Annals of PDE (2021) 7:18
https://doi.org/10.1007/s40818-021-00110-5

MANUSCRIPT

Asymptotic Stability of Equilibria for Screened Vlasov–Poisson Systems via Pointwise Dispersive Estimates

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Received: 9 November 2019 / Accepted: 4 August 2021 / Published online: 20 August 2021

Abstract
We revisit the proof of Landau damping near stable homogenous equilibria of Vlasov–Poisson systems with screened interactions in the whole space $\mathbb{R}^d$ (for $d \geq 3$) that was first established by Bedrossian, Masmoudi and Mouhot in [5]. Our proof follows a Lagrangian approach and relies on precise pointwise in time dispersive estimates in the physical space for the linearized problem that should be of independent interest. This allows to cut down the smoothness of the initial data required in [5] (roughly, we only need Lipschitz regularity). Moreover, the time decay estimates we prove are essentially sharp, being the same as those for free transport, up to a logarithmic correction.

Keywords Vlasov equations · Non-linear stability · Spatially homogeneous steady states · Dispersive estimates

1 Introduction

In this paper, we are interested in the large time behavior of solutions to the Vlasov–Poisson system with screening

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$$\begin{cases} \partial_t f_i + v \cdot \nabla_x f_i + E \cdot \nabla_v f_i = 0, \\ E = -\nabla_x (1 - \Delta_x)^{-1} (\rho_i - 1), \qquad \rho_i(t,x) = \displaystyle\int_{\mathbb{R}^d} f_i(t,x,v)\, dv, \end{cases} \tag{1.1}$$

on the whole space $x \in \mathbb{R}^d$, $v \in \mathbb{R}^d$, $d \geq 3$, where $f_i = f_i(t,x,v) \geq 0$ and $E = E(t,x)$. The screening effect comes from the fact that the interaction potential associated to $\mathrm{Id} - \Delta$ is exponentially decaying as opposed to the Coulomb potential associated to $-\Delta$. This system is sometimes referred to as Vlasov–Yukawa and can also be seen as the Vlasov–Poisson system describing the dynamics of ions, in a background of electrons that satisfy a linearization of the Maxwell-Boltzmann law (we refer for example to [3,6,11]).

The global regularity of finite energy solutions for the Vlasov–Poisson system in the case of three or lower spatial dimension is by now classical ([9,14,16,19,20]). The asymptotic behavior of solutions for initial data near the trivial equilibrium 0 has also been the topic of many studies. This was first established in dimension $d \geq 3$ in the unscreened case by Bardos and Degond in [2], following a Lagrangian approach. More recently the sharp faster decay of derivatives was established in [15]. This was extended to (1.1) in dimension $d \geq 2$ in [8] making use of the better decay of the electric field in the case of screened interactions. Other approaches based on vector fields [21] or Fourier analysis [23] (space–time resonances) were also developed recently.

We are interested in the stability and the large time behavior of solutions near spatially homogeneous stationary states $\mu(v)$ such that $\int_{\mathbb{R}^d} \mu(v)\, dv = 1$. Namely, we look for a solution under the form $f_i(t,x,v) = \mu(v) + f(t,x,v)$, where f solves the perturbed system

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v \mu = -E \cdot \nabla_v f, \\ E = -\nabla_x (1 - \Delta_x)^{-1} \rho, \qquad \rho(t,x) = \displaystyle\int_{\mathbb{R}^d} f(t,x,v)\, dv, \\ f|_{t=0} = f_0. \end{cases} \tag{1.2}$$

For a class of stable equilibria, we shall study the large time behavior of solutions to (1.2) for suitably small initial data f_0. The dynamics of solutions is expected to asymptotically approach that of solutions to free transport, a scattering phenomenon that is often referred to as *Landau damping*. Landau damping was proved on the torus $\mathbb{T}^d \times \mathbb{R}^d$ ($d \geq 1$) for data with Gevrey regularity [4,17], while on the whole space $\mathbb{R}^3 \times \mathbb{R}^3$ it was recently established for the screened Vlasov–Poisson system (1.2) by Bedrossian, Masmoudi and Mouhot in [5], for data with finite Sobolev regularity. The proof in [5] is inspired by that for the torus case. Though dispersion on the whole space is used at some crucial points in order to close the estimates in finite regularity, the approach is much more related to the one of [4] for the torus than that of Bardos-Degond (it is actually dispersive properties of the free transport in the frequency space that are used to control the so-called echoes of [17]).

In this paper, we prove Landau damping and derive dispersive estimates for solutions to (1.2) via a Lagrangian approach that is closer to the Bardos-Degond analysis [2] for the $\mu = 0$ case. Roughly speaking, after proving precise pointwise estimates for the

linearized equation, the proof of nonlinear stability can be obtained almost in the same way as in the $\mu = 0$ case. This will allow to strongly cut down the needed regularity on the initial data, as compared to [5].

Let us now specify our assumptions on the equilibrium μ and state the main result of this paper. We assume

- **(H1)** $\mu \in L^1(\mathbb{R}^d)$ is a smooth decaying function satisfying $\langle v\rangle^k \nabla_v \mu \in W^{2,\infty}$ and $\langle v\rangle^{4d+6}\nabla_v \mu \in W^{2d+5,1}$ for some $k > d$.
- **(H2)** μ satisfies the Penrose stability criterion:

$$\inf_{\gamma\geq 0}\inf_{\tau\in\mathbb{R},\,\xi\in\mathbb{R}^d}\left|1-\int_0^{+\infty} e^{-(\gamma+i\tau)s}\frac{1}{1+|\xi|^2}i\xi\cdot\widehat{\nabla_v\mu}(s\xi)\,ds\right|\geq\kappa,$$

  for some constant $\kappa > 0$, where $\widehat{\nabla_v\mu}$ is the Fourier transform of $\nabla_v\mu$ (see (1.7) for the convention we use).

Assumption **(H2)**, called Penrose stability criterion in plasma physics, is a classical linear stability condition which is also used in [4,5,17]. If this assumption is violated in the sense that the function inside the inf vanishes for some $\gamma > 0$, the equilibrium μ is linearly unstable, there exist smooth localized perturbations with exponential growth, and also nonlinearly unstable; we refer for example to [12] Theorem 3.1 for a proof of nonlinear instability in dimension 1 for periodic boundary conditions (the proof could be adapted to handle higher dimensions, see [13], and localized perturbations on the whole space, following [10]). There are many known more tractable sufficient conditions on μ in order for **(H2)** to be satisfied, we refer for example to [17, Section 2.2]. In particular, this assumption is satisfied for any sufficiently smooth and localized positive radial equilibrium $\mu(v) = F(|v|)$ in dimension $d \geq 3$ and for any positive radial equilibrium with $F' < 0$ in any dimension. Assumption **(H1)** simply states that the equilibrium μ needs to be sufficiently smooth and localized - we did not try to optimize on this assumption for the sake of readability.

1.1 Main Result

Our main result is as follows. We recall that we consider $d \geq 3$.

Theorem 1.1 *Assume that (H1) and (H2) are satisfied.*

Let $k > d$, σ, $p \in (1,+\infty)$ satisfying $p(\sigma-1) > 2d$ and p' such that $p^{-1}+p'^{-1} = 1$. Let $f_0 \in W^{1,\infty}\cap W^{1,1}$ be an initial condition for (1.2) with

$$\|\langle v\rangle^{k/p'} f_0\|_{W^{\sigma,p}} < +\infty, \qquad \text{for some } \sigma,\, p \in (1,+\infty) \text{ satisfying } p(\sigma-1) > 2d \tag{1.3}$$

and

$$\|\langle v\rangle^k f_0\|_{W^{1,\infty}} + \|f_0\|_{W^{1,1}} + \|f_0\|_{L^1_x L^\infty_v} + \|\nabla_{x,v} f_0\|_{L^1_x L^\infty_v} \leq \varepsilon_0. \tag{1.4}$$

Then if $\varepsilon_0 > 0$ is small enough, there exists a unique global solution of (1.2) such that

$$\|\rho(t)\|_{L^1} + \langle t\rangle\|\nabla_x\rho(t)\|_{L^1} + \langle t\rangle^d\|\rho(t)\|_{L^\infty} + \langle t\rangle^{d+1}\|\nabla_x\rho(t)\|_{L^\infty} \lesssim \varepsilon_0\log(2+t), \tag{1.5}$$

for all $t \geq 0$, *with* $\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}$.

Theorem 1.1 proves that the solution of (1.2) enjoys the same decay properties as the free transport up to a logarithmic correction. Note that we also establish the sharp higher decay of derivatives, still up to a logarithmic correction. The same decay holds for $E = -\nabla_x (1 - \Delta_x)^{-1} \rho$ via standard elliptic estimates. Improved decay of higher derivatives can be obtained by iterating the method introduced in this paper, since the submission of the paper this analysis has been performed in [18]. Note that the low dimension cases where dispersion effects are weaker are left open. In dimension 2, when $\mu = 0$, the decay properties of the density (with weaker estimates for the highest order derivatives in order to close the argument) have been established in [8] it would be very interesting to study the influence of a non-trivial μ on this behaviour. The behavior of the highest derivatives is usually hard to establish when studying decay rates in nonlinear dispersive problems. A common approach is to have non sharp estimates in a high norm and to recover the linear dispersive rate for much lower norms, this is in some sense the approach chosen in [5]. Here, since we have tried to minimize the maximal number of derivatives, we have allowed a logarithmic loss and could obtain an almost sharp estimate of the L^∞ decay for the highest derivatives. It seems possible to suppress the logarithmic loss in the L^∞ estimates by involving higher derivatives with weaker estimates.

Note that we have not tried to optimize the needed regularity of the initial data in this statement, (1.3) means that we ask slightly more than Lipschitz regularity for the initial condition f_0. Since we can take p arbitrarily large, this means that σ can be taken arbitrarily close to one. We could actually take $\sigma = 1 + \frac{2d}{p}$ by replacing $W^{\sigma,p}$ by the Besov space $B^\sigma_{p,1}$ (this would change in particular the local existence result of Proposition 3.1). Note that we do not ask for any smallness in (1.3). This is used only in order to ensure that a certain quantity (namely $\mathcal{N}(t)$ defined in (3.4)), that we shall use for a bootstrap argument, is continuous in time.

We could actually even only ask $\sigma > \frac{2d}{p}$, so that loosely speaking, merely smallness in Hölder norm (instead of Lipschitz) would be needed, but this would require to replace the $W^{1,\infty}$ and $W^{1,1}$ estimates for the density ρ proved in the paper by more technical $\mathscr{C}^{0,\alpha} = B^\alpha_{\infty,\infty}$ and $B^\alpha_{1,\infty}$ estimates. The main part of the analysis of this paper, namely the linearized estimates for the density ρ, is based on a Littlewood–Paley decomposition and is thus amenable to such a generalization. For the non-linear estimates, one should replace differentiation by finite differences.

As a consequence of Theorem 1.1, we obtain the aforemetioned scattering property for the solution to (1.2).

Corollary 1.1 *With the same assumptions and notations as in Theorem 1.1, there is* $f_\infty \in W^{1,\infty}$ *given by*

$$f_\infty(x,v) = f_0\left(x + Y_\infty(x,v), v + W_\infty(x,v)\right) + \mu\left(v + W_\infty(x,v)\right) - \mu(v)$$

such that

$$\|f(t, x + tv, v) - f_\infty(x,v)\|_{L^\infty_{x,v}} \lesssim \varepsilon_0 \frac{\log(2+t)}{\langle t \rangle^{d-1}}, \tag{1.6}$$

for all $t \geq 0$. *Moreover, we also have that* $\|Y_\infty\|_{L^\infty_{x,v}} + \|W_\infty\|_{L^\infty_{x,v}} \lesssim \varepsilon_0$.

The proof which is an easy consequence of (1.5) will be given in Section 8.

The remaining of the paper is devoted to the proof of Theorem 1.1 and is organized as follows. Linear estimates are derived in Section 2. This is the main ingredient in the proof: we shall establish in particular that the density ρ of the solution of the linearized equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v \mu = 0, \\ E = -\nabla_x (1 - \Delta_x)^{-1} \rho, \qquad \rho(t,x) = \displaystyle\int_{\mathbb{R}^d} f(t,x,v)\, dv, \\ f|_{t=0} = f_0 \end{cases}$$

enjoys the same pointwise in time L^p decay as that of the free transport equation (that corresponds to $\mu = 0$), up to a logarithmic correction.

The bootstrap argument allowing to get Theorem 1.1 is introduced in Section 3. Section 4 and 5 are devoted to stability estimates for characteristics. In view of applying the linear estimates of Section 2, several source terms need to be estimated. The contribution from the initial data is studied in Section 6, while the estimates on the terms due to the reaction term $-E \cdot \nabla_v \mu$ are established in Section 7. Note that in order to establish the higher decay of derivatives, we shall use a different, more straightforward change of variables than in [15]. This approach actually allows to recover the estimate of [15] without logarithmic loss. The logarithmic loss in our main result only comes from the linear estimates in the presence of a non-trivial μ. The paper ends with a reminder of a few classical estimates related to the Littlewood–Paley decomposition.

1.2 Notations

We use $\widehat{\cdot}$ for the "space" Fourier transform on $\mathbb{R}^d$ and $\widetilde{\cdot}$ for the "space–time" Fourier transform on $\mathbb{R}^{d+1}$ with the convention:

$$\widehat{g}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} g(x)\, dx, \qquad \widetilde{h}(\tau,\xi) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} e^{-i\tau t} e^{-ix\cdot\xi} h(t,x)\, dxdt. \quad (1.7)$$

Throughout the paper, functions depending on time are extended by zero for $t < 0$.

We shall use the homogeneous Littlewood–Paley decomposition in $\mathbb{R}^n$ with $n = d$ or $d+1$. We write for $u \in \mathcal{S}'(\mathbb{R}^n)$,

$$u = \sum_{q\in\mathbb{Z}} u_q$$

where

$$\overline{u_q}(\zeta) = \overline{u}(\zeta)\chi_q(\zeta), \quad \chi_q(\zeta) = \chi(\frac{\zeta}{2^q}), \qquad \bar{\cdot} = \widehat{\cdot} \text{ or } \widetilde{\cdot}, \quad (1.8)$$

and $\chi \in [0,1]$ is a fixed smooth compactly supported function in the annulus $\frac{3}{4} \le |\zeta| \le 8/3$ which is equal to one in the annulus $\frac{4}{3} \le |\zeta| \le 3/2$. More precisely, we

take

$$\chi(\zeta) = \phi(\zeta/2) - \phi(\zeta)$$

where $\zeta \mapsto \phi(\zeta) \in [0,1]$ is a smooth nonnegative function which is supported in the ball $B(0,4/3)$ and which is identically one in the ball $B(0,3/4)$. The classical Bernstein Lemma is recalled in Lemma A.1.

2 Linear Estimates

In this section we study the linear equation

$$\rho(t,x) = \int_0^t \int_{\mathbb{R}^d} [\nabla_x (1-\Delta_x)^{-1} \rho](s, x-(t-s)v) \cdot \nabla_v \mu(v)\, dv ds + S(t,x), \quad t \geq 0, \tag{2.1}$$

with S being a given source term. In what follows, we extend ρ and S by zero for $t < 0$ so that the equation (2.1) is satisfied for $t \in \mathbb{R}$. The main result of this section is the following.

Theorem 2.1 *Assume that (H1) and (H2) are satisfied. Then for all $S \in L^1(\mathbb{R}, L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$, there exists a unique solution of (2.1) in $L^1_{loc}(\mathbb{R}, L^2(\mathbb{R}^d))$ that can be expressed in the following way:*

$$\rho = S + G *_{t,x} S, \tag{2.2}$$

where the kernel $G(t,x)$ satisfies $G|_{t<0} = 0$ and there exists $C > 0$ such that the following uniform estimates hold:

$$\|G(t)\|_{L^1} \leq \frac{C}{1+t}, \qquad \|G(t)\|_{L^\infty} \leq \frac{C}{t^{d-1+\delta} + t^{d+1}}, \quad \forall t > 0, \tag{2.3}$$

where $\delta \in (0,1)$ can be chosen arbitrarily small. Furthermore, its spatial derivatives satisfy

$$\|\nabla_x G(t)\|_{L^1} \leq \frac{C}{t^2}, \qquad \|\nabla_x G(t)\|_{L^\infty} \leq \frac{C}{t^{d+2}}, \quad \forall t \geq 1. \tag{2.4}$$

As a corollary of Theorem 2.1, we immediately obtain decay estimates for the solution of (2.1).

Corollary 2.1 *Assume that (H1) and (H2) are satisfied. Then, there exists $M > 0$ such that for all $S \in L^1(\mathbb{R}, L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$, the solution of (2.1) satisfies the estimates*

$$\|\rho(t)\|_{L^1} + t^d \|\rho(t)\|_{L^\infty} \leq M \log(1+t) \|S\|_{Y^0_t},$$
$$t\|\nabla\rho(t)\|_{L^1} + t^{d+1} \|\nabla\rho(t)\|_{L^\infty} \leq M \log(1+t) \|S\|_{Y^1_t},$$

for $t \geq 1$, where the norms Y_t^0, Y_t^1 are defined by

$$\begin{aligned} \|S\|_{Y_t^0} &= \sup_{[0,t]} \left(\|S(s)\|_{L^1} + (1+s)^d \|S(s)\|_{L^\infty} \right), \\ \|S\|_{Y_t^1} &= \sup_{[0,t]} \left(\|S(s)\|_{L^1} + (1+s)\|\nabla S(s)\|_{L^1} + (1+s)^{d+1} \|\nabla S(s)\|_{L^\infty} \right). \end{aligned} \tag{2.5}$$

In Corollary 2.1, we state only large time estimates, since the estimates for $t \leq 1$ can be obtained in a straightforward way.

Note that derivatives decay at a t^{-1} faster rate. In particular, Corollary 2.1 immediately yields decay estimates for the linearized Vlasov–Poisson system of (1.2) around $\mu(v)$, namely for the system

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v \mu = 0, \\ E = -\nabla_x (1 - \Delta_x)^{-1} \rho, \qquad \rho(t,x) = \displaystyle\int_{\mathbb{R}^d} f(t,x,v)\, dv, \\ f|_{t=0} = f_0. \end{cases} \tag{2.6}$$

Indeed, using the method of characteristics, we obtain that ρ solves (2.1) with $S(t,x)$ given by

$$S(t,x) = \int_{\mathbb{R}^d} f_0(x - tv, v)\, dv.$$

Assuming that $f_0 \in L^1_{x,v}$ and $f_0 \in L^1_x(L^\infty_v)$, we have from the standard dispersive estimates for free transport (see [2]) that

$$\|S(t)\|_{L^1} \leq \|f_0\|_{L^1}, \quad \|S(t)\|_{L^\infty} \leq \frac{1}{t^d} \|f_0\|_{L^1_x L^\infty_v}.$$

Similar estimates hold for derivatives:

$$\|\nabla S(t)\|_{L^1} \leq \frac{1}{t} \|\nabla_v f_0\|_{L^1}, \quad \|\nabla S(t)\|_{L^\infty} \leq \frac{1}{t^{d+1}} \|\nabla_v f_0\|_{L^1_x L^\infty_v}.$$

Therefore, we obtain from Corollary 2.1 that the pointwise behavior of the density of the linearized equation (2.6) is the same as the one of the free transport, up to a logarithmic loss.

Let us right away provide the proof of Corollary 2.1.

Proof of Corollary 2.1 Using the representation (2.2) and the fact that S and G vanish for negative times we obtain

$$\rho(t) = S(t) + \int_0^t G(t-s) *_x S(s)\, ds.$$

Therefore, using (2.3), we obtain

$$\begin{aligned}\|\rho(t)\|_{L^1} &\lesssim \|S(t)\|_{L^1} + \int_0^t \frac{1}{1+(t-s)}\|S(s)\|_{L^1}\,ds \\ &\lesssim \left(1+\int_0^t \frac{1}{1+s}\,ds\right)\sup_{[0,t]}\|S(s)\|_{L^1}.\end{aligned}$$

In a similar way, we have

$$\begin{aligned}\|\rho(t)\|_{L^\infty} &\lesssim \|S(t)\|_{L^\infty} + \int_0^{\frac{t}{2}} \|G(t-s)\|_{L^\infty}\|S(s)\|_{L^1}\,ds \\ &+ \int_{\frac{t}{2}}^t \|G(t-s)\|_{L^1}\|S(s)\|_{L^\infty}\,ds.\end{aligned}$$

Therefore, using again (2.3), we get that

$$\begin{aligned}\|\rho(t)\|_{L^\infty} &\lesssim \|S(t)\|_{L^\infty} + \frac{1}{t^{d+1}}\int_0^{\frac{t}{2}} \|S(s)\|_{L^1}\,ds + \int_{\frac{t}{2}}^t \frac{1}{1+t-s}\|S(s)\|_{L^\infty}\,ds \\ &\lesssim \|S(t)\|_{L^\infty} + \frac{1}{t^d}\sup_{[0,t/2]}\|S(s)\|_{L^1} + \frac{1}{t^d}\sup_{[t/2,t]}(1+s^d)\|S(s)\|_{L^\infty}\int_0^{\frac{t}{2}}\frac{1}{1+s}\,ds \\ &\lesssim t^{-d}\log(1+t)\|S\|_{Y_t^0},\end{aligned}$$

upon recalling the notation (2.5). This proves the desired estimates for $\rho(t)$. Similarly, we compute

$$\begin{aligned}t\|\nabla\rho(t)\|_{L^p} &\lesssim t\|\nabla S(t)\|_{L^p} + t\int_0^{\frac{t}{2}} \|\nabla G(t-s)\|_{L^p}\|S(s)\|_{L^1}\,ds \\ &+ t\int_{\frac{t}{2}}^t \|G(t-s)\|_{L^1}\|\nabla S(s)\|_{L^p}\,ds\end{aligned}$$

for $p=1$ and $p=\infty$. By using (2.3) and (2.4), the estimates for derivatives follow. □

Before giving the proof of Theorem 2.1 it will be useful to establish some properties of the kernel of the integral equation (2.1).

Let us set $w = x-(t-s)v$ and then integrate by parts to get the equivalent formulation

$$\begin{aligned}\rho(t,x) &= \int_0^t\int_{\mathbb{R}^d}\rho(s,w)\left(\left(1-\frac{1}{(t-s)^2}\Delta_v\right)^{-1}\Delta_v\mu\right) \\ &\times\left(\frac{x-w}{t-s}\right)\frac{1}{(t-s)^{d+1}}\,dwds + S(t,x), \quad t\geq 0. \end{aligned} \tag{2.7}$$

Since ρ and S by zero for $t < 0$, the equation (2.1) is satisfied for $t \in \mathbb{R}$ and can be rewritten as the convolution equation

$$\rho(t,x) = (K *_{t,x} \rho)(t,x) + S(t,x), \quad t \in \mathbb{R}, \tag{2.8}$$

where the kernel K is given by

$$K(t,x) = \frac{1}{t^{d+1}} \left(\left(1 - \frac{1}{t^2} \Delta_v \right)^{-1} \Delta_v \mu \right) \left(\frac{x}{t} \right) 1_{t>0}.$$

Note that we have

$$\begin{aligned} (K *_{t,x} \rho)(t,x) &= \int_{\mathbb{R}} \int_{\mathbb{R}^d} K(t-s, x-w) \rho(s,w) dw ds \\ &= \int_0^t \left(K(t-s,\cdot) *_x \rho(s,\cdot) \right)(x)\, ds, \end{aligned}$$

where we use the notation $*_{t,x}$ for the space–time convolution and $*_x$ for the space convolution.

We have the following properties for the kernel K:

Lemma 2.1 *Assuming (H1), there exists $C > 0$ such that the following estimates hold:*

$$\|K(t)\|_{L^1} \leq \frac{C}{1+t}, \quad \|K(t)\|_{L^\infty} \leq \frac{C}{t^d(1+t)}, \quad \forall t > 0.$$

Note that we get in particular from this lemma that $K \in L^1_{loc}(\mathbb{R}, L^1(\mathbb{R}^d))$ so that its Fourier transform on $\mathbb{R}^{d+1}$ is well defined (at least as a tempered distribution). We shall not use explicitly these precise properties of K besides the fact that its Fourier transform makes sense.

Proof We clearly get that

$$\|K(t,\cdot)\|_{L^\infty} \leq \frac{C}{t^{d+1}} \|F_t\|_{L^\infty},$$

and by change of variables, that

$$\|K(t,\cdot)\|_{L^1} \leq \frac{C}{t} \|F_t\|_{L^1},$$

where

$$F_t(v) = \left(\left(1 - \frac{1}{t^2} \Delta_v \right)^{-1} \Delta_v \mu \right)(v),$$

so that it only remains to estimate the L^1 and L^∞ norms of F_t. We use the homogeneous Littlewood–Paley decomposition and the Bernstein inequality (A.1) to get that for every $p \in [1, +\infty]$,

$$\|F_t\|_{L^p} \lesssim \sum_{q\in\mathbb{Z}} \frac{2^{2q}}{1+\frac{2^{2q}}{t^2}} \|\mu_q\|_{L^p}.$$

We can thus obtain from that uniformly for $t > 0$,

$$\|F_t\|_{L^p} \lesssim \sum_{q\le 0} 2^{2q}\|\mu_q\|_{L^p} + \sum_{q\ge 0} 2^{-q}\|\mu_q\|_{W^{3,p}} \lesssim \|\mu\|_{W^{3,p}}.$$

This yields the estimates for $t \ge 1$. It remains to improve the estimates for $t \in (0, 1]$. This time, we additionally rely on the bound $(1+\frac{|\xi|^2}{t^2}) \ge \max\left(2\frac{|\xi|}{t}, \frac{|\xi|^2}{t^2}\right)$ to obtain

$$\begin{aligned}\|F_t\|_{L^p} &\lesssim \sum_{q\le 0} t2^q\|\mu_q\|_{L^p} + \sum_{q\ge 0} t^2\|\mu_q\|_{L^p}\\ &\lesssim \sum_{q\le 0} t2^q\|\mu_q\|_{L^p} + \sum_{q\ge 0} t^2 2^{-q}\|\mu_q\|_{W^{1,p}} \lesssim t\|\mu\|_{W^{1,p}},\end{aligned}$$

hence concluding the proof.

Proof of Theorem 2.1

We now give the proof of Theorem 2.1 which we shall split in several steps. As already justified in [5,17] (see in particular the proof of Proposition 2.2 in [5] and the proof of Theorem 3.1 in [17]), we can express the solution of (2.1) through its space–time Fourier transform by

$$\widetilde{\rho}(\tau,\xi) = \frac{1}{1-\widetilde{K}(\tau,\xi)}\widetilde{S}(\tau,\xi)$$

in which $\widetilde{K}(\tau,\xi)$ is given by

$$\widetilde{K}(\tau,\xi) = \int_0^{+\infty} e^{-i\tau t}\frac{i\xi}{1+|\xi|^2}\cdot\widehat{\nabla_v\mu}(t\xi)\,dt. \tag{2.9}$$

The fact that the inverse Fourier transform of $\frac{1}{1-\widetilde{K}(\tau,\xi)}\widetilde{S}(\tau,\xi)$ vanishes for $t < 0$ comes from a Paley-Wiener type argument and uses the fact that $1-\widetilde{K}(z,\xi)$ does not cancel in the half-plane $\Im z \le 0$, which is precisely the Penrose stability condition (H2).

We can then write

$$\widetilde{\rho}(\tau,\xi)=\widetilde{S}(\tau,\xi)+\frac{\widetilde{K}(\tau,\xi)}{1-\widetilde{K}(\tau,\xi)}\widetilde{S}(\tau,\xi)$$

and hence the expression (2.2) follows by setting

$$G(t,x)=\mathcal{F}^{-1}_{(\tau,\xi)\to(t,x)}\Big(\frac{\widetilde{K}(\tau,\xi)}{1-\widetilde{K}(\tau,\xi)}\Big). \tag{2.10}$$

It thus remains to check the claimed properties of G.

At first, we observe that for $\xi\neq 0$, $\frac{\widetilde{K}(z,\xi)}{1-\widetilde{K}(z,\xi)}$ is a holomorphic function in $\Im z<0$ thanks to (H1) and (H2):

- the Penrose condition (H2) ensures that $1-\widetilde{K}(z,\xi)$ is away from 0,
- and (H1) entails that $\widehat{\nabla_v\mu}$ and its derivatives are decreasing sufficiently fast, so that one can apply Lebesgue dominated convergence theorem.

Now, note that thanks to (H1) and (H2), $\frac{\widetilde{K}(z,\xi)}{1-\widetilde{K}(z,\xi)}$ is uniformly bounded in $\Im z\leq 0$. Indeed, by (H1), $\nabla_v\mu\in W^{2,1}$ and therefore we get

$$\forall\tau\in\mathbb{R},\gamma\geq 0,\quad |\widetilde{K}(\tau-i\gamma,\xi)|\lesssim\int_0^{+\infty}e^{-\gamma t}\frac{|\xi|}{(1+t|\xi|)^2}\,dt\lesssim\int_0^{+\infty}\frac{ds}{(1+s)^2}. \tag{2.11}$$

In addition, by integration by parts, we can compute

$$\begin{aligned}\widetilde{K}(\tau,\xi)&=\int_0^{+\infty}\frac{(1-\partial_t^2)(e^{-i\tau t})}{1+\tau^2}\frac{i\xi}{1+|\xi|^2}\cdot\widehat{\nabla_v\mu}(t\xi)\,dt\\&=\int_0^{+\infty}\frac{e^{-i\tau t}}{1+\tau^2}\frac{i\xi}{1+|\xi|^2}\cdot(1-\partial_t^2)\widehat{\nabla_v\mu}(t\xi)\,dt\\&\quad-\frac{1}{1+\tau^2}\frac{i\xi}{1+|\xi|^2}\xi\cdot\nabla_\xi[\widehat{\nabla_v\mu}](0).\end{aligned}$$

Note that we have used that $\widehat{\nabla_v\mu}(0)=0$, which ensures that a boundary term cancel. By (H1), $\langle v\rangle^2\nabla_v\mu\in W^{2,1}$. This yields, arguing similarly as for (2.11), for any fixed $\xi\neq 0$,

$$|\widetilde{K}(\tau,\xi)|\lesssim\frac{1}{1+\tau^2}.$$

Since a similar expression holds for the derivatives $\partial_\tau^\alpha\widetilde{K}(\tau,\xi)$ and by (H1), $\langle v\rangle^2\nabla_v\mu\in W^{4,1}$, for any fixed $\xi\neq 0$, we have

$$|\partial_\tau^\alpha\widetilde{K}(\tau,\xi)|\lesssim\frac{1}{1+\tau^2},\quad\alpha\leq 2, \tag{2.12}$$

uniformly in τ.

This entails in particular that $\frac{\widetilde{K}(\tau,\xi)}{1-\widetilde{K}(\tau,\xi)}$ and its inverse Fourier transform in time have moderate decrease. Therefore, thanks to an adequate version of the Paley-Wiener theorem (see Theorem 3.5 in [22]), we get that $\mathcal{F}^{-1}_{\tau\to t}\left(\frac{\widetilde{K}}{1-\widetilde{K}}\right)$ vanishes for $t<0$ for every ξ. We conclude that G vanishes for $t<0$.

It remains to prove the pointwise decay estimates (2.3)-(2.4). We need the following properties of $\widetilde{K}(\tau,\xi)$.

Lemma 2.2 *We can write*

$$\widetilde{K}(\tau,\xi)=\frac{1}{1+|\xi|^2}\widetilde{K}^{h,1}(\tau,\xi) \tag{2.13}$$

where $\widetilde{K}^{h,1}(\tau,\xi)$ is positively homogeneous of degree zero and $\widetilde{K}^{h,1}\in\mathscr{C}^{2d+3}(\mathbb{R}^{d+1}\backslash\{0\})$. Moreover, there exists $C>0$ such that

$$|\partial_\tau^\alpha\partial_\xi^\beta\widetilde{K}^{h,1}(\tau,\xi)|\le C,\qquad\forall(\alpha,\beta),\ |\alpha|+|\beta|\le 2d+3,\quad\forall(\tau,\xi)\in\mathbb{S}^d, \tag{2.14}$$

where $\mathbb{S}^d$ is the unit sphere of $\mathbb{R}^{d+1}$.

Proof From the definition (2.9), we have

$$\widetilde{K}^{h,1}(\tau,\xi)=\int_0^{+\infty}e^{-i\tau t}i\xi\cdot\widehat{\nabla_v\mu}(t\xi)\,dt.$$

We observe that for $\lambda>0$,

$$\widetilde{K}^{h,1}(\lambda\tau,\lambda\xi)=\int_0^{+\infty}e^{-i\lambda\tau t}i\lambda\xi\cdot\widehat{\nabla_v\mu}(t\lambda\xi)\,dt=\widetilde{K}^{h,1}(\tau,\xi),$$

upon using the change of variable $s=\lambda t$. Note that by using again (2.11), we have

$$\sup_{(\tau,\xi)\in\mathbb{R}^{d+1}}|\widetilde{K}^{h,1}(\tau,\xi)|<+\infty.$$

To estimate the derivatives on the sphere, we first handle the case when $|\xi|\ge\frac12$. Thanks to (H1), $\langle v\rangle^{2d+3}\nabla_v\mu\in W^{2d+5,1}$ and consequently, we have

$$|\partial_\tau^\alpha\partial_\xi^\beta\widetilde{K}^{h,1}(\tau,\xi)|\lesssim\int_0^{+\infty}\frac{\langle t\rangle^{|\alpha|+|\beta|}}{(1+t|\xi|)^N}\,dt$$

for $N=2d+5$, and therefore, for $|\xi|\ge\frac12$ and $|\alpha|+|\beta|\le 2d+3$, we have

$$|\partial_\tau^\alpha\partial_\xi^\beta\widetilde{K}^{h,1}(\tau,\xi)|\lesssim 1,\qquad(\tau,\xi)\in\mathbb{S}^d,\quad|\xi|\ge\frac12. \tag{2.15}$$

Let us next consider the case when $|\xi| \le \frac{1}{2}$, in which we make use of the fact that τ is bounded below away from zero, recalling $(\tau, \xi) \in \mathbb{S}^d$. Integrating by parts, we get for every $n \ge 2$

$$\widetilde{K}^{h,1}(\tau,\xi) = \sum_{k=2}^{n} \frac{1}{(i\tau)^k} \mathscr{P}_k(\xi) + \frac{1}{(i\tau)^n} R_n(\tau,\xi) \tag{2.16}$$

where

$$\mathscr{P}_k(\xi) = i\xi \cdot (D_\xi^{k-1}\widehat{\nabla_v \mu})(0) : \xi^{\otimes k-1},$$
$$R_n(\tau,\xi) = \int_0^{+\infty} e^{-i\tau t} r_n(t,\xi)\, dt, \quad r_n(t,\xi) = i\xi \cdot (D_\xi^n \widehat{\nabla_v \mu})(t\xi)) : \xi^{\otimes n},$$

with the definition

$$\xi \cdot (D_\xi^k \widehat{\nabla_v \mu})(\zeta) : \xi^{\otimes k} = \sum_{j_0, j_1, \cdots j_k} \xi_{j_0} \xi_{j_1} \cdots \xi_{j_k} \frac{\partial^k}{\partial_{\xi_{j_1}} \cdots \partial_{\xi_{j_k}}} \widehat{\partial_{v_{j_0}} \mu}(\zeta).$$

Note that $\mathscr{P}_k$ is a homogeneous polynomial of degree k. Thanks to (H1), $\langle v \rangle^{2d+3} \nabla_v \mu \in W^{2d+5,1}$ and thus we have for all $n \le 2d+3$,

$$|r_n(t,\xi)| \lesssim \frac{|\xi|^{n+1}}{(1+t|\xi|)^N},$$

for $N = 2d+5$. More generally, using $\langle v \rangle^{4d+6} \nabla_v \mu \in W^{2d+5,1}$, we have for all $n \le 2d+3$ and $|\beta| \le 2d+3$,

$$|\partial_\xi^\beta r_n(t,\xi)| \lesssim \frac{|\xi|^{n+1-|\beta|}}{(1+t|\xi|)^{N-|\beta|}}.$$

Consequently, applying derivatives to the expansion (2.16), we get for $|\xi| \le \frac{1}{2}$,

$$|\partial_\tau^\alpha \partial_\xi^\beta \widetilde{K}^{h,1}(\tau,\xi)| \lesssim 1 + \int_0^{+\infty} \frac{t^{|\alpha|}|\xi|^{n+1-|\beta|}}{(1+t|\xi|)^{N-|\beta|}}\, dt \lesssim 1 + \int_0^{+\infty} \frac{s^{|\alpha|}|\xi|^{n-|\beta|-|\alpha|}}{(1+s)^{N-|\beta|}}\, ds.$$

Thus, we can fix $n = 2d+3$. We get for $|\alpha| + |\beta| \le 2d+3$,

$$|\partial_\tau^\alpha \partial_\xi^\beta \widetilde{K}^{h,1}(\tau,\xi)| \lesssim 1, \qquad (\tau,\xi) \in \mathbb{S}^d, \quad |\xi| \le \frac{1}{2}.$$

This, together with (2.15), concludes the proof.

Remark 2.1 We can also express the expansion (2.16) for $n = 2$ in a slightly different way. We write

$$\widetilde{K}^{h,1}(\tau,\xi) = \frac{1}{(i\tau)^2} \mathscr{P}_2(\xi) + \frac{\xi^{\otimes 2}}{(i\tau)^2} : \widetilde{K}^{h,2}(\tau,\xi) \tag{2.17}$$

where $\mathscr{P}_2$ is a homogeneous polynomial of degree two in ξ and

$$\widetilde{K}^{h,2}(\tau,\xi) = \int_0^{+\infty} e^{-i\tau t} i\xi \cdot (\widehat{D_\xi^2 \nabla_v \mu})(t\xi)\, dt.$$

From the same arguments as in the proof of Lemma 2.2, we get that $\widetilde{K}^{h,2}(\tau,\xi)$ is homogeneous of degree zero on $\mathbb{R}^{d+1}$ and that

$$|\partial_\tau^\alpha \partial_\xi^\beta \widetilde{K}^{h,2}(\tau,\xi)| \leq C, \qquad \forall(\alpha,\beta),\ |\alpha|+|\beta| \leq 2d+3, \quad \forall(\tau,\xi) \in \mathbb{S}^d. \tag{2.18}$$

Combining Remark 2.1 and Lemma 2.2, we obtain

Corollary 2.2 *We can write the expansion*

$$\widetilde{K}(\tau,\xi) = \frac{1}{1+|\xi|^2+\tau^2}\left(\widetilde{K}^{h,1}(\tau,\xi) - \frac{\mathscr{P}_2(\xi)}{1+|\xi|^2} - \frac{\xi^{\otimes 2}}{1+|\xi|^2} : \widetilde{K}^{h,2}(\tau,\xi)\right)$$

where $\widetilde{K}^{h,1}$, $\widetilde{K}^{h,2}$ are positively homogeneous of degree zero and satisfy the estimates (2.14), (2.18) and $\mathscr{P}_2(\xi)$ is a homogeneous polynomial of degree 2.

Proof The corollary follows from a combination of (2.13) and (2.17).

Let us now use the properties of $\widetilde{K}(\tau,\xi)$ to derive the pointwise decay estimates for $G(t,x)$, recalling (2.10). We first use the homogeneous Littlewood–Paley decomposition of $\mathbb{R}^{d+1}$ to decompose $G(t,x)$, yielding

$$G(t,x) = \sum_{q\in\mathbb{Z}} G_q(t,x),$$

recalling (1.8). The general strategy will consist of treating differently the contribution of high and low frequencies. We first deal with high frequencies:

Lemma 2.3 *There exist $A \geq 1$ and $C > 0$ such that for every $\delta \in (0,1]$ and every q with $2^q \geq A$, we have the estimates*

$$\begin{aligned} \|G_q(t)\|_{L^1} &\leq C\frac{2^{q(1+\delta)}}{1+2^{2q}}\frac{1}{(1+2^q|t|)^N}, \\ \|G_q(t)\|_{L^\infty} &\leq C\frac{2^{q(d+1+\delta)}}{1+2^{2q}}\frac{1}{(1+2^q|t|)^N}, \quad \forall t \in \mathbb{R}, \end{aligned} \tag{2.19}$$

for $N = d+3$.

Proof We first observe that we can rewrite (2.10) under the form

$$\widetilde{G} = \widetilde{K} + \widetilde{K}\widetilde{G}.$$

Let φ be a smooth cut-off function that is supported in an annulus slightly larger than that of χ, so that $\varphi = 1$ on the support of χ. Setting

$$\widetilde{\mathcal{K}}_q = \varphi\left(\frac{(\tau,\xi)}{2^q}\right)\widetilde{K},$$

we hence get the convolution equation

$$G_q = K_q + \mathcal{K}_q *_{t,x} G_q. \tag{2.20}$$

Let us first bound K_q. We shall prove that for all $q > 1$,

$$\|K_q(t)\|_{L^1_x} \lesssim \frac{2^{q\delta}}{1+2^{2q}}\frac{2^q}{(1+2^q|t|)^N}, \tag{2.21}$$

for $N = d+3$. Using the expansion in Corollary 2.2, we write

$$\widetilde{K}_q = \widetilde{K}_{q,1} + \widetilde{K}_{q,2},$$

with

$$\widetilde{K}_{q,1} = \frac{1}{1+|\xi|^2+\tau^2}\widetilde{K}^{h,1}(\tau,\xi)\chi_q(\tau,\xi),$$
$$\widetilde{K}_{q,2} = \frac{-1}{1+|\xi|^2+\tau^2}\left(\frac{\xi^{\otimes 2}}{1+|\xi|^2} : \widetilde{K}^{h,2}(\tau,\xi) + \frac{\mathscr{P}_2(\xi)}{1+|\xi|^2}\right)\chi_q(\tau,\xi).$$

We first check (2.21) for $K_{q,1}$. By a scaling argument, we can write

$$K_{q,1}(t,x) = 2^{q(d+1)}k_{q,1}(2^q t, 2^q x)$$

where

$$k_{q,1}(T,X) = \mathcal{F}^{-1}_{(\tau,\xi)\to(T,X)}\widetilde{K}_{q,1}(2^q\tau, 2^q\xi).$$

We claim that

$$|\partial_\tau^\alpha \partial_\xi^\beta \widetilde{k}_{q,1}(\tau,\xi)| \lesssim \frac{1}{1+2^{2q}}, \tag{2.22}$$

uniformly in q for $q > 1$ for $|\alpha|+|\beta| \le N+d$. This yields (2.21) for $K_{q,1}$. Indeed, from (2.22), we obtain from taking integration by parts

$$|k_{q,1}(T,X)| \lesssim \frac{1}{1+2^{2q}}\frac{1}{(1+|T|+|X|)^{N+d}},$$

which in turn implies

$$\|k_{q,1}(T)\|_{L^1_X} \lesssim \frac{1}{1+2^{2q}}\frac{1}{(1+|T|)^N}.$$

Therefore, by a change of variables, we obtain

$$\|K_{q,1}(t)\|_{L^1_x} \lesssim \frac{1}{1+2^{2q}} \frac{2^q}{(1+2^q|t|)^N}.$$

We now prove the claim (2.22). By homogeneity of $\widetilde{K}^{h,1}$, we can write

$$\widetilde{k}_{q,1}(\tau,\xi) = \frac{1}{1+|2^q\xi|^2+(2^q\tau)^2}\widetilde{K}^{h,1}(\tau,\xi)\chi(\tau,\xi).$$

Since $\widetilde{K}^{h,1}$ is a smooth homogeneous function of degree zero that satisfies (2.14), that is

$$|\partial^\gamma_{\tau,\xi}\widetilde{K}^{h,1}(\tau,\xi)| \lesssim |(\tau,\xi)|^{-|\gamma|} \lesssim 1,$$

since on support of χ, $|\xi|+|\tau|$ is bounded from below by a strictly positive number. On the other hand,

$$\left|\partial^\gamma_{\tau,\xi}\left(\frac{1}{1+|2^q\xi|^2+(2^q\tau)^2}\right)\right| \lesssim \frac{2^{q|\gamma|}}{(1+2^{2q}(|\xi|^2+\tau^2))^{1+|\gamma|/2}} \lesssim \frac{1}{1+2^{2q}}$$

on the support of χ. We thus deduce the estimate (2.22), and hence (2.21) for $K_{q,1}$.

For what concerns $K_{q,2}$, using (A.3), for any $\delta \in (0,1]$, we have

$$\begin{aligned}\|K_{q,2}(t)\|_{L^1_x} \lesssim\ & 2^{q\delta}\left\|\mathcal{F}^{-1}_{(\tau,\xi)\to(t,x)}\left(\frac{1}{1+|\xi|^2+\tau^2}\widetilde{K}^{h,2}(\tau,\chi)\chi_q(\tau,\xi)\right)\right\|_{L^1_x} \\ & + 2^{q\delta}\left\|\mathcal{F}^{-1}_{(\tau,\xi)\to(t,x)}\left(\frac{1}{1+|\xi|^2+\tau^2}\chi_q(\tau,\xi)\right)\right\|_{L^1_x},\end{aligned}$$

Again, note that $\widetilde{K}^{h,2}$ is a smooth homogeneous function of degree zero that satisfies (2.18) and therefore we have

$$|\partial^\gamma_{\tau,\xi}\widetilde{K}^{h,2}(\tau,\xi)| \lesssim |(\tau,\xi)|^{-|\gamma|} \lesssim 1,$$

on the support of χ. As a result, arguing as for $K_{q,1}$ with a scaling argument, we deduce (2.21).

Moreover, exactly as above, we obtain for all $q > 1$,

$$\|\mathcal{K}_q(t)\|_{L^1_x} \lesssim \frac{2^{q\delta}}{1+2^{2q}} \frac{2^q}{(1+2^q|t|)^N}. \tag{2.23}$$

Therefore, rewriting (2.20) as

$$G_q(t,x) = K_q(t,x) + \int_{\mathbb{R}} \left(\mathcal{K}_q(t-s,\cdot) *_x G_q(s,\cdot)\right)(x)\,ds, \tag{2.24}$$

and taking the L^1 norm in x and by using (2.21) and (2.23), we thus obtain that

$$\|G_q(t)\|_{L^1} \lesssim \frac{2^{q\delta}}{1+2^{2q}} \frac{2^q}{(1+2^q|t|)^N} + \int_{\mathbb{R}} \frac{2^{q\delta}}{1+2^{2q}} \frac{2^q}{(1+2^q|t-s|)^N} \|G_q(s)\|_{L^1}\, ds.$$

Let us set for all $T \in \mathbb{R}$

$$|||G_q|||_{1,T} = (1+|T|)^N \|G_q(\frac{T}{2^q})\|_{L^1}.$$

We deduce after a change of variables that

$$\begin{aligned} |||G_q|||_{1,T} &\lesssim \frac{2^q 2^{q\delta}}{1+2^{2q}} + \frac{2^{q\delta}}{1+2^{2q}} \int_{\mathbb{R}} \frac{(1+|T|)^N}{(1+|T-S|)^N} \frac{1}{(1+|S|)^N} |||G_q|||_{1,S}\, dS \\ &\lesssim \frac{2^q 2^{q\delta}}{1+2^{2q}} + \frac{2^{q\delta}}{1+2^{2q}} \sup_{S\in\mathbb{R}} |||G_q|||_{1,S}, \end{aligned} \tag{2.25}$$

where we have used that for $N > 1$,

$$\sup_{T\in\mathbb{R}} \int_{\mathbb{R}} \frac{(1+|T|)^N}{(1+|T-S|)^N} \frac{1}{(1+|S|)^N}\, dS < +\infty.$$

Consequently, after taking the sup in T, we can find $A > 1$ sufficiently large such that for all q satisfying $2^q > A$, the last term on the right of (2.25) is absorbed into the left, yielding

$$\sup_T |||G_q|||_{1,T} \lesssim \frac{2^q 2^{q\delta}}{1+2^{2q}}.$$

This proves the L^1 estimate in (2.19).

It remains to estimate the L^∞ norm. The proof follows the same lines. Arguing as for (2.21), we obtain

$$\|K_q(t)\|_{L^\infty(\mathbb{R}^d)} + \|\mathcal{K}_q(t)\|_{L^\infty(\mathbb{R}^d)} \lesssim \frac{2^{q\delta}}{1+2^{2q}} \frac{2^{q(d+1)}}{(1+2^q|t|)^N},$$

for any $\delta \in (0, 1]$. We then get by using (2.20) and (2.21) that

$$\begin{aligned} \|G_q(t)\|_{L^\infty} &\lesssim \frac{2^{q\delta}}{1+2^{2q}} \frac{2^{q(d+1)}}{(1+2^q|t|)^N} + \int_{\mathbb{R}} \|\mathcal{K}_q(t-s)\|_{L^1} \|G_q(s)\|_{L^\infty} ds \\ &\lesssim \frac{2^{q\delta}}{1+2^{2q}} \frac{2^{q(d+1)}}{(1+2^q|t|)^N} + \int_{\mathbb{R}} \frac{2^{q\delta}}{1+2^{2q}} \frac{2^q}{(1+2^q|t-s|)^N} \|G_q(s)\|_{L^\infty}\, ds. \end{aligned}$$

We then conclude as before by setting

$$|||G_q|||_{\infty,T} = (1+|T|)^N \|G_q(\frac{T}{2^q})\|_{L^\infty},$$

that

$$\sup_T |||G_q|||_{\infty,T} \lesssim \frac{2^{q\delta}2^{q(d+1)}}{1+2^{2q}} + \frac{2^{q\delta}}{1+2^{2q}} \sup_S |||G_q|||_{\infty,S}.$$

The estimate (2.19) for the L^∞ norm thus follows by choosing A sufficiently large.

From now on, $A > 1$ is fixed and there remains to estimate G_q for $2^q \leq A$. This is the content of the next lemma.

Lemma 2.4 *For $A > 1$, there exists $C > 0$ such that for every $q \in \mathbb{Z}$ with $2^q \leq A$, we have the estimate*

$$\|G_q(t)\|_{L^1} \leq C\frac{2^q}{(1+2^q|t|)^N}, \qquad \|G_q(t)\|_{L^\infty} \leq C\frac{2^{q(d+1)}}{(1+2^q|t|)^N}, \quad \forall t \in \mathbb{R}, \quad (2.26)$$

for $N = d+3$.

Proof We use directly the expression (2.10) and argue as in the proof of Lemma 2.3. By a scaling argument we can write that

$$G_q(t,x) = 2^{q(d+1)} g_q(2^q t, 2^q x)$$

where

$$g_q(T,X) = \mathcal{F}^{-1}_{(\tau,\xi)\to(T,X)} \frac{\widetilde{K}(2^q\tau, 2^q\xi)}{1-\widetilde{K}(2^q\tau,2^q\xi)}\chi(\tau,\xi).$$

To get the result, it is sufficient to prove that $\partial_\tau^\alpha \partial_\xi^\beta \widetilde{g}_q$ is bounded on the support of χ uniformly in q for $2^q \leq A$ for $|\alpha|+|\beta| \leq N+d$. From the Penrose stability condition (H2), such an estimate for $\widetilde{g}_q$ follows from a similar one bearing on

$$\widetilde{k}_q(\tau,\xi) = \widetilde{K}(2^q\tau, 2^q\xi)\chi(\tau,\xi).$$

From Corollary 2.2, we can write by homogeneity that

$$\widetilde{k}_q(\tau,\xi) = \frac{1}{1+|2^q\xi|^2+(2^q\tau)^2} \left(\widetilde{K}^{h,1}(\tau,\xi) - \frac{\mathscr{P}_2(\xi)}{2^{-2q}+|\xi|^2} - \frac{\xi^{\otimes 2}}{2^{-2q}+|\xi|^2} : \widetilde{K}^{h,2}(\tau,\xi)\right)\chi(\tau,\xi).$$

Since on the support of χ, $|\xi|+|\tau|$ is bounded below by a positive number, we have that uniformly in q,

$$\left|\partial^\gamma_{\tau,\xi}\left(\frac{1}{1+|2^q\xi|^2+(2^q\tau)^2}\right)\right| \lesssim \frac{2^{q|\gamma|}}{(1+2^{2q}(|\xi|^2+\tau^2))^{1+|\gamma|/2}} \lesssim 1.$$

We also observe that on the support of χ, ξ belongs to a ball so that we have

$$\left|\partial_\xi^\alpha\left(\frac{\mathscr{P}_2(\xi)}{2^{-2q}+|\xi|^2}\right)\right| + \left|\partial_\xi^\alpha\left(\frac{\xi^{\otimes 2}}{2^{-2q}+|\xi|^2}\right)\right| \lesssim 2^{2q|\alpha|} \lesssim 1.$$

This is for the control of these terms that we use that we are in the low frequency regime $2^q \leq A$. Since $\widetilde{K}^{h,1}(\tau,\xi)$ and $\widetilde{K}^{h,2}(\tau,\xi)$ satisfy (2.14), (2.18), the uniform estimate for $\partial_\tau^\alpha \partial_\xi^\beta \widetilde{k}_q$ follows.

End of the Proof of Theorem 2.1

We shall combine Lemma 2.3 and Lemma 2.4. Let A be given by Lemma 2.3.

We have

$$\|G(t)\|_{L^1} \leq \sum_{2^q \leq A} \|G_q(t)\|_{L^1} + \sum_{2^q \geq A} \|G_q(t)\|_{L^1} \tag{2.27}$$

Let us first consider large time estimates that is to say for $t \geq A$. In this case, the second sum gives

$$\sum_{2^q \geq A} \|G_q(t)\|_{L^1} \lesssim \sum_{2^q \geq A} \frac{2^{q(1+\delta)}}{1+2^{2q}} \frac{1}{(1+2^q|t|)^N} \lesssim \frac{1}{|t|^N},$$

since $\delta \leq 1$. For the first sum, we split

$$\sum_{2^q \leq A} \|G_q(t)\|_{L^1} \lesssim \sum_{2^q \leq A} \frac{2^q}{(1+2^q|t|)^N} \lesssim \sum_{2^q \leq t^{-1}} \frac{2^q}{(1+2^q|t|)^N} + \sum_{t^{-1} \leq 2^q \leq A} \frac{2^q}{(1+2^q|t|)^N}.$$

The first term above contains only negative q so that

$$\sum_{2^q \leq t^{-1}} \frac{2^q}{(1+2^q|t|)^N} \lesssim \sum_{2^q \leq t^{-1}} 2^q \lesssim \frac{1}{t}.$$

For the second term, we write

$$\sum_{t^{-1} \leq 2^q \leq A} \frac{2^q}{(1+2^q|t|)^N} \lesssim \frac{1}{t^N} \sum_{t^{-1} \leq 2^q \leq 1} 2^{q(1-N)} + \frac{1}{t^N} \sum_{1 \leq 2^q \leq A} 2^{q(1-N)} \lesssim \frac{1}{t} + \frac{1}{t^N},$$

since $N > 1$. We have thus proven that for t large enough

$$\|G(t)\|_{L^1} \lesssim \frac{1}{t}.$$

The estimates for the L^∞ norm follows the same lines using that $N > d+1$.

Let us explain how we obtain the estimates for $t \leq A$. We use again (2.27). For the first sum, we just use that

$$\sum_{2^q \leq A} \|G_q(t)\|_{L^1} \lesssim \sum_{2^q \leq A} \frac{2^q}{(1+2^q|t|)^N} \lesssim \sum_{2^q \leq A} 2^q \lesssim 1.$$

For the second sum, we write

$$\sum_{2^q \geq A} \|G_q(t)\|_{L^1} \lesssim \sum_{2^q \geq A} \frac{2^{q(1+\delta)}}{1+2^{2q}} \frac{1}{(1+2^q|t|)^N} \lesssim \sum_{2^q \geq A} 2^{q(-1+\delta)} \lesssim 1.$$

To get the short time estimates for the L^∞ norm, we only handle in a slightly different way the second sum. We write

$$\begin{aligned}\sum_{2^q \geq A} \|G_q(t)\|_{L^\infty} &\lesssim \sum_{2^q \geq A} \frac{2^{q(d+1+\delta)}}{1+2^{2q}} \frac{1}{(1+2^q|t|)^N} \\ &\lesssim \sum_{t^{-1} \geq 2^q \geq A} 2^{q(d-1+\delta)} + \frac{1}{t^N} \sum_{2^q \geq t^{-1}} 2^{q(d-1+\delta-N)} \lesssim \frac{1}{t^{d-1+\delta}}.\end{aligned}$$

We can also estimate derivatives of G using the Bernstein inequality (A.1). Note that we will use only large time estimates (that is to say for $t \geq 1$). We write

$$\begin{aligned}\|\nabla G(t)\|_{L^1} &\leq \sum_{2^q \leq A} 2^q \|G_q(t)\|_{L^1} + \sum_{2^q \geq A} 2^q \|G_q(t)\|_{L^1} \\ &\lesssim \sum_{2^q \leq t^{-1}} 2^{2q} + \frac{1}{t^N} \sum_{t^{-1} \leq 2^q \leq A} 2^{q(2-N)} + \frac{1}{t^N} \sum_{2^q \geq A} \frac{2^{q(2+\delta)}}{2^{q(2+N)}} \lesssim \frac{1}{t^2} + \frac{1}{t^N}.\end{aligned}$$

The estimate for the L^∞ norm of the derivatives follows the same lines, using $N > d+2$. This concludes the proof of Theorem 2.2.

3 The Bootstrap Argument

Equipped with the linear estimates (in the form of Corollary 2.1), we are now in position to introduce the continuation argument that we shall use to establish Theorem 1.1. As usual, the characteristics $(X_{s,t}(x,v), V_{s,t}(x,v))$ associated to the transport equation with the vector field $(v, E(t,x))$ are defined as the solution to the ODE system:

$$\begin{cases} \dfrac{d}{ds} X_{s,t}(x,v) = V_{s,t}(x,v), & X_{t,t}(x,v) = x, \\ \dfrac{d}{ds} V_{s,t}(x,v) = E(s, X_{s,t}(x,v)), & V_{t,t}(x,v) = v. \end{cases} \tag{3.1}$$

By the method of characteristics, the solution to the Vlasov–Poisson system (1.2) must satisfy

$$f(t,x,v) = f_0(X_{0,t}(x,v), V_{0,t}(x,v)) - \int_0^t E(s, X_{s,t}(x,v)) \cdot \nabla_v \mu(V_{s,t}(x,v))\, ds. \tag{3.2}$$

Consequently, $\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\, dv$ solves the equation

$$\rho(t,x) - \int_0^t \int_{\mathbb{R}^d} [\nabla_x (1-\Delta_x)^{-1} \rho](s, x-(t-s)v) \cdot \nabla_v \mu(v)\, dv ds = S(t,x), \tag{3.3}$$

with

$$\begin{aligned} S(t,x) = &\int_{\mathbb{R}^d} f_0(X_{0,t}(x,v), V_{0,t}(x,v))\, dv \\ &+ \int_0^t \int_{\mathbb{R}^d} E(s, x-(t-s)v) \cdot \nabla_v \mu(v)\, dv ds \\ &- \int_0^t \int_{\mathbb{R}^d} E(s, X_{s,t}(x,v)) \cdot \nabla_v \mu(V_{s,t}(x,v))\, dv ds. \end{aligned}$$

To study (3.3), let us introduce the following weighted in time norm:

$$\begin{aligned} \mathcal{N}(t) = &\sup_{[0,t]} \frac{1}{\log(2+s)} \\ &\Big(\|\rho(s)\|_{L^1} + \langle s \rangle^d \|\rho(s)\|_{L^\infty} + \langle s \rangle \|\nabla \rho(s)\|_{L^1} + \langle s \rangle^{d+1} \|\nabla \rho(s)\|_{L^\infty} \Big). \end{aligned} \tag{3.4}$$

First, we recall the following local well-posedness result for (1.2) whose proof is standard and therefore omitted. In what follows, we shall say that $f \in W^{\sigma,p}_k$ if $\langle v \rangle^k f \in W^{\sigma,p}_{x,v}$.

Proposition 3.1 *[Local well-posedness] Let $f_0 \in W^{1,1} \cap W^{\sigma,p}_{k/p'}$ with $k > d$, $p(\sigma - 1) > 2d$ and $p^{-1} + p'^{-1} = 1$. Then, there exists $T_0 > 0$ and a unique classical solution $f(t) \in \mathscr{C}([0,T_0], W^{1,1} \cap W^{\sigma,p}_{k/p'})$. Denote by $T^\star > 0$ the maximal existence time; if $T^\star < +\infty$, then*

$$\|\rho\|_{L^1(0,T^\star, W^{1,\infty})} = +\infty. \tag{3.5}$$

We thus apply Proposition 3.1, to obtain a unique local solution $f(t) \in \mathscr{C}([0,T_0], W^{1,1} \cap W^{\sigma,p}_{k/p'})$ of (1.2) that can be continued as long as $\|\rho(t)\|_{W^{1,\infty}}$ remains bounded. We note that the continuity in time of f and the Sobolev embedding entail that the function $t \mapsto \mathcal{N}(t)$ is continuous as well.

Let $\varepsilon \in (0,1]$ to be fixed later. We define

$$T = \sup \Big\{ t \in (0, T^\star),\ \mathcal{N}(t) \le \varepsilon \Big\}, \tag{3.6}$$

where $T^\star$ is the maximal time of existence. By continuity of $\mathcal{N}$ and thanks to the blow-up criterion (3.5), there is $\varepsilon_0 > 0$ such that if

$$\|\langle v\rangle^k f_0\|_{W^{1,\infty}} + \|f_0\|_{W^{1,1}} \leq \varepsilon_0,$$

then $T^\star > 2$ and moreover we can ensure that the time defined in (3.6) also satisfies $T > 2$.

Now applying the linear theory, namely Corollary 2.1, for the equation (3.3), we have

$$\mathcal{N}(t) \leq M' \left(\|S\|_{Y_t^0} + \|S\|_{Y_t^1} \right), \tag{3.7}$$

for $t \leq T$, where the norms Y_t^0, Y_t^1 are defined as in (2.5).

The main task is therefore to estimate $\|S\|_{Y_t^0} + \|S\|_{Y_t^1}$. We are going to prove the following result.

Theorem 3.1 *There exist $\varepsilon_1 \in (0, 1]$, $C_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_1)$, for all $t \leq T$ (where T is defined as in (3.6)),*

$$\|S\|_{Y_t^0} + \|S\|_{Y_t^1} \leq C_0\varepsilon_0 + C_0\varepsilon^2. \tag{3.8}$$

With Theorem 3.1 at hand, we can conclude the proof of Theorem 1.1.

Proof We choose ε_0 and ε small enough so that $\varepsilon_0 \leq \varepsilon \leq \varepsilon_1$ and

$$C_0\varepsilon_0 + C_0\varepsilon^2 \leq \frac{1}{2M'}\varepsilon, \tag{3.9}$$

where M' is the constant appearing in (3.7).

Assume that $T < T^\star$. We deduce by Theorem 3.1, (3.7) and (3.9) that

$$\mathcal{N}(T) \leq \frac{1}{2}\varepsilon.$$

This is a contradiction with the definition of T as by continuity of $\mathcal{N}$, we would then find $T_1 \in (T, T^\star)$ such that

$$\mathcal{N}(T_1) \leq \varepsilon.$$

Once we know that $T = T^\star$, we also get that $T^\star = +\infty$: indeed from the blow-up criterion of Proposition 3.1, we cannot have $T^\star < +\infty$ since then $\mathcal{N}(T^\star) \leq \varepsilon$. □

The rest of this paper is devoted to the proof of Theorem 3.1. We shall therefore work on the interval $[0, T]$, on which we have $\mathcal{N}(t) \leq \varepsilon$, a property which we will refer to as the bootstrap assumption. All subsequent estimates will be independent of T.

4 Decay Estimates for Characteristics

In this section, we study the characteristics $(X_{s,t}(x,v), V_{s,t}(x,v))$ defined as the solutions to the ODEs (3.1). Integrating (3.1), we have for all $0 \le s \le t \le T$,

$$\begin{aligned} X_{s,t}(x,v) &= x - v(t-s) + \int_s^t (\tau - s) E\Big(\tau, X_{\tau,t}(x,v)\Big)\, d\tau \\ V_{s,t}(x,v) &= v - \int_s^t E\Big(\tau, X_{\tau,t}(x,v)\Big)\, d\tau. \end{aligned} \tag{4.1}$$

We have the following pointwise bounds on characteristics.

Proposition 4.1 *There is $\varepsilon_1 \in (0,1)$ such that the following holds for all $\varepsilon \le \varepsilon_1$. For all $0 \le s, t \le T$ and for $x, v \in \mathbb{R}^d$, the map $x \mapsto X_{s,t}(x,v)$ and $v \mapsto V_{s,t}(x,v)$ are $\mathscr{C}^1$ diffeomorphisms, and we can write*[1]

$$X_{s,t}(x,v) = x - (t-s)v + Y_{s,t}(x - vt, v), \qquad V_{s,t}(x,v) = v + W_{s,t}(x - vt, v), \tag{4.2}$$

where $Y_{s,t}(x,v)$, $W_{s,t}(x,v)$ satisfy the following uniform estimates

$$\begin{aligned} &\sup_{0\le s\le t\le T} \frac{1+s^{d-1}}{\log(2+s)} (\|Y_{s,t}\|_{L^\infty_{x,v}} + \|\nabla_x Y_{s,t}\|_{L^\infty_{x,v}}) \\ &\quad + \sup_{0\le s\le t\le T} \frac{1+s^{d-2}}{\log(2+s)} \|\nabla_v Y_{s,t}\|_{L^\infty_{x,v}} \lesssim \varepsilon, \\ &\sup_{0\le s\le t\le T} \frac{1+s^{d}}{\log(2+s)} (\|W_{s,t}\|_{L^\infty_{x,v}} + \|\nabla_x W_{s,t}\|_{L^\infty_{x,v}}) \\ &\quad + \sup_{0\le s\le t\le T} \frac{1+s^{d-1}}{\log(2+s)} \|\nabla_v W_{s,t}\|_{L^\infty_{x,v}} \lesssim \varepsilon. \end{aligned} \tag{4.3}$$

Proof Let $0 \le s \le t \le T$. By definition of $Y_{s,t}$, we have from (4.1)

$$Y_{s,t}(x,v) = \int_s^t (\tau - s) E\Big(\tau, x + \tau v + Y_{\tau,t}(x,v)\Big)\, d\tau. \tag{4.4}$$

Differentiating this identity with respect to x and taking the sup norm in x and v, we get

$$\begin{aligned} \|\nabla_x Y_{s,t}\|_{L^\infty_{x,v}} &\lesssim \int_s^t (\tau - s) \|\nabla_x E(\tau, x + \tau v + Y_{\tau,t}(x,v))\|_{L^\infty_{x,v}} (1 + \|\nabla_x Y_{\tau,t}\|_{L^\infty_{x,v}})\, d\tau \\ &\lesssim (1 + \sup_{0\le\tau\le t} \|\nabla_x Y_{\tau,t}\|_{L^\infty_{x,v}}) \int_s^t \tau \|\nabla_x E(\tau)\|_{L^\infty}\, d\tau. \end{aligned} \tag{4.5}$$

[1] We have chosen to write the remainder as functions of $(x - vt, v)$, instead of (x,v), in view of the expected large time behavior, which is that of free transport.

Next, since $\nabla_x E = -\nabla_x (1-\Delta_x)^{-1}(\nabla_x \rho)$, we obtain from (A.2) and the bootstrap assumption that

$$\|E(\tau)\|_{L^\infty} + \|\nabla_x E(\tau)\|_{L^\infty} \lesssim \|\nabla_x \rho(\tau)\|_{L^\infty} \lesssim \frac{\log(2+\tau)}{1+\tau^{1+d}}\varepsilon, \tag{4.6}$$

and thus $\tau \|\nabla_x E(\tau)\|_{L^\infty}$ is integrable in time. This yields

$$\|\nabla_x Y_{s,t}\|_{L^\infty_{x,v}} \lesssim \varepsilon(1 + \|\nabla_x Y_{s,t}\|_{L^\infty_{x,v}}).$$

Thus, for ε sufficiently small, we have $\|\nabla_x Y_{s,t}\|_{L^\infty_{x,v}} \lesssim \varepsilon$, for all $0 \le s \le t \le T$. Therefore, taking ε small enough, we deduce that for all $v \in \mathbb{R}^d$, the map $x \mapsto X_{s,t}(x,v)$ is a $\mathscr{C}^1$ diffeomorphism (since we have just shown that it is a small perturbation of $x \mapsto x-(t-s)v$). As a consequence, for any integrable function $H = H(x)$, we have

$$\sup_{0\le s,t\le T} \|H(X_{s,t})\|_{L^\infty_v L^1_x} \lesssim \|H\|_{L^1_x}. \tag{4.7}$$

We are ready to derive the uniform estimates in (4.3). Using the pointwise decay in (4.6), it follows directly from (4.4) and (4.5) that

$$\|Y_{s,t}\|_{L^\infty_{x,v}} + \|\nabla_x Y_{s,t}\|_{L^\infty_{x,v}} \lesssim \frac{\log(2+s)}{1+s^{d-1}}\varepsilon.$$

In addition, we also obtain from (4.4) that

$$\begin{aligned}\|\nabla_v Y_{s,t}\|_{L^\infty_{x,v}} &\lesssim \int_s^t (\tau - s)\|\nabla_x E(\tau)\|_{L^\infty}(\tau + \|\nabla_v Y_{\tau,t}\|_{L^\infty})\, d\tau \\ &\lesssim \varepsilon \int_s^t \frac{\log(2+\tau)}{1+\tau^{d-1}}\, d\tau + \varepsilon \int_s^t \frac{\log(2+\tau)}{1+\tau^d}\|\nabla_v Y_{\tau,t}\|_{L^\infty}\, d\tau.\end{aligned}$$

In dimension $d \ge 3$, since $\frac{\log(2+\tau)}{1+\tau^{d-1}}$ is integrable, we get from the same argument as before that

$$\|\nabla_v Y_{s,t}\|_{L^\infty_{x,v}} \lesssim \varepsilon \frac{\log(2+s)}{1+s^{d-2}} \tag{4.8}$$

which is particular bounded, giving the claimed estimates on $Y_{s,t}(x,v)$. Similarly, by construction, we have

$$W_{s,t}(x,v) = -\int_s^t E\Big(\tau, x + \tau v + Y_{\tau,t}(x,v)\Big)\, d\tau,$$

which first directly yields

$$\|W_{s,t}\|_{L^\infty_{x,v}} \lesssim \varepsilon \int_s^t \frac{\log(2+\tau)}{1+\tau^{d+1}}\, d\tau \lesssim \varepsilon \frac{\log(2+s)}{1+s^d}.$$

Moreover, using the estimates already proved for $Y_{s,t}$, we have

$$\|\nabla_x W_{s,t}\|_{L^\infty_{x,v}} \lesssim \varepsilon \int_s^t \frac{\log(2+\tau)}{1+\tau^{d+1}}(1+\|\nabla_x Y_{\tau,t}\|_{L^\infty_{x,v}})\,d\tau \lesssim \varepsilon \frac{\log(2+s)}{1+s^d}.$$
$$\|\nabla_v W_{s,t}\|_{L^\infty_{x,v}} \lesssim \varepsilon \int_s^t \frac{\tau\log(2+\tau)}{1+\tau^{d+1}}(1+\|\nabla_v Y_{\tau,t}\|_{L^\infty_{x,v}})\,d\tau \lesssim \varepsilon \frac{\log(2+s)}{1+s^{d-1}}.$$

Imposing again ε small enough, we deduce that for all $x \in \mathbb{R}^d$, the map $v \mapsto V_{s,t}(x,v)$ is a $\mathscr{C}^1$ diffeomorphism. The proposition follows.

Remark 4.1 In dimension 2, derivatives in v of characteristics have a slow growth in time (see in particular the estimates leading to (4.8)), which prevents from performing the same nonlinear stability analysis.

5 Straightening Characteristics

In order to boil down to the case of free transport, we shall rely on a change of variables in velocity, that is close to the identity on the interval $[0,T]$. This is the content of the following proposition.

Proposition 5.1 *There is $\varepsilon_1 \in (0,1)$ such that the following holds for all $\varepsilon \le \varepsilon_1$. For all $0 \le s,t \le T$, there exists a $\mathscr{C}^1$ map $(x,v) \mapsto \Psi_{s,t}(x,v)$ such that*

$$X_{s,t}(x, \Psi_{s,t}(x,v)) = x - (t-s)v, \tag{5.1}$$

for all $x,v \in \mathbb{R}^d$. In addition, for every $x \in \mathbb{R}^d$, $v \mapsto \Psi_{s,t}(x,v)$ is a diffeomorphism, and there hold the following uniform estimates

$$\begin{aligned} \sup_{0\le s,t\le T} \frac{1+s^d}{\log(2+s)} \Big(\|\Psi_{s,t}(x,v) - v\|_{L^\infty_{x,v}} + \|\nabla_x \Psi_{s,t}(x,v)\|_{L^\infty_{x,v}} \Big) \lesssim \varepsilon, \\ \sup_{0\le s,t\le T} \frac{1+s^{d-1}}{\log(2+s)} \|\nabla_v(\Psi_{s,t}(x,v) - v)\|_{L^\infty_{x,v}} \lesssim \varepsilon. \end{aligned} \tag{5.2}$$

Proof From (4.1), we write

$$X_{s,t}(x,v) = x - (t-s)\Big(v + \Phi_{s,t}(x,v)\Big)$$

where

$$\Phi_{s,t}(x,v) = -\frac{1}{t-s}\int_s^t (\tau - s) E\Big(\tau, x - (t-\tau)v + Y_{\tau,t}(x - vt, v)\Big)\,d\tau.$$

Using (4.6), we obtain

$$\|\Phi_{s,t}\|_{L^\infty_{x,v}} \lesssim \int_s^t \|E(\tau)\|_{L^\infty_{x,v}}\,d\tau \lesssim \varepsilon \int_s^t \frac{\log(2+\tau)}{1+\tau^{d+1}}\,d\tau \lesssim \varepsilon \frac{\log(2+s)}{1+s^d}.$$

In addition, using the bounds on $Y_{s,t}(x,v)$ obtained in the previous section, we get

$$\begin{aligned}\|\nabla_x \Phi_{s,t}\|_{L^\infty_{x,v}} &\lesssim \frac{1}{t-s}\int_s^t (\tau-s)\|\nabla_x E(\tau)\|_{L^\infty_x}\Big(1+\|\nabla_x Y_{\tau,t}\|_{L^\infty_{x,v}}\Big)\,d\tau\\ &\lesssim \int_s^t \|\nabla_x E(\tau)\|_{L^\infty_x}\,d\tau \lesssim \varepsilon\frac{\log(2+s)}{1+s^d},\end{aligned}$$

using (4.6). Finally, we compute

$$\begin{aligned}\|\nabla_v \Phi_{s,t}\|_{L^\infty_{x,v}} &\lesssim \frac{1}{t-s}\int_s^t (\tau-s)\|\nabla_x E(\tau)\|_{L^\infty_x}\Big(t-\tau+\|(\nabla_v - t\nabla_x)Y_{\tau,t}\|_{L^\infty_{x,v}}\Big)\,d\tau\\ &\lesssim \int_s^t (1+\tau)\|\nabla_x E(\tau)\|_{L^\infty_x}\,d\tau + \int_s^t \|\nabla_x E(\tau)\|_{L^\infty_x}\|(\nabla_v-\tau\nabla_x)Y_{\tau,t}\|_{L^\infty_{x,v}}\,d\tau\end{aligned}$$

in which we have written $t\nabla_x Y_{\tau,t} = (t-\tau)\nabla_x Y_{\tau,t} + \tau\nabla_x Y_{\tau,t}$ and used the boundedness of $\nabla_x Y_{\tau,t}(x,v)$. Therefore, using again (4.6) and (4.3), we have

$$\|\nabla_v \Phi_{s,t}\|_{L^\infty_{x,v}} \lesssim \varepsilon\int_s^t \frac{\log(2+\tau)}{1+\tau^d}\,d\tau \lesssim \varepsilon\frac{\log(2+s)}{1+s^{d-1}}.$$

We deduce that the map

$$(x,v)\mapsto (x, v+\Phi_{s,t}(x,v))$$

is a $\mathscr{C}^1$ diffeomorphism from $\mathbb{R}^{2d}$ to $\mathbb{R}^{2d}$ with Jacobian determinant close to one. As a consequence, there exists a $\mathscr{C}^1$ diffeomorphism $\Psi_{s,t}(x,v)$ such that

$$X_{s,t}(x,\Psi_{s,t}(x,v)) = x-(t-s)v,$$

for all x, v. By construction, note that we have

$$\Psi_{s,t}(x,v) = v - \Phi_{s,t}(x,\Psi_{s,t}(x,v)). \tag{5.3}$$

The estimates (5.2) for $\Psi_{s,t}(x,v)$ thus follow from the estimates already proved for $\Phi_{s,t}(x,v)$. This ends the proof of the proposition.

6 Contribution of the Initial Data

In this section, we estimate the contribution of

$$\mathcal{I}(t,x) = \int_{\mathbb{R}^d} f_0(X_{0,t}(x,v), V_{0,t}(x,v))\,dv. \tag{6.1}$$

Namely, we prove the following proposition.

Proposition 6.1 *For $0 \le t \le T$, there holds*

$$\|\mathcal{I}(t)\|_{L^1} + \langle t\rangle^d \|\mathcal{I}(t)\|_{L^\infty} + \langle t\rangle \|\nabla_x \mathcal{I}(t)\|_{L^1} + \langle t\rangle^{d+1} \|\nabla_x \mathcal{I}(t)\|_{L^\infty} \le C_0 \varepsilon_0, \quad (6.2)$$

for some positive constant C_0.

Proof Since

$$(x, v) \mapsto (X_{0,t}(x, v), V_{0,t}(x, v))$$

is a measure preserving diffeomorphism, we have that

$$\|\mathcal{I}(t)\|_{L^1} \le \|f_0\|_{L^1_{x,v}} \le \varepsilon_0.$$

Next, to prove the L^∞ estimate, we aim at getting as close as possible to the dynamics of free transport, which corresponds to $f_0(x - tv, v)$. To proceed, we first straighten the characteristics following Proposition 5.1, yielding

$$\mathcal{I}(t, x) = \int_{\mathbb{R}^d} f_0(x - tv, V_{0,t}(x, \Psi_{0,t}(x, v))) \det(\nabla_v \Psi_{0,t}(x, v))\, dv,$$

from which we deduce by using the change of variables $w = x - tv$

$$\mathcal{I}(t, x) = \int_{\mathbb{R}^d} f_0 \left(w, V_{0,t} \left(x, \Psi_{0,t} \left(x, \frac{x - w}{t} \right) \right) \right) \det \left(\nabla_v \Psi_{0,t} \right) \left(x, \frac{x - w}{t} \right) \frac{dw}{t^d}. \quad (6.3)$$

Then, by using (5.2) that yields

$$\| \det \left(\nabla_v \Psi_{0,t} \right) \|_{L^\infty_{x,v}} \lesssim 1,$$

we get that

$$\|\mathcal{I}(t)\|_{L^\infty} \lesssim \frac{1}{t^d} \int_{\mathbb{R}^d} \|f_0(w, \cdot)\|_{L^\infty_v}\, dw \lesssim \frac{1}{t^d} \varepsilon_0.$$

Next, to compute the derivative of $\mathcal{I}$, we shall introduce $(Y_{0,t}, W_{0,t})$ as defined in (4.2) on the characteristics and use the change of variables $w = x - vt$ to get

$$\begin{aligned} \mathcal{I}(t, x) &= \int_{\mathbb{R}^d} f_0 \left(x - vt + Y_{0,t}(x - tv, v), v + W_{0,t}(x - tv, v) \right) dv \\ &= \int_{\mathbb{R}^d} f_0 \left(w + Y_{0,t} \left(w, \frac{x - w}{t} \right), \frac{x - w}{t} + W_{0,t} \left(w, \frac{x - w}{t} \right) \right) \frac{dw}{t^d}. \end{aligned} \quad (6.4)$$

Therefore, we obtain that

$$\begin{aligned}
&\nabla_x \mathcal{I}(t,x) \\
&= \int_{\mathbb{R}^d} \Bigg[(\nabla_x f_0)\left(w + Y_{0,t}\left(w, \frac{x-w}{t}\right), \frac{x-w}{t} + W_{0,t}\left(w, \frac{x-w}{t}\right)\right) \\
&\quad \cdot (\nabla_v Y_{0,t})\left(w, \frac{x-w}{t}\right) \\
&\quad + (\nabla_v f_0)\left(w + Y_{0,t}\left(w, \frac{x-w}{t}\right), \frac{x-w}{t} + W_{0,t}\left(w, \frac{x-w}{t}\right)\right) \\
&\quad \cdot \left(e + (\nabla_v W_{0,t})\left(w, \frac{x-w}{t}\right)\right)\Bigg] \frac{dw}{t^{d+1}},
\end{aligned}$$

where e is the vector $(1, \cdots, 1)^t$.

Consequently, from Proposition 4.1, we obtain that

$$|\nabla_x \mathcal{I}(t,x)| \lesssim \int_{\mathbb{R}^d} \left|\nabla_{x,v} f_0\right| \left(w + Y_{0,t}\left(w, \frac{x-w}{t}\right), \frac{x-w}{t} + W_{0,t}\left(w, \frac{x-w}{t}\right)\right) \frac{dw}{t^{d+1}}. \tag{6.5}$$

Going back to the original coordinates, we get

$$|\nabla_x \mathcal{I}(t,x)| \lesssim \frac{1}{t} \int_{\mathbb{R}^d} |\nabla_{x,v} f_0|(X_{0,t}(x,v), V_{0,t}(x,v))\, dv.$$

By integrating in x, this yields

$$\|\nabla_x \mathcal{I}(t)\|_{L^1} \lesssim \frac{1}{t} \|\nabla_{x,v} f_0\|_{L^1} \lesssim \frac{\varepsilon_0}{t}.$$

For the L^∞ norm, we use again the straightening change of variables $v = \Psi_{0,t}(x, \widetilde{v})$ and $w = x - t\widetilde{v}$ to obtain

$$|\nabla_x \mathcal{I}(t,x)| \lesssim \frac{1}{t^{d+1}} \int_{\mathbb{R}^d} \|\nabla_{x,v} f_0(w, \cdot)\|_{L^\infty_v}\, dw \lesssim \frac{\varepsilon_0}{t^{d+1}}.$$

This yields the proposition for the case $t \geq 1$. For $0 \leq t \leq 1$, the estimates are straightforward, using directly the bounds on the characteristics from Proposition 4.1. □

7 Contribution of the Reaction Term

We next turn to the reaction term

$$\mathcal{R}(t,x) = \mathcal{R}_{\mathrm{L}}(t,x) - \mathcal{R}_{\mathrm{NL}}(t,x) \tag{7.1}$$

where

$$\mathcal{R}_{\mathrm{NL}}(t,x) = \int_0^t \int_{\mathbb{R}^d} E(s, X_{s,t}(x,v)) \cdot \nabla_v \mu(V_{s,t}(x,v))\, dvds,$$
$$\mathcal{R}_{\mathrm{L}}(t,x) = \int_0^t \int_{\mathbb{R}^d} E(s, x-(t-s)v) \cdot \nabla_v \mu(v)\, dvds.$$

In order to estimate $\mathcal{R}$, we shall first establish the following general fact. Let us set for any given F and ν:

$$\mathcal{T}[F,\nu](t,x) = -\int_0^t \int_{\mathbb{R}^d} F(s, X_{s,t}(x,v)) \cdot \nabla_v \nu(V_{s,t}(x,v))\, dvds + \int_0^t \int_{\mathbb{R}^d} F(s, x-(t-s)v) \cdot \nabla_v \nu(v)\, dvds.$$

Note that in particular, we have $\mathcal{R}(t,x) = \mathcal{T}[E,\mu](t,x)$. We have

Lemma 7.1 *Assume that $\nu \in W^{3,\infty}_v$ with*

$$|\partial_v^\alpha \nabla \nu| \lesssim \frac{1}{\langle v\rangle^N}, \quad |\alpha| = 1, 2, \tag{7.2}$$

for some $N > d$. We have the decomposition

$$\mathcal{T}[F,\nu](t,x) = \mathcal{T}_1[F,\nu](t,x) + \mathcal{T}_2[F,\nu](t,x), \tag{7.3}$$

where for $j = 1, 2$, we have

$$\mathcal{T}_j[F,\nu](t,x) = \int_0^t \int_{\mathbb{R}^d} F(s, x-(t-s)v) \cdot H_{0,j}[\nu](s,t,x,v),$$

in which

$$\begin{aligned} H_{0,1}[\nu](s,t,x,v) &= \nabla_v \nu(v) - \nabla_v \nu(V_{s,t}(x, \Psi_{s,t}(x,v))) \\ H_{0,2}[\nu](s,t,x,v) &= \nabla_v \nu(V_{s,t}(x, \Psi_{s,t}(x,v)))\Big[1 - \det(\nabla_v \Psi_{s,t}(x,v))\Big]. \end{aligned} \tag{7.4}$$

Moreover, for $0 \le s \le t \le T$, the kernels enjoy the uniform estimate

$$|H_{0,j}[\nu](s,t,x,v)| \lesssim \frac{\varepsilon}{(1+|v|)^N} \frac{\log(2+s)}{1+s^{d-1}} \tag{7.5}$$

and there exists C such that for every F and for every $t \in (0,T]$,

$$\|\mathcal{T}_j[F,\nu](t)\|_{L^1} \le C\varepsilon \sup_{[0,t]} \frac{\|F(s)\|_{L^1}}{\log(2+s)}, \tag{7.6}$$

$$\|\mathcal{T}_j[F,v](t)\|_{L^\infty} \leq \frac{C\varepsilon}{t^d} \sup_{[0,t]} \left(\frac{\|F(s)\|_{L^1}}{\log(2+s)} + \frac{1+s^d\|F(s)\|_{L^\infty}}{\log(2+s)} \right). \tag{7.7}$$

Proof Using the change of variables provided by Proposition 5.1, we obtain

$$\begin{aligned}\mathcal{T}[F,v](t,x) = &-\int_0^t \int_{\mathbb{R}^d} F(s, x-(t-s)v) \\ &\cdot \nabla_v v(V_{s,t}(x, \Psi_{s,t}(x,v))) \det(\nabla_v \Psi_{s,t}(x,v))\, dvds \\ &+ \int_0^t \int_{\mathbb{R}^d} F(s, x-(t-s)v) \cdot \nabla_v v(v)\, dvds.\end{aligned}$$

The decomposition (7.3) follows. Moreover, for $0 \leq s \leq t \leq T$, the kernels enjoy the estimate (7.5). This follows from Propositions 4.1 and 5.1: by the mean value inequality, (7.2) and (4.3)–(5.2), we have

$$\begin{aligned}|H_{0,1}[v](s,t,x,v)| &\lesssim \frac{1}{\langle v\rangle^N} |v - V_{s,t}(x, \Psi_{s,t}(x,v))| \\ &\lesssim \frac{1}{\langle v\rangle^N} \left[|v - V_{s,t}(x,v)| + |V_{s,t}(x,v) - V_{s,t}(x, \Psi_{s,t}(x,v))| \right] \\ &\lesssim \frac{1}{\langle v\rangle^N} \left[|v - V_{s,t}(x,v)| + \|\nabla_v V_{s,t}\|_{L^\infty_{x,v}} |v - \Psi_{s,t}(x,v)| \right] \\ &\lesssim \frac{\varepsilon}{\langle v\rangle^N} \frac{\log(2+s)}{1+s^d}.\end{aligned}$$

We handle $H_{0,2}$ similarly.

Next, using the change of variable $w = x - (t-s)v$ and the bounds on the kernels, we obtain

$$|\mathcal{T}_j[F,v](t,x)| \lesssim \varepsilon \int_0^t \int_{\mathbb{R}^d} |F(s,w)| \left(1 + \left|\frac{x-w}{t-s}\right|\right)^{-N} \frac{\log(2+s)}{1+s^{d-1}} \frac{dwds}{(t-s)^d}.$$

This yields

$$\begin{aligned}\|\mathcal{T}_j[F,v](t)\|_{L^1} &\lesssim \varepsilon \int_0^t \|F(s)\|_{L^1} \frac{\log(2+s)}{1+s^{d-1}}\, ds \\ &\lesssim \varepsilon \sup_{[0,t]} \frac{\|F(s)\|_{L^1}}{\log(2+s)} \int_0^t \frac{\log^2(2+s)}{1+s^{d-1}}\, ds,\end{aligned}$$

hence the L^1 estimate. For the L^∞ norm, we use

$$\|\mathcal{T}_j[F,v](t)\|_{L^\infty} \lesssim \frac{\varepsilon}{t^d} \int_0^{\frac{t}{2}} \|F(s)\|_{L^1} \frac{\log(2+s)}{1+s^{d-1}}\, ds + \varepsilon \int_{\frac{t}{2}}^t \|F(s)\|_{L^\infty} \frac{\log(2+s)}{1+s^{d-1}}\, ds,$$

which gives

$$\|\mathcal{T}_j[F,v](t)\|_{L^\infty} \lesssim \frac{\varepsilon}{t^d}\left(\sup_{[0,t/2]}\frac{\|F\|_{L^1}}{\log(2+s)} + \sup_{[t/2,t]}\frac{1+s^d}{\log(2+s)}\|F(s)\|_{L^\infty}\right) \int_0^t \frac{\log^2(2+s)}{1+s^{d-1}}\,ds.$$

This concludes the proof of the lemma. □

As an application of Lemma 7.1, we can now derive appropriate estimates for $\mathcal{R}$.

Proposition 7.1 *For all* $0 \le t \le T$,

$$\|\mathcal{R}(t)\|_{L^1} + (1+t^d)\|\mathcal{R}(t)\|_{L^\infty} \lesssim \varepsilon^2.$$

Proof Since we have $\mathcal{R}(t,x) = \mathcal{T}[E,\mu](t,x)$, the estimate for $1 \le t \le T$ follows from (H1), (7.6), (7.7), the fact that

$$\|E(t)\|_{L^p} \lesssim \|\rho(t)\|_{L^p},$$

for $p \in [1,\infty]$, and the bootstrap assumption. The estimates are straightforward for $0 \le t \le 1$, bounding directly (7.1) using the bounds on the characteristics from Proposition 4.1. □

We shall now estimate derivatives of $\mathcal{R}$.

Proposition 7.2 *There exists* $C > 0$ *such that for every* $0 \le t \le T$

$$(1+t)\|\nabla\mathcal{R}(t)\|_{L^1} + (1+t^{d+1})\|\nabla\mathcal{R}(t)\|_{L^\infty} \le C\varepsilon^2.$$

Proof The estimates for $0 \le t \le 1$ follow directly from (7.1) and the bounds on the characteristics from Proposition 4.1. We therefore focus on the case $t \ge 1$. We shall first express $\nabla\mathcal{R}_{\mathrm{NL}}$ in an appropriate way. Using (4.2), we have

$$\begin{aligned}\mathcal{R}_{\mathrm{NL}}(t,x) &= \int_0^t\int_{\mathbb{R}^d} E(s, Y_{s,t}(x-tv,v)+x-tv+sv) \\ &\quad\cdot \nabla_v\mu(W_{s,t}(x-tv,v)+v)\,dvds \\ &= \int_0^t\int_{\mathbb{R}^d} E\left(s, Y_{s,t}(w,\frac{x-w}{t})+w+\frac{s}{t}(x-w)\right) \\ &\quad\cdot \nabla_v\mu\left(W_{s,t}(w,\frac{x-w}{t})+\frac{x-w}{t}\right)\frac{dwds}{t^d}.\end{aligned}$$

By taking the gradient in x, we next obtain that

$$\partial_j\mathcal{R}_{\mathrm{NL}}(t,x) = \mathcal{R}^1_j + \mathcal{R}^2_j + \mathcal{R}^3_j + \mathcal{R}^4_j,$$

where

$$\mathcal{R}_j^1 = \int_0^t \int_{\mathbb{R}^d} s\partial_j E\left(s, Y_{s,t}(w, \frac{x-w}{t}) + w + \frac{s}{t}(x-w)\right) \cdot \nabla_v \mu\left(W_{s,t}(w, \frac{x-w}{t}) + \frac{x-w}{t}\right) \frac{dwds}{t^{d+1}},$$

$$\mathcal{R}_j^2 = \int_0^t \int_{\mathbb{R}^d} E \cdot \partial_{v_j} \nabla_v \mu\left(W_{s,t}(w, \frac{x-w}{t}) + \frac{x-w}{t}\right) \frac{dwds}{t^{d+1}},$$

$$\mathcal{R}_j^3 = \int_0^t \int_{\mathbb{R}^d} \left(\partial_{v_j} Y_{s,t}(w, \frac{x-w}{t}) \cdot \nabla_x E\left(s, Y_{s,t}(w, \frac{x-w}{t}) + w + \frac{s}{t}(x-w)\right)\right) \cdot \nabla_v \mu\left(W_{s,t}(w, \frac{x-w}{t}) + \frac{x-w}{t}\right) \frac{dwds}{t^{d+1}},$$

$$\mathcal{R}_j^4 = \int_0^t \int_{\mathbb{R}^d} E\left(s, Y_{s,t}(w, \frac{x-w}{t}) + w + \frac{s}{t}(x-w)\right) \cdot \left(\partial_{v_j} W_{s,t}(w, \frac{x-w}{t}) \cdot \nabla_v\right) \nabla_v \mu\left(W_{s,t}(w, \frac{x-w}{t}) + \frac{x-w}{t}\right) \frac{dwds}{t^{d+1}}.$$

We can now estimate directly $\mathcal{R}_j^3, \mathcal{R}_j^4$ that are indeed nonlinear terms. Going back to the v variable and recalling (4.2), we have that

$$\mathcal{R}_j^3 = \int_0^t \int_{\mathbb{R}^d} \left(\partial_{v_j} Y_{s,t}(x - vt, v) \cdot \nabla_x E\left(s, X_{s,t}(x,v)\right)\right) \cdot \nabla_v \mu\left(V_{s,t}(x,v)\right) \frac{dvds}{t}.$$

From (4.3), we then obtain the pointwise estimate

$$\begin{aligned} |\mathcal{R}_j^3(t,x)| &\lesssim \varepsilon \int_0^t \int_{\mathbb{R}^d} \left|\nabla_x E(s, X_{s,t}(x,v))\right| |\nabla_v \mu(V_{s,t}(x,v)| \frac{\log(2+s)}{1+s^{d-2}} \frac{dvds}{t} \\ &\lesssim \varepsilon \int_0^t \|\nabla_x E(s)\|_{L^\infty} \frac{\log(2+s)}{1+s^{d-2}} \int_{\mathbb{R}^d} |\nabla_v \mu(V_{s,t}(x,v)| \frac{dvds}{t}. \end{aligned}$$

By Proposition 4.1, for all $x \in \mathbb{R}^d$, $v \mapsto V_{s,t}(x,v)$ is a $\mathscr{C}^1$ diffeomorphism and therefore by (4.3) and (H1),

$$\sup_x \int_{\mathbb{R}^d} |\nabla_v \mu(V_{s,t}(x,v)| dv \lesssim 1.$$

By (A.2) and the bootstrap assumption, we have

$$\|\nabla_x E(s)\|_{L^1} \lesssim \|\nabla_x \rho(s)\|_{L^1} \lesssim \varepsilon \frac{\log(2+s)}{1+s}$$

and thus the above yields

$$\|\mathcal{R}_j^3(t)\|_{L^1} \lesssim \frac{\varepsilon}{t} \int_0^t \|\nabla_x E(s)\|_{L^1} \frac{\log(2+s)}{1+s^{d-2}}\, ds \lesssim \frac{\varepsilon^2}{t} \int_0^t \frac{\log^2(2+s)}{1+s^{d-1}}\, ds \lesssim \frac{\varepsilon^2}{t}.$$

For the L^∞ norm, we use again (H1) and the change of variable given by Proposition 5.1 to obtain that

$$|\mathcal{R}_j^3(t,x)| \lesssim \frac{\varepsilon}{t}\int_0^t\int_{\mathbb{R}^d} |\nabla_x E(s,w)| \left(1+\left|\frac{x-w}{t-s}\right|\right)^{-N} \frac{\log(2+s)}{1+s^{d-2}} \frac{dwds}{(t-s)^d}.$$

Using the bootstrap assumption, we find

$$\begin{aligned}|\mathcal{R}_j^3(t,x)| &\lesssim \frac{\varepsilon}{t^{d+1}}\int_0^{\frac{t}{2}} \|\nabla_x E(s)\|_{L^1} \frac{\log(2+s)}{1+s^{d-2}} + \frac{\varepsilon}{t}\int_{\frac{t}{2}}^t \|\nabla_x E(s)\|_{L^\infty} \frac{\log(2+s)}{1+s^{d-2}} \\ &\lesssim \frac{\varepsilon^2}{t^{d+1}}\int_0^{\frac{t}{2}} \frac{\log^2(2+s)}{1+s^{d-1}} \lesssim \frac{\varepsilon^2}{t^{d+1}}.\end{aligned}$$

The estimates for $\mathcal{R}_j^4$ are obtained in the same way.

We shall now estimate $\mathcal{R}_j^1$, $\mathcal{R}_j^2$, together with $\partial_j \mathcal{R}_{\mathrm{L}}$. Recalling (4.2) and going back to the variable $v = (x-w)/t$, we write

$$\begin{aligned}\mathcal{R}_j^1 &= \int_0^t\int_{\mathbb{R}^d} s\partial_j E\left(s, X_{s,t}(x,v)\right)\cdot \nabla_v \mu\left(V_{s,t}(x,v)\right) \frac{dwds}{t}, \\ \mathcal{R}_j^2 &= \int_0^t\int_{\mathbb{R}^d} E\left(s, X_{s,t}(x,v)\right)\cdot \partial_{v_j}\nabla_v \mu\left(V_{s,t}(x,v)\right) \frac{dwds}{t}.\end{aligned}$$

Next, using again the straightening change of variable of Proposition 5.1, we obtain

$$\begin{aligned}\mathcal{R}_j^1 &= \int_0^t\int_{\mathbb{R}^d} s\partial_j E\left(s, x-(t-s)v\right)\cdot \nabla_v \mu\left(V_{s,t}(x,\Psi_{s,t}(x,v)\right) \det(\nabla_v \Psi_{s,t}(x,v)) \frac{dvds}{t}, \\ \mathcal{R}_j^2 &= \int_0^t\int_{\mathbb{R}^d} E\left(s, x-(t-s)v\right)\cdot \partial_{v_j}\nabla_v \mu\left(V_{s,t}(x,\Psi_{s,t}(x,v)\right) \det(\nabla_v \Psi_{s,t}(x,v)) \frac{dvds}{t}.\end{aligned}$$

On the other hand, using also the change of variable $w = x - tv$, we obtain that

$$\mathcal{R}_{\mathrm{L}}(t,x) = \int_0^t\int_{\mathbb{R}^d} E(s, w+\frac{s}{t}(x-w))\cdot\nabla_v\mu(\frac{x-w}{t})\frac{dwds}{t^d}$$

and therefore

$$\partial_j \mathcal{R}_{\mathrm{L}}(t,x) = \mathcal{R}_{\mathrm{L},j}^1 + \mathcal{R}_{\mathrm{L},j}^2$$

where

$$\begin{aligned}\mathcal{R}_{\mathrm{L},j}^1 &= \int_0^t\int_{\mathbb{R}^d} (s\partial_j E)(s, w+\frac{s}{t}(x-w))\cdot\nabla_v\mu(\frac{x-w}{t})\frac{dwds}{t^{d+1}} \\ &= \int_0^t\int_{\mathbb{R}^d} (s\partial_j E)(s, x-(t-s)v)\cdot\nabla_v\mu(v)\frac{dwds}{t}\end{aligned}$$

$$\mathcal{R}^2_{\mathrm{L},j} = \int_0^t \int_{\mathbb{R}^d} E(s, w + \frac{s}{t}(x-w)) \cdot \partial_{v_j} \nabla_v \mu(\frac{x-w}{t}) \, \frac{dwds}{t^{d+1}}$$
$$= \int_0^t \int_{\mathbb{R}^d} E(s, x-(t-s)v) \cdot \partial_{v_j} \nabla_v \mu(v) \, \frac{dwds}{t}.$$

Consequently, we observe that

$$-\mathcal{R}^1_j + \mathcal{R}^1_{\mathrm{L},j} = \frac{1}{t}\mathcal{T}[s\partial_j E, \mu](t,x), \quad -\mathcal{R}^2_j + \mathcal{R}^2_{\mathrm{L},j} = \frac{1}{t}\mathcal{T}[E, \partial_j \mu](t,x),$$

so that by (H1), we can apply Lemma 7.1. Therefore, by (7.6), (7.7) and the bootstrap assumption, we get that

$$\| -\mathcal{R}^i_j(t) + \mathcal{R}^i_{\mathrm{L},j}(t)\|_{L^1} \lesssim \frac{\varepsilon^2}{t}, \quad \| -\mathcal{R}^i_j(t) + \mathcal{R}^i_{\mathrm{L},j}(t)\|_{L^\infty} \lesssim \frac{\varepsilon^2}{t^{d+1}}, \quad i = 1, 2$$

which allows to end the proof. □

We are finally in position to conclude.

Proof This follows from the estimates of Proposition 6.1, Proposition 7.1, and Proposition 7.2. □

8 Proof of Corollary 1.1

We can use again that f is given by (3.2) from the characteristic method. We can also observe that

$$\int_0^t E(s, X_{s,t}(x,v)) \cdot \nabla_v \mu(V_{s,t}(x,v)) \, ds = \int_0^t \frac{d}{ds} V_{s,t}(x,v) \cdot \nabla_v \mu(V_{s,t}(x,v)) \, ds$$
$$= \mu(v) - \mu(V_{0,t}(x,v)).$$

By using $(Y_{s,t}, W_{s,t})$ that are defined in (4.2), we can thus write that

$$f(t, x+tv, v) = f_0(x+Y_{0,t}(x,v), v+W_{0,t}(x,v)) + \mu(v+W_{0,t}(x,v)) - \mu(v). \quad (8.1)$$

Consequently, the result follows immediately if we prove that there exists $(Y_\infty(x,v), W_\infty(x,v))$ such that

$$\|Y_{0,t}(x,v) - Y_\infty(x,v)\|_{L^\infty_{x,v}} \lesssim \varepsilon_0 \frac{\log(2+t)}{1+t^{d-1}},$$
$$\|W_{0,t}(x,v) - W_\infty(x,v)\|_{L^\infty_{x,v}} \lesssim \varepsilon_0 \frac{\log(2+t)}{1+t^d}.$$

We give the proof for $Y_{0,t}$, the one for $W_{0,t}$ being similar.

Let us define $\mathbb{Y}_t(s,x,v) = Y_{s,t}(x,v)1_{0\le s\le t}$. From the integral equation (4.4) and the decay estimates (1.5), we get that $\mathbb{Y}_t \in \mathscr{C}_b(\mathbb{R}^+ \times \mathbb{R}^{2d})$ and that for every $t_1 \ge t_2 \ge 1$,

$$\|\mathbb{Y}_{t_1} - \mathbb{Y}_{t_2}\|_{\mathscr{C}_b(\mathbb{R}^+\times\mathbb{R}^{2d})} \lesssim \varepsilon_0 \frac{\log(2+t_2)}{1+t_2^{d-1}} + \varepsilon_0 \|\mathbb{Y}_{t_1} - \mathbb{Y}_{t_2}\|_{\mathscr{C}_b(\mathbb{R}^+\times\mathbb{R}^{2d})}.$$

For ε_0 sufficiently small, this yields by the Cauchy criterion that $\lim_{t\to+\infty} \mathbb{Y}_t := \mathbb{Y}_\infty$ exists in $\mathscr{C}_b(\mathbb{R}^+ \times \mathbb{R}^{2d})$ and then that

$$\|\mathbb{Y}_t - \mathbb{Y}_\infty\|_{\mathscr{C}_b(\mathbb{R}^+\times\mathbb{R}^{2d})} \lesssim \varepsilon_0 \frac{\log(2+t)}{1+t^{d-1}}.$$

We conclude by setting $Y_\infty = \mathbb{Y}_\infty(0,\cdot)$.

Acknowledgements TN was partially supported by the NSF under grant DMS-1764119 and an AMS Centennial Fellowship, DHK by the grant ANR-19-CE40-0004 and FR by the grants ANR ODA and Singflows. Part of this work was done while TN was visiting Princeton University.

Appendix A

We recall (1.8) for the Littlewood–Paley decomposition in $\mathbb{R}^n$. In the paper, we use it (for $n = d$ and for $n = d+1$). Let us state the classical Bernstein Lemma.

Lemma A.1 *For every $p \in [1,+\infty]$ and any multi-index α, there exist $c > 0$, $C > 0$ such that for every $u \in L^p$, we have Bernstein's inequalities:*

$$c2^{|\alpha|q}\|u_q\|_{L^p} \le \|\partial^\alpha(u_q)\|_{L^p} \le C2^{|\alpha|q}\|u_q\|_{L^p}, \quad \forall q \in \mathbb{Z}. \tag{A.1}$$

We refer for example to [1, Chapter 2, Lemma 2.1] for the proof. As an application, we get

Lemma A.2 *Let P_1 and P_2 be homogeneous polynomials of degree 1 and 2. For all $p \in [1,+\infty]$, for all $u \in L^p$, for all $\ell \in \mathbb{N}$,*

$$\|P_1(D)(1-\Delta)^{-1}u\|_{L^p} \lesssim \|u\|_{L^p}, \tag{A.2}$$
$$\|P_2(D)(1-\Delta)^{-1}u_\ell\|_{L^p} \lesssim 2^{\ell\delta}\|u_\ell\|_{L^p}, \tag{A.3}$$

where u_ℓ is defined as in (1.8) and $\delta \in (0,1)$ is arbitrarily small.

Proof By using the homogeneous Littlewood–Paley decomposition and the Bernstein inequalities, we get

$$\|P_1(D)(1-\Delta)^{-1}u\|_{L^p} \lesssim \sum_{q\in\mathbb{Z}} \frac{2^q}{1+2^{2q}}\|u_q\|_{L^p} \lesssim \|u\|_{L^p}\Big(\sum_{q\le 0} 2^q + \sum_{q\ge 0} 2^{-q}\Big).$$

For the second estimate, we write

$$\begin{aligned}\|P_2(D)(1-\Delta)^{-1}w\|_{L^p} &\lesssim \sum_{q\in\mathbb{Z}} \frac{2^{2q}}{1+2^{2q}}\|w_q\|_{L^p} \\ &\lesssim \|w\|_{L^p}\sum_{q<0} 2^{2q} + \sup_{q\geq 0} 2^{q\delta}\|w_q\|_{L^p}\sum_{q\geq 0} 2^{-q\delta}\end{aligned}$$

and apply it for $w = u_\ell$, which ends the proof. □

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