DECOMPOSITION NUMBERS AND GLOBAL PROPERTIES

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ABSTRACT. If G is a finite group, p is a prime, and B is a Brauer p-block with decomposition matrix D_B, we study non-zero columns of D_B and possible interactions with the structure of B.

Dedicated to the memory of Jan Saxl

1. Introduction

Recall that if G is a finite group, p is a prime, $\mathrm{IBr}(G)$ is a set of irreducible p-Brauer characters of G and $\chi \in \mathrm{Irr}(G)$ is an irreducible complex character of G, then the restriction χ° of χ to the p-regular elements of G uniquely decomposes as

$$\chi^\circ = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi,\varphi}\varphi,$$

for uniquely determined non-negative integers $d_{\chi,\varphi}$, called the *decomposition numbers*. These decomposition numbers constitute nothing less than the main connection between the representations of characteristic zero and characteristic p.

There are recent spectacular results on decomposition numbers of groups of Lie type [BDT]; there are new and old exciting conjectures on the decomposition numbers of groups of Lie type and of symmetric groups; and there are classical theorems which completely determine them in blocks with certain defect groups. The Fong-Swan theorem, in addition, tells us that given $\varphi \in \mathrm{IBr}(G)$, then there is a $\chi \in \mathrm{Irr}(G)$ such that $d_{\chi,\varphi} = 1$ and $d_{\chi,\mu} = 0$ for $\mu \neq \varphi$, whenever G is p-solvable. But all these facts and speculations only occur in the realm of specific families of groups: there does not seem to be general results on decomposition numbers for arbitrary general finite groups.

Our objectives here are far more modest, and yet, we cannot completely solve them. In [NT], for $p > 3$, we studied the relation between certain columns of the

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decomposition matrix

$$D = (d_{\chi,\varphi})$$

of G and local properties of G. Specifically, we gave strong evidence that $d_{\chi,1} \neq 0$ for all $\chi \in \mathrm{Irr}(G)$ of degree not divisible by p if and only if G has a self-normalizing Sylow p-subgroup. Furthermore, if B is a Brauer p-block of G, we proved in several cases that there is a $\varphi \in \mathrm{IBr}(G)$ such that $d_{\chi,\varphi} \neq 0$ for all $\chi \in \mathrm{Irr}(B)$ of height zero if and only if the Brauer correspondent b of B has a unique modular representation. (We write that $l(b) = 1$, in that situation.)

For the sake of further discussion, perhaps it is worthwhile to go a step further and ask if the following apparently basic questions are happening in the decomposition matrices of arbitrary finite groups.

Conjecture A. *Let G be a finite group, and $p > 3$ a prime.*

(a) *If B is the principal block of G, we have that $d_{\chi,1} \neq 0$ for all $\chi \in \mathrm{Irr}(B)$ if and only if G has a normal p-complement.*
(b) *If B is any p-block, then $l(B) = 1$ if and only if there is some $\varphi \in \mathrm{IBr}(B)$ such that $d_{\chi,\varphi} \neq 0$ for all $\chi \in \mathrm{Irr}(B)$.*
(c) *If p is any prime, then B is nilpotent, if and only if, there is some $\varphi \in \mathrm{IBr}(B)$ such that $d_{\chi,\varphi}$ is a power of p for all $\chi \in \mathrm{Irr}(B)$.*

If $p = 2$ and $G = M_{22}$, then $\{d_{\chi,1} \mid \chi \in \mathrm{Irr}(G)\} = \{1, 2, 3, 4, 5\}$; if $p = 3$ and $G = {}^2G_2(27)$, then G has two 3-blocks, one of 3-defect zero, and $d_{\chi,1} \neq 0$ for all χ in the principal block of G. Hence, we do have to restrict ourselves to primes $p > 3$.

Our main results on Conjecture A in this paper are Theorem 3.6 and Theorem 4.1. In particular, we prove that Conjecture A(a) holds for any finite group G (and the prime p), as long as no composition factor of G is a simple group of Lie type in characteristic p. However, we are not able to decide on Conjecture A(a) when G is a simple group of Lie type in defining characteristic.

Let us finally remark that one direction of Conjecture A(a) is easy. Indeed, if G has a normal p-complement N, then N is contained in the kernel of the characters in the principal block, and N is the set of the p-regular elements of G. Therefore $d_{\chi,1} \neq 0$ for all $\chi \in \mathrm{Irr}(B)$. We also notice that Conjecture A(c) follows from Conjecture A(b) and the main conjecture in [MN]. Also, Conjecture A(a) follows from Conjecture A(b) and Corollary 6.13 of [N]. Although we have checked many cases (and proved it for p-solvable groups or blocks with cyclic defect groups in Theorem 4.1), we should point out that the evidence that we have for Conjecture A(b) is not overwhelming.

2. Reduction to Simple Groups

In this section, we reduce Conjecture A(a) to a question on simple groups.

Theorem 2.1. *Conjecture A(a) is true if and only if it is true for simple groups.*

Proof. The "if" direction is clear. For the "only if" direction, we argue by induction on $|G|$. Let us write $B_0(G)$ for the principal p-block of G. By induction, we have that G has a unique minimal normal subgroup N such that G/N has a normal p-complement K/N, and that $\mathbf{O}_{p'}(G) = 1$. Let $\theta \in \mathrm{Irr}(B_0(N))$, and let $\chi \in \mathrm{Irr}(B_0(G))$ lie over θ. Suppose that G is not simple, and therefore that $N < G$. By using the hypothesis, and Clifford's theorem, we have that $d_{\theta^g,1} \neq 0$ for some $g \in G$. Then $d_{\theta,1} \neq 0$. By induction, we have that N has a normal p-complement. Therefore, N is a p'-group or a p-group. In the first case, we have that K is a normal p-complement for G. In the second case, N is a p-group, and G is p-solvable. Since $\mathbf{O}_{p'}(G) = 1$, we have that G has a unique p-block. Hence, $K = G$ and G/N is a p'-group. Now, if $\chi \in \mathrm{Irr}(G/N)$, then $\chi^\circ \in \mathrm{IBr}(G/N)$. By hypothesis, χ° contains the trivial character. Thus $\chi^\circ = 1$, and we deduce that G/N is trivial. □

Also, let us record the following fact.

Theorem 2.2. *Conjecture A(a) is true for finite groups with cyclic Sylow p-subgroups.*

Proof. We use Theorem 1 of [D]. If e is the number of non-exceptional characters in the principal block B of G, and t is the number of exceptional characters, then we know that $|\mathrm{Irr}(B)| = e + t$, and that all the decomposition numbers $d_{\chi,\varphi}$ in B are either 0 or 1. If χ is exceptional, by hypothesis we have that $d_{\chi,1} = 1$. By Theorem 1 (Part 2) of [D], then there is exactly one non-exceptional character ψ such that $d_{\psi,1} = 1$. Using the hypothesis, we deduce that $e = 1$. By Corollary 6.13 of [N], we have that G has a normal p-complement. □

3. Simple groups

In this section, we will prove Conjecture A(a) for most of simple groups. Specifically, we will aim to show that the given simple group S admits an irreducible character χ in the principal p-block B_0 with $d_{\chi,1} = 0$, if $3 < p$ divides $|S|$. Of course, if $p \nmid |S|$, then Conjecture A(a) holds trivially.

Proposition 3.1. *Conjecture A(a) holds for any alternating group $S = \mathsf{A}_n$ with $n \geq 5$ and any prime $p > 2$.*

Proof. View $S = [H, H]$ with $H = \mathsf{S}_n$. It suffices to find $\chi \in \mathrm{Irr}(H)$ such that χ belongs to the principal p-block $B_0(H)$ of H, but χ° does not contain any Brauer character of degree 1. As in [J], let χ^λ be the irreducible character of H labeled by the partition $\lambda \vdash n$. The cases $n = 5, 6$ can be checked directly using [GAP], so we will assume $n \geq 7$.

First we consider the case $p|n$ and let $\chi = \chi^{(n-2,1^2)}$. By Peel's theorem [J, Theorem 24.1], $\chi \in B_0(H)$ and χ° is the sum of two irreducible Brauer characters, $\psi^{(n-1,1)}$ of degree $n-2$ and $\psi^{(n-2,1^2)}$ of degree $(n-2)(n-3)/2$. Hence we are done in this case.

Next suppose that $p \nmid n$ but $p \leq n-2$, and consider $\chi = \chi^{(n-p,1^p)}$. Again by Peel's theorem, $\chi \in B_0(H)$ and χ° is irreducible, of degree $\geq n-1 \geq 6$, and we are done again.

It remains to consider the case $p = n-1$; in particular, $n \geq 8$ and $p \geq 7$. Take $\chi = \chi^{(n-3,2,1)}$, so that $\chi \in B_0(H)$. As shown in the proof of [NT, Lemma 3.1], χ° is the sum of two irreducible Brauer characters, $\psi^{(n-3,2,1)}$ of degree $(2n^3-15n^2+25n+6)/6$ and $\psi^{(n-2,2)}$ of degree $(n^2-3n-2)/2$, and thus χ has the desired property. $\square$

Proposition 3.2. *Conjecture A(a) holds for $p > 3$ and S any of the 26 sporadic simple groups or the Tits group ${}^2F_4(2)'$.*

Proof. By Theorem 2.2, we may assume that the Sylow p-subgroups of S are not cyclic. Also, the cases where the decomposition matrix for the prime p is known in [GAP] can be checked easily. In what follows, we consider the cases where this matrix is not completely known for a given prime $p > 3$. In most of these cases, we can find an irreducible character $\chi \in \mathrm{Irr}(S)$ such that $\chi(1) = \mathfrak{d}_p(S)$, the smallest dimension of nontrivial p-Brauer characters of S, which is known thanks to [Jan], and that χ belongs to B_0. In such a case, χ° is irreducible and so $d_{\chi,1} = 0$, and we are done. This argument applies to all but three cases, which we now analyze.

Suppose $S = \mathrm{B}$ and $p = 7$. Then we take $\chi \in B_0(S)$ of degree 9550635. The restriction $\chi|_M$ to a maximal subgroup $M \cong \mathrm{Fi}_{23}$ is

$$\theta_{782} + 2\cdot\theta_{3588} + \theta_{5083} + \theta_{25806} + \theta_{60996} + 2\cdot\theta_{279565} + \theta_{789360} + \theta_{1951872} + \theta_{2236520} + \theta_{3913910},$$

where the subscripts indicate the degrees of the constituents. Now, using the decomposition matrix for M at $p = 7$ [ModAt], one can check that the restriction of none of these constituents to p'-elements in M contains 1_M. It follows that $d_{\chi,1} = 0$.

Suppose $S = \mathrm{Ly}$ and $p = 5$. Then the character $\chi \in \mathrm{Irr}(S)$ of degree 2480 belongs to B_0 [GAP]. Note that S has a maximal subgroup $M \cong Q \rtimes R$, where $Q = 3^5$ and $R \cong 2 \times \mathrm{M}_{11}$. Using [GAP] we can check that the Q-fixed point part of $\chi|_M$ (i.e. the part of $\chi|_M$ that is trivial at Q), viewed as an M_{11}-character, is an irreducible character of degree 16 which is irreducible modulo p. It follows that $d_{\chi,1} = 0$.

Suppose $S = \mathrm{Fi}_{24}'$ and $p = 5$. Then we take $\chi \in B_0(S)$ of degree 1666833. The restriction $\chi|_M$ to a maximal subgroup $M \cong \mathrm{Fi}_{23}$ is

$$\theta_{3588} + \theta_{60996} + \theta_{789360} + \theta_{812889},$$

where the subscripts again indicate the degrees of the constituents. Using the decomposition matrix for M at $p = 5$ [ModAt], one can check that the restriction of none of these constituents to p'-elements in M contains 1_M. It follows that $d_{\chi,1} = 0$. $\square$

The next statement proves Conjecture A(a) for some finite simple classical groups in the defining characteristic p, using known results on decomposition numbers modulo p of irreducible Weil characters.

Proposition 3.3. *Conjecture A(a) holds for the following simple groups S of Lie type in characteristic $p > 2$:*
(i) $S = \mathrm{PSp}_{2n}(q)$, $q = p^f$, $n \geq 1$, *and* $p > 3$ *if* $2 \nmid n$.
(ii) $S = \mathrm{PSL}_n(q)$, $q = p^f$, $n \geq 3$, *and* $\gcd(n, (q-1)/(p-1)) = 1$.
(iii) $S = \mathrm{PSU}_n(q)$, $q = p^f$, *and* $n \geq 3$ *is not divisible by the* 2*-part of* $q + 1$.

Proof. It is well known, see [Hu], that S has only two p-blocks, the principal block B_0, and one more block consisting of only the Steinberg character St. Hence it suffices to find $\chi \in \mathrm{Irr}(S)$ with $\chi \neq \mathsf{St}$ and $d_{\chi,1} = 0$.

(i) This has already been done in [NT, Proposition 3.11].

(ii) Let χ be the unipotent character of S of degree $(q^n - q)/(q-1)$, and let $q = p^d$. By [ZS, Theorem 1.11], $d_{\chi,1} + 2$ is the number N of d-tuples $(x_0, \ldots, x_{d-1})$ with $x_i \in \{0, 1\}$ and

$$n(p-1)\sum_{s=0}^{d-1} x_s p^s \equiv 0 \pmod{(q-1)}.$$

Since $\gcd(n, (q-1)/(p-1)) = 1$, the latter condition is equivalent to that $\sum_{s=0}^{d-1} p^s$ divides $\sum_{s=0}^{d-1} x_s p^s$. It follows that $N = 2$, and so $d_{\chi,1} = 0$, as desired.

(iii) Let χ be the unipotent character of S of degree $(q^n - q)/(q + 1)$, and let $q = p^d$. By [Z, Main Theorem], $d_{\chi,1}$ is the number N' of d-tuples $(x_0, \ldots, x_{d-1})$ with $x_i \in \{0, 1\}$ and

$$n\left((p-1)\sum_{s=0}^{d-1} x_s p^s + 1\right) \equiv 0 \pmod{(q+1)}.$$

Since $(q+1)_2 \nmid n$ and $p > 2$, the left-hand-side in the latter congruence is not divisible by $(q + 1)_2$. It follows that $N' = 0$, and so $d_{\chi,1} = 0$, as desired. □

Unfortunately, the current state of knowledge of decomposition matrices of simple groups of Lie type in the defining characteristic does not allow one to handle the remaining groups of Lie type (of large rank).

From now on, we will assume that the simple group S is $S = [G, G]$ with $G = \mathcal{G}^F$, where $\mathcal{G}$ is a simple algebraic group of adjoint type, defined over a field of characteristic $r \neq p$, and $F : \mathcal{G} \to \mathcal{G}$ a Steinberg endomorphism. Let the pair $(\mathcal{G}^*, F^*)$ be dual to $(\mathcal{G}, F)$, and let $G^* := (\mathcal{G}^*)^{F^*}$ be the dual group. Suppose $1 \neq t \in G^*$ be any (semisimple) p-element, say of order p^a. As $\mathbf{Z}(\mathcal{G}) = 1$, we can consider the semisimple character χ_t labeled by the G^*-conjugacy class of t, see [DM, Corollary 14.47].

Let $\mathcal{T}_0$ be an F-stable, maximally split, maximal torus of $\mathcal{G}$, so that $B = UT_0$ is a Borel subgroup of G, with $T_0 = \mathcal{T}_0^F$ and $U = \mathbf{O}_r(B)$ a Sylow r-subgroup of G.

Lemma 3.4. *In the above notation, let ψ be any irreducible constituent of $(\chi_t)|_S$. Then $\psi \in B_0(S)$. Moreover, $d_{\psi,1_S} = 0$ if at least one of the following two conditions holds.*

(a) χ_t *does not belong to the principal series, that is,* $(\chi_t)|_U$ *does not contain* 1_U.
(b) $p^a = |t|$ *does not divide* $|T_0|$.

Proof. By [BNOT, Lemma 3.1], every irreducible constituent ψ of $(\chi_t)|_S$ belongs to $B_0(S)$. Furthermore, G/S is abelian of order coprime. In particular, if $(\chi_t)^\circ$ does not contain any Brauer character of G of degree 1, then $d_{\psi,1_S} = 0$.

Next, any linear p-Brauer character of G is trivial at U as $U \leq S$. Hence, any linear constituent of χ_t° is trivial at U, and so we are done in the case of (a).

Now we claim that (b) implies (a). Indeed, assume that $(\chi_t)|_U$ contains 1_U. This means that

$$^*R^{\mathcal{G}}_{\mathcal{T}_0}(\chi_t) \neq 0 \tag{3.1}$$

for the Harish-Chandra restriction of χ_t. By [C, Proposition 8.4.6], χ_t is a linear combination of the Deligne-Lusztig characters $R^{\mathcal{G}}_{\mathcal{T}}(\theta)$, where $\mathcal{T}$ is an F-stable maximal torus of $\mathcal{G}$, $\theta \in \mathrm{Irr}(\mathcal{T}^F)$, and the pair $(\mathcal{T}, \theta)$ belongs to the geometric conjugacy class labeled by t. By Mackey's formula [DM, Theorem 11.13], $^*R^{\mathcal{G}}_{\mathcal{T}_0} \circ R^{\mathcal{G}}_{\mathcal{T}}(\theta)$ can be nonzero only when there is some $x \in G = \mathcal{G}^F$ such that $\mathcal{T}_0 \cap x\mathcal{T}x^{-1}$ contains a maximal torus of $\mathcal{G}$. In such a case, we actually have $\mathcal{T}_0 = x\mathcal{T}x^{-1}$ and hence $\mathcal{T}$ is G-conjugate to $\mathcal{T}_0$; in fact, $\mathcal{T}^F = (x^{-1}\mathcal{T}_0 x)^F = x^{-1}T_0 x$. We have shown that (3.1) implies that there is a pair $(\mathcal{T}, \theta)$, corresponding to $(\mathcal{T}^*, t)$ with $\mathcal{T}^F = x^{-1}T_0 x$ for some $x \in G$. In this case we have $|t| = |\theta|$, see [H2, Lemma 2.1(a)]. As $|\theta|$ divides $|\mathcal{T}^F| = |T_0|$, we deduce that $|t|$ divides $|T_0|$. □

Theorem 3.5. *Let* $p > 3$ *and let* $S = [G, G]$ *be a finite simple group of Lie type in characteristic* $r \neq p$, *with* $G = \mathcal{G}^F$ *as described above. Then Conjecture A(a) holds for* S.

Proof. By Theorem 2.2, we may assume that G has a non-cyclic Sylow p-subgroup.

(i) First we deal with classical groups. Suppose that $S = \mathrm{PSL}_n(q)$ with $n \geq 2$. Then the split torus T_0 of $G = \mathrm{PGL}_n(q)$ has order $(q-1)^{n-1}$. Now, if $p \nmid (q-1)$ then, choosing $t \in G$ of order p, we see by Lemma 3.4 that any irreducible constituent ψ of $(\chi_t)|_S$ belongs to $B_0(S)$ and $d_{\psi,1_S} = 0$. Assume $p|(q-1)$. Then the Steinberg character St of S belongs to $B_0(S)$ by [LS, Theorem 3.1]. On the other hand, as $p > 2$, $p \nmid (q+1)$, whence $d_{\mathsf{St},1_S} = 0$ by [H1, Theorem B].

Next suppose that $S = \mathrm{PSU}_n(q)$ with $n \geq 3$. Then the split torus T_0 of $G = \mathrm{PGU}_n(q)$ has order dividing $(q^2-1)^{n-1}$. Hence, if $p \nmid (q^2-1)$ then, choosing $t \in G$ of order p, we see by Lemma 3.4 that any irreducible constituent ψ of $(\chi_t)|_S$ belongs to $B_0(S)$ and $d_{\psi,1_S} = 0$. Assume $p|(q^2-1)$. Then the Steinberg character St of S belongs to $B_0(S)$ by [LS, Theorem 3.1]. If $n \geq 4$, then $d_{\mathsf{St},1_S} = 0$ by [H1, Theorem B]. If $n = 3$ and $p \nmid (q+1)$, then $p \nmid (q^3+1)$, and so $d_{\mathsf{St},1_S} = 0$ by [H1, Theorem B]. If $n = 3$ and $2 < p|(q+1)$, then, according to Theorems 4.3 and 4.5 of [Ge], the smallest unipotent character of S, of degree $q^2 - q$, belongs to $B_0(S)$ and is irreducible modulo p.

Consider the cases where $S = \mathrm{PSp}_{2n}(q)$ or $\Omega_{2n+1}(q)$ with $n \geq 2$, or $S = P\Omega^+_{2n}(q)$ with $n \geq 4$. Then $|T_0| = (q-1)^n$, and so, when $p \nmid (q-1)$ we can finish using Lemma 3.4. Assume now that $p|(q-1)$. Then the Steinberg character St of S belongs to $B_0(S)$ by [LS, Theorem 3.1], and $d_{\mathsf{St},1_S} = 0$ by [H1, Theorem B] (since $p \nmid (q+1)$).

Let $S = P\Omega^-_{2n}(q)$ with $n \geq 4$. Then $|T_0|$ divides $(q^2-1)^n$, and so, when $p \nmid (q^2-1)$ we can again finish using Lemma 3.4. Assume now that $p|(q^2-1)$. Then the Steinberg character St of S belongs to $B_0(S)$ by [LS, Theorem 3.1], and $d_{\mathsf{St},1_S} = 0$ by [H1, Theorem B].

(ii) Next we consider low-rank exceptional groups. Suppose $S = {}^2B_2(q)$ with $q = 2^{2a+1} \geq 8$. If $p|(q \pm \sqrt{2q}+1)$, then the two characters $\Gamma_{1,2}$ of degree $(q-1)\sqrt{q/2}$ belong to $B_0(S)$ and are irreducible modulo p, see [Bu]. If $p|(q-1)$, then the Steinberg character belongs to $B_0(S)$ and is irreducible modulo p by [Bu]. (Alternatively, in both cases we can also apply Theorem 2.2.)

Suppose $S = {}^2G_2(q)$ with $q = 3^{2a+1} \geq 27$. If $2 < p|(q+1)$, or $p|(q+\sqrt{3q}+1)$, then the two characters $\xi_{5,6}$ of degree $(q-1)(q-\sqrt{3q}+1)\sqrt{q/3}$ belong to $B_0(S)$ and are irreducible modulo p, see Satz D.2.3 and Satz D.2.4 of [H3]. If $p|(q-\sqrt{3q}+1)$, then the two characters $\xi_{7,8}$ of degree $(q-1)(q+\sqrt{3q}+1)\sqrt{q/3}$ belong to $B_0(S)$ and are irreducible modulo p, see [H3, Satz D.2.5]. If $p|(q-1)$, then the Steinberg character belongs to $B_0(S)$ and is irreducible modulo p by [H3, Satz D.2.2]. (Alternatively, in all cases we can also apply Theorem 2.2.)

Suppose $S = {}^2F_4(q)$ with $q = 2^{2a+1} \geq 8$. By the main result of [H4], the Steinberg character St belongs to $B_0(S)$, and $d_{\mathsf{St},1_S} = 0$ by [H1, Theorem B].

Suppose $S = {}^3D_4(q)$ with $q \geq 2$. By the main result of [H4], the Steinberg character St belongs to $B_0(S)$. Now, if $p \nmid (q+1)$, then $d_{\mathsf{St},1_S} = 0$ by [H1, Theorem B]. If $2 < p|(q+1)$, then p is coprime to $|T_0| = (q-1)(q^3-1)$, and so we can apply Lemma 3.4.

Suppose $S = F_4(q)$ with $q \geq 2$ or $S = G_2(q)$ with $q > 2$. By the main result of [H4], the Steinberg character St belongs to $B_0(S)$. Now, if $p \nmid (q+1)$, then $d_{\mathsf{St},1_S} = 0$ by [H1, Theorem B]. If $2 < p|(q+1)$, then p is coprime to $|T_0| = (q-1)^4$, respectively $(q-1)^2$, and so we are done by Lemma 3.4.

(iii) Finally, we consider the four large-rank exceptional groups, of type $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$, or $E_8(q)$. Here, $|T_0|$ divides $(q^2-1)^6$ in the second case, and $(q-1)^8$ in the other cases. Hence, if $p \nmid (q^2-1)$ in the second case, and $p \nmid (q-1)$ in the other cases, then we are done by applying Lemma 3.4. Thus we may assume $p|(q^2-1)$ in the second case, and $p|(q-1)$ in the other three cases. As $p > 2$, in all these cases $d_{\mathsf{St},1_S} = 0$, and we are done if $\mathsf{St} \in B_0(S)$.

Next we recall the notation of *regular numbers*, in the sense of Springer [Sp]. If $W < \mathrm{GL}(V)$ is a finite reflection group acting on a finite dimentional complex space V, then $v \in V$ is called *regular*, if $\mathrm{Stab}_W(v) = 1$. An element $g \in W$ is called *regular*, if it has a regular eigenvector; in particular, the identity element in W is regular. An integer $e \in \mathbb{N}$ is called *regular*, if W admits a regular element of order e, see [Sp, §5].

We let d denote the order of q modulo p. By results mainly of Broué, Malle, and Michel, and of Cabanes and Enguehard, summarised in [KM, Theorem A], the unipotent characters in a block of G are unions of d-Harish-Chandra series. Moreover, individual d-Harish-Chandra series are in bijection with irreducible characters of the corresponding relative Weyl groups (see [KM, Theorem B]). Thus by the degree formula for Lusztig induction, the blocks of maximal defect are those parametrized by cuspidal pairs (L, λ) with d-cuspidal $\lambda \in \mathrm{Irr}(L)$ of degree coprime to p, hence with the d-split Levi subgroup L having a d-torus in its center, so with L being the centralizer $\mathbf{C}_{\mathcal{G}}(\mathcal{T})$ of a Sylow d-torus $\mathcal{T}$ of $\mathcal{G}$. In particular if $\mathbf{C}_{\mathcal{G}}(\mathcal{T})$ is a maximal torus of $\mathcal{G}$, that is, if d is a regular number, in the aforementioned sense of Springer, for the Weyl group W of G, then there is just one such block, which must be the principal block. In this case, the Steinberg character St lies in the principal block of G, hence $\mathsf{St}|_S \in B_0(S)$.

In the four cases listed above, $d \leq 2$, whence d is regular by [Sp, §§5.4, 6.12], and so we are done. $\square$

Altogether, we have proved the following

Theorem 3.6. *Conjecture A(a) holds for any finite non-abelian simple group S, unless possibly S is a simple group of Lie type in characteristic p. Furthermore, Conjecture A(a) holds for any finite group G (and the prime p), as long as no composition factor of G is a simple group of Lie type in characteristic p.*

The remainder of this section is to supply some additional details for the proof of [NTV, Theorem 5.2], which we reformulate below (namely, for the part (ii) of the proof that deals with exceptional groups of Lie type).

Theorem 3.7. *Let $S \not\cong {}^2F_4(2)'$ be a finite simple group of Lie type in characteristic r and $p \neq r$ an odd prime. Then there exists a non-trivial, rational-valued,* $\mathrm{Aut}(S)$*-invariant, unipotent character of p'-degree that belongs to the principal p-block of S.*

Proof. Part (ii) of the proof of [NTV, Theorem 5.2] deals with $S = [G, G]$, where G is of exceptional type, but not a Suzuki-Ree group. We will continue to use the notation of part (iii) of the proof of Theorem 3.5.

By the results of [Sp, §§5.4, 6.12], all relevant numbers d are regular for W, unless $(G, d) = (E_6(q), 5)$, $({}^2E_6(q), 10)$, or $G = E_7(q)$ and $d = 4, 5, 8, 10, 12$, or $G = E_8(q)$ and $d = 7, 9, 14, 18$.

(a) First assume that $G = E_7(q)$. The non-regular numbers are $d = 4, 5, 8, 10, 12$. Here, eight unipotent characters are irrational (those lying in the Harish-Chandra series above the two cuspidal unipotent characters of E_6, those two in the principal series belonging to the non-rational characters of the Hecke algebra, and the two cuspidal unipotent characters). It is immediate from the explicit list of d-Harish-Chandra series in [BMM, Tab. 2] that in each case there exists a unipotent character of p'-degree in the principal block that is $\mathrm{Aut}(S)$-invariant, since it is the unique unipotent character of that degree. This concerns the lines 20, 24, 30, 34, 37 in loc. cit.

(b) Next, assume that $G = E_8(q)$. The non-regular numbers are $d = 7, 9, 14, 18$. Arguing as in (a), in each case we can find a unipotent character of p'-degree in the principal block that is $\mathrm{Aut}(S)$-invariant; namely, $\phi_{8,91}$, $\phi_{560,47}$, $\phi_{8,91}$, $\phi_{560,47}$, respectively, in the notation of [C, §13.9], which are on the lines 58, 66, 75, and 77 of [BMM, Tab. 2].

Suppose now that $(G, d) = (E_6(q), 5)$. Then we can consider the character $\phi_{24,6}$, in the notation of [C, §13.9], which is listed on line 7 of [BMM, Tab. 2]. Finally, if $(G, d) = ({}^2E_6(q), 10)$, then we consider the character $\phi''_{2,16}$, in the notation of [C, §13.9], which is listed on line 12 of [BMM, Tab. 2]. □

4. Further evidence

Theorem 4.1. *Conjecture A(b) is true if G is p-solvable or if B has a cyclic defect group.*

Proof. Assume that G is p-solvable, and that $l(B) > 1$. Let $\mu \in \mathrm{IBr}(B)$ be different from φ. By the Fong-Swan theorem, we have that there is $\chi \in \mathrm{Irr}(B)$ such that $\chi^\circ = \mu$. Since $d_{\chi,\varphi} \neq 0$, we conclude that $\mu = \varphi$, a contradiction.

If B has a cyclic defect group, we know that $l(b) = 1$, by Theorem C of [NT]. Since $l(B) = l(b)$, the result follows. □

Below we write down the table for the sporadic groups, which do not have cyclic Sylow p-subgroups, and whose decomposition matrices are available. The last column shows the smallest degree of a character in the principal block such that $d_{\chi,1} = 0$.

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Table 1. Sporadic Groups

G	p	degree
Co_1	7	276
Co_2	5	23
Co_3	5	23
Fi_{22}	5	78
Fi_{23}	5	3588
HN	5	133
He	5	51
He	7	153
J_2	5	14
McL	5	231
O′N	7	10944
Ru	5	378
Suz	5	143

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