta \mapsto P_{\mathrm{top}}(\beta\phi)$ is continuous, as $\beta_2 \to \beta_1$, the upper and lower bounds of inequality (12) approach each other. Therefore, if we can find β_1' and β_2' so that $\beta_1 \leqslant \beta_1' < \beta_2' \leqslant \beta_2$ and the upper and lower bounds of inequality (12) are within 2^{-n}, any rational number satisfying the inequalities of inequality (12) can be used to approximate $h_\mu(f)$.

We recall that the pressure function $\beta \mapsto P_{\mathrm{top}}(\beta\phi)$ is computable, see [6, 34]. Therefore, the upper and lower bounds in inequality (12) are also computable. We consider a sequence $(\beta_{1,m}', \beta_{2,m}')$ of pairs of rational numbers so that $\beta_1 \leqslant \beta_{1,m}' < \beta_{2,m}' \leqslant \beta_2$ and $\beta_{2,m}' - \beta_{1,m}'$ decreases to zero as $m \to \infty$. By approximating the upper and lower bounds of inequality (12) sufficiently well for each m, we may compute an m so that the upper and lower bounds of inequality (12), when applied to $\beta_{1,m}'$ and $\beta_{2,m}'$, are within 2^{-n}. □

Next, we present the proof of theorem A, which is broken into the following two statements:

Theorem 3.4. *The function $\phi \mapsto h_{\infty,\phi}$ is upper semi-computable on $C(X, \mathbb{R})$.*

Proof. Suppose that $\phi \in C(X, \mathbb{R})$ is given by an oracle χ. We can compute a lower bound q of ϕ by computing a lower bound of $\chi(n)$. We observe that $\mathrm{ES}(\beta\phi) = \mathrm{ES}(\beta(\phi - q))$ and $\phi - q \geqslant 0$. By applying lemma 3.3 to a strictly increasing sequence $(\beta_n)_n$ converging to ∞, we compute a sequence of entropies $h_{\mu_n}(f)$ for $\mu_n \in \mathrm{ES}(\beta(\phi - q))$ with $\beta_n < \beta < \beta_{n+1}$. Then, by applying lemma 2.1, we conclude that these entropies approach the residual entropy from above. □

Next, we characterize the continuity of the residual entropy.

Proposition 3.5. *The function $\phi \mapsto h_{\infty,\phi}$ is continuous at $\phi_0 \in C(X, \mathbb{R})$ if and only if $h_{\infty,\phi_0} = 0$.*

Proof. Let $\phi_0 \in C(X, \mathbb{R})$. If $h_{\infty,\phi_0} = 0$ then $\phi \mapsto h_{\infty,\phi}$ is continuous at ϕ_0 by lemma 2.2. Assume now $h_{\infty,\phi_0} > 0$. We recall the definition of the set of uniquely maximizing locally constant potentials $\mathcal{O} = \bigcup_k \mathcal{O}_k$ from section 1.2. We observe that since $\mathcal{O}_k$ is dense in $\mathrm{LC}_k(X, \mathbb{R})$,

[6] We note that the lemma does not require the computability of β, only its existence.

it follows that $\mathcal{O}$ is dense in $\mathrm{LC}(X, \mathbb{R})$. Since $\mathcal{O}$ consists of the locally constant potentials with a uniquely maximizing periodic point, for all $\phi \in \mathcal{O}$, $h_{\infty,\phi} = 0$. Finally, since $\mathrm{LC}(X, \mathbb{R})$ is dense in $C(X, \mathbb{R})$, we conclude that the map $\phi \mapsto h_{\infty,\phi}$ is not continuous at ϕ_0. □

Finally, we use the previous two results to prove corollary 1.1

Corollary 3.6. *The function $\phi \mapsto h_{\infty,\phi}$ is computable at ϕ_0 if and only if $h_{\infty,\phi_0} = 0$.*

Proof. If $h_{\infty,\phi_0} > 0$, then by proposition 3.5, the map $\phi \mapsto h_{\infty,\phi}$ is not continuous at ϕ_0, so the function cannot be computable at ϕ_0. On the other hand, suppose that $h_{\infty,\phi_0} = 0$. By theorem 3.4, there is a Turing machine χ so that for any oracle ψ of ϕ_0, $(\chi(\psi, m))_m$ is a sequence of rational numbers decreasing to zero. By taking m_n sufficiently large, $\chi(\psi, m_n) < 2^{-n}$. Let ℓ_n be the largest precision to which the oracle ψ is queried within χ and let $\phi' \in C(X, \mathbb{R})$ be a function such that there exists an oracle ψ' for ϕ' that agrees with ψ up to precision ℓ_n. Then $\chi(\psi, m_n) = \chi(\psi', m_n)$ is an upper bound on $h_{\phi',\infty}$. Since the entropy is nonnegative, $|\chi(\psi, m_n) - h_{\phi',\infty}| < 2^{-n}$ and the function $\phi \mapsto h_{\infty,\phi}$ is computable at ϕ_0. □

4. Computability of zero temperature measures for locally constant potentials: the case of bounded cylinder length

In this section, we prove theorem B. We assume that θ in equation (2) is a computable real number. Moreover, we assume, whenever necessary, that $\phi \in \mathrm{LC}_k(X, \mathbb{R})$ is a potential given by an oracle.

We observe that by using lemma 3.2, we can compute a superset of $\mathrm{EPer}^k_{\max}(\phi)$. In particular, for $x \in \mathrm{EPer}^k(f)$, we can approximate $\mu_x(\phi)$ using formula (4). Then, $\mathrm{EPer}^k_{\max}(\phi)$ is a subset of those k-elementary points for which the approximations of $\mu_x(\phi)$ and b_ϕ permit the possibility of equality. By increasing the accuracy of these approximations, the computed superset of $\mathrm{EPer}^k_{\max}(\phi)$ shrinks. By studying the number and k-cylinder supports of this superset, we may identify potentials in $\mathcal{O}_k$ or $\mathcal{O}_k \dot\cup \mathcal{U}_k$. In particular, we conclude that there are Turing machines $\psi_{\mathcal{O}_k}$ and $\psi_{\mathcal{O}_k \dot\cup \mathcal{U}_k}$ which take a potential as input and terminate if and only if $\phi \in \mathcal{O}_k$ or $\phi \in \mathcal{O}_k \dot\cup \mathcal{U}_k$, respectively.

Proposition 4.1. *The sets $\mathcal{O}_k$ and $\mathcal{O}_k \dot\cup \mathcal{U}_k$ are recursively open sets.*

Proof. We use the Turing machines $\psi_{\mathcal{O}_k}$ and $\psi_{\mathcal{O}_k \dot\cup \mathcal{U}_k}$ identified above. If $\phi \in \mathcal{O}_k$, then there is a positive gap between the approximation to b_ϕ and the second-largest value of $\mu_y(\phi)$ for a k-elementary periodic point y. Similarly, if $\phi \in \mathcal{O}_k \dot\cup \mathcal{U}_k$, then there is a gap between the approximation to b_ϕ and the largest $\mu_x(\phi)$ of a k-elementary periodic point x which is not included in the superset of $\mathrm{EPer}^k_{\max}(\phi)$ constructed above. Perturbations of ϕ by no more than half these gaps remain within $\mathcal{O}_k$ and $\mathcal{O}_k \dot\cup \mathcal{U}_k$, respectively. By using more accurate approximations, we can identify more potentials and refine the radii of the constructed balls, so that, in the limit, the constructed open sets cover $\mathcal{O}_k$ or $\mathcal{O}_k \dot\cup \mathcal{U}_k$. □

We now discuss the computability of the entropy and the zero-temperature measure. These propositions are the main computability statements of theorem B.

Proposition 4.2. *The map $\phi \mapsto h_{\infty,\phi}$ is computable on $\mathcal{O}_k \dot\cup \mathcal{U}_k$. Moreover, the map $\phi \mapsto \mu_{\infty,\phi}$ is computable on $\mathcal{O}_k$.*

Proof. Suppose that $\phi \in \mathcal{O}_k \dot\cup \mathcal{U}_k$. Since the entropy of all zero-temperature measures of potentials in $\mathcal{O}_k \dot\cup \mathcal{U}_k$ is zero, which is computable, the entropy function is computable. Suppose now that we know that $\phi \in \mathcal{O}_k$. By inspecting the proof of proposition 4.1, we find that for all ϕ'

produced by small perturbations in the proof, the same k-elementary orbit x maximizes $\mu(\phi')$. Therefore, for every ϕ' in the ball, the zero-temperature measure is $\mu_x = \mu_{\infty,\phi}$. This measure is computable since the supporting k-elementary periodic point is computable. □

We now complete the proof of theorem B by showing that the functions $\phi \mapsto \mu_{\infty,\phi}$ and $\phi \mapsto h_{\infty,\phi}$ are not continuous, and, hence, not computable on the complement of the points in proposition 4.2.

Proposition 4.3. *The map $\phi \mapsto \mu_{\infty,\phi}$ is not continuous at any $\phi_0 \in \mathcal{U}_k \dot{\cup} \mathcal{V}_k$. Moreover, the map $\phi \mapsto h_{\infty,\phi}$ is not continuous at any $\phi_0 \in \mathcal{V}_k$. In particular, the corresponding maps are not computable at ϕ_0.*

Proof. Suppose that $\phi_0 \in \mathcal{V}_k$, then $h_{\infty,\phi_0} > 0$. The proof of proposition 3.5 shows that, when restring to $\mathrm{LC}_k(X,\mathbb{R})$, the residual entropy map is not continuous, and, hence, not computable at ϕ_0. Additionally, using the density of $\mathcal{O}_k$, there is an infinite sequence $(\phi_n)_{n\in\mathbb{N}}$ in $\mathcal{O}_k$ whose limit is ϕ_0. Since there are only finitely many k-elementary periodic points, by passing to a subsequence, we may assume that there is a k-elementary periodic point x so that $\mu_x = \mu_{\infty,\phi_n}$ for all n. If the zero-temperature measure map were continuous in the Wasserstein–Kantorovich metric, then μ_{∞,ϕ_0} would be μ_x, but this is not possible since the entropy of a periodic point measure is 0. Thus, the map $\phi \mapsto \mu_{\infty,\phi}$ is not continuous, and, hence, not computable at ϕ_0.

To show that the map $\phi \mapsto \mu_{\infty,\phi}$ is not continuous at $\phi_0 \in \mathcal{U}_k$, we find two sequences of potentials converging to ϕ_0 where the corresponding sequences of zero-temperature measures have different limits. In particular, we construct two sequences of potentials $(\phi_{1,n})_n$ and $(\phi_{2,n})_n$ where, for all n, $\mu_{\infty,\phi_{1,n}} = \mu_x$ and $\mu_{\infty,\phi_{2,n}} = \mu_y$ with x and y distinct k-elementary periodic points. Since $\phi_0 \in \mathcal{U}_k$, there are two k-elementary periodic points $x, y \in \mathrm{EPer}^k_{\max}(\phi)$ with disjoint k-cylinder support. Thus, there exists a k-cylinder $\mathcal{C}(\tau)$ in the support of x, but not in the support of y. Similarly, there is a k-cylinder $\mathcal{C}(\tau')$ that is not in the support of x, but is in the support of y. By (slightly) increasing ϕ_0 on $\mathcal{C}(\tau)$ or $\mathcal{C}(\tau')$, we can make $\mathrm{EPer}^k_{\max}(\phi)$ consist of a single k-elementary periodic point x or y. Thus, the function $\phi \mapsto \mu_{\infty,\phi}$ is not continuous, and, hence, not computable at ϕ_0. □

5. Computability of zero temperature measures for locally constant potentials: the case of unbounded cylinder length

It is natural to ask whether proposition 4.1 requires k to be given or if the statements can be generalized to the sets $\mathcal{O} = \bigcup_k \mathcal{O}_k$ and $\mathcal{O} \dot{\cup} \mathcal{U}$, where $\mathcal{U} = \bigcup_k \mathcal{U}_k$. We give a negative answer to this question by showing the fact that the sets $\mathcal{O}_k$ and $\mathcal{O}_k \dot{\cup} \mathcal{U}_k$ are recursively open does not extend to $\mathcal{O}$ and $\mathcal{O} \dot{\cup} \mathcal{U}$. In particular, we prove, in theorem C, that $\mathcal{O}$ has no interior points in $\mathrm{LC}(X,\mathbb{R})$. We begin with an illustrative example where $\mathcal{O}$ is not open and which provides the motivation for the proof of theorem C.

Example 5.1. Consider the SFT with alphabet $\{0, 1, 2, 3\}$ and transition matrix given in figure 1. Let $\phi \in \mathrm{LC}_2(X,\mathbb{R})$ be the potential whose value on cylinders $\mathcal{C}_2(01)$ and $\mathcal{C}_2(10)$ is 2, while its value on any other cylinder of length 2 is 1. In other words, ϕ is defined by the following matrix:

$$\begin{pmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

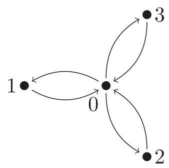

Figure 1. The transition matrix and corresponding directed graph illustrating the allowable transitions between states.

For each $n \in \mathbb{N}$, we define a potential $\phi_n \in \mathrm{LC}_{2n+2}(X, \mathbb{R})$ which is a perturbation of ϕ. For a segment τ we denote by $\#_2(\tau)$ and $\#_3(\tau)$ the number of 2's or 3's appearing in τ, respectively. We define

$$\phi_n(w) = \begin{cases} 2 + \dfrac{2}{n} & w \in \mathcal{C}_2(01) \cup \mathcal{C}_2(10),\ \#_2(\pi_{2n+2}(w)) + \#_3(\pi_{2n+2}(w)) = 1 \\ \phi(w) & \text{otherwise} \end{cases}.$$

In other words, $\phi_n(w) = \phi(w)$ unless $\tau = \pi_{2n+2}(w)$ begins with 01 or 10 and contains either (exactly one 2 and no 3's) or (exactly one 3 and no 2's). We see that $\|\phi - \phi_n\|_\infty = \frac{2}{n}$. Moreover, $\mathrm{EPer}^2_{\max}(\phi) = \mathrm{EPer}^{2n+2}_{\max}(\phi)$ consists of the single 2-elementary periodic orbit of $x = \mathrm{Or}(01)$. We observe that both ϕ and ϕ_n are constant on the orbit of x, so $\mu_x(\phi) = 2 = \mu_x(\phi_n)$.

On the other hand, $\mathrm{EPer}^{2n+2}_{\max}(\phi_n)$ contains at least three $(2n+2)$-elementary periodic orbits: the orbits generated by $(01)^n02$, $(01)^n03$, and $(01)^n02(01)^n03$. Here, $(01)^n$ represents the sequence of length $2n$ consisting of 01 repeated n times. Let $z_1 = \mathrm{Or}((01)^n02)$, $z_2 = \mathrm{Or}((01)^n03)$, and $z_3 = \mathrm{Or}((01)^n02(01)^n03)$. We observe that $\mu_{z_i}(\phi_n) = 2 + \frac{1}{n+1} > 2$ for $i = 1, 2, 3$. On the other hand, we observe that $\mu_{z_i}(\phi) = 2 - \frac{1}{n+1}$.

Putting this together, we note that since $\mathrm{EPer}^2_{\max}(\phi)$ consists of a single periodic orbit, $\phi \in \mathcal{O}_2$ with $h_{\infty,\phi} = 0$ and $\mu_{\infty,\phi} = \mu_x$. On the other hand, since z_1, z_2, and z_3 have overlapping cylinders, $\phi_n \in \mathcal{V}_{2n+2}$ with $h_{\infty,\phi_n} > 0$. We, therefore, conclude that since $\phi_n \to \phi$, $\mathcal{O}$ is not open in the supremum norm topology on $\mathrm{LC}(X, \mathbb{R})$, so, in particular, $\mathcal{O}$ is not a recursively open set. We observe, however, that by lemma 2.2, $h_{\infty,\phi_n} \to 0$ as $n \to \infty$.

This example shows that, in general, $\mathcal{O}$ is not open in $\mathrm{LC}(X, \mathbb{R})$. Moreover, we note that, in our example, the maximal $(2n+2)$-elementary periodic orbits of ϕ_n do not include the maximal 2-elementary periodic orbits of ϕ. In other words, the set of maximizing elementary periodic orbits may change considerably under perturbations once the cylinder length is not fixed.

Using this example as a guide, we show that the set $\mathcal{U} \dot\cup \mathcal{V}$ is dense in $\mathrm{LC}(X, \mathbb{R})$, where $\mathcal{V} = \bigcup_k \mathcal{V}_k$. This shows that the proof of proposition 4.1 does not directly extend to the sets $\mathcal{O}$ and $\mathcal{O} \dot\cup \mathcal{U}$.

Proposition 5.2. *Let $f\colon X \to X$ be transitive SFT with positive topological entropy. Then, the set $\mathcal{U} \dot\cup \mathcal{V}$ is dense in $\mathrm{LC}(X, \mathbb{R})$ with respect to the supremum norm topology.*

Proof. Let $\phi \in \mathcal{O}$. We show that in every neighborhood of ϕ there exists $\phi' \in \mathcal{U} \dot\cup \mathcal{V}$. Since $\mathcal{O}$ is dense, the density of $\mathcal{U} \dot\cup \mathcal{V}$ follows.

Let $k \in \mathbb{N}$ be such that $\phi \in \mathcal{O}_k$, and let $x \in \mathrm{EPer}^k_{\max}(\phi)$ correspond to the unique maximal k-elementary periodic orbit for ϕ with period ℓ_x and generating segment τ_x. We now consider periodic points of the form $z_m = \mathcal{O}(\tau_x^m y)$, where τ_x^m denotes the m-times concatenation of τ_x and y is a segment of length ℓ_y. By transitivity and positive topological entropy of f we may assume that $y_i \neq x_i$ for some $i \in \{1, \ldots, \min\{\ell_x, \ell_y\}\}$.

In the following, we fix the segment y and vary $m \geqslant 2$. Let $\ell = \ell(m)$ be the smallest cylinder length so that z_m is ℓ-elementary periodic. We observe that $(m-1)\ell_x < \ell$ since the $(m-1)\ell_x$-cylinders starting at the first two copies of τ_x in z_m are identical. On the other hand, $\ell \leqslant m\ell_x + \ell_y$ is a consequence of the construction of z_m. We restrict our attention to cylinders of length ℓ throughout the remainder of this proof. The fact that x is a k-elementary periodic point with period ℓ_x implies that $|\mathcal{S}_\ell(x)| = \ell_x$, see equation (3). Moreover, since z_m is ℓ-elementary periodic with period $m\ell_x + \ell_y$, $|\mathcal{S}_\ell(z_m)| = m\ell_x + \ell_y$. We define potentials $\phi_{\varepsilon,\ell}$ as follows:

$$\phi_{\varepsilon,\ell}(w) = \begin{cases} \phi(w) + \varepsilon & \mathcal{C}_\ell(w) \in \mathcal{S}_\ell(z_m)\backslash\mathcal{S}_\ell(x) \\ \phi(w) & \text{otherwise} \end{cases}.$$

We observe that $\|\phi - \phi_{\varepsilon,\ell}\|_\infty = \varepsilon$. By construction, $\mu_x\left(\phi_{\varepsilon,\ell}\right) = \mu_x(\phi)$. On the other hand, since $|\mathcal{S}_\ell(z_m)\backslash\mathcal{S}_\ell(x)| \geqslant (m-1)\ell_x + \ell_y$, it follows that

$$\mu_{z_m}(\phi) + \frac{(m-1)\ell_x + \ell_y}{m\ell_x + \ell_y}\varepsilon \leqslant \mu_{z_m}\left(\phi_{\varepsilon,\ell}\right) \leqslant \mu_{z_m}(\phi) + \varepsilon.$$

Furthermore, since z_m begins with m copies of τ_x, for $0 \leqslant i < m\ell_x - k$, $\mathcal{C}_k(f^i(z_m)) \in \mathcal{S}_k(x)$. Let $m' = \left\lfloor m - \frac{\ell_y + k}{\ell_x} \right\rfloor$. Then, $z_m = \mathcal{O}(\tau_x^{m'}\tau_x^{m-m'}y)$ and

$$\mu_{z_m}(\phi) = \frac{m'\ell_x}{m\ell_x + \ell_y}\mu_x(\phi) + \frac{(m-m')\ell_x + \ell_y}{m\ell_x + \ell_y}\mu_{\mathcal{O}(\tau_x^{m-m'}y)}(\phi).$$

For fixed $\varepsilon > 0$, we observe that as m (and hence m') increases,

$$\mu_{z_m}(\phi_{\varepsilon,\ell}) \to \mu_{z_m}(\phi) + \varepsilon \quad \text{and} \quad \mu_{z_m}(\phi) \to \mu_x(\phi).$$

Therefore, for any fixed $\varepsilon > 0$, there exists an m so that $\mu_{z_m}(\phi_{\varepsilon,\ell}) > \mu_x(\phi_{\varepsilon,\ell}) = \mu_x(\phi)$. For the remainder of the proof, fix such ε, m and ℓ.

Finally, we consider the family of potentials $\phi_{t,\ell}$, where $0 \leqslant t \leqslant \varepsilon$. Let $t_0 = \sup\left\{t : x \in \mathrm{EPer}^\ell_{\max}\left(\phi_{t,\ell}\right)\right\}$. We observe that $t_0 > 0$ since $x \in \mathrm{EPer}^\ell_{\max}\left(\phi_{0,\ell}\right)$ and $\mathcal{O}_\ell$ is open. On the other hand, $t_0 < \varepsilon$ since $x \notin \mathrm{EPer}^\ell_{\max}\left(\phi_{\varepsilon,\ell}\right)$. At t_0, $\mathrm{EPer}^\ell_{\max}\left(\phi_{t_0,\ell}\right)$ must contain at least two elementary periodic orbits, x and some other orbit. Therefore, $\phi_{t_0,\ell} \notin \mathcal{O}$ and $\|\phi - \phi_{t_0,\ell}\|_\infty = t < \varepsilon$. Therefore, $\phi_{t_0,\ell} \in \mathcal{U} \,\dot{\cup}\, \mathcal{V}$, and, by allowing ε decreasing to zero, the conclusion follows. □

Acknowledgments

The authors thank the anonymous referees for many helpful suggestions, which greatly improved the paper.

ORCID iDs

Michael Burr https://orcid.org/0000-0001-8921-4870
Christian Wolf https://orcid.org/0000-0002-7976-3574

References

[1] Binder I, Braverman M, Rojas C and Yampolsky M 2011 Computability of Brolin–Lyubich measure *Commun. Math. Phys.* **308** 743–71

[2] Bissacot R, Garibaldi E and Thieullen P 2018 Zero-temperature phase diagram for double-well type potentials in the summable variation class *Ergod. Theor. Dynam. Syst.* **38** 863–85
[3] Brattka V, Hertling P and Weihrauch K 2008 A tutorial on computable analysis *New Computational Paradigms* (Berlin: Springer) pp 425–91
[4] Braverman M and Yampolsky M 2009 *Computability of Julia Sets* (*Algorithms and Computation in Mathematics* vol 23) (Berlin: Springer)
[5] Brémont J 2003 Gibbs measures at temperature zero *Nonlinearity* **16** 419–26
[6] Burr M, Schmoll M and Wolf C 2020 On the computability of rotation sets and their entropies *Ergod. Theor. Dynam. Syst.* **40** 367–401
[7] Chazottes J, Gambaudo J-M and Ugalde E 2011 Zero-temperature limit of one-dimensional Gibbs states via renormalization: the case of locally constant functions *Ergod. Theor. Dynam. Syst.* **31** 1109–61
[8] Chazottes J and Hochman M 2010 On the zero-temperature limit of Gibbs states *Commun. Math. Phys.* **297** 265–81
[9] Collet P 2015 On the complexity of some geometrical objects *Nonlinear Dynamics New Directions: Theoretical Aspects* ed H González-Aguilar and E Ugalde (Berlin: Springer) pp 29–45
[10] Contreras G 2016 Ground states are generically a periodic orbit *Invent Math.* **205** 383–412
[11] Contreras G, Lopes A O and Thieullen P 2001 Lyapunov minimizing measures for expanding maps of the circle *Ergod. Theor. Dynam. Syst.* **21** 1379–409
[12] Coronel D and Rivera-Letelier J 2015 Sensitive dependence of gibbs measures at low temperatures *J. Stat. Phys.* **160** 1658–83
[13] Galatolo S, Hoyrup M and Rojas C 2011 Dynamics and abstract computability: computing invariant measures *Discrete Continuous Dyn. Syst.* **29** 193–212
[14] Gangloff S, Herrera A, Rojas C and Sablik M 2020 Computability of topological entropy: from general systems to transformations on cantor sets and the interval *Discrete Continuous Dyn. Syst.-Ser.* A **40** 4259–86
[15] Hertling P and Spandl C 2008 Shifts with decidable language and non-computable entropy *Discrete Math. Theor. Comput. Sci.* **10** 75–93
[16] Hochman M and Meyerovitch T 2010 A characterization of the entropies of multidimensional shifts of finite type *Ann. Math.* **171** 2011–38
[17] Jenkinson O 2001 Rotation, entropy, and equilibrium states *Trans. Am. Math. Soc.* **353** 3713–39
[18] Jenkinson O 2019 Ergodic optimization in dynamical systems *Ergod. Theor. Dynam. Syst.* **39** 2593–618
[19] Jenkinson O and Pollicott M 2002 Calculating Hausdorff dimensions of Julia sets and Kleinian limit sets *Am. J. Math.* **124** 495–545
[20] Jenkinson O and Pollicott M 2004 Entropy, exponents and invariant densities for hyperbolic systems: dependence and computation *Modern Dynamical Systems and Applications* (Cambridge: Cambridge University Press) pp 365–84
[21] Kitchens B 1998 *Symbolic Dynamics: One-Sided, Two-Sided and Countable State Markov Shifts* (Berlin: Springer)
[22] Kucherenko T and Wolf C 2014 The geometry and entropy of rotation sets *Isr. J. Math.* **1999** 791–829
[23] Leplaideur R 2005 A dynamical proof for the convergence of gibbs measures at temperature zero *Nonlinearity* **18** 2847–80
[24] Morris I D 2008 Maximizing measures of generic Hölder functions have zero entropy *Nonlinearity* **21** 993–1000
[25] Parthasarathy K R 1961 On the category of ergodic measures *Illinois J. Math.* **5** 648–56
[26] Pavlov R 2014 Shifts of finite type with nearly full entropy *Proc. Lond. Math. Soc.* **108** 103–32
[27] Pavlov R and Schraudner M 2015 Entropies realizable by block gluing $\mathbb{Z}^d$ shifts of finite type *J. d'Analyse Math.* **126** 113–74
[28] Anthony Q and Siefken J 2012 Ergodic optimization of super-continuous functions on shift spaces *Ergod. Theor. Dynam. Syst.* **32** 2071–82
[29] Rettinger R and Weihrauch K 2002 The computational complexity of some Julia sets *Electron. Notes Theor. Comput. Sci.* **66** 154–64
[30] Tyrrell R 1970 *Convex Analysis* (Princeton, NJ: Princeton University Press)
[31] Rojas C and Yampolsky M 2017 Computable geometric complex analysis and complex dynamics *Technical Report* To appear in Handbook on Computability in Complex Analysis (arXiv:1703.06459 [math.CV])

[32] Shamos M I 1978 Computational geometry *PhD Thesis* Yale University

[33] Spandl C 2007 Computing the topological entropy of shifts *Electron. Notes Theor. Comput. Sci.* **167** 131–55

[34] Spandl C 2008 Computability of topological pressure for sofic shifts with applications in statistical physics *J. Univers. Comput. Sci.* **14** 876–95

[35] Walters P 1981 *An Introduction to Ergodic Theory* (*Graduate Texts in Mathematics* vol 79) (Berlin: Springer)

[36] Walters P 1992 Differentiability properties of the pressure of a continuous transformation on a compact metric space *J. Lond. Math. Soc.* **46** 471–81

[37] Weihrauch K 2000 *Computable Analysis: An Introduction Texts in Theoretical Computer Science* (*An EATCS Series*) (Berlin: Springer)

[38] Wolf C and Yang Y 2019 A topological classification of locally constant potentials via zero-temperature measures *Trans. Am. Math. Soc.* **372** 3113–40

[39] Yap C 2007 Is it really zero? *KIAS Magazine* vol 34

[40] Ziemian K 1995 Rotation sets for subshifts of finite type *Fundam. Math.* **146** 189–201