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Note

Generalized rainbow Turán numbers of odd cycles

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ABSTRACT

Given graphs F and H, the *generalized rainbow Turán number* $\mathrm{ex}(n, F, \text{rainbow-}H)$ is the maximum number of copies of F in an n-vertex graph with a proper edge-coloring that contains no rainbow copy of H. B. Janzer determined the order of magnitude of $\mathrm{ex}(n, C_s, \text{rainbow-}C_t)$ for all $s \geq 4$ and $t \geq 3$, and a recent result of O. Janzer implied that $\mathrm{ex}(n, C_3, \text{rainbow-}C_{2k}) = O(n^{1+1/k})$. We prove the corresponding upper bound for the remaining cases, showing that $\mathrm{ex}(n, C_3, \text{rainbow-}C_{2k+1}) = O(n^{1+1/k})$. This matches the known lower bound for k even and is conjectured to be tight for k odd.

1. Introduction

The *Turán number* of a graph H is the maximum number of edges in an H-free graph on n vertices, denoted $\mathrm{ex}(n, H)$. This has been generalized in many different ways. For example, the *rainbow Turán number* $\mathrm{ex}^*(n, H)$, introduced in [14], is the maximum number of edges in a graph on n vertices which can be properly edge-colored with no rainbow copy of H. Another natural variation is the *generalized Turán number* $\mathrm{ex}(n, F, H)$, which is the maximum number of copies of a graph F in an n-vertex graph that contains no copy of H, and was first studied systematically by Alon and Shikhelman [1]. Both of these problems have been extensively studied, see for example [12,14] and [1–4,7,8,10,15].

Gerbner, Mészáros, Methuku and Palmer [6] considered the following generalized problem which unites the two concepts above. Given two graphs F and H, the *generalized rainbow Turán number*, denoted by $\mathrm{ex}(n, F, \text{rainbow-}H)$, is the maximum number of copies of F in an n-vertex graph which can be properly edge-colored to avoid a rainbow copy of H. Note that trivially we have $\mathrm{ex}(n, F, \text{rainbow-}H) \geq \mathrm{ex}(n, F, H)$. The question for $F = H$ has been studied for paths, trees, cycles and cliques, see [6,9,11].

Recently, B. Janzer [11] determined the order of magnitude of $\mathrm{ex}(n, C_s, \text{rainbow-}C_t)$ for all cases except for $s = 3$. In the case $s = 3$, he gave the following bounds.

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Theorem 1.1 *(Janzer [11]). If $k \geq 2$ is odd then $\mathrm{ex}(n, C_3, \text{rainbow-}C_{2k}) = \Omega(n^{1+1/k})$, and if k is even then $\mathrm{ex}(n, C_3, \text{rainbow-}C_{2k+1}) = \Omega(n^{1+1/k})$. Furthermore, for every integer $k \geq 2$, we have*

$$\mathrm{ex}(n, C_3, \text{rainbow-}C_{2k}) = O(\mathrm{ex}^*(n, C_{2k})),$$

and

$$\mathrm{ex}(n, C_3, \text{rainbow-}C_{2k}) \geq \mathrm{ex}(n, C_3, C_{2k}) = \Omega(\mathrm{ex}(n, \{C_4, C_6, \dots, C_{2k}\})),$$
$$\mathrm{ex}(n, C_3, \text{rainbow-}C_{2k+1}) \geq \mathrm{ex}(n, C_3, C_{2k+1}) = \Omega(\mathrm{ex}(n, \{C_4, C_6, \dots, C_{2k}\})).$$

Very recently, O. Janzer [12] settled a well-known conjecture of Keevash, Mubayi, Sudakov and Verstraëte [14], proving that

$$\mathrm{ex}^*(n, C_{2k}) = \Theta(n^{1+1/k}). \tag{1}$$

Together with Theorem 1.1, this yields $\mathrm{ex}(n, C_3, \text{rainbow-}C_{2k}) = O(n^{1+1/k})$, which is tight for k odd.

In this paper, we study the remaining open case for cycles, proving an upper bound on $\mathrm{ex}(n, C_3, \text{rainbow-}C_{2k+1})$ which matches the lower bound given by B. Janzer [11] for k even and which is expected to be sharp for k odd as well.

Theorem 1.2. *For $k \geq 2$, we have*

$$\mathrm{ex}(n, C_3, \text{rainbow-}C_{2k+1}) = O\left(n^{1+1/k}\right).$$

In Section 2, we give a self-contained proof of the upper bound on $\mathrm{ex}(n, C_3, \text{rainbow-}C_5)$ which gives an explicit constant, although it is likely not best possible. In Section 3, we prove Theorem 1.2 by applying (1) to a subgraph in which every rainbow copy of C_{2k} extends to a rainbow copy of C_{2k+1} in the original graph. Throughout the paper, we use P_k to denote the path with k edges.

2. No rainbow C_5

Theorem 2.1. *We have* $\mathrm{ex}(n, C_3, \text{rainbow-}C_5) \leq 32n^{3/2}$.

Proof. Let G be an n-vertex graph with a proper edge-coloring c containing no rainbow copy of C_5. First, we will show that G contains at most $4|E(G)|$ triangles.

Fix a vertex $v \in V(G)$ and let d denote the degree of v. We will count in two ways the pairs (S, e) where $S \subset N(v)$ contains $\lceil d/2 \rceil$ neighbors of v and $e \in E(G[S])$ satisfies $c(e) \neq c(vu)$ for every $u \in S$. There are $\binom{d}{\lceil d/2 \rceil}$ ways to choose the set S. Throw out any edge in $G[S]$ whose color appears on an edge incident with v and S. Let E' denote the set of remaining edges in $G[S]$. Now E' must be rainbow P_3-free, otherwise we can find a rainbow copy of C_5 in G. Therefore, since Johnston, Palmer, and Sarkar [13] showed that $\mathrm{ex}^*(n, P_3) = \frac{3}{2}n$, we have $|E'| \leq \frac{3}{2}\left\lceil \frac{d}{2} \right\rceil$. Thus, the number of triangles containing v which are formed in this way, that is, the number of pairs of such a set S and an edge from E' of a different color, is at most $\frac{3}{2}\left\lceil \frac{d}{2} \right\rceil \binom{d}{\lceil d/2 \rceil}$.

On the other hand, we could instead first choose an edge e in the neighborhood of v to form our triangle, which can be done in $|E(G[N(v)])|$ ways, and then select an additional $\lceil d/2 \rceil - 2$ vertices from $N(v)$ to form the rest of S. However, we do not want the color of e to appear on an edge incident to v. Since the edge coloring is proper, this only requires us to throw out at most one vertex from $N(v)$ since at most one edge incident to v can have the same color as e. Thus, we can pick the remaining vertices of S in at least $\binom{d-3}{\lceil d/2 \rceil - 2}$ ways. Therefore, we have

$$|E(G[N(v)])| \cdot \binom{d-3}{\lceil d/2 \rceil - 2} \leq \frac{3}{2}\left\lceil \frac{d}{2} \right\rceil \cdot \binom{d}{\lceil d/2 \rceil}.$$

Thus, $|E(G[N(v)])| \leq 6d$, so the number of triangles in G is at most

$$\frac{1}{3}\sum_{v \in V(G)} |E(G[N(v)])| \leq \frac{1}{3}\sum_{v \in V(G)} 6d(v) = 4|E(G)|.$$

We may assume that every edge of G is in a triangle, otherwise we could delete an edge without decreasing the number of triangles. Now assume towards a contradiction that $|E(G)| \geq 8n^{3/2}$.

One-by-one, delete vertices of degree less than $4\sqrt{n}$ in G. Note that not all vertices are deleted, since otherwise $|E(G)| < 4n^{3/2}$. Denote by G' the remaining induced subgraph with minimum degree $\delta(G') \geq 4\sqrt{n}$, and let $n' = |V(G')|$. Fix an arbitrary vertex $v \in V(G')$.

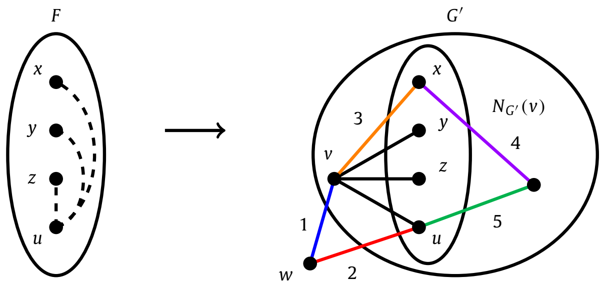

Fig. 1. If there is a vertex of degree at least 3 in F, then G contains a rainbow copy of C_5. Edges in F are dashed while edges in G are solid.

A *cherry* is a path of length 2. We will form an auxiliary graph F with vertex set $V(F) = N_{G'}(v)$ which contains an edge uw if and only if there are at least seven cherries of the form uxw in G'. We will show that F must contain a vertex of degree at least 3 and use this vertex to find a rainbow copy of C_5 in G.

Let $S \subseteq N_{G'}(v)$ be a set of $\lfloor\sqrt{n}\rfloor$ vertices. We will count the cherries in G' with endpoints in S. Since each vertex x in $V(G')$ is the center vertex in exactly $\binom{d_S(x)}{2}$ cherries[4] of this form, we can count the desired cherries as follows:

$$\sum_{x\in V(G')}\binom{d_S(x)}{2} \ge n'\binom{\frac{1}{n'}\sum d_S(x)}{2} = n'\binom{\frac{1}{n'}\sum_{s\in S} d_{G'}(s)}{2} \ge n'\binom{\frac{1}{n'}\cdot\delta(G')|S|}{2}$$

$$\ge \frac{(\delta(G')|S|)^2}{4n'} \ge \frac{16n|S|^2}{4n'} > 7\binom{|S|}{2},$$

where we use convexity and the fact that $\delta(G') \ge 4\sqrt{n}$. Thus, there must be some pair of vertices in S which are the endpoints of at least seven cherries in G', and hence, these vertices are adjacent in F. Since S was an arbitrary set of $\lfloor\sqrt{n}\rfloor$ vertices in $N_{G'}(v) = V(F)$, we have shown that $\alpha(F) \le \sqrt{n}$, where $\alpha(F)$ denotes the independence number of F. This gives $\Delta(F) \ge |F|/\alpha(F) - 1 \ge 4\sqrt{n}/\sqrt{n} - 1 \ge 3$, so there is a vertex u in F of degree at least 3. Let x, y, z be neighbors of u in F, as in Fig. 1. By assumption, the edge uv is in at least one triangle in G, so there is a vertex $w \in V(G)$, possibly in $\{x, y, z\}$, which forms a triangle with uv.

Let $c(vw) = 1$ and $c(uw) = 2$. Then since c is a proper coloring, at least one of vx, vy, and vz is colored with a new color, say $c(vx) = 3$. Since ux is an edge in F, there are at least seven cherries in G' with endpoints u and x, and hence, at least one with new colors 4 and 5 which avoids v and w. Thus, there is a rainbow copy of C_5 in G, and we reach a contradiction. Therefore, G must contain at most $8n^{3/2}$ edges, and hence, at most $32n^{3/2}$ triangles, as desired. ■

3. No rainbow C_{2k+1}

Proof of Theorem 1.2. Let G be an n-vertex graph with a proper edge-coloring $f : E(G) \to C$ containing no rainbow copy of C_{2k+1}. We may assume each edge in G is in at least one triangle.

As in Section 2, we begin by giving a bound on the number of triangles in G in terms of the number of edges in G. Fix a vertex $v \in V(G)$ and let d denote the degree of v. Pick a set $S \subset N(v)$ containing $\lceil d/2\rceil$ neighbors of v, and throw out any edge in $G[S]$ which is colored using some color which appears on an edge from v to S.

Then $G[S]$ cannot contain a rainbow copy of P_{2k-1}. A result of Ergemlidze, Győri, and Methuku [5] showed that $\mathrm{ex}^*(n, P_{k+1}) < \left(\frac{9k}{7}+2\right)n$. Therefore, we obtain

$$|E(G[S])| \le \mathrm{ex}^*\left(\left\lceil\frac{d}{2}\right\rceil, P_{2k-1}\right) \le \frac{18k-4}{7}\cdot\left\lceil\frac{d}{2}\right\rceil.$$

By counting the triangles containing v and two adjacent vertices in S in two ways, we get

$$|E(G[N(v)])|\cdot\binom{d-3}{\lceil d/2\rceil-2} \le \frac{18k-4}{7}\cdot\left\lceil\frac{d}{2}\right\rceil\cdot\binom{d}{\lceil d/2\rceil}.$$

Thus, $|E(G[N(v)])| \le \frac{72k-16}{7}d$, and the number of triangles in G is at most

$$\frac{1}{3}\sum_{v\in V(G)}|E(G[N(v)])| \le \frac{72k-16}{21}\sum_{v\in V(G)} d(v) = \frac{144k-32}{21}|E(G)|.$$

[4] $d_S(x) = |N_G(x)\cap S|$.

We will show that $|E(G)| = O(n^{1+1/k})$.

Assume towards a contradiction that G has more edges. For each edge $uv \in E(G)$, arbitrarily fix a vertex $w = w(uv)$ such that u, v, and w form a triangle in G. We will find a subgraph of G in which any rainbow copy of C_{2k} can be extended to a rainbow copy of C_{2k+1} in G. To this end, randomly select a partition of $V(G)$ into parts A of size $\lfloor \frac{n}{2} \rfloor$ and B of size $\lceil \frac{n}{2} \rceil$. Similarly, take a random partition of C into parts X and Y of sizes $\lfloor \frac{|C|}{2} \rfloor$ and $\lceil \frac{|C|}{2} \rceil$, respectively.

Let F be the subgraph with vertex set B which contains an edge $uv \in E(G[B])$ if and only if the vertex $w = w(uv)$ is in A and $f(uv) \in X$ while $f(uw), f(vw) \in Y$. Then the expected number of edges in F is least $|E(G)|/64$, so we can fix partitions (A, B) and (X, Y) such that the corresponding graph F has at least this many edges.

Note that F inherits the proper edge-coloring f from G, so we can apply (1) to this subgraph. Since $\mathrm{ex}^*(n, C_{2k}) = O(n^{1+1/k})$, there must be a rainbow copy of C_{2k} in F. This cycle contains only vertices in B, so we can replace an arbitrary edge uv in the cycle by a pair of edges uw and wv with $w \in A$ to create a copy of C_{2k+1} in G. Furthermore, the cycle in F is colored only with colors from X, while the new edges have colors from Y, so we have found a rainbow copy of C_{2k+1}. But this is a contradiction, since G contains no rainbow copies of C_{2k+1}. Thus, $|E(G)| = O(n^{1+1/k})$, which implies that the number of triangles in G is $O(n^{1+1/k})$, as desired. ■

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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