

Equivariant incidence algebras and equivariant Kazhdan–Lusztig–Stanley theory

Nicholas Proudfoot

ABSTRACT. We establish a formalism for working with incidence algebras of posets with symmetries, and we develop equivariant Kazhdan–Lusztig–Stanley theory within this formalism. This gives a new way of thinking about the equivariant Kazhdan–Lusztig polynomial and equivariant Z -polynomial of a matroid.

1. INTRODUCTION

The incidence algebra of a locally finite poset was first introduced by Rota, and has proved to be a natural formalism for studying such notions as Möbius inversion [11], generating functions [4], and Kazhdan–Lusztig–Stanley polynomials [12, Section 6].

A special class of Kazhdan–Lusztig–Stanley polynomials that have received a lot of attention recently is that of Kazhdan–Lusztig polynomials of matroids, where the relevant poset is the lattice of flats [5, 9]. If a finite group W acts on a matroid M (and therefore on the lattice of flats), one can define the W -equivariant Kazhdan–Lusztig polynomial of M [7]. This is a polynomial whose coefficients are virtual representations of W , and has the property that taking dimensions recovers the ordinary Kazhdan–Lusztig polynomial of M . In the case of the uniform matroid of rank d on n elements, it is actually much easier to describe the S_n -equivariant Kazhdan–Lusztig polynomial, which admits a nice description in terms of partitions of n , than it is to describe the non-equivariant Kazhdan–Lusztig polynomial [7, Theorem 3.1].

While the definition of Kazhdan–Lusztig–Stanley polynomials is greatly clarified by the language of incidence algebras, the definition of the equivariant Kazhdan–Lusztig polynomial of a matroid is completely *ad hoc* and not nearly as elegant. The purpose of this note is to define the equivariant incidence algebra of a poset with a finite group of symmetries, and to show that the basic constructions of Kazhdan–Lusztig–Stanley theory make sense in this more general setting. In the case of a matroid, we show that this approach recovers the same equivariant Kazhdan–Lusztig polynomials that were defined in [7].

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KEYWORDS. Incidence algebra, Kazhdan–Lusztig–Stanley polynomial, matroid.

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2. THE EQUIVARIANT INCIDENCE ALGEBRA

Fix once and for all a field k . Let P be a locally finite poset equipped with the action of a finite group W . We consider the category $\mathcal{C}^W(P)$ whose objects consist of

- a k -vector space V
- a direct product decomposition $V = \prod_{x \leq y \in P} V_{xy}$, each V_{xy} finite dimensional
- an action of W on V compatible with the decomposition.

More concretely, for any $\sigma \in W$ and any $x \leq y \in P$, we have a linear map

$$\varphi_{xy}^\sigma : V_{xy} \rightarrow V_{\sigma(x)\sigma(y)},$$

and we require that $\varphi_{xy}^e = \text{id}_{V_{xy}}$ and that $\varphi_{\sigma(x)\sigma(y)}^{\sigma'} \circ \varphi_{xy}^\sigma = \varphi_{xy}^{\sigma'\sigma}$. Morphisms in $\mathcal{C}^W(P)$ are defined to be linear maps that are compatible with both the decomposition and the action. This category admits a monoidal structure, with tensor product given by

$$(U \otimes V)_{xz} := \bigoplus_{x \leq y \leq z} U_{xy} \otimes V_{yz}.$$

Let $I^W(P)$ be the Grothendieck ring of $\mathcal{C}^W(P)$; we call $I^W(P)$ the **equivariant incidence algebra** of P with respect to the action of W .

EXAMPLE 2.1. If W is the trivial group, then $I^W(P)$ is isomorphic to the usual incidence algebra of P with coefficients in $\mathbb{Z}$. That is, it is isomorphic as an abelian group to a direct product of copies of $\mathbb{Z}$, one for each interval in P , and multiplication is given by convolution.

REMARK 2.2. If W acts on P and $\psi : W' \rightarrow W$ is a group homomorphism, then ψ induces a functor $F_\psi : \mathcal{C}^W(P) \rightarrow \mathcal{C}^{W'}(P)$ and a homomorphism $R_\psi : I^W(P) \rightarrow I^{W'}(P)$.

We now give a second, more down to earth description of $I^W(P)$. Let $\text{VRep}(W)$ denote the ring of finite dimensional virtual representations of W over the field k . A group homomorphism $\psi : W' \rightarrow W$ induces a ring homomorphism

$$\Lambda_\psi : \text{VRep}(W) \rightarrow \text{VRep}(W').$$

For any $x \in P$, let $W_x \subset W$ be the stabilizer of x . We also define $W_{xy} := W_x \cap W_y$ and $W_{xyz} := W_x \cap W_y \cap W_z$. Note that, for any $x, y \in P$ and $\sigma \in W$, conjugation by σ gives a group isomorphism

$$\psi_{xy}^\sigma : W_{xy} \rightarrow W_{\sigma(x)\sigma(y)},$$

which induces a ring isomorphism

$$\Lambda_{\psi_{xy}^\sigma} : \text{VRep}(W_{\sigma(x)\sigma(y)}) \rightarrow \text{VRep}(W_{xy}).$$

An element $f \in I^W(P)$ is uniquely determined by a collection

$$\{f_{xy} \mid x \leq y \in P\},$$

where $f_{xy} \in \text{VRep}(W_{xy})$ and for any $\sigma \in W$ and $x \leq y \in P$, $f_{xy} = \Lambda_{\psi_{xy}^\sigma}(f_{\sigma(x)\sigma(y)})$. The unit $\delta \in I^W(P)$ is characterized by the property that δ_{xx} is the 1-dimensional trivial representation of W_x for all $x \in P$ and $\delta_{xy} = 0$ for all $x < y \in P$. The following proposition describes the product structure on $I^W(P)$ in this representation.

PROPOSITION 2.3. *For any $f, g \in I^W(P)$,*

$$(fg)_{xz} := \sum_{x \leq y \leq z} \frac{|W_{xyz}|}{|W_{xz}|} \text{Ind}_{W_{xyz}}^{W_{xz}} \left(\left(\text{Res}_{W_{xyz}}^{W_{xy}} f_{xy} \right) \otimes \left(\text{Res}_{W_{xyz}}^{W_{yz}} g_{yz} \right) \right).$$

REMARK 2.4. It may be surprising to see the fraction $\frac{|W_{xyz}|}{|W_{xz}|}$ in the statement of Proposition 2.3, since $\text{VRep}(W_{xy})$ is not a vector space over the rational numbers. We could in fact replace the sum over $[x, z]$ with a sum over one representative of each W_{xz} -orbit in $[x, z]$ and then eliminate the factor of $\frac{|W_{xyz}|}{|W_{xz}|}$. Including the fraction in the equation allows us to avoid choosing such representatives.

REMARK 2.5. Proposition 2.3 could be taken as the definition of $I^W(P)$. It is not so easy to prove associativity directly from this definition, though it can be done with the help of Mackey’s restriction formula (see for example [3, Corollary 32.2]).

REMARK 2.6. Suppose that $\psi : W' \rightarrow W$ is a group homomorphism, and for any $x, y \in P$, consider the induced group homomorphism $\psi_{xy} : W'_{xy} \rightarrow W_{xy}$. For any $f \in I^W(P)$, we have, $R_\psi(f)_{xy} = \Lambda_{\psi_{xy}}(f_{xy})$. In particular, if W' is the trivial group, then $R_\psi(f)_{xy}$ is equal to the dimension of the virtual representation $f_{xy} \in \text{VRep}(W_{xy})$.

Before proving Proposition 2.3, we state the following standard lemma in representation theory.

LEMMA 2.7. Suppose that $E = \bigoplus_{s \in S} E_s$ is a vector space that decomposes as a direct sum of pieces indexed by a finite set S . Suppose that G acts linearly on E and acts by permutations on S such that, for all $s \in S$ and $\gamma \in G$, $\gamma \cdot E_s = E_{\gamma \cdot s}$. For each $x \in S$, let $G_x \subset G$ denote the stabilizer of x . Then there exists an isomorphism

$$E \cong \bigoplus_{s \in S} \frac{|G_s|}{|G|} \text{Ind}_{G_s}^G (E_s)$$

of representations of G ⁽¹⁾

Proof of Proposition 2.3. By linearity, it is sufficient to prove the proposition in the case where we have objects U and V of $\mathcal{C}^W(P)$ with $f = [U]$ and $g = [V]$. This means that, for all $x \leq y \leq z \in P$, $f_{xy} = [U_{xy}] \in \text{VRep}(W_{xy})$, $g_{yz} = [V_{yz}] \in \text{VRep}(W_{yz})$, and

$$(fg)_{xz} = [(U \otimes V)_{xz}] = \left[\bigoplus_{x \leq y \leq z} U_{xy} \otimes V_{yz} \right] \in \text{VRep}(W_{xz}).$$

The proposition then follows from Lemma 2.7 by taking $E = (U \otimes V)_{xz}$, $S = [x, z]$, and $G = W_{xz}$. $\square$

Let R be a commutative ring. Given an element $f \in I^W(P) \otimes R$ and a pair of elements $x \leq y \in P$, we will write f_{xy} to denote the corresponding element of $\text{VRep}(W_{xy}) \otimes R$.

PROPOSITION 2.8. An element $f \in I^W(P) \otimes R$ is (left or right) invertible if and only if $f_{xx} \in \text{VRep}(W_x) \otimes R$ is invertible for all $x \in P$. In this case, the left and right inverses are unique and they coincide.

Proof. By Proposition 2.3, an element g is a right inverse to f if and only if $g_{xx} = f_{xx}^{-1}$ for all $x \in P$ and

$$\sum_{x \leq y \leq z} \frac{|W_{xyz}|}{|W_{xz}|} \text{Ind}_{W_{xyz}}^{W_{xz}} \left(\left(\text{Res}_{W_{xyz}}^{W_{xy}} f_{xy} \right) \otimes \left(\text{Res}_{W_{xyz}}^{W_{yz}} g_{yz} \right) \right) = 0$$

⁽¹⁾ As in Remark 2.4, we may eliminate the fraction at the cost of choosing one representative of each W -orbit in S .

for all $x < z \in P$.⁽²⁾ The second condition can be rewritten as

$$\left(\text{Res}_{W_{xz}}^{W_x} f_{xx}\right) \otimes g_{xz} = - \sum_{x < y \leq z} \frac{|W_{xyz}|}{|W_{xz}|} \text{Ind}_{W_{xyz}}^{W_{xz}} \left(\left(\text{Res}_{W_{xyz}}^{W_{xy}} f_{xy}\right) \otimes \left(\text{Res}_{W_{xyz}}^{W_{yz}} g_{yz}\right) \right).$$

If f_{xx} is invertible in $\text{VRep}(W_x) \otimes R$, then $\text{Res}_{W_{xz}}^{W_x} f_{xx}$ is invertible in $\text{VRep}(W_{xz}) \otimes R$, and this equation has a unique solution for g . Thus f has a right inverse if and only if $f_{xx} \in \text{VRep}(W_x) \otimes R$ is invertible for all $x \in P$. The argument for left inverses is identical, so it remains only to show that left and right inverses coincide.

Let g be right inverse to f . Then g is also left inverse to some function, which we will denote h . We then have

$$f = f\delta = f(gh) = (fg)h = \delta h = h,$$

so g is left inverse to f , as well. $\square$

3. EQUIVARIANT KAZHDAN–LUSZTIG–STANLEY THEORY

In this section we take R to be the ring $\mathbb{Z}[t]$ and for each $f \in I^W(P) \otimes \mathbb{Z}[t]$ and $x \leq y \in P$, we write $f_{xy}(t)$ for the corresponding component of f . One can regard $f_{xy}(t)$ as a polynomial whose coefficients are virtual representations of W_{xy} , or equivalently as a graded virtual representation of W_{xy} . We assume that P is equipped with a W -invariant **weak rank function** in the sense of [2, Section 2]. This is a collection of natural numbers $\{r_{xy} \in \mathbb{N} \mid x \leq y \in P\}$ with the following properties:

- $r_{xy} > 0$ if $x < y$
- $r_{xy} + r_{yz} = r_{xz}$ if $x \leq y \leq z$
- $r_{xy} = r_{\sigma(x)\sigma(y)}$ if $x \leq y$ and $\sigma \in W$.

Following the notation of [9, Section 2.1], we define

$$\mathcal{J}^W(P) := \left\{ f \in I^W(P) \otimes \mathbb{Z}[t] \mid \deg f_{xy}(t) \leq r_{xy} \text{ for all } x \leq y \right\}$$

along with

$$\mathcal{J}_{1/2}^W(P) := \left\{ f \in I^W(P) \otimes \mathbb{Z}[t] \mid \deg f_{xy}(t) < r_{xy}/2 \text{ and } f_{xx}(t) = \delta_{xx}(t) \right\}.$$

Note that $\mathcal{J}^W(P)$ is a subalgebra of $I^W(P)$, and we define an involution $f \mapsto \bar{f}$ of $\mathcal{J}^W(P)$ by putting $\bar{f}_{xy}(t) := t^{r_{xy}} f_{xy}(t^{-1})$. An element $\kappa \in \mathcal{J}^W(P)$ is called a **P -kernel** if $\kappa_{xx}(t) = \delta_{xx}(t)$ for all $x \in P$ and $\bar{\kappa} = \kappa^{-1}$.

THEOREM 3.1. *If $\kappa \in \mathcal{J}^W(P)$ is a P -kernel, there exists a unique pair of functions $f, g \in \mathcal{J}_{1/2}^W(P)$ such that $\bar{f} = \kappa f$ and $\bar{g} = g\kappa$.*

Proof. We follow the proof in [9, Theorem 2.2]. We will prove existence and uniqueness of f ; the proof for g is identical. Fix elements $x < w \in P$. Suppose that $f_{yw}(t)$ has been defined for all $x < y \leq w$ and that the equation $\bar{f} = \kappa f$ holds where defined. Let

$$Q_{xw}(t) := \sum_{x < y \leq w} \frac{|W_{xyw}|}{|W_{xw}|} \text{Ind}_{W_{xyw}}^{W_{xw}} \left(\left(\text{Res}_{W_{xyw}}^{W_{xy}} \kappa_{xy}(t)\right) \otimes \left(\text{Res}_{W_{xyw}}^{W_{yw}} f_{yw}(t)\right) \right),$$

which is an element of $\text{VRep}(W_{xw}) \otimes \mathbb{Z}[t]$. The equation $\bar{f} = \kappa f$ for the interval $[x, w]$ translates to

$$\bar{f}_{xw}(t) - f_{xw}(t) = Q_{xw}(t).$$

⁽²⁾ If the ring R has integer torsion, then we rewrite this equation without the fractions as described in Remark 2.4.

It is clear that there is at most one polynomial $f_{xw}(t)$ of degree strictly less than $r_{xw}/2$ satisfying this equation. The existence of such a polynomial is equivalent to the statement

$$t^{r_{xw}} Q_{xw}(t^{-1}) = -Q_{xw}(t).$$

To prove this, we observe that

$$\begin{aligned} & t^{r_{xw}} Q_{xw}(t^{-1}) \\ &= t^{r_{xw}} \sum_{x < y \leq w} \frac{|W_{xyw}|}{|W_{xw}|} \operatorname{Ind}_{W_{xyw}}^{W_{xw}} \left(\left(\operatorname{Res}_{W_{xyw}}^{W_{xy}} \kappa_{xy}(t^{-1}) \right) \otimes \left(\operatorname{Res}_{W_{xyw}}^{W_{yw}} f_{yw}(t^{-1}) \right) \right) \\ &= \sum_{x < y \leq w} \frac{|W_{xyw}|}{|W_{xw}|} \operatorname{Ind}_{W_{xyw}}^{W_{xw}} \left(\left(\operatorname{Res}_{W_{xyw}}^{W_{xy}} t^{r_{xy}} \kappa_{xy}(t^{-1}) \right) \otimes \left(\operatorname{Res}_{W_{xyw}}^{W_{yw}} t^{r_{yw}} f_{yw}(t^{-1}) \right) \right) \\ &= \sum_{x < y \leq w} \frac{|W_{xyw}|}{|W_{xw}|} \operatorname{Ind}_{W_{xyw}}^{W_{xw}} \left(\left(\operatorname{Res}_{W_{xyw}}^{W_{xy}} \bar{\kappa}_{xy}(t) \right) \otimes \left(\operatorname{Res}_{W_{xyw}}^{W_{yw}} \bar{f}_{yw}(t) \right) \right) \\ &= \sum_{x < y \leq w} \frac{|W_{xyw}|}{|W_{xw}|} \operatorname{Ind}_{W_{xyw}}^{W_{xw}} \left(\left(\operatorname{Res}_{W_{xyw}}^{W_{xy}} \bar{\kappa}_{xy}(t) \right) \otimes \left(\operatorname{Res}_{W_{xyw}}^{W_{yw}} (\kappa f)_{yw}(t) \right) \right). \end{aligned}$$

This is formally equal to the expression for $(\bar{\kappa}(\kappa f))_{xw} - (\kappa f)_{xw}$, which by associativity is equal to the expression for

$$((\bar{\kappa}\kappa)f)_{xw} - (\kappa f)_{xw} = f_{xw} - (\kappa f)_{xw}.$$

Thus we have

$$\begin{aligned} t^{r_{xw}} Q_{xw}(t^{-1}) &= - \sum_{x < y \leq w} \frac{|W_{xyw}|}{|W_{xw}|} \operatorname{Ind}_{W_{xyw}}^{W_{xw}} \left(\left(\operatorname{Res}_{W_{xyw}}^{W_{xy}} \kappa_{xy}(t) \right) \otimes \left(\operatorname{Res}_{W_{xyw}}^{W_{yw}} f_{yw}(t) \right) \right) \\ &= -Q_{xw}(t). \end{aligned}$$

Thus there is a unique choice of polynomial $f_{xw}(t)$ consistent with the equation $\bar{f} = \kappa f$ on the interval $[x, w]$. $\square$

We will refer to the element $f \in \mathcal{S}_{1/2}^W(P)$ from Theorem 3.1 is the **right equivariant KLS-function** associated with κ , and to g as the **left equivariant KLS-function** associated with κ . For any $x \leq y$, we will refer to the graded virtual representations $f_{xy}(t)$ and $g_{xy}(t)$ as (right or left) **equivariant KLS-polynomials**. When W is the trivial group, these definitions specialize to the ones in [9] Section 2].

EXAMPLE 3.2. Let $\zeta \in \mathcal{S}^W(P)$ be the element defined by letting $\zeta_{xy}(t)$ be the trivial representation of W_{xy} in degree zero for all $x \leq y$, and let $\chi := \zeta^{-1}\zeta$. The function χ is called the **equivariant characteristic function** of P with respect to the action of W . We have $\chi^{-1} = \bar{\zeta}^{-1}\zeta = \bar{\chi}$, so χ is a P -kernel. Since $\bar{\zeta} = \zeta\chi$, ζ is equal to the left KLS-function associated with χ . However, the right KLS-function f associated with χ is much more interesting! See Propositions 4.1 and 4.3 for a special case of this construction.

We next introduce the equivariant analogue of the material in [9] Section 2.3]. If κ is a P -kernel with right and left KLS-functions f and g , we define $Z := g\kappa f \in \mathcal{S}^W(P)$, which we call the **equivariant Z-function** associated with κ . For any $x \leq y$, we will refer to the graded virtual representation $Z_{xy}(t)$ as an **equivariant Z-polynomial**.

PROPOSITION 3.3. *We have $\bar{Z} = Z$.*

Proof. Since $\bar{g} = g\kappa$, we have $Z = g\kappa f = \bar{g}f$. Since $\bar{f} = \kappa f$, we have $Z = g\kappa f = g\bar{f}$. Thus $\bar{Z} = \bar{\bar{g}f} = \bar{\bar{g}}\bar{f} = g\bar{f} = Z$. $\square$

REMARK 3.4. Suppose that $\kappa \in I^W(P)$ is a P -kernel and $f, g, Z \in I^W(P)$ are the associated equivariant KLS-functions and equivariant Z -function. It is immediate from the definitions that, if $\psi : W' \rightarrow W$ is a group homomorphism, then $R_\psi(f), R_\psi(g), R_\psi(Z) \in I^{W'}(P)$ are the equivariant KLS-functions and equivariant Z -function associated with the P -kernel $R_\psi(\kappa) \in I^{W'}(P)$. In particular, if we take W' to be the trivial group, then Remark 2.6 tells us that the ordinary KLS-polynomials and Z -polynomials are recovered from the equivariant KLS-polynomials and Z -polynomials by sending virtual representations to their dimensions.

4. MATROIDS

Let M be a matroid, let L be the lattice of flats of M equipped with the usual weak rank function, and let W be a finite group acting on L . Let $OS_M^W(t)$ be the Orlik–Solomon algebra of M [8], regarded as a graded representation of W . Following [7, Section 2], we define

$$H_M^W(t) := t^{\text{rk } M} OS_M^W(-t^{-1}) \in \text{VRep}(W) \otimes \mathbb{Z}[t].$$

If W is trivial, then $H_M^W(t) \in \mathbb{Z}[t]$ is equal to the characteristic polynomial of M . For any $F \leq G \in L$, let M_{FG} be the minor of M with lattice of flats $[F, G]$ obtained by deleting the complement of G and contracting F ; this matroid inherits an action of the stabilizer group $W_{FG} \subset W$. Define $H \in \mathcal{J}^W(L)$ by putting $H_{FG}(t) = H_{M_{FG}}^{W_{FG}}(t)$ for all $F \leq G$.

PROPOSITION 4.1. *The function H is the equivariant characteristic function of L .*

Proof. It is proved in [7, Lemma 2.5] that $\zeta H = \bar{\zeta}$. Multiplying on the left by ζ^{-1} , we have $H = \zeta^{-1} \bar{\zeta}$, which is the definition of the equivariant characteristic function. $\square$

REMARK 4.2. The proof of [7, Lemma 2.5] is surprisingly difficult.⁽³⁾ Consequently, Proposition 4.1 is a deep fact about Orlik–Solomon algebras, not just a formal consequence of the definitions.

The **equivariant Kazhdan–Lusztig polynomial** $P_M^W(t) \in \text{VRep}(W) \otimes \mathbb{Z}[t]$ was introduced in [7, Section 2.2]. Define $P \in \mathcal{J}_{1/2}^W(L)$ by putting $P_{FG}(t) = P_{M_{FG}}^{W_{FG}}(t)$ for all $F \leq G$. The defining recursion for $P_M^W(t)$ in [7, Theorem 2.8] translates to the formula $\bar{P} = HP$, which immediately implies the following proposition.

PROPOSITION 4.3. *The function P is the right equivariant KLS-function associated with H .*

The **equivariant Z -polynomial** $Z_M^W(t) \in \text{VRep}(W) \otimes \mathbb{Z}[t]$ was introduced in [10, Section 6]. Define $Z \in \mathcal{J}^W(L)$ by putting $Z_{FG}(t) = Z_{M_{FG}}^{W_{FG}}(t)$ for all $F \leq G$. The defining recursion for $Z_M^W(t)$ in [10, Section 6] translates to the formula $Z = \bar{\zeta}P$.

PROPOSITION 4.4. *The function Z is the Z -function associated with H .*

Proof. Example 3.2 tells us that the right KLS-function associated with H is ζ and Proposition 4.3 tells us that the left KLS-function associated with H is P , thus the Z -function is equal $\zeta HP = \bar{\zeta}P = Z$. $\square$

The following corollary was asserted without proof in [10, Section 6], and follows immediately from Propositions 3.3 and 4.4.

⁽³⁾ The difficult part appears in the proof of Lemma 2.4, which is then used to prove Lemma 2.5.

COROLLARY 4.5. *The polynomial $Z_M^W(t)$ is palindromic. That is,*

$$t^{\text{rk } M} Z_M^W(t^{-1}) = Z_M^W(t).$$

When W is the trivial group, Gao and Xie define polynomials $Q_M(t)$ and $\hat{Q}_M(t) = (-1)^{\text{rk } M} Q_M(t)$ with the property that $(P^{-1})_{FG}(t) = \hat{Q}_{M_{FG}}(t)$ [6]. If $\hat{0}$ and $\hat{1}$ are the minimal and maximal flats of M , this is equivalent to the statement that $Q_M(t) = (-1)^{\text{rk } M} (P^{-1})_{\hat{0}\hat{1}}(t)$. The polynomial $Q_M(t)$ is called the **inverse Kazhdan–Lusztig polynomial of M** [4]. Using the machinery of this paper, we may extend their definition to the equivariant setting by defining the **equivariant inverse Kazhdan–Lusztig polynomial**

$$Q_M^W(t) := (-1)^{\text{rk } M} (P^{-1})_{\hat{0}\hat{1}}(t).$$

If we then define $\hat{Q} \in \mathcal{S}_{1/2}^W(L)$ by putting $\hat{Q}_{FG}(t) = (-1)^{r_{FG}} Q_{M_{FG}}^{W_{FG}}(t)$ for all $F \leq G$, we immediately obtain the following proposition.

PROPOSITION 4.6. *The functions P and $\hat{Q}$ are mutual inverses in $I^W(L)$.*

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UNIVERSITY OF OREGON, DEPARTMENT OF MATHEMATICS, EUGENE, OR 97403 (USA)
 Email address: njp@uoregon.edu

(4) The reason for bestowing this name on $Q_M(t)$ rather than $\hat{Q}_M(t)$ is that $Q_M(t)$ has non-negative coefficients; this was conjectured in [6] Conjecture 4.1] and proved in [1] Theorem 1.4].