

PLASMA ECHOES NEAR STABLE PENROSE DATA*

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Abstract. In this paper we construct particular solutions to the classical Vlasov–Poisson system near stable Penrose initial data on $\mathbb{T} \times \mathbb{R}$ that are a combination of elementary waves with arbitrarily high frequencies. These waves mutually interact giving birth, eventually, to an infinite cascade of echoes of smaller and smaller amplitude. The echo solutions do not belong to the analytic or Gevrey classes studied by Mouhot and Villani but do, nonetheless, exhibit damping phenomena for large times.

Key words. Landau damping, plasma echoes, Vlasov–Poisson

AMS subject classifications. 35QXX, 82XX

DOI. 10.1137/21M1392553

1. Introduction. In the physical literature, the large time behavior of a plasma modeled by the classical Vlasov–Poisson system is characterized by

- *Landau damping.* Decay of the electric field for large times.
- *Plasma echoes.* An elementary wave, arising as a *free transport* of initial data of the form $\varepsilon f_1(v)e^{ik_1x+i\eta_1v}$, will generate an electric field of order ε that is localized near the critical time $\tau_1 = \eta_1/k_1$ and decays¹ for larger times. When two elementary waves $\varepsilon f_j(v)e^{ik_jx+i\eta_jv}$, with arbitrarily large frequencies k_j, η_j , $j = 1, 2$ and the associated critical times $\tau_j = \eta_j/k_j$, interact, a third wave of the same form is created. The electric field of this third wave is localized but of order ε^2 , near the “echo time” $\tau = (\eta_1 + \eta_2)/(k_1 + k_2)$, which could be long after the first two waves have died away. The phenomenon is often referred to as an “echo” [3].

From the discussion above, an echo is of a higher order (ε^2) in amplitude. The created wave again interacts with the other two waves, creating higher order waves and higher order echoes and so on. That is, starting from two waves, an infinite number of waves, of smaller and smaller amplitudes, appear, with an infinite number of echoes of smaller and smaller amplitudes.

In this context, the fundamental question is to understand the described heuristic picture and analyze large time *nonlinear* behavior of such an infinite cascade. While the linear Landau damping was discovered and fully understood by Landau [6], the nonlinear analogue has been largely elusive. However, important progress has been

*Received by the editors January 19, 2021; accepted for publication (in revised form) November 4, 2021; published electronically February 8, 2022.

<https://doi.org/10.1137/21M1392553>

Funding: The work of the second author was partially supported by the National Science Foundation grant DMS-1764119, an AMS Centennial fellowship, and a Simons Foundation fellowship. The work of the third author was partially supported by the National Science Foundation grant DMS-1709270 and a Simons Foundation Investigator Award.

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¹polynomially or exponentially small, depending on the regularity of $f_1(v)$.

made by Mouhot and Villani in their celebrated work [5], where the problem was solved in the case of analytic or Gevrey data. Their proof has then been simplified in [2]. The damping for data with finite Sobolev regularity remains largely open due to plasma echoes [3] and high frequency instabilities [1, 8], while it is known to be false for data with very low regularity [7].

In the companion paper [4], we give an elementary proof of the nonlinear Landau damping for analytic and Gevrey data [5, 2]. In this paper, we construct an exact solution of the classical Vlasov–Poisson system, starting from an infinite number of elementary waves of amplitude ε . Provided that ε is sufficiently small and each wave has an analytic regularity, we are able to track all the interactions and to construct solutions which display an infinite number of echoes, which are of a smaller and smaller amplitude as time evolves. The associated electric field decays for large times, and therefore *Landau damping* holds for such data. As we are allowed to take the frequencies of each elementary wave to be arbitrarily large, the solutions do not belong to the class of analytic or Gevrey solutions constructed in [5, 2].

Precisely, we consider the following classical Vlasov–Poisson system²

$$(1.1) \quad \partial_t f + v \partial_x f + E \partial_v f + E \partial_v \mu = 0, \quad \partial_x E = \rho = \int_{\mathbb{R}} f \, dv$$

on the torus $\mathbb{T} \times \mathbb{R}$ for small initial data $f^0(x, v)$ satisfying

$$(1.2) \quad \iint_{\mathbb{T} \times \mathbb{R}} f^0(x, v) \, dx dv = 0.$$

In (1.1), $\mu(v)$ is a stable Penrose equilibrium. Precisely, we require that $\mu(v)$ is such that

- $\mu(v)$ is real analytic and satisfies

$$(1.3) \quad |\widehat{\langle v \rangle^2 \mu}(\eta)| + |\widehat{\mu}(\eta)| \leq C_0 e^{-\theta_0 |\eta|}.$$

- $\mu(v)$ satisfies the Penrose stability condition, namely,

$$(1.4) \quad \inf_{k \in \mathbb{Z}^d \setminus \{0\}; \Re \lambda \geq 0} \left| 1 + \int_0^\infty e^{-\lambda t} t \widehat{\mu}(kt) \, dt \right| \geq \kappa_0 > 0.$$

The condition is to ensure that the linearized system of (1.1) (obtained by dropping the nonlinear term $E \partial_v f$) is solvable. It holds for a variety of equilibria including the Gaussian $\mu(v) = e^{-|v|^2/2}$. In three or higher dimensions, the condition is valid for any positive and radially symmetric equilibria [5].

We consider the initial data which are a sum of highly oscillatory simple modes of the form

$$(1.5) \quad f^0(x, v) = \sum_{(k, \eta) \in \mathbb{Z} \times \mathbb{Z}} \varepsilon f_{k, \eta}^0(v) e^{iKkx + iL\eta v}$$

for arbitrary parameters K and L and for small ε . In view of (1.2), we take $f_{k, \eta}^0(v) = 0$ for $k = 0$. We stress that f^0 is rapidly oscillating in x and v when K, L are sufficiently

²obtained from the standard Vlasov–Poisson equations

$$\partial_t \tilde{f} + v \partial_x \tilde{f} + E \partial_v \tilde{f} = 0, \quad \partial_x E = \rho = \int_{\mathbb{R}} \tilde{f} \, dv - 1$$

by writing $\tilde{f} = f + \mu$.

large. In particular, since K, L are allowed to be arbitrarily large, the initial data is of order $\epsilon \langle K, L \rangle^s$ in Sobolev spaces $W^{s, \infty}(\mathbb{T} \times \mathbb{R})$, which is also arbitrarily large for any $s > 0$.

Our main result asserts that Landau damping holds for data of the form (1.5). Precisely, we prove the following.

THEOREM 1.1. *Let $\lambda_0, C_0 > 0$, and K, L be arbitrary so that*

$$(1.6) \quad L \leq C_0 K.$$

Assume that (1.3)–(1.4) hold. Then, for sufficiently small ϵ independent of K, L and for any initial data of the form (1.5) with the analytic coefficients $f_{k, \eta}^0(v)$ satisfying³

$$(1.7) \quad |\hat{f}_{k, \eta}^0(\eta')| \leq e^{-2\lambda_0 \langle k, \eta, \eta' \rangle},$$

uniformly in k, η, η' , there exists a unique global solution to the Vlasov–Poisson system (1.1). In addition, the solution can be written in the form

$$(1.8) \quad f(t, x, v) = \sum_{(k, \eta, p) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{N}^*} \varepsilon^p f_{k, \eta, p}(t, v) e^{iKkx + i(L\eta - Kkt)v},$$

where the coefficients $f_{k, \eta, p}(t, v)$ are analytic in v and satisfy

$$(1.9) \quad |\hat{f}_{k, \eta, p}(t, \eta')| \leq C_1^p e^{-\lambda_0 \langle k, \eta, p, \eta' \rangle}, \quad t \geq 0,$$

uniformly in k, η, p, η' , for some universal constant C_1 that is independent of K, L , and ϵ —in particular, the associated electric field

$$E(t, x) = \sum_{(k, \eta, p) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{N}^*} \varepsilon^p \hat{f}_{k, \eta, p}(t, Kkt - L\eta) \frac{e^{iKkx}}{iKk} \longrightarrow 0$$

exponentially fast in any Sobolev spaces $W^{s, q}$, $s \geq 0$ and $q \geq 1$, as $t \rightarrow \infty$.

We note that the results easily generalize to higher dimensions. Condition $L \leq C_0 K$, coupled with the assumptions on the (k, η) dependence of the initial data, imply that the “echos” occur at times which are, essentially, uniformly bounded.

Finally, we note that the parameters K, L are allowed to be arbitrary (satisfying (1.6)), which in particular includes the analytic data studied by Mouhot and Villani [5] as a special case for $K = 1$ and $L = 0$.

2. Linear theory. In this section, we recall the linear Landau damping theory developed in the companion paper [4]. Precisely, let $k \in \mathbb{Z}$, and let $\hat{\rho}(t)$ satisfy

$$(2.1) \quad \hat{\rho}(t) + \int_0^t (t-s) \hat{\mu}(k(t-s)) \hat{\rho}(s) \, ds = \hat{S}(t)$$

with a source term $\hat{S}(t)$. Taking the Laplace transform of (2.1) in time, we get

$$(2.2) \quad \mathcal{L}[\hat{\rho}(t)](\lambda) = \frac{\mathcal{L}[\hat{S}(t)](\lambda)}{1 + \mathcal{L}[t\hat{\mu}(kt)](\lambda)},$$

where $\mathcal{L}[F(t)](\lambda)$ denotes the usual Laplace transform of $F(t)$. The Penrose stability condition (1.4) ensures that the symbol $1 + \mathcal{L}[t\hat{\mu}(kt)](\lambda)$ never vanishes.

We then have the following.

³We use the notation $\langle x_1, \dots, x_n \rangle = \sqrt{1 + x_1^2 + \dots + x_n^2}$. Depending on the context, we also use $\hat{\cdot}$ to denote the Fourier transform in x, v or both.

PROPOSITION 2.1. Assume that (1.3)–(1.4) hold. Then, the solution $\widehat{\rho}(t)$ to (2.1) exists and satisfies

$$(2.3) \quad \widehat{\rho}(t) = \widehat{S}(t) + \int_0^t G_k(t-s) \widehat{S}(s) ds,$$

where $|G_k(t)| \leq C_1 e^{-\theta_1 |kt|}$ for some positive constants θ_1, C_1 .

Proof. From (2.2), we can write

$$(2.4) \quad \mathcal{L}[\widehat{\rho}](\lambda) = \mathcal{L}[\widehat{S}](\lambda) + \widetilde{G}_k(\lambda) \mathcal{L}[\widehat{S}](\lambda),$$

where we denote

$$(2.5) \quad \widetilde{G}_k(\lambda) := -\frac{\mathcal{L}[t\widehat{\mu}(kt)](\lambda)}{1 + \mathcal{L}[t\widehat{\mu}(kt)](\lambda)}.$$

The integral formulation (2.3) thus follows, where $G_k(t)$ is the inverse Laplace transform of $\widetilde{G}_k(\lambda)$. It remains to prove the estimate on $G_k(t)$. First, we note by definition that

$$\mathcal{L}[t\widehat{\mu}(kt)](\lambda) = \int_0^\infty e^{-\lambda t} t \widehat{\mu}(kt) dt.$$

Thus, the Penrose condition (1.4) ensures that the denominator $1 + \mathcal{L}[t\widehat{\mu}(kt)](\lambda)$ never vanishes for $\Re \lambda \geq 0$. Furthermore, using (1.3), we in fact have

$$(2.6) \quad |\mathcal{L}[t\widehat{\mu}(kt)](\lambda)| \leq C_0 \int_0^\infty e^{-\Re \lambda t} t e^{-\theta_0 |kt|} dt \leq C_1 |k|^{-2}$$

for $\Re \lambda \geq -\theta_1 |k|$ and for any $\theta_1 < \theta_0$. On the other hand, for $\Re \lambda = -\theta_1 |k|$, integrating by parts in time, we get

$$\begin{aligned} \mathcal{L}[t\widehat{\mu}(kt)](\lambda) &= \int_0^\infty \frac{(M_k^2 - \partial_t^2)(e^{-\lambda t})}{M_k^2 - \lambda^2} t \widehat{\mu}(kt) dt \\ &= \int_0^\infty \frac{e^{-\lambda t}}{M_k^2 - \lambda^2} (M_k^2 - \partial_t^2)(t \widehat{\mu}(kt)) dt - \frac{\widehat{\mu}(0)}{M_k^2 - \lambda^2} \end{aligned}$$

for any constant $M_k \neq \lambda$. Taking $M_k = 2\theta_1 |k|$, we have

$$|\mathcal{L}[t\widehat{\mu}(kt)](\lambda)| \leq C_0 \int_0^\infty \frac{e^{\frac{1}{2}\theta_0 |kt|}}{\theta_1^2 |k|^2 + |\Im \lambda|^2} (|k| + |k|^2 t) e^{-\theta_0 |kt|} dt + \frac{C_0}{\theta_1^2 |k|^2 + |\Im \lambda|^2},$$

which gives

$$(2.7) \quad |\mathcal{L}[t\widehat{\mu}(kt)](\lambda)| \leq C_1 (1 + |k|^2 + |\Im \lambda|^2)^{-1}$$

for any λ on the line $\{\Re \lambda = -\theta_1 |k|\}$. This proves that $\mathcal{L}[t\widehat{\mu}(kt)](\lambda)$ is analytic in $\{\Re \lambda \geq -\theta_1 |k|\}$ for any $\theta_1 < \theta_0$.

We next prove that there is a positive $\theta_1 < \theta_0$ so that $\widetilde{G}_k(\lambda)$ is analytic in $\{\Re \lambda \geq -\theta_1 |k|\}$ and the estimate (2.7) also holds for $\widetilde{G}_k(\lambda)$, possibly with a different constant C_1 . Indeed, the estimate (2.7) shows that there are k_0, τ_0 so that $|\mathcal{L}[t\widehat{\mu}(kt)](\lambda)| \leq \frac{1}{2}$ for all $|k| \geq k_0$ and $\lambda \geq -\theta_1 |k|$ or for $\Re \lambda = -\theta_1 |k|$ and $|\Im \lambda| \geq \tau_0$. While for $|\Im \lambda| \leq \tau_0$

and $|k| \leq k_0$, since the Penrose condition (1.4) holds for $\Re \lambda = 0$, there is a small positive constant θ_1 so that

$$(2.8) \quad |1 + \mathcal{L}[t\hat{\mu}(kt)](\lambda)| \geq \frac{1}{2}\kappa_0$$

for $\Re \lambda = -\theta_1|k|$ (recalling that $1 \leq |k| \leq k_0$). Combining, we have that $1 + \mathcal{L}[t\hat{\mu}(kt)](\lambda)$ is bounded below away from zero on $\{\Re \lambda \geq -\theta_1|k|\}$ for all $k \in \mathbb{Z}$. The estimates on $\tilde{G}_k(\lambda)$ thus follows from those on $\mathcal{L}[t\hat{\mu}(kt)](\lambda)$.

By definition, we have

$$G_k(t) = \frac{1}{2\pi i} \int_{\{\Re \lambda = \gamma_0\}} e^{\lambda t} \tilde{G}_k(\lambda) d\lambda$$

for some large positive constant γ_0 . Since $\tilde{G}_k(\lambda)$ is analytic in $\{\Re \lambda \geq -\theta_1|k|\}$, and thus we can apply the Cauchy theory to deform the complex contour of integration from $\{\Re \lambda = \gamma_0\}$ into $\{\Re \lambda = -\theta_1|k|\}$, on which both estimates (2.7) and (2.8) hold. Therefore,

$$|G_k(t)| \leq C_1 \int_{\{\Re \lambda = -\theta_1|k|\}} e^{-\theta_1|kt|} (1 + |k|^2 + |\Im \lambda|^2)^{-1} d\lambda \leq C_1 e^{-\theta_1|kt|}.$$

The proposition follows. \square

3. Construction.

3.1. Setup. Let us first detail the construction of the profiles $f_{k,\eta,p}$ in the infinite series (1.8). Indeed, we formally plug the Ansatz (1.8) to the Vlasov–Poisson system (1.1) with the corresponding electric field

$$(3.1) \quad E(t, x) = \sum_{(k,\eta,p) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{N}^*} \varepsilon^p \hat{f}_{k,\eta,p}(t, Kkt - L\eta) \frac{e^{iKkx}}{iKk}$$

and match the terms of order ε^p . At each step we consider the term $E\partial_v f$ of (1.1) as a source term for the linear Vlasov–Poisson near the equilibrium μ . This yields, for each $(k, \eta, p) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{N}^*$, the following linearized equation for $f_{k,\eta,p}(t, v)$ and $\hat{E}_{k,\eta,p}(t)$,

$$(3.2) \quad \partial_t f_{k,\eta,p} + \hat{E}_{k,\eta,p} e^{-i(L\eta - Kkt)v} \partial_v \mu = N_{k,\eta,p},$$

in which

- For $p = 1$, we take $f_{k,\eta,1}(0, v) = f_{k,\eta}^0(v)$ and $N_{k,\eta,1} = 0$.
- For $p \geq 2$, we take $f_{k,\eta,p}(0, v) = 0$ and

$$N_{k,\eta,p}(t, v) = \sum_{(k_1, \eta_1, k_2, \eta_2, p_1, p_2) \in A_{k,\eta,p}} e^{-i(L\eta_1 - Kk_1 t)v} \hat{E}_{k_1, \eta_1, p_1}(t) \times [\partial_v + i(L\eta_2 - Kk_2 t)] f_{k_2, \eta_2, p_2}(t, v),$$

where $A_{k,\eta,p}$ denotes the set of sextets in $\mathbb{Z} \times \mathbb{Z} \times \mathbb{N}^*$:

$$A_{k,\eta,p} := \left\{ k_1 + k_2 = k, \eta_1 + \eta_2 = \eta, p_1 + p_2 = p \right\}.$$

The electric field is defined by

$$\widehat{E}_{k,\eta,0}(t) = 0$$

and for $p > 0$, by a direct computation,

$$(3.3) \quad \widehat{E}_{k,\eta,p}(t) = \frac{1}{iKk} \widehat{\rho}_{k,\eta,p}(t) = \frac{1}{iKk} \widehat{f}_{k,\eta,p}(t, Kkt - L\eta).$$

As a consequence of the assumption (1.2) on the initial data, we will prove in the next sections that $\widehat{E}_{0,\eta,p}(t) = 0$ for all η and p . Note that taking the Fourier transform of (1.8) in x and v , we have

$$(3.4) \quad \widehat{f}(t, Kk, \eta') = \sum_{(\eta,p) \in \mathbb{Z} \times \mathbb{N}^*} \varepsilon^p \widehat{f}_{k,\eta,p}(t, \eta' - L\eta + Kkt)$$

and $\widehat{f}(t, k', \eta') = 0$ for $k' \notin K\mathbb{Z}$.

It remains to derive estimates on the Fourier transform $\widehat{f}_{k,\eta,p}(t, \eta')$ of each functions $f_{k,\eta,p}(t, v)$ in order to ensure the convergence of the infinite series (3.4).

3.2. Resolution using Penrose's kernel. We begin by converting (3.2) to an integral equation.

LEMMA 3.1. *Let $f_{k,\eta,p}(t, v)$ be constructed as indicated above. Set*

$$(3.5) \quad \widehat{S}_{k,\eta,p}(t, \eta') = \widehat{f}_{k,\eta,p}(0, \eta') + \int_0^t \widehat{N}_{k,\eta,p}(s, \eta') \, ds.$$

Then, there holds

$$\widehat{f}_{k,\eta,p}(t, \eta') = \widehat{S}_{k,\eta,p}(t, \eta') - \int_0^t \widehat{E}_{k,\eta,p}(s) \widehat{\partial_v \mu}(\eta' + L\eta - Kks) \, ds.$$

In addition,

$$(3.6) \quad \widehat{\rho}_{k,\eta,p}(t) = \widehat{S}_{k,\eta,p}(t, Kkt - L\eta) + \int_0^t G_k(t-s) \widehat{S}_{k,\eta,p}(s, Kks - L\eta) \, ds,$$

where $|G_k(t)| \leq C_0 e^{-\theta_0 |Kkt|}$.

Proof. Integrating (3.2) in time, we obtain

$$f_{k,\eta,p}(t, v) = - \int_0^t \widehat{E}_{k,\eta,p}(s) e^{-i(L\eta - Kks)v} \partial_v \mu(v) \, ds + S_{k,\eta,p},$$

where

$$S_{k,\eta,p}(t, v) = f_{k,\eta,p}(0, v) + \int_0^t N_{k,\eta,p}(s, v) \, ds.$$

Taking the Fourier transform yields the expression for $\widehat{f}_{k,\eta,p}(t, \eta')$. In particular, using (3.3), we have

$$\widehat{\rho}_{k,\eta,p}(t) + \int_0^t (t-s) \widehat{\mu}(Kk(t-s)) \widehat{\rho}_{k,\eta,p}(s) \, ds = \widehat{S}_{k,\eta,p}(t, Kkt - L\eta).$$

Using the linear theory developed in Proposition 2.1, the lemma follows. \square

3.3. Inductive estimates. In this section, we shall inductively derive estimates on $\widehat{f}_{k,\eta,p}(t, \eta')$. In what follows, we fix $\lambda_0 > 0$ and K, L to be arbitrarily large so that

$$(3.7) \quad L \lesssim K.$$

Then, we have the following.

PROPOSITION 3.2. *Let $\sigma > 1$ be fixed. There is some universal constant C_0 so that*

$$(3.8) \quad |\widehat{f}_{k,\eta,p}(t, \eta')| \leq C_0^p e^{-\lambda_p(t)\langle k, \eta, p, \eta' \rangle} \langle k \rangle^{-1},$$

$$(3.9) \quad |\widehat{\rho}_{k,\eta,p}(t)| \leq C_0^p e^{-\lambda_p(t)\langle k, \eta, p, L\eta - Kkt \rangle} \langle t \rangle^{-\sigma},$$

uniformly in k, η, p, η' , and $t \geq 0$, where $\lambda_p(t)$ is defined by

$$(3.10) \quad \lambda_p(t) = \lambda_0 + \langle t \rangle^{-\delta} + p^{-\delta}$$

for some $0 < \delta \ll 1$. In addition, $\widehat{\rho}_{0,\eta,p}(t) = 0$ for all η and p .

Note that all the estimates are uniform in the large parameters K and L . The following subsections are devoted to the proof of this proposition, which will be done by induction on $p \geq 1$.

3.4. Estimates for $p = 1$. We first estimate $\widehat{S}_{k,\eta,p}(t, \eta')$ for $p = 1$. By construction, $N_{k,\eta,1} = 0$, and thus we have

$$\widehat{S}_{k,\eta,1}(t, \eta') = \widehat{f}_{k,\eta,1}(0, \eta') = \widehat{f}_{k,\eta}^0(\eta').$$

By assumption on the initial data $f_{0,\eta}^0(v) = 0$ in (1.5), we have $\widehat{S}_{0,\eta,1}(t, \eta') = 0$, and so $\widehat{\rho}_{0,\eta,1}(t) = 0$ due to the representation (3.6). We now consider $k \neq 0$. Using the assumption (1.7) in (3.6), we obtain

$$\begin{aligned} |\widehat{\rho}_{k,\eta,1}(t)| &\leq |\widehat{S}_{k,\eta,1}(t, Kkt - L\eta)| + \int_0^t |G_k(t-s)\widehat{S}_{k,\eta,1}(s, Kks - L\eta)| ds \\ &\leq e^{-2\lambda_0\langle k, \eta, L\eta - Kkt \rangle} + C_0 \int_0^t e^{-\theta_0|Kk(t-s)|} e^{-2\lambda_0\langle k, \eta, L\eta - Kks \rangle} ds. \end{aligned}$$

Using $\lambda_0 \leq \theta_0/4$ and the triangle inequality, we bound

$$e^{-\frac{1}{2}\theta_0|Kk(t-s)|} e^{-2\lambda_0|L\eta - Kks|} \leq e^{-2\lambda_0|L\eta - Kkt|}.$$

Hence,

$$\begin{aligned} |\widehat{\rho}_{k,\eta,1}(t)| &\leq e^{-2\lambda_0\langle k, \eta, L\eta - Kkt \rangle} + C_0 e^{-2\lambda_0\langle k, \eta, L\eta - Kkt \rangle} \int_0^t e^{-\frac{1}{2}\theta_0|Kk(t-s)|} ds \\ &\leq C_0 e^{-2\lambda_0\langle k, \eta, L\eta - Kkt \rangle}. \end{aligned}$$

To complete the proof of (3.9) for $p = 1$, we need to check the decay in time. Indeed, using the triangle inequality

$$|Kkt| \leq |Kkt - L\eta| + |L\eta|$$

and the fact that $K \geq 1$ and $L \lesssim K$, we have

$$(3.11) \quad |kt| \leq K^{-1}|Kkt - L\eta| + LK^{-1}|\eta| \leq |Kkt - L\eta| + |\eta|.$$

This proves that

$$|\widehat{\rho}_{k,\eta,1}(t)| \leq C_0 e^{-\lambda_0 \langle k, \eta, L\eta - Kkt \rangle} e^{-\lambda_0 \langle kt \rangle},$$

which proves (3.9) for $p = 1$, since $k \neq 0$. To estimate (3.8), we use Lemma 3.1 to estimate

$$\begin{aligned} |\widehat{f}_{k,\eta,1}(t, \eta')| &\leq |\widehat{S}_{k,\eta,1}(t, \eta')| + \int_0^t |\widehat{E}_{k,\eta,1}(s) \widehat{\partial}_v \mu(\eta' + L\eta - Kks)| ds \\ &\leq e^{-2\lambda_0 \langle k, \eta, \eta' \rangle} \\ &\quad + C_0 \langle Kk \rangle^{-1} \int_0^t e^{-\lambda_0 \langle k, \eta, L\eta - Kks \rangle} e^{-\theta_0 |\eta' + L\eta - Kks|} \langle s \rangle^{-\sigma} ds \\ &\leq e^{-2\lambda_0 \langle k, \eta, \eta' \rangle} + C_0 \langle Kk \rangle^{-1} e^{-\lambda_0 \langle k, \eta, \eta' \rangle} \\ &\leq C_0 \langle k \rangle^{-1} e^{-\lambda_0 \langle k, \eta, \eta' \rangle}, \end{aligned}$$

where we used the exponential decay of the electric field, proven above, to insert an extra factor $\langle s \rangle^{-\sigma}$ with $\sigma > 1$. This proves Proposition 3.2 for $p = 1$.

3.5. Estimates on $\widehat{E}_{k,\eta,p}$. In this section, we shall prove the estimates (3.9) on $\widehat{E}_{k,\eta,p}$ for $p > 1$, under the inductive assumption that the estimates (3.8)–(3.9) on \widehat{f}_{k,η,p_1} and \widehat{E}_{k,η,p_1} hold for all $p_1 \leq p - 1$. In particular, $\widehat{E}_{0,\eta_1,p_1}(t) = 0$. Precisely, we prove the following lemma.

LEMMA 3.3. *Let $P > 1$. Assume that (3.8)–(3.9) hold true for any k, η , and $p \leq P - 1$. Then (3.9) is true for any k, η , and $p = P$, with $\widehat{\rho}_{0,\eta,P}(t) = 0$.*

In view of Lemma 3.1, we first prove the following.

LEMMA 3.4. *Under the assumption of Lemma 3.3, there holds*

$$|\widehat{S}_{k,\eta,p}(t, Kkt - L\eta)| \leq C_0^p e^{-\lambda_p(t) \langle k, \eta, p, L\eta - Kkt \rangle} \langle t \rangle^{-\sigma},$$

where $S_{k,\eta,p}$ is defined as in (3.5).

Proof of Lemma 3.3 using Lemma 3.4. By Lemma 3.1, we have

$$(3.12) \quad \widehat{\rho}_{k,\eta,p}(t) = \widehat{S}_{k,\eta,p}(t, Kkt - L\eta) + \int_0^t G_k(t-s) \widehat{S}_{k,\eta,p}(s, Kkt - L\eta) ds,$$

where $|G_k(t)| \leq C_0 e^{-\theta_0 |Kkt|}$. Using Lemma 3.4, we have

$$\begin{aligned} |\widehat{\rho}_{k,\eta,p}(t)| &\leq C_0^p e^{-\lambda_p(t) \langle k, \eta, p, L\eta - Kkt \rangle} \langle t \rangle^{-\sigma} \\ &\quad + C_0^p \int_0^t e^{-\theta_0 |Kk(t-s)|} e^{-\lambda_p(s) \langle k, \eta, p, L\eta - Kks \rangle} \langle s \rangle^{-\sigma} ds. \end{aligned}$$

Using $\lambda_p(t) \leq \lambda_p(s) \leq \frac{1}{2}\theta_0$, we have

$$\begin{aligned} e^{-\frac{1}{2}\theta_0 |Kk(t-s)|} e^{-\lambda_p(s) \langle k, \eta, p, L\eta - Kks \rangle} &\leq e^{-\lambda_p(t) |Kk(t-s)|} e^{-\lambda_p(t) \langle k, \eta, p, L\eta - Kks \rangle} \\ &\leq e^{-\lambda_p(t) \langle k, \eta, p, L\eta - Kkt \rangle}. \end{aligned}$$

On the other hand, since $k \neq 0$, we easily bound

$$\int_0^t e^{-\frac{1}{2}\theta_0|Kk(t-s)|} \langle s \rangle^{-\sigma} ds \leq C_0 \langle t \rangle^{-\sigma}.$$

The desired estimates on $\widehat{\rho}_{k,\eta,p}(t)$ follow. \square

Proof of Lemma 3.4. By construction, for $p > 1$, $\widehat{f}_{k,\eta,p}(0, \eta') = 0$, and thus we have

$$\widehat{S}_{k,\eta,p}(t, \eta') = \int_0^t \widehat{N}_{k,\eta,p}(s, \eta') ds,$$

where the nonlinear interaction $\widehat{N}_{k,\eta,p}(s, \eta')$ is computed by

$$\widehat{N}_{k,\eta,p}(s, \eta') = i \sum_{A_{k,\eta,p}} \widehat{E}_{k_1,\eta_1,p_1}(s) [\eta' + L\eta - Kks] \widehat{f}_{k_2,\eta_2,p_2}(s, \eta' + L\eta_1 - Kk_1s).$$

Note that by induction, $\widehat{E}_{0,\eta_1,p_1}(t) = 0$, and so the above summation is for $k_1 \neq 0$. In addition, observe that at $\eta' = Kkt - L\eta$, $\widehat{N}_{k,\eta,p}(s, Kkt - L\eta)$ has a prefactor $Kk(t-s)$, which vanishes at $k = 0$. Hence, by construction (3.12), $\widehat{\rho}_{0,\eta,p}(t) = 0$ for any η, p . Therefore, in what follows, we focus on $k, k_1 \neq 0$. By induction, for $p_1, p_2 \leq p-1$, we have

$$\begin{aligned} |\widehat{E}_{k_1,\eta_1,p_1}(t)| &\leq C_0^{p_1} e^{-\lambda_{p_1}(t)\langle k_1,\eta_1,p_1,L\eta_1-Kk_1t \rangle} |Kk_1|^{-1} \langle t \rangle^{-\sigma}, \\ |\widehat{f}_{k_2,\eta_2,p_2}(t, \eta')| &\leq C_0^{p_2} e^{-\lambda_{p_2}(t)\langle k_2,\eta_2,p_2,\eta' \rangle} \langle k_2 \rangle^{-1}. \end{aligned}$$

Hence, recalling the definition of $A_{k,\eta,p}$, we have

$$\begin{aligned} |\widehat{S}_{k,\eta,p}(t, \eta')| &\leq C_0^p \sum_{A_{k,\eta,p}} |Kk_1|^{-1} \langle k_2 \rangle^{-1} \int_0^t e^{-\lambda_{p_1}(s)\langle k_1,\eta_1,p_1,L\eta_1-Kk_1s \rangle} \langle s \rangle^{-\sigma} \\ &\quad \times |\eta' + L\eta - Kks| e^{-\lambda_{p_2}(s)\langle k_2,\eta_2,p_2,\eta'+L\eta_1-Kk_1s \rangle} ds. \end{aligned}$$

It is crucial to note that $\lambda_p(t)$ is strictly decreasing in both p and t . We will use this monotonicity in order to gain time decay in the estimates. Using $k = k_1 + k_2$, $\eta = \eta_1 + \eta_2$, and $p = p_1 + p_2$, we note that

$$\begin{aligned} &e^{-\lambda_{p_1}(s)\langle k_1,\eta_1,p_1,L\eta_1-Kk_1s \rangle} e^{-\lambda_{p_2}(s)\langle k_2,\eta_2,p_2,\eta'+L\eta_1-Kk_1s \rangle} \\ &\leq C_{k,\eta,p,\eta',0}(s, t) C_{k,\eta,p,\eta',1}(s, t) C_{k,\eta,p,\eta',2}(s, t) e^{-\lambda_p(t)\langle k,\eta,p,\eta' \rangle}, \end{aligned}$$

where the factors $C_{k,\eta,p,j}(s, t)$ are defined by

$$\begin{aligned} (3.13) \quad C_{k,\eta,p,\eta',0}(s, t) &:= e^{-(s^{-\delta}-t^{-\delta})\langle k,\eta,p,\eta' \rangle}, \\ C_{k,\eta,p,\eta',1}(s, t) &:= e^{-(p_1^{-\delta}-p^{-\delta})\langle k_1,\eta_1,p_1,L\eta_1-Kk_1s \rangle}, \\ C_{k,\eta,p,\eta',2}(s, t) &:= e^{-(p_2^{-\delta}-p^{-\delta})\langle k_2,\eta_2,p_2,\eta'+L\eta_1-Kk_1s \rangle}, \end{aligned}$$

each of which is smaller than one. These factors may be seen as gains coming from the monotonicity of $\lambda_p(t)$. Combining and noting $|k| \leq 2\langle k_1 \rangle \langle k_2 \rangle$, we thus obtain

$$\begin{aligned} (3.14) \quad |\widehat{S}_{k,\eta,p}(t, \eta')| &\leq C_0^p e^{-\lambda_p(t)\langle k,\eta,p,\eta' \rangle} \langle Kk \rangle^{-1} \sum_{A_{k,\eta,p}} \int_0^t |\eta' + L\eta - Kks| \\ &\quad \times C_{k,\eta,p,\eta',0}(s, t) C_{k,\eta,p,\eta',1}(s, t) C_{k,\eta,p,\eta',2}(s, t) \langle s \rangle^{-\sigma} ds. \end{aligned}$$

Evaluating at $\eta' = Kkt - L\eta$, we get

$$(3.15) \quad |\hat{S}_{k,\eta,p}(t, Kkt - L\eta)| \leq C_0^p e^{-\lambda_p(t)\langle k,\eta,p,L\eta-Kkt \rangle} \sum_{A_{k,\eta,p}} \int_0^t (t-s) \times C_{k,\eta,p,0} C_{k,\eta,p,1} C_{k,\eta,p,2}(s,t) \langle s \rangle^{-\sigma} ds$$

with $C_{k,\eta,p,j} = C_{k,\eta,p,\eta',j}(s,t)$ for $\eta' = Kkt - L\eta$. The lemma thus follows from the following claim:

$$(3.16) \quad \sum_{A_{k,\eta,p}} \int_0^t (t-s) C_{k,\eta,p,0} C_{k,\eta,p,1} C_{k,\eta,p,2}(s,t) \langle s \rangle^{-\sigma} ds \leq C_0 \langle t \rangle^{-\sigma}.$$

Let us first bound the factors $C_{k,\eta,p,j}(s,t)$.

LEMMA 3.5. *Setting $C_{k,\eta,p,j} = C_{k,\eta,p,\eta',j}(s,t)$ as in (3.13) for $\eta' = Kkt - L\eta$, we have*

$$\begin{aligned} C_{k,\eta,p,0}(s,t) &\leq e^{-\theta_0(s^{-\delta}-t^{-\delta})\langle k,\eta,p,kt \rangle}, \\ C_{k,\eta,p,1}(s,t) &\leq e^{-\theta_0(p_1^{-\delta}-p^{-\delta})\langle k_1,\eta_1,p_1,k_1s \rangle}, \\ C_{k,\eta,p,2}(s,t) &\leq e^{-\theta_0(p_2^{-\delta}-p^{-\delta})\langle k_2,\eta_2,p_2,kt-k_1s \rangle} \end{aligned}$$

for some positive constant θ_0 .

Proof. Recalling the inequality (3.11), $|kt| \leq |Kkt - L\eta| + |\eta|$, we have

$$\begin{aligned} C_{k,\eta,p,1}(s,t) &= e^{-(s^{-\delta}-t^{-\delta})\langle k,\eta,p,L\eta-Kkt \rangle} \\ &\leq e^{-(s^{-\delta}-t^{-\delta})\langle k,\eta,p \rangle/2} e^{-(s^{-\delta}-t^{-\delta})\langle \eta,L\eta-Kkt \rangle/2} \\ &\leq e^{-\theta_0(s^{-\delta}-t^{-\delta})\langle k,\eta,p,kt \rangle}, \end{aligned}$$

provided θ_0 is small enough. The bounds on $C_{k,\eta,p,1}(s,t)$ and $C_{k,\eta,p,2}(s,t)$ are similar. \square

Let us now prove the claim (3.16). To estimate the time integral, we consider two cases: $p_1 \leq p/2$ and $p_2 \leq p/2$.

Case 1: $p_1 \leq p/2$. In this case, we note that

$$p_1^{-\delta} - p^{-\delta} \geq \theta_\delta p_1^{-\delta}$$

for some positive constant θ_δ . This and the estimate from Lemma 3.5 yield

$$C_{k,\eta,p,1}(s,t) \leq e^{-\theta_\delta p_1^{-\delta} \langle k_1,\eta_1,p_1,k_1s \rangle}.$$

Let us further bound the exponent. Using the standard Young's inequality $ab \lesssim a^q + b^{q'}$, with $q = 1/(1-\delta)$ and $q' = q/(q-1)$, we have

$$(3.17) \quad |a|^{1-\delta} = (ap_1^{-\delta})^{1-\delta} |p_1|^{\delta(1-\delta)} \leq C_\delta \left(|p_1|^{-\delta} |a| + |p_1|^{1-\delta} \right)$$

for some constant C_δ . Using this with $a = \langle k_1,\eta_1,k_1s \rangle$, we have

$$p_1^{-\delta} \langle k_1,\eta_1,p_1,k_1s \rangle \geq |p_1|^{1-\delta} + |p_1|^{-\delta} \langle k_1,\eta_1,k_1s \rangle \geq \frac{1}{C_\delta} \langle k_1,\eta_1,k_1s \rangle^{1-\delta}.$$

Clearly, we also have $p_1^{-\delta} \langle k_1, \eta_1, p_1, k_1 s \rangle \geq \langle p_1 \rangle^{1-\delta}$, recalling $p_1 \in \mathbb{N}^*$. This yields

$$(3.18) \quad p_1^{-\delta} \langle k_1, \eta_1, p_1, k_1 s \rangle \geq \frac{1}{2C_\delta} \langle k_1, \eta_1, p_1, k_1 s \rangle^{1-\delta}.$$

Therefore,

$$(3.19) \quad C_{k,\eta,p,1}(s, t) \leq e^{-\theta_\delta p_1^{-\delta} \langle k_1, \eta_1, p_1, k_1 s \rangle} \leq e^{-\theta'_\delta \langle k_1, \eta_1, p_1 \rangle^{1-\delta}} e^{-\theta'_\delta \langle k_1 s \rangle^{1-\delta}}$$

for some positive constant θ'_δ .

On the other hand, we simply bound

$$C_{k,\eta,p,0}(s, t) \leq e^{-\theta_0(s^{-\delta} - t^{-\delta}) \langle k, \eta, p, kt \rangle} \leq e^{-\theta_0(s^{-\delta} - t^{-\delta}) \langle t \rangle},$$

noting $k \neq 0$. We also bound $C_{k,\eta,p,2}(s, t) \leq 1$. Inserting these estimates into (3.16), we have

$$\begin{aligned} & \sum_{A_{k,\eta,p}} \int_0^t (t-s) C_{k,\eta,p,0}(s, t) C_{k,\eta,p,1}(s, t) C_{k,\eta,p,2}(s, t) \langle s \rangle^{-\sigma} ds \\ & \leq \sum_{A_{k,\eta,p}} e^{-\theta'_\delta \langle k_1, \eta_1, p_1 \rangle^{1-\delta}} \int_0^t (t-s) e^{-\theta_0(s^{-\delta} - t^{-\delta}) \langle t \rangle} e^{-\theta'_\delta \langle k_1 s \rangle^{1-\delta}} \langle s \rangle^{-\sigma} ds \\ & \lesssim \int_0^t (t-s) e^{-\theta_0(s^{-\delta} - t^{-\delta}) \langle t \rangle} e^{-\theta'_\delta \langle s \rangle^{1-\delta}} \langle s \rangle^{-\sigma} ds, \end{aligned}$$

in which we used $e^{-\theta'_\delta \langle k_1 s \rangle^{1-\delta}} \leq e^{-\theta'_\delta \langle s \rangle^{1-\delta}}$, since $k_1 \neq 0$. It remains to bound the time integral

$$\int_0^t (t-s) e^{-\theta_0(s^{-\delta} - t^{-\delta}) \langle t \rangle} e^{-\theta'_\delta \langle s \rangle^{1-\delta}} \langle s \rangle^{-\sigma} ds \leq C_0 \langle t \rangle^{-\sigma}.$$

Indeed, the estimate is clear for $s \geq t/2$, using the exponential term $e^{-\theta'_\delta \langle s \rangle^{1-\delta}}$ in the integrand. On the other hand, for $s \leq t/2$, we make use of the fact that $s^{-\delta} - t^{-\delta} \geq \theta_\delta t^{-\delta}$, yielding again an exponential decaying term

$$e^{-(s^{-\delta} - t^{-\delta}) \langle t \rangle} \leq e^{-\theta_\delta \langle t \rangle^{1-\delta}}.$$

The claim (3.16) follows.

Case 2: $p_2 \leq p/2$. Similarly, in this case, we use

$$p_2^{-\delta} - p^{-\delta} \geq \theta_\delta p_2^{-\delta}$$

for some positive constant θ_δ , which implies

$$C_{k,\eta,p,2}(s, t) \leq e^{-\theta_\delta p_2^{-\delta} \langle k_2, \eta_2, p_2, kt - k_1 s \rangle}.$$

Estimating the exponent exactly as done in (3.19), we thus obtain

$$(3.20) \quad C_{k,\eta,p,2}(s, t) \leq e^{-\theta_\delta \langle k_2, \eta_2, p_2 \rangle^{1-\delta}} e^{-\theta_\delta \langle kt - k_1 s \rangle^{1-\delta}}.$$

In the case when $|kt - k_1 s| \geq t/2$, the above yields an exponential decay term in (k_2, η_2, p_2, t) . The claim (3.16) thus follows.

It remains to consider the case when $|kt - k_1 s| \leq t/2$. It suffices to treat the case $k > 0$, the other being similar. In this case, we note that $k_1 > 0$ and $s \in [k_1^{-1}(k - 1/2)t, k_1^{-1}(k + 1/2)t]$. In particular, as $s < t$, we have $k_1 \geq k$. We treat two cases $k_1 = k$ and $k_1 > k$, separately.

Consider first the case when $k_1 = k \neq 0$. We then have

$$C_{k,\eta,p,2}(s, t) \leq e^{-\theta_\delta \langle k_2, \eta_2, p_2, k(t-s) \rangle^{1-\delta}} \leq e^{-\theta_\delta \langle k_2, \eta_2, p_2 \rangle^{1-\delta}} e^{-\theta_\delta \langle t-s \rangle^{1-\delta}},$$

while we simply bound $C_{k,\eta,p,0}(s, t) \leq 1$ and $C_{k,\eta,p,1}(s, t) \leq 1$. Let us now check the claim (3.16) for this case. We have

$$\begin{aligned} & \sum_{A_{k,\eta,p}} \int_0^t (t-s) C_{k,\eta,p,0}(s, t) C_{k,\eta,p,1}(s, t) C_{k,\eta,p,2}(s, t) \langle s \rangle^{-\sigma} ds \\ & \leq \sum_{A_{k,\eta,p}} e^{-\theta_\delta \langle k_2, \eta_2, p_2 \rangle^{1-\delta}} \int_0^t (t-s) e^{-\theta_\delta \langle t-s \rangle^{1-\delta}} \langle s \rangle^{-\sigma} ds \\ & \lesssim \int_0^t e^{-\frac{1}{2}\theta_\delta \langle t-s \rangle^{1-\delta}} \langle s \rangle^{-\sigma} ds, \end{aligned}$$

which is clearly bounded by $C_0 \langle t \rangle^{-\sigma}$.

Next, we consider the case when $k_1 > k > 0$. In this case, recalling (3.20), we have

$$C_{k,\eta,p,2}(s, t) \leq e^{-\theta_\delta \langle k_2, \eta_2, p_2 \rangle^{1-\delta}},$$

while we use the following bound on $C_{k,\eta,p,0}(s, t)$:

$$C_{k,\eta,p,0}(s, t) \leq e^{-\theta_0(s^{-\delta} - t^{-\delta}) \langle kt \rangle}.$$

Since $s \in [k_1^{-1}(k - 1/2)t, k_1^{-1}(k + 1/2)t]$ and $k_1 > k > 0$, we bound

$$s^{-\delta} - t^{-\delta} \geq \left(\frac{k_1^\delta}{(k + 1/2)^\delta} - 1 \right) \frac{1}{t^\delta} \geq \theta_\delta t^{-\delta} |k|^{-1}$$

for some positive constant θ_δ independent of k, k_1 . This proves

$$C_{k,\eta,p,0}(s, t) \leq e^{-\theta_\delta \langle t \rangle^{1-\delta}}.$$

We also bound $C_{k,\eta,p,1}(s, t) \leq 1$. Combining the estimates into (3.16), we thus have

$$\begin{aligned} & \sum_{A_{k,\eta,p}} \int_0^t (t-s) C_{k,\eta,p,0}(s, t) C_{k,\eta,p,1}(s, t) C_{k,\eta,p,2}(s, t) \langle s \rangle^{-\sigma} ds \\ & \leq \sum_{A_{k,\eta,p}} e^{-\theta_\delta \langle k_2, \eta_2, p_2 \rangle^{1-\delta}} \int_0^t (t-s) e^{-\theta_\delta \langle t \rangle^{1-\delta}} \langle s \rangle^{-\sigma} ds, \end{aligned}$$

which is again bounded by $C_0 \langle t \rangle^{-\sigma}$. The claim (3.16) follows. \square

3.6. Estimates on $\widehat{f}_{k,\eta,p}$. In this section, we prove the estimates (3.8) on $\widehat{f}_{k,\eta,p}$:

$$(3.21) \quad |\widehat{f}_{k,\eta,p}(t, \eta')| \leq C_0^p e^{-\lambda_p(t) \langle k, \eta, p, \eta' \rangle} \langle k \rangle^{-1},$$

assuming that the estimates (3.8) on \widehat{f}_{k,η,p_1} hold for all $p_1 \leq p-1$ and the estimates (3.9) on \widehat{E}_{k,η,p_1} hold for all $p_1 \leq p$. This will end the proof of Proposition 3.2. By Lemma 3.1, we have

$$\widehat{f}_{k,\eta,p}(t, \eta') = \widehat{S}_{k,\eta,p}(t, \eta') - \int_0^t \widehat{E}_{k,\eta,p}(s) \widehat{\partial_v \mu}(\eta' + L\eta - Kks) ds.$$

Using (3.9) and the analyticity assumption on $\mu(v)$, we get

$$\begin{aligned} & \int_0^t |\widehat{E}_{k,\eta,p}(s) \widehat{\partial_v \mu}(\eta' + L\eta - Kks)| ds \\ & \leq C_0^p \langle Kk \rangle^{-1} \int_0^t e^{-\lambda_p(s) \langle k, \eta, p, L\eta - Kks \rangle} e^{-\theta_0 |\eta' + L\eta - Kks|} \langle s \rangle^{-\sigma} ds. \\ & \leq C_0^p \langle Kk \rangle^{-1} e^{-\lambda_p(t) \langle k, \eta, p, \eta' \rangle} \int_0^t \langle s \rangle^{-\sigma} ds \\ & \leq C_0^p \langle k \rangle^{-1} e^{-\lambda_p(t) \langle k, \eta, p, \eta' \rangle} \end{aligned}$$

in which we used that $\lambda(t) \leq \lambda(s) \leq \theta_0$.

It remains to give estimates on $\widehat{S}_{k,\eta,p}(t, \eta')$. Recall from (3.14) that

$$\begin{aligned} |\widehat{S}_{k,\eta,p}(t, \eta')| & \leq C_0^p e^{-\lambda_p(t) \langle k, \eta, p, \eta' \rangle} \langle Kk \rangle^{-1} \sum_{A_{k,\eta,p}} \int_0^t |\eta' + L\eta - Kks| \\ & \quad \times C_{k,\eta,p,\eta',0}(s, t) C_{k,\eta,p,\eta',1}(s, t) C_{k,\eta,p,\eta',2}(s, t) \langle s \rangle^{-\sigma} ds, \end{aligned}$$

where the factors $C_{k,\eta,p,\eta',j}(s, t)$ are defined as in (3.13). Since $K \geq 1$ and $L \lesssim K$, we have

$$K^{-1} |\eta' + L\eta - Kks| \leq |\eta'| + |\eta| + |ks| \lesssim \langle s \rangle \langle k, \eta, p, \eta' \rangle.$$

The claim (3.21) will follow from the following estimates, which we will now prove:

$$(3.22) \quad \sum_{A_{k,\eta,p}} C_{k,\eta,p,\eta',1}(s, t) C_{k,\eta,p,\eta',2}(s, t) \lesssim 1,$$

and

$$(3.23) \quad \int_0^t e^{-(s^{-\delta} - t^{-\delta}) \langle k, \eta, p, \eta' \rangle} \langle k, \eta, p, \eta' \rangle \langle s \rangle^{-\sigma+1} ds \lesssim 1,$$

uniformly in k, η, p, η' , and t .

Let us start with (3.22). As argued above, we have, for $p_1 \leq p/2$,

$$C_{k,\eta,p,\eta',1}(s, t) \leq e^{-\theta_\delta p_1^{-\delta} \langle k_1, \eta_1, p_1, L\eta_1 - Kk_1s \rangle} \leq e^{-\theta_\delta \langle k_1, \eta_1, p_1 \rangle^{1-\delta}}.$$

Similarly, for $p_2 \leq p/2$, we have

$$C_{k,\eta,p,\eta',2}(s, t) \leq e^{-\theta_\delta p_2^{-\delta} \langle k_2, \eta_2, p_2, \eta' - L\eta_1 + Kk_1s \rangle} \leq e^{-\theta_\delta \langle k_2, \eta_2, p_2 \rangle^{1-\delta}}.$$

In both cases, the claim (3.22) follows in view of the definition of $A_{k,\eta,p}$. Finally, we check (3.23). We have $\langle s \rangle^{-\sigma+1} \lesssim |\frac{d}{ds} s^{-\delta}|$. Therefore,

$$\begin{aligned} & \int_0^t e^{-(s^{-\delta} - t^{-\delta}) \langle k, \eta, p, \eta' \rangle} \langle k, \eta, p, \eta' \rangle \langle s \rangle^{-\sigma+1} ds \\ & \lesssim \int_0^t e^{-(s^{-\delta} - t^{-\delta}) \langle k, \eta, p, \eta' \rangle} \langle k, \eta, p, \eta' \rangle \left| \frac{d}{ds} s^{-\delta} \right| ds, \end{aligned}$$

which is bounded. This ends the proof of Proposition 3.2.

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