

Dedekind sums arising from newform Eisenstein series

T. Stucker

University of Idaho, Moscow, ID 83844, USA
stuc7464@vandals.uidaho.edu

A. Vennos

Salisbury University, Salisbury, MD 21801, USA
avennos3@gulls.salisbury.edu

M. P. Young*

Department of Mathematics, Texas A&M University
College Station, TX 77843-3368, USA
myoung@math.tamu.edu

Received 5 August 2019

Accepted 22 March 2020

Published 21 August 2020

For primitive nontrivial Dirichlet characters χ_1 and χ_2 , we study the weight zero newform Eisenstein series $E_{\chi_1, \chi_2}(z, s)$ at $s = 1$. The holomorphic part of this function has a transformation rule that we express in finite terms as a generalized Dedekind sum. This gives rise to the explicit construction (in finite terms) of elements of $H^1(\Gamma_0(N), \mathbb{C})$. We also give a short proof of the reciprocity formula for this Dedekind sum.

Keywords: Dedekind sums; reciprocity formula; Eisenstein series; newform.

Mathematics Subject Classification 2020: 11F20, 11F03, 11F67

1. Introduction

1.1. Background and statement of result

Let χ_1, χ_2 be primitive Dirichlet characters modulo q_1, q_2 , respectively, with $\chi_1 \chi_2(-1) = 1$. The weight zero newform Eisenstein series attached to χ_1 and χ_2 is defined (initially) as

$$E_{\chi_1, \chi_2}(z, s) = \frac{1}{2} \sum_{(c, d)=1} \frac{(q_2 y)^s \chi_1(c) \chi_2(d)}{|cq_2 z + d|^{2s}}, \quad \operatorname{Re}(s) > 1.$$

*Corresponding author.

Here, E_{χ_1, χ_2} is an automorphic form on the congruence subgroup $\Gamma_0(q_1 q_2)$ with central character $\psi = \chi_1 \overline{\chi_2}$. Precisely, for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q_1 q_2)$,

$$E_{\chi_1, \chi_2}(\gamma z, s) = \psi(\gamma) E_{\chi_1, \chi_2}(z, s), \quad (1.1)$$

where $\psi(\gamma) = \psi(d)$. Moreover, E_{χ_1, χ_2} is an eigenfunction of all the Hecke operators (see (2.4)), which indicates why it is called a newform. We refer to [15] for the properties of the newform Eisenstein series used in this paper.

The classical Kronecker limit formula relates the constant term in the Laurent expansion of $E_{1,1}(z, s)$ at $s = 1$ to $\log \eta$, where η is the Dedekind η -function given by

$$\eta(z) = e^{\pi i z / 12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}).$$

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(1)$, with $c > 0$, $\log \eta$ obeys the transformation formula

$$\log \eta(\gamma z) = \log \eta(z) + \pi i \left(\frac{a+d}{12c} + s(-d, c) \right) + \frac{1}{2} \log(-i(cz + d)),$$

where $s(h, k)$ is the classical Dedekind sum given by

$$s(h, k) = \sum_{r=1}^{k-1} \frac{r}{k} \left(\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2} \right).$$

See [1] for more background on the η -function and the classical Dedekind sums.

Consider the “completed” Eisenstein series defined by

$$E_{\chi_1, \chi_2}^*(z, s) = \frac{(q_2/\pi)^s}{\tau(\chi_2)} \Gamma(s) L(2s, \chi_1 \chi_2) E_{\chi_1, \chi_2}(z, s). \quad (1.2)$$

Here, τ denotes the Gauss sum given by $\tau(\chi) = \sum_{n \pmod{q}} \chi(n) e_q(n)$, where $e_q(n) = e(n/q)$, $e(x) = \exp(2\pi i x)$, and χ is a Dirichlet character modulo q . The Fourier expansion for E_{χ_1, χ_2}^* is conveniently stated in [15] (see also [10]). When $q_1, q_2 \neq 1$, the Fourier expansion simplifies as

$$E_{\chi_1, \chi_2}^*(z, s) = 2\sqrt{y} \sum_{n \neq 0} \lambda_{\chi_1, \chi_2}(n, s) e(nx) K_{s-\frac{1}{2}}(2\pi|n|y), \quad (1.3)$$

where K_ν is the K -Bessel function and

$$\lambda_{\chi_1, \chi_2}(n, s) = \chi_2(\operatorname{sgn}(n)) \sum_{ab=|n|} \chi_1(a) \overline{\chi_2}(b) \left(\frac{b}{a} \right)^{s-\frac{1}{2}}. \quad (1.4)$$

The Fourier expansion gives the analytic continuation of $E_{\chi_1, \chi_2}^*(z, s)$ to $s \in \mathbb{C}$. In particular, there is no pole at $s = 1$, and (1.3) specializes as

$$E_{\chi_1, \chi_2}^*(z, 1) = f_{\chi_1, \chi_2}(z) + \chi_2(-1) \overline{f_{\chi_1, \chi_2}}(z), \quad (1.5)$$

where

$$f_{\chi_1, \chi_2}(z) = \sum_{n=1}^{\infty} \frac{\lambda_{\chi_1, \chi_2}(n, 1)}{\sqrt{n}} e(nz), \quad (1.6)$$

using $K_{1/2}(2\pi y) = 2^{-1}y^{-1/2} \exp(-2\pi y)$. Because $E_{\chi_1, \chi_2}^*(z, s)$ has no pole at $s = 1$, (1.5) is the analogue of the Kronecker limit formula and the function f_{χ_1, χ_2} is the analogue of $\log \eta$.

Define

$$\phi_{\chi_1, \chi_2}(\gamma, z) = \phi_{\chi_1, \chi_2}(\gamma) = f_{\chi_1, \chi_2}(\gamma z) - \psi(\gamma) f_{\chi_1, \chi_2}(z),$$

for $\gamma \in \Gamma_0(q_1 q_2)$ and $z \in \mathbb{H}$; in Lemma 2.1, we show that ϕ_{χ_1, χ_2} is independent of z .

Definition 1. Let χ_1, χ_2 be primitive Dirichlet characters of conductors q_1, q_2 , respectively, with $q_1, q_2 > 1$, and $\chi_1 \chi_2(-1) = 1$. For $\gamma \in \Gamma_0(q_1 q_2)$, define the Dedekind sum S_{χ_1, χ_2} associated to the newform Eisenstein series E_{χ_1, χ_2} by

$$S_{\chi_1, \chi_2}(\gamma) = \frac{\tau(\overline{\chi_1})}{\pi i} \phi_{\chi_1, \chi_2}(\gamma). \quad (1.7)$$

Let B_1 denote the first Bernoulli function given by

$$B_1(x) = \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

The first main result in this paper is an evaluation of S_{χ_1, χ_2} in finite terms.

Theorem 1.1. Let χ_1, χ_2 be primitive Dirichlet characters of conductors q_1, q_2 , respectively, with $q_1, q_2 > 1$, and $\chi_1 \chi_2(-1) = 1$. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q_1 q_2)$. For $c \geq 1$, then

$$S_{\chi_1, \chi_2}(\gamma) = \sum_{j \pmod{c}} \sum_{n \pmod{q_1}} \overline{\chi_2}(j) \overline{\chi_1}(n) B_1\left(\frac{j}{c}\right) B_1\left(\frac{n}{q_1} + \frac{aj}{c}\right). \quad (1.8)$$

Our second main result gives a simple proof of the following reciprocity formula.

Theorem 1.2. For $\gamma = \begin{pmatrix} a & b \\ c q_1 q_2 & d \end{pmatrix} \in \Gamma_0(q_1 q_2)$, let $\gamma' = \begin{pmatrix} d & -b \\ -b q_1 q_2 & a \end{pmatrix} \in \Gamma_0(q_1 q_2)$. If χ_1 and χ_2 are even, then

$$S_{\chi_1, \chi_2}(\gamma) = S_{\chi_2, \chi_1}(\gamma'). \quad (1.9)$$

If χ_1 and χ_2 are odd, then

$$S_{\chi_1, \chi_2}(\gamma) = -S_{\chi_2, \chi_1}(\gamma') + (1 - \psi(\gamma)) \frac{\tau(\overline{\chi_1}) \tau(\overline{\chi_2})}{(\pi i)^2} L(1, \chi_1) L(1, \chi_2). \quad (1.10)$$

The main step in the proof of Theorem 1.2 is to study the action of the Fricke involution $\omega_{q_1 q_2} = \begin{pmatrix} 0 & -1 \\ q_1 q_2 & 0 \end{pmatrix}$. Since E_{χ_1, χ_2} is a pseudo-eigenvector of all the Atkin–Lehner operators (see [15, Sec. 9]), it seems plausible that an adaptation of the proof can give a family of reciprocity formulas, one for each Atkin–Lehner operator.

Many authors have investigated generalized Dedekind sums arising from various types of Eisenstein series. Goldstein [8] studies the Eisenstein series attached to cusps for the principal congruence subgroup $\Gamma(N)$. Nagasaka [13] and Goldstein

and Razar [9] investigate functions essentially equivalent, in our notation, to $f_{\chi, \chi}$; they derive the transformation properties of $f_{\chi, \chi}$ (including the reciprocity formula) by relation to the Mellin transform of the product of Dirichlet L -functions instead of via properties of Eisenstein series.

The generalized Dedekind sums attached to pairs of Dirichlet characters have appeared in the literature in connection with certain Eisenstein-type series. Berndt [3, Sec. 6] defines generalized Dedekind sums which essentially correspond to the right-hand side of (1.8) when $q_1 = 1$ or $q_2 = 1$. Berndt derives properties of his Dedekind sums using a different variant of Eisenstein series than what is used in this paper; Berndt's Eisenstein-type series have more complicated transformation properties than E_{χ_1, χ_2} (compare [2, Theorem 2] to (1.1)). Many authors have studied generalized Dedekind sums, such as [5, 6, 12, 14], based ultimately on Berndt's transformation formulas.

Reciprocity formulas for variants of S_{χ_1, χ_2} , with general pairs of characters χ_1, χ_2 have appeared in [6]. However, it appears that Theorem 1.2 is new (e.g., [6, Theorem 1] excludes the case $p = 1$ which would correspond to Theorem 1.2).

In Sec. 5, we connect S_{χ_1, χ_2} to the Eisenstein component of the Eichler–Shimura isomorphism in weight 2.

2. Basic Properties of S_{χ_1, χ_2}

Lemma 2.1. *The function ϕ_{χ_1, χ_2} is independent of z .*

Proof. Since $E_{\chi_1, \chi_2}^*(\gamma z, 1) = \psi(\gamma)E_{\chi_1, \chi_2}^*(z, 1)$ and $E_{\chi_1, \chi_2}^*(z, 1) = f_{\chi_1, \chi_2}(z) + \chi_2(-1)\overline{f_{\chi_1, \chi_2}}(z)$, it immediately follows that

$$\phi_{\chi_1, \chi_2}(\gamma, z) = -\chi_2(-1)\overline{\phi_{\chi_1, \chi_2}}(\gamma, z). \quad (2.1)$$

Since ϕ_{χ_1, χ_2} is holomorphic and $\overline{\phi_{\chi_1, \chi_2}}$ is antiholomorphic, ϕ_{χ_1, χ_2} must be constant in z . $\square$

For later reference, we point out a symmetrized form for ϕ_{χ_1, χ_2} following from (2.1):

$$\phi_{\chi_1, \chi_2}(\gamma) = \frac{1}{2}(\phi_{\chi_1, \chi_2}(\gamma) - \chi_2(-1)\overline{\phi_{\chi_1, \chi_2}}(\gamma)). \quad (2.2)$$

Lemma 2.2. *Let $\gamma_1, \gamma_2 \in \Gamma_0(q_1 q_2)$. Then*

$$S_{\chi_1, \chi_2}(\gamma_1 \gamma_2) = S_{\chi_1, \chi_2}(\gamma_1) + \psi(\gamma_1)S_{\chi_1, \chi_2}(\gamma_2). \quad (2.3)$$

Remarks. It is obvious from the definition that $S_{\chi_1, \chi_2}(\gamma) = 0$ if $\gamma = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, for $n \in \mathbb{Z}$, and consequently $S_{\chi_1, \chi_2}(\gamma)$ only depends on the lower row of γ (or, alternatively, the first column of γ).

Let $G = \Gamma_0(q_1 q_2)$ and $M = \mathbb{C}$, and consider the action $\gamma.z$ of G on M given by $\gamma.z = \psi(\gamma)z$. Note G acts via automorphisms on M (as a module). With this notation, Lemma 2.2 shows that S_{χ_1, χ_2} is a 1-cocycle (or a crossed homomorphism)

for this group action of G on M . Hence, S_{χ_1, χ_2} gives rise to an element of $H^1(G, M)$. In particular, if ψ is trivial then $H^1(G, M) = \text{Hom}(\Gamma_0(q_1 q_2), \mathbb{C})$ (i.e. S_{χ_1, χ_2} is a group homomorphism). Note also that ψ is trivial on $\Gamma_1(q_1 q_2)$ so S_{χ_1, χ_2} may always be viewed as an element of $\text{Hom}(\Gamma_1(q_1 q_2), \mathbb{C})$.

Proof. Since ψ is multiplicative, and by Lemma 2.1, we have

$$\begin{aligned} \phi_{\chi_1, \chi_2}(\gamma_1 \gamma_2) &= \underbrace{f_{\chi_1, \chi_2}(\gamma_1 \gamma_2 z) - \psi(\gamma_1) f_{\chi_1, \chi_2}(\gamma_2 z)}_{\phi_{\chi_1, \chi_2}(\gamma_1)} \\ &\quad + \psi(\gamma_1) \underbrace{(f_{\chi_1, \chi_2}(\gamma_2 z) - \psi(\gamma_2) f_{\chi_1, \chi_2}(z))}_{\phi_{\chi_1, \chi_2}(\gamma_2)}. \end{aligned}$$

□

Let T_n be the Hecke operator acting on weight 0 periodic functions, with character χ (cf. [11, (6.13)]), defined by

$$T_n f(z) = \frac{1}{\sqrt{n}} \sum_{ad=n} \chi(a) \sum_{b \pmod{d}} f\left(\frac{az+b}{d}\right).$$

It is easy to check that

$$T_n E_{\chi_1, \chi_2}^*(z, s) = \lambda_{\chi_1, \chi_2}(n, s) E_{\chi_1, \chi_2}^*(z, s), \quad (2.4)$$

for any $n \geq 1$. We remark in passing that

$$T_n f_{\chi_1, \chi_2} = \lambda_{\chi_1, \chi_2}(n, 1) f_{\chi_1, \chi_2}, \quad (2.5)$$

which follows immediately from (1.5) and the fact that the Hecke operators preserve holomorphicity (and anti-holomorphicity).

3. Proof of Theorem 1.1

Our goal for the proof of Theorem 1.1 is to use properties of f_{χ_1, χ_2} in order to simplify ϕ_{χ_1, χ_2} and write it in finite terms. Our process loosely follows the methodology of Goldstein [8]. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q_1 q_2)$, with $c \geq 1$, and let $z = \frac{-d}{c} + \frac{i}{c^2 u}$ for some $u > 0$. Then $\gamma z = \frac{a}{c} + iu$, and

$$\phi_{\chi_1, \chi_2}(\gamma) = \lim_{u \rightarrow 0^+} \left[f_{\chi_1, \chi_2}\left(\frac{a}{c} + iu\right) - \psi(\gamma) f_{\chi_1, \chi_2}\left(\frac{-d}{c} + \frac{i}{c^2 u}\right) \right].$$

From the Fourier expansion of E_{χ_1, χ_2}^* , it is clear that $\lim_{u \rightarrow 0^+} f_{\chi_1, \chi_2}\left(\frac{-d}{c} + \frac{i}{c^2 u}\right) = 0$. Thus,

$$\phi_{\chi_1, \chi_2}(\gamma) = \lim_{u \rightarrow 0^+} f_{\chi_1, \chi_2}\left(\frac{a}{c} + iu\right). \quad (3.1)$$

This is the “constant term” in the Fourier expansion of f_{χ_1, χ_2} around the cusp a/c .

To evaluate this limit, we begin by writing f_{χ_1, χ_2} as

$$f_{\chi_1, \chi_2}(z) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{\chi_1(l) \overline{\chi_2}(k)}{l} e(klz).$$

Then

$$f_{\chi_1, \chi_2}(z) = \sum_{l=1}^{\infty} \frac{\chi_1(l)}{l} \theta_{\chi_2}(z, l), \quad \text{where } \theta_{\chi}(z, l) := \sum_{k=1}^{\infty} \bar{\chi}(k) e(klz). \quad (3.2)$$

The following lemma will be used in several of the proofs as follows.

Lemma 3.1. *Let χ be a character of conductor q . Let $a, c, l \in \mathbb{Z}$ with $c \geq 1$, $c \equiv 0 \pmod{q}$, $(a, c) = 1$, and $l \not\equiv 0 \pmod{\frac{c}{q}}$. Then*

$$\sum_{j \pmod{c}} \bar{\chi}(j) e_c(alj) = 0.$$

Proof. Let $j = A + qB$ where A runs modulo q and B runs modulo c/q . Then

$$\sum_{j \pmod{c}} \bar{\chi}(j) e_c(alj) = \sum_{A \pmod{q}} \bar{\chi}(A) e_c(alA) \sum_{B \pmod{c/q}} e_{c/q}(alB).$$

Since $\frac{c}{q} \nmid al$, the sum over B vanishes. $\square$

Lemma 3.2. *Let χ be a character of conductor q . Let $a, c, l \in \mathbb{Z}$ with $c \geq 1$, $c \equiv 0 \pmod{q}$, $(a, c) = 1$, and $l \not\equiv 0 \pmod{\frac{c}{q}}$. Then*

$$\theta_{\chi}\left(\frac{a}{c} + iu, l\right) = \sum_{j=1}^{c-1} \bar{\chi}(j) e_c(alj) \frac{x^j - 1}{1 - x^c}, \quad \text{where } x = e(iul).$$

Proof. We have

$$\theta_{\chi}\left(\frac{a}{c} + iu, l\right) = \sum_{k=0}^{\infty} \bar{\chi}(k) e_c(akl) x^k.$$

Now, let $k = j + mc$ where $0 \leq j < c$ and m runs over non-negative integers. Then

$$\theta_{\chi}\left(\frac{a}{c} + iu, l\right) = \sum_{j=0}^{c-1} \bar{\chi}(j) e_c(ajl) x^j \sum_{m=0}^{\infty} x^{mc} = \sum_{j=1}^{c-1} \bar{\chi}(j) e_c(ajl) \frac{x^j}{1 - x^c}. \quad (3.3)$$

Using Lemma 3.1 and adding $0 = \sum_{j=1}^{c-1} \bar{\chi}(j) e_c(ajl) \frac{-1}{1 - x^c}$ to (3.3) completes the proof. $\square$

Corollary 3.3. *Under the same assumptions as Lemma 3.2,*

$$\lim_{u \rightarrow 0^+} \theta_{\chi}\left(\frac{a}{c} + iu, l\right) = - \sum_{j \pmod{c}} \bar{\chi}(j) B_1\left(\frac{j}{c}\right) e_c(alj).$$

Proof. As u approaches 0, $x = e(iul)$ approaches 1, and $\lim_{x \rightarrow 1} \frac{x^j - 1}{1 - x^c} = \frac{-j}{c}$. Thus,

$$\begin{aligned} \lim_{u \rightarrow 0} \theta_{\chi}\left(\frac{a}{c} + iu, l\right) &= \sum_{j=1}^{c-1} \frac{-j}{c} \bar{\chi}(j) e_c(alj) \\ &= - \sum_{j \pmod{c}} \bar{\chi}(j) \left(\frac{j}{c} - \left\lfloor \frac{j}{c} \right\rfloor - \frac{1}{2} + \frac{1}{2}\right) e_c(alj). \end{aligned}$$

Note $\overline{\chi}(j)(\frac{j}{c} - \lfloor \frac{j}{c} \rfloor - \frac{1}{2}) = \overline{\chi}(j)B_1(\frac{j}{c})$, since $\overline{\chi}(j) = 0$ when $\frac{j}{c} \in \mathbb{Z}$, so using Lemma 3.1 again finishes the proof. $\square$

Remark. We need a definition of the generalized Bernoulli function for a (primitive) Dirichlet character χ modulo q , which is stated in [4, Definition 1]. One may easily unify Berndt's formulas as

$$B_{1,\chi}(x) = \frac{-\tau(\overline{\chi})}{2\pi i} \sum_{\substack{l \in \mathbb{Z} \\ l \neq 0}} \frac{\chi(l)}{l} e_q(lx). \quad (3.4)$$

Proof of Theorem 1.1. We apply (3.2) to (3.1). Provided that we can interchange the limits (see Lemma 3.4),

$$\phi_{\chi_1, \chi_2}(\gamma) = \lim_{u \rightarrow 0^+} \sum_{l=1}^{\infty} \frac{\chi_1(l)}{l} \theta_{\chi_2} \left(\frac{a}{c} + iu, l \right) = \sum_{l=1}^{\infty} \frac{\chi_1(l)}{l} \lim_{u \rightarrow 0^+} \theta_{\chi_2} \left(\frac{a}{c} + iu, l \right). \quad (3.5)$$

Then by Corollary 3.3,

$$\phi_{\chi_1, \chi_2}(\gamma) = - \sum_{l=1}^{\infty} \frac{\chi_1(l)}{l} \sum_{j \pmod{c}} \overline{\chi_2}(j) B_1 \left(\frac{j}{c} \right) e_c(alj).$$

Applying (2.2), we obtain

$$\begin{aligned} \phi_{\chi_1, \chi_2}(\gamma) &= - \frac{1}{2} \sum_{l=1}^{\infty} \frac{\chi_1(l)}{l} \sum_{j \pmod{c}} \overline{\chi_2}(j) B_1 \left(\frac{j}{c} \right) e_c(alj) \\ &\quad + \frac{\chi_2(-1)}{2} \sum_{l=1}^{\infty} \frac{\chi_1(l)}{l} \sum_{j \pmod{c}} \overline{\chi_2}(j) B_1 \left(\frac{j}{c} \right) e_c(-alj). \end{aligned}$$

Changing variables $l \rightarrow -l$ and using $\chi_1 \chi_2(-1) = 1$, this simplifies as

$$\phi_{\chi_1, \chi_2}(\gamma) = - \frac{1}{2} \sum_{j \pmod{c}} \overline{\chi_2}(j) B_1 \left(\frac{j}{c} \right) \sum_{l \neq 0} \frac{\chi_1(l)}{l} e_c(alj).$$

Letting $c = c' q_1$ and substituting (3.4), we obtain

$$\phi_{\chi_1, \chi_2}(\gamma) = \frac{\pi i}{\tau(\overline{\chi_1})} \sum_{j \pmod{c}} \overline{\chi_2}(j) B_1 \left(\frac{j}{c} \right) B_{1, \chi_1} \left(\frac{aj}{c'} \right). \quad (3.6)$$

Next, we use [4, Theorem 3.1] which states

$$B_{1,\chi}(x) = \sum_{n=1}^{q-1} \overline{\chi}(n) B_1 \left(\frac{x+n}{q} \right). \quad (3.7)$$

Substituting (3.7) into (3.6) completes the proof. $\square$

Lemma 3.4. *The interchange of limits in (3.5) is justified.*

Proof. Applying Lemma 3.2 to the left-hand side of (3.5), we have

$$\phi_{\chi_1, \chi_2}(\gamma) = \lim_{u \rightarrow 0^+} \sum_{l=1}^{\infty} \frac{\chi_1(l)}{l} \sum_{j=0}^{c-1} \overline{\chi_2}(j) e_c(alj) \frac{x^j - 1}{1 - x^c}.$$

Let $R(x) = R_{j,c}(x) = \frac{x^j - 1}{1 - x^c}$. Note that R is a rational function (in x) with no poles on $0 \leq x \leq 1$, so it is smooth on this interval.

Let $a_l = \chi_1(l) e_c(alj)$, $b_l = \frac{1}{l} R(e^{-2\pi l u})$, and $S(N) = \sum_{l=1}^N a_l$. By Lemma 3.1, $\sum_{l \pmod{c}} a_l = 0$ (since we may assume $(j, q_2) = 1$ whence $j \not\equiv 0 \pmod{c/q_1}$), so $S(N)$ is bounded (independently of u , of course). Therefore, by partial summation, $\sum_{l=1}^{\infty} a_l b_l = \sum_{l=1}^{\infty} S(l)(b_l - b_{l+1})$. We claim $|b_l - b_{l+1}| = O(l^{-2})$, with an implied constant independent of u . Given this claim, the Weierstrass M -test shows the sum converges uniformly in u which justifies the interchange of limits.

Now, we show the claim. We have

$$|b_{l+1} - b_l| = \frac{1}{l} \left| R(e^{-2\pi(l+1)u}) - R(e^{-2\pi l u}) - \frac{R(e^{-2\pi(l+1)u})}{l+1} \right|. \quad (3.8)$$

Here, $\frac{|R(e^{-2\pi(l+1)u})|}{l+1} \leq \frac{C_1}{l}$ for some constant C_1 independent of l and u . By the mean value theorem,

$$R(e^{-2\pi l u}) - R(e^{-2\pi(l+1)u}) = (e^{-2\pi l u} - e^{-2\pi(l+1)u}) R'(t)$$

for some $t \in [0, 1]$. Since $R(t)$ is smooth on $[0, 1]$, then $|R'(t)| \leq C_2$ for some constant C_2 independent of l and u . Additionally,

$$e^{-2\pi l u} - e^{-2\pi(l+1)u} = e^{-2\pi l u} (1 - e^{-2\pi u}) \leq \frac{C_3}{l} u e^{-2\pi l u} \leq \frac{C_4}{l},$$

for some constants C_3, C_4 , since $x e^{-x}$ is bounded for $0 \leq x < \infty$. Putting everything together proves the claim. $\square$

4. Proof of Theorem 1.2

Let $\omega = \omega_{q_1 q_2} = \begin{pmatrix} 0 & -1 \\ q_1 q_2 & 0 \end{pmatrix}$ be the Fricke involution. An easy calculation shows that if $\gamma = \begin{pmatrix} a & b \\ c q_1 q_2 & d \end{pmatrix} \in \Gamma_0(q_1 q_2)$, then

$$\omega \gamma = \gamma' \omega, \quad (4.1)$$

where $\gamma' = \begin{pmatrix} d & -c \\ -b q_1 q_2 & a \end{pmatrix} \in \Gamma_0(q_1 q_2)$. Note the map $\gamma \rightarrow \gamma'$ is an involution. The newform Eisenstein series is a generalized eigenfunction of the Fricke involution, precisely it satisfies (see [15, Sec. 9.2])

$$E_{\chi_1, \chi_2}(\omega z, s) = \chi_2(-1) E_{\chi_2, \chi_1}(z, s).$$

For the completed Eisenstein series, using (1.2) we deduce

$$E_{\chi_1, \chi_2}^*(\omega z, 1) = \delta_{\chi_1, \chi_2} E_{\chi_2, \chi_1}^*(z, 1), \quad \text{where } \delta_{\chi_1, \chi_2} = \chi_2(-1) \frac{\tau(\chi_1) q_2}{\tau(\chi_2) q_1}.$$

Define $\phi_{\chi_1, \chi_2}(\omega) = f_{\chi_1, \chi_2}(\omega z) - \delta_{\chi_1, \chi_2} f_{\chi_2, \chi_1}(z)$, and similarly define

$$S_{\chi_1, \chi_2}(\omega) = \frac{\tau(\overline{\chi_1})}{\pi i} \phi_{\chi_1, \chi_2}(\omega). \quad (4.2)$$

An easy modification of the proof of Lemma 2.1 shows that $\phi_{\chi_1, \chi_2}(\omega)$ is independent of z (justifying the notation).

Lemma 4.1. *Let χ_1, χ_2 be primitive Dirichlet characters of conductors q_1, q_2 , respectively, with $q_1, q_2 > 1$, and $\chi_1 \chi_2(-1) = 1$. Then*

$$S_{\chi_1, \chi_2}(\omega) = \begin{cases} \frac{\tau(\overline{\chi_1})\tau(\overline{\chi_2})}{(\pi i)^2} L(1, \chi_1) L(1, \chi_2), & \chi_1(-1) = \chi_2(-1) = -1, \\ 0, & \chi_1(-1) = \chi_2(-1) = 1. \end{cases} \quad (4.3)$$

Proof. The ideas are similar to the proof of Theorem 1.2, so we will be brief. We have $\phi_{\chi_1, \chi_2}(\omega) = \lim_{u \rightarrow 0^+} f_{\chi_1, \chi_2}(iu)$. Then following the idea of proof in Lemma 3.2, we have

$$f_{\chi_1, \chi_2}(iu) = \sum_{\ell=1}^{\infty} \frac{\chi_1(\ell)}{\ell} \sum_{0 \leq j < q_2} \overline{\chi_2}(j) \frac{x^j - 1}{1 - x^{q_2}}, \quad x = e(\ell i u).$$

Letting $u \rightarrow 0^+$ (using a variant on Lemma 3.4 to change the limits) gives

$$\phi_{\chi_1, \chi_2}(\omega) = -L(1, \chi_1) B_{1, \chi_2}(0),$$

using (3.7). Finally, we use (3.4) to complete the proof. $\square$

Now, we calculate $f_{\chi_1, \chi_2}(\omega \gamma z) - \delta_{\chi_1, \chi_2} \overline{\psi}(\gamma) f_{\chi_2, \chi_1}(z)$ in two ways. One expression is

$$\underbrace{f_{\chi_1, \chi_2}(\omega \gamma z) - \delta_{\chi_1, \chi_2} f_{\chi_2, \chi_1}(\gamma z)}_{\phi_{\chi_1, \chi_2}(\omega)} + \delta_{\chi_1, \chi_2} \underbrace{[f_{\chi_2, \chi_1}(\gamma z) - \overline{\psi}(\gamma) f_{\chi_2, \chi_1}(z)]}_{\phi_{\chi_2, \chi_1}(\gamma)}.$$

Alternatively, using (4.1), it equals

$$\underbrace{f_{\chi_1, \chi_2}(\gamma' \omega z) - \psi(\gamma') f_{\chi_1, \chi_2}(\omega z)}_{\phi_{\chi_1, \chi_2}(\gamma')} + \overline{\psi}(\gamma) \underbrace{[f_{\chi_1, \chi_2}(\omega z) - \delta_{\chi_1, \chi_2} f_{\chi_1, \chi_2}(z)]}_{\phi_{\chi_1, \chi_2}(\omega)},$$

where we have used $\psi(\gamma') = \psi(a) = \overline{\psi}(d) = \overline{\psi}(\gamma)$. Equating the two expressions, we derive

$$\phi_{\chi_1, \chi_2}(\gamma') - \delta_{\chi_1, \chi_2} \phi_{\chi_2, \chi_1}(\gamma) = (1 - \overline{\psi}(\gamma)) \phi_{\chi_1, \chi_2}(\omega).$$

Converting the notation using (1.7), and using $\delta_{\chi_1, \chi_2} \frac{\tau(\overline{\chi_1})}{\tau(\overline{\chi_2})} = \chi_1(-1)$, we derive

$$S_{\chi_1, \chi_2}(\gamma') - \chi_1(-1) S_{\chi_2, \chi_1}(\gamma) = (1 - \overline{\psi}(\gamma)) S_{\chi_1, \chi_2}(\omega).$$

Using Lemma 4.1 and switching the roles of γ and γ' completes the proof of Theorem 1.2.

5. Remarks on the Eichler–Shimura Isomorphism

Let $E_{2,\chi_1,\chi_2}(z)$ be the holomorphic weight 2 Eisenstein series attached to the primitive nontrivial characters χ_1, χ_2 , defined by (using the notation (1.4))

$$E_{2,\chi_1,\chi_2}(z) = 2 \sum_{n=1}^{\infty} n^{1/2} \lambda_{\chi_1,\chi_2}(n, 1) q^n, \quad q = e^{2\pi iz}.$$

See [7, Sec. 4.6] for more details. The Eichler–Shimura map applied to E_{2,χ_1,χ_2} is defined by

$$\gamma \mapsto \int_{\infty}^{\gamma(\infty)} E_{2,\chi_1,\chi_2}(z) dz,$$

for $\gamma \in \Gamma_0(q_1 q_2)$. By direct calculation with (1.6), we have

$$\frac{d}{dz} \frac{1}{\pi i} f_{\chi_1,\chi_2}(z) = E_{2,\chi_1,\chi_2}(z).$$

Therefore the Eichler–Shimura map applied to E_{2,χ_1,χ_2} is precisely $\tau(\overline{\chi_1}) S_{\chi_1,\chi_2}$.

Acknowledgments

The third author thanks Riad Masri and Ian Petrow for thoughtful comments.

This work was conducted in summer 2018 during an REU conducted at Texas A&M University. The authors thank the Department of Mathematics at Texas A&M and the NSF for supporting the REU. In addition, this material is based upon work supported by the National Science Foundation under agreement No. DMS-170222 (M.Y.). Any opinions, findings and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

References

- [1] T. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, 2nd edn., Graduate Texts in Mathematics, Vol. 41 (Springer-Verlag, New York, 1990), x+204 pp.
- [2] B. Berndt, *Character Transformation Formulae Similar to Those for The Dedekind Eta-Function*, Proceedings of Symposia in Pure Mathematics, Vol. 24 (American Mathematical Society, Providence, RI, 1973), pp. 9–30.
- [3] B. Berndt, On Eisenstein series with characters and the values of Dirichlet L -functions, *Acta Arith.* **28**(3) (1975/1976) 299–320.
- [4] B. Berndt, Character analogues of the Poisson and Euler–MacLaurin summation formulas with applications, *J. Number Theory* **7**(4) (1975) 413–445.
- [5] M. Cenkci, M. Can and V. Kurt, Degenerate and character Dedekind sums, *J. Number Theory* **124**(2) (2007) 346–363.
- [6] M. C. Dağlı and M. Can, On reciprocity formula of character Dedekind sums and the integral of products of Bernoulli polynomials, *J. Number Theory* **156** (2015) 105–124.
- [7] F. Diamond and J. Shurman, *A First Course in Modular Forms*, Graduate Texts in Mathematics, Vol. 228 (Springer-Verlag, New York, 2005), pp. xvi+436.

- [8] L. Goldstein, Dedekind sums for a Fuchsian group, I., *Nagoya Math. J.* **50** (1973) 21–47.
- [9] L. J. Goldstein and M. Razar, The theory of Hecke integrals, *Nagoya Math. J.* **63** (1976) 93–121.
- [10] M. N. Huxley, *Scattering Matrices for Congruence Subgroups*, Modular Forms (Horwood, Chichester, Durham, 1983), pp. 141–156.
- [11] H. Iwaniec, *Topics in Classical Automorphic Forms*, *Graduate Studies in Mathematics*, Vol. 17 (American Mathematical Society, Providence, RI, 1997).
- [12] J. Meyer, Character analogues of Dedekind sums and transformations of analytic Eisenstein series, *Pacific J. Math.* **194**(1) (2000) 137–164.
- [13] C. Nagasaka, On generalized Dedekind sums attached to Dirichlet characters, *J. Number Theory* **19**(3) (1984) 374–383.
- [14] C. Sekine, On Eisenstein series with characters and Dedekind sums, *Acta Arithmetica* **116**(1) (2005) 1–11.
- [15] M. Young, Explicit calculations with Eisenstein series, *J. Number Theory* **199** (2019) 1–48.