Maximum Number of Almost Similar Triangles in the Plane

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Abstract

A triangle T' is ε-similar to another triangle T if their angles pairwise differ by at most ε. Given a triangle T, $\varepsilon > 0$ and $n \in \mathbb{N}$, Bárány and Füredi asked to determine the maximum number of triangles $h(n, T, \varepsilon)$ being ε-similar to T in a planar point set of size n. We show that for almost all triangles T there exists $\varepsilon = \varepsilon(T) > 0$ such that $h(n, T, \varepsilon) = (1 + o(1))n^3/24$. Exploring connections to hypergraph Turán problems, we use flag algebras and stability techniques for the proof.

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1 Introduction

Let T, T' be triangles with angles $\alpha \le \beta \le \gamma$ and $\alpha' \le \beta' \le \gamma'$ respectively. The triangle T' is ε-*similar* to T if $|\alpha - \alpha'| < \varepsilon, |\beta - \beta'| < \varepsilon$, and $|\gamma - \gamma'| < \varepsilon$. Bárány and Füredi [5], motivated by Conway, Croft, Erdős and Guy [7], studied the maximum number $h(n, T, \varepsilon)$ of triangles in a planar set of n points that are ε-similar to a triangle T. For every T and $\varepsilon = \varepsilon(T) > 0$ sufficiently small, Bárány and Füredi [5] found the following lower bound construction: Place the n points in three groups with as equal sizes as possible, and each group very close to a (different) vertex of the triangle T. Now, iterate this by splitting each of the three groups into three further subgroups of
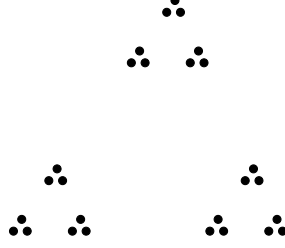
Figure 1: Sketch of the construction on 27 vertices.

points, see Figure 1 for an illustration of this construction. Define a sequence $h(n)$ by $h(0) = h(1) = h(2) = 0$ and for $n \geq 3$

$$h(n) := \max\{abc + h(a) + h(b) + h(c) : a + b + c = n, \ a, b, c \in \mathbb{N}\}.$$

By the above described construction, this sequence $h(n)$ is a lower bound on $h(n, T, \varepsilon)$. When T is an equilateral triangle then equality holds.

Theorem 1.1 (Bárány, Füredi [5])**.** *Let T be an equilateral triangle. Then there exists $\varepsilon_0 \geq 1°$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and all n we have $h(n, T, \varepsilon) = h(n)$. In particular, when n is a power of 3, $h(n, T, \varepsilon) = \frac{1}{24}(n^3 - n)$.*

Bárány and Füredi [5] also found various examples of triangles T (e.g. the isosceles right angled triangle) where $h(n, T, \varepsilon)$ is larger than $h(n)$.

The space of triangle shapes $S \subseteq \mathbb{R}^3$ can be represented with triples $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ of angles $\alpha, \beta, \gamma > 0$ with $\alpha + \beta + \gamma = \pi$. When we make statements about almost every triangle, we mean it in a measure theoretic sense, i.e. that there exists a set $S' \subseteq S$ with the 2-dimensional Lebesgue measure being 0 such that the statements holds for all triangles $T \in S \setminus S'$. In [5] it also was proved that $h(n, T, \varepsilon)$ can only be slightly larger than $h(n)$ for almost every triangle T.

Theorem 1.2 (Bárány, Füredi [5])**.** *For almost every triangle T there is an $\varepsilon > 0$ such that*

$$h(n, T, \varepsilon) \leq 0.25072 \binom{n}{3}(1 + o(1)).$$

The previously described construction of Bárány and Füredi gives a lower bound of $0.25\binom{n}{3}(1 + o(1))$. Bárány and Füredi [5] reduced the problem of determining $h(n, T, \varepsilon)$ to a hypergraph Turán problem and used the method of flag algebras, to get an upper bound on the corresponding Turán problem. Flag algebras is a powerful tool invented by Razborov [14], which has been used to solve problems in various different areas, including graph theory [10, 13], permutations [3, 15] and discrete geometry [4, 11]. An obstacle Bárány and Füredi [5] encountered is that the conjectured extremal example is an iterative construction and flag algebras tend to struggle with those. We will overcome this issue by using flag algebras only to prove a weak stability result and then use cleaning techniques to identify the recursive structure. Similar ideas have been used in [1] and [2]. This allows us to prove the asymptotic result and for large enough n an exact recursion.

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Theorem 1.3. *For almost every triangle T there is an $\varepsilon = \varepsilon(T) > 0$ such that*

$$h(n, T, \varepsilon) = \frac{1}{4}\binom{n}{3}(1 + o(1)). \tag{1}$$

Theorem 1.4. *There exists n_0 such that for all $n \geq n_0$ and for almost every triangle T there is an $\varepsilon = \varepsilon(T) > 0$ such that*

$$h(n, T, \varepsilon) = a \cdot b \cdot c + h(a, T, \varepsilon) + h(b, T, \varepsilon) + h(c, T, \varepsilon), \tag{2}$$

where $n = a + b + c$ and a, b, c are as equal as possible.

We will observe that Theorem 1.4 implies the exact result when n is a power of 3.

Corollary 1.5. *Let n be a power of 3. Then, for almost every triangle T there is an $\varepsilon = \varepsilon(T) > 0$ such that*

$$h(n, T, \varepsilon) = \frac{1}{24}(n^3 - n).$$

The paper is organized as follows. In Section 2 we introduce terminology and notation that we use, we establish a connection from maximizing the number of similar triangles to Turán problems; and we apply flag algebras in our setting to derive a weak stability result. In Section 3 we apply cleaning techniques to refine the stability result and derive our main result, Theorem 1.3. In Section 4 we prove our exact results, Theorem 1.4 and Corollary 1.5. Finally, in Section 5 we discuss further questions.

2 Preparation

2.1 Terminology and Notation

Definition 2.1. Let G be a 3-uniform hypergraph (shortly a 3-graph), $\mathcal{H}$ be a family of 3-graphs, $v \in V(G)$ and $A, B \subseteq V(G)$. Then,

- G is $\mathcal{H}$-*free*, if it does not contain a copy of any $H \in \mathcal{H}$,

- a 3-graph G on n vertices is *extremal* with respect to $\mathcal{H}$, if G is $\mathcal{H}$-free and $e(G') \leq e(G)$ for every $\mathcal{H}$-free 3-graph G' on n vertices. If $\mathcal{H}$ is clear from context, we only say G is extremal,

- for $a, b \in V(G)$, denote $N(a, b)$ the *neighborhood* of a and b, i.e. the set of vertices $c \in V(G)$ such that $abc \in E(G)$,

- we write $L(v)$ for the *linkgraph* of v, that is the graph G' with $V(G') = V(G) \setminus \{v\}$ and $E(G')$ being the set of all pairs a, b with $abv \in E(G)$,

- we write $L_A(v)$ for the linkgraph of v on A, that is the graph G' with $V(G') = A \setminus \{v\}$ and $E(G')$ being the set of all pairs $a, b \subseteq A \setminus \{v\}$ with $abv \in E(G)$,

- we write $L_{A,B}(v)$ for the (bipartite) linkgraph of v on $A \cup B$, that is the graph G' with $V(G') = A \cup B \setminus \{v\}$ and $E(G')$ being the set of all pairs a, b with $a \in A, b \in B$ and $abv \in E(G)$,

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- we denote by $|L(v)|, |L_A(v)|$ and $|L_{A,B}(v)|$ the number of edges of the linkgraphs $L(v), L_A(v)$ and $L_{A,B}(v)$ respectively.

Define a 3-graph $S(n)$ on n vertices recursively. For $n = 1, 2$, let $S(n)$ be the 3-graph on n vertices with no edges. For $n \geq 3$, choose integers $a \geq b \geq c$ as equal as possible such that $n = a + b + c$. Then, define $S(n)$ to be the 3-graph constructed by taking vertex disjoint copies of $S(a), S(b)$ and $S(c)$ and adding edges with each of the 3 vertices coming from a different copy. Bárány and Füredi [5] observed that $|S(n)| \geq \frac{1}{24}n^3 - O(n \log n)$.

Given a set $B \subseteq \mathbb{C}$ and $\delta > 0$, we call the set $U_\delta(B) := \{z : |z - b| < \delta$ for some $b \in B\}$ the δ-*neighborhood* of B. If $B = \{b\}$ for some $b \in \mathbb{C}$, abusing notation, we write $U_\delta(b)$ for it.

2.2 Forbidden subgraphs

Given a finite point set $P \subseteq \mathbb{R}^2$ in the plane, a triangle $T \in S$ and an $\varepsilon > 0$, we denote by $G(P, T, \varepsilon)$ the 3-graph with vertex set $V(G(P, T, \varepsilon)) = P$ and triples abc being an edge in $G(P, T, \varepsilon)$ iff abc forms a triangle ε-similar to T. A 3-graph H is *forbidden* if $|V(H)| \leq 12$ and for almost every triangle shape $T \in S$ there exists an $\varepsilon = \varepsilon(T) > 0$ such that for every point set $P \subseteq \mathbb{R}^2$, $G(P, T, \varepsilon)$ is H-free. Denote $\mathcal{F}$ the family of all forbidden 3-graphs and $\mathcal{T}_\mathcal{F} \subseteq S$ the set of all triangles T such that there exists $\varepsilon = \varepsilon(T) > 0$ such that for every point set $P \subseteq \mathbb{R}^2$, $G(P, T, \varepsilon)$ is $\mathcal{F}$-free. Given $T \in \mathcal{T}_\mathcal{F}$, we let $\varepsilon(T) > 0$ be a real number such that for every point set $P \subseteq \mathbb{R}^2$, $G(P, T, \varepsilon(T))$ is $\mathcal{F}$-free.

In our definition of forbidden 3-graphs we restrict the size to be at most 12. The reason we choose the number 12 is that the largest forbidden subgraph we need for our proof has size 12 and we try to keep the family $\mathcal{F}$ as small as possible.

We will prove Theorem 1.3, Theorem 1.4 and Corollary 1.5 for all triangles $T \in \mathcal{T}_\mathcal{F}$. Note that by the definition of $\mathcal{F}$, almost all triangles are in $\mathcal{T}_\mathcal{F}$. Bárány and Füredi [5] determined the following hypergraphs to be members of $\mathcal{F}$.

Lemma 2.2 (Bárány and Füredi [5], see Lemma 11.2)**.** *The following hypergraphs are members of $\mathcal{F}$.*

- $K_4^- = \{123, 124, 134\}$
- $C_5^- = \{123, 124, 135, 245\}$
- $C_5^+ = \{126, 236, 346, 456, 516\}$
- $L_2 = \{123, 124, 125, 136, 456\}$
- $L_3 = \{123, 124, 135, 256, 346\}$

- $L_4 = \{123, 124, 156, 256, 345\}$
- $L_5 = \{123, 124, 135, 146, 356\}$
- $L_6 = \{123, 124, 145, 346, 356\}$
- $P_7^- = \{123, 145, 167, 246, 257, 347\}$.

For the non-computer assisted part our proof, we will need to extend this list. For the computer assisted part, we excluded additional graphs on 7 and 8 vertices.

Lemma 2.3. *The following hypergraphs are members of $\mathcal{F}$.*

- $L_7 = \{123, 124, 125, 136, 137, 458, 678\}$

- $L_8 = \{123, 124, 125, 136, 137, 468, 579, 289\}$
- $L_9 = \{123, 124, 125, 136, 237, 469, 578, 189\}$
- $L_{10} = \{123, 124, 125, 126, 137, 138, 239, 58a, 47b, 69c, abc\}$.

Note that this is not the complete list. To verify that those hypergraphs are forbidden, we will we use the same method as Bárány and Füredi [5] used to show that the hypergraphs from Lemma 2.2 are forbidden. For sake of completeness, we repeat their argument here.

Proof. We call a 3-graph H on r vertices *dense* if there exists a vertex ordering $v_1, v_2, \ldots, v_r$ such that for every $3 \leq i \leq r - 1$ there exists exactly one edge $e_i \in E(H[\{v_1, \ldots, v_i\}])$ containing v_i, and there exists exactly two edges e_r, e_r' containing v_r. Note that L_7, L_8, L_9 and L_{10} are dense.

For convenience, we will work with a different representation of triangles shapes. A triangle shape $T \in S$ is characterized by a complex number $z \in \mathbb{C} \setminus \mathbb{R}$ such that the triangle with vertices $0, 1, z$ is similar to T. Note that there are at most twelve complex numbers w such that the triangle $\{0, 1, w\}$ is similar to T.

Let H be a dense hypergraph on r vertices with vertex ordering $v_1, \ldots, v_r$ and let $P = \{p_1, \ldots, p_r\} \subseteq \mathbb{R}^2$ be a point set such that $G(P, T, \varepsilon)$ contains H (with p_i corresponding to v_i), where ε is small enough such that the following argument holds. Let $\delta > 0$ be sufficiently small. Without loss of generality, we can assume that $p_1 = (0, 0)$ and $p_2 = (1, 0)$. Now, since H is dense, $v_1 v_2 v_3 \in E(H)$ and therefore $p_1 p_2 p_3$ forms a triangle ε-similar to T. Therefore, there exists at most 12 points (which are functions in z) such that p_3 is in a δ-neighborhood of one of them. Since, v_4 is contained in some edge with vertices from $\{v_1, v_2, v_3, v_4\}$, there are at most $12 \cdot 12 = 144$ points (which are functions in z) such that q_4 is in a δ-neighborhood of one of them. Continuing this argument, we find functions $f_{i,j}(z)$ in z where $3 \leq i \leq r - 1$ and $j \leq 12^{r-3}$ such that

$$(p_3, p_4, \ldots, p_r) \in U_\delta(f_{3,j}(z)) \times U_\delta(f_{4,j}(z)) \times \ldots \times U_\delta(f_{r-1,j}(z))$$

for some $j \leq 12^{r-3}$. Since H is dense, v_r is contained in exactly two edges e_r and e_r'. For each $j \leq 12^{r-3}$, because $v_k \in e_r$, there exists at most 12 points $f_{r,j,\ell}(z)$ where $\ell \leq 12$ such that

$$p_k \in U_\delta\left(f_{r,j,\ell}(z)\right).$$

Similarly, because $v_k \in e_r'$, there exists at most 12 points $g_{r,j,\ell}(z)$ where $\ell' \leq 12$ such that

$$p_k \in U_\delta\left(g_{r,j,\ell'}(z)\right).$$

Thus,

$$p_k \in \bigcup_{\ell,\ell' \leq 12} U_\delta\left(f_{r,j,\ell}(z)\right) \cap U_\delta\left(g_{r,j,\ell'}(z)\right). \tag{3}$$

Note that if there exists a z such that for each $1 \leq j \leq 12^{r-3}$ none of the equations

$$f_{r,j,\ell}(z) = g_{r,j,\ell'}(z), \qquad 1 \leq \ell, \ell' \leq 12 \tag{4}$$

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holds, then we can choose $\varepsilon > 0$ such that

$$\delta < \frac{1}{3} \max_{\ell, \ell'} |f_{r,j,\ell}(z) - g_{r,j,\ell'}(z)|, \tag{5}$$

and therefore the set in (3) is empty, contradicting that $G(P, T, \varepsilon)$ contains a copy of H. Note that, because of (5), ε depends on z and therefore on T. If we could find one $z \in \mathbb{C}$ not satisfying any of the equations in (4), then each of the equations is non-trivial (the solution space is not $\mathbb{C}$). Thus, for each equation the solution set has Lebesgue measure 0. Since there are only at most 12^{r-2} equations, the union of the solution sets still has measure 0. Thus, we can conclude that for almost all triangles T there exists ε such that $G(P, T, \varepsilon)$ is H-free for every point set P.

It remains to show that for $H \in \{L_7, L_8, L_9, L_{10}\}$ there exists $z \in \mathbb{C}$ not satisfying any of the equations in (4). We will show this for a z corresponding to the equilateral triangle ($z = \frac{1}{2} + i \cdot \frac{\sqrt{3}}{2}$). For T being the equilateral triangle, there are at most 2^{r-2} equations to check. Because of the large amount of cases, we will use a computer to verify it.

Our computer program is a simple brute force recursive approach. It starts by embedding $p_1 = (0,0)$ and $p_2 = (1,0)$. For each subsequent $3 \le i \le r$ it tries both options for embedding p_i dictated by e_i. Finally, it checks if the points forming e'_r form an equilateral triangle. If in none of the 2^{r-2} generated point configurations the points of e'_r form an equilateral triangle, then H is a member of $\mathcal{F}$. An implementation of this algorithm in python is available at `http://lidicky.name/pub/triangle`. This completes the proof of Lemma 2.3. ∎

Instead of Theorem 1.3, we will actually prove the following stronger result.

Theorem 2.4. *We have*

$$\mathrm{ex}(n, \mathcal{F}) = 0.25 \binom{n}{3}(1 + o(1)).$$

First, we observe that Theorem 2.4 implies Theorem 1.3. Let $P \subseteq \mathbb{R}^2$ be a point set of size n and let $T \in \mathcal{T}_{\mathcal{F}}$. Then, $G(P, T, \varepsilon(T))$ is $\mathcal{F}$-free. Now, the number of ε-similar triangles T equals the number of edges in $G(P, T, \varepsilon(T))$. Since $G(P, T, \varepsilon(T))$ is $\mathcal{F}$-free, we have

$$h(n, T, \varepsilon) \le \mathrm{ex}(n, \mathcal{F}).$$

Therefore, Theorem 2.4 implies Theorem 1.3.

2.3   A structural result via Flag Algebras

It is a standard application of flag algebras to determine an upper bound for $\mathrm{ex}(n, \mathcal{G})$ given a family $\mathcal{G}$ of 3-uniform hypergraphs. Running the method of flag algebras on 7 vertices, Bárány and Füredi [5] obtained

$$\mathrm{ex}(n, \mathcal{F}) \le \mathrm{ex}(n, \{K_4^-, C_5^-, C_5^+, L_2, L_3, L_4, L_5, L_6, P_7^-\}) \le 0.25072 \binom{n}{3}(1 + o(1)). \tag{6}$$

We note that when running flag algebras on 8 vertices and forbidding more 3-graphs in $\mathcal{F}$, then we can obtain the following improved bound.

$$\mathrm{ex}(n, \mathcal{F}) \leq 0.2502 \binom{n}{3}(1 + o(1)). \tag{7}$$

It is conjectured in [9] that $\mathrm{ex}(n, \{K_4^-, C_5\}) = 0.25\binom{n}{3}(1 + o(1))$. Note that this conjecture is a significant strengthening of (6) and (7).

We use flag algebras to prove a stability result. For an excellent explanation of flag algebras in the setting of 3-graphs see [9]. Here, we will focus on the formulation of the problem rather than providing a formal explanation of the general method. As a consequence, we obtain the following lemma, which gives the first rough structure of extremal constructions. This approach was developed in [1] and [2].

Lemma 2.5. *Let $n \in \mathbb{N}$ be sufficiently large and let G be an $\mathcal{F}$-free 3-graph on n vertices and $|E(G)| \geq 1/24n^3(1 + o(1))$ edges. Then there exists an edge $x_1 x_2 x_3 \in E(G)$ such that for n large enough*

(i) *the neighborhoods $N(x_1, x_2), N(x_2, x_3)$, and $N(x_1, x_3)$ are pairwise disjoint.*

(ii) *$\min\{|N(x_1, x_2)|, |N(x_2, x_3)|, |N(x_1, x_3)|\} \geq 0.26n$.*

(iii) *$n - |N(x_1, x_2)| - |N(x_2, x_3)| - |N(x_1, x_3)| \leq 0.012n$.*

Proof. Property (i) is in fact true for any edge $e = x_1 x_2 x_3 \in E(G)$. Indeed, without loss of generality, assume $N(x_1, x_2) \cap N(x_1, x_3) \neq \emptyset$. Let $v \in N(x_1, x_2) \cap N(x_1, x_3)$. Then v, x_1, x_2, x_3 spans at least 3 edges and therefore G contains a copy of K_4^-, a contradiction.

Denote $T_{i,j,k}$ the family of 3-graphs that are obtained from a complete 3-partite 3-graph with part sizes i, j and k by adding $\mathcal{F}$-free 3-graphs in each of the three parts. Let X be a subgraph of G isomorphic to $T_{2,2,1}$ on vertices $x_1, x_1', x_2, x_2', x_3$ with edges $x_1 x_2 x_3, x_1 x_2' x_3, x_1' x_2 x_3, x_1' x_2' x_3$. Further, define

$$B_1 := N(x_2, x_3) \cap N(x_2', x_3), \quad B_3 := N(x_1, x_2) \cap N(x_1', x_2) \cap N(x_1, x_2') \cap N(x_1', x_2'),$$
$$B_2 := N(x_1, x_3) \cap N(x_1', x_3), \quad J := V(G) \setminus (B_1 \cup B_2 \cup B_3).$$

Let $b_i := |B_i|/n$ for $1 \leq i \leq 3$. Note that $V(G) = B_1 \cup B_2 \cup B_3 \cup J$ is a partition because of property (i).

We choose X such that

$$b_1 b_2 + b_1 b_3 + b_2 b_3 - \frac{1}{4}\left(b_1^2 + b_2^2 + b_3^2\right) \tag{8}$$

is maximized. Flag algebras can be used to give a lower bound on the expected value of (8) for X chosen uniformly at random and therefore also a lower bound on (8) when X is chosen to maximize (8).

Let Z be a fixed *labeled* subgraph of G belonging to $T_{i', j', k'}$. Denote by $T_{i,j,k}(Z)$ the family of subgraphs of G that contain Z, belong to $T_{i,j,k}$, where $i' \leq i$, $j' \leq j$, and

$k' \leq k$, and the natural three parts of Z are mapped to the same 3 parts in $T_{i,j,k}(Z)$. The normalized number of $T_{i,j,k}(Z)$ is

$$t_{i,j,k}(Z) := \frac{|T_{i,j,k}(Z)|}{\binom{n-|V(Z)|}{i+j+k-|V(Z)|}}.$$

The subgraphs of G isomorphic to $T_{i,j,k}$ are denoted by $T_{i,j,k}(\emptyset)$. The normalized number is

$$t_{i,j,k} := \frac{|T_{i,j,k}(\emptyset)|}{\binom{n}{i+j+k}}.$$

Notice that $b_1 = t_{3,2,1}(X) + o(1)$, $2b_1 b_2 = t_{3,3,1}(X) + o(1)$, and $b_1^2 = t_{4,3,1}(X) + o(1)$. We start with (8) and obtain the following.

$$\left(b_1 b_2 + b_1 b_3 + b_2 b_3 - \frac{1}{4}\left(b_1^2 + b_2^2 + b_3^2 \right) \right) n^2$$

$$= \left(2 b_1 b_2 + 2 b_1 b_3 + 2 b_2 b_3 - \frac{1}{2}\left(b_1^2 + b_2^2 + b_3^2 \right) \right) \binom{n-5}{2} + o(n^2)$$

$$= \left(t_{3,3,1}(X) + t_{3,2,2}(X) + t_{2,3,2}(X) - \frac{1}{2}\left(t_{4,2,1}(X) + t_{2,4,1}(X) + t_{2,2,3}(X) \right) \right) \binom{n-5}{2}$$

$$+ o(n^2)$$

$$\geq \frac{1}{t_{2,2,1}\binom{n}{5}} \left(\sum_{Y \in T_{2,2,1}(\emptyset)} \left(t_{3,3,1}(Y) + t_{3,2,2}(Y) + t_{2,3,2}(Y) \right.\right.$$

$$\left.\left. - \frac{1}{2}\left(t_{4,2,1}(Y) + t_{2,4,1}(Y) + t_{2,2,3}(Y) \right) \right) \binom{n-5}{2} + o(n^2) \right.$$

$$\geq \frac{1}{t_{2,2,1}\binom{n}{5}} \left(9\, t_{3,3,1} + 12\, t_{3,2,2} - \frac{1}{2}\left(6\, t_{4,2,1} + 3\, t_{2,2,3} \right) \right) \binom{n}{7} + o(n^2)$$

$$= \frac{1}{7\, t_{2,2,1}}\left(3\, t_{3,3,1} + 3.5\, t_{3,2,2} - t_{4,2,1} \right) \binom{n-5}{2} + o(n^2).$$

Claim 2.6. *Using flag algebras, we get that if $t_{1,1,1} \geq 0.25$ then*

$$\frac{1}{7\, t_{2,2,1}}\left(3\, t_{3,3,1} + 3.5\, t_{3,2,2} - t_{4,2,1} \right) \geq \frac{1.2814228}{7 \cdot 0.37502377} > 0.48813.$$

The calculations for Claim 2.6 are computer assisted; we use CSDP [6] to calculate numerical solutions of semidefinite programs. The data files and programs for the calculations are available at `http://lidicky.name/pub/triangle`. Claim 2.6 gives a lower bound on (8) as follows

$$b_1 b_2 + b_1 b_3 + b_2 b_3 - \frac{1}{4}\left(b_1^2 + b_2^2 + b_3^2 \right) \geq \frac{1.2814228}{14 \cdot 0.37502377} > 0.24406. \tag{9}$$

Notice that if $b_1 = b_2 = b_3 = \frac{1}{3}$, then (8), which is the left hand side of (9), is 0.25. The conclusions (ii) and (iii) of Lemma 2.5 can be obtained from (9). Indeed, assume

$b_1 < 0.26$. Then,

$$b_1 b_2 + b_1 b_3 + b_2 b_3 - \frac{1}{4}\left(b_1^2 + b_2^2 + b_3^2\right)$$

$$\leq b_1(1 - b_1) + \left(\frac{1 - b_1}{2}\right)^2 - \frac{1}{4}\left(b_1^2 + 2\left(\frac{1 - b_1}{2}\right)^2\right)$$

$$= -\frac{9}{8}b_1^2 + \frac{3}{4}b_1 + \frac{1}{8} < -\frac{9}{8}0.26^2 + \frac{3}{4}0.26 + \frac{1}{8} = 0.24325,$$

contradicting (9). Thus, we have $b_1 \geq 0.26$, concluding (ii). Next, assume $b_1 + b_2 + b_3 \leq 0.988$. Then,

$$b_1 b_2 + b_1 b_3 + b_2 b_3 - \frac{1}{4}\left(b_1^2 + b_2^2 + b_3^2\right)$$

$$\leq b_1(0.988 - b_1) + \left(\frac{0.988 - b_1}{2}\right)^2 - \frac{1}{4}\left(b_1^2 + 2\left(\frac{0.988 - b_1}{2}\right)^2\right)$$

$$= -\frac{9}{8}b_1^2 + 0.741 b_1 + 0.122018 \leq \frac{61009}{250000} < 0.244037,$$

where in the last step we used that the maximum is obtained at $b_1 = 247/750$. This contradicts (9). Thus, we have $b_1 + b_2 + b_3 \geq 0.988$, concluding (iii). ∎

In the proof of Lemma 2.5, we chose a suitable copy of $T_{2,2,1}$ to find the initial 3-partition. One could do the same approach by starting with base $T_{1,1,1}$ instead. However, the resulting bounds would be weaker and not sufficient for the rest of the proof. This is caused by obtaining a lower bound on (8) by taking a random base.

3   Proof of Theorem 2.4

In the first subsection of this section, we apply cleaning techniques to prove refinements, namely Lemmas 3.1 and 3.2, of our flag algebra result Lemma 2.5. Having those structural results at hand, we prove Theorem 2.4 in Subsection 3.2. Recall that the hypergraph Turán result, Theorem 2.4, implied our main result, Theorem 1.3.

3.1   Refinements of the stability result

Lemma 3.1. *Let G be an $\mathcal{F}$-free 3-graph on n vertices and $|E(G)| \geq 1/24 n^3(1+o(1))$, satisfying $|L(w)| \geq \frac{1}{8}n^2(1 + o(1))$ for every $w \in V(G)$. Then there exists an edge $x_1 x_2 x_3 \in E(G)$ such that for*

$$A_1 := N(x_2, x_3), \quad A_2 := N(x_1, x_3), \quad A_3 := N(x_1, x_2), \quad J := V(G) \setminus (A_1 \cup A_2 \cup A_3),$$
$$A_1' := A_1 \setminus \{x_1\}, \quad A_2' := A_2 \setminus \{x_2\}, \quad and \quad A_3' := A_3 \setminus \{x_3\}$$

we have for n sufficiently large

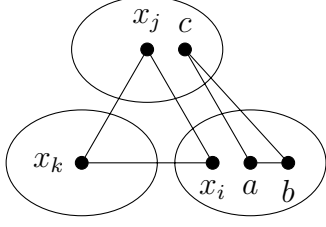*(a) $0.26n \leq |A_i| \leq 0.48n$ for $i \in [3]$.*
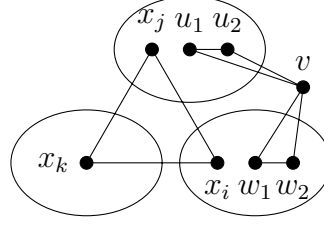
(b) $|J| \leq 0.012n$.

Figure 2: Situation for (c).

Figure 3: Situation for (d).

(c) *No triple abc with $a, b \in A'_i$ and $c \in A'_j$ for some $i, j \in [3], i \neq j$ forms an edge.*

(d) *For $v \in V(G) \setminus \{x_1, x_2, x_3\}$, $w_1, w_2 \in A'_i$, $u_1, u_2 \in A'_j$ with $i, j \in [3]$ and $i \neq j$ we have $vw_1w_2 \notin E(G)$ or $vu_1u_2 \notin E(G)$.*

(e) *For every $v \in V(G) \setminus \{x_1, x_2, x_3\}$, there exists $i \in [3]$ such that $|L_{A_j, A_k}(v)| \geq 0.001n^2$, where $j, k \in [3], j \neq k, j \neq i, k \neq i$.*

Proof. Apply Lemma 2.5 and get an edge $x_1x_2x_3$ with the properties from Lemma 2.5. The sets A_1, A_2, A_3 are pairwise disjoint.

Note that (a) and (b) hold by Lemma 2.5. To prove (c), assume that there exists $abc \in E(G)$ with $a, b \in A'_i$ and $c \in A'_j$ for some $i, j \in [3], i \neq j$. Let $k \in [3], k \neq i, k \neq j$, see Figure 2 for an illustration. Now,

$$x_ix_jx_k, abc, x_jx_ka, x_jx_kb, cx_ix_k \in E(G).$$

Therefore G contains a copy of L_2 on $\{x_1, x_2, x_3, a, b, c\}$. Since $L_2 \in \mathcal{F}$, by Lemma 2.2 this contradicts that G is $\mathcal{F}$-free.

To prove (d) by way of contradiction, assume that there exist $v \in V(G) \setminus \{x_1, x_2, x_3\}, w_1, w_2 \in A'_i, u_1, u_2 \in A'_j$ for $i, j \in [3]$ with $i \neq j$ such that $vw_1w_2 \in E(G)$ and $vu_1u_2 \in E(G)$. Let $k \in [3], k \neq i, k \neq j$, see Figure 3 for an illustration. Now, $\{x_1, x_2, x_3, v, u_1, u_2, w_1, w_2\}$ spans a copy of L_7 because

$$x_ix_jx_k, vw_1w_2, vu_1u_2, x_jx_kw_1, x_jx_kw_2, x_ix_ku_1, x_ix_ku_2 \in E(G).$$

However, $L_7 \in \mathcal{F}$ by Lemma 2.3, contradicting that G is $\mathcal{F}$-free.

We finally prove (e). Let $v \in V(G) \setminus \{x_1, x_2, x_3\}$. Towards contradiction, assume

$$|L_{A_1, A_2}(v)| < 0.001n^2 \quad \text{and} \quad |L_{A_1, A_3}(v)| < 0.001n^2 \quad \text{and} \quad |L_{A_2, A_3}(v)| < 0.001n^2.$$

By property (d), there exists $i \in [3]$ such that $|L_{A'_j}(v)| = |L_{A'_k}(v)| = 0$ for $j, k \in [3] \setminus \{i\}$ with $j \neq k$. Note, that $|L_{A_i}(v)| \leq |A_i|^2/4$, since $L_{A_i}(v)$ is triangle-free, because otherwise there would be a copy of K_4^- in G. We have

$$\begin{aligned}
|L(v)| &\leq |J| \cdot n + |L_{A_1, A_2}(v)| + |L_{A_2, A_3}(v)| + |L_{A_1, A_3}(v)| \\
&\quad + |L_{A_1}(v)| + |L_{A_2}(v)| + |L_{A_3}(v)| \\
&\leq |J| \cdot n + 0.003n^2 + \frac{|A_i|^2}{4} + 2n \leq 0.012n^2 + 0.003n^2 + 0.06n^2 + 2n \\
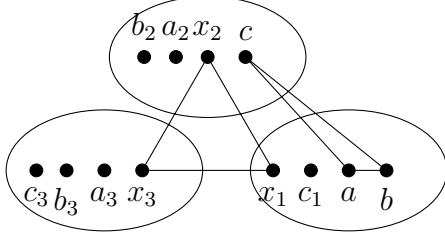&< 0.08n^2(1 + o(1)),
\end{aligned}$$

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Figure 4: Case 1.

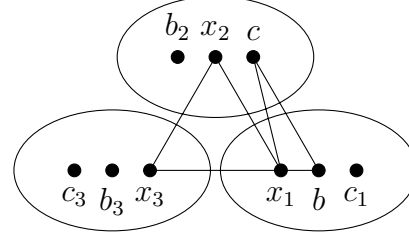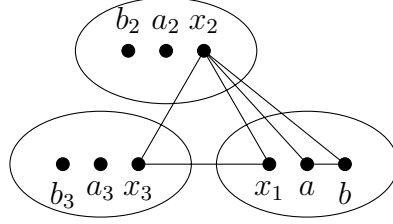Figure 5: Case 2.

Figure 6: Case 4.

contradicting the assumption $|L(v)| \geq \frac{1}{8}n^2(1 + o(1))$. Note that we used $|A_i| \leq 0.48n$ and $|J| \leq 0.012n$ from properties (a) and (b). This completes the proof of Lemma 3.1. ∎

The following lemma is a refinement of Lemma 3.1. It gives a clean structure on the whole vertex set instead of only 98.8% of the vertices by assigning each of the vertices of J to one of the three parts.

Lemma 3.2. *Let $n \in \mathbb{N}$ be sufficiently large and let G be an $\mathcal{F}$-free 3-graph on n vertices and $|E(G)| \geq 1/24n^3(1 + o(1))$, satisfying $|L(w)| \geq \frac{1}{8}n^2(1 + o(1))$ for every $w \in V(G)$. Then there exists a vertex partition $V(G) = X_1 \cup X_2 \cup X_3$ with $|X_i| \geq 0.26n$ for $i \in [3]$ such that no triple abc with $a, b \in X_i$ and $c \in X_j$ for some $i, j \in [3]$ with $i \neq j$ forms an edge.*

Proof. Let $x_1x_2x_3 \in E(G)$ be an edge with the properties from Lemma 3.1. By property (e) we can partition $J = J_1 \cup J_2 \cup J_3$ such that for every $v \in J_i$ we have $|L_{A_j,A_k}(v)| \geq 0.001n^2$, where $j, k \in [3], j \neq k, j \neq i, k \neq i$. Set $X_i := A_i \cup J_i$. Note that by properties (c) and (e) for every $v \in X_i \setminus \{x_i\}$ we have $|L_{A_j,A_k}(v)| \geq 0.001n^2$, where $j, k \in [3], j \neq k, j \neq i, k \neq i$. Further, by property (a) and definition of X_i we have $|X_i| \geq 0.26n$ for n large enough.

Towards contradiction, assume that there exists $a, b \in X_1$ and $c \in X_2$ with $abc \in E(G)$. For each a, b, c we find their neighbors in $A_1 \cup A_2 \cup A_3$ that put them to J_1 and J_2. These neighbors are in $A_1 \cup A_2 \cup A_3$ because they were adjacent to some of x_1, x_2, x_3. This will eventually form one of the forbidden subgraphs. We will distinguish cases depending on how a, b, c coincide with x_1, x_2, x_3.

Case 1: $a, b \neq x_1$ and $c \neq x_2$.
Since

$$|L_{A_2,A_3}(a)| \geq 0.001n^2, \quad |L_{A_2,A_3}(b)| \geq 0.001n^2 \quad \text{and} \quad |L_{A_a,A_3}(c)| \geq 0.001n^2,$$

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there exists distinct vertices $a_3, b_3, c_3 \in A_3, a_2, b_2 \in A_2 \setminus \{c\}$ and $c_1 \in A_1 \setminus \{a, b\}$ such that $aa_2a_3, bb_2b_3, cc_1c_3 \in E(G)$, see Figure 4 for an illustration. We have

$$x_1x_2x_3, abc, aa_2a_3, bb_2b_3, cc_1c_3, c_1x_2x_3, a_2x_1x_3, b_2x_1x_3, c_3x_1x_2, b_3x_1x_2, a_3x_1x_2 \in E(G),$$

and therefore the vertices $\{x_1, x_2, x_3, a, b, c, c_1, a_2, b_2, a_3, b_3, c_3\}$ span a copy of L_{10}, a contradiction.

Case 2: $a = x_1$ and $c \neq x_2$.

By property (d), there exists distinct vertices $b_3, c_3 \in A_3, b_2 \in A_2 \setminus \{c\}$ and $c_1 \in A_1 \setminus \{a, b\}$ such that $bb_2b_3, cc_1c_3 \in E(G)$, see Figure 5 for an illustration. We have

$$x_1x_2x_3, x_1bc, bb_2b_3, cc_1c_3, c_1x_2x_3, b_2x_1x_3, c_3x_1x_2, b_3x_1x_2 \in E(G),$$

and therefore the vertices $\{x_1, x_2, x_3, b, c, c_1, b_2, b_3, c_3\}$ span a copy of L_9, a contradiction.

Case 3: $b = x_1$ and $c \neq x_2$.

This case is similar to Case 2, we omit the proof.

Case 4: $a, b \neq x_1$ and $c = x_2$.

By property (d), there exists distinct vertices $a_3, b_3 \in A_3, a_2, b_2 \in A_2 \setminus \{c\}$ such that $aa_2a_3, bb_2b_3 \in E(G)$, see Figure 6 for an illustration. We have

$$x_1x_2x_3, abx_2, aa_2a_3, bb_2b_3, a_2x_1x_3, b_2x_1x_3, b_3x_1x_2, a_3x_1x_2 \in E(G),$$

and therefore the vertices $\{x_1, x_2, x_3, a, b, a_2, b_2, a_3, b_3\}$ span a copy of L_8, a contradiction.

Case 5: $a = x_1$ and $c = x_2$.

This means that $b \in N(x_1, x_2) = A_3$, contradicting $b \in X_1$.

Case 6: $b = x_1$ and $c = x_2$.

This case is similar to Case 5, we omit the proof.

We conclude that for $a, b \in X_1, c \in X_3$, we have $abc \notin E(G)$. Similarly, for $a, b \in X_i, c \in X_j$ with $i \neq j$, we have $abc \notin E(G)$.

∎

3.2 The asymptotic result

In this subsection we will prove Theorem 2.4. We first observe that an extremal $\mathcal{F}$-free 3-graph satisfies a minimum degree condition.

Lemma 3.3. *Let G be an $\mathcal{F}$-free 3-graph and $u, v \in V(G)$. Denote $G_{u,v}$ the 3-graph constructed from G by adding a copy w of v and deleting u, i.e.*

$$V(G_{u,v}) = V(G) \cup \{w\} \setminus \{u\}, \quad E(G_{u,v}) = E(G[V(G) \setminus \{u\}]) \cup \{wab \mid abv \in E(G)\}.$$

Then, $G_{u,v}$ is also $\mathcal{F}$-free.

Proof. Towards contradiction assume that $G_{u,v}$ contains a copy of some $F \in \mathcal{F}$. Since G is $\mathcal{F}$-free, this copy F' of F contains the vertices v and w. $F' - w$ is a subgraph of G and thus $\mathcal{F}$-free, in particular $F' - w \notin \mathcal{F}$. Thus, there exists a set of triangle

shape $\mathcal{T}$ of positive measure such that for $T \in \mathcal{T}$ and $\varepsilon > 0$ there exists a point set $P = P(T, \varepsilon) \subseteq \mathbb{R}^2$ with $F' - w$ being isomorphic to $G(P, T, \varepsilon)$. Construct a new point set P' from $P(T, \varepsilon)$ by adding a new point p_w very close to the point corresponding to v, so that v and p_w have the same linkgraph in $G(P', T, \varepsilon)$ and that there is no edge in $G(P', T, \varepsilon)$ containing both p_w and v. Now, $G(P', T, \varepsilon)$ contains a copy of F, contradicting that $F \in \mathcal{F}$.

∎

Lemma 3.4. *Let G be an extremal $\mathcal{F}$-free 3-graph on n vertices. Then for every $u \in V(G)$, we have $|L(u)| \geq \frac{1}{8}n^2(1 + o(1))$.*

Proof. Assume that there exists $u \in V(G)$ with $|L(u)| < \frac{1}{8}n^2 - n^{3/2}$ for n sufficiently large. Let $v \in V(G)$ be a vertex maximizing $|L(v)|$. The 3-graph $G_{u,v}$ is $\mathcal{F}$-free by Lemma 3.3 and has more edges than G:

$$|E(G_{u,v})| - |E(G)| \geq -|L(u)| + |L(v)| - d(u,v) \geq -\frac{1}{8}n^2 + n^{3/2} + \frac{3|E(G)|}{n} - n$$

$$\geq -\frac{1}{8}n^2 + n^{3/2} + \frac{3|E(S(n))|}{n} - n \geq -\frac{1}{8}n^2 + n^{3/2} + \frac{1}{8}n^3 - O(n \log n) > 0,$$

for n sufficiently large. This contradicts the extremality of G. Thus for every $u \in V(G)$, we have $|L(u)| \geq \frac{1}{8}n^2 - n^{3/2} = \frac{1}{8}n^2(1 + o(1))$.

∎

Proof of Theorem 2.4. For the lower bound, we have

$$\mathrm{ex}(n, \mathcal{F}) \geq e(S(n)) = \frac{1}{24}n^3(1 + o(1)).$$

For the upper bound, let n_0 be large enough such that the following reasoning holds. For $n \geq n_0$, $\mathrm{ex}(n, \mathcal{F}) \leq 0.251\binom{n}{3}$ by (6). We will prove by induction on n that $\mathrm{ex}(n, \mathcal{F}) \leq \frac{1}{24}n^3 + n \cdot n_0^2$. This trivially holds for $n \leq n_0$, because

$$\mathrm{ex}(n, \mathcal{F}) \leq \binom{n}{3} \leq \frac{1}{24}n^3 + n \cdot n_0^2.$$

For $n_0 \leq n \leq 4n_0$, we have

$$\mathrm{ex}(n, \mathcal{F}) \leq 0.251\binom{n}{3} \leq \frac{1}{24}n^3 + 0.001\frac{n^3}{6} \leq \frac{1}{24}n^3 + n \cdot n_0^2.$$

Now, let G be an extremal $\mathcal{F}$-free 3-graph on $n \geq 4n_0$ vertices. By Lemma 3.4 we have $|L(w)| \geq \frac{1}{8}n^2(1 + o(1))$ for every $w \in V(G)$. Therefore, the assumptions for Lemma 3.2 hold. Take a vertex partition $V(G) = X_1 \cup X_2 \cup X_2$ with the properties from Lemma 3.2. Now, for all $i \in [3]$, $|X_i| \geq 0.26n \geq n_0$ and since $G[X_i]$ is $\mathcal{F}$-free, we have

$$e(G[X_i]) \leq \frac{1}{24}|X_i|^3 + |X_i| \cdot n_0^2$$

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by the induction assumption. We conclude

$$e(G) \leq |X_1||X_2||X_3| + \sum_{i=1}^{3} e(G[X_i]) \leq |X_1||X_2||X_3| + n \cdot n_0^2 + \frac{1}{24} \sum_{i=1}^{3} |X_i|^3$$

$$\leq \frac{1}{24}n^3 + n \cdot n_0^2,$$

where in the last step we used that the function $g(x_1, x_2, x_3) := x_1 x_2 x_3 + 1/24(x_1^3 + x_2^3 + x_3^3)$ with domain $\{(x_1, x_2, x_3) \in [0.26, 0.48]^3 : x_1 + x_2 + x_3 = 1\}$ achieves its maximum at $x_1 = x_2 = x_3 = 1/3$. This can be verified quickly using basic calculus or simply by using a computer, we omit the details. ∎

Analyzing the previous proof actually gives a stability result.

Lemma 3.5. *Let G be an $\mathcal{F}$-free 3-graph on n vertices and $|E(G)| = 1/24n^3(1+o(1))$, satisfying $|L(w)| \geq \frac{1}{8}n^2(1 + o(1))$ for every $w \in V(G)$. Then G has a vertex partition $V(G) = X_1 \cup X_2 \cup X_3$ such that*

- *there is no edge $e = xyz$ with $x, y \in X_i$ and $z \notin X_i$ for $i \in [3]$,*
- *$|X_i| = \frac{n}{3}(1 + o(1))$ for every $i \in [3]$.*

Proof. Take a vertex partition $V(G) = X_1 \cup X_2 \cup X_3$ from Lemma 3.2. In particular, the first property holds. Since $G[X_i]$ is $\mathcal{F}$-free, we have by Theorem 2.4 that $e(G[X_i]) \leq \frac{1}{24}|X_i|^3(1 + o(1))$. Now, again

$$\frac{1}{24}n^3(1 + o(1)) = e(G) \leq |X_1||X_2||X_3| + \sum_{i=1}^{3} e(G[X_i])$$

$$\leq |X_1||X_2||X_3| + \frac{1}{24} \sum_{i=1}^{3} |X_i|^3(1 + o(1).$$

Again, since the polynomial g with domain $\{(x_1, x_2, x_3) \in [0.26, 0.48]^3 : x_1 + x_2 + x_3 = 1\}$ achieves its unique maximum at $x_1 = x_2 = x_3 = 1/3$, we get $|X_i| = (1/3 + o(1))n$. ∎

4 Exact results

In the first subsection of this section we prove Lemma 4.3, a refinement of Lemma 3.5. In Subsection 4.2 we deduce Theorem 1.4 from Lemma 4.3 and finally, in Subsection 4.3, we deduce Corollary 1.5 from Theorem 1.4.

4.1 Further refinement

Lemma 4.1. *Let $T \in \mathcal{T}_{\mathcal{F}}$ and $P \subseteq \mathbb{R}^2$ be a point set. Denote $G = G(P, T, \varepsilon(T))$. For every $u, v \in V(G)$ there exists a point set P' such that $G_{u,v} = G(P', T, \varepsilon(T))$.*

Proof. Let $u, v \in V(G)$. Construct P' from P by removing the point corresponding to u and adding a point close enough to the point corresponding to v. This point set satisfies $G_{u,v} = G(P', T, \varepsilon(T))$. ∎

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Lemma 4.2. *Let $T \in \mathcal{T}_\mathcal{F}$ be a triangle shape and let $P \subseteq \mathbb{R}^2$ be an n-element point set maximizing the number of triangles being $\varepsilon(T)$-similar to T. Denote $G = G(P, T, \varepsilon(T))$. Then for every $w \in V(G)$, we have $|L(w)| \geq \frac{1}{8}n^2(1 + o(1))$.*

Proof. We have that G is $\mathcal{F}$-free. Assume that there exists $u \in V(G)$ with

$$|L(u)| < \frac{1}{8}n^2 - n^{3/2}.$$

Let $v \in V(G)$ be a vertex maximizing $|L(v)|$. By Lemma 4.1 there exists a point set P' such that $G_{u,v} = G(P', T, \varepsilon(T))$. We have $|E(G_{u,v})| > |E(G)|$ by the same calculation as in the proof of Lemma 3.4. This contradicts the maximality of P. ∎

Now, we will strengthen the previous stability result, Lemma 3.5.

Lemma 4.3. *Let $T \in \mathcal{T}_\mathcal{F}$. There exists n_0 such that for every $n \geq n_0$ the following holds. Let P be an n-element point set maximizing the number of triangles being $\varepsilon(T)$-similar to T. Then, the 3-graph $G = G(P, T, \varepsilon(T))$ has a vertex partition $V(G) = X_1 \cup X_2 \cup X_2$ such that*

(i) there is no edge $e = xyz$ with $x, y \in X_i$ and $z \notin X_i$ for $i \in [3]$,

(ii) $xyz \in E(G)$ for all $x \in X_1, y \in X_2, z \in X_3$,

(iii) $|X_i| - |X_j| \leq 1$ for all $i, j \in [3]$.

Proof. By Lemma 4.2, for every $w \in V(G)$, $|L(w)| \geq \frac{1}{8}n^2(1 + o(1))$. Further, we have

$$e(G) \geq e(S(n)) = \frac{1}{4}\binom{n}{3}(1 + o(1)).$$

Therefore, the assumptions from Lemma 3.5 hold. Let $V(G) = X_1 \cup X_2 \cup X_3$ be a partition having the properties from Lemma 3.5. In particular, property (i) holds. Towards contradiction, assume that there exists $x \in X_1, y \in X_2, z \in X_3$ with $xyz \notin E(G)$. For $i \in [3]$, let P_i be the point set corresponding to the set X_i. We have,

$$e(G[X_i]) = e(G(P_i, T, \varepsilon(T))).$$

Construct a new point set P' by taking a large enough triangle of shape T and placing each of the point sets P_i close to one of the three vertices of T. Using condition (i), this new point set P' satisfies

$$e(G(P', T, \varepsilon(T))) = |X_1||X_2||X_3| + \sum_{i=1}^{3} e(G(P_i, T, \varepsilon(T)))$$

$$= |X_1||X_2||X_3| + \sum_{i=1}^{3} e(G[X_i]) > e(G),$$

contradicting the maximality of P. Therefore, for all $x \in X_1, y \in X_2, z \in X_3$ we have $xyz \in E(G)$.

By Theorem 2.4, we have

$$\frac{e(G[X_1])}{\binom{|X_1|}{3}} = \frac{1}{4} + o(1) \quad \text{and} \quad \frac{e(G[X_2])}{\binom{|X_2|}{3}} = \frac{1}{4} + o(1).$$

Next, towards contradiction, assume that without loss of generality $|X_1| \geq |X_2| + 2$. Let $v_1 \in X_1$ be minimizing $|L_{X_1}(v_1)|$ and let $v_2 \in X_2$ be maximizing $|L_{X_2}(v_2)|$. By the choice of v_1 and v_2,

$$|L_{X_1}(v_1)| \leq \frac{3e(G[X_1])}{|X_1|} \quad \text{and} \quad |L_{X_2}(v_2)| \geq \frac{3e(G[X_2])}{|X_2|}.$$

The hypergraph G_{v_1,v_2} is still $\mathcal{F}$-free by Lemma 3.3 and has more edges than G:

$$|E(G_{v_1,v_2})| - |E(G)| = |X_1||X_3| + |L_{X_2}(v_2)| - |L_{X_1}(v_1)| - |X_2||X_3| - |X_3|$$

$$\geq \frac{3e(G[X_2])}{|X_2|} - \frac{3e(G[X_1])}{|X_1|} + |X_3|(|X_1| - |X_2| - 1)$$

$$= \frac{3|e(G[X_2])|X_1| - 3e(G[X_1])|X_2|}{|X_1||X_2|} + |X_3|(|X_1| - |X_2| - 1)$$

$$= \left(\frac{1}{4} + o(1)\right) \frac{3\binom{|X_2|}{3}|X_1| - 3\binom{|X_1|}{3}|X_2|}{|X_1||X_2|} + |X_3|(|X_1| - |X_2| - 1)$$

$$\geq \left(\frac{1}{8} + o(1)\right) \frac{|X_2|^3|X_1| - |X_1|^3|X_2|}{|X_1||X_2|} + |X_3|(|X_1| - |X_2| - 1)$$

$$= \left(\frac{1}{8} + o(1)\right) (|X_2|^2 - |X_1|^2) + |X_3|(|X_1| - |X_2| - 1)$$

$$= (|X_1| - |X_2|) \left(|X_3| - (|X_1| + |X_2|) \left(\frac{1}{8} + o(1)\right)\right) - |X_3|$$

$$= (|X_1| - |X_2|) \left(\frac{n}{4} + o(n)\right) - |X_3| \geq n \left(\frac{1}{2} + o(1)\right) - \left(\frac{1}{3} + o(1)\right) n > 0.$$

We obtain $|E(G_{v_1,v_2})| > |E(G)|$. By Lemma 4.1 there exists a point set P' such that $G_{v_1,v_2} = G(P', T, \varepsilon(T))$. This contradicts the maximality of P. Therefore property (iii) holds, completing the proof of this lemma. ∎

4.2   Proof of Theorem 1.4

Let $T \in \mathcal{T}_{\mathcal{F}}$ and P be an n-element point set maximizing the number of triangles being $\varepsilon(T)$-similar to T. Denote $G = G(P, T, \varepsilon(T))$. By Lemma 4.3, the 3-graph G has a vertex partition $V(G) = X_1 \cup X_2 \cup X_2$ such that $|X_i| - |X_j| \leq 1$ for all $i, j \in [3]$ and there is no edge $e = xyz$ with $xy \in X_i$ and $z \notin X_i$ for $i \in [3]$. Since the sets X_1, X_2, X_3 correspond to point sets of the same sizes, we have $e(G[X_i]) \leq h(|X_i|, T, \varepsilon(T))$ for $i \in [3]$. Let $a = |X_1|, b = |X_2|$ and $c = |X_3|$. Now,

$$h(n, T, \varepsilon(T)) = e(G) \leq a \cdot b \cdot c + e(G[X_1]) + e(G[X_2]) + e(G[X_3])$$

$$\leq a \cdot b \cdot c + h(a, T, \varepsilon(T)) + h(b, T, \varepsilon(T)) + h(c, T, \varepsilon(T)).$$

It remains to show

$$h(n, T, \varepsilon(T)) \geq a \cdot b \cdot c + h(a, T, \varepsilon(T)) + h(b, T, \varepsilon(T)) + h(c, T, \varepsilon(T)).$$

There exists point sets $P_a, P_b, P_c \subseteq \mathbb{R}^2$ of sizes a, b, c respectively, such that

$$e(G(P_a, T, \varepsilon(T))) = h(a, T, \varepsilon(T)), \qquad e(G(P_b, T, \varepsilon(T))) = h(b, T, \varepsilon(T))$$
$$\text{and} \qquad e(G(P_c, T, \varepsilon(T))) = h(c, T, \varepsilon(T)).$$

Note that we can assume that $\text{diam}(P_a) = 1$, $\text{diam}(P_b) = 1$ and $\text{diam}(P_c) = 1$, where $\text{diam}(Q)$ of a point set Q is the largest distance between two points in the point set. By arranging the three point sets P_a, P_b, P_c in shape of a large enough triangle T, we get a point set P such that

$$h(n, T, \varepsilon(T)) \geq e(G(P, T, \varepsilon(T))) = a \cdot b \cdot c + h(a, T, \varepsilon(T)) + h(b, T, \varepsilon(T)) + h(c, T, \varepsilon(T)),$$

completing the proof of Theorem 1.4.

4.3   Proof of Corollary 1.5

Let T be a triangle shape for which there is an $\varepsilon(T)$ such that (2) holds. By Theorem 1.4, (2) holds for almost all triangles. Take a point set P on $3^\ell \geq n_0$ points maximizing the number of triangles being $\varepsilon(T)$-similar to T. Denote $H = G(P, T, \varepsilon(T))$. Note that because of scaling invariance we can assume that $\text{diam}(P)$ is arbitrary small. By applying (2) iteratively, we have

$$h(3^{\ell+i}, T, \varepsilon(T)) = 3^i \cdot e(H) + 3^{3\ell} \frac{1}{24} \left(3^{3i} - 3^i \right) \tag{10}$$

for all $i \geq 0$.

Now, towards contradiction, assume that there exists a point set $P' \subseteq \mathbb{R}^2$ of 3^k points such that the number of triangles similar to $\varepsilon(T)$ is more than $e(S(3^k))$. Let $G = G(P', T, \varepsilon(T))$. Then,

$$e(G) > e(S(3^k)) = \frac{1}{24} \left(3^{3k} - 3^k \right).$$

Construct a point set $\bar{P} \subseteq \mathbb{R}^2$ of $3^{\ell+k}$ points by taking all points $p_G + p_H$, $p_G \in P'$, $p_H \in P$ where addition is coordinate-wise. Let $\bar{G} := G(\bar{P}, T, \varepsilon(T))$. Since we can assume that $\text{diam}(P')$ is arbitrary small, $\bar{G}$ is the 3-graph constructed from G by replacing every vertex by a copy of H. Now,

$$e(\bar{G}) = e(G) \cdot 3^{3\ell} + e(H) \cdot 3^k > 3^k \cdot e(H) + 3^{3\ell} \frac{1}{24} \left(3^{3k} - 3^k \right),$$

contradicting (10). This completes the proof of Corollary 1.5.

5   Concluding remarks

When carefully reading the proof, one can observe that also the following Turán type results hold. Recall that $\mathcal{F}$ is the set of forbidden 3-graphs defined in Section 2.2.

Theorem 5.1. *The following statements hold.*

(a) *There exists n_0 such that for all $n \geq n_0$*

$$\mathrm{ex}(n, \mathcal{F}) = a \cdot b \cdot c + \mathrm{ex}(a, \mathcal{F}) + \mathrm{ex}(b, \mathcal{F}) + \mathrm{ex}(c, \mathcal{F}),$$

where $n = a + b + c$ and a, b, c are as equal as possible.

(b) *Let n be a power of 3. Then,*

$$\mathrm{ex}(n, \mathcal{F}) = \frac{1}{24}(n^3 - n).$$

It would be interesting to prove the Turán type results, Theorem 1.3 and Theorem 5.1, for a smaller family of hypergraphs than $\mathcal{F}$. Potentially the following conjecture by Falgas-Ravry and Vaughan could be tackled in a similar way.

Conjecture 5.2 (Falgas-Ravry and Vaughan [9])**.**

$$\mathrm{ex}(n, \{K_4^-, C_5\}) = \frac{1}{4}\binom{n}{3}(1 + o(1)).$$

Considering that for our proof it was particularly important that K_4^- and $L_2 = \{123, 124, 125, 136, 456\}$ are forbidden, we conjecture that $S(n)$ has asymptotically the most edges among $\{K_4^-, L_2\}$-free 3-graphs.

Conjecture 5.3.

$$\mathrm{ex}(n, \{K_4^-, L_2\}) = \frac{1}{4}\binom{n}{3}(1 + o(1)).$$

Note that a standard application of flag algebras on 7 vertices shows

$$\mathrm{ex}(n, \{K_4^-, L_2\}) \leq 0.25074\binom{n}{3}$$

for n sufficiently large. It is conjectured (see [12]) that $\mathrm{ex}(n, C_5^-) = 0.25\binom{n}{3}$.

Theorem 1.3 determines $h(n, T, \varepsilon)$ asymptotically for almost all triangles T and $\varepsilon > 0$ sufficiently small. It remains an open problem to determine $h(n, T, \varepsilon)$ for some triangles $T \in S$. Bárány and Füredi [5] provided asymptotically better bounds stemming from recursive constructions for some of those triangles. Potentially a similar proof technique to ours could be used to determine $h(n, T, \varepsilon)$ for some of those triangle shapes.

Another interesting research direction is to change the space, and study point sets in $\mathbb{R}^3$ or even $\mathbb{R}^d$ instead of the plane. Given a triangle $T \in S$, $\varepsilon > 0$, $d \geq 2$ and $n \in \mathbb{N}$,
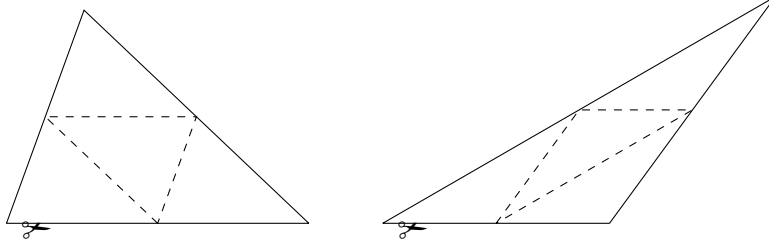
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Figure 7: A cutout of a tetrahedron using an acute triangle on the left. A cutout not giving a tetrahedron coming from an obtuse triangle on the right. Bend along the dashed lines.

denote $g_d(n, T, \varepsilon)$ the maximum number of triangles in a set of n points from $\mathbb{R}^d$ that are ε-similar to a triangle T. Being allowed to use one more dimension might help us to find constructions with more triangles being ε-similar to T.

For an acute triangle T and $d = 3$, we can group the n points into four roughly equal sized groups and place each group very close to a vertex of a tetrahedron with each face being similar to T. For a crafty reader, we are including a cutout that leads to a tetrahedron with all sides being the same triangle in Figure 7 on the left. Each group can again be split up in the same way. If we keep doing this iteratively, we obtain

$$g_3(n, T, \varepsilon) \geq \frac{1}{15} n^3 (1 + o(1))$$

for some $\varepsilon > 0$. Note that for almost all acute triangles T,

$$g_2(n, T, \varepsilon) = h(n, T, \varepsilon) = \frac{1}{24} n^3 (1 + o(1)) < g_3(n, T, \varepsilon).$$

For T being an equilateral triangle and $d \geq 4$ we can find a better construction. There is a d-simplex with all faces forming equilateral triangles. Grouping the n points into $d + 1$ roughly equal sized groups and placing each group very close to the vertex of the d-simplex and then iterating this, gives us

$$g_d(n, T, \varepsilon) \geq \sum_{i \geq 1} \left(\frac{n}{(d+1)^i} \right)^3 \binom{d+1}{3} (d+1)^{i-1} (1 + o(1)) = \frac{1}{6} \frac{d-1}{d+2} n^3 (1 + o(1)).$$

The following variation of the problem could also be interesting. We say that two triangles are ε-*isomorphic* if their side lengths are $a \leq b \leq c$ and $a' \leq b' \leq c'$ and $|a - a'|, |b - b'|, |c - c'| < \varepsilon$. Maximizing the number of ε-isomorphic triangles has the following upper bound. Denote the side lengths of a triangle T by a, b, and c. Now color edges of K_n with colors a, b, and c such that the number of rainbow triangles is maximized. Note that rainbow triangles would correspond to triangles isomorphic to T, if there exists an embedding of K_n in some R^d such that the distances correspond to the colors. The problem of maximizing the number of rainbow triangles in a 3-edge-colored K_n is a problem of Erdős and Sós (see [8]) that was solved by flag

algebras [2]. The asymptotic construction is an iterated blow-up of a properly 3-edge-colored K_4. Properly 3-edge-colored K_4 can be embedded as a tetrahedron in $\mathbb{R}^3$. This gives $\frac{1}{16}n^3(1+o(1))$ ε-isomorphic triangles in $\mathbb{R}^3$. This heuristics suggests that increasing the dimension beyond 3 may allow us to embed slightly more ε-isomorphic triangles by making it possible to embed more of the iterated blow-up of K_4 construction. The number of rainbow triangles the iterated blow-up of a properly 3-edge-colored K_4 is $\frac{1}{15}n^3(1+o(1))$ which is an upper bound on the number of ε-isomorphic triangles for any d.

In our construction maximizing the number of ε-similar triangles for $d = 3$, the majority of triangles are actually ε-isomorphic. Already for $d = 3$, we can embed $\frac{1}{15}n^3(1+o(1))$ ε-similar triangles, which is the upper bound on the number of ε-isomorphic triangles for any d. This suggests that increasing the dimension beyond $d = 3$ may result in only very small increases on the number ε-isomorphic triangles or a very different construction is needed.

The above heuristic does not apply to isosceles triangles. Maximizing the number of ε-isomorphic triangles would correspond to a 2-edge-coloring of K_n and maximizing the number of induced path on 3 vertices in one of the two colors. The extremal construction is a balanced complete bipartite graph in one color. Increasing the dimension helps with embedding a bigger 2-edge-coloring of K_n and in turn obtaining a larger number of ε-isomorphic triangles with $\frac{1}{8}n^3(1+o(1))$ being the upper bound.

In general, the number of obtuse triangles do not seem to benefit as much from higher dimensions. Embedding three ε-similar obtuse triangles on 4 points is not possible for any d for almost all obtuse triangles. This contrasts with acute triangles, where 4 points can give four ε-isomorphic triangles for dimension at least 3. The reader may try it for ε-isomorphic triangles with cutouts in Figure 7. We have not explored the above problems for obtuse triangles further.

Acknowledgements

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