ON IRREDUCIBLE PRODUCTS OF CHARACTERS

GABRIEL NAVARRO AND PHAM HUU TIEP

Abstract. We study the problem when the product of two non-linear Galois conjugate characters of a finite group is irreducible. We also prove new results on irreducible tensor products of cross-characteristic Brauer characters of quasisimple groups of Lie type.

1. Introduction

In character theory we soon learn that the product of complex non-linear characters is rarely irreducible. If G is a finite group and $\chi \in \mathrm{Irr}(G)$ is non-linear, we know that χ^2 is not irreducible (because the tensor product $V \otimes V$ of any G-module has the symmetric submodule). And, if $\bar{\chi}$ is the complex-conjugate of χ, then $\chi\bar{\chi}$ is also not irreducible, simply because it contains the trivial character. What might be perhaps a surprise is that there are examples of non-linear characters χ such that $\chi\chi^\sigma$ is irreducible, where $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is a Galois automorphism. In our first result in this paper, we show that essentially there are only five examples illustrating this phenomenon.

Theorem A. *Let G be a finite group, and let $\chi \in \mathrm{Irr}(G)$ be faithful. If $\chi\chi^\sigma \in \mathrm{Irr}(G)$, then $\mathbf{F}(G) = \mathbf{Z}(G)$. If G is quasi-simple and $\chi(1) > 1$, then $G = 2 \cdot \mathsf{A}_5$, $3 \cdot \mathsf{A}_6$, $2 \cdot J_2$, $3 \cdot J_3$, or $4_1 \cdot \mathrm{PSL}_3(4)$.*

Notice that the first part of Theorem A implies that there are no solvable examples of irreducible products of faithful non-linear Galois conjugate characters, using that $\mathbf{C}_G(\mathbf{F}(G)) \leq \mathbf{F}(G)$ in a solvable group G. (This consequence can also be deduced from the main results on irreducible product of characters in solvable groups in [Is2].) Once these five examples among quasi-simple groups are discovered, we can easily construct many groups having non-linear faithful Galois conjugate characters whose product is irreducible, by using central products of those, extensions, wreath products, etc. It might well be that the semisimple layer $\mathbf{E}(G)$ in Theorem A is a central product of a number of copies of these five groups, but this seems difficult to prove, and perhaps, the result is not totally worth the effort.

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In the case of quasisimple groups, Theorem A follows from the following stronger result:

Theorem B. *Let G be a finite quasisimple group, and let α, β be irreducible characters of G of the same degree $\alpha(1) = \beta(1) > 1$. Suppose that $\alpha\beta$ is irreducible. Then* $(G/(\mathrm{Ker}(\alpha) \cap \mathrm{Ker}(\beta)), \alpha(1))$ *is* $(2 \cdot \mathsf{A}_5, 2)$, $(3 \cdot \mathsf{A}_6, 3)$, $(6 \cdot \mathsf{A}_7, 6)$, $(2 \cdot J_2, 6)$, $(3 \cdot J_3, 18)$, $((2^2 \times 3) \cdot \mathrm{PSL}_3(4), 6)$, $(4_1 \cdot \mathrm{PSL}_3(4), 8)$, $(4^2 \cdot \mathrm{PSL}_3(4), 8)$, $((3^2 \times 2) \cdot \mathrm{PSU}_4(3), 6)$, *or* $(3 \cdot G_2(3), 27)$.

The study of irreducible products of ordinary (and ℓ-Brauer) characters of quasisimple groups was initiated by I. Zisser in [Zi], Bessenrodt and Kleshchev and collaborators for alternating groups and their covers [B], [BK1], [BK2], [BK3], [KT1], and continued by K. Magaard and the second author in [MT] for groups of Lie type. This problem is an important part of the Aschbacher–Scott program [A] on classifying maximal subgroups of finite classical groups. The main result of [MT] solved the problem for all finite groups of Lie type over fields $\mathbb{F}_q$ with $q > 5$, *except* for the symplectic groups and groups of type F_4 and ${}^2F_4$ in characteristic 2. In the second result in this paper, we complete the classification for the symplectic series, still leaving open the case of $\mathrm{Sp}_{2n}(2)$.

Theorem C. *Let $n \geq 2$ and let q be a power of 2. Let $\mathbb{F}$ be an algebraically closed field of characteristic $\ell = 0$ or $\ell \neq 2$. Suppose that $G = \mathrm{Sp}_{2n}(q)$ admits nontrivial irreducible $\mathbb{F}G$-modules V and W such that $V \otimes W$ is irreducible. Then $q = 2$.*

Together with the main results of [MT] and [KT2, Theorem 8.7], Theorem C implies the following result on irreducible tensor products of cross characteristic representations of finite quasisimple groups of Lie type.

Theorem D. *Let G be a finite quasisimple group of Lie type, of simply connected type, defined over a field $\mathbb{F}_q$ of characteristic p. Suppose that, for some $\ell = 0$ or not equal to p, G admits ℓ-Brauer characters α and β, both of degree > 1, such that $\alpha\beta$ is irreducible. Then one of the following holds:*

(i) *$q \leq 3$, but $G \not\cong \mathrm{SL}_n(q)$.*
(ii) *$G = \mathrm{Sp}_{2n}(5)$, at least one of α, β is a Weil character, but $\alpha(1) \neq \beta(1)$.*
(iii) *$2|q$, $G = F_4(q)$ or ${}^2F_4(q)$, and ℓ divides $|G|$.*

As discussed in Remark 2.3 below, $G = \mathrm{SU}_n(2)$ with $n \geq 4$, and $\mathrm{Sp}_{2n}(q)$ with $q = 2$ and $n \geq 3$, and with $q = 3, 5$ with $n \geq 2$, indeed occur in Theorem D, at least when $\ell = 0$.

We end this note with a question. When studying irreducible product of characters and normal constituents, a problem naturally shows up: if G is a quasi-simple group and $\alpha, \beta \in \mathrm{Irr}(G)$ are faithful, when is $\alpha\beta = m\gamma$ for some $\gamma \in \mathrm{Irr}(G)$? (Or more generally, when is $\alpha\beta$ a sum of $\mathrm{Aut}(G)$-conjugates of some γ?) Although there are (very few) quasi-simple examples of this, we conjecture that this never happens in simple groups.

Conjecture E. *Suppose that G is a simple group, and let $\alpha, \beta, \gamma \in \mathrm{Irr}(G)$. If $\alpha\beta = m\gamma$ for certain integer m, then $m = 1$.*

2. Proofs of Theorems C and D

Proposition 2.1. *Let $q = 2^f \geq 4$ be a power of 2, $n \geq 3$, $G := \mathrm{Sp}_{2n}(q)$, and let N_1 be any of the integers*

$$\frac{(q^n+1)(q^n-q)}{2(q-1)} \text{ or } \frac{(q^n-1)(q^n+q)}{2(q-1)}.$$

(i) *Let N_2 be any of the integers*

$$\frac{(q^n-1)(q^n-q)}{2(q+1)} \text{ or } \frac{(q^n+1)(q^n+q)}{2(q+1)} \text{ or } \frac{(q^n+1)(q^n+q)}{2(q+1)} - 1 \text{ or } \frac{q^{2n}-1}{q+1}.$$

Then G has no irreducible complex character of degree $(N_1-1)N_2$.

(ii) *Let $N_3 := (q^n+1)(q^n+q)/2(q+1)$. Then G has no irreducible complex character of degree $N_1(N_3-1)$.*

Proof. (a) Note that when $(n, q) = (3, 4)$, $N_1 - 1$ is divisible by 31 or 59, which is not a divisor of $|\mathrm{Sp}_6(4)|$, and $N_3 - 1$ is divisible by 7^2, which is again not a divisor of $|\mathrm{Sp}_6(4)|$. Likewise, when $(n, q) = (4, 4)$, $N_1 - 1$ is divisible by 251 or 127, and $N_3 - 1$ is divisible by 131, and none of these primes is a divisor of $|\mathrm{Sp}_8(4)|$. Similarly, when $(n, q) = (3, 8)$, $N_1 - 1$ is divisible by 313 or 18979, and $N_3 - 1$ is divisible by 29, and none of these primes is a divisor of $|\mathrm{Sp}_6(8)|$. Hence the statements follow in these cases.

From now on we will assume that $n \geq 5$ when $q = 4$ and $n \geq 4$ when $q = 8$. These conditions ensure by [Zs] that $2^k - 1$ has a primitive prime divisor $\ell(2, k)$, i.e. a prime that divides $2^k - 1$ but not $\prod_{i=1}^{k-1}(2^i - 1)$ for $k \in \{(2n-2)f, (n-1)f\}$.

(b) To prove (i), assume by way of contradiction that there is $\chi \in \mathrm{Irr}(G)$ such that χ has degree $D = (N_1 - 1)N_2$. We choose $n_0 \in \{n, n-1\}$ to be odd. A direct computation shows that, for each N_2, there is a prime

$$\ell \in \{\ell(2, 2nf), \ell(2, (2n-2)f), \ell(2, n_0 f)\} \tag{2.1}$$

that does not divide $\chi(1)$. Thus

$$\ell \nmid \chi(1),\ \chi(1)_2 \leq q/2. \tag{2.2}$$

We will use (2.2) to derive a contradiction, using Lusztig's classification of irreducible characters of G [C, DM]. Since the dual group of G can be identified with G, we can find a semisimple element $s \in G$ and a unipotent character ψ of $\mathbf{C}_G(s)$ such that

$$\chi(1) = \psi(1) \cdot [G : \mathbf{C}_G(s)]_{2'}.$$

If $s = 1$, i.e. $\chi(1)$ is unipotent, then since $\chi(1)_2 \leq q/2$ by (2.2), by [MMT, Lemma 7.2] we have that

$$\chi(1) \in \left\{1,\ \frac{(q^n+\gamma)(q^n+\gamma q)}{2(q+1)},\ \frac{(q^n-\delta)(q^n+\delta q)}{2(q-1)} \mid \gamma, \delta = \pm 1\right\},$$

and so $\chi(1) < D$, a contradiction. Hence $s \neq 1$, and

$$\mathbf{C}_G(s) \cong \mathrm{Sp}_{2a}(q) \times \mathrm{GL}_{b_1}(q^{r_1}) \times \ldots \times \mathrm{GL}_{b_k}(q^{r_k}) \times \mathrm{GU}_{c_1}(q^{s_1}) \times \ldots \times \mathrm{GU}_{b_m}(q^{s_m}), \tag{2.3}$$

where $a, k, m \in \mathbb{Z}_{\geq 0}$, $b_i, r_i, c_j, s_j \in \mathbb{Z}_{\geq 1}$, and

$$n = a + \sum_{i=1}^{k} b_i r_i + \sum_{j=1}^{m} c_j s_j, \ a \leq n-1.$$

Now, if $2 \leq a \leq n-2$, or $2 \leq b_i r_i \leq n-2$ for some i, or $2 \leq c_j s_j \leq n-2$ for some j, then

$$\mathbf{C}_G(s) \leq \mathrm{Sp}_{2d}(q) \times \mathrm{Sp}_{2n-2d}(q),$$

with $2 \leq d \leq n-2$ and $d = a$, $d = b_i r_i$, or $d = c_j s_j$. In such a case, the choice (2.1) of ℓ implies that ℓ divides $[G : \mathbf{C}_G(s)]_{2'}$, contradicting (2.2). Thus

$$a, b_i r_i, c_j s_j \in \{0, 1, n-1, n\}.$$

Moreover, the same argument rules out that case where $a = 1$ and, in addition, some $b_i r_i$ or $c_j s_j$ equals 1.

(b1) Suppose $b_i r_i = n$ for some i or $c_j s_j = n$ for some j. If $n \geq 7$, then

$$\chi(1) \geq [G : \mathbf{C}_G(s)]_{2'} \geq (q-1)(q^2-1)\dots(q^n-1) > q^{4n-2} > D,$$

a contradiction. Consider the case $3 \leq n \leq 6$. Here, if $\psi(1) > 1$, then $q|\psi(1)$ by [MMT, Lemma 7.2], contradicting (2.2). Hence $\psi(1) = 1$, and so $2 \nmid \chi(1)$ and

$$N_2 = \frac{(q^n+1)(q^n+q)}{2(q+1)} - 1, \ \frac{q^{2n}-1}{q+1}.$$

In particular, we can choose $\ell = \ell(2, (2n-2)f)\}$ to fulfill (2.2). For brevity, we can write

$$\mathbf{C}_G(s) = \mathrm{GL}^{\epsilon}_b(q^r),$$

where $br = n$, and GL^{ϵ} stands for GL when $\epsilon = +$ and for GU when $\epsilon = -$. Now, if $r \geq 2$, then ℓ divides $[G : \mathbf{C}_G(s)]_{2'}$, contradicting (2.2). Hence $r = 1$. In this case, (2.2) is fulfilled for both two choices $\ell_+ := \ell(2, (n-1)f)$ and $\ell_- := \ell(2, (2n-2)f)$. On the other hand, at least one of ℓ_+ and ℓ_- divides $[G : \mathbf{C}_G(s)]_{2'} = [\mathrm{Sp}_{2n}(q) : \mathrm{GL}^{\epsilon}_n(q)]_{2'}$, again contradicting (2.2).

(b2) Suppose $b_i r_i = n-1$ for some i or $c_j s_j = n-1$ for some j. Then

$$\chi(1) \geq [G : \mathbf{C}_G(s)]_{2'} \geq D' := \frac{q^{2n}-1}{q^2-1} \cdot (q-1)(q^2-1)\dots(q^{n-1}-1).$$

Now, if $n \geq 6$, then $D' > (q^{2n}-1)^2/(q^2-1) > D$, a contradiction. If $n = 5$, then $D' > D$ since $q \geq 4$, again a contradiction.

Suppose $n = 4$. In this case, $D = \chi(1)$ is divisible by $[G : \mathbf{C}_G(s)]_{2'}$, a multiple of $[\mathrm{Sp}_8(q) : \mathrm{Sp}_2(q) \times \mathrm{GL}_3(q)]_{2'}$ when $b_i r_i = 3$ and of $[\mathrm{Sp}_8(q) : \mathrm{Sp}_2(q) \times \mathrm{GU}_3(q)]_{2'}$ when $c_j s_j = 3$. It follows that D is divisible by $(q^4+1)(q^2+1)^2$. On the other hand, N_1 is congruent to 0 or -1 modulo q^4+1, and N_1 is congruent to 0 or -1 modulo q^2+1. Hence $N_1 - 1$ is coprime to $(q^4+1)(q^2+1)^2$. As $D = (N_1-1)N_2$, N_2 is divisible by $(q^4+1)(q^2+1)^2$, again leading to a contradiction since $N_2 < q^8$.

Suppose $n = 3$, in which case $q \geq 16$ by our assumption. Then D is greater than $(q^6-1)(q^4-1)(q^2-1)/(q-1)^3$, which is the upper bound for the degree of irreducible characters of $\mathrm{Sp}_6(q)$ [Lu], again a contradiction.

(b3) In the remaining case, we must then have $a = n-1$, and so

$$\mathbf{C}_G(s) = \mathrm{Sp}_{2n-2}(q) \times \mathrm{GL}_1(q) \text{ or } \mathrm{Sp}_{2n-2}(q) \times \mathrm{GU}_1(q).$$

Since $\chi(1)_2 \leq q/2$ by (2.2), we must have by [MMT, Lemma 7.2] that

$$\psi(1) \in \left\{1,\ \frac{(q^{n-1}+\gamma)(q^{n-1}+\gamma q)}{2(q+1)},\ \frac{(q^{n-1}-\delta)(q^{n-1}+\delta q)}{2(q-1)} \mid \gamma,\delta = \pm 1\right\},$$

whence $\chi(1) \leq \psi(1)(q^{2n}-1)/(q-1) < D$, a contradiction.

(c) To prove (ii), assume by way of contradiction that there is $\chi \in \mathrm{Irr}(G)$ such that χ has degree $D' = N_1(N_3-1)$. Note that for each choice of N_1, we can find $\ell \in \{\ell(2,2nf), \ell(2,(2n-2)f)\}$ such that

$$\ell \nmid \chi(1),\ \chi(1)_2 = q/2. \tag{2.4}$$

As above, we can find a semisimple element $s \in G$ and a unipotent character ψ of $\mathbf{C}_G(s)$ such that

$$\chi(1) = \psi(1) \cdot [G : \mathbf{C}_G(s)]_{2'}.$$

If $s = 1$, i.e. $\chi(1)$ is unipotent, then since $\chi(1)_2 = q/2$ by (2.2), by [MMT, Lemma 7.2] we have that

$$\chi(1) \in \left\{\frac{(q^n+\gamma)(q^n+\gamma q)}{2(q+1)},\ \frac{(q^n-\delta)(q^n+\delta q)}{2(q-1)} \mid \gamma,\delta = \pm 1\right\},$$

and so $\chi(1) < D'$, a contradiction. Hence $s \neq 1$, and we can represent $\mathbf{C}_G(s)$ as in (2.3). We also note that $\psi(1)_2 = q/2$ by (2.4); in particular, $\psi(1) > 1$. Now we can repeat the arguments in (b) verbatim (noting in the case $n = 4$ of (b2) that now we have $(q^4+1)(q^2+1)^2$ divides $\chi(1)$ but not $N_1(N_3-1)$). $\square$

Now we prove Theorem C, which we reformulate below:

Theorem 2.2. *Let $n \geq 2$ and let q be a power of 2. Let $\mathbb{F}$ be an algebraically closed field of characteristic $\ell = 0$ or $\ell \neq 2$. Suppose that $G = \mathrm{Sp}_{2n}(q)$ admits nontrivial irreducible $\mathbb{F}G$-modules V and W such that $V \otimes W$ is irreducible. Then $q = 2$.*

Proof. (i) First we deal with the case $n = 2$ (and $q \geq 4$). By [GT, Theorem 1.1], $\dim(V), \dim(W) \geq q(q-1)^2/2$. On the other hand, if $q > 4$ then the largest degree of irreducible characters of $\mathrm{Sp}_4(q)$ is $(q+1)^2(q^2+1)$ [E], which is then smaller than $(q(q-1)^2/2)^2$, hence $V \otimes W$ cannot be irreducible. If $q = 4$, then the largest degree of $\mathrm{Sp}_4(4)$ is 340, so the irreducibility of $V \otimes W$ forces $\dim(V) = \dim(W) = 18$, whence $V \cong W$ and is self-dual, so $V \otimes W$ cannot be irreducible. From now on we will assume $n \geq 3$ and $q \geq 4$.

The proof crucially relies on the characterization of the so-called *linear-Weil* and *unitary-Weil* ℓ-Brauer characters of G, as introduced in [GT, Table I], based on some local properties $(\mathcal{W}_2^\varepsilon)$, $\varepsilon = \pm$, as defined in [GT, §3].

Let $\mathcal{N} = \mathbb{F}_q^{2n}$ be the natural module for G, endowed with a G-invariant non-degenerate alternating form (so that $G = \mathrm{Sp}(\mathcal{N})$), and let the parabolic subgroup P be the stabilizer in G of a totally singular 2-dimensional subspace of $\mathcal{N}$. Then, as shown in [GT, §3], $Q = \mathbf{O}_2(P)$ has order q^{4n-5} with center $Z =: \mathbf{Z}(Q) > [Q,Q]$ elementary abelian of order q^3. Next, $P = Q \rtimes L$, where $L \cong \mathrm{GL}_2(q) \times \mathrm{Sp}_{2n-2}(q)$ is a Levi subgroup. Then P has four orbits on $\mathrm{IBr}_\ell(Z)$:

- $\mathcal{O}_0 := \{1_Z\}$;
- $\mathcal{O}_1$ of length q^2-1 (all the characters in this orbit are trivial at $[Q,Q]$); and
- $\mathcal{O}_2^\varepsilon$ of length $q(q-1)(q+\varepsilon)/2$ for $\varepsilon = \pm$ – each character λ in the orbit $(\mathcal{W}_2^\varepsilon)$ has stabilizer

$$K_\lambda = Q \rtimes \big(\mathrm{O}_2^\varepsilon(q) \times \mathrm{Sp}_{2n-4}(q)\big)$$

in P.

Now $V \in \mathrm{IBr}_\ell(G)$ is said to have property $(\mathcal{W}_2^\varepsilon)$ for some $\varepsilon = \pm$ if the Brauer character of every irreducible constituent of $V|_Z$ belongs to $\mathcal{O}_0 \cup \mathcal{O}_1 \cup \mathcal{O}_2^\varepsilon$. One of the main results, Theorem 1.2, of [GT] characterizes the linear-Weil modules of G as the only nontrivial irreducible modules that have property $(\mathcal{W}_2^+)$, and similarly, the unitary-Weil modules of G as the only nontrivial irreducible modules that have property $(\mathcal{W}_2^-)$.

(ii) Now we return to $V, W \in \mathrm{IBr}_\ell(G)$, being nontrivial and having irreducible tensor product. Here we assume that there is some $\varepsilon = \pm$ such that both $V|_Z$ and $W|_Z$ afford an irreducible constituent with character $\lambda \in \mathcal{O}_2^\varepsilon$. Consider the corresponding isotypic component V_λ of $V|_Z$, which is certainly stabilized by $K_\lambda = \mathrm{Stab}_P(\lambda)$. By [GT, Lemma 9.2] and its proof, there is a unique irreducible Brauer character μ of Q that lies above λ; in fact, $\mu|_Z = q^{2n-4}\lambda$ and $Q_\lambda := Q/\mathrm{Ker}(\lambda)$ is an extraspecial 2-group of order $2q^{4n-8}$. Moreover, there is an irreducible $\mathbb{F}K_\lambda$-module E_λ of dimension q^{2n-4} such that E_λ affords the Q-character μ, and the traces of elements of K_λ acting on E_λ are controlled by [GT, Lemma 2.4]. It follows from Gallagher's theorem that

$$V_\lambda \cong E_\lambda \otimes A_\lambda,$$

for some $\mathbb{F}(K_\lambda/Q)$-module A_λ.

Since Z is elementary abelian 2-group, $\lambda = \bar\lambda$ and $\mu = \bar\mu$ by uniqueness. Hence the dual module E_λ^* also affords the Q-character μ, and so we can write

$$W_\lambda \cong E_\lambda^* \otimes B_\lambda,$$

for some $\mathbb{F}(K_\lambda/Q)$-module B_λ. Now, the socle of $A_\lambda \otimes B_\lambda$ contains a simple submodule $C \otimes D$, where $C \in \mathrm{IBr}_\ell(\mathrm{O}_2^\varepsilon(q))$ and $D \in \mathrm{IBr}_\ell(\mathrm{Sp}_{2n-4}(q))$. In fact, we can view C as a (K_λ/Q)-module that is trivial on $\mathrm{Sp}_{2n-4}(q)$, and D as a P/Q-module that is trivial on $\mathrm{GL}_2(q)$ (recall that $P/Q \cong \mathrm{GL}_2(q) \times \mathrm{Sp}_{2n-4}(q)$). Hence, working in P/K, we have

$$\mathrm{Ind}_{K_\lambda}^P(C \otimes D) \cong \mathrm{Ind}_{K_\lambda}^P(C) \otimes D.$$

As $E_\lambda \otimes E_\lambda^*$ contains the trivial submodule $\mathbb{F}$, it follows that $V_\lambda \otimes W_\lambda$ contains the simple K_λ-submodule $C \otimes D$, which is trivial on Q. Applying Frobenius' reciprocity, we have

$$\begin{aligned} 0 &\neq \mathrm{Hom}_{\mathbb{F}K_\lambda}\big(C \otimes D, (V \otimes W)|_{K_\lambda}\big) \\ &\cong \mathrm{Hom}_{\mathbb{F}G}\big(\mathrm{Ind}_{K_\lambda}^G(C \otimes D), V \otimes W\big) \\ &= \mathrm{Hom}_{\mathbb{F}G}\big(\mathrm{Ind}_P^G(\mathrm{Ind}_{K_\lambda}^P(C \otimes D)), V \otimes W\big) \\ &\cong \mathrm{Hom}_{\mathbb{F}G}\big(\mathrm{Ind}_P^G(\mathrm{Ind}_{K_\lambda}^P(C) \otimes D), V \otimes W\big). \end{aligned}$$

Since $V \otimes W$ is irreducible, this implies that there exists a simple subquotient X of $\mathrm{Ind}_{K_\lambda}^P(C)$ such that $V \otimes W$ is a simple subquotient of $\mathrm{Ind}_P^G(X \otimes D)$. Recalling C is trivial on $Q \rtimes \mathrm{Sp}_{2n-4}(q)$ and working in $P/(Q \rtimes \mathrm{Sp}_{2n-4}(q)) \cong \mathrm{GL}_2(q)$, we can view X as a simple $\mathbb{F}\mathrm{GL}_2(q)$-module, whence $\dim(X) \leq q+1$. Thus we have shown that

$$\dim(V)\dim(W) \leq \dim\bigl(\mathrm{Ind}_P^G(X \otimes D)\bigr) \leq \frac{(q^{2n}-1)(q^{2n-2}-1)}{(q-1)(q^2-1)}(q+1)\dim(D). \tag{2.5}$$

Next, $V|_Z$ affords the entire orbit $\mathcal{O}_2^\varepsilon$, and so does $W|_Z$. Using the transitive action of P, we obtain

$$\begin{aligned} \dim(V) &\geq |\mathcal{O}_2^\varepsilon| \cdot \dim(V_\lambda) = |\mathcal{O}_2^\varepsilon| \cdot q^{2n-4}\dim(A_\lambda), \\ \dim(W) &\geq |\mathcal{O}_2^\varepsilon| \cdot \dim(W_\lambda) = |\mathcal{O}_2^\varepsilon| \cdot q^{2n-4}\dim(B_\lambda), \end{aligned} \tag{2.6}$$

whence

$$\dim(V)\dim(W) \geq |\mathcal{O}_2^\varepsilon|^2 \cdot q^{4n-8}\dim(A_\lambda \otimes B_\lambda) \geq |\mathcal{O}_2^\varepsilon|^2 \cdot q^{4n-8}\dim(C \otimes D). \tag{2.7}$$

Together with (2.5), we have shown

$$\bigl(q(q-1)(q+\epsilon)/2\bigr)^2 q^{4n-8} \leq (q^{2n}-1)(q^{2n-2}-1)/(q-1)^2. \tag{2.8}$$

(iii) Now, if $q \geq 8$, then (2.8) implies that

$$\frac{(q-1)^6}{4q^4} \leq \bigl(1-\frac{1}{q^{2n}}\bigr)\cdot\bigl(1-\frac{1}{q^{2n-2}}\bigr),$$

a contradiction, since $n \geq 2$. Furthermore, if $q=4$ and $\varepsilon=+$, then (2.8) implies that

$$\frac{(q-1)^4(q+1)^2}{4q^4} \leq \bigl(1-\frac{1}{q^{2n}}\bigr)\cdot\bigl(1-\frac{1}{q^{2n-2}}\bigr),$$

again a contradiction.

Thus we have shown that, when $q \geq 8$, $V|_Z$ and $W|_Z$ cannot both afford $\mathcal{O}_2^\varepsilon$ for any $\varepsilon = \pm$, and when $q=4$, $V|_Z$ and $W|_Z$ cannot both afford $\mathcal{O}_2^+$.

Note that $\mathcal{O}_0 \cup \mathcal{O}_1 = \mathrm{IBr}_\ell(Z/[Q,Q])$. Hence the faithfulness of V implies that $V|_Z$ must afford $\mathcal{O}_2^\kappa$ for some $\kappa = \pm$. Using [GT, Theorem 1.2], when $q \geq 8$, we have ruled out the cases where at least one of V, W is not a Weil (linear or unitary) module, or when both V, W are linear-Weil, or when both V, W are unitary-Weil. Thus when $q=8$, we may assume that V is linear-Weil and W is unitary-Weil.

Likewise, when $q=4$, we have ruled out the cases where both V, W are non-Weil, or when one of V, W is non-Weil and the other is linear-Weil, or when both V, W are linear-Weil. Thus when $q=4$, we may assume that W is unitary-Weil.

Thus in the rest of the proof we may assume that $q \geq 4$, W is unitary-Weil; in particular,

$$\begin{gathered} \text{either } \dim(W) \in \left\{\tfrac{(q^n-1)(q^n-q)}{2(q+1)}, \tfrac{(q^n+1)(q^n+q)}{2(q+1)}, \tfrac{(q^n+1)(q^n+q)}{2(q+1)}-1\right\} \text{ and } \dim(B_\lambda)=1, \\ \text{or } \dim(W) = \tfrac{q^{2n}-1}{q+1} \text{ and } \dim(B_\lambda) \leq 2. \end{gathered}$$

Indeed, $\dim(W)$ is listed in [GT, Table I], and the bound on $\dim(B_\lambda)$ follows from (2.6) with $\varepsilon = -$. It follows that

$$\dim(W) \geq \frac{(q^n-1)(q^n-q)}{2(q+1)}\dim(B_\lambda). \tag{2.9}$$

(iv) Here we consider the case where $q = 4$, $n \geq 4$, and W is unitary-Weil, and V is non-Weil or unitary-Weil. Then we can write $V|_P = V_1 \oplus V_2$, where

$$V_2 := \bigoplus_{\lambda \in \mathcal{O}_2^-} E_\lambda \otimes A_\lambda$$

and V_1 is some $\mathbb{F}P$-module that does not afford $\mathcal{O}_2^-$ on restriction to Z. Fix a transvection $t \in Z$ and let ψ, ψ_j denote the Brauer character of V and of V_j, $j = 1, 2$. Then

$$\psi(t) = \psi_1(t) + q^{2n-4} \dim(A_\lambda) \sum_{\lambda' \in \mathcal{O}_2^-} \lambda'(t) = \psi_U(t) - 6 \cdot 4^{2n-4} \dim(A_\lambda),$$

where the equality $\sum_{\lambda' \in \mathcal{O}_2^-} \lambda'(t) = -q(q-1)/2$ follows from the proof of [GT, Proposition 4.1]. Since $\dim(V_2) = |\mathcal{O}_2^-| \cdot q^{2n-4} \dim(A_\lambda) = 18 \cdot 4^{2n-4} \dim(A_\lambda)$, we obtain

$$\psi(t) = \psi_1(t) - \dim(V_2)/3. \tag{2.10}$$

Now, we can find a G-conjugate t_1 of t which is contained (as a transvection) in the subgroup $\mathrm{Sp}_{2n-4}(q)$. Then t_1 acts on $Q_\lambda/\mathbf{Z}(Q_\lambda)$, viewed as a $(4n-8)$-dimensional vector space over $\mathbb{F}_q$, with a fixed point subspace of codimension 2. The aforementioned remark about the character of the K_λ-module E_λ in the first paragraph of (ii) shows that the trace of t_1 on E_λ has absolute value 0 or q^{2n-5}. It follows that

$$|\psi(t)| = |\psi(t_1)| \leq \dim(V_1) + q^{2n-5} \cdot |\mathcal{O}_2^-| \cdot \dim(A_\lambda) = \dim(V_1) + \dim(V_2)/4. \tag{2.11}$$

Note that $|t| = 2$, and so $\psi_1(t) \in \mathbb{Z}$ and $-\dim(V_1) \leq \psi_1(t) \leq \dim(V_1) = \psi_1(1)$. Suppose in addition that $\dim(V_1) = \psi_1(1) < \dim(V_2)/3$. Then together with (2.10) and (2.11), we obtain

$$\dim(V_1) + \dim(V_2)/4 \geq |\psi(t)| = \dim(V_2)/3 - \psi_1(t) \geq \dim(V_2)/3 - \dim(V_1),$$

and so $\dim(V_1) \geq \dim(V_2)/24$. Thus we always have $\dim(V_1) \geq \dim(V_2)/24$, whence

$$\dim(V) \geq \frac{25}{24} \dim(V_2) = \frac{25}{24} \cdot 18 \cdot 4^{2n-4} \dim(A_\lambda). \tag{2.12}$$

Now we apply (2.9) and (2.12) to (2.5) to obtain

$$\frac{25}{24} \cdot 18 \cdot 4^{2n-4} \dim(A_\lambda) \cdot \frac{(4^n-1)(4^n-4)}{10} \dim(B_\lambda) \leq \frac{(4^{2n}-1)(4^{2n-2}-1)}{9} \dim(D).$$

As $\dim(D) \leq \dim(A_\lambda)\dim(B_\lambda)$, this implies

$$\frac{135}{2 \cdot 4^3} \leq \left(1 + \frac{1}{4^n}\right)\left(1 + \frac{1}{4^{n-1}}\right),$$

which is a contradiction since $n \geq 4$.

(v) Here we consider the case $(n, q) = (3, 4)$. Then $\dim(W) \geq 378$. Since the largest degree of irreducible characters of $G = \mathrm{Sp}_6(4)$ is 371280 [GAP], it follows from the irreducibility of $V \otimes W$ that $\dim(V) \leq 982$. This implies by [GT, Theorem 1.1] that V is a Weil module. Leaving out the case V is linear-Weil to the next parts (vi) and (vii) of the proof, we assume here that V is unitary-Weil. Then, in addition to (2.9) we also have that

$$\dim(V) \geq \frac{(q^n-1)(q^n-q)}{2(q+1)} \dim(A_\lambda).$$

Applying this and (2.9) to (2.5), we obtain

$$\frac{(4^n-1)(4^n-4)}{10} \dim(A_\lambda) \cdot \frac{(4^n-1)(4^n-4)}{10} \dim(B_\lambda) \leq \frac{(4^{2n}-1)(4^{2n-2}-1)}{9} \dim(D).$$

As $\dim(D) \leq \dim(A_\lambda)\dim(B_\lambda)$ and $(n,q) = (3,4)$, this is a contradiction.

(vi) The rest of the proof is to handle the case where V is linear-Weil and W is unitary-Weil, and $q \geq 4$ as above.

First we consider the case where $\dim(V) = (q^{2n}-1)/(q-1)$. According to [GT, Table I and Proposition 7.9], there is a one-dimensional $\mathbb{F}P_1$-module X such that $V \cong \mathrm{Ind}_{P_1}^G(X)$, where P_1 is the stabilizer in G of a one-dimensional subspace of $\mathcal{N}$. It follows that

$$V \otimes W \cong \mathrm{Ind}_{P_1}^G\big(X \otimes W|_{P_1}\big),$$

forcing $W|_{P_1}$ to be irreducible. But this contradicts [GT, Proposition 7.4].

According to [GT, Table I], it remains to consider the case where V is inside the reduction modulo ℓ of a complex module $V_{\mathbb{C}}$ which affords the linear-Weil character ρ_n^i for some $i = 1, 2$. Suppose that $V_{\mathbb{C}}(\mathrm{mod}\,\ell) = V$. By [L, Theorem 1.1], in this case the simple self-dual module V is a (graph) submodule of $\mathrm{Ind}_{P_1}^G(\mathbb{F})$, where $\mathbb{F}$ denotes the trivial $\mathbb{F}P_1$-module. By duality, V is also a quotient of $\mathrm{Ind}_{P_1}^G(\mathbb{F})$, whence $V|_{P_1}$ contains $\mathbb{F}$. On the other hand, by [GT, Proposition 7.4], the fixed-point submodule Y for $W|_{Q_1}$ is nonzero and has dimension at most $(q^{2n-2}-1)/(q+1)$, where $Q_1 := \mathbf{O}_2(P_1)$, and Y is stabilized by P_1. Thus $V \otimes W$ contains $\mathbb{F} \otimes Y \cong Y$ as a P_1-submodule, and so, by irreducibility and Frobenius' reciprocity, $V \otimes W$ is a quotient of $\mathrm{Ind}_{P_1}^G(Y)$, whence

$$\begin{aligned}\frac{(q^n+1)(q^n-q)}{2(q-1)} \cdot \frac{(q^n-1)(q^n-q)}{2(q+1)} &\leq \dim(V)\dim(W) \\ &\leq [G : P_1]\dim(Y) \leq \frac{q^{2n}-1}{q-1} \cdot \frac{q^{2n-2}-1}{q+1},\end{aligned}$$

a contradiction. In particular, we have completed the proof in the case $\ell = 0$.

(vii) By [GT, Table I], it remains to consider the case where either $\ell|(q^n-1)/(q-1)$ and $V_{\mathbb{C}}$ affords the character ρ_n^1 of degree $(q^n+1)(q^n-q)/2(q-1)$, or $\ell|(q^n+1)$ and $V_{\mathbb{C}}$ affords the character ρ_n^2 of degree $(q^n-1)(q^n+q)/2(q-1)$; in either case, $\dim(V) = \dim(V_{\mathbb{C}}) - 1$. We will let ρ denote the corresponding character ρ_n^i of $V_{\mathbb{C}}$.

First we assume that W is obtained by reducing modulo ℓ a $\mathbb{C}G$-module $W_{\mathbb{C}}$, which then affords a unitary-Weil character say θ, by [GT, Table I]. As we mentioned at the end of (vi), $\rho\theta$ is *reducible*. On the other hand, if χ° denotes the restriction to ℓ'-elements of a complex character χ of G, then $\rho^0 - 1_G$ is the Brauer character of V and so $(\rho\theta)^0 - \theta^0$ is the Brauer character of the irreducible $\mathbb{F}G$-module $V \otimes W$, and we are assuming that θ^0 is the Brauer character of W. It follows that $\rho\theta$ must be the sum of two irreducible complex characters, one of degree $\theta(1)$ and the other of degree $(\rho(1)-1)\theta(1)$. The latter contradicts Lemma 2.1(i) applied to $N_1 = \rho(1)$ and $N_2 = \theta(1)$.

According to [GT, Table I], the only case left to consider for W is that $\ell|(q+1)$ and W is inside the reduction modulo ℓ of a complex module $W_{\mathbb{C}}$ which affords the unitary-Weil character $\beta = \beta_n$ of degree $(q^n+1)(q^n+q)/2(q+1)$. Now we have

$$(\rho\beta)^{\circ} = (\rho^0 - 1_G)(\beta^0 - 1_G) + (\rho^0 - 1_G) + (\beta^0 - 1_G) + 1_G, \tag{2.13}$$

a sum of 4 irreducible Brauer characters (of $V \otimes W$, V, W, and $\mathbb{F}$, respectively. Again by the conclusion at the end of (vi), $\rho\beta = \gamma_1 + \ldots + \gamma_m$ is the sum of $m \geq 2$ complex irreducible characters of G. Because $[\rho\beta, 1_G] = [\rho, \bar{\beta}]_G = 0$, 1_G is not a constituent of $\rho\beta$, and so (2.13) implies that $m \leq 3$. Furthermore, by [GT, Theorem 6.1],

$$\text{None of } \gamma_i(1) \text{ can be } \rho(1) - 1 \text{ or } \beta(1) - 1. \tag{2.14}$$

It follows that $m = 2$.

Using Lemma 2.1(i) for $(N_1, N_2) = (\rho(1), \beta(1) - 1)$ and (2.14), we now have that $\{\gamma_1(1), \gamma_2(1)\}$ must be either

$$\{(\rho(1) - 1)(\beta(1) - 1) + 1,\ \rho(1) + \beta(1) - 2\},$$

or

$$\{\beta(1)(\rho(1) - 1),\ \beta(1)\},$$

or

$$\{\rho(1)(\beta(1) - 1),\ \rho(1)\}.$$

The first case where $\gamma_i(1) = \rho(1) + \beta(1) - 2$ is ruled out by [GT, Theorem 6.1], since

$$\max\left(\frac{q^{2n} - 1}{q + 1}, \frac{(q^n - 1)(q^n + q)}{2(q - 1)}\right) < \rho(1) + \beta(1) - 2 < \frac{q^{2n} - 1}{q - 1}.$$

The second case is impossible by Lemma 2.1(i) applied to $N_1 = \rho(1)$ and $N_2 = \beta(1)$. The third case is impossible by Lemma 2.1(ii) applied to $N_1 = \rho(1)$ and $N_3 = \beta(1)$. $\square$

Proof of Theorem D. The fact that either $q \leq 3$, or G must be one of the groups described in (ii)–(iii) of Theorem D follows from [MT, Theorems 1.1 and 1.2] and Theorem 2.2. Next, the case $G = \mathrm{SL}_n(2)$ or $\mathrm{SL}_n(3)$ is ruled out by [MT, Proposition 3.3] and [KT2, Theorem 8.8], respectively. Now we consider the case $G = \mathrm{Sp}_{2n}(5)$. By [MT, Proposition 5.2], it must be the case that at least one of α and β, say α, is a Weil character. Assume now that $\beta(1) = \alpha(1)$. By [GMST, Theorem 2.1], β is also a Weil character; moreover, α is obtained by reducing modulo ℓ a complex Weil character $\alpha_{\mathbb{C}}$, and likewise, β is obtained by reducing modulo ℓ a complex Weil character $\beta_{\mathbb{C}}$, furthermore, $\alpha_C\beta_C$ is irreducible. By [MT, Proposition 5.4], we have that $\{\alpha(1), \beta(1)\} = \{(5^n - 1)/2, (5^n + 1)/2\}$, i.e. $\alpha(1) \neq \beta(1)$, a contradiction. $\square$

Remark 2.3. (i) The case $q = 2$, i.e. $G = \mathrm{Sp}_{2n}(2)$, can indeed occur in Theorem D and Theorem 2.2: as shown in [MMT, Proposition 7.4], when $n \geq 3$, $\alpha_n\beta_n$ and $\alpha_n\gamma_n$ are irreducible, where $\alpha_n, \beta_n, \gamma_n \in \mathrm{Irr}(\mathrm{Sp}_{2n}(2))$ are *unitary-Weil* characters (as defined in [GT, Table I]) of degree

$$(2^n - 1)(2^{n-1} - 1)/3,\ (2^n + 1)(2^{n-1} + 1)/3,\ (2^{2n} - 1)/3.$$

It is plausible that these are the only irreducible tensor products of nontrivial complex characters of $\mathrm{Sp}_{2n}(2)$ when $n \geq 4$.

(ii) The cases $G = \mathrm{SU}_n(2)$ with $n \geq 4$, and $\mathrm{Sp}_{2n}(3)$, $\mathrm{Sp}_{2n}(5)$ with $n \geq 2$, can indeed occur in Theorem D, see [LST, Proposition 3.3(iii)] and [MT, Proposition 5.4]. In all these exhibited examples, both of the characters α and β are Weil characters. On the other hand, [GMT, Theorem 1.3] offers further examples of irreducible tensor products $\alpha\beta$ of $\mathrm{Sp}_{2n}(3)$ (with $n \geq 3$), where exactly one of α and β is a Weil character.

Remark 2.4. Note that $\mathrm{Aut}(\mathrm{Sp}_4(4))$ admits two irreducible complex characters of degree 18 and 50 whose tensor product is irreducible, whereas $\mathrm{Sp}_4(4)$ has no such example. Thus the almost simple groups may behave differently than the simple groups with respect to the irreducible tensor product problem.

3. Proof of Theorems A and B

First we prove Theorem B, which we restate below:

Theorem 3.1. *Let G be a finite quasisimple group, and let α, β be irreducible characters of G of the same degree $\alpha(1) = \beta(1) > 1$. Suppose that $\alpha\beta$ is irreducible. Then $(G/(\mathrm{Ker}(\alpha) \cap \mathrm{Ker}(\beta)), \alpha(1))$ is $(2 \cdot \mathsf{A}_5, 2)$, $(3 \cdot \mathsf{A}_6, 3)$, $(6 \cdot \mathsf{A}_7, 6)$, $(2 \cdot J_2, 6)$, $(3 \cdot J_3, 18)$, $((2^2 \times 3) \cdot \mathrm{PSL}_3(4), 6)$, $(4_1 \cdot \mathrm{PSL}_3(4), 8)$, $(4^2 \cdot \mathrm{PSL}_3(4), 8)$, $((3^2 \times 2) \cdot \mathrm{PSU}_4(3), 6)$, or $(3 \cdot G_2(3), 27)$.*

Proof. (i) Let $S = G/\mathbf{Z}(G)$ be the non-abelian simple quotient of G. Then the small cases $S = \mathsf{A}_n$ with $n \leq 10$, or S is one of the 26 sporadic simple groups, or

$$\begin{aligned} S = \ & \mathrm{SL}_3(2),\ \mathrm{PSL}_3(4),\ \mathrm{SL}_6(2),\ \mathrm{SL}_7(2),\ \mathrm{SU}_3(3),\ \mathrm{SU}_3(4), \mathrm{PSU}_3(8), \\ & \mathrm{SU}_4(2),\ \mathrm{PSU}_4(3),\ \mathrm{PSU}_6(2),\ \mathrm{Sp}_4(4),\ \mathrm{Sp}_6(2),\ \Omega_7(3),\ \mathrm{Sp}_8(2), \\ & \Omega_8^{\pm}(2),\ {}^2B_2(8),\ G_2(3),\ G_2(4),\ {}^2F_4(2)',\ F_4(2),\ {}^2E_6(2) \end{aligned}$$

are checked using [GAP].

Next we consider the case $S = \mathsf{A}_n$ with $n \geq 11$. If $G = S$, then by [Zi, Theorem 10] and its proof, we must have that $n = k^2$ for some $k \in \mathbb{N}$, and, say α, is obtained by restricting the S_n-character labeled by the partition $(n-1, 1)$, whereas the other is one of the two constituent of the S_n-character labeled by the partition $(k, k, \ldots, k)$; in particular, $\alpha(1) < \beta(1)$.

Hence we may assume that $G = 2 \cdot \mathsf{A}_n$ and α is faithful. Assume β is faithful. If neither α nor β is a basic spin character, then $\alpha\beta$ is reducible by [KT1, Theorem F]. So we may assume that α is a basic spin character, in which case, by [KT1, Theorem A], β is also basic spin as it has the same degree. Now $\gamma := \alpha\beta$ is an irreducible character of A_n, of degree $D_1 := 2^{2\lfloor n/2-1 \rfloor} \geq 2^{n-3}$, and it lies under an irreducible character δ of S_n of degree $D = D_1$ or $2D_1$, a 2-power. As $n \geq 9$, it follows from [BBOO, Theorem 2.4] that $n = D + 1$, a contradiction.

It remains to consider the case α is faithful but β is not. If moreover α is basic spin, then $\beta(1) = \alpha(1) = 2^{\lfloor n/2-1 \rfloor} \geq 2^{(n-3)/2} > n - 1$, and so β cannot be lying under an irreducible character of S_n by the same result [BBOO, Theorem 2.4]. Hence we may assume that α is not basic spin, and $\alpha(1) = \beta(1) > 2^{\lfloor n/2-1 \rfloor} \geq 2^{(n-3)/2} > n - 1$. This final case is ruled out by the recent result [Mo, Theorem 1.2].

(ii) From now on we let S be a simple group of Lie type defined over $\mathbb{F}_q$, $q = p^f$, not isomorphic to any of the small groups handled using [GAP] in (i). (One can use [MT, Theorem 1.2] to slightly reduce the number of subcases for S to be considered here, but we will give a uniform treatment of all possibilities.) The main idea is to show that, in most cases, any irreducible character of G of degree $\alpha(1)$ has ℓ-defect 0 for some prime ℓ. In particular, β also has ℓ-defect 0, but then $\alpha\beta$ has degree divisible by $|G|_\ell^2$ and so cannot be irreducible.

To exhibit the above ℓ, we will rely on the arguments in [M], which also use the existence of *primitive prime divisors* $\ell(m) := \ell(q,m)$, i.e. a prime divisor of $q^m - 1$ that does not divide $\prod_{i=1}^{m-1}(q^i - 1)$ [Zs], for suitable m.

First we consider the case $S = \mathrm{PSL}_n(q)$ with $n \geq 4$, $(n,q) \neq (4,2), (6,2), (7,2)$; in particular, both $\ell(n)$ and $\ell(n-1)$ exist. As shown in [M], α has defect 0 for at least one of the primes $\ell(n)$, $\ell(n-1)$, or p, whence the above observation applies.

If $S = \mathrm{PSL}_2(q)$ with $q \geq 8$ and $q \neq 9$, then $\alpha(1) \geq (q-1)/\gcd(2, q-1)$ and so $\alpha\beta$ has degree too big to be an irreducible character of G. Assume $S = \mathrm{PSL}_3(q)$ and $q \neq 2, 4$. Then $\ell(3)$ exists, and if α does not have $\ell(3)$-defect 0, then $\alpha(1) \geq q(q+1)$, and again $\alpha\beta$ has too big degree.

Next we consider the case $S = \mathrm{PSU}_n(q)$ with $n \geq 4$, $(n,q) \neq (4,2), (4,3), (6,2)$; in particular, the primitive prime divisors ℓ_1 and ℓ_2 as indicated in [M, Table 3.5] exist. As shown in [M], α has defect 0 for at least one of the primes ℓ_1, ℓ_2, or p, whence we are done. Assume $S = \mathrm{PSU}_3(q)$ and $q \neq 2, 3, 4, 8$. Then $\ell(6)$ exists, and if α does not have $\ell(6)$-defect 0, then $\alpha(1) \geq q(q-1)$, and $\alpha\beta$ has too big degree.

(iii) Assume $S = \Omega_{2n+1}(q)$ with $n \geq 3$, $(n,q) \neq (3,2), (3,3), (4,2)$; in particular, the primitive prime divisors ℓ_1 and ℓ_2 as indicated in [M, Table 3.5] exist. As shown in [M], α has defect 0 for at least one of the primes ℓ_1, ℓ_2, or p, or else $2|n$ and α has $\ell(n-1)$-defect 0, and so we are done again. Assume $S = \Omega_5(q)$ and $q \geq 5$. Then $\ell(4)$ exists, and if α does not have $\ell(4)$-defect 0, then $\alpha(1) \geq (q^2-1)/2$, and $\alpha\beta$ has too big degree to be irreducible.

Next we consider the case $S = \mathrm{PSp}_{2n}(q)$ with $n \geq 3$ and $2 \nmid q$; in particular, the primitive prime divisors ℓ_1 and ℓ_2 as indicated in [M, Table 3.5] exist. If moreover α is unipotent, then one can argue as in the above case of $\Omega_{2n+1}(q)$. Assume α is not unipotent. As shown in [M], if α does not have defect 0 for at least one of the primes ℓ_1, ℓ_2, or p, then $2|n$ and $\alpha(1) = q^{n(n-1)}(q^n-1)/2$ (and so $\alpha\beta$ has too big degree), or $\alpha(1) = (q^n-1)/2$. In the latter case, both α and β are Weil characters by [TZ, Theorem 5.2], and $\alpha\beta$ is reducible by [MT, Proposition 5.4].

Assume now that $S = P\Omega_{2n}^\epsilon(q)$ with $n \geq 4$, $\epsilon = \pm$, and $(n,q) \neq (4,2)$; in particular, the primitive prime divisors ℓ_1 and ℓ_2 as indicated in [M, Table 3.5] exist. As shown in [M], if α does not have defect 0 for at least one of the primes ℓ_1, ℓ_2, or p, then $2|n$, $\epsilon = +$, and α has $\ell(2n-4)$-defect 0, and so we are done again.

(iv) Now we consider exceptional groups of Lie type. We will again use the primes ℓ_1, ℓ_2, ℓ_3 as indicated in [M, §4]. First let $S = {}^2B_2(q)$ with $q \geq 8$. If α does not have defect 0 neither for ℓ_1 nor for ℓ_2 or 2, then α is one of the two, complex conjugate, unipotent characters of degree $(q-1)\sqrt{q/2}$. Hence $\alpha\beta = \alpha^2$ or $\alpha\overline{\alpha}$, none of which can be irreducible. The same arguments apply to the case $S = {}^2G_2(q)$ with $q \geq 27$.

Suppose $S = {}^2F_4(q)$ with $q \geq 8$. Then, as shown in [M], either α has defect 0 for at least one of ℓ_1, ℓ_2, ℓ_3, or 2, or else $\alpha(1) = q^2(q^4-1)^2/3$, in which case $\alpha\beta(1)$ is too big. Next assume that $S = G_2(q)$ with $q \geq 5$. Then, as shown in [M], either α has defect 0 for at least one of ℓ_1, ℓ_2, or p, or else $\alpha(1) = q(q^2-1)^2/3$, in which case $\alpha\beta(1)$ is too big. If $S = {}^3D_4(q)$, then all α of positive $\ell(12)$-defect have too big degree for $\alpha\beta$ to be irreducible. Suppose $S = F_4(q)$ with $q > 2$. Then, as shown in [M], either α has defect 0 for at least one of ℓ_1, ℓ_2, or p, or else $\alpha(1)$ is again too big.

Finally, suppose that $S = {}^2E_6(q)$ with $q > 2$, or $S = E_6(q)$, $E_7(q)$, or $E_8(q)$. In all these cases, as shown in [M], α has defect 0 for at least one of the primes ℓ_1, ℓ_2, ℓ_3, or p, and so we are done. $\square$

Remark 3.2. We note that, in fact, out treatment of *generic* (that is, not the ones considered in (i), plus a few addtional small exceptions) Lie-type groups in Theorem 3.1 also applies to the case where $L := E(G)$ is quasisimple, $F(G) = \mathbf{Z}(G)$, and $\alpha, \beta \in \mathrm{Irr}(G)$ are nontrivial on L. Indeed, the arguments show that, if θ is an irreducible constituent of $\alpha|_L$, then either θ has ℓ-defect 0 for some prime ℓ that does not divide $|\mathrm{Out}(L)|$, hence also coprime to $|G/\mathbf{Z}(G)L|$, or it is the Steinberg character of L. Apart from a small list of exceptions, it follows that $\alpha\beta(1)$ does not divide $|G/\mathbf{Z}(G)|$, and so $\alpha\beta$ cannot be irreducible. The case of $2 \cdot \mathsf{S}_n$ is handled in [B], [BK1], [BK3], see also [BK2] for the case of A_n.

Now we can prove Theorem A. As the reader will see, for the first part, we reproduce some arguments in Theorem 2.3 of [Is2] for not necessarily solvable groups.

Proof of Theorem A. Suppose that $\mu^G = \chi$, where $\mu \in \mathrm{Irr}(H)$, and H is a subgroup of G. Then
$$\chi\chi^\sigma = \mu^G\chi^\sigma = (\mu(\chi^\sigma)_H)^G$$
and we deduce that $(\chi^\sigma)_H$ is irreducible. Since
$$[(\chi^\sigma)_H, (\chi^\sigma)_H] = [\chi_H, \chi_H]$$
we deduce that $\chi_H \in \mathrm{Irr}(H)$. Then $\chi_H = \mu$ and by degrees we have that $G = H$. Therefore, we have that χ is primitive. In particular, by the Clifford correspondence, if $N \triangleleft G$, then χ_N is a multiple of an irreducible character τ of N. If furthermore this character τ is linear, then $N \leq \mathbf{Z}(G)$ is cyclic (using that χ is faithful). In particular, every abelian normal subgroup of G is central and cyclic.

Assume that $\mathbf{Z}(G) < \mathbf{F}(G)$, and we look for a contradiction. As we said, we simply rearrange some of the arguments in Theorem 2.3 of [Is2] in our particular case, and check that we can apply them when G is not necessarily solvable, to obtain a contradiction.

Let $E \triangleleft G$ be nilpotent and minimal such that E is not contained in $\mathbf{Z}(G)$, and let $Z = E \cap \mathbf{Z}(G)$. By the first paragraph in the proof, we have that E is not abelian. (In the situation of Theorem 2.3 of [Is2], to obtain that E is non-abelian takes a few paragraphs and a previous lemma on solvable groups.) Arguing as in the first paragraph of the proof of Theorem 2.3 of [Is2], we have that E is a p-group of nilpotent class 2, $Z > 1$, and that E/Z is an abelian chief factor of G. Write $\chi_E = d\theta$, for some faithful $\theta \in \mathrm{Irr}(E)$ and $\chi_Z = \chi(1)\lambda$, where $\lambda \in \mathrm{Irr}(Z)$. By Theorem 6.18 of [Is3], we have that θ is fully ramified with respect to E/Z. (Notice that if θ extends λ, then θ is linear and faithful, so E is abelian.) Hence $\lambda^E = e\theta$, where $e^2 = |E : Z|$. Also, $(\lambda^\sigma)^E = e\theta^\sigma$. Write $\nu = \lambda\lambda^\sigma$, and notice that

$$\nu^E = \theta\theta^\sigma \, .$$

If ν extends to E, then, by Problem 6.12 of [Is3], we have that

$$\theta\theta^\sigma = \sum_{\substack{\mu\in\mathrm{Irr}(E)\\ \mu_Z=\nu}} \mu \, .$$

Since $\chi\chi^\sigma$ is irreducible, it follows that all the extensions of ν to E are G-conjugate, by Clifford's theorem. We deduce that $\nu = \lambda\lambda^\sigma \neq 1$, because, otherwise, $\lambda^\sigma = \bar\lambda$ and $\theta^\sigma = \bar\theta$. Then $\theta\theta^\sigma$ would countain the trivial character 1_E, and thus $\theta\theta^\sigma = \theta(1)^2 1_E$, which is not possible, since θ vanishes off Z. In particular, we deduce that $|Z| > 2$ (since λ and λ^σ are non-trivial.)

Now, Isaacs' arguments in the last paragraph of page 636 and first paragraph of page 637 in [Is2], show that either $|Z| = p$ is odd or else $p = 2$ and $|Z| = 4$ (and in this case E' has order 2 and E/E' is elementary abelian). This latter case is solved by the clever argument in the last three paragraphs in Theorem 2.3 of [Is2]. So we are left with the case where $|Z| = p$, and p odd.

In this final case, the theory in [Is1] (Theorem 9.1) applies, and produces a complement U of E/Z in G/Z, a character $\Psi^{(\lambda)} \in \mathrm{Char}(G)$, and a bijection of characters $\mathrm{Irr}(G|\theta) \to \mathrm{Irr}(U|\lambda)$. Theorem 9.1 of [Is1] only requires that E/Z has odd order, so we can apply this theorem even if G is non-solvable. Arguing as in the p odd case of Theorem 2.3 of [Is2], we finish the first part of Theorem A. The second part follows from the more general result Theorem 3.1. (Note that the characters α and β in the extra examples of $6\cdot\mathsf{A}_7$, $(2^2\times 3)\cdot\mathrm{PSL}_3(4)$, $4^2\cdot\mathrm{PSL}_3(4)$, $(3^2\times 2)\cdot\mathrm{PSU}_4(3)$, and $3\cdot G_2(3)$, are not Galois conjugate.) $\square$

Notice that Theorem A does not have an analog in characteristic $\ell > 0$ since, outside ℓ-solvable groups, $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ does not necessarily act on Brauer characters. For ℓ-solvable groups, the result easily follows from the Fong-Swan theorem.

Corollary 3.3. *Suppose that G is ℓ-solvable. If $\phi \in \mathrm{IBr}(G)$ is faithful non-linear and $\sigma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$, then $\phi\phi^\sigma$ is not irreducible.*

Proof. By the Fong-Swan theorem (Theorem 10.1 of [N]), let $\chi \in \mathrm{Irr}(G)$ be such that $\chi^\circ = \phi$. Since ϕ is faithful, notice that χ is faithful (using the definition for faithful Brauer characters). Now, $(\chi\chi^\sigma)^\circ = \phi\phi^\sigma$. If $\phi\phi^\sigma$ is irreducible, then $\chi\chi^\sigma$ is irreducible. But his is not possible by Theorem A. $\square$

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