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HYDRODYNAMIC LIMIT OF 3DIMENSIONAL EVOLUTIONARY BOLTZMANN EQUATION IN CONVEX DOMAINS*

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Abstract. This is the second half of our work on the hydrodynamic limit (the first half [L. Wu and Z. Ouyang, manuscript] focuses on the stationary problem). We consider the 3D evolutionary Boltzmann equation in convex domains with diffusive-reflection boundary condition. We rigorously derive the unsteady incompressible Navier–Stokes–Fourier system and justify the asymptotic convergence as the Knudsen number ε shrinks to zero. The proof is based on an innovative remainder estimate and an intricate analysis of boundary layers with geometric correction.

Key words. boundary layer, remainder estimates, geometric correction

AMS subject classifications. 82C40, 35Q20

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1. Introduction.

1.1. Problem presentation. We consider the evolutionary Boltzmann equation in a three-dimensional smooth convex domain $\Omega \ni x = (x_1, x_2, x_3)$ with velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. The density function $\mathfrak{F}^\varepsilon(t, x, v)$ satisfies

$$
\left\{
\begin{aligned}
&\varepsilon^2 \partial_t \mathfrak{F}^\varepsilon + \varepsilon v \cdot \nabla_x \mathfrak{F}^\varepsilon = Q[\mathfrak{F}^\varepsilon, \mathfrak{F}^\varepsilon] \ \text{ in } \ \mathbb{R}_+ \times \Omega \times \mathbb{R}^3,\\
&\mathfrak{F}^\varepsilon(0, x, v) = \mathfrak{F}^\varepsilon_0(x, v) \ \text{ in } \ \Omega \times \mathbb{R}^3,\\
&\mathfrak{F}^\varepsilon(t, x_0, v) = P^\varepsilon[\mathfrak{F}^\varepsilon](t, x_0, v) \ \text{ for } \ t \in \mathbb{R}_+, \ \ x_0 \in \partial\Omega, \text{ and } \ v \cdot n(x_0) < 0,
\end{aligned}
\right.
\tag{1.1}
$$

where $n(x_0)$ is the unit outward normal vector at x_0.

The Knudsen number ε characterizes the average distance a particle might travel between two collisions, and we assume $0 < \varepsilon << 1$. Intuitively, as $\varepsilon \to 0$, the collisions occur more and more frequently and the overall behaviors of this particle system get closer and closer to that of the fluids.

In this paper, we assume that Q is the hard-sphere collision operator (see [15, Chapter 1] and the following subsections), and in the diffusive-reflection boundary condition

$$
P^\varepsilon[\mathfrak{F}^\varepsilon](t, x_0, v) := \mu^\varepsilon_b(t, x_0, v) \int_{\mathfrak{u}\cdot n(x_0) > 0} \mathfrak{F}^\varepsilon(t, x_0, \mathfrak{u}) \, |\mathfrak{u} \cdot n(x_0)| \, \mathrm{d}\mathfrak{u}. \tag{1.2}
$$

It describes that the particles are absorbed by the boundary and then reemitted based on a boundary Maxwellian μ^ε_b.

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Boundary assumption: The boundary Maxwellian,

$$\mu_b^\varepsilon(t,x_0,v) := \frac{\rho_b^\varepsilon(t,x_0)}{2\pi\Big(\theta_b^\varepsilon(t,x_0)\Big)^2}\exp\left(-\frac{\left|v-u_b^\varepsilon(t,x_0)\right|^2}{2\theta_b^\varepsilon(t,x_0)}\right), \tag{1.3}$$

is a perturbation of the standard Maxwellian

$$\mu(v) = \frac{1}{2\pi}\exp\left(-\frac{|v|^2}{2}\right). \tag{1.4}$$

We assume that both μ_b^ε and μ satisfies the normalization condition

$$\int_{v\cdot n(x_0)>0}\mu_b^\varepsilon(t,x_0,v)\,|v\cdot n(x_0)|\,\mathrm{d}v = \int_{v\cdot n(x_0)>0}\mu(v)\,|v\cdot n(x_0)|\,\mathrm{d}v = 1. \tag{1.5}$$

In addition, we require that the particles are only reflected on $\partial\Omega$ without in-flow or out-flow, i.e.,

$$\int_{\mathbb{R}^3}\mu_b^\varepsilon(t,x_0,v)\big(v\cdot n(x_0)\big)\mathrm{d}v = \int_{\mathbb{R}^3}\mu(v)\big(v\cdot n(x_0)\big)\mathrm{d}v = 0. \tag{1.6}$$

We also assume that $(\rho_b^\varepsilon, u_b^\varepsilon, \theta_b^\varepsilon) \in C^3(\mathbb{R}^+\times\partial\Omega)$ can be expanded into a power series with respect to ε,

$$\begin{aligned}\rho_b^\varepsilon(t,x_0) &:= 1+\sum_{k=1}^\infty \varepsilon^k\rho_{b,k}(t,x_0), \quad u_b^\varepsilon(t,x_0) := 0+\sum_{k=1}^\infty\varepsilon^k u_{b,k}(t,x_0),\\ \theta_b^\varepsilon(t,x_0) &:= 1+\sum_{k=1}^\infty\varepsilon^k\theta_{b,k}(t,x_0),\end{aligned} \tag{1.7}$$

i.e., $\Big(\rho_b^\varepsilon, u_b^\varepsilon, \theta_b^\varepsilon\Big)$ is a perturbation of $(1,0,1)$. Naturally, we know $\rho_{b,k}, u_{b,k}, \theta_{b,k} \in C^3$. Hence, we may also expand the boundary Maxwellian μ_b^ε into a power series with respect to ε,

$$\mu_b^\varepsilon(t,x_0,v) = \mu(v)+\mu^{\frac12}(v)\left(\sum_{k=1}^\infty\varepsilon^k\mu_k(t,x_0,v)\right) \tag{1.8}$$

with $\mu_k \in C^3$. In particular, we have

$$\mu_1(t,x_0,v) := \mu^{\frac12}(v)\bigg(\rho_{b,1}(t,x_0)+u_{b,1}(t,x_0)\cdot v+\theta_{b,1}(t,x_0)\frac{|v|^2-3}{2}\bigg). \tag{1.9}$$

We further assume that

$$\left|\mathrm{e}^{K_0t}\langle v\rangle^\vartheta\,\mathrm{e}^{\varrho|v|^2}\frac{\mu_b^\varepsilon-\mu}{\mu^{\frac12}}\right| + \left|\mathrm{e}^{K_0t}\langle v\rangle^\vartheta\,\mathrm{e}^{\varrho|v|^2}\frac{\partial_t(\mu_b^\varepsilon-\mu)}{\mu^{\frac12}}\right| \le C_0\varepsilon \tag{1.10}$$

for any $0\le\varrho<\frac14$ and $3<\vartheta\le\vartheta_0$ with some given large ϑ_0. Here $C_0, K_0>0$ are constants and $C_0>0$ is sufficiently small. This indicates that the boundary Maxwellian μ_b^ε is very close to the global Maxwellian μ and its time derivative is also very small.

Based on (1.5), (1.6), and (1.8), we know

$$\int_{\mathbb{R}^3} \mu_k(t,x_0,v)\mu^{\frac{1}{2}}(v)\,|v\cdot n(x_0)|\,\mathrm{d}v = 0 \quad \text{for} \quad k\geq 1, \tag{1.11}$$
$$\int_{v\cdot n(x_0)\lesseqgtr 0} \mu_k(t,x_0,v)\mu^{\frac{1}{2}}(v)\,|v\cdot n(x_0)|\,\mathrm{d}v = 0 \quad \text{for} \quad k\geq 1.$$

Remark 1.1. In particular for $k=1$, we know $u_{b,1}\cdot n=0$. In fluid mechanics, this corresponds to a nonpenetration boundary condition.

Initial assumption: We assume that the initial data $\mathfrak{F}_0\geq 0$ are a perturbation of the standard Maxwellian

$$\mathfrak{F}_0^\varepsilon(x,v) := \mu(v) + \mu^{\frac{1}{2}}(v)f_0(x,v) := \mu(v) + \mu^{\frac{1}{2}}(v)\sum_{k=1}^\infty \varepsilon^k f_{0,k}(x,v). \tag{1.12}$$

We assume that $f_0\in C^3(\Omega)$ for any v and satisfies

$$\iint_{\Omega\times\mathbb{R}^3} \mu^{\frac{1}{2}}(v)f_0(x,v)\mathrm{d}v\mathrm{d}x = 0, \tag{1.13}$$

which means that $f_{0,k}\in C^3(\Omega)$ for any v and

$$\iint_{\Omega\times\mathbb{R}^3} \mu^{\frac{1}{2}}(v)f_{0,k}(x,v)\mathrm{d}v\mathrm{d}x = 0 \quad \text{for} \quad k\geq 1. \tag{1.14}$$

In particular, we assume that the initial data $f_{0,1}\in\mathcal{N}$, i.e.,

$$f_{0,1}(x,v) := \mu^{\frac{1}{2}}(v)\left(\rho_{0,1}(x) + u_{0,1}(x)\cdot v + \theta_{0,1}(x)\frac{|v|^2-3}{2}\right) \tag{1.15}$$

for some smooth function $(\rho_{0,1},u_{0,1},\theta_{0,1})$ satisfying the Boussinesq relation $\rho_{0,1}+\theta_{0,1}=\text{constant}$.

Remark 1.2. The assumption on $f_{0,1}$ is designed to simplify the discussion of the initial layer and highlight the boundary effects. For example, if $\mathfrak{F}_0^\varepsilon$ is a local Maxwellian like μ_b^ε in (1.3), then this requirement is naturally verified.

Also, we assume the smallness of initial perturbation

$$\left|(1+|v|^2)^{\frac{\vartheta}{2}}\mathrm{e}^{\varrho|v|^2}f_0\right| \leq C_0\varepsilon \tag{1.16}$$

for any $0\leq\varrho<\frac{1}{4}$ and $3<\vartheta\leq\vartheta_0$. Here the constant $C_0>0$ is sufficiently small.

Compatibility assumption: Also, the initial and boundary data satisfy the compatibility conditions at $t=0$ and $x_0\in\partial\Omega$:

(1.17)
$$\mu_k(0,x_0,v)=0,\quad \partial_t\mu_k(0,x_0,v)=0 \quad \text{for} \quad k\geq 1,$$
$$f_{0,k}(x_0,v)=\rho_{0,k}(x_0)\mu^{\frac{1}{2}},\quad \nabla_x f_{0,k}(x_0,v)=0,\quad \nabla_x^2 f_{0,k}(x_0,v)=0 \quad \text{for} \quad k\geq 1.$$

Remark 1.3. Roughly speaking, the compatibility conditions require that $\mu_b^\varepsilon\sim\mu$ and $\mathfrak{F}_0^\varepsilon\sim C\mu$ at $(0,x_0,v)$, i.e., the initial data and boundary data do not have severe variations at the intersection point. They are designed to simplify the interaction of the initial layer and boundary layer.

We may directly check that the solution $\mathfrak{F}^\varepsilon$ satisfies the mass conservation

$$\iint_{\Omega\times\mathbb{R}^3}\mathfrak{F}^\varepsilon(t,x,v)\mathrm{d}v\mathrm{d}x=\iint_{\Omega\times\mathbb{R}^3}\mathfrak{F}_0(x,v)\mathrm{d}v\mathrm{d}x=\iint_{\Omega\times\mathbb{R}^3}\mu(v)\mathrm{d}v\mathrm{d}x=\sqrt{2\pi}\left|\Omega\right|. \tag{1.18}$$

We intend to study the behavior of $\mathfrak{F}^\varepsilon$ as $\varepsilon\to 0$.

1.2. Perturbation equation. We rewrite the solution $\mathfrak{F}^\varepsilon$ as a perturbation of the standard Maxwellian

$$\mathfrak{F}^\varepsilon(t,x,v)=\mu(v)+\mu^{\frac{1}{2}}(v)f^\varepsilon(t,x,v). \tag{1.19}$$

(1.18) implies that f^ε satisfies the conservation law

$$\iint_{\Omega\times\mathbb{R}^3}f^\varepsilon(t,x,v)\mu^{\frac{1}{2}}(v)\mathrm{d}v\mathrm{d}x=0, \tag{1.20}$$

and the equation

$$\begin{cases}\varepsilon^2\partial_t f^\varepsilon+\varepsilon v\cdot\nabla_x f^\varepsilon+\mathcal{L}[f^\varepsilon]=\Gamma[f^\varepsilon,f^\varepsilon] \ \text{ in } \ \mathbb{R}_+\times\Omega\times\mathbb{R}^3,\\ f^\varepsilon(0,x,v)=f_0(x,v) \ \text{ in } \ \Omega\times\mathbb{R}^3,\\ f^\varepsilon(t,x_0,v)=\mathcal{P}^\varepsilon[f^\varepsilon](t,x_0,v) \ \text{ for } \ t\in\mathbb{R}_+,\ \ x_0\in\partial\Omega, \text{and} \ \ v\cdot n(x_0)<0,\end{cases} \tag{1.21}$$

where

$$\mathcal{L}[f^\varepsilon]:=-2\mu^{-\frac{1}{2}}Q\Big[\mu,\mu^{\frac{1}{2}}f^\varepsilon\Big],\quad \Gamma[f^\varepsilon,f^\varepsilon]:=\mu^{-\frac{1}{2}}Q\Big[\mu^{\frac{1}{2}}f^\varepsilon,\mu^{\frac{1}{2}}f^\varepsilon\Big], \tag{1.22}$$

and

$$\begin{aligned}\mathcal{P}^\varepsilon[f^\varepsilon](t,x_0,v):={}&\mu_b^\varepsilon(t,x_0,v)\mu^{-\frac{1}{2}}(v)\int_{\mathfrak{u}\cdot n(x_0)>0}\mu^{\frac{1}{2}}(\mathfrak{u})f^\varepsilon(t,x_0,\mathfrak{u})\left|\mathfrak{u}\cdot n(x_0)\right|\mathrm{d}\mathfrak{u}\\ &+\mu^{-\frac{1}{2}}(v)\Big(\mu_b^\varepsilon(t,x_0,v)-\mu(v)\Big).\end{aligned} \tag{1.23}$$

Hence, in order to study $\mathfrak{F}^\varepsilon$, it suffices to consider f^ε.

1.3. Linearized Boltzmann operator. To clarify, we specify the hard-sphere collision operator Q in (1.1) and (1.21),

$$Q[F,G]:=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}q(\omega,|\mathfrak{u}-v|)\Big(F(\mathfrak{u}_*)G(v_*)-F(\mathfrak{u})G(v)\Big)\mathrm{d}\omega\mathrm{d}\mathfrak{u} \tag{1.24}$$

with

$$\mathfrak{u}_*:=\mathfrak{u}+\omega\big((v-\mathfrak{u})\cdot\omega\big),\qquad v_*:=v-\omega\big((v-\mathfrak{u})\cdot\omega\big), \tag{1.25}$$

and the hard-sphere collision kernel

$$q(\omega,|\mathfrak{u}-v|):=q_0\left|\omega\cdot(v-\mathfrak{u})\right| \tag{1.26}$$

for a positive constant q_0.

Based on [15, Chapter 3], the linearized Boltzmann operator $\mathcal{L}$ is

$$\mathcal{L}[f]=-2\mu^{-\frac{1}{2}}Q\big[\mu,\mu^{\frac{1}{2}}f\big]:=\nu(v)f-K[f], \tag{1.27}$$

where

(1.28)
$$\nu(v)=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}q(\omega,|\mathfrak{u}-v|)\mu(\mathfrak{u})\mathrm{d}\omega\mathrm{d}\mathfrak{u}=\pi^2q_0\bigg(\Big(2\,|v|+\frac{1}{|v|}\Big)\int_0^{|v|}\mathrm{e}^{-z^2}\mathrm{d}z+\mathrm{e}^{-|v|^2}\bigg),$$

(1.29)
$$K[f](v)=\ K_2[f](v)-K_1[f](v)=\int_{\mathbb{R}^3}k(\mathfrak{u},v)f(\mathfrak{u})\mathrm{d}\mathfrak{u},$$

(1.30)
$$K_1[f](v)=\ \mu^{\frac{1}{2}}(v)\int_{\mathbb{R}^3}\int_{\mathbb{S}^1}q(\omega,|\mathfrak{u}-v|)\mu^{\frac{1}{2}}(\mathfrak{u})f(\mathfrak{u})\mathrm{d}\omega\mathrm{d}\mathfrak{u}=\int_{\mathbb{R}^3}k_1(\mathfrak{u},v)f(\mathfrak{u})\mathrm{d}\mathfrak{u},$$

(1.31)
$$\begin{aligned}K_2[f](v)&=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}q(\omega,|\mathfrak{u}-v|)\mu^{\frac{1}{2}}(\mathfrak{u})\Big(\mu^{\frac{1}{2}}(v_*)f(\mathfrak{u}_*)+\mu^{\frac{1}{2}}(\mathfrak{u}_*)f(v_*)\Big)\mathrm{d}\omega\mathrm{d}\mathfrak{u}\\&=\int_{\mathbb{R}^3}k_2(\mathfrak{u},v)f(\mathfrak{u})\mathrm{d}\mathfrak{u}\end{aligned}$$

for some kernels

$$k(\mathfrak{u},v)=k_2(\mathfrak{u},v)-k_1(\mathfrak{u},v),\tag{1.32}$$

$$k_1(\mathfrak{u},v)=\pi q_0\,|\mathfrak{u}-v|\exp\bigg(-\frac{1}{2}\,|\mathfrak{u}|^2-\frac{1}{2}\,|v|^2\bigg),\tag{1.33}$$

$$k_2(\mathfrak{u},v)=\frac{2\pi q_0}{|\mathfrak{u}-v|}\exp\bigg(-\frac{1}{4}\,|\mathfrak{u}-v|^2-\frac{1}{4}\frac{(|\mathfrak{u}|^2-|v|^2)^2}{|\mathfrak{u}-v|^2}\bigg).\tag{1.34}$$

In particular, $\mathcal{L}$ is self-adjoint in $L^2(\mathbb{R}^3)$ and the null space $\mathcal{N}$ is a five-dimensional space spanned by the orthonormal basis

$$\mu^{\frac{1}{2}}\bigg\{1,v,\frac{|v|^2-3}{2}\bigg\}.\tag{1.35}$$

We denote $\mathcal{N}^\perp$ as the orthogonal complement of $\mathcal{N}$ in $L^2(\mathbb{R}^3)$.

1.4. Main result. Let $\langle\cdot,\cdot\rangle$ be the standard L^2 inner product for $v\in\mathbb{R}^3$. Define the L^p and L^∞ norms in $\mathbb{R}^3$:

$$|f(t,x)|_p:=\bigg(\int_{\mathbb{R}^3}|f(t,x,v)|^p\,\mathrm{d}v\bigg)^{\frac{1}{p}},\quad |f(t,x)|_\infty:=\operatorname*{ess\,sup}_{v\in\mathbb{R}^3}|f(t,x,v)|\,.\tag{1.36}$$

Furthermore, we define the L^p and L^∞ norms in $\Omega\times\mathbb{R}^3$:

(1.37)
$$\|f(t)\|_p:=\bigg(\iint_{\Omega\times\mathbb{R}^3}|f(t,x,v)|^p\,\mathrm{d}v\mathrm{d}x\bigg)^{\frac{1}{p}},\quad \|f(t)\|_\infty:=\operatorname*{ess\,sup}_{(x,v)\in\Omega\times\mathbb{R}^3}|f(t,x,v)|\,.$$

Moreover, we define the L^p and L^∞ norms in $\mathbb{R}_+\times\Omega\times\mathbb{R}^3$:

(1.38)
$$|||f|||_p:=\bigg(\int_{\mathbb{R}_+}\iint_{\Omega\times\mathbb{R}^3}|f(x,v)|^p\,\mathrm{d}v\mathrm{d}x\bigg)^{\frac{1}{p}},\quad |||f|||_\infty:=\operatorname*{ess\,sup}_{(t,x,v)\in\mathbb{R}_+\times\Omega\times\mathbb{R}^3}|f(t,x,v)|\,.$$

Define the weighted L^2 norms

$$|f(t,x)|_{\nu} := \left|\nu^{\frac{1}{2}} f(t,x)\right|_2, \quad \|f(t)\|_{\nu} := \left\|\nu^{\frac{1}{2}} f(t)\right\|_2, \quad |||f|||_{\nu} := \left|\left|\left|\nu^{\frac{1}{2}} f\right|\right|\right|_2. \tag{1.39}$$

Denote the Japanese bracket by

$$\langle v\rangle := \left(1+|v|^2\right)^{\frac{1}{2}}. \tag{1.40}$$

Define the weighted L^∞ norm for $\varrho,\vartheta\geq 0$:

$$\begin{aligned} |f(t,x)|_{\infty,\vartheta,\varrho} &:= \operatorname*{ess\,sup}_{v\in\mathbb{R}^3}\left(\langle v\rangle^{\vartheta}\mathrm{e}^{\varrho|v|^2}|f(t,x,v)|\right),\\ \|f(t)\|_{\infty,\vartheta,\varrho} &:= \operatorname*{ess\,sup}_{(x,v)\in\Omega\times\mathbb{R}^3}\left(\langle v\rangle^{\vartheta}\mathrm{e}^{\varrho|v|^2}|f(t,x,v)|\right),\\ |||f|||_{\infty,\vartheta,\varrho} &:= \operatorname*{ess\,sup}_{(t,x,v)\in\mathbb{R}_+\times\Omega\times\mathbb{R}^3}\left(\langle v\rangle^{\vartheta}\mathrm{e}^{\varrho|v|^2}|f(t,x,v)|\right). \end{aligned} \tag{1.41}$$

In (1.1) and (1.21), based on the flow direction, we can divide the boundary $\gamma := \{(x_0,v):\ x_0\in\partial\Omega, v\in\mathbb{R}^3\}$ into the in-flow boundary γ_-, the out-flow boundary γ_+, and the grazing set γ_0:

$$\gamma_- := \{(x_0,v):\ x_0\in\partial\Omega,\ v\cdot n(x_0)<0\}, \tag{1.42}$$

$$\gamma_+ := \{(x_0,v):\ x_0\in\partial\Omega,\ v\cdot n(x_0)>0\}, \tag{1.43}$$

$$\gamma_0 := \{(x_0,v):\ x_0\in\partial\Omega,\ v\cdot n(x_0)=0\}. \tag{1.44}$$

It is easy to see $\gamma=\gamma_+\cup\gamma_-\cup\gamma_0$. In particular, the boundary condition is only given on $\mathbb{R}_+\times\gamma_-$.

Define $\mathrm{d}\gamma = |v\cdot n|\,\mathrm{d}\varpi\mathrm{d}v$ on γ for the surface measure ϖ. Define the L^p and L^∞ norms on the boundary:

$$\|f(t)\|_{\gamma,p} := \left(\iint_{\gamma}|f(t,x,v)|^p\,\mathrm{d}\gamma\right)^{\frac{1}{p}}, \quad \|f(t)\|_{\gamma,\infty} := \operatorname*{ess\,sup}_{(x,v)\in\gamma}|f(t,x,v)|. \tag{1.45}$$

Define the L^p and L^∞ norms on the boundary with time:

$$|||f|||_{\gamma,p} := \left(\int_{\mathbb{R}_+}\iint_{\gamma}|f(t,x,v)|^p\,\mathrm{d}\gamma\right)^{\frac{1}{p}}, \quad |||f|||_{\gamma,\infty} := \operatorname*{ess\,sup}_{(t,x,v)\in\mathbb{R}_+\times\gamma}|f(x,v)|. \tag{1.46}$$

Also, define the weighted L^∞ norm for $\varrho,\vartheta\geq 0$:

$$\begin{aligned} \|f(t)\|_{\gamma,\infty,\varrho,\vartheta} &:= \operatorname*{ess\,sup}_{(x,v)\in\gamma}\left(\langle v\rangle^{\vartheta}\mathrm{e}^{\varrho|v|^2}|f(t,x,v)|\right),\\ |||f|||_{\gamma,\infty,\varrho,\vartheta} &:= \operatorname*{ess\,sup}_{(t,x,v)\in\mathbb{R}_+\times\gamma}\left(\langle v\rangle^{\vartheta}\mathrm{e}^{\varrho|v|^2}|f(t,x,v)|\right). \end{aligned} \tag{1.47}$$

The similar notation also applies to $\gamma_\pm$. In all the above notation, we can replace $\mathbb{R}_+$ by $[0,t]$ or even $[s,t]$, and it can be understood from the context without confusion.

Now we are ready to state our main theorem.

THEOREM 1.1. *For given μ_b^ε and $\mathfrak{F}_0^\varepsilon$ satisfying the assumptions in section* 1.1, *there exists a unique positive solution $\mathfrak{F}^\varepsilon = \mu + \mu^{\frac{1}{2}} f^\varepsilon \geq 0$ to the evolutionary Boltzmann equation* (1.1). *In particular, f^ε satisfies* (1.21) *with* (1.20), *and fulfils that for $0 \leq \varrho < \frac{1}{4}$ and $3 < \vartheta \leq \vartheta_0$, there exists $K > 0$, such that*

$$\left\|\left|\mathrm{e}^{Kt}\Big(f^\varepsilon - \varepsilon F\Big)\right|\right\|_{\infty,\vartheta,\varrho} \lesssim_\delta \varepsilon^{\frac{4}{3}-\delta} \tag{1.48}$$

for any $0 < \delta << 1$, where

$$F = \mu^{\frac{1}{2}} \left(\rho + u\cdot v + \theta \frac{|v|^2 - 3}{2} \right) \tag{1.49}$$

in which (ρ, u, θ) satisfies the unsteady Navier–Stokes–Fourier system

$$\begin{cases} \partial_t u + u\cdot\nabla_x u - \gamma_1 \Delta_x u + \nabla_x p = 0, \\ \nabla_x \cdot u = 0, \\ \partial_t \theta + u\cdot\nabla_x\theta - \gamma_2 \Delta_x \theta = 0 \end{cases} \tag{1.50}$$

with initial and boundary data

$$\rho(0,x) = \rho_{0,1}, \quad u(0,x) = u_{0,1}, \quad \theta(0,x) = \theta_{0,1}, \tag{1.51}$$

$$\rho(t,x_0) = \rho_{b,1}(t,x_0) + M(t,x_0), \quad u(t,x_0) = u_{b,1}(t,x_0), \quad \theta(t,x_0) = \theta_{b,1}(t,x_0). \tag{1.52}$$

Here $\gamma_1 > 0$ and $\gamma_2 > 0$ are some constants, $M(t,x_0)$ is a function chosen such that the Boussinesq relation

$$\nabla_x(\rho + \theta) = 0, \tag{1.53}$$

and the conservation law (1.20) *hold for all time t.*

Remark 1.4. The Boussinesq relation implies that $\rho(t,x) + \theta(t,x) = C(t)$ for some time-dependent function $C(t)$ in the whole domain Ω. However, at each t, the boundary data $\rho_{b,1}(t,x_0)$ and $\theta_{b,1}(t,x_0)$ do not necessarily have the same sum at different x_0. Hence, $M(t,x_0)$ is designed to fill this gap. Note that we are still free to choose the $C(t)$ (i.e., M still has one dimension of freedom at each t) and it is eventually determined by the conservation law (1.20).

Remark 1.5. From the above theorem, we know $f^\varepsilon \sim \varepsilon F$ is of order $O(\varepsilon)$. The difference $f^\varepsilon - \varepsilon F = o(\varepsilon)$ as $\varepsilon \to 0$.

Remark 1.6. The case $\rho_{b,1}(t,x_0) = 0$, $u_{b,1}(t,x_0) = 0$, and $\theta_{b,1}(t,x_0) \neq 0$ is called the nonisothermal model, which represents a system that only has heat transfer through the boundary but has no particle exchange and no work done between the environment and the system. Based on the above theorem, the hydrodynamic limit is an unsteady Navier–Stokes–Fourier system with nonslip boundary condition. This provides a rigorous derivation of this important fluid model.

Remark 1.7. In the smallness assumption (1.10), if $K_0 = 0$, then the main theorem still holds with $K = 0$. Exponential decay in time is not necessary.

Remark 1.8. Our proof of the main theorem relies on the assumptions in section 1.1. To remove these technical requirements will be a main topic of our future research.

1.5. History and motivation.

1.5.1. Previous results. The hydrodynamic limit is central to connecting the kinetic theory and fluid mechanics. It provides rigorous derivation of fluid equations (like Euler equations or Navier–Stokes equations, etc.) from the kinetic equations (like Boltzmann equations, Landau equations, etc.). As an integrated step to tackle the well-known Hilbert's sixth problem, since the early 20th century, these types of problems have been extensively studied in many different settings: stationary or evolutionary, linear or nonlinear, strong solution or weak solution, etc.

The early result [26] by Hilbert dates back to 1916, using the so-called Hilbert's expansion, i.e., an expansion of the density function $\mathfrak{F}^{\varepsilon}$ as a power series of the Knudsen number ε.

The general theory of initial-boundary-value problems for hydrodynamic limits was first developed by Grad [18], and then extended by Sone [34, 35, 36] and Sone and Aoki [39], for both the evolutionary and stationary equations. The classical books by Sone [37, 38] provide a comprehensive summary of previous results and give a complete analysis of such approaches. However, the results in [37, 38] are only formal and lack rigorous justifications.

On a large time scale $\sim \varepsilon^{-1}$, the diffusion effects dominate and the formal derivation reveals that the Boltzmann solution is close to that of the incompressible Navier–Stokes–Fourier system. A lot of works for $\mathbb{R}^n$ or $\mathbb{T}^n$ domains have been presented, e.g., Golse and Saint-Raymond [17], Bardos, Golse, and Levermore [2, 3, 4], Bardos and Ukai [5], Briant [6], Briant, Merino-Aceituno, and Mouhot [7], Gallagher and Tristani [14], Guo [19], for either smooth solutions or renormalized solutions.

For other time scales and models, due to the huge number, it is almost impossible to give a complete list of all the related publications. Reader may refer to Golse and Saint-Raymond [16], Saint-Raymond [32], Masmoudi and Saint-Raymond [31], Masi, Esposito, and Lebowitz [30], Guo [19], Guo and Jang [21], Guo, Jang, and Jiang [22, 23] and the references therein. It is also worth noting that the book by Saint-Raymond [33] and the references therein provide a nice summary of the progress.

Due to its physical significance, the study of the kinetic equation in bounded domains has attracted a lot of attention recently. In the case $\varepsilon = 1$ and $\mu_b^{\varepsilon} = \mu$, Guo [20] justifies the global well-posedness and decay of the Boltzmann equation under various boundary conditions. In particular, for specular boundary, Guo [20] requires analyticity of the domain, which is removed by Kim and Lee [28, 27]. Esposito et al. [10] handles the case when $\varepsilon = 1$ and μ_b^{ε} is an $O(1)$ small perturbation of μ. The idea is also adapted to treat the Vlasov–Poisson–Boltzmann system in Cao, Kim, and Lee [8].

For the hydrodynamic limits $\varepsilon \to 0$ in bounded domains, the geometry of the domain will play a key role in the analysis. We refer to Esposito, Lebowitz, and Marra [13], Esposito et al. [11], Esposito, Guo, and Marra [12], Arkeryd et al. [1], Wu [42, 44] and the references therein.

Note that the boundary layer plays a significant role in proving the asymptotic convergence in the L^{∞} sense. If instead we consider L^p convergence for $1 \leq p < \infty$ which is technically easier, then the boundary layer is of order $\varepsilon^{\frac{1}{p}}$ due to rescaling, which is negligible compared with the interior solution as $\varepsilon \to 0$. As far as we are aware of, at this stage the best result of hydrodynamic limits for the three-dimensional (3D) evolutionary problem in bounded domains is [11], which justifies the L^2 convergence without boundary layer analysis. As for the stationary problem, the best result is our paper [46], which justifies the L^{∞} convergence with a detailed discussion of boundary layers.

In this paper, we will fill the last piece and focus on the 3D evolutionary problem with L^∞ convergence.

1.5.2. Asymptotic analysis. For the evolutionary Boltzmann equation where the state of gas is close to a uniform state at rest, the expansion of the perturbation $f^\varepsilon = O(\varepsilon)$ consists of three parts: the interior solution $f^\varepsilon_{\rm in}$ which is based on a hierarchy of linearized Boltzmann equations and satisfies a steady Navier–Stokes–Fourier system, the initial layer $f^\varepsilon_{\rm il}$ which is based on a nonlocal ODE and decays rapidly when it is away from $t = 0$, and the boundary layer $f^\varepsilon_{\rm bl}$ which is based on a half-space kinetic equation and decays rapidly when it is away from the boundary.

The justification of hydrodynamic limits usually involves two steps: well-posedness of expansion and remainder estimates:

- Step 1: Expanding $f^\varepsilon_{\rm in} = \sum_{k=1}^\infty \varepsilon^k F_k$, $f^\varepsilon_{\rm il} = \sum_{k=1}^\infty \varepsilon^k \mathcal{F}_k$, and $f^\varepsilon_{\rm bl} = \sum_{k=1}^\infty \varepsilon^k \mathscr{F}_k$ as power series of ε and proving the coefficients F_k, $\mathcal{F}_k$, and $\mathscr{F}_k$ are well-defined. This is doable by inserting the above expansion ansatz into the Boltzmann equation and comparing the order of ε to get a hierarchy of equations for F_k, $\mathcal{F}_k$, and $\mathscr{F}_k$. Traditionally, the estimates of interior solutions F_k and initial layers $\mathcal{F}_k$ are relatively straightforward. On the other hand, boundary layers $\mathscr{F}_k$ satisfy one-dimensional half-space problems which lose some key structures of the original equations. The well-posedness of boundary layer equations is sometimes extremely difficult to prove and it is possible that they are actually ill-posed (e.g., certain types of Prandtl layers).
- Step 2: Proving that $R = f^\varepsilon - \varepsilon F_1 - \varepsilon\mathcal{F}_1 - \varepsilon\mathscr{F}_1 = o(\varepsilon)$ as $\varepsilon \to 0$. Ideally, this should be done just by expanding to the leading-order level F_1, $\mathcal{F}_1$, and $\mathscr{F}_1$. However, in singular perturbation problems, the estimates of the remainder R usually involve negative powers of ε, which requires expansion to higher-order terms F_N and $\mathscr{F}_N$ for $N \geq 2$ in order to provide sufficient power of ε. In other words, we define $R = f^\varepsilon - \sum_{k=1}^N \varepsilon^k F_k - \sum_{k=1}^N \varepsilon^k \mathcal{F}_k - \sum_{k=1}^N \varepsilon^k \mathscr{F}_k$ for $N \geq 2$ instead of $R = f^\varepsilon - \varepsilon F_1 - \varepsilon\mathcal{F}_1 - \varepsilon\mathscr{F}_1$ to get a better estimate of R.

1.6. Methodology. The geometric effects in boundary layer analysis have been observed for a long time (see [13]). Inspired by [9], a new formulation of boundary layer based on the Milne problem with geometric correction was proposed in [45] to study a simple kinetic model—neutron transport equations, in a two-dimensional (2D) plate domain. The key component of the proof is the L^∞ well-posedness and decay of the boundary layer equation. Furthermore, through a careful discussion of the weighted $W^{1,\infty}$ regularity and quasi-$W^{2,\infty}$ regularity, such results were extended in [24, 25, 43, 41] to treat more general 2D/3D domains and boundary conditions.

The neutron transport equation is a linear equation with homogeneous collision kernels. In contrast, the Boltzmann equation poses more technical complications due to the higher dimension of null space and more singular collision kernels. As far as we are aware of, the best result for the Boltzmann boundary layer is the weighted $W^{1,\infty}$ regularity obtained in our paper [46].

1.6.1. Upshots of the paper. While the sister paper [46] contains a complete analysis of the well-posedness and regularity of the boundary layer equation, in this paper, we will focus on the remainder estimates. This is equivalent to considering the linearized Boltzmann equation with diffusive boundary. The first such result was proposed in [20] by a novel $L^2 - L^\infty$ framework, and the argument is improved in [10]. Such a method is further adapted to treat hydrodynamic limits to form the $L^2-L^6-L^\infty$ framework in [11] for the Boltzmann equation and [24, 41] for the neutron transport equation. All of these frameworks have both stationary and evolutionary versions.

The major difficulty of our paper lies in the fact that the remainder estimates of the evolutionary problem are much worse compared to the stationary problem in [44, 46]. Neither the $L^2 - L^\infty$ nor $L^2 - L^6 - L^\infty$ framework provides sufficient control of the remainder R_e.

To be more specific, the validity of our main theorem requires $\varepsilon^{1+\delta}$ convergence for the remainder R_e. The stationary remainder estimate reads

$$\|R_s\|_{L^\infty(\Omega\times\mathbb{R}^3)} \lesssim \varepsilon^{-\frac{5}{2}} \|S_s\|_{L^{\frac{6}{5}}(\Omega\times\mathbb{R}^3)} + \text{good terms}, \tag{1.54}$$

where R_s is the stationary remainder and $S_s = (v\cdot\nabla_x + \varepsilon^{-1}\mathcal{L})[R_s] \sim \varepsilon^3$. Also, the main contribution of S_s—the boundary layer term contains a rescaling $\eta \sim \varepsilon^{-1}\mathfrak{N}$ for normal distance $\mathfrak{N}$ which provides additional $\varepsilon^{\frac{5}{6}}$ under $L^{\frac{6}{5}}$ norm. Eventually, we get $\|R_s\|_{L^\infty} \lesssim \varepsilon^{-\frac{5}{2}} \times \varepsilon^{3+\frac{5}{6}} = \varepsilon^{\frac{4}{3}}$ convergence. However, for the evolutionary problem, the $L^2 - L^\infty$ framework (as in [42]) justifies

$$\|R_e\|_{L^\infty(\mathbb{R}^+\times\Omega\times\mathbb{R}^3)} \lesssim \varepsilon^{-\frac{7}{2}} \|S_e(t)\|_{L^2(\Omega\times\mathbb{R}^3)} + \text{good terms}, \tag{1.55}$$

where $S_e(t) = (\varepsilon\partial_t + v\cdot\nabla_x + \varepsilon^{-1}\mathcal{L})[R_e(t)] \sim \varepsilon^3$ and we only obtain an extra $\varepsilon^{\frac{1}{2}}$ from the boundary layer rescaling under the L^2 norm. Hence, we have $\|R_e\|_{L^\infty} \lesssim \varepsilon^{-\frac{7}{2}} \times \varepsilon^{3+\frac{1}{2}} = \varepsilon^0 \sim 1$ which is far from closing the proof. On the other hand, the $L^2 - L^6 - L^\infty$ framework (as in [11]) justifies

$$\|R_e\|_{L^\infty(\Omega\times\mathbb{R}^3)} \lesssim \varepsilon^{-\frac{5}{2}} \|S_e\|_{L^2(\Omega\times\mathbb{R}^3)} + \text{good terms}. \tag{1.56}$$

Hence, we have $\|R_e\|_{L^\infty} \lesssim \varepsilon^{-\frac{5}{2}} \times \varepsilon^{3+\frac{1}{2}} = \varepsilon$ which is still not enough. Unfortunately, the strategy in [41] for the neutron transport equation also will not work for our case. In [41], the proof requires the coercivity bound

$$\left\langle \mathcal{L}[R_e], R_e^5 \right\rangle \gtrsim \|(\mathbb{I}-\mathbb{P})[R_e]\|_{L^6}^6, \tag{1.57}$$

which is absent in the Boltzmann equation.

In summary, we have to develop new ideas to tackle this difficulty. Our strategy mainly includes two steps, and we need to make significant modifications and improvements in both the remainder estimates and boundary layer construction.

First, we introduce a modified $L^2 - L^6 - L^\infty$ boostrap framework. This is rooted in the nonlinear energy method and we use an intricate energy-dissipation structure to bound both the instantaneous and accumulative R_e with mutual dependence. In detail, we justify L^2 bounds of R_e and $\partial_t R_e$ with a nonstandard energy method and prove the L^6 bound of $R_e(t)$ with a fresh kernel estimate with interpolation argument. To be more specific, we show that

$$\|R_e\|_{L^\infty(\mathbb{R}^+\times\Omega\times\mathbb{R}^3)} \lesssim \varepsilon^{-\frac{5}{2}} \Big(\|S_e(t)\|_{L^{\frac{6}{5}}(\Omega\times\mathbb{R}^3)} + \|\partial_t S_e\|_{L^2(\mathbb{R}^+\times\Omega\times\mathbb{R}^3)} \Big) + \text{good terms}. \tag{1.58}$$

Next, our central idea is to smartly utilize the "good" stationary remainder estimates. We design the highest-order boundary layer in a rather unusual way. Specifically, we reformulate the ε-Milne problem with geometric correction, such that it recovers the stationary equation as in [46]. This allows us to use the stationary remainder estimate (1.54) to control boundary layers, leaving several nontrivial remainder terms which can be delicately handled. In this fashion, we get

$$\|S_e(t)\|_{L^{\frac{6}{5}}(\Omega\times\mathbb{R}^3)} + \|\partial_t S_e\|_{L^2(\mathbb{R}^+\times\Omega\times\mathbb{R}^3)} \lesssim \varepsilon^{3+\frac{5}{6}}. \tag{1.59}$$

Hence, we get the same $\varepsilon^{\frac{4}{3}}$ convergence as in the stationary case.

1.6.2. Notation and convention. Throughout this paper, $C > 0$ denotes a constant that only depends on the domain Ω, but does not depend on the data or ε. It is referred to as universal and can change from one inequality to another. When we write $C(z)$, it means a certain positive constant depending on the quantity z. We write $a \lesssim b$ to denote $a \leq Cb$.

This paper is organized as follows: in section 2, we perform the asymptotic expansion and matching procedure; in section 3, we record the main theorems proved in [46] on the well-posedness and regularity of the boundary layer equation, i.e., the ε-Milne problem with geometric correction; in sections 4 and 5, we study the remainder estimates for both the stationary and evolutionary equations; finally, in section 6, we prove the main theorem.

2. Asymptotic expansion.

2.1. Interior expansion. We define the interior expansion

$$f^{\varepsilon}_{\text{in}}(t,x,v) = \sum_{k=1}^{3} \varepsilon^k F_k(t,x,v). \tag{2.1}$$

Plugging it into (1.21) and comparing the orders of ε, we obtain

$$\mathcal{L}[F_1] = 0, \tag{2.2}$$
$$\mathcal{L}[F_2] = -v\cdot\nabla_x F_1 + \Gamma[F_1, F_1], \tag{2.3}$$
$$\mathcal{L}[F_3] = -\partial_t F_1 - v\cdot\nabla_x F_2 + 2\Gamma[F_1, F_2]. \tag{2.4}$$

The analysis of F_k solvability is standard and well known. Note that the null space $\mathcal{N}$ of the operator $\mathcal{L}$ is spanned by

$$\mu^{\frac{1}{2}}\left\{1, v_1, v_2, v_3, \frac{|v|^2-3}{2}\right\} = \{\varphi_0, \varphi_1, \varphi_2, \varphi_3, \varphi_4\}. \tag{2.5}$$

Then $\mathcal{L}[f] = S$ is solvable if and only if $S \in \mathcal{N}^{\perp}$, the orthogonal complement of $\mathcal{N}$ in $L^2(\mathbb{R}^3)$. In the spirit of [37, Chapter 4] and [38, Chapter 3], similar to stationary problems in [46, section 2.1], each F_k consists of three parts:

$$F_k(t,x,v) := A_k(t,x,v) + B_k(t,x,v) + C_k(t,x,v). \tag{2.6}$$

- Principal contribution $A_k := \sum_{i=0}^{4} A_{k,i}\varphi_i \in \mathcal{N}$, where the coefficients $A_{k,i}$ must be determined at each order k independently.
- Connecting contribution $B_k := \sum_{i=0}^{4} B_{k,i}\varphi_i \in \mathcal{N}$, where the coefficients $B_{k,i}$ depend on A_s for $1 \leq s \leq k-1$. In other words, B_k is accumulative information from previous orders and thus is not independent. This term is present due to the nonlinearity in Γ.

- Orthogonal contribution $C_k \in \mathcal{N}^{\perp}$ satisfying

$$\mathcal{L}[C_k] = \partial_t F_{k-2} - v \cdot \nabla_x F_{k-1} + \sum_{i=1}^{k-1} \Gamma[F_i, F_{k-i}], \tag{2.7}$$

  which can be uniquely determined. Similar to B_k, here C_k is also accumulative information from previous orders and thus is not independent.

All in all, we will focus on how to determine A_k. Traditionally, we write

$$A_k = \mu^{\frac{1}{2}} \left(\rho_k + u_k \cdot v + \theta_k \left(\frac{|v|^2 - 3}{2} \right) \right), \tag{2.8}$$

where the coefficients ρ_k, u_k, and θ_k represent density, velocity, and temperature on the macroscopic scale. In particular, (ρ_k, u_k, θ_k) satisfies as follows:

(2.3) implies

$$p_1 - (\rho_1 + \theta_1) = 0, \tag{2.9}$$

$$\nabla_x p_1 = 0, \tag{2.10}$$

$$\nabla_x \cdot u_1 = 0; \tag{2.11}$$

(2.4) implies

$$p_2 - (\rho_2 + \theta_2 + \rho_1\theta_1) = 0, \tag{2.12}$$

$$\partial_t u_1 + u_1 \cdot \nabla_x u_1 - \gamma_1 \Delta_x u_1 + \nabla_x p_2 = 0, \tag{2.13}$$

$$\partial_t \theta_1 + u_1 \cdot \nabla_x \theta_1 - \gamma_2 \Delta_x \theta_1 = 0, \tag{2.14}$$

$$\nabla_x \cdot u_2 + u_1 \cdot \nabla_x \rho_1 = 0. \tag{2.15}$$

Here p_1 and p_2 represent the pressure, γ_1 and γ_2 are constants. The higher-order expansion produces more complicated fluid equations, which can be found in [37, Chapter 4]. If the interior solution F_k cannot satisfy the initial and boundary conditions, then we have to introduce initial layer $\mathcal{F}_k$ and boundary layer $\mathscr{F}_k$ to handle the gap.

2.2. Initial-layer expansion (temporal substitution). We define the rescaled time variable τ by making the scaling transform $\tau = \frac{t}{\varepsilon^2}$, which implies $\frac{\partial}{\partial t} = \frac{1}{\varepsilon^2}\frac{\partial}{\partial \tau}$. Then, under the substitution $t \to \tau$, (1.21) is transformed into

$$\begin{cases} \partial_\tau f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon + \mathcal{L}[f^\varepsilon] = \Gamma[f^\varepsilon, f^\varepsilon] \ \text{ in } \ \mathbb{R}_+ \times \Omega \times \mathbb{R}^3, \\ f^\varepsilon(0, x, v) = f_0(x, v) \ \text{ in } \ \Omega \times \mathbb{R}^3, \\ f^\varepsilon(\tau, x_0, v) = \mathcal{P}^\varepsilon[f^\varepsilon](\tau, x_0, v) \ \text{ for } \ \tau \in \mathbb{R}_+, \ \ x_0 \in \partial\Omega, \ \text{ and } \ v \cdot n(x_0) < 0. \end{cases} \tag{2.16}$$

We define the initial layer expansion,

$$f_{\rm il}^{\varepsilon}(\tau,x,v)=\sum_{k=1}^{4}\varepsilon^k\mathcal{F}_k(\tau,x,v), \tag{2.17}$$

where $\mathcal{F}_k$ can be determined by comparing the order of ε via plugging (2.17) into (2.16). Thus, we have

$$\partial_\tau\mathcal{F}_1+\mathcal{L}[\mathcal{F}_1]=0, \tag{2.18}$$

$$\partial_\tau\mathcal{F}_2+\mathcal{L}[\mathcal{F}_2]=-v\cdot\nabla_x\mathcal{F}_1+\Gamma[\mathcal{F}_1,\mathcal{F}_1]+2\Gamma[F_1,\mathcal{F}_1], \tag{2.19}$$

$$\partial_\tau\mathcal{F}_3+\mathcal{L}[\mathcal{F}_3]=-v\cdot\nabla_x\mathcal{F}_2+2\Gamma[\mathcal{F}_1,\mathcal{F}_2]+2\Gamma[F_1,\mathcal{F}_2]+2\Gamma[F_2,\mathcal{F}_1], \tag{2.20}$$

$$\begin{aligned}\partial_\tau\mathcal{F}_4+\mathcal{L}[\mathcal{F}_4]=&-v\cdot\nabla_x\mathcal{F}_3+2\Gamma[\mathcal{F}_1,\mathcal{F}_3]+\Gamma[\mathcal{F}_2,\mathcal{F}_2]+2\Gamma[F_1,\mathcal{F}_3]+2\Gamma[F_3,\mathcal{F}_1]\\&+2\Gamma[F_2,\mathcal{F}_2].\end{aligned} \tag{2.21}$$

2.3. Boundary layer expansion. This is very similar to the stationary problem in [46, section 2.2]. We need to introduce several geometric substitutions.

1. In a neighborhood of $x_0\in\partial\Omega$ define an orthogonal curvilinear coordinates system (ι_1,ι_2) such that at x_0 the coordinate lines coincide with the principal directions. Let $\mathfrak{N}$ be the normal distance to the boundary. Then $(\mathfrak{N},\iota_1,\iota_2)$ forms a local orthogonal coordinate system.
   Assume $\partial\Omega$ is parameterized by $r=r(\iota_1,\iota_2)$. Denote $P_i=|\partial_ir|$ for $i=1,2$. Then define the two orthogonal unit tangential vectors
   $$\varsigma_1:=\frac{\partial_1r}{P_1},\quad \varsigma_2:=\frac{\partial_2r}{P_2}. \tag{2.22}$$
   Also, the outward unit normal vector is
   $$n:=\frac{\partial_1r\times\partial_2r}{|\partial_1r\times\partial_2r|}=\varsigma_1\times\varsigma_2. \tag{2.23}$$
   Let κ_1 and κ_2 denote two principal curvatures and R_1 and R_2 two radii of principal curvatures.
2. We also decompose the velocity into normal and tangential directions
   $$\left\{\begin{array}{rcl}-v\cdot n&=&v_\eta,\\-v\cdot\varsigma_1&=&v_\phi,\\-v\cdot\varsigma_2&=&v_\psi.\end{array}\right. \tag{2.24}$$
   Denote $\mathfrak{v}=(v_\eta,v_\phi,v_\psi)$.
3. Define the scaled variable $\eta=\frac{\mathfrak{N}}{\varepsilon}$, which implies $\frac{\partial}{\partial\mathfrak{N}}=\frac{1}{\varepsilon}\frac{\partial}{\partial\eta}$.

Under these substitutions $(x,v)\to(\eta,\iota_1,\iota_2,\mathfrak{v})$, (1.21) is transformed into

(2.25)
$$\begin{cases} \varepsilon^2\partial_t f^\varepsilon + v_\eta\dfrac{\partial f^\varepsilon}{\partial\eta} - \dfrac{\varepsilon}{R_1-\varepsilon\eta}\left(v_\phi^2\dfrac{\partial f^\varepsilon}{\partial v_\eta} - v_\eta v_\phi\dfrac{\partial f^\varepsilon}{\partial v_\phi}\right) - \dfrac{\varepsilon}{R_2-\varepsilon\eta}\left(v_\psi^2\dfrac{\partial f^\varepsilon}{\partial v_\eta} - v_\eta v_\psi\dfrac{\partial f^\varepsilon}{\partial v_\psi}\right) \\ \quad - \dfrac{\varepsilon}{P_1P_2}\left(\dfrac{\partial_{11}r\cdot\partial_2 r}{P_1(\varepsilon\kappa_1\eta-1)}v_\phi v_\psi + \dfrac{\partial_{12}r\cdot\partial_2 r}{P_2(\varepsilon\kappa_2\eta-1)}v_\psi^2\right)\dfrac{\partial f^\varepsilon}{\partial v_\phi} \\ \quad - \dfrac{\varepsilon}{P_1P_2}\left(\dfrac{\partial_{22}r\cdot\partial_1 r}{P_2(\varepsilon\kappa_2\eta-1)}v_\phi v_\psi + \dfrac{\partial_{12}r\cdot\partial_1 r}{P_1(\varepsilon\kappa_1\eta-1)}v_\phi^2\right)\dfrac{\partial f^\varepsilon}{\partial v_\psi} \\ \quad - \varepsilon\left(\dfrac{v_\phi}{P_1(\varepsilon\kappa_1\eta-1)}\dfrac{\partial f^\varepsilon}{\partial\iota_1} + \dfrac{v_\psi}{P_2(\varepsilon\kappa_2\eta-1)}\dfrac{\partial f^\varepsilon}{\partial\iota_2}\right) + \mathcal{L}[f^\varepsilon] = \Gamma[f^\varepsilon,f^\varepsilon] \text{ in } \mathbb{R}_+\times\Omega\times\mathbb{R}^3, \\ f^\varepsilon(0,\eta,\iota_1,\iota_2,\mathfrak{v}) = f_0(\eta,\iota_1,\iota_2,\mathfrak{v}) \ \text{ in } \ \Omega\times\mathbb{R}^3, \\ f^\varepsilon(t,0,\iota_1,\iota_2,\mathfrak{v}) = \mathcal{P}^\varepsilon[f^\varepsilon](t,0,\iota_1,\iota_2,\mathfrak{v}) \ \text{ for } \ v_\eta>0. \end{cases}$$

We define the boundary layer expansion as follows:

$$f^\varepsilon_{\mathrm{bl}}(t,\eta,\iota_1,\iota_2,\mathfrak{v}) = \sum_{k=1}^{3}\varepsilon^k\mathscr{F}_k(t,\eta,\iota_1,\iota_2,\mathfrak{v}), \tag{2.26}$$

where $\mathscr{F}_k$ can be defined by comparing the order of ε via plugging (2.26) into (2.25). Thus, in a neighborhood of the boundary, we have

(2.27)
$$v_\eta\frac{\partial\mathscr{F}_1}{\partial\eta} - \frac{\varepsilon}{R_1-\varepsilon\eta}\left(v_\phi^2\frac{\partial\mathscr{F}_1}{\partial v_\eta} - v_\eta v_\phi\frac{\partial\mathscr{F}_1}{\partial v_\phi}\right) - \frac{\varepsilon}{R_2-\varepsilon\eta}\left(v_\psi^2\frac{\partial\mathscr{F}_1}{\partial v_\eta} - v_\eta v_\psi\frac{\partial\mathscr{F}_1}{\partial v_\psi}\right) + \mathcal{L}[\mathscr{F}_1] = 0,$$

(2.28)
$$v_\eta\frac{\partial\mathscr{F}_2}{\partial\eta} - \frac{\varepsilon}{R_1-\varepsilon\eta}\left(v_\phi^2\frac{\partial\mathscr{F}_2}{\partial v_\eta} - v_\eta v_\phi\frac{\partial\mathscr{F}_2}{\partial v_\phi}\right) - \frac{\varepsilon}{R_2-\varepsilon\eta}\left(v_\psi^2\frac{\partial\mathscr{F}_2}{\partial v_\eta} - v_\eta v_\psi\frac{\partial\mathscr{F}_2}{\partial v_\psi}\right) + \mathcal{L}[\mathscr{F}_2] = Z_1,$$

where $Z_1 = Z_1\left[F_1,\mathscr{F}_1,\dfrac{\partial\mathscr{F}_1}{\partial v_\phi},\dfrac{\partial\mathscr{F}_1}{\partial v_\psi},\dfrac{\partial\mathscr{F}_1}{\partial\iota_1},\dfrac{\partial\mathscr{F}_1}{\partial\iota_2}\right]$ as

(2.29)
$$\begin{aligned} Z_1 :=\ & 2\Gamma[F_1,\mathscr{F}_1] + \Gamma[\mathscr{F}_1,\mathscr{F}_1] + \frac{1}{P_1P_2}\left(\frac{\partial_{11}r\cdot\partial_2 r}{P_1(\varepsilon\kappa_1\eta-1)}v_\phi v_\psi + \frac{\partial_{12}r\cdot\partial_2 r}{P_2(\varepsilon\kappa_2\eta-1)}v_\psi^2\right)\frac{\partial\mathscr{F}_1}{\partial v_\phi} \\ & + \frac{1}{P_1P_2}\left(\frac{\partial_{22}r\cdot\partial_1 r}{P_2(\varepsilon\kappa_2\eta-1)}v_\phi v_\psi + \frac{\partial_{12}r\cdot\partial_1 r}{P_1(\varepsilon\kappa_1\eta-1)}v_\phi^2\right)\frac{\partial\mathscr{F}_1}{\partial v_\psi} + \frac{v_\phi}{P_1(\varepsilon\kappa_1\eta-1)}\frac{\partial\mathscr{F}_1}{\partial\iota_1} \\ & + \frac{v_\psi}{P_2(\varepsilon\kappa_2\eta-1)}\frac{\partial\mathscr{F}_1}{\partial\iota_2}. \end{aligned}$$

However, we define $\mathscr{F}_3$ in a completely different fashion. Let $\mathscr{F}_3$ satisfy

$$\begin{aligned}(2.30)\quad & v_\eta \frac{\partial \mathscr{F}_3}{\partial \eta} - \frac{\varepsilon}{R_1-\varepsilon\eta}\left(v_\phi^2 \frac{\partial \mathscr{F}_3}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathscr{F}_3}{\partial v_\phi}\right) - \frac{\varepsilon}{R_2-\varepsilon\eta}\left(v_\psi^2 \frac{\partial \mathscr{F}_3}{\partial v_\eta} - v_\eta v_\psi \frac{\partial \mathscr{F}_3}{\partial v_\psi}\right)\\ & - \frac{\varepsilon}{P_1P_2}\left(\frac{\partial_{11}r\cdot\partial_2 r}{P_1(\varepsilon\kappa_1\eta-1)} v_\phi v_\psi + \frac{\partial_{12}r\cdot\partial_2 r}{P_2(\varepsilon\kappa_2\eta-1)} v_\psi^2\right)\frac{\partial \mathscr{F}_3}{\partial v_\phi}\\ & - \frac{\varepsilon}{P_1P_2}\left(\frac{\partial_{22}r\cdot\partial_1 r}{P_2(\varepsilon\kappa_2\eta-1)} v_\phi v_\psi + \frac{\partial_{12}r\cdot\partial_1 r}{P_1(\varepsilon\kappa_1\eta-1)} v_\phi^2\right)\frac{\partial \mathscr{F}_3}{\partial v_\psi}\\ & - \varepsilon\left(\frac{v_\phi}{P_1(\varepsilon\kappa_1\eta-1)}\frac{\partial \mathscr{F}_3}{\partial \iota_1} + \frac{v_\psi}{P_2(\varepsilon\kappa_2\eta-1)}\frac{\partial \mathscr{F}_3}{\partial \iota_2}\right) + \mathcal{L}[\mathscr{F}_3] = Z_2,\end{aligned}$$

where

$$\begin{aligned}(2.31)\qquad Z_2 := {} & 2\Gamma[\mathscr{F}_1,\mathscr{F}_2] + 2\Gamma[F_1,\mathscr{F}_2] + 2\Gamma[F_2,\mathscr{F}_1]\\ & + \frac{1}{P_1P_2}\left(\frac{\partial_{11}r\cdot\partial_2 r}{P_1(\varepsilon\kappa_1\eta-1)} v_\phi v_\psi + \frac{\partial_{12}r\cdot\partial_2 r}{P_2(\varepsilon\kappa_2\eta-1)} v_\psi^2\right)\frac{\partial \mathscr{F}_2}{\partial v_\phi}\\ & + \frac{1}{P_1P_2}\left(\frac{\partial_{22}r\cdot\partial_1 r}{P_2(\varepsilon\kappa_2\eta-1)} v_\phi v_\psi + \frac{\partial_{12}r\cdot\partial_1 r}{P_1(\varepsilon\kappa_1\eta-1)} v_\phi^2\right)\frac{\partial \mathscr{F}_2}{\partial v_\psi}\\ & + \frac{v_\phi}{P_1(\varepsilon\kappa_1\eta-1)}\frac{\partial \mathscr{F}_2}{\partial \iota_1} + \frac{v_\psi}{P_2(\varepsilon\kappa_2\eta-1)}\frac{\partial \mathscr{F}_2}{\partial \iota_2}.\end{aligned}$$

Obviously, (2.30) actually contains all terms in (2.25) except the time derivative, so it is essentially

$$(2.32)\qquad \varepsilon v\cdot\nabla_x \mathscr{F}_3 + \mathcal{L}[\mathscr{F}_3] = Z_2.$$

Hence, we will resort to the well-posedness and decay theory of the linearized stationary problem instead of that of the half-space boundary layer equation (the so-called ε-Milne problem with geometric correction).

2.4. Initial condition expansion. The bridge between the interior solution and initial layer is the initial condition. Plugging the combined expansion from (2.1) and (2.17),

$$(2.33)\qquad f^\varepsilon \sim \sum_{k=1}^{3}\varepsilon^k F_k + \sum_{k=1}^{4}\varepsilon^k \mathcal{F}_k,$$

into the initial condition (1.21), and comparing the order of ε, we obtain

$$(2.34)\qquad F_1 + \mathcal{F}_1 = f_{0,1},$$

$$(2.35)\qquad F_2 + \mathcal{F}_2 = f_{0,2},$$

$$(2.36)\qquad F_3 + \mathcal{F}_3 = f_{0,3}.$$

Since we do not expand the interior solution $f_{\text{in}}^\varepsilon$ to higher order, we simply require the initial condition such that $\mathcal{F}_4$ decays to zero as $\tau\to\infty$.

2.5. Boundary condition expansion. The bridge between the interior solution and boundary layer is the boundary condition. Define

$$\mathcal{P}[f](t,x_0,v) := \mu^{\frac{1}{2}}(v)\int_{\mathfrak{u}\cdot n(x_0)>0}\mu^{\frac{1}{2}}(\mathfrak{u})f(t,x_0,\mathfrak{u})\left|\mathfrak{u}\cdot n(x_0)\right|\mathrm{d}\mathfrak{u}. \tag{2.37}$$

Plugging the combined expansion from (2.1) and (2.26),

$$f^\varepsilon \sim \sum_{k=1}^{3}\varepsilon^k F_k + \sum_{k=1}^{2}\varepsilon^k \mathscr{F}_k, \tag{2.38}$$

into the boundary condition (1.21) and (1.23), and comparing the order of ε, we obtain

$$F_1+\mathscr{F}_1 = \mathcal{P}[F_1+\mathscr{F}_1] + \mu_1(x_0,v), \tag{2.39}$$

$$F_2+\mathscr{F}_2 = \mathcal{P}[F_2+\mathscr{F}_2]+\mu_1(x_0,v)\int_{\mathfrak{u}\cdot n(x_0)>0}\mu^{\frac{1}{2}}(\mathfrak{u})(F_1+\mathscr{F}_1)\left|\mathfrak{u}\cdot n(x_0)\right|\mathrm{d}\mathfrak{u}+\mu_2(x_0,v). \tag{2.40}$$

For F_3 and $\mathscr{F}_3$, since the boundary layer $\mathscr{F}_3$ is defined differently, we can assign the stronger version

$$\begin{aligned} F_3+\mathscr{F}_3 = {}& \mathcal{P}[F_3+\mathscr{F}_3] + \varepsilon^{-2}\Big(\mu_b^\varepsilon - \mu - \varepsilon\mu^{\frac{1}{2}}\mu_1\Big)\mu^{-1}\mathcal{P}[F_1+\mathscr{F}_1] \\ & + \varepsilon^{-1}\Big(\mu_b^\varepsilon - \mu\Big)\mu^{-1}\mathcal{P}[F_2+\mathscr{F}_2] \\ & + \varepsilon^{-3}\mu^{-\frac{1}{2}}\Big(\mu_b^\varepsilon - \mu - \varepsilon\mu^{\frac{1}{2}}\mu_1 - \varepsilon^2\mu^{\frac{1}{2}}\mu_2\Big). \end{aligned} \tag{2.41}$$

2.6. Matching procedure. Define the length of boundary layer $L=\varepsilon^{-\frac{1}{2}}$. Also, denote $\mathscr{R}[v_\eta,v_\phi,v_\psi] = (-v_\eta,v_\phi,v_\psi)$.

Step 1: Construction of F_1, $\mathcal{F}_1$, and $\mathscr{F}_1$.

A direct computation reveals that $F_1 = A_1+B_1+C_1$, where $B_1=C_1=0$. Define

$$F_1 = \mu^{\frac{1}{2}}\left(\rho_1 + u_1\cdot v + \theta_1\frac{|v|^2-3}{2}\right), \tag{2.42}$$

where (ρ_1,u_1,θ_1) satisfies the Navier–Stokes–Fourier system

$$\begin{cases} \partial_t u_1 + u_1\cdot\nabla_x u_1 - \gamma_1\Delta_x u_1 + \nabla_x p_2 = 0, \\ \nabla_x\cdot u_1 = 0, \\ \partial_t\theta_1 + u_1\cdot\nabla_x\theta_1 - \gamma_2\Delta_x\theta_1 = 0 \end{cases} \tag{2.43}$$

with the initial condition

$$\rho_1(0,x) = \rho_{0,1}(x), \quad u_1(0,x) = u_{0,1}(x), \quad \theta_1(0,x) = \theta_{0,1}(x), \tag{2.44}$$

and the boundary condition

$$\rho_1(t,x_0) = \rho_{b,1}(t,x_0) + M_1(t,x_0), \quad u_1(t,x_0) = u_{b,1}(t,x_0), \quad \theta_1(t,x_0) = \theta_{b,1}(t,x_0). \tag{2.45}$$

Here $M_1(t,x_0)$ is such that the Boussinesq relation

$$\nabla_x(\rho_1+\theta_1)=0 \tag{2.46}$$

is satisfied. Note that the above requirement means that for fixed t, M_1 still has one dimension of freedom. It is eventually fully determined by enforcing the conservation law

$$\iint_{\Omega\times\mathbb{R}^3} F_1(t,x,v)\mu^{\frac{1}{2}}(v)\mathrm{d}v\mathrm{d}x=0. \tag{2.47}$$

Then based on the compatibility condition of μ_1 which is

$$\int_{\mathfrak{u}\cdot n(x_0)>0}\mu^{\frac{1}{2}}(\mathfrak{u})\mu_1(t,x_0,\mathfrak{u})\left|\mathfrak{u}\cdot n(x_0)\right|\mathrm{d}\mathfrak{u}=0, \tag{2.48}$$

we naturally obtain $\mathcal{P}[F_1]=M_1\mu^{\frac{1}{2}}$, which means

$$F_1=\mathcal{P}[F_1]+\mu_1\ \text{ on }\ \partial\Omega. \tag{2.49}$$

Therefore, compared with (2.39), it is not necessary to introduce the boundary layer at this order and we simply take $\mathscr{F}_1=0$. Also, the interior solution can already satisfy the initial data, so it is not necessary to introduce the initial layer at this order and we simply take $\mathcal{F}_1=0$.

Step 2: Construction of F_2, $\mathcal{F}_2$, and $\mathscr{F}_2$.

Define $F_2=A_2+B_2+C_2$, where B_2 and C_2 can be uniquely determined following previous analysis in section 2.1 and [46, section 2.1], and

$$A_2=\mu^{\frac{1}{2}}\left(\rho_2+u_2\cdot v+\theta_2\frac{|v|^2-3}{2}\right), \tag{2.50}$$

satisfying a linear fluid-type equation provided F_1 is known. Now F_2 does not satisfy (2.40) alone, so we have to introduce the boundary layer. Let $\mathscr{F}_2$ satisfy the ε-Milne problem with geometric correction

(2.51)
$$\begin{cases}v_\eta\dfrac{\partial\mathscr{F}_2}{\partial\eta}-\dfrac{\varepsilon}{R_1-\varepsilon\eta}\left(v_\phi^2\dfrac{\partial\mathscr{F}_2}{\partial v_\eta}-v_\eta v_\phi\dfrac{\partial\mathscr{F}_2}{\partial v_\phi}\right)-\dfrac{\varepsilon}{R_2-\varepsilon\eta}\left(v_\psi^2\dfrac{\partial\mathscr{F}_2}{\partial v_\eta}-v_\eta v_\psi\dfrac{\partial\mathscr{F}_2}{\partial v_\psi}\right)\\ \qquad +\mathcal{L}[\mathscr{F}_2]=0,\\ \mathscr{F}_2(t,0,\iota_1,\iota_2,\mathfrak{v})=h(t,\iota_1,\iota_2,,\mathfrak{v})-\tilde{h}(t,\iota_1,\iota_2,,\mathfrak{v})\ \text{ for }\ v_\eta>0,\\ \mathscr{F}_2(t,L,\iota_1,\iota_2,,\mathfrak{v})=\mathscr{F}_2(t,L,\iota_1,\iota_2,,\mathscr{R}[\mathfrak{v}])\end{cases}$$

with the in-flow boundary data

$$h(t,\iota_1,\iota_2,\mathfrak{v})=M_1\mu_1(t,x_0,v)+\mu_2(t,x_0,v)-\Big((B_2+C_2)-\mathcal{P}[B_2+C_2]\Big). \tag{2.52}$$

Based on Theorems 3.2 and 3.4, there exists a unique

$$\tilde{h}(t,\iota_1,\iota_2,\mathfrak{v})=\mu^{\frac{1}{2}}\sum_{k=0}^{4}\tilde{D}_k(t,\iota_1,\iota_2)\mathbf{e}_k, \tag{2.53}$$

such that (2.51) is well-posed and the solution decays exponentially fast to zero (here $\mathbf{e}_k$ with $k = 0, 1, 2, 3, 4$ form a basis of null space $\mathcal{N}$ of $\mathcal{L}$). In particular, $\tilde{D}_1 = 0$. Then we further require that A_2 satisfies the boundary condition

$$A_2(t, x_0, v) = \tilde{h}(t, \iota_1, \iota_2, \mathfrak{v}) + M_2(t, x_0)\mu^{\frac{1}{2}}(v). \tag{2.54}$$

Here x_0 corresponds to (ι_1, ι_2) and v corresponds to $\mathfrak{v}$, based on substitution in section 2.3. Here, the constant $M_2(t, x_0)$ is chosen to enforce the Boussinesq relation

$$p_2 - (\rho_2 + \theta_2 + \rho_1\theta_1) = 0, \tag{2.55}$$

where p_2 is the pressure solved from (2.43) and it is determined up to a time-dependent function. Similar to the construction of F_1, due to (1.20), we can choose M_2 to satisfy the conservation law

$$\iint_{\Omega\times\mathbb{R}^3} (F_2 + \mathscr{F}_2 + \mathcal{F}_2)(t, x, v)\mu^{\frac{1}{2}}(v)\mathrm{d}v\mathrm{d}x = 0. \tag{2.56}$$

We can verify that such construction satisfies the boundary condition (2.40) as in [46, section 2.5].

Also, the initial layer is no longer zero at this order. It satisfies

$$\begin{cases} \partial_\sigma \mathcal{F}_2 + \mathcal{L}[\mathcal{F}_2] = 0, \\ \mathcal{F}_2(0, x, v) = (B_2 + C_2)(0, x, v) - \mathcal{F}_{2,\infty}, \end{cases} \tag{2.57}$$

where $\mathcal{F}_{2,\infty}(x, v) \in \mathcal{N}$ is chosen based on Theorem 6.4 such that

$$\lim_{\tau\to\infty} \mathcal{F}_2(\tau, x, v) = 0. \tag{2.58}$$

Then we further require that A_2 satisfies the initial condition

$$A_2(0, x, v) = \mathcal{F}_{2,\infty}(x, v). \tag{2.59}$$

Step 3: Construction of F_3, $\mathcal{F}_3$, and $\mathscr{F}_3$.

This is almost the same as in Step 2. Define $F_3 = A_3 + B_3 + C_3$, where B_3 and C_3 can be uniquely determined following previous analysis, and

$$A_3 = \mu^{\frac{1}{2}}\left(\rho_3 + u_3 \cdot v + \theta_3 \frac{|v|^2 - 3}{2}\right), \tag{2.60}$$

satisfying a linear fluid-type equation provided F_1 and F_2 are known. In particular, since the boundary layer at this order is defined in a trickier way, we simply define the boundary condition

$$A_3(t, x_0, v) = 0. \tag{2.61}$$

On the other hand, define the boundary layer $\mathscr{F}_3$ by

$$v_\eta \frac{\partial \mathscr{F}_3}{\partial \eta} - \frac{\varepsilon}{R_1 - \varepsilon\eta}\left(v_\phi^2 \frac{\partial \mathscr{F}_3}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathscr{F}_3}{\partial v_\phi}\right) - \frac{\varepsilon}{R_2 - \varepsilon\eta}\left(v_\psi^2 \frac{\partial \mathscr{F}_3}{\partial v_\eta} - v_\eta v_\psi \frac{\partial \mathscr{F}_3}{\partial v_\psi}\right) \tag{2.62}$$

$$- \frac{\varepsilon}{P_1P_2}\left(\frac{\partial_{11}r \cdot \partial_2 r}{P_1(\varepsilon\kappa_1\eta - 1)} v_\phi v_\psi + \frac{\partial_{12}r \cdot \partial_2 r}{P_2(\varepsilon\kappa_2\eta - 1)} v_\psi^2\right)\frac{\partial \mathscr{F}_3}{\partial v_\phi} \tag{2.63}$$

$$- \frac{\varepsilon}{P_1P_2}\left(\frac{\partial_{22}r \cdot \partial_1 r}{P_2(\varepsilon\kappa_2\eta - 1)} v_\phi v_\psi + \frac{\partial_{12}r \cdot \partial_1 r}{P_1(\varepsilon\kappa_1\eta - 1)} v_\phi^2\right)\frac{\partial \mathscr{F}_3}{\partial v_\psi} \tag{2.64}$$

$$- \varepsilon\left(\frac{v_\phi}{P_1(\varepsilon\kappa_1\eta - 1)}\frac{\partial \mathscr{F}_3}{\partial \iota_1} + \frac{v_\psi}{P_2(\varepsilon\kappa_2\eta - 1)}\frac{\partial \mathscr{F}_3}{\partial \iota_2}\right) + \mathcal{L}[\mathscr{F}_3] = Z, \tag{2.65}$$

where

$$Z := 2\Gamma[F_1, \mathscr{F}_2] + \frac{1}{P_1P_2}\left(\frac{\partial_{11}r\cdot\partial_2 r}{P_1(\varepsilon\kappa_1\eta-1)}v_\phi v_\psi + \frac{\partial_{12}r\cdot\partial_2 r}{P_2(\varepsilon\kappa_2\eta-1)}v_\psi^2\right)\frac{\partial\mathscr{F}_2}{\partial v_\phi}$$
$$+ \frac{1}{P_1P_2}\left(\frac{\partial_{22}r\cdot\partial_1 r}{P_2(\varepsilon\kappa_2\eta-1)}v_\phi v_\psi + \frac{\partial_{12}r\cdot\partial_1 r}{P_1(\varepsilon\kappa_1\eta-1)}v_\phi^2\right)\frac{\partial\mathscr{F}_2}{\partial v_\psi}$$
$$+ \frac{v_\phi}{P_1(\varepsilon\kappa_1\eta-1)}\frac{\partial\mathscr{F}_2}{\partial\iota_1} + \frac{v_\psi}{P_2(\varepsilon\kappa_2\eta-1)}\frac{\partial\mathscr{F}_2}{\partial\iota_2}. \tag{2.66}$$

This is essentially,

$$\varepsilon v\cdot\nabla_x\mathscr{F}_3 + \mathcal{L}[\mathscr{F}_3] = Z. \tag{2.67}$$

The boundary condition is taken as

(2.68)
$$\mathscr{F}_3 = \mathcal{P}[\mathscr{F}_3] + \varepsilon^{-2}\left(\mu_b^\varepsilon - \mu - \varepsilon\mu^{\frac12}\mu_1\right)\mu^{-1}\mathcal{P}[F_1+\mathscr{F}_1] + \varepsilon^{-1}\left(\mu_b^\varepsilon-\mu\right)\mu^{-1}\mathcal{P}[F_2+\mathscr{F}_2]$$
$$+ \varepsilon^{-3}\mu^{-\frac12}\left(\mu_b^\varepsilon - \mu - \varepsilon\mu^{\frac12}\mu_1 - \varepsilon^2\mu^{\frac12}\mu_2\right) - \Big((B_3+C_3) - \mathcal{P}[B_3+C_3]\Big).$$

Also, the initial layer satisfies

$$\begin{cases} \partial_\sigma\mathcal{F}_3 + \mathcal{L}[\mathcal{F}_3] = -v\cdot\nabla_x\mathcal{F}_2 + 2\Gamma[F_1,\mathcal{F}_2], \\ \mathcal{F}_3(0,x,v) = (B_3+C_3)(0,x,v) - \mathcal{F}_{3,\infty}, \end{cases} \tag{2.69}$$

where $\mathcal{F}_{3,\infty}(x,v)\in\mathcal{N}$ is chosen based on Theorem 6.4 such that

$$\lim_{\tau\to\infty}\mathcal{F}_3(\tau,x,v) = 0. \tag{2.70}$$

Then we further require that A_3 satisfies the initial condition

$$A_3(0,x,v) = \mathcal{F}_{3,\infty}(x,v). \tag{2.71}$$

In a similar fashion, we can define $\mathcal{F}_4$

$$\begin{cases} \partial_\sigma\mathcal{F}_4 + \mathcal{L}[\mathcal{F}_4] = -v\cdot\nabla_x\mathcal{F}_3 + 2\Gamma[F_1,\mathcal{F}_3] + 2\Gamma[F_3,\mathcal{F}_1] + 2\Gamma[F_2,\mathcal{F}_2], \\ \mathcal{F}_4(0,x,v) = -\mathcal{F}_{4,\infty}, \end{cases} \tag{2.72}$$

where $\mathcal{F}_{4,\infty}(x,v)\in\mathcal{N}$ is chosen based on Theorem 6.4 such that

$$\lim_{\tau\to\infty}\mathcal{F}_4(\tau,x,v) = 0. \tag{2.73}$$

3. Well-posedness and regularity of boundary layers. This section has been well studied in [46, sections 3 and 4], so we just present the formulation and notation, and record the main theorems without giving detailed proofs.

3.1. Well-posedness and decay. In this subsection, we will study the well-posedness and decay of the ε-Milne problem with geometric correction. Let the null space $\mathcal{N}$ of the operator $\mathcal{L}$ be spanned by $\mu^{\frac{1}{2}}\left\{1, v_\eta, v_\phi, v_\psi, \frac{|\mathfrak{v}|^2-3}{2}\right\} = \{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$. Given data $h(\mathfrak{v})$ and $S(\eta, \mathfrak{v})$, we intend to find

$$\tilde{h}(\mathfrak{v}) := \sum_{i=0}^{4} \tilde{D}_i \mathbf{e}_i \in \mathcal{N} \tag{3.1}$$

with $\tilde{D}_1 = 0$ and the other $\tilde{D}_i$ are constants such that the ε-Milne problem with geometric correction for $\mathcal{G}(\eta, \mathfrak{v})$ in the domain $(\eta, \mathfrak{v}) \in [0, L] \times \mathbb{R}^3$ as

$$\begin{cases} v_\eta \dfrac{\partial \mathcal{G}}{\partial \eta} - \dfrac{\varepsilon}{R_1 - \varepsilon\eta}\left(v_\phi^2 \dfrac{\partial \mathcal{G}}{\partial v_\eta} - v_\eta v_\phi \dfrac{\partial \mathcal{G}}{\partial v_\phi}\right) - \dfrac{\varepsilon}{R_2 - \varepsilon\eta}\left(v_\psi^2 \dfrac{\partial \mathcal{G}}{\partial v_\eta} - v_\eta v_\psi \dfrac{\partial \mathcal{G}}{\partial v_\psi}\right) + \mathcal{L}[\mathcal{G}] = S, \\ \mathcal{G}(0, \mathfrak{v}) = h(\mathfrak{v}) - \tilde{h}(\mathfrak{v}) \quad \text{for} \quad v_\eta > 0, \\ \mathcal{G}(L, \mathfrak{v}) = \mathcal{G}(L, \mathscr{R}[\mathfrak{v}]), \end{cases} \tag{3.2}$$

is well-posed, and $\mathcal{G}$ decays exponentially fast to zero as η becomes larger and larger. Here $\mathscr{R}[\mathfrak{v}] = (-v_\eta, v_\phi, v_\psi)$ and $L = \varepsilon^{-\frac{1}{2}}$. For simplicity, we temporarily ignore the dependence of ι_1, ι_2, but our estimates are uniform in these variables. Also, the estimates and decaying rate should be uniform in ε.

In this section, we introduce some special notation to describe the norms for $(\eta, \mathfrak{v}) \in [0, L] \times \mathbb{R}^3$. Define the L^2 norms as follows:

$$\|f\|_2 := \left(\int_0^L \int_{\mathbb{R}^3} |f(\eta, \mathfrak{v})|^2 \,\mathrm{d}\mathfrak{v}\mathrm{d}\eta\right)^{\frac{1}{2}}. \tag{3.3}$$

Define the weighted L^∞ norms as follows:

$$\|f\|_{\infty,\vartheta,\varrho} := \operatorname*{ess\,sup}_{(\eta,\mathfrak{v})\in[0,L]\times\mathbb{R}^3} \left(\langle\mathfrak{v}\rangle^{\vartheta} \mathrm{e}^{\varrho|\mathfrak{v}|^2} |f(\eta, \mathfrak{v})|\right), \tag{3.4}$$

Here, we require $0 \leq \varrho < \frac{1}{4}$ and $\vartheta > 3$.

Since the boundary data $h(\mathfrak{v})$ are only defined on $v_\eta > 0$, we naturally extend the above definitions to this half-domain as follows:

$$|h|_{\infty,\vartheta,\varrho} := \sup_{v_\eta > 0} \left(\langle\mathfrak{v}\rangle^{\vartheta} \mathrm{e}^{\varrho|\mathfrak{v}|^2} |h(\mathfrak{v})|\right). \tag{3.5}$$

Throughout this section, we assume

$$|h|_{\infty,\vartheta,\varrho} \lesssim 1, \quad \left\|\mathrm{e}^{K\eta} S\right\|_{\infty,\vartheta,\varrho} \lesssim 1 \tag{3.6}$$

for some constant $K > 0$ uniform in ε.

THEOREM 3.1 (L^2 well-posedness of $\mathcal{G}$). *Assume* (3.6) *holds. Then there exists $\tilde{h} \in \mathcal{N}$ such that there exists a unique solution $\mathcal{G}(\eta, \mathfrak{v})$ to the ε-Milne problem with geometric correction* (3.2) *satisfying*

$$\|\mathcal{G}\|_2 \lesssim 1. \tag{3.7}$$

THEOREM 3.2 (L^2 decay). *Assume* (3.6) *holds. Then there exists* $0 < K_0 < K$ *such that the solution* $g(\eta, \mathfrak{v})$ *to* (3.2) *satisfying*

$$\left\| \mathrm{e}^{K_0\eta} \mathcal{G} \right\|_2 \lesssim 1. \tag{3.8}$$

THEOREM 3.3 (L^∞ well-posedness of $\mathcal{G}$). *Assume* (3.6) *holds. Then there exists a unique solution* $\mathcal{G}(\eta, \mathfrak{v})$ *to the* ε*-Milne problem with geometric correction* (3.2) *satisfying for* $\varrho \geq 0$ *and integer* $\vartheta \geq 3$,

$$\|\mathcal{G}\|_{\infty,\vartheta,\varrho} \lesssim 1. \tag{3.9}$$

THEOREM 3.4 (L^∞ decay). *Assume* (3.6) *holds. Then there exists* $0 < K_0 < K$ *such that the solution* $g(\eta, \mathfrak{v})$ *to* (3.2) *satisfying for* $\varrho \geq 0$ *and* $\vartheta > 3$,

$$\left\| \mathrm{e}^{K_0\eta} \mathcal{G} \right\|_{\infty,\vartheta,\varrho} \lesssim 1. \tag{3.10}$$

3.2. Regularity estimates. Now we begin to study the regularity of the solution $\mathcal{G}$ to (3.2). In this subsection, denote the boundary data $p = h - \tilde{h}$. Besides (3.6), throughout this section, we further require the regularity bound that for $\varrho \geq 0$ and $\vartheta > 3$

$$|\nabla_{\mathfrak{v}} p|_{\infty,\vartheta,\varrho} \lesssim 1, \quad \left\| \mathrm{e}^{K\eta} \partial_\eta S \right\|_{\infty,\vartheta,\varrho} + \left\| \mathrm{e}^{K\eta} \nabla_{\mathfrak{v}} S \right\|_{\infty,\vartheta,\varrho} \lesssim 1. \tag{3.11}$$

Define a weight function

$$\zeta(\eta; \mathfrak{v}) = \left(\left(v_\eta^2 + v_\phi^2 + v_\psi^2\right) - \left(\frac{R_1 - \varepsilon\eta}{R_1}\right)^2 v_\phi^2 - \left(\frac{R_2 - \varepsilon\eta}{R_2}\right)^2 v_\psi^2 \right)^{\frac{1}{2}}. \tag{3.12}$$

It is easy to see that the closer a point $(\eta; v_\eta, v_\phi, v_\psi)$ is to the grazing set $(\eta; v_\eta, v_\phi, v_\psi) = (0; 0, v_\phi, v_\psi)$, the smaller ζ is. In particular, at the grazing set, $\zeta(0; 0, v_\phi, v_\psi) = 0$.

LEMMA 3.5 (weight function in ε-Milne problem). *Let* ζ *be defined as in* (3.12). *We have*

$$v_\eta \frac{\partial \zeta}{\partial \eta} - \frac{\varepsilon}{R_1 - \varepsilon\eta} \left(v_\phi^2 \frac{\partial \zeta}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \zeta}{\partial v_\phi} \right) - \frac{\varepsilon}{R_2 - \varepsilon\eta} \left(v_\psi^2 \frac{\partial \zeta}{\partial v_\eta} - v_\eta v_\psi \frac{\partial \zeta}{\partial v_\psi} \right) = 0. \tag{3.13}$$

THEOREM 3.6. *Assume* (3.6) *and* (3.11) *hold. For* $K_0 > 0$ *sufficiently small, we have*

$$\left\| \mathrm{e}^{K_0\eta} \zeta \frac{\partial \mathcal{G}}{\partial \eta} \right\|_{\infty,\vartheta,\varrho} + \left\| \mathrm{e}^{K_0\eta} \nu\zeta \frac{\partial \mathcal{G}}{\partial v_\eta} \right\|_{\infty,\vartheta,\varrho} \lesssim |\ln(\varepsilon)|^8, \tag{3.14}$$

$$\left\| \mathrm{e}^{K_0\eta} \nu\zeta \frac{\partial \mathcal{G}}{\partial v_\phi} \right\|_{\infty,\vartheta,\varrho} + \left\| \mathrm{e}^{K_0\eta} \nu\zeta \frac{\partial \mathcal{G}}{\partial v_\psi} \right\|_{\infty,\vartheta,\varrho} \lesssim 1. \tag{3.15}$$

COROLLARY 3.7. *Assume* (3.6) *and* (3.11) *hold. We have*

$$\varepsilon \left\| \mathrm{e}^{K_0\eta} v_\phi^2 \frac{\partial \mathcal{G}}{\partial v_\eta} \right\|_{\infty,\vartheta,\varrho} + \varepsilon \left\| \mathrm{e}^{K_0\eta} v_\psi^2 \frac{\partial \mathcal{G}}{\partial v_\eta} \right\|_{\infty,\vartheta,\varrho} \lesssim |\ln(\varepsilon)|^8. \tag{3.16}$$

THEOREM 3.8. *Assume* (3.6) *and* (3.11) *hold. We have*

$$\left\| \mathrm{e}^{K_0\eta} \frac{\partial \mathcal{G}}{\partial \iota_1} \right\|_{\infty,\vartheta,\varrho} \lesssim |\ln(\varepsilon)|^8, \quad \left\| \mathrm{e}^{K_0\eta} \frac{\partial \mathcal{G}}{\partial \iota_2} \right\|_{\infty,\vartheta,\varrho} \lesssim |\ln(\varepsilon)|^8. \tag{3.17}$$

THEOREM 3.9. *Assume* (3.6) *and* (3.11) *hold. We have*

$$\left\|\mathrm{e}^{K_0\eta}\nu\frac{\partial\mathcal{G}}{\partial v_\phi}\right\|_{\infty,\vartheta,\varrho}\lesssim\left|\ln(\varepsilon)\right|^8,\quad \left\|\mathrm{e}^{K_0\eta}\nu\frac{\partial\mathcal{G}}{\partial v_\psi}\right\|_{\infty,\vartheta,\varrho}\lesssim\left|\ln(\varepsilon)\right|^8. \tag{3.18}$$

Remark 3.1. Theorem 3.6, Corollary 3.7, Theorems 3.8 and 3.9 provide bounds of all kinds of normal and velocity derivatives. However, note that the $\frac{\partial\mathcal{G}}{\partial\eta}$ estimate must be accompanied by the weight ζ since it may have a singularity near the grazing set. Similarly, the $\frac{\partial\mathcal{G}}{\partial v_\eta}$ estimate should be with either ζ or ε. On the other hand, $\frac{\partial\mathcal{G}}{\partial\iota_i}$, $\frac{\partial\mathcal{G}}{\partial v_\phi}$, and $\frac{\partial\mathcal{G}}{\partial v_\psi}$ can avoid the introduction of ζ or ε, since they do not directly interact with the grazing set.

4. Stationary remainder estimates. Our analysis of boundary layers relies on the study of the linearized stationary Boltzmann equation

$$\begin{cases}\varepsilon v\cdot\nabla_x f+\mathcal{L}[f]=S(x,v) & \text{in } \Omega\times\mathbb{R}^3,\\ f(x_0,v)=\mathcal{P}[f](x_0,v)+h(x_0,v) & \text{for } x_0\in\partial\Omega \text{ and } v\cdot n<0,\end{cases} \tag{4.1}$$

where

$$\mathcal{P}[f](x_0,v)=\mu^{\frac12}(v)\int_{\gamma_+}f(x_0,v)\mu^{\frac12}(v)\mathrm{d}\gamma. \tag{4.2}$$

The data S and h satisfy the compatibility condition

$$\iint_{\Omega\times\mathbb{R}^3}S(x,v)\mu^{\frac12}(v)\mathrm{d}v\mathrm{d}x+\int_{\gamma_-}h(x,v)\mu^{\frac12}(v)\mathrm{d}\gamma=0. \tag{4.3}$$

It is easy to see that if f is a solution to (4.1), then $f+C\mu^{\frac12}$ is also a solution for arbitrary $C\in\mathbb{R}$. Hence, to guarantee uniqueness, the solution should satisfy the normalization condition

$$\iint_{\Omega\times\mathbb{R}^3}f(x,v)\mu^{\frac12}(v)\mathrm{d}v\mathrm{d}x=0. \tag{4.4}$$

This problem has been well studied in [44, section 4], so we will only present the main theorems.

LEMMA 4.1 (time-independent Green's identity). *Assume* $f(x,v),\ g(x,v)\in L^2(\Omega\times\mathbb{R}^2)$ *and* $v\cdot\nabla_x f,\ v\cdot\nabla_x g\in L^2(\Omega\times\mathbb{R}^2)$ *with* $f,\ g\in L^2(\gamma)$. *Then*

$$\iint_{\Omega\times\mathbb{R}^2}\Big((v\cdot\nabla_x f)g+(v\cdot\nabla_x g)f\Big)\mathrm{d}x\mathrm{d}v=\iint_{\gamma_+}fg\mathrm{d}\gamma-\iint_{\gamma_-}fg\mathrm{d}\gamma. \tag{4.5}$$

Proof. See [10, Lemma 2.2] for the proof. ∎

LEMMA 4.2. *For Boltzmann collision operator k, we have*

$$|k(\mathfrak{u},\mathfrak{v})|\lesssim\left(|\mathfrak{u}-\mathfrak{v}|+\frac{1}{|\mathfrak{u}-\mathfrak{v}|}\right)\mathrm{e}^{-\frac18|\mathfrak{u}-\mathfrak{v}|^2-\frac18\frac{\left||\mathfrak{u}|^2-|\mathfrak{v}|^2\right|^2}{|\mathfrak{u}-\mathfrak{v}|^2}}. \tag{4.6}$$

Proof. See [20, Lemma 3] for the proof. ∎

LEMMA 4.3. *Let* $0\leq\varrho<\frac14$ *and* $\vartheta\geq0$. *Then for* $\delta>0$ *sufficiently small and any* $\mathfrak{v}\in\mathbb{R}^3$,

$$\int_{\mathbb{R}^3}\mathrm{e}^{\delta|\mathfrak{u}-\mathfrak{v}|^2}|k(\mathfrak{u},\mathfrak{v})|\frac{\langle\mathfrak{v}\rangle^\vartheta\mathrm{e}^{\varrho|\mathfrak{v}|^2}}{\langle\mathfrak{u}\rangle^\vartheta\mathrm{e}^{\varrho|\mathfrak{u}|^2}}\mathrm{d}\mathfrak{u}\lesssim\frac{1}{\langle\mathfrak{v}\rangle}. \tag{4.7}$$

Proof. See [20, Lemma 3] for the proof. ∎

4.1. L^{2m} estimates.

LEMMA 4.4. *The solution $f(x,v)$ to (4.1) satisfies*

$$\varepsilon\|\mathbb{P}[f]\|_{2m} \lesssim \varepsilon\left\|(1-\mathcal{P})[f]\right\|_{\gamma_+,\frac{4m}{3}} + \|(\mathbb{I}-\mathbb{P})[f]\|_2 + \varepsilon\|(\mathbb{I}-\mathbb{P})[f]\|_{2m} + \left\|\nu^{-\frac{1}{2}}S\right\|_2 + \varepsilon\left\|h\right\|_{\gamma_-,\frac{4m}{3}}. \tag{4.8}$$

THEOREM 4.5. *The solution $f(x,v)$ to (4.1) satisfies the estimate*

$$\begin{aligned}&\frac{1}{\varepsilon^{\frac{1}{2}}}\left\|(1-\mathcal{P})[f]\right\|_{\gamma_+,2} + \frac{1}{\varepsilon}\left\|(\mathbb{I}-\mathbb{P})[f]\right\|_{\nu} + \|\mathbb{P}[f]\|_{2m} \lesssim o(1)\varepsilon^{\frac{3}{2m}}\left(\|f\|_{\gamma_+,\infty} + \|f\|_\infty\right)\\ &\quad + \frac{1}{\varepsilon^2}\|\mathbb{P}[S]\|_{\frac{2m}{2m-1}} + \frac{1}{\varepsilon}\left\|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[S]\right\|_2 + \|h\|_{\gamma_-,\frac{4m}{3}} + \frac{1}{\varepsilon}\|h\|_{\gamma_-,2}.\end{aligned} \tag{4.9}$$

4.2. L^∞ estimates.

THEOREM 4.6. *Assume (4.3) and (4.4) hold. The solution $f(x,v)$ to (4.1) satisfies for $\vartheta \geq 0$ and $0 \leq \varrho < \frac{1}{4}$,*

$$\begin{aligned}&\|f\|_{\infty,\vartheta,\varrho} + \|f\|_{\gamma_+,\infty,\varrho,\vartheta}\\ &\lesssim \frac{1}{\varepsilon^{2+\frac{3}{2m}}}\|\mathbb{P}[S]\|_{\frac{2m}{2m-1}} + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\left\|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[S]\right\|_2 + \left\|\nu^{-1}S\right\|_{\infty,\vartheta,\varrho}\\ &\quad + \frac{1}{\varepsilon^{\frac{3}{2m}}}\|h\|_{\gamma_-,\frac{4m}{3}} + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\|h\|_{\gamma_-,2} + \|h\|_{\gamma_-,\infty,\varrho,\vartheta}.\end{aligned} \tag{4.10}$$

Remark 4.1. Inserting Theorem 4.6 into Theorem 4.5, we actually have

$$\begin{aligned}&\frac{1}{\varepsilon^{\frac{1}{2}}}\left\|(1-\mathcal{P})[f]\right\|_{\gamma_+,2} + \frac{1}{\varepsilon}\left\|(\mathbb{I}-\mathbb{P})[f]\right\|_{\nu} + \|\mathbb{P}[f]\|_{2m}\\ &\lesssim \frac{1}{\varepsilon^2}\|\mathbb{P}[S]\|_{\frac{2m}{2m-1}} + \frac{1}{\varepsilon}\left\|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[S]\right\|_2 + \left\|\nu^{-1}S\right\|_{\infty,\vartheta,\varrho}\\ &\quad + \|h\|_{\gamma_-,\frac{4m}{3}} + \frac{1}{\varepsilon}\|h\|_{\gamma_-,2} + \|h\|_{\gamma_-,\infty,\varrho,\vartheta}.\end{aligned} \tag{4.11}$$

5. Evolutionary remainder estimates. We consider the linearized evolutionary Boltzmann equation

$$\begin{cases}\varepsilon^2\partial_t f + \varepsilon v\cdot\nabla_x f + \mathcal{L}[f] = S(t,x,v) & \text{in } \mathbb{R}_+\times\Omega\times\mathbb{R}^3,\\ f(0,x,v) = z(x,v) & \text{in } \Omega\times\mathbb{R}^3,\\ f(t,x_0,v) = \mathcal{P}[f](t,x_0,v) + h(t,x_0,v) & \text{on } \mathbb{R}_+\times\gamma_-,\end{cases} \tag{5.1}$$

where

$$\mathcal{P}[f](t,x_0,v) = \mu^{\frac{1}{2}}(v)\int_{\mathfrak{u}\cdot n(x_0)>0}\mu^{\frac{1}{2}}(\mathfrak{u})f(t,x_0,\mathfrak{u})\left|\mathfrak{u}\cdot n(x_0)\right|\mathrm{d}\mathfrak{u}. \tag{5.2}$$

The data z, S, and h satisfy the additional requirement

$$\begin{aligned}&\iint_{\Omega\times\mathbb{R}^3} z(x,v)\mu^{\frac{1}{2}}(v)\mathrm{d}v\mathrm{d}x = 0,\\ &\iint_{\Omega\times\mathbb{R}^3} S(t,x,v)\mu^{\frac{1}{2}}(v)\mathrm{d}v\mathrm{d}x + \int_{\gamma_-} h(t,x,v)\mu^{\frac{1}{2}}(v)\mathrm{d}\gamma = 0.\end{aligned} \tag{5.3}$$

Then we can easily derive

$$\iint_{\Omega\times\mathbb{R}^3} f(t,x,v)\mu^{\frac{1}{2}}(v)\mathrm{d}v\mathrm{d}x = 0. \tag{5.4}$$

Our analysis is based on the ideas in [10], [20], [42], and [44]. In particular, we will invoke the results of the stationary problem in the previous section. Since proof of the well-posedness of (5.1) is standard, we will focus on the a priori estimates here.

5.1. Preliminaries. We first introduce the well-known micro-macro decomposition. Define $\mathbb{P}$ as the orthogonal projection onto the null space of $\mathcal{L}$:

$$\mathbb{P}[f] := \mu^{\frac{1}{2}}(v)\bigg(a_f(t,x) + v\cdot b_f(t,x) + \frac{|v|^2-3}{2}c_f(t,x)\bigg) \in \mathcal{N}, \tag{5.5}$$

where a_f, b_f, and c_f are coefficients. When there is no confusion, we will simply write a,b,c. Definitely, $\mathcal{L}\Big[\mathbb{P}[f]\Big]=0$. Then the operator $\mathbb{I}-\mathbb{P}$ is naturally

$$(\mathbb{I}-\mathbb{P})[f] := f - \mathbb{P}[f], \tag{5.6}$$

which satisfies $(\mathbb{I}-\mathbb{P})[f]\in\mathcal{N}^{\perp}$, i.e., $\mathcal{L}[f] = \mathcal{L}\Big[(\mathbb{I}-\mathbb{P})[f]\Big]$.

LEMMA 5.1. *The linearized collision operator $\mathcal{L} = \nu I - K$ defined in* (1.27) *is self-adjoint in L^2. It satisfies*

$$\langle v\rangle \lesssim \nu(v) \lesssim \langle v\rangle\,, \tag{5.7}$$

$$\langle f, \mathcal{L}[f]\rangle(t,x) = \Big\langle (\mathbb{I}-\mathbb{P})[f], \mathcal{L}\Big[(\mathbb{I}-\mathbb{P})[f]\Big]\Big\rangle(t,x), \tag{5.8}$$

$$\big|(\mathbb{I}-\mathbb{P})[f(t,x)]\big|_{\nu}^2 \lesssim \langle f,\mathcal{L}[f]\rangle(t,x) \lesssim \big|(\mathbb{I}-\mathbb{P})[f(t,x)]\big|_{\nu}^2\,. \tag{5.9}$$

Proof. These are standard properties of $\mathcal{L}$. See [15, Chapter 3] and [20, Lemma 3]. ∎

LEMMA 5.2. *For $0<\delta<<1$, define the near-grazing set of $\gamma_{\pm}$,*

$$\gamma_{\pm}^{\delta} := \left\{(x,v)\in\gamma_{\pm} : |n(x)\cdot v|\leq\delta \quad or \quad |v|\geq\frac{1}{\delta} \quad or \quad |v|\leq\delta\right\}. \tag{5.10}$$

Then

$$\int_s^t \Big\| f\mathbf{1}_{\gamma_{\pm}\backslash\gamma_{\pm}^{\delta}}\Big\|_{\gamma,1} \leq C(\delta)\bigg(\varepsilon\|f(s)\|_1 + \int_s^t \Big(\|f\|_1 + \|\varepsilon\partial_t f + v\cdot\nabla_x f\|_1\Big)\bigg). \tag{5.11}$$

Proof. See [10, Lemma 2.1] with a standard time rescaling argument. ∎

LEMMA 5.3 (time-dependent Green's identity). *Assume $f(t,x,v),\ g(t,x,v)\in L^2(\mathbb{R}_+\times\Omega\times\mathbb{R}^3)$ and $\partial_t f + v\cdot\nabla_x f,\ \partial_t g + v\cdot\nabla_x g\in L^2(\mathbb{R}_+\times\Omega\times\mathbb{R}^3)$ with $f,\ g\in L^2(\mathbb{R}_+\times\gamma)$. Then for almost all $s,t\in\mathbb{R}_+$,*

$$\begin{aligned}\int_s^t\iint_{\Omega\times\mathbb{R}^3}\Big(&(\partial_t f + v\cdot\nabla_x f)g + (\partial_t g + v\cdot\nabla_x g)f\Big)\\ &= \int_s^t\iint_{\gamma_+} fg\mathrm{d}\gamma - \int_s^t\iint_{\gamma_-} fg\mathrm{d}\gamma + \iint_{\Omega\times\mathbb{R}^3} f(t)g(t) - \iint_{\Omega\times\mathbb{R}^3} f(s)g(s).\end{aligned} \tag{5.12}$$

Proof. See [10, Lemma 2.2] for the proof. ∎

5.2. L^2 estimates.

LEMMA 5.4. *Assume* (5.3) *and* (5.4) *hold. The solution* $f(t,x,v)$ *to the equation* (5.1) *satisfies*

$$\varepsilon|||\mathbb{P}[f]|||_2 \lesssim \varepsilon^{\frac{3}{2}}\|f(t)\|_2 + \varepsilon|||(1-\mathcal{P})[f]|||_{\gamma_+,2} + |||(\mathbb{I}-\mathbb{P})[f]|||_2 + \left|\left|\left|\nu^{-\frac{1}{2}}S\right|\right|\right|_2 + \varepsilon^{\frac{3}{2}}\|z\|_2 + \varepsilon|||h|||_{\gamma_-,2}. \tag{5.13}$$

Proof. Apply Green's identity in Lemma 5.3 to (5.1). Then for any $\psi \in L^2(\mathbb{R}_+ \times \Omega \times \mathbb{R}^3)$ satisfying $\varepsilon\partial_t\psi + v\cdot\nabla_x\psi \in L^2(\mathbb{R}_+ \times \Omega \times \mathbb{R}^3)$ and $\psi \in L^2(\mathbb{R}_+ \times \gamma)$, we have

$$\begin{aligned}\varepsilon\int_0^t\iint_{\gamma_+} f\psi\mathrm{d}\gamma - \varepsilon\int_0^t\iint_{\gamma_-} f\psi\mathrm{d}\gamma - \varepsilon\int_0^t\iint_{\Omega\times\mathbb{R}^3}(v\cdot\nabla_x\psi)f \\ = \varepsilon^2\int_0^t\iint_{\Omega\times\mathbb{R}^3} f\partial_t\psi - \varepsilon^2\iint_{\Omega\times\mathbb{R}^3} f(t)\psi(t) + \varepsilon^2\iint_{\Omega\times\mathbb{R}^3} f(0)\psi(0) \\ - \int_0^t\iint_{\Omega\times\mathbb{R}^3}\psi\mathcal{L}\Big[(\mathbb{I}-\mathbb{P})[f]\Big] + \int_0^t\iint_{\Omega\times\mathbb{R}^3} S\psi.\end{aligned} \tag{5.14}$$

The proof follows the same idea as in the proof of the stationary version (see Lemma 4.4) and [44, Lemma 4.3] with $m=1$. Actually, we use almost the same test function $\psi \sim \mu^{\frac{1}{2}}v\cdot\nabla_x\phi$ to estimate a, b, and c, where ϕ satisfies proper elliptic equations. Hence, we will omit the details and only present the main result. Compared with the stationary estimate, the new terms only show up on the right-hand side of (5.14). Using Hölder's inequality, we know

$$\left|\varepsilon^2\int_0^t\iint_{\Omega\times\mathbb{R}^3} f\partial_t\psi\right| \lesssim \varepsilon^2|||f|||_2|||\partial_t\psi|||_2 \lesssim \varepsilon^2|||\partial_t f|||_2|||\partial_t\nabla_x\phi|||_2. \tag{5.15}$$

In a similar fashion, we have

$$\left|\varepsilon^2\iint_{\Omega\times\mathbb{R}^3} f(t)\psi(t)\right| \lesssim \varepsilon^2\|f(t)\|_2\|\psi(t)\|_2 \lesssim \varepsilon^2\|f(t)\|_2\|\nabla_x\phi(t)\|_2, \tag{5.16}$$

$$\left|\varepsilon^2\iint_{\Omega\times\mathbb{R}^3} f(0)\psi(0)\right| \lesssim \varepsilon^2\|f(0)\|_2\|\psi(0)\|_2 \lesssim \varepsilon^2\|z\|_2\|\nabla_x\phi(0)\|_2. \tag{5.17}$$

Step 1: Estimates of c.

We choose the test function

$$\psi = \psi_c = \mu^{\frac{1}{2}}(v)\left(|v|^2 - \beta_c\right)\left(v\cdot\nabla_x\phi_c(t,x)\right), \tag{5.18}$$

where for fixed t,

$$\begin{cases} -\Delta_x\phi_c = c(t,x) \text{ in } \Omega, \\ \phi_c = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.19}$$

and constant $\beta_c \in \mathbb{R}$ is determined as in the proof of [44, Lemma 4.3]. Based on the standard elliptic estimates (see [29]), we have

$$\|\phi_c(t)\|_{H^2(\Omega)} \lesssim \|c(t)\|_{L^2(\Omega)}. \tag{5.20}$$

Eventually, we have

(5.21)
$$\begin{aligned}&\varepsilon\|c\|^2_{L^2([0,t]\times\Omega)}\\&\quad\lesssim\bigg(|||(\mathbb{I}-\mathbb{P})[f]|||_2+\Big|\Big|\Big|\nu^{-\frac12}S\Big|\Big|\Big|_2+\varepsilon|||(1-\mathcal{P})[f]|||_{\gamma_+,2}+\varepsilon|||h|||_{\gamma_-,2}\bigg)\|c\|_{L^2([0,t]\times\Omega)}\\&\qquad+\varepsilon^2\bigg(|||f|||_2|||\partial_t\nabla_x\phi_c|||_2+\|f(t)\|_2\|c(t)\|_{L^2(\Omega)}+\|z\|_2\|c(0)\|_{L^2(\Omega)}\bigg).\end{aligned}$$

Step 2: Estimates of b.

We further divide this step into several substeps:

Substep 2.1: Estimates of $\Big(\partial_i\partial_j\Delta_x^{-1}b_j\Big)b_i$ for $i,j=1,2,3$.

Let $b=(b_1,b_2,b_3)$. We choose the test functions for $i,j=1,2,3$,

$$\psi=\psi_{b,i,j}=\mu^{\frac12}(v)\left(v_i^2-\beta_{b,i,j}\right)\partial_j\phi_{b,j},\tag{5.22}$$

where

$$\begin{cases}-\Delta_x\phi_{b,j}=b_j(t,x)\ \text{ in }\ \Omega,\\ \phi_{b,j}=0\ \text{ on }\ \partial\Omega,\end{cases}\tag{5.23}$$

and constant $\beta_{b,i,j}\in\mathbb{R}$ is determined as in the proof of [44, Lemma 4.3]. Eventually, we obtain

(5.24)
$$\begin{aligned}&\varepsilon\left|\int_0^t\int_\Omega\Big(\partial_i\partial_j\Delta_x^{-1}b_j\Big)b_i\right|\\&\quad\lesssim\bigg(|||(\mathbb{I}-\mathbb{P})[f]|||_2+\Big|\Big|\Big|\nu^{-\frac12}S\Big|\Big|\Big|_2+\varepsilon|||(1-\mathcal{P})[f]|||_{\gamma_+,2}+\varepsilon|||h|||_{\gamma_-,2}\bigg)\|b\|_{L^2([0,t]\times\Omega)}\\&\qquad+\varepsilon^2\bigg(|||f|||_2|||\partial_t\nabla_x\phi_{b,j}|||_2+\|f(t)\|_2\|b(t)\|_{L^2(\Omega)}+\|z\|_2\|b(0)\|_{L^2(\Omega)}\bigg).\end{aligned}$$

Substep 2.2: Estimates of $\Big(\partial_i\partial_i\Delta_x^{-1}b_j\Big)b_j$ for $i\neq j$.

Notice that the $i=j$ case is included in Substep 2.1. We choose the test function

$$\psi=\tilde\psi_{b,i,j}=\mu^{\frac12}(v)\,|v|^2\,v_iv_j\partial_i\phi_{b,j}\quad\text{for}\quad i\neq j.\tag{5.25}$$

Eventually, we obtain

(5.26)
$$\begin{aligned}&\varepsilon\left|\int_0^t\int_\Omega\Big(\partial_i\partial_i\Delta_x^{-1}b_j\Big)b_j\right|\\&\quad\lesssim\bigg(|||(\mathbb{I}-\mathbb{P})[f]|||_2+\Big|\Big|\Big|\nu^{-\frac12}S\Big|\Big|\Big|_2+\varepsilon|||(1-\mathcal{P})[f]|||_{\gamma_+,2}+\varepsilon|||h|||_{\gamma_-,2}\bigg)\|b\|_{L^2([0,t]\times\Omega)}\\&\qquad+\varepsilon^2\bigg(|||f|||_2|||\partial_t\nabla_x\phi_{b,j}|||_2+\|f(t)\|_2\|b(t)\|_{L^2(\Omega)}+\|z\|_2\|b(0)\|_{L^2(\Omega)}\bigg).\end{aligned}$$

Substep 2.3: Synthesis.

Summarizing (5.24) and (5.26), we may sum up over $j=1,2,3$ to obtain, for any $i=1,2,3$,

(5.27)

$$\begin{aligned}\varepsilon&\|b_i\|^2_{L^2([0,t]\times\Omega)}\\ &\lesssim\left(|||(\mathbb{I}-\mathbb{P})[f]|||_2+\left|\left|\left|\nu^{-\frac{1}{2}}S\right|\right|\right|_2+\varepsilon|||(1-\mathcal{P})[f]|||_{\gamma_+,2}+\varepsilon|||h|||_{\gamma_-,2}\right)\|b\|_{L^2([0,t]\times\Omega)}\\ &\quad+\varepsilon^2\left(|||f|||_2\sum_{j=1}^3|||\partial_t\nabla_x\phi_{b,j}|||_2+\|f(t)\|_2\|b(t)\|_{L^2(\Omega)}+\|z\|_2\|b(0)\|_{L^2(\Omega)}\right),\end{aligned}$$

which further implies

(5.28)

$$\begin{aligned}\varepsilon&\|b\|^2_{L^2([0,t]\times\Omega)}\\ &\lesssim\left(|||(\mathbb{I}-\mathbb{P})[f]|||_2+\left|\left|\left|\nu^{-\frac{1}{2}}S\right|\right|\right|_2+\varepsilon|||(1-\mathcal{P})[f]|||_{\gamma_+,2}+\varepsilon|||h|||_{\gamma_-,2}\right)\|b\|_{L^2([0,t]\times\Omega)}\\ &\quad+\varepsilon^2\left(|||f|||_2\sum_{j=1}^3|||\partial_t\nabla_x\phi_{b,j}|||_2+\|f(t)\|_2\|b(t)\|_{L^2(\Omega)}+\|z\|_2\|b(0)\|_{L^2(\Omega)}\right).\end{aligned}$$

Step 3: Estimates of a.

We choose the test function

$$\psi=\psi_a=\mu^{\frac{1}{2}}(v)\left(|v|^2-\beta_a\right)\left(v\cdot\nabla_x\phi_a(t,x)\right), \tag{5.29}$$

where

$$\begin{cases}-\Delta_x\phi_a=a(t,x)\ \text{ in }\ \Omega,\\ \dfrac{\partial\phi_a}{\partial n}=0\ \text{ on }\ \partial\Omega,\end{cases} \tag{5.30}$$

and constant $\beta_a\in\mathbb{R}$ is determined as in the proof of [44, Lemma 4.3]. Eventually, we get

(5.31)

$$\begin{aligned}\varepsilon&\|a\|^2_{L^2([0,t]\times\Omega)}\\ &\lesssim\left(|||(\mathbb{I}-\mathbb{P})[f]|||_2+\left|\left|\left|\nu^{-\frac{1}{2}}S\right|\right|\right|_2+\varepsilon|||(1-\mathcal{P})[f]|||_{\gamma_+,2}+\varepsilon|||h|||_{\gamma_-,2}\right)\|a\|_{L^2([0,t]\times\Omega)}\\ &\quad+\varepsilon^2\left(|||f|||_2|||\partial_t\nabla_x\phi_a|||_2+\|f(t)\|_2\|a(t)\|_{L^2(\Omega)}+\|z\|_2\|a(0)\|_{L^2(\Omega)}\right).\end{aligned}$$

Step 4: First synthesis.

Collecting (5.21), (5.31), and (5.31), we deduce

(5.32)

$$\begin{aligned}\varepsilon|||\mathbb{P}[f]|||_2^2\lesssim&\left(|||(\mathbb{I}-\mathbb{P})[f]|||_2+\left|\left|\left|\nu^{-\frac{1}{2}}S\right|\right|\right|_2+\varepsilon|||(1-\mathcal{P})[f]|||_{\gamma_+,2}+\varepsilon|||h|||_{\gamma_-,2}\right)|||\mathbb{P}[f]|||_2\\ &+\varepsilon^2|||f|||_2\left(|||\partial_t\nabla_x\phi_a|||_2+\sum_{j=1}^3|||\partial_t\nabla_x\phi_{b,j}|||_2+|||\partial_t\nabla_x\phi_c|||_2\right)\\ &+\varepsilon^2\|f(t)\|_2^2+\varepsilon^2\|z\|_2^2.\end{aligned}$$

In order to close the proof, we must bound $|||\partial_t\nabla_x\phi_a|||_2$, $|||\partial_t\nabla_x\phi_{b,j}|||_2$, and $|||\partial_t\nabla_x\phi_c|||_2$.

Apply Green's identity in Lemma 4.1 to (5.1). Then for any $\psi \in L^2(\Omega\times\mathbb{R}^3)$ independent of time t satisfying $v\cdot\nabla_x\psi \in L^2(\Omega\times\mathbb{R}^3)$ and $\psi\in L^2(\gamma)$, we have

(5.33)
$$\varepsilon^2\iint_{\Omega\times\mathbb{R}^3}\partial_t f(t)\psi = -\varepsilon\iint_{\gamma_+} f(t)\psi \mathrm{d}\gamma + \varepsilon\iint_{\gamma_-} f(t)\psi\mathrm{d}\gamma + \varepsilon\iint_{\Omega\times\mathbb{R}^3}(v\cdot\nabla_x\psi)f(t) - \iint_{\Omega\times\mathbb{R}^3}\psi\mathcal{L}\Big[(\mathbb{I}-\mathbb{P})[f](t)\Big] + \iint_{\Omega\times\mathbb{R}^3}S(t)\psi.$$

Step 5: Estimate of $\partial_t\nabla_x\phi_c$.

For fixed t, taking $\psi = -\mu^{\frac12}\frac{|v|^2-3}{2}\partial_t\phi_c(t)$, using integration by parts, we have

(5.34)
$$\begin{aligned}\varepsilon^2\iint_{\Omega\times\mathbb{R}^3}\partial_t f(t)\psi &= -\varepsilon^2\iint_{\Omega\times\mathbb{R}^3}\partial_t f(t)\mu^{\frac12}\frac{|v|^2-3}{2}\partial_t\phi_c(t) = -\varepsilon^2\int_\Omega \partial_t c(t)\partial_t\phi_c(t)\\ &= -\varepsilon^2\int_\Omega \Delta_x\partial_t\phi_c(t)\partial_t\phi_c(t) = \varepsilon^2\int_\Omega|\partial_t\nabla_x\phi_c(t)|^2\\ &= \|\partial_t\nabla_x\phi_c(t)\|^2_{L^2(\Omega)}.\end{aligned}$$

Following a similar argument as in Steps 1–3, we have

$$\varepsilon^2|||\partial_t\nabla_x\phi_c|||_2 \lesssim \varepsilon\|b\|_{L^2([0,t]\times\Omega)} + \varepsilon|||(\mathbb{I}-\mathbb{P})[f]|||_2 + \left|\left|\left|\nu^{-\frac12}S\right|\right|\right|_2. \tag{5.35}$$

Step 6: Estimate of $\partial_t\nabla_x\phi_{b,j}$.

For fixed t, taking $\psi = -\mu^{\frac12}v_j\partial_t\phi_{b,j}(t)$, using integration by parts, we have

(5.36)
$$\begin{aligned}\varepsilon^2\iint_{\Omega\times\mathbb{R}^3}\partial_t f(t)\psi &= -\varepsilon^2\iint_{\Omega\times\mathbb{R}^3}\partial_t f(t)\mu^{\frac12}v_j\partial_t\phi_{b,j}(t) = -\varepsilon^2\int_\Omega\partial_t\phi_{b,j}(t)\partial_t\phi_{b,j}(t)\\ &= -\varepsilon^2\int_\Omega\Delta_x\partial_t\phi_{b,j}(t)\partial_t\phi_b^i(t) = \varepsilon^2\int_\Omega|\partial_t\nabla_x\phi_{b,j}(t)|^2\\ &= \|\partial_t\nabla_x\phi_{b,j}\|^2_{L^2(\Omega)}.\end{aligned}$$

Following a similar argument as in Steps 1–3, we have

(5.37)
$$\varepsilon^2|||\partial_t\nabla_x\phi_{b,j}|||_2 \lesssim \varepsilon\|a\|_{L^2([0,t]\times\Omega)} + \varepsilon\|c\|_{L^2([0,t]\times\Omega)} + \varepsilon|||(\mathbb{I}-\mathbb{P})[f]|||_2 + \left|\left|\left|\nu^{-\frac12}S\right|\right|\right|_2.$$

Step 7: Estimate of $\partial_t\nabla_x\phi_a$.

For fixed t, taking $\psi = -\mu^{\frac12}\partial_t\phi_a(t)$, using integration by parts, we have

$$\begin{aligned}\varepsilon^2\iint_{\Omega\times\mathbb{R}^3}\partial_t f(t)\psi &= -\varepsilon^2\iint_{\Omega\times\mathbb{R}^3}\partial_t f(t)\mu^{\frac12}\partial_t\phi_a(t) = -\varepsilon^2\int_\Omega\partial_t a(t)\partial_t\phi_a(t)\\ &= -\varepsilon^2\int_\Omega\Delta_x\partial_t\phi_a(t)\partial_t\phi_a(t) = \varepsilon^2\int_\Omega|\partial_t\nabla_x\phi_a(t)|^2\\ &= \varepsilon^2\|\partial_t\nabla_x\phi_a(t)\|^2_{L^2(\Omega)}.\end{aligned} \tag{5.38}$$

Following a similar argument as in Steps 1–3, we have

$$\varepsilon^2 |||\partial_t \nabla_x \phi_a|||_2 \lesssim \varepsilon \|b\|_{L^2([0,t]\times\Omega)} + \left|\left|\left|\nu^{-\frac{1}{2}} S\right|\right|\right|_2. \tag{5.39}$$

Step 8: Second synthesis.

Inserting (5.35), (5.37), and (5.39) into (5.32), we have

$$\begin{aligned}\varepsilon |||\mathbb{P}[f]|||_2^2 \lesssim& \left(|||(\mathbb{I}-\mathbb{P})[f]|||_2 + \left|\left|\left|\nu^{-\frac{1}{2}} S\right|\right|\right|_2 + \varepsilon |||(1-\mathcal{P})[f]|||_{\gamma_+,2} + \varepsilon |||h|||_{\gamma_-,2} \right) |||\mathbb{P}[f]|||_2 \\ &+ |||f|||_2 \left(\varepsilon |||\mathbb{P}[f]|||_2 + \varepsilon |||(\mathbb{I}-\mathbb{P})[f]|||_2 + \left|\left|\left|\nu^{-\frac{1}{2}} S\right|\right|\right|_2 \right) + \varepsilon^2 \|f(t)\|_2^2 + \varepsilon^2 \|z\|_2^2.\end{aligned} \tag{5.40}$$

Applying Cauchy's inequality, we have

$$\begin{aligned}\varepsilon |||\mathbb{P}[f]|||_2^2 \lesssim& \, o(1)\varepsilon |||\mathbb{P}[f]|||_2^2 + \varepsilon^2 \|f(t)\|_2^2 + \varepsilon |||(1-\mathcal{P})[f]|||_{\gamma_+,2}^2 + \frac{1}{\varepsilon} |||(\mathbb{I}-\mathbb{P})[f]|||_2^2 \\ &+ \frac{1}{\varepsilon} \left|\left|\left|\nu^{-\frac{1}{2}} S\right|\right|\right|_2^2 + \varepsilon^2 \|z\|_2^2 + \varepsilon |||h|||_{\gamma_-,2}^2.\end{aligned} \tag{5.41}$$

Hence, absorbing $o(1)\varepsilon |||\mathbb{P}[f]|||_2^2$ into the left-hand side, we have

$$\begin{aligned}\varepsilon |||\mathbb{P}[f]|||_2 \lesssim& \, \varepsilon^{\frac{3}{2}} \|f(t)\|_2 + \varepsilon |||(1-\mathcal{P})[f]|||_{\gamma_+,2} + |||(\mathbb{I}-\mathbb{P})[f]|||_2 \\ &+ \left|\left|\left|\nu^{-\frac{1}{2}} S\right|\right|\right|_2 + \varepsilon^{\frac{3}{2}} \|z\|_2 + \varepsilon |||h|||_{\gamma_-,2}.\end{aligned} \tag{5.42}$$

This completes our proof. ∎

THEOREM 5.5. *Assume* (5.3) *and* (5.4) *hold. The solution* $f(t,x,v)$ *to* (5.1) *satisfies*

$$\begin{aligned}&\|f(t)\|_2 + \frac{1}{\varepsilon^{\frac{1}{2}}} |||(1-\mathcal{P})[f]|||_{\gamma_+,2} + \frac{1}{\varepsilon} |||(\mathbb{I}-\mathbb{P})[f]|||_\nu + |||\mathbb{P}[f]|||_2 \\ &\lesssim \frac{1}{\varepsilon^2} |||\mathbb{P}[S]|||_2 + \frac{1}{\varepsilon} \left|\left|\left|\nu^{-\frac{1}{2}} (\mathbb{I}-\mathbb{P})[S]\right|\right|\right|_2 + \frac{1}{\varepsilon} |||h|||_{\gamma_-,2} + \|z\|_2.\end{aligned} \tag{5.43}$$

Proof.

Step 1: Energy estimate.

Multiplying f on both sides of (5.1) and applying Green's identity in Lemma 5.3 imply

$$\frac{\varepsilon^2}{2} \|f(t)\|_2^2 + \frac{\varepsilon}{2} |||f|||_{\gamma_+,2}^2 - \frac{\varepsilon}{2} |||\mathcal{P}[f]+h|||_{\gamma_-,2}^2 + \int_0^t \int_{\Omega\times\mathbb{R}^3} f\mathcal{L}[f] = \frac{\varepsilon^2}{2} \|z\|_2^2 + \int_0^t \int_{\Omega\times\mathbb{R}^3} fS. \tag{5.44}$$

Direct computation reveals that

$$\begin{aligned}&\frac{\varepsilon}{2} |||f|||_{\gamma_+,2}^2 - \frac{\varepsilon}{2} |||\mathcal{P}[f]+h|||_{\gamma_-,2}^2 \\ &\quad = \frac{\varepsilon}{2} |||f|||_{\gamma_+,2}^2 - \frac{\varepsilon}{2} |||\mathcal{P}[f]|||_{\gamma_-,2}^2 - \frac{\varepsilon}{2} |||h|||_{\gamma_-,2}^2 + \varepsilon \int_0^t \int_{\gamma_-} h\mathcal{P}[f] \mathrm{d}\gamma \\ &\quad = \frac{\varepsilon}{2} |||(1-\mathcal{P})[f]|||_{\gamma_+,2}^2 - \frac{\varepsilon}{2} |||h|||_{\gamma_-,2}^2 + \varepsilon \int_0^t \int_{\gamma_-} h\mathcal{P}[f] \mathrm{d}\gamma \\ &\quad \gtrsim \varepsilon |||(1-\mathcal{P})[f]|||_{\gamma_+,2}^2 - \frac{1}{\eta} |||h|||_{\gamma_-,2}^2 - \varepsilon^2 \eta |||\mathcal{P}[f]|||_{\gamma_+,2},\end{aligned} \tag{5.45}$$

where $0 < \eta << 1$ will be determined later. On the other hand, based on Lemma 5.1, we know

$$\int_0^t \int_{\Omega\times\mathbb{R}^3} f\mathcal{L}[f] \gtrsim |||(\mathbb{I}-\mathbb{P})[f]|||_\nu^2. \tag{5.46}$$

Inserting (5.45) and (5.46) into (5.44), we have

$$\begin{aligned} &\varepsilon^2\|f(t)\|_2^2 + \varepsilon|||(1-\mathcal{P})[f]|||_{\gamma_+,2}^2 + |||(\mathbb{I}-\mathbb{P})[f]|||_\nu^2 \\ &\quad\lesssim \eta\varepsilon^2|||\mathcal{P}[f]|||_{\gamma_+,2} + \varepsilon^2\|z\|_2^2 + \frac{1}{\eta}|||h|||_{\gamma_-,2}^2 + \int_0^t\int_{\Omega\times\mathbb{R}^3} fS. \end{aligned} \tag{5.47}$$

Step 2: $|||\mathcal{P}[f]|||_{\gamma_+,2}$.

Multiplying f on both sides of (5.1), we have

$$\varepsilon\partial_t(f^2) + v\cdot\nabla_x(f^2) = \frac{2}{\varepsilon}\Big(-f\mathcal{L}[f] + fS\Big). \tag{5.48}$$

Taking the absolute value and integrating (5.48) over $[0,t]\times\Omega\times\mathbb{R}^3$, using Lemma 5.1, we deduce

$$\Big|\Big|\Big|\varepsilon\partial_t(f^2) + v\cdot\nabla_x(f^2)\Big|\Big|\Big|_1 \lesssim \frac{1}{\varepsilon}\left(|||(\mathbb{I}-\mathbb{P})[f]|||_2^2 + \left|\int_0^t\int_{\Omega\times\mathbb{R}^3} fS\right|\right). \tag{5.49}$$

On the other hand, applying Lemma 5.2 to f^2, for near-grazing set γ^δ, we have

(5.50)

$$\begin{aligned} \Big|\Big|\Big|\mathbf{1}_{\gamma\backslash\gamma^\delta} f\Big|\Big|\Big|_{\gamma,2}^2 &= \Big|\Big|\Big|\mathbf{1}_{\gamma\backslash\gamma^\delta} f^2\Big|\Big|\Big|_{\gamma,1} \le C(\delta)\left(\varepsilon\left\|z^2\right\|_1 + \Big|\Big|\Big|f^2\Big|\Big|\Big|_1 + \Big|\Big|\Big|\varepsilon\partial_t(f^2) + v\cdot\nabla_x(f^2)\Big|\Big|\Big|_1\right) \\ &= C(\delta)\left(\varepsilon\|z\|_2^2 + |||f|||_2^2 + \Big|\Big|\Big|\varepsilon\partial_t(f^2) + v\cdot\nabla_x(f^2)\Big|\Big|\Big|_1\right) \\ &\lesssim C(\delta)\left(\varepsilon\|z\|_2^2 + |||f|||_2^2 + \frac{1}{\varepsilon}|||(\mathbb{I}-\mathbb{P})[f]|||_2^2 + \frac{1}{\varepsilon}\left|\int_0^t\int_{\Omega\times\mathbb{R}^3} fS\right|\right). \end{aligned}$$

We can rewrite $\mathcal{P}[f](t,x_0,v) = y(t,x)\mu^{\frac12}(v)$. Then for δ small, we deduce

(5.51)

$$\begin{aligned} \Big|\Big|\Big|\mathcal{P}[\mathbf{1}_{\gamma\backslash\gamma^\delta} f]\Big|\Big|\Big|_{\gamma,2}^2 &= \int_0^t\int_{\partial\Omega} |y(t,x)|^2\left(\int_{v\cdot n(x)\ge\delta,\delta\le|v|\le\delta^{-1}} \mu(v)\,|v\cdot n(x)|\,\mathrm{d}v\right)\mathrm{d}x \\ &\ge \frac{1}{2}\left(\int_0^t\int_{\partial\Omega}|y(t,x)|^2\right)\left(\int_{\gamma_+}\mu(v)\,|v\cdot n(x)|\,\mathrm{d}v\right) = \frac{1}{2}|||\mathcal{P}[f]|||_{\gamma_+,2}^2, \end{aligned}$$

where we utilize the bounds that

$$\int_{v\cdot n(x)\le\delta}\mu(v)\,|v\cdot n(x)|\,\mathrm{d}v \lesssim \delta, \tag{5.52}$$

$$\int_{|v|\le\delta \text{ or } |v|\ge\delta^{-1}}\mu(v)\,|v\cdot n(x)|\,\mathrm{d}v \lesssim \delta. \tag{5.53}$$

Therefore, from (5.51) and the fact

$$\Big|\Big|\Big|\mathcal{P}[\mathbf{1}_{\gamma\backslash\gamma^\delta} f]\Big|\Big|\Big|_{\gamma_+,2} \lesssim \Big|\Big|\Big|\mathbf{1}_{\gamma\backslash\gamma^\delta} f\Big|\Big|\Big|_{\gamma_+,2} \lesssim \Big|\Big|\Big|\mathbf{1}_{\gamma\backslash\gamma^\delta} f\Big|\Big|\Big|_{\gamma,2}, \tag{5.54}$$

we conclude

$$\|\|\mathcal{P}[f]\|\|^2_{\gamma_+,2} \lesssim \left\|\left\|\mathcal{P}[\mathbf{1}_{\gamma\setminus\gamma^\delta} f]\right\|\right\|_{\gamma_+,2} \lesssim \left\|\left\|\mathbf{1}_{\gamma\setminus\gamma^\delta} f\right\|\right\|_{\gamma,2}. \tag{5.55}$$

Considering (5.50), we have

$$\|\|\mathcal{P}[f]\|\|^2_{\gamma_+,2} \lesssim C(\delta)\left(\varepsilon\|z\|_2^2 + \|\|f\|\|_2^2 + \frac{1}{\varepsilon}\|\|(\mathbb{I}-\mathbb{P})[f]\|\|_2^2 + \frac{1}{\varepsilon}\left|\int_0^t\int_{\Omega\times\mathbb{R}^3} fS\right|\right). \tag{5.56}$$

For fixed $0<\delta<<1$ and using $f=\mathbb{P}[f]+(\mathbb{I}-\mathbb{P})[f]$, we obtain

$$\|\|\mathcal{P}[f]\|\|^2_{\gamma_+,2} \lesssim \varepsilon\|z\|_2^2 + \|\|\mathbb{P}[f]\|\|_2^2 + \frac{1}{\varepsilon}\|\|(\mathbb{I}-\mathbb{P})[f]\|\|_2^2 + \frac{1}{\varepsilon}\left|\int_0^t\int_{\Omega\times\mathbb{R}^3} fS\right|. \tag{5.57}$$

Step 3: Synthesis.

Plugging (5.57) into (5.47) with ε sufficiently small to absorb $\|\|(\mathbb{I}-\mathbb{P})[f]\|\|_2^2$ into the left-hand side, we obtain

$$\begin{aligned}\varepsilon^2\|f(t)\|_2^2 + \varepsilon\|\|(1-\mathcal{P})[f]\|\|^2_{\gamma_+,2} + \|\|(\mathbb{I}-\mathbb{P})[f]\|\|_\nu^2 &\lesssim \eta\varepsilon^2\|\|\mathbb{P}[f]\|\|_2^2 + \varepsilon^2\|z\|_2^2 \\ &\quad + \frac{1}{\eta}\|\|h\|\|^2_{\gamma_-,2} + \left|\int_0^t\int_{\Omega\times\mathbb{R}^3} fS\right|.\end{aligned} \tag{5.58}$$

We square on both sides of (5.13) to obtain

$$\begin{aligned}\varepsilon^2\|\|\mathbb{P}[f]\|\|_2^2 &\lesssim \varepsilon^3\|f(t)\|_2^2 + \varepsilon^2\|\|(1-\mathcal{P})[f]\|\|^2_{\gamma_+,2} + \|\|(\mathbb{I}-\mathbb{P})[f]\|\|_2^2 + \left\|\left\|\nu^{-\frac12}S\right\|\right\|_2^2 \\ &\quad + \varepsilon^3\|z\|_2^2 + \varepsilon^2\|\|h\|\|^2_{\gamma_-,2}.\end{aligned} \tag{5.59}$$

Multiplying a small constant on both sides of (5.59) and adding to (5.58) with η sufficiently small to absorb $\varepsilon^2\|\|(1-\mathcal{P})[f]\|\|^2_{\gamma_+,2}$, $\|\|(\mathbb{I}-\mathbb{P})[f]\|\|_2^2$, $\varepsilon^3\|f(t)\|_2^2$ and $\eta\varepsilon^2\|\|\mathbb{P}[f]\|\|_2^2$ into the left-hand side, we obtain

$$\begin{aligned}&\varepsilon^2\|f(t)\|_2^2 + \varepsilon\|\|(1-\mathcal{P})[f]\|\|^2_{\gamma_+,2} + \|\|(\mathbb{I}-\mathbb{P})[f]\|\|_\nu^2 + \varepsilon^2\|\|\mathbb{P}[f]\|\|_2^2 \\ &\quad\lesssim \|\|h\|\|^2_{\gamma_-,2} + \varepsilon^2\|z\|_2^2 + \left\|\left\|\nu^{-\frac12}S\right\|\right\|_2^2 + \left|\int_0^t\int_{\Omega\times\mathbb{R}^3} fS\right|.\end{aligned} \tag{5.60}$$

Applying Cauchy's inequality, we have

$$\begin{aligned}\left|\int_0^t\int_{\Omega\times\mathbb{R}^3} fS\right| &\lesssim \left|\int_0^t\int_{\Omega\times\mathbb{R}^3}(\mathbb{I}-\mathbb{P})[f](\mathbb{I}-\mathbb{P})[S]\right| + \left|\int_0^t\int_{\Omega\times\mathbb{R}^3}\mathbb{P}[f]\mathbb{P}[S]\right| \\ &\lesssim o(1)\|\|(\mathbb{I}-\mathbb{P})[f]\|\|_\nu^2 + \left\|\left\|\nu^{-\frac12}(\mathbb{I}-\mathbb{P})[S]\right\|\right\|_2^2 \\ &\quad + o(1)\varepsilon^2\|\|\mathbb{P}[f]\|\|_2^2 + \frac{1}{\varepsilon^2}\|\|\mathbb{P}[S]\|\|_2^2.\end{aligned} \tag{5.61}$$

Inserting (5.61) into (5.60) to absorb $o(1)\|\|(\mathbb{I}-\mathbb{P})[f]\|\|_2^2$ and $o(1)\varepsilon^2\|\|\mathbb{P}[f]\|\|_2^2$ into the left-hand side, we obtain

$$\begin{aligned}&\varepsilon^2\|f(t)\|_2^2 + \varepsilon\|\|(1-\mathcal{P})[f]\|\|^2_{\gamma_+,2} + \|\|(\mathbb{I}-\mathbb{P})[f]\|\|_\nu^2 + \varepsilon^2\|\|\mathbb{P}[f]\|\|_2^2 \\ &\quad\lesssim \frac{1}{\varepsilon^2}\|\|\mathbb{P}[S]\|\|_2^2 + \left\|\left\|\nu^{-\frac12}(\mathbb{I}-\mathbb{P})[S]\right\|\right\|_2^2 + \|\|h\|\|^2_{\gamma_-,2} + \varepsilon^2\|z\|_2^2.\end{aligned} \tag{5.62}$$

Hence, our desired result naturally follows. ∎

COROLLARY 5.6. *Since* (5.1) *is a linear equation, taking the time derivative on both sides, we know* $\partial_t f$ *satisfies*

$$\left\{\begin{array}{l}\varepsilon^2\partial_t(\partial_t f)+\varepsilon v\cdot\nabla_x(\partial_t f)+\mathcal{L}[\partial_t f]=\partial_t S(t,x,v)\quad in\ \ \mathbb{R}_+\times\Omega\times\mathbb{R}^3,\\[2mm] \partial_t f(0,x,v)=-\dfrac{1}{\varepsilon^2}\mathcal{L}[z(x,v)]-\dfrac{1}{\varepsilon}v\cdot\nabla_x z(x,v)+\dfrac{1}{\varepsilon^2}S(0,x,v)\quad in\ \ \Omega\times\mathbb{R}^3,\\[2mm] \partial_t f(t,x_0,v)=\mathcal{P}[\partial_t f](t,x_0,v)+\partial_t h(t,x_0,v)\quad on\ \ \mathbb{R}_+\times\gamma_-,\end{array}\right. \tag{5.63}$$

where we solve the initial data $\partial_t f(0,x,v)$ *from* (5.1). *Then applying Lemma* 5.5 *to* (5.63), *we obtain*

$$\begin{aligned}&\|\partial_t f(t)\|_2+\frac{1}{\varepsilon^{\frac12}}|||(1-\mathcal{P})[\partial_t f]|||_{\gamma_+,2}+\frac{1}{\varepsilon}|||(\mathbb{I}-\mathbb{P})[\partial_t f]|||_{\nu}+|||\mathbb{P}[\partial_t f]|||_2\\ &\lesssim\frac{1}{\varepsilon^2}|||\mathbb{P}[\partial_t S]|||_2+\frac{1}{\varepsilon}\left|\left|\left|\nu^{-\frac12}(\mathbb{I}-\mathbb{P})[\partial_t S]\right|\right|\right|_2+\frac{1}{\varepsilon}|||\partial_t h|||_{\gamma_-,2}+\frac{1}{\varepsilon^2}\|\nu z\|_2\\ &\quad+\frac{1}{\varepsilon}\|v\cdot\nabla_x z\|_2+\frac{1}{\varepsilon^2}\|S(0)\|_2.\end{aligned}\tag{5.64}$$

5.3. L^{2m} estimates. Throughout this section, we need $\frac32<m<3$. Let $o(1)$ denote a sufficiently small constant.

LEMMA 5.7. *Assume* (5.3) *and* (5.4) *hold. The solution* $f(t,x,v)$ *to* (5.1) *satisfies*

$$\begin{aligned}&\varepsilon\|\mathbb{P}[f(t)]\|_{2m}\lesssim\varepsilon\left\|(1-\mathcal{P})[f(t)]\right\|_{\gamma_+,\frac{4m}{3}}+\|(\mathbb{I}-\mathbb{P})[f(t)]\|_2+\varepsilon\|(\mathbb{I}-\mathbb{P})[f(t)]\|_{2m}\\ &\quad+\left\|\nu^{-\frac12}S(t)\right\|_2+\varepsilon\left\|h(t)\right\|_{\gamma_-,\frac{4m}{3}}+\varepsilon^2\|\partial_t f(t)\|_2.\end{aligned}\tag{5.65}$$

Proof. This is very similar to the proof of Lemma 5.4 and the stationary version (see Lemma 4.4) and [44, Lemma 4.3]. We apply Green's identity to (5.1) and choose particular test functions to control a, b, and c. However, there is no simple way to get around the $\partial_t\nabla_x\phi$ terms as in Steps 5–7 of the proof of Lemma 5.4. Here, we resort to stationary techniques, i.e., to use a time-independent Green's identity instead of time-dependent one.

Apply Green's identity in Lemma 4.1 to (5.1). Then for any $\psi(t)\in L^2(\Omega\times\mathbb{R}^3)$ satisfying $v\cdot\nabla_x\psi(t)\in L^2(\Omega\times\mathbb{R}^3)$ and $\psi(t)\in L^2(\gamma)$, we have

$$\begin{aligned}&\varepsilon\iint_{\gamma_+}f(t)\psi(t)\mathrm{d}\gamma-\varepsilon\iint_{\gamma_-}f(t)\psi(t)\mathrm{d}\gamma-\varepsilon\iint_{\Omega\times\mathbb{R}^3}\Big(v\cdot\nabla_x\psi(t)\Big)f(t)\\ &=-\iint_{\Omega\times\mathbb{R}^3}\psi(t)\mathcal{L}\Big[(\mathbb{I}-\mathbb{P})[f](t)\Big]+\iint_{\Omega\times\mathbb{R}^3}S(t)\psi(t)-\varepsilon^2\iint_{\Omega\times\mathbb{R}^3}\partial_t f(t)\psi(t).\end{aligned}\tag{5.66}$$

Then except from $-\varepsilon^2\iint_{\Omega\times\mathbb{R}^3}\partial_t f(t)\psi(t)$, this is exactly the same as the stationary estimates in Lemma 4.4, so we just mimick the proof there and that of Lemma 5.4, and point out the major differences. In particular, we always use the bound

$$\left|\varepsilon^2\iint_{\Omega\times\mathbb{R}^3}\partial_t f(t)\psi(t)\right|\lesssim\varepsilon^2|||\partial_t f(t)|||_2|||\psi(t)|||_2.\tag{5.67}$$

Step 1: Estimates of c.

We choose the test function

$$\psi(t)=\psi_c(t)=\mu^{\frac{1}{2}}(v)\left(|v|^2-\beta_c\right)\left(v\cdot\nabla_x\phi_c(t,x)\right), \tag{5.68}$$

where

$$\begin{cases} -\Delta_x\phi_c(t)=c\,|c|^{2m-2}(t,x) \ \text{ in } \ \Omega,\\ \phi_c(t)=0 \ \text{ on } \ \partial\Omega, \end{cases} \tag{5.69}$$

and constant $\beta_c\in\mathbb{R}$ is determined as in the proof of [44, Lemma 4.3]. Based on the standard elliptic estimates in [29], we have

$$\|\phi_c(t)\|_{W^{2,\frac{2m}{2m-1}}(\Omega)}\lesssim\left\||c(t)|^{2m-1}\right\|_{L^{\frac{2m}{2m-1}}(\Omega)}\lesssim\|c(t)\|_{L^{2m}(\Omega)}^{2m-1}. \tag{5.70}$$

Hence, by the Sobolev embedding theorem, we know

$$\|\psi_c(t)\|_2\lesssim\|\phi_c\|_{H^1(\Omega)}\lesssim\|\phi_c(t)\|_{W^{2,\frac{2m}{2m-1}}(\Omega)}\lesssim\|c(t)\|_{L^{2m}(\Omega)}^{2m-1}, \tag{5.71}$$

$$\|\phi_c(t)\|_{W^{1,\frac{2m}{2m-1}}(\Omega)}\lesssim\|\phi_c(t)\|_{W^{2,\frac{2m}{2m-1}}(\Omega)}\lesssim\|c(t)\|_{L^{2m}(\Omega)}^{2m-1}. \tag{5.72}$$

Also, for $1\leq m\leq 3$, using the Sobolev embedding theorem and trace estimates, we have

$$\begin{aligned} |\nabla_x\phi_c(t)|_{L^{\frac{4m}{4m-3}}(\partial\Omega)}&\lesssim|\nabla_x\phi_c(t)|_{W^{\frac{1}{2m},\frac{2m}{2m-1}}(\partial\Omega)}\lesssim\|\nabla_x\phi_c(t)\|_{W^{1,\frac{2m}{2m-1}}(\Omega)}\\ &\lesssim\|\phi_c(t)\|_{W^{2,\frac{2m}{2m-1}}(\Omega)}\lesssim\|c(t)\|_{L^{2m}(\Omega)}^{2m-1}. \end{aligned} \tag{5.73}$$

Eventually, we have

$$\begin{aligned} \varepsilon\|c(t)\|_{L^{2m}(\Omega)}&\lesssim\varepsilon\,\|(1-\mathcal{P})[f(t)]\|_{\gamma_+,\frac{4m}{3}}+\|(\mathbb{I}-\mathbb{P})[f(t)]\|_2+\varepsilon\|(\mathbb{I}-\mathbb{P})[f(t)]\|_{2m}\\ &\quad+\left\|\nu^{-\frac{1}{2}}S(t)\right\|_2+\varepsilon\,\|h(t)\|_{\gamma_-,\frac{4m}{3}}+\varepsilon^2\|\partial_tf(t)\|_2. \end{aligned} \tag{5.74}$$

Step 2: Estimates of b.

We further divide this step into several substeps:

Substep 2.1: Estimates of $(\partial_i\partial_j\Delta_x^{-1}(b_j\,|b_j|^{2m-2}))b_i$ for $i,j=1,2,3$.

Let $b=(b_1,b_2,b_3)$. We choose the test functions for $i,j=1,2,3$,

$$\psi(t)=\psi_{b,i,j}(t)=\mu^{\frac{1}{2}}(v)\left(v_i^2-\beta_{b,i,j}\right)\partial_j\phi_{b,j}, \tag{5.75}$$

where

$$\begin{cases} -\Delta_x\phi_{b,j}(t)=b_j\,|b_j|^{2m-2}(t,x) \ \text{ in } \ \Omega,\\ \phi_{b,j}(t)=0 \ \text{ on } \ \partial\Omega, \end{cases} \tag{5.76}$$

and constant $\beta_{b,i,j} \in \mathbb{R}$ is determined as in the proof of [44, Lemma 4.3]. We can recover the elliptic estimates and trace estimates. Eventually, we have

$$\begin{aligned}
&\varepsilon\left|\int_\Omega \left(\partial_i\partial_j\Delta_x^{-1}\Big(b_j\left|b_j\right|^{2m-2}\Big)\right)b_i\right| \\
&\lesssim \|b(t)\|_{L^{2m}(\Omega)}^{2m-1}\bigg(\varepsilon\left\|(1-\mathcal{P})[f(t)]\right\|_{\gamma_+,\frac{4m}{3}} + \|(\mathbb{I}-\mathbb{P})[f(t)]\|_2 + \varepsilon\|(\mathbb{I}-\mathbb{P})[f(t)]\|_{2m} \\
&\quad + \left\|\nu^{-\frac12}S(t)\right\|_2 + \varepsilon\left\|h(t)\right\|_{\gamma_-,\frac{4m}{3}} + \varepsilon^2\|\partial_t f(t)\|_2\bigg).
\end{aligned} \tag{5.77}$$

Substep 2.2: Estimates of $\left(\partial_i\partial_i\Delta_x^{-1}\Big(b_j\left|b_j\right|^{2m-2}\Big)\right)b_j$ for $i \neq j$.

Notice that the $i = j$ case is included in Substep 2.1. We choose the test function

$$\psi(t) = \tilde{\psi}_{b,i,j}(t) = \mu^{\frac12}(v)\left|v\right|^2 v_iv_j\partial_i\phi_{b,j} \quad \text{for} \quad i \neq j. \tag{5.78}$$

Eventually, we have

$$\begin{aligned}
&\varepsilon\left|\int_\Omega \left(\partial_i\partial_i\Delta_x^{-1}\Big(b_j\left|b_j\right|^{2m-2}\Big)\right)b_j\right| \\
&\lesssim \|b(t)\|_{L^{2m}(\Omega)}^{2m-1}\bigg(\varepsilon\left\|(1-\mathcal{P})[f(t)]\right\|_{\gamma_+,\frac{4m}{3}} + \|(\mathbb{I}-\mathbb{P})[f(t)]\|_2 + \varepsilon\|(\mathbb{I}-\mathbb{P})[f(t)]\|_{2m} \\
&\quad + \left\|\nu^{-\frac12}S(t)\right\|_2 + \varepsilon\left\|h(t)\right\|_{\gamma_-,\frac{4m}{3}} + \varepsilon^2\|\partial_t f(t)\|_2\bigg).
\end{aligned} \tag{5.79}$$

Substep 2.3: Synthesis.

Summarizing (5.77) and (5.79), we may sum up over $j = 1, 2, 3$ to obtain, for any $i = 1, 2, 3$,

$$\begin{aligned}
\varepsilon\|b_i(t)\|_{L^{2m}(\Omega)}^{2m} \lesssim \|b(t)\|_{L^{2m}(\Omega)}^{2m-1}\bigg(&\varepsilon\left\|(1-\mathcal{P})[f(t)]\right\|_{\gamma_+,\frac{4m}{3}} + \|(\mathbb{I}-\mathbb{P})[f(t)]\|_2 \\
&+ \varepsilon\|(\mathbb{I}-\mathbb{P})[f(t)]\|_{2m} \\
&+ \left\|\nu^{-\frac12}S(t)\right\|_2 + \varepsilon\left\|h(t)\right\|_{\gamma_-,\frac{4m}{3}} + \varepsilon^2\|\partial_t f(t)\|_2\bigg),
\end{aligned} \tag{5.80}$$

which further implies

$$\begin{aligned}
\varepsilon\|b(t)\|_{L^{2m}(\Omega)} \lesssim\ &\varepsilon\left\|(1-\mathcal{P})[f(t)]\right\|_{\gamma_+,\frac{4m}{3}} + \|(\mathbb{I}-\mathbb{P})[f(t)]\|_2 + \varepsilon\|(\mathbb{I}-\mathbb{P})[f(t)]\|_{2m} \\
&+ \left\|\nu^{-\frac12}S(t)\right\|_2 + \varepsilon\left\|h(t)\right\|_{\gamma_-,\frac{4m}{3}} + \varepsilon^2\|\partial_t f(t)\|_2.
\end{aligned} \tag{5.81}$$

Step 3: Estimates of a.

We choose the test function

$$\psi(t) = \psi_a(t) = \mu^{\frac12}(v)\left(\left|v\right|^2 - \beta_a\right)\Big(v\cdot\nabla_x\phi_a(t,x)\Big), \tag{5.82}$$

where

$$
\begin{cases} -\Delta_x\phi_a(t) = a\left|a\right|^{2m-2}(t,x) - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega a\left|a\right|^{2m-2}(t,x)\mathrm{d}x \ \text{ in } \ \Omega, \\ \dfrac{\partial\phi_a(t)}{\partial n} = 0 \ \text{ on } \ \partial\Omega, \end{cases} \tag{5.83}
$$

and constant $\beta_a \in \mathbb{R}$ is determined as in the proof of [44, Lemma 4.3]. We can recover the elliptic estimates and trace estimates. Eventually, we have

(5.84)

$$
\begin{aligned} \varepsilon\|a(t)\|_{L^{2m}(\Omega)} \lesssim{}& \varepsilon\left\|(1-\mathcal{P})[f(t)]\right\|_{\gamma_+,\frac{4m}{3}} + \|(\mathbb{I}-\mathbb{P})[f(t)]\|_2 + \varepsilon\|(\mathbb{I}-\mathbb{P})[f(t)]\|_{2m} \\ &+ \left\|\nu^{-\frac{1}{2}}S(t)\right\|_2 + \varepsilon\left\|h(t)\right\|_{\gamma_-,\frac{4m}{3}} + \varepsilon^2\|\partial_t f(t)\|_2. \end{aligned}
$$

Step 4: Synthesis.

Collecting (5.74), (5.81), and (5.84), we deduce

(5.85)

$$
\begin{aligned} \varepsilon\|\mathbb{P}[f(t)]\|_{2m} \lesssim{}& \varepsilon\left\|(1-\mathcal{P})[f(t)]\right\|_{\gamma_+,\frac{4m}{3}} + \|(\mathbb{I}-\mathbb{P})[f(t)]\|_2 + \varepsilon\|(\mathbb{I}-\mathbb{P})[f(t)]\|_{2m} \\ &+ \left\|\nu^{-\frac{1}{2}}S(t)\right\|_2 + \varepsilon\left\|h(t)\right\|_{\gamma_-,\frac{4m}{3}} + \varepsilon^2\|\partial_t f(t)\|_2. \end{aligned}
$$

□

THEOREM 5.8. *Assume* (5.3) *and* (5.4) *hold. The solution* $f(t,x,v)$ *to* (5.1) *satisfies*

(5.86)

$$
\begin{aligned} &\frac{1}{\varepsilon^{\frac{1}{2}}}\left\|(1-\mathcal{P})[f(t)]\right\|_{\gamma_+,2} + \frac{1}{\varepsilon}\left\|(\mathbb{I}-\mathbb{P})[f(t)]\right\|_\nu + \|\mathbb{P}[f(t)]\|_{2m} \\ &\quad + \frac{1}{\varepsilon^{\frac{1}{2}}}|||(1-\mathcal{P})[\partial_t f]|||_{\gamma_+,2} + \frac{1}{\varepsilon}|||(\mathbb{I}-\mathbb{P})[\partial_t f]|||_\nu + |||\mathbb{P}[\partial_t f]|||_2 \\ &\lesssim o(1)\varepsilon^{\frac{3}{2m}}\Big(\left\|f(t)\right\|_{\gamma_+,\infty} + \left\|f(t)\right\|_\infty\Big) + \frac{1}{\varepsilon^2}\|\mathbb{P}[S(t)]\|_{\frac{2m}{2m-1}} \\ &\quad + \frac{1}{\varepsilon}\left\|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[S(t)]\right\|_2 + \frac{1}{\varepsilon^2}|||\mathbb{P}[\partial_t S]|||_2 + \frac{1}{\varepsilon}\left|\left|\left|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[\partial_t S]\right|\right|\right|_2 + \left\|h(t)\right\|_{\gamma_-,\frac{4m}{3}} \\ &\quad + \frac{1}{\varepsilon}\left\|h(t)\right\|_{\gamma_-,2} + \frac{1}{\varepsilon}|||\partial_t h|||_{\gamma_-,2} + \frac{1}{\varepsilon^2}\|\nu z\|_2 + \frac{1}{\varepsilon}\|v\cdot\nabla_x z\|_2 + \frac{1}{\varepsilon^2}\|S(0)\|_2. \end{aligned}
$$

Proof.

Step 1: Energy estimate.

Multiplying f on both sides of (5.1) and using similar estimates as in the proof of Lemma 5.5, the stationary energy structure implies

(5.87)

$$
\begin{aligned} &\varepsilon\left\|(1-\mathcal{P})[f(t)]\right\|^2_{\gamma_+,2} + \left\|(\mathbb{I}-\mathbb{P})[f(t)]\right\|^2_\nu \\ &\quad\lesssim \eta\varepsilon^2\|\mathbb{P}[f(t)]\|_2^2 + \frac{1}{\eta}\left\|h(t)\right\|^2_{\gamma_-,2} + \left|\int_{\Omega\times\mathbb{R}^3} f(t)S(t)\right| + \varepsilon^2\left|\int_{\Omega\times\mathbb{R}^3} f(t)\partial_t f(t)\right|. \end{aligned}
$$

We square on both sides of (5.65) to obtain

(5.88)

$$
\begin{aligned} \varepsilon^2\|\mathbb{P}[f(t)]\|^2_{2m} \lesssim{}& \varepsilon^2\left\|(1-\mathcal{P})[f(t)]\right\|^2_{\gamma_+,\frac{4m}{3}} + \|(\mathbb{I}-\mathbb{P})[f(t)]\|_2^2 + \varepsilon^2\|(\mathbb{I}-\mathbb{P})[f(t)]\|^2_{2m} \\ &+ \left\|\nu^{-\frac{1}{2}}S(t)\right\|_2^2 + \varepsilon^2\left\|h(t)\right\|^2_{\gamma_-,\frac{4m}{3}} + \varepsilon^4\|\partial_t f(t)\|_2^2. \end{aligned}
$$

Hölder's inequality implies

$$\|\!|\mathbb{P}[f(t)]|\!\|_2 \lesssim \|\!|\mathbb{P}[f(t)]|\!\|_{2m}. \tag{5.89}$$

Multiplying a small constant on both sides of (5.88) and adding to (5.87) with η sufficiently small to absorb $\eta\varepsilon^2 \|\!|\mathbb{P}[f(t)]|\!\|_2^2$, and $\|\!|(\mathbb{I}-\mathbb{P})[f(t)]|\!\|_2^2$ into the left-hand side, we obtain

(5.90)

$$\begin{aligned}
&\varepsilon \left\|(1-\mathcal{P})[f(t)]\right\|_{\gamma_+,2}^2 + \|(\mathbb{I}-\mathbb{P})[f(t)]\|_{\nu}^2 + \varepsilon^2 \|\mathbb{P}[f(t)]\|_{2m}^2 \\
&\quad\lesssim \varepsilon^2 \left\|(1-\mathcal{P})[f(t)]\right\|_{\gamma_+,\frac{4m}{3}}^2 + \varepsilon^2 \|(\mathbb{I}-\mathbb{P})[f(t)]\|_{2m}^2 + \varepsilon^4 \|\partial_t f(t)\|_2^2 + \left\|\nu^{-\frac{1}{2}} S(t)\right\|_2^2 \\
&\qquad + \varepsilon^2 \left\|h(t)\right\|_{\gamma_-,\frac{4m}{3}}^2 + \left\|h(t)\right\|_{\gamma_-,2}^2 + \left|\int_{\Omega\times\mathbb{R}^3} f(t)S(t)\right| + \varepsilon^2 \left|\int_{\Omega\times\mathbb{R}^3} f(t)\partial_t f(t)\right|.
\end{aligned}$$

Now we have to handle $\varepsilon^2 \|\!|(1-\mathcal{P})[f(t)]|\!\|_{\gamma_+,\frac{4m}{3}}^2$ and $\varepsilon^2 \|\!|(\mathbb{I}-\mathbb{P})[f(t)]|\!\|_{2m}^2$ on the right-hand side.

Step 2: Interpolation argument.

By an interpolation estimate and Young's inequality, we have

(5.91)

$$\begin{aligned}
\left\|(1-\mathcal{P})[f(t)]\right\|_{\gamma_+,\frac{4m}{3}} &\leq \left\|(1-\mathcal{P})[f(t)]\right\|_{\gamma_+,2}^{\frac{3}{2m}} \left\|(1-\mathcal{P})[f(t)]\right\|_{\gamma_+,\infty}^{\frac{2m-3}{2m}} \\
&= \left(\frac{1}{\varepsilon^{\frac{6m-9}{4m^2}}} \left\|(1-\mathcal{P})[f(t)]\right\|_{\gamma_+,2}^{\frac{3}{2m}}\right)\left(\varepsilon^{\frac{6m-9}{4m^2}} \left\|(1-\mathcal{P})[f(t)]\right\|_{\gamma_+,\infty}^{\frac{2m-3}{2m}}\right) \\
&\lesssim \left(\frac{1}{\varepsilon^{\frac{6m-9}{4m^2}}} \left\|(1-\mathcal{P})[f(t)]\right\|_{\gamma_+,2}^{\frac{3}{2m}}\right)^{\frac{2m}{3}} \\
&\quad + o(1)\left(\varepsilon^{\frac{6m-9}{4m^2}} \left\|(1-\mathcal{P})[f(t)]\right\|_{\gamma_+,\infty}^{\frac{2m-3}{2m}}\right)^{\frac{2m}{2m-3}} \\
&\leq \frac{1}{\varepsilon^{\frac{2m-3}{2m}}} \left\|(1-\mathcal{P})[f(t)]\right\|_{\gamma_+,2} + o(1)\varepsilon^{\frac{3}{2m}} \left\|(1-\mathcal{P})[f(t)]\right\|_{\gamma_+,\infty} \\
&\leq \frac{1}{\varepsilon^{\frac{2m-3}{2m}}} \left\|(1-\mathcal{P})[f(t)]\right\|_{\gamma_+,2} + o(1)\varepsilon^{\frac{3}{2m}} \left\|(1-\mathcal{P})[f(t)]\right\|_{\gamma_+,\infty}.
\end{aligned}$$

Similarly, we have

(5.92)

$$\begin{aligned}
\|(\mathbb{I}-\mathbb{P})[f(t)]\|_{2m} &\leq \|(\mathbb{I}-\mathbb{P})[f(t)]\|_2^{\frac{1}{m}} \|(\mathbb{I}-\mathbb{P})[f(t)]\|_\infty^{\frac{m-1}{m}} \\
&= \left(\frac{1}{\varepsilon^{\frac{3m-3}{2m^2}}} \|(\mathbb{I}-\mathbb{P})[f(t)]\|_2^{\frac{1}{m}}\right)\left(\varepsilon^{\frac{3m-3}{2m^2}} \|(\mathbb{I}-\mathbb{P})[f(t)]\|_\infty^{\frac{m-1}{m}}\right) \\
&\lesssim \left(\frac{1}{\varepsilon^{\frac{3m-3}{2m^2}}} \|(\mathbb{I}-\mathbb{P})[f(t)]\|_2^{\frac{1}{m}}\right)^m + o(1)\left(\varepsilon^{\frac{3m-3}{2m^2}} \|(\mathbb{I}-\mathbb{P})[f(t)]\|_\infty^{\frac{m-1}{m}}\right)^{\frac{m}{m-1}} \\
&\leq \frac{1}{\varepsilon^{\frac{3m-3}{2m}}} \|(\mathbb{I}-\mathbb{P})[f(t)]\|_2 + o(1)\varepsilon^{\frac{3}{2m}} \|(\mathbb{I}-\mathbb{P})[f(t)]\|_\infty.
\end{aligned}$$

We need this extra $\varepsilon^{\frac{3}{2m}}$ for the convenience of the L^∞ estimate. Then we know for sufficiently small ε and $\frac{3}{2} < m < 3$,

(5.93)
$$\begin{aligned}\varepsilon^2 \left\|(1-\mathcal{P})[f(t)]\right\|^2_{\gamma_+,\frac{4m}{3}} &\lesssim \varepsilon^{2-\frac{2m-3}{m}} \left\|(1-\mathcal{P})[f(t)]\right\|^2_{\gamma_+,2} + o(1)\varepsilon^{2+\frac{3}{m}} \left\|(1-\mathcal{P})[f(t)]\right\|^2_{\gamma_+,\infty}\\ &\lesssim o(1)\varepsilon \left\|(1-\mathcal{P})[f(t)]\right\|^2_{\gamma_+,2} + o(1)\varepsilon^{2+\frac{3}{m}} \left\|f(t)\right\|^2_{\gamma_+,\infty}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\varepsilon^2\|(\mathbb{I}-\mathbb{P})[f(t)]\|^2_{2m} &\lesssim \varepsilon^{2-\frac{3m-3}{m}}\|(\mathbb{I}-\mathbb{P})[f(t)]\|^2_2 + o(1)\varepsilon^{2+\frac{3}{m}}\|(\mathbb{I}-\mathbb{P})[f(t)]\|^2_\infty\\ &\lesssim o(1)\|(\mathbb{I}-\mathbb{P})[f(t)]\|^2_2 + o(1)\varepsilon^{2+\frac{3}{m}}\|f(t)\|^2_\infty.\end{aligned} \tag{5.94}$$

Inserting (5.93) and (5.94) into (5.90), and absorbing $o(1)\varepsilon \left\|(1-\mathcal{P})[f(t)]\right\|^2_{\gamma_+,2}$ and $o(1)\|(\mathbb{I}-\mathbb{P})[f(t)]\|^2_2$ into the left-hand side, we obtain

(5.95)
$$\begin{aligned}&\varepsilon \left\|(1-\mathcal{P})[f(t)]\right\|^2_{\gamma_+,2} + \|(\mathbb{I}-\mathbb{P})[f(t)]\|^2_\nu + \varepsilon^2\|\mathbb{P}[f(t)]\|^2_{2m}\\ &\quad\lesssim o(1)\varepsilon^{2+\frac{3}{m}}\left(\|f(t)\|^2_{\gamma_+,\infty} + \|f(t)\|^2_\infty\right) + \varepsilon^4\|\partial_t f(t)\|^2_2 + \left\|\nu^{-\frac{1}{2}}S(t)\right\|^2_2 + \varepsilon^2\left\|h(t)\right\|^2_{\gamma_-,\frac{4m}{3}}\\ &\qquad + \|h(t)\|^2_{\gamma_-,2} + \left|\int_{\Omega\times\mathbb{R}^3} f(t)S(t)\right| + \varepsilon^2\left|\int_{\Omega\times\mathbb{R}^3} f(t)\partial_t f(t)\right|.\end{aligned}$$

Step 3: Synthesis.

We can decompose

$$\int_{\Omega\times\mathbb{R}^3} f(t)S(t) = \iint_{\Omega\times\mathbb{R}^3} \mathbb{P}[f(t)]\mathbb{P}[S(t)] + \iint_{\Omega\times\mathbb{R}^3} (\mathbb{I}-\mathbb{P})[f(t)](\mathbb{I}-\mathbb{P})[S(t)]. \tag{5.96}$$

Hölder's inequality and Cauchy's inequality imply

$$\begin{aligned}\iint_{\Omega\times\mathbb{R}^3} \mathbb{P}[f(t)]\mathbb{P}[S(t)] \leq \|\mathbb{P}[f(t)]\|_{2m}\|\mathbb{P}[S(t)]\|_{\frac{2m}{2m-1}} &\lesssim o(1)\varepsilon^2\|\mathbb{P}[f(t)]\|^2_{2m}\\ &\quad + \frac{1}{\varepsilon^2}\|\mathbb{P}[S(t)]\|^2_{\frac{2m}{2m-1}},\end{aligned} \tag{5.97}$$

and

(5.98)
$$\iint_{\Omega\times\mathbb{R}^3} (\mathbb{I}-\mathbb{P})[f(t)](\mathbb{I}-\mathbb{P})[S(t)] \lesssim o(1)\left\|(\mathbb{I}-\mathbb{P})[f(t)]\right\|^2_\nu + \left\|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[S(t)]\right\|^2_2.$$

Inserting (5.97) and (5.98) into (5.96) and further (5.90), absorbing $o(1)\varepsilon^2\|\mathbb{P}[f](t)\|^2_{2m}$ and $o(1)\left\|(\mathbb{I}-\mathbb{P})[f(t)]\right\|^2_\nu$ into the left-hand side, we get

(5.99)
$$\begin{aligned}&\varepsilon \left\|(1-\mathcal{P})[f(t)]\right\|^2_{\gamma_+,2} + \|(\mathbb{I}-\mathbb{P})[f(t)]\|^2_\nu + \varepsilon^2\|\mathbb{P}[f(t)]\|^2_{2m}\\ &\quad\lesssim o(1)\varepsilon^{2+\frac{3}{m}}\left(\|f(t)\|^2_{\gamma_+,\infty} + \|f(t)\|^2_\infty\right) + \varepsilon^4\|\partial_t f(t)\|^2_2 + \frac{1}{\varepsilon^2}\|\mathbb{P}[S(t)]\|^2_{\frac{2m}{2m-1}}\\ &\qquad + \left\|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[S(t)]\right\|^2_2 + \varepsilon^2\left\|h(t)\right\|^2_{\gamma_-,\frac{4m}{3}} + \|h(t)\|^2_{\gamma_-,2} + \varepsilon^2\left|\int_{\Omega\times\mathbb{R}^3} f(t)\partial_t f(t)\right|.\end{aligned}$$

Now we handle the most difficult term:

$$\varepsilon^2\left|\int_{\Omega\times\mathbb{R}^3} f(t)\partial_t f(t)\right| \lesssim \varepsilon^2\|f(t)\|_2\|\partial_t f(t)\|_2 \lesssim o(1)\varepsilon^2\|f(t)\|_2^2+\varepsilon^2\|\partial_t f(t)\|_2^2. \tag{5.100}$$

Here $o(1)\varepsilon^2\|f(t)\|_2^2$ can be absorbed into the left-hand side of (5.99). Then we resort to (5.64) to tackle $\varepsilon^2\|\partial_t f(t)\|_2^2$:

$$\begin{aligned}&\varepsilon^2\|\partial_t f(t)\|_2^2+\varepsilon|||(1-\mathcal{P})[\partial_t f]|||_{\gamma_+,2}^2+|||(\mathbb{I}-\mathbb{P})[\partial_t f]|||_{\nu}^2+\varepsilon^2|||\mathbb{P}[\partial_t f]|||_2^2\\ &\quad\lesssim \frac{1}{\varepsilon^2}|||\mathbb{P}[\partial_t S]|||_2^2+\left|\left|\left|\nu^{-\frac12}(\mathbb{I}-\mathbb{P})[\partial_t S]\right|\right|\right|_2^2+|||\partial_t h|||_{\gamma_-,2}^2+\frac{1}{\varepsilon^2}\|\nu z\|_2^2\\ &\qquad+\|v\cdot\nabla_x z\|_2^2+\frac{1}{\varepsilon^2}\|S(0)\|_2^2.\end{aligned} \tag{5.101}$$

Multiplying a small constant on (5.99) and adding it to (5.101) to absorb $\varepsilon^2\|\partial_t f(t)\|_2^2$, we have

(5.102)

$$\begin{aligned}&\varepsilon\,\|(1-\mathcal{P})[f(t)]\|_{\gamma_+,2}^2+\|(\mathbb{I}-\mathbb{P})[f(t)]\|_{\nu}^2+\varepsilon^2\|\mathbb{P}[f(t)]\|_{2m}^2\\ &\quad+\varepsilon|||(1-\mathcal{P})[\partial_t f]|||_{\gamma_+,2}^2+|||(\mathbb{I}-\mathbb{P})[\partial_t f]|||_{\nu}^2+\varepsilon^2|||\mathbb{P}[\partial_t f]|||_2^2\\ &\lesssim o(1)\varepsilon^{2+\frac{3}{m}}\Big(\|f(t)\|_{\gamma_+,\infty}^2+\|f(t)\|_\infty^2\Big)\\ &\quad+\frac{1}{\varepsilon^2}\|\mathbb{P}[S(t)]\|_{\frac{2m}{2m-1}}^2+\left\|\nu^{-\frac12}(\mathbb{I}-\mathbb{P})[S(t)]\right\|_2^2+\frac{1}{\varepsilon^2}|||\mathbb{P}[\partial_t S]|||_2^2+\left|\left|\left|\nu^{-\frac12}(\mathbb{I}-\mathbb{P})[\partial_t S]\right|\right|\right|_2^2\\ &\quad+\varepsilon^2\,\|h(t)\|_{\gamma_-,\frac{4m}{3}}^2+\|h(t)\|_{\gamma_-,2}^2+|||\partial_t h|||_{\gamma_-,2}^2+\frac{1}{\varepsilon^2}\|\nu z\|_2^2+\|v\cdot\nabla_x z\|_2^2+\frac{1}{\varepsilon^2}\|S(0)\|_2^2.\end{aligned}$$

Then our desired result follows. ∎

Remark 5.1. Roughly speaking, Theorem 5.8 justifies that in order to bound instantaneously f in L^{2m}, we need the accumulative bound for f and $\partial_t f$ in L^2.

5.4. L^∞ estimates. Now we begin to consider the mild formulation. When tracking the solution backward along the characteristics, once it hits the in-flow boundary or initial time, it either terminates (when hitting the initial time) or is diffusively reflected (when hitting the boundary). Following this idea, we may define the backward stochastic cycles, with multiple hitting times and out-flow integrals.

DEFINITION 5.9 (hitting time and position). *For any $(t,x,v)\in\mathbb{R}_+\times\Omega\times\mathbb{R}^3$ with $(x,v)\notin\gamma_0$, define the backward hitting time*

$$t_b(t,x,v):=\inf\{s>0: x-\varepsilon sv\notin\Omega \quad or \quad t=\varepsilon^2 s\}. \tag{5.103}$$

Also, define the hitting position

$$x_b := x-\varepsilon t_b(x,v)v. \tag{5.104}$$

Note that $x_b\in\Omega$ means the characteristic already hit the initial time, and $x_b\in\partial\Omega$ means the characteristic hits the boundary, so it can be reflected and continue moving.

DEFINITION 5.10 (stochastic cycle). *For any $(t,x,v)\in\mathbb{R}_+\times\Omega\times\mathbb{R}^3$ with $(x,v)\notin\gamma_0$, let $(t_0,x_0,v_0)=(t,x,v)$. Define the first stochastic triple*

$$(t_1,x_1,v_1):=\Big(t-\varepsilon^2t_b(x_0,v_0),x_b(x_0,v_0),v_1\Big) \tag{5.105}$$

for some v_1 satisfying $v_1\cdot n(x_1)>0$.

Inductively, assume we know the kth *stochastic triple (t_k,x_k,v_k) with $t_k>0$ (i.e., $x_k\in\partial\Omega$). Define the* $(k+1)$th *stochastic triple*

$$(t_{k+1},x_{k+1},v_{k+1}):=\Big(t_k-\varepsilon^2t_b(x_k,v_k),x_k(x_k,v_k),v_{k+1}\Big) \tag{5.106}$$

for some v_{k+1} satisfying $v_{k+1}\cdot n(x_{k+1})>0$.

Remark 5.2. Roughly speaking, this definition describes one characteristic line with reflection (alternatively the so-called stochastic cycle), starting from $(t_k,x_k,v_k)\in\gamma_+$, then tracking back to $(t_{k+1},x_{k+1},v_k)\in\{0\}\times\Omega\times\mathbb{R}^3$ which will terminate, or $(t_{k+1},x_{k+1},v_k)\in(0,\infty)\times\gamma_-$, diffusively reflected to $(t_{k+1},x_{k+1},v_{k+1})\in\gamma_+$, and beginning a new cycle. t_k is the actual time the characteristic moves backward. Note that we are free to choose any $v_k\cdot n(x_k)>0$, so a different sequence $\{v_k\}_{k=1}^{\infty}$ represents different stochastic cycles.

DEFINITION 5.11 (diffusive reflection integral). *Let $\mathcal{V}_k=\{v\in\mathbb{R}^3: v\cdot n(x_k)>0\}$, so the stochastic cycle must satisfy $v_k\in\mathcal{V}_k$. Let the iterated integral for $k\geq 2$ be defined as*

$$\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\prod_{j=1}^{k-1}\mathrm{d}\sigma_j:=\int_{\mathcal{V}_1}\ldots\bigg(\int_{\mathcal{V}_{k-1}}\mathrm{d}\sigma_{k-1}\bigg)\cdots\mathrm{d}\sigma_1, \tag{5.107}$$

where $\mathrm{d}\sigma_j:=\mu(v_j)\,|v_j\cdot n(x_j)|\,\mathrm{d}v_j$ is a probability measure.

We define a weight function scaled with parameter ξ for $0\leq\varrho<\frac14$ and $\vartheta\geq 0$,

$$w(v):=\langle v\rangle^{\vartheta}\,\mathrm{e}^{\varrho|v|^2} \tag{5.108}$$

and

$$\tilde{w}(v):=\frac{1}{\mu^{\frac12}(v)w(v)}=\sqrt{2\pi}\frac{\mathrm{e}^{\left(\frac14-\varrho\right)|v|^2}}{\Big(1+|v|^2\Big)^{\frac{\vartheta}{2}}}. \tag{5.109}$$

LEMMA 5.12. *For $T_0>0$ sufficiently large, there exists constants $C_1,C_2>0$ independent of T_0, such that for $k=C_1T_0^{\frac54}$ and $(x,v)\in\times\bar{\Omega}\times\mathbb{R}^3$,*

$$\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\mathbf{1}_{\left\{\frac{t-t_k(x,v,v_1,\ldots,v_{k-1})}{\varepsilon^2}<\frac{T_0}{\varepsilon}\right\}}\prod_{j=1}^{k-1}\mathrm{d}\sigma_j\leq\left(\frac12\right)^{C_2T_0^{\frac54}}. \tag{5.110}$$

Proof. This is a rescaled version of [10, Lemma 4.1]. Since our hitting time in (5.103) is rescaled with ε, we should rescale back in the statement of the lemma. ◻

Remark 5.3. Roughly speaking, Lemma 5.12 states that even though we have the freedom to choose v_k in each stochastic cycle, in the long run, the accumulative time will not be too small. After enough reflections $\sim k$, most characteristics have the accumulative time that will exceed any set threshold T_0.

THEOREM 5.13. *Assume* (5.3) *and* (5.4) *hold. The solution* $f(t,x,v)$ *to* (5.1) *satisfies for* $\vartheta \geq 0$ *and* $0 \leq \varrho < \frac{1}{4}$,

(5.111)
$$\begin{aligned}
&|||f|||_{\infty,\vartheta,\varrho} + |||f|||_{\gamma_+,\infty,\varrho,\vartheta} \\
&\quad\lesssim \frac{1}{\varepsilon^{2+\frac{3}{2m}}}\|\mathbb{P}[S(t)]\|_{\frac{2m}{2m-1}} + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\left\|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[S(t)]\right\|_2 + \frac{1}{\varepsilon^{2+\frac{3}{2m}}}|||\mathbb{P}[\partial_t S]|||_2 \\
&\qquad + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\left|\left|\left|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[\partial_t S]\right|\right|\right|_2 + \left|\left|\left|\nu^{-1}S\right|\right|\right|_{\infty,\vartheta,\varrho} + \frac{1}{\varepsilon^{\frac{3}{2m}}}\|h(t)\|_{\gamma_-,\frac{4m}{3}} \\
&\qquad + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\|h(t)\|_{\gamma_-,2} + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}|||\partial_t h|||_{\gamma_-,2} + |||h|||_{\gamma_-,\infty,\varrho,\vartheta} \\
&\qquad + \frac{1}{\varepsilon^{2+\frac{3}{2m}}}\|\nu z\|_2 + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\|v\cdot\nabla_x z\|_2 + \|z\|_{\infty,\vartheta,\varrho} + \frac{1}{\varepsilon^{2+\frac{3}{2m}}}\|S(0)\|_2.
\end{aligned}$$

Proof.

Step 1: Mild formulation.

Denote the weighted solution

$$g(t,x,v) := w(v)f(t,x,v), \tag{5.112}$$

and the weighted nonlocal operator

$$K_{w(v)}[g](v) := w(v)K\left[\frac{g}{w}\right](v) = \int_{\mathbb{R}^3} k_{w(v)}(v,\mathfrak{u})g(\mathfrak{u})\mathrm{d}\mathfrak{u}, \tag{5.113}$$

where

$$k_{w(v)}(v,\mathfrak{u}) := k(v,\mathfrak{u})\frac{w(v)}{w(\mathfrak{u})}. \tag{5.114}$$

Multiplying w on both sides of (5.1), we have

$$\begin{cases}
\varepsilon^2\partial_t g + \varepsilon v\cdot\nabla_x g + \nu g = K_w(t,x,v) + w(v)S(t,x,v) \quad \text{in} \quad \mathbb{R}_+\times\Omega\times\mathbb{R}^3, \\
g(0,x,v) = w(v)z(x,v) \quad \text{in} \quad \Omega\times\mathbb{R}^3, \\
g(t,x_0,v) = w(v)\mu^{\frac{1}{2}}(v)\displaystyle\int_{\mathfrak{u}\cdot n>0}\tilde{w}(\mathfrak{u})g(t,x_0,\mathfrak{u})\mathrm{d}\mathfrak{u} + wh(t,x_0,v) \\
\quad \text{for} \quad x_0\in\partial\Omega \quad \text{and} \quad v\cdot n<0,
\end{cases} \tag{5.115}$$

We introduce the indicator function $\mathbf{1}_{\{t_k=0\}}$ which implies the characteristic hits the initial time and $\mathbf{1}_{\{t_k>0\}}$ which implies the characteristic hits the boundary. We can rewrite the solution of (5.1) along the characteristics by Duhamel's principle as

(5.116)
$$\begin{aligned}
g(t,x,v) = &\left(\mathbf{1}_{\{t_1=0\}}w(v)z(x_1,v)\mathrm{e}^{-\nu(v)\frac{t-t_1}{\varepsilon^2}} + \mathbf{1}_{\{t_1>0\}}w(v)h(t_1,x_1,v)\mathrm{e}^{-\nu(v)\frac{t-t_1}{\varepsilon^2}}\right) \\
&+ \int_0^{\frac{t-t_1}{\varepsilon^2}} w(v)S\Big(t-\varepsilon^2 s, x-\varepsilon s v, v\Big)\mathrm{e}^{-\nu(v)s}\mathrm{d}s \\
&+ \int_0^{\frac{t-t_1}{\varepsilon^2}} K_{w(v)}[g]\Big(t-\varepsilon^2 s, x-\varepsilon s v, v\Big)\mathrm{e}^{-\nu(v)s}\mathrm{d}s \\
&+ \frac{\mathrm{e}^{-\nu(v)\frac{t-t_1}{\varepsilon^2}}}{\tilde{w}(v)}\int_{\mathcal{V}_1} g(t_1,x_1,v_1)\tilde{w}(v_1)\mathrm{d}\sigma_1,
\end{aligned}$$

where the last term refers to $\mathcal{P}[f]$. We may further rewrite the last term using (5.116) along the stochastic cycle by applying Duhamel's principle k times as

(5.117)
$$\begin{aligned}
g(t,x,v) = &\Big(\mathbf{1}_{\{t_k=0\}}w(v)z(x_1,v)\mathrm{e}^{-\nu(v)\frac{t-t_1}{\varepsilon^2}} + \mathbf{1}_{\{t_k>0\}}w(v)h(t_1,x_1,v)\mathrm{e}^{-\nu(v)\frac{t-t_1}{\varepsilon^2}}\Big)\\
&+\int_0^{\frac{t-t_1}{\varepsilon^2}} w(v)S\Big(t-\varepsilon^2 s, x-\varepsilon s v, v\Big)\mathrm{e}^{-\nu(v)s}\mathrm{d}s\\
&+\int_0^{\frac{t-t_1}{\varepsilon^2}} K_{w(v)}[g]\Big(t-\varepsilon^2 s, x-\varepsilon s v, v\Big)\mathrm{e}^{-\nu(v)s}\mathrm{d}s\\
&+\frac{\mathrm{e}^{-\nu(v)\frac{t-t_1}{\varepsilon^2}}}{\tilde{w}(v)}\sum_{\ell=1}^{k-1}\int_{\prod_{j=1}^{\ell}\mathcal{V}_j}\Big(G_\ell[t,x,v]+H_\ell[t,x,v]\Big)\tilde{w}(v_\ell)\\
&\quad\times\Bigg(\prod_{j=1}^{\ell}\mathrm{e}^{-\nu(v_j)\frac{t_j-t_{j+1}}{\varepsilon^2}}\mathrm{d}\sigma_j\Bigg)\\
&+\frac{\mathrm{e}^{-\nu(v)\frac{t-t_1}{\varepsilon^2}}}{\tilde{w}(v)}\int_{\prod_{j=1}^{k}\mathcal{V}_j} g(t_k,x_k,v_k)\tilde{w}(v_k)\Bigg(\prod_{j=1}^{k}\mathrm{e}^{-\nu(v_j)\frac{t_j-t_{j+1}}{\varepsilon^2}}\mathrm{d}\sigma_j\Bigg),
\end{aligned}$$

where

$$\begin{aligned}
G_\ell[t,x,v] := &\mathbf{1}_{\{t_{\ell+1}=0\}}w(v_\ell)z(x_{\ell+1},v_\ell) + \mathbf{1}_{\{t_{\ell+1}>0\}}w(v_\ell)h(t_{\ell+1},x_{\ell+1},v_\ell)\\
&+\int_0^{\frac{t_\ell-t_{\ell+1}}{\varepsilon^2}}\Big(w(v_\ell)S\Big(t_\ell-\varepsilon^2 s, x_\ell-\varepsilon s v_\ell, v_\ell\Big)\mathrm{e}^{\nu(v_\ell)s}\Big)\mathrm{d}s,
\end{aligned}\tag{5.118}$$

$$H_\ell[t,x,v] := \int_0^{\frac{t_\ell-t_{\ell+1}}{\varepsilon^2}}\Big(K_{w(v_\ell)}[g]\Big(t_\ell-\varepsilon^2 s, x_\ell-\varepsilon s v_\ell, v_\ell\Big)\mathrm{e}^{\nu(v_\ell)s}\Big)\mathrm{d}s.\tag{5.119}$$

Step 2: Estimates of source terms, initial terms, and boundary terms.

We set $k=CT_0^{\frac{5}{4}}$ for T_0 defined in Lemma 5.12. Consider all terms in (5.117) related to h and S.

Since $t_1\leq t$, we have

$$\begin{aligned}
&\left|\mathbf{1}_{\{t_k=0\}}w(v)z(x_1,v)\mathrm{e}^{-\nu(v)\frac{t-t_1}{\varepsilon^2}} + \mathbf{1}_{\{t_k>0\}}w(v)h(t_1,x_1,v)\mathrm{e}^{-\nu(v)\frac{t-t_1}{\varepsilon^2}}\right|\\
&\quad\leq \|wz\|_{\infty} + |||wh|||_{\gamma_-,\infty}.
\end{aligned}\tag{5.120}$$

Also,

(5.121)
$$\begin{aligned}
\left|\int_0^{\frac{t-t_1}{\varepsilon^2}} w(v)S\Big(t-\varepsilon^2 s, x-\varepsilon s v, v\Big)\mathrm{e}^{-\nu(v)s}\mathrm{d}s\right| &\leq \left|\left|\left|\nu^{-1}wS\right|\right|\right|_{\infty}\left|\int_0^{\frac{t-t_1}{\varepsilon^2}}\nu(v)\mathrm{e}^{-\nu(v)s}\mathrm{d}s\right|\\
&\leq \left|\left|\left|\nu^{-1}wS\right|\right|\right|_{\infty}.
\end{aligned}$$

Then we turn to terms defined in G_ℓ of (5.118). Noting that $\frac{1}{\tilde{w}} \lesssim 1$, we know

(5.122)
$$\begin{aligned}&\left|\frac{\mathrm{e}^{-\nu(v)\frac{t-t_1}{\varepsilon^2}}}{\tilde{w}(v)}\sum_{\ell=1}^{k-1}\int_{\prod_{j=1}^{\ell}\mathcal{V}_j}\mathbf{1}_{\{t_{\ell+1}=0\}}w(v_\ell)z(x_{\ell+1},v_\ell)\tilde{w}(v_\ell)\bigg(\prod_{j=1}^{\ell}\mathrm{e}^{-\nu(v_j)\frac{t_j-t_{j+1}}{\varepsilon^2}}\mathrm{d}\sigma_j\bigg)\right|\\
&\quad\lesssim\left\|wz\right\|_\infty\left|\sum_{\ell=1}^{k-1}\int_{\prod_{j=1}^{\ell}\mathcal{V}_j}\tilde{w}(v_\ell)\prod_{j=1}^{\ell}\mathrm{d}\sigma_j\right|\lesssim\left\|wz\right\|_\infty\left|\sum_{\ell=1}^{k-1}\int_{\mathcal{V}_\ell}\tilde{w}(v_\ell)\mathrm{d}\sigma_\ell\right|\lesssim CT_0^{\frac{5}{4}}\left\|wz\right\|_\infty,\end{aligned}$$

and

(5.123)
$$\begin{aligned}&\left|\frac{\mathrm{e}^{-\nu(v)\frac{t-t_1}{\varepsilon^2}}}{\tilde{w}(v)}\sum_{\ell=1}^{k-1}\int_{\prod_{j=1}^{\ell}\mathcal{V}_j}\mathbf{1}_{\{t_{\ell+1}>0\}}w(v_\ell)h(t_{\ell+1},x_{\ell+1},v_\ell)\tilde{w}(v_\ell)\bigg(\prod_{j=1}^{\ell}\mathrm{e}^{-\nu(v_j)\frac{t_j-t_{j+1}}{\varepsilon^2}}\mathrm{d}\sigma_j\bigg)\right|\\
&\quad\lesssim\left|\!\left|\!\left|wh\right|\!\right|\!\right|_{\gamma_-,\infty}\left|\sum_{\ell=1}^{k-1}\int_{\prod_{j=1}^{\ell}\mathcal{V}_j}\tilde{w}(v_\ell)\prod_{j=1}^{\ell}\mathrm{d}\sigma_j\right|\\
&\quad\lesssim\left|\!\left|\!\left|wh\right|\!\right|\!\right|_{\gamma_-,\infty}\left|\sum_{\ell=1}^{k-1}\int_{\mathcal{V}_\ell}\tilde{w}(v_\ell)\mathrm{d}\sigma_\ell\right|\lesssim CT_0^{\frac{5}{4}}\left|\!\left|\!\left|wh\right|\!\right|\!\right|_{\gamma_-,\infty}.\end{aligned}$$

Similarly,

(5.124)
$$\begin{aligned}&\left|\frac{\mathrm{e}^{-\nu(v)\frac{t-t_1}{\varepsilon^2}}}{\tilde{w}(v)}\sum_{\ell=1}^{k-1}\int_{\prod_{j=1}^{\ell}\mathcal{V}_j}\int_0^{\frac{t_\ell-t_{\ell+1}}{\varepsilon^2}}\Big(w(v_\ell)S\Big(t_\ell-\varepsilon^2 s,x_\ell-\varepsilon s v_\ell,v_\ell\Big)\mathrm{e}^{\nu(v_\ell)s}\Big)\mathrm{d}s\tilde{w}(v_\ell)\right.\\
&\qquad\left.\times\bigg(\prod_{j=1}^{\ell}\mathrm{e}^{-\nu(v_j)\frac{t_j-t_{j+1}}{\varepsilon^2}}\mathrm{d}\sigma_j\bigg)\right|\\
&\quad\lesssim\left|\!\left|\!\left|\nu^{-1}wS\right|\!\right|\!\right|_\infty\sum_{\ell=1}^{k-1}\int_{\prod_{j=1}^{\ell}\mathcal{V}_j}\left|\int_0^{\frac{t_\ell-t_{\ell+1}}{\varepsilon^2}}\nu(v_\ell)\mathrm{e}^{\nu(v_\ell)\big(s-\frac{t_\ell-t_{\ell+1}}{\varepsilon^2}\big)}\mathrm{d}s\right|\tilde{w}(v_\ell)\prod_{j=1}^{\ell}\mathrm{d}\sigma_j\bigg)\\
&\quad\lesssim CT_0^{\frac{5}{4}}\left|\!\left|\!\left|\nu^{-1}wS\right|\!\right|\!\right|_\infty.\end{aligned}$$

Collecting all terms in (5.120), (5.121), (5.122), (5.123), and (5.124), we have

(5.125)
$$\begin{aligned}\text{initial term and boundary term contribution}&\lesssim CT_0^{\frac{5}{4}}\Big(\left\|wz\right\|_\infty+\left|\!\left|\!\left|wh\right|\!\right|\!\right|_{\gamma_-,\infty}\Big)\\
&\lesssim\left\|wz\right\|_\infty+\left|\!\left|\!\left|wh\right|\!\right|\!\right|_{\gamma_-,\infty}\end{aligned}$$

and

$$\text{source term contribution}\lesssim CT_0^{\frac{5}{4}}\left|\!\left|\!\left|\nu^{-1}wS\right|\!\right|\!\right|_\infty\lesssim\left|\!\left|\!\left|\nu^{-1}wS\right|\!\right|\!\right|_\infty. \tag{5.126}$$

Step 3: Estimates of multiple reflection.

We focus on the last term in (5.117), which can be decomposed based on accumulative time t_{k+1}:

(5.127)
$$\begin{aligned}&\left|\frac{\mathrm{e}^{-\nu(v)\frac{t-t_1}{\varepsilon^2}}}{\tilde{w}(v)}\int_{\prod_{j=1}^k\mathcal{V}_j} g(t_k,x_k,v_k)\tilde{w}(v_k)\bigg(\prod_{j=1}^k \mathrm{e}^{-\nu(v_j)\frac{t_j-t_{j+1}}{\varepsilon^2}}\mathrm{d}\sigma_j\bigg)\right|\\ &\leq\left|\frac{\mathrm{e}^{-\nu(v)\frac{t-t_1}{\varepsilon^2}}}{\tilde{w}(v)}\int_{\prod_{j=1}^k\mathcal{V}_j}\mathbf{1}_{\{\frac{t-t_k}{\varepsilon^2}\leq\frac{T_0}{\varepsilon}\}} g(t_k,x_k,v_k)\tilde{w}(v_k)\bigg(\prod_{j=1}^k \mathrm{e}^{-\nu(v_j)\frac{t_j-t_{j+1}}{\varepsilon^2}}\mathrm{d}\sigma_j\bigg)\right|\\ &\quad+\left|\frac{\mathrm{e}^{-\nu(v)\frac{t-t_1}{\varepsilon^2}}}{\tilde{w}(v)}\int_{\prod_{j=1}^k\mathcal{V}_j}\mathbf{1}_{\{\frac{t-t_k}{\varepsilon^2}\geq\frac{T_0}{\varepsilon}\}} g(t_k,x_k,v_k)\tilde{w}(v_k)\bigg(\prod_{j=1}^k \mathrm{e}^{-\nu(v_j)\frac{t_j-t_{j+1}}{\varepsilon^2}}\mathrm{d}\sigma_j\bigg)\right|\\ &:=J_1+J_2.\end{aligned}$$

Based on Lemma 5.12, we have

$$\begin{aligned}J_1&\lesssim|||g|||_\infty\left|\int_{\Pi_{j=1}^{k-1}\mathcal{V}_j}\mathbf{1}_{\{\frac{t-t_k}{\varepsilon^2}\leq\frac{T_0}{\varepsilon}\}}\bigg(\int_{\mathcal{V}_k}\tilde{w}(v_k)\mathrm{d}\sigma_k\bigg)\bigg(\prod_{j=1}^{k-1}\mathrm{d}\sigma_j\bigg)\right|\\ &\lesssim|||g|||_\infty\left|\int_{\Pi_{j=1}^{k-1}\mathcal{V}_j}\mathbf{1}_{\{\frac{t-t_k}{\varepsilon^2}\leq\frac{T_0}{\varepsilon}\}}\bigg(\prod_{j=1}^{k-1}\mathrm{d}\sigma_j\bigg)\right|\lesssim\left(\frac{1}{2}\right)^{C_2T_0^{\frac{5}{4}}}|||g|||_\infty.\end{aligned}\tag{5.128}$$

On the other hand, when t_k is large, the exponential terms become extremely small, so we obtain

(5.129)
$$\begin{aligned}J_2&\lesssim|||g|||_\infty\left|\mathrm{e}^{-\nu(v)\frac{t-t_1}{\varepsilon^2}}\int_{\Pi_{j=1}^{k-1}\mathcal{V}_j}\mathbf{1}_{\{\frac{t-t_k}{\varepsilon^2}\geq\frac{T_0}{\varepsilon}\}}\bigg(\int_{\mathcal{V}_k}\tilde{w}(v_k)\mathrm{d}\sigma_k\bigg)\bigg(\prod_{j=1}^{k-1}\mathrm{e}^{-\nu(v_j)\frac{t_j-t_{j+1}}{\varepsilon^2}}\mathrm{d}\sigma_j\bigg)\right|\\ &\lesssim|||g|||_\infty\left|\mathrm{e}^{-\nu(v)\frac{t-t_1}{\varepsilon^2}}\int_{\Pi_{j=1}^{k-1}\mathcal{V}_j}\mathbf{1}_{\{\frac{t-t_k}{\varepsilon^2}\geq\frac{T_0}{\varepsilon}\}}\bigg(\prod_{j=1}^{k-1}\mathrm{e}^{-\nu(v_j)\frac{t_j-t_{j+1}}{\varepsilon^2}}\mathrm{d}\sigma_j\bigg)\right|\lesssim\mathrm{e}^{-\frac{T_0}{\varepsilon}}|||g|||_\infty.\end{aligned}$$

Summarizing (5.128) and (5.129), we get for δ arbitrarily small

$$\text{multiple reflection term contribution}\lesssim\delta\|g\|_\infty.\tag{5.130}$$

Step 4: Estimates of K_w terms.

So far, the only remaining terms in (5.117) are related to K_w. We focus on

(5.131)
$$\left|\int_0^{\frac{t-t_1}{\varepsilon^2}}K_{w(v)}[g]\Big(t-\varepsilon^2s,x-\varepsilon sv,v\Big)\mathrm{e}^{-\nu(v)s}\mathrm{d}s\right|\lesssim\left|\left|\left|K_{w(v)}[g]\Big(t-\varepsilon^2s,x-\varepsilon sv,v\Big)\right|\right|\right|_\infty.$$

Denote $T(s;t,x,v):=t-\varepsilon^2s$ and $X(s;t,x,v):=x-\varepsilon(t_1-s)v$. Define the back-time stochastic cycle from (T,X,v') as (t_i',x_i',v_i') with $(t_0',x_0',v_0')=(T,X,v')$. Then we can rewrite K_w along the stochastic cycle as (5.117):

(5.132)
$$\left|K_{w(v)}[g]\Big(t-\varepsilon^2s,x-\varepsilon(t_1-s)v,v\Big)\right|=\left|K_{w(v)}[g](T,X,v)\right|$$

$$
\begin{aligned}
&= \left|\int_{\mathbb{R}^3} k_{w(v)}(v,v')g(T,X,v')\mathrm{d}v'\right|\\
&\leq \left|\int_{\mathbb{R}^3}\int_0^{\frac{T-t_1'}{\varepsilon^2}} k_{w(v)}(v,v')K_{w(v')}[g]\Big(T-\varepsilon^2 r, X-\varepsilon r v', v'\Big)\mathrm{e}^{-\nu(v')r}\mathrm{d}r\mathrm{d}v'\right|\\
&\quad + \left|\int_{\mathbb{R}^3}\frac{\mathrm{e}^{-\nu(v')\frac{T-t_1'}{\varepsilon^2}}}{\tilde{w}(v')}\sum_{\ell=1}^{k-1}\int_{\prod_{j=1}^{\ell}\mathcal{V}_j'} k_{w(v)}(v,v')H_\ell[T,X,v']\tilde{w}(v_\ell')\Big(\prod_{j=1}^{\ell}\mathrm{e}^{-\nu(v_j')\frac{t_j'-t_{j+1}'}{\varepsilon^2}}\mathrm{d}\sigma_j'\Big)\mathrm{d}v'\right|\\
&\quad + \bigg|\int_{\mathbb{R}^3} k_{w(v)}(v,v')\,(\text{initial terms} + \text{boundary terms}\\
&\quad + \text{source terms} + \text{multiple reflection terms})\,\mathrm{d}v'\bigg|\\
&:= I + II + III.
\end{aligned}
$$

Using estimates (5.125), (5.126), (5.130) from Steps 2 and 3, and Lemma 4.3, we can bound III directly:

$$
III \lesssim \|wz\|_\infty + |||wh|||_{\gamma_-,\infty} + \Big|\Big|\Big|\nu^{-1}wS\Big|\Big|\Big|_\infty + \delta|||g|||_\infty. \tag{5.133}
$$

I and II are much more complicated. We may further rewrite I as

(5.134)

$$
I = \left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_0^{\frac{T-t_1'}{\varepsilon^2}} k_{w(v)}(v,v')k_{w(v')}(v',v'')g\Big(T-\varepsilon^2 r, X-\varepsilon r v', v''\Big)\mathrm{e}^{-\nu(v')r}\mathrm{d}r\mathrm{d}v'\mathrm{d}v''\right|,
$$

which will estimated in four cases:

$$
I := I_1 + I_2 + I_3 + I_4. \tag{5.135}
$$

Case I: $I_1 : |v| \geq N$.

Based on Lemma 4.3, we have

$$
\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} k_{w(v)}(v,v')k_{w(v')}(v',v'')\mathrm{d}v'\mathrm{d}v''\right| \lesssim \frac{1}{1+|v|} \lesssim \frac{1}{N}. \tag{5.136}
$$

Hence, we get

$$
I_1 \lesssim \frac{1}{N}|||g|||_\infty. \tag{5.137}
$$

Case II: $I_2 : |v| \leq N$, $|v'| \geq 2N$, or $|v'| \leq 2N$, $|v''| \geq 3N$.

Notice this implies either $|v'-v| \geq N$ or $|v'-v''| \geq N$. Hence, either of the following is correspondingly valid:

$$
\left|k_{w(v)}(v,v')\right| \leq C\mathrm{e}^{-\delta N^2}\left|k_{w(v)}(v,v')\right|\mathrm{e}^{\delta|v-v'|^2}, \tag{5.138}
$$

$$
\left|k_{w(v')}(v',v'')\right| \leq C\mathrm{e}^{-\delta N^2}\left|k_{w(v')}(v',v'')\right|\mathrm{e}^{\delta|v'-v''|^2}. \tag{5.139}
$$

Based on Lemma 4.3, we know

$$
\int_{\mathbb{R}^3}\left|k_{w(v)}(v,v')\right|\mathrm{e}^{\delta|v-v'|^2}\mathrm{d}v' < \infty, \tag{5.140}
$$

$$
\int_{\mathbb{R}^3}\left|k_{w(v')}(v',v'')\right|\mathrm{e}^{\delta|v'-v''|^2}\mathrm{d}v'' < \infty. \tag{5.141}
$$

Hence, we have

$$I_2 \lesssim \mathrm{e}^{-\delta N^2} |||g|||_\infty. \tag{5.142}$$

Case III: $I_3 : 0 \le r \le \delta$ and $|v| \le N$, $|v'| \le 2N$, $|v''| \le 3N$.

In this case, since the integral with respect to r is restricted in a very short interval, there is a small contribution as

$$I_3 \lesssim \left| \int_0^\delta \mathrm{e}^{-r} \mathrm{d}r \right| |||g|||_\infty \lesssim \delta |||g|||_\infty. \tag{5.143}$$

Case IV: $I_4 : r \ge \delta$ and $|v| \le N$, $|v'| \le 2N$, $|v''| \le 3N$.

This is the most complicated case. Since $k_{w(v)}(v, v')$ has a possible integrable singularity of $\frac{1}{|v-v'|}$, we can introduce the truncated kernel $k_N(v, v')$ which is smooth and has a compactly supported range such that

$$\sup_{|v|\le 3N} \int_{|v'|\le 3N} \left| k_N(v,v') - k_{w(v)}(v,v') \right| \mathrm{d}v' \le \frac{1}{N}. \tag{5.144}$$

Then we can split

$$\begin{aligned} k_{w(v)}(v,v')k_{w(v')}(v',v'') = {}& k_N(v,v')k_N(v',v'') + \big(k_{w(v)}(v,v') - k_N(v,v')\big)\, k_{w(v')}(v',v'') \\ &+ \big(k_{w(v')}(v',v'') - k_N(v',v'')\big)\, k_N(v,v'). \end{aligned} \tag{5.145}$$

This means that we further split I_4 into

$$I_4 := I_{4,1} + I_{4,2} + I_{4,3}. \tag{5.146}$$

Based on (5.144), we have

$$I_{4,2} \lesssim \frac{1}{N} \|g\|_\infty, \quad I_{4,3} \lesssim \frac{1}{N} \|g\|_\infty. \tag{5.147}$$

Therefore, the only remaining term is $I_{4,1}$. Note that we always have $X - \varepsilon r v' \in \Omega$. Hence, we define the change of variable $v' \to y$ as $y = (y_1, y_2, y_3) = X - \varepsilon r v'$. Then the Jacobian

$$\left| \frac{\mathrm{d}y}{\mathrm{d}v'} \right| = \left\| \begin{array}{ccc} -\varepsilon r & 0 & 0 \\ 0 & -\varepsilon r & 0 \\ 0 & 0 & -\varepsilon r \end{array} \right\| = \varepsilon^3 r^3 \ge \varepsilon^3 \delta^3. \tag{5.148}$$

Considering $|v|, |v'|, |v''| \le 3N$, we know $|g| \lesssim |f|$. Also, since k_N is bounded, we estimate

$$I_{4,1} \lesssim \int_0^{\frac{T-t_1'}{\varepsilon^2}} \int_{|v'|\le 2N} \int_{|v''|\le 3N} \mathbf{1}_{\{X-\varepsilon r v' \in \Omega\}} \left| f(T-\varepsilon^2 r, X-\varepsilon r v', v'') \right| \mathrm{e}^{-\nu(v')r} \mathrm{d}r \mathrm{d}v' \mathrm{d}v''. \tag{5.149}$$

Using the decomposition $f = \mathbb{P}[f] + (\mathbb{I} - \mathbb{P})[f]$, (5.148), and Hölder's inequality, we estimate them separately:

$$\int_0^{\frac{T-t_1'}{\varepsilon^2}} \int_{|v'|\le 2N} \int_{|v''|\le 3N} \mathbf{1}_{\{X-\varepsilon r v' \in \Omega\}} \left| \mathbb{P}[f](T-\varepsilon^2 r, X-\varepsilon r v', v'') \right| \mathrm{e}^{-\nu(v')r} \mathrm{d}r \mathrm{d}v' \mathrm{d}v'' \tag{5.150}$$

$$
\begin{aligned}
&\leq \int_0^{\frac{T-t_1'}{\varepsilon^2}} \left(\int_{|v'|\leq 2N}\int_{|v''|\leq 3N} \mathbf{1}_{\{X-\varepsilon rv'\in\Omega\}}\mathrm{d}v'\mathrm{d}v''\right)^{\frac{2m-1}{2m}}\\
&\quad\times\bigg(\int_{|v'|\leq 2N}\int_{|v''|\leq 3N} \mathbf{1}_{\{X-\varepsilon rv'\in\Omega\}}\Big(\mathbb{P}[f]\Big)^{2m}\\
&\quad\times(T-\varepsilon^2 r, X-\varepsilon rv', v'')\mathrm{e}^{-\nu(v')r}\mathrm{d}v'\mathrm{d}v''\bigg)^{\frac{1}{2m}}\mathrm{e}^{-r}\mathrm{d}r\\
&\lesssim \int_0^{\frac{T-t_1'}{\varepsilon^2}} \left(\frac{1}{\varepsilon^3\delta^3}\int_{|v''|\leq 3N}\int_\Omega \mathbf{1}_{\{y\in\Omega\}}\Big(\mathbb{P}[f]\Big)^{2m}(T-\varepsilon^2 r, y, v'')\mathrm{d}y\mathrm{d}v''\right)^{\frac{1}{2m}}\mathrm{e}^{-r}\mathrm{d}r\\
&\lesssim \frac{1}{\varepsilon^{\frac{3}{2m}}\delta^{\frac{3}{2m}}}\sup_{[0,T]}\|\mathbb{P}[f(t)]\|_{2m},
\end{aligned}
$$

and

(5.151)

$$
\begin{aligned}
&\int_0^{\frac{T-t_1'}{\varepsilon^2}}\int_{|v'|\leq 2N}\int_{|v''|\leq 3N} \mathbf{1}_{\{X-\varepsilon rv'\in\Omega\}}\left|(\mathbb{I}-\mathbb{P})[f](T-\varepsilon^2 r, X-\varepsilon rv', v'')\right|\\
&\quad\times\mathrm{e}^{-\nu(v')r}\mathrm{d}v'\mathrm{d}v''\mathrm{d}r\\
&\leq \int_0^{\frac{T-t_1'}{\varepsilon^2}} \left(\int_{|v'|\leq 2N}\int_{|v''|\leq 3N} \mathbf{1}_{\{X-\varepsilon rv'\in\Omega\}}\mathrm{d}v'\mathrm{d}v''\right)^{\frac{1}{2}}\\
&\quad\times\left(\int_{|v'|\leq 2N}\int_{|v''|\leq 3N} \mathbf{1}_{\{X-\varepsilon rv'\in\Omega\}}\Big((\mathbb{I}-\mathbb{P})[f]\Big)^{2}(T-\varepsilon^2 r, X-\varepsilon rv', v'')\mathrm{d}v'\mathrm{d}v''\right)^{\frac{1}{2}}\mathrm{e}^{-r}\mathrm{d}r\\
&\lesssim \int_0^{\frac{T-t_1'}{\varepsilon^2}} \left(\frac{1}{\varepsilon^3\delta^3}\int_{|v''|\leq 3N}\int_\Omega \mathbf{1}_{\{y\in\Omega\}}\Big((\mathbb{I}-\mathbb{P})[f]\Big)^{2}(T-\varepsilon^2 r, y, v'')\mathrm{d}y\mathrm{d}v''\right)^{\frac{1}{2}}\mathrm{e}^{-r}\mathrm{d}r\\
&\lesssim \frac{1}{\varepsilon^{\frac{3}{2}}\delta^{\frac{3}{2}}}\sup_{[0,T]}\|(\mathbb{I}-\mathbb{P})[f(t)]\|_{2}.
\end{aligned}
$$

Inserting (5.150) and (5.151) into (5.149), we obtain

$$
I_{4,1}\lesssim \frac{1}{\varepsilon^{\frac{3}{2m}}\delta^{\frac{3}{2m}}}\sup_{[0,T]}\|\mathbb{P}[f(t)]\|_{2m}+\frac{1}{\varepsilon^{\frac{3}{2}}\delta^{\frac{3}{2}}}\sup_{[0,T]}\|(\mathbb{I}-\mathbb{P})[f(t)]\|_{2}. \tag{5.152}
$$

Combined with (5.147), we know

$$
I_4\lesssim \frac{1}{N}|||g|||_\infty+\frac{1}{\varepsilon^{\frac{3}{2m}}\delta^{\frac{3}{2m}}}\sup_{[0,T]}\|\mathbb{P}[f(t)]\|_{2m}+\frac{1}{\varepsilon^{\frac{3}{2}}\delta^{\frac{3}{2}}}\sup_{[0,T]}\|(\mathbb{I}-\mathbb{P})[f(t)]\|_{2}. \tag{5.153}
$$

Summarizing all four cases in (5.137), (5.142), (5.143), and (5.153), we obtain

(5.154)

$$
I\lesssim \left(\frac{1}{N}+\mathrm{e}^{-\delta N^2}+\delta\right)|||g|||_\infty+\frac{1}{\varepsilon^{\frac{3}{2m}}\delta^{\frac{3}{2m}}}\sup_{[0,t]}\|\mathbb{P}[f(t)]\|_{2m}+\frac{1}{\varepsilon^{\frac{3}{2}}\delta^{\frac{3}{2}}}\sup_{[0,t]}\|(\mathbb{I}-\mathbb{P})[f(t)]\|_{2}.
$$

Choosing δ sufficiently small and then taking N sufficiently large, we have

$$
I\lesssim \delta|||g|||_\infty+\frac{1}{\varepsilon^{\frac{3}{2m}}\delta^{\frac{3}{2m}}}\sup_{[0,t]}\|\mathbb{P}[f(t)]\|_{2m}+\frac{1}{\varepsilon^{\frac{3}{2}}\delta^{\frac{3}{2}}}\sup_{[0,t]}\|(\mathbb{I}-\mathbb{P})[f(t)]\|_{2}. \tag{5.155}
$$

By a similar but tedious computation, we arrive at

$$II \lesssim \delta|||g|||_\infty + \frac{1}{\varepsilon^{\frac{3}{2m}}\delta^{\frac{3}{2m}}}\sup_{[0,t]}\|\mathbb{P}[f(t)]\|_{2m} + \frac{1}{\varepsilon^{\frac{3}{2}}\delta^{\frac{3}{2}}}\sup_{[0,t]}\|(\mathbb{I}-\mathbb{P})[f(t)]\|_2. \tag{5.156}$$

Combined with (5.133), we have

$$\begin{aligned}\left|\int_0^{\frac{t-t_1}{\varepsilon^2}} K_{w(v)}[g]\Big(t-\varepsilon^2 s, x-\varepsilon s v, v\Big)\mathrm{e}^{-\nu(v)s}\mathrm{d}s\right| &\lesssim \delta|||g|||_\infty + \frac{1}{\varepsilon^{\frac{3}{2m}}\delta^{\frac{3}{2m}}}\sup_{[0,t]}\|\mathbb{P}[f(t)]\|_{2m} + \frac{1}{\varepsilon^{\frac{3}{2}}\delta^{\frac{3}{2}}}\sup_{[0,t]}\|(\mathbb{I}-\mathbb{P})[f(t)]\|_2\\ &\quad + \|wz\|_\infty + |||wh|||_{\gamma_-,\infty} + \left|\left|\left|\nu^{-1}wS\right|\right|\right|_\infty.\end{aligned} \tag{5.157}$$

All the other terms in (5.117) related to K_w can be estimated in a similar fashion. At the end of the day, we have

$$\begin{aligned}&K_w \text{ term contribution}\\ &\quad\lesssim \delta|||g|||_\infty + \frac{1}{\varepsilon^{\frac{3}{2m}}\delta^{\frac{3}{2m}}}\sup_{[0,t]}\|\mathbb{P}[f(t)]\|_{2m} + \frac{1}{\varepsilon^{\frac{3}{2}}\delta^{\frac{3}{2}}}\sup_{[0,t]}\|(\mathbb{I}-\mathbb{P})[f(t)]\|_2\\ &\qquad + \|wz\|_\infty + |||wh|||_{\gamma_-,\infty} + \left|\left|\left|\nu^{-1}wS\right|\right|\right|_\infty.\end{aligned} \tag{5.158}$$

Step 5: Synthesis.

Summarizing all of above and inserting (5.125), (5.126), (5.130), and (5.158) into (5.117), we obtain for any $(t,x,v)\in\mathbb{R}_+\times\bar{\Omega}\times\mathbb{R}^3$,

$$\begin{aligned}|g(t,x,v)| &\lesssim \delta|||g|||_\infty + \frac{1}{\varepsilon^{\frac{3}{2m}}\delta^{\frac{3}{2m}}}\sup_{[0,t]}\|\mathbb{P}[f(t)]\|_{2m} + \frac{1}{\varepsilon^{\frac{3}{2}}\delta^{\frac{3}{2}}}\sup_{[0,t]}\|(\mathbb{I}-\mathbb{P})[f(t)]\|_2\\ &\quad + \|wz\|_\infty + |||wh|||_{\gamma_-,\infty} + \left|\left|\left|\nu^{-1}wS\right|\right|\right|_\infty.\end{aligned} \tag{5.159}$$

Taking the supremum over $[0,t]\times\gamma_+$ in (5.159), we have

$$\begin{aligned}\sup_{[0,t]}\|g(t)\|_{\gamma_+,\infty} &\lesssim \delta|||g|||_\infty + \frac{1}{\varepsilon^{\frac{3}{2m}}\delta^{\frac{3}{2m}}}\sup_{[0,t]}\|\mathbb{P}[f(t)]\|_{2m} + \frac{1}{\varepsilon^{\frac{3}{2}}\delta^{\frac{3}{2}}}\sup_{[0,t]}\|(\mathbb{I}-\mathbb{P})[f(t)]\|_2\\ &\quad + \|wz\|_\infty + |||wh|||_{\gamma_-,\infty} + \left|\left|\left|\nu^{-1}wS\right|\right|\right|_\infty.\end{aligned} \tag{5.160}$$

Based on Theorem 5.8, for $\frac{3}{2}<m<3$, we obtain

$$\begin{aligned}\sup_{[0,t]}\|g(t)\|_{\gamma_+,\infty} &\lesssim \delta|||g|||_\infty + o(1)\Big(\sup_{[0,t]}\|f(t)\|_{\gamma_+,\infty} + \sup_{[0,t]}\|f(t)\|_\infty\Big) + E\\ &\lesssim \delta\|g\|_\infty + o(1)\Big(\sup_{[0,t]}\|g(t)\|_{\gamma_+,\infty} + \sup_{[0,t]}\|g(t)\|_\infty\Big) + E,\end{aligned} \tag{5.161}$$

where

$$\begin{aligned}E &:= \frac{1}{\varepsilon^{2+\frac{3}{2m}}}\|\mathbb{P}[S(t)]\|_{\frac{2m}{2m-1}} + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\left\|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[S(t)]\right\|_2 + \frac{1}{\varepsilon^{2+\frac{3}{2m}}}|||\mathbb{P}[\partial_t S]|||_2\\ &\quad + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\left|\left|\left|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[\partial_t S]\right|\right|\right|_2 + \left|\left|\left|\nu^{-1}wS\right|\right|\right|_\infty + \frac{1}{\varepsilon^{\frac{3}{2m}}}\|h(t)\|_{\gamma_-,\frac{4m}{3}}\\ &\quad + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\|h(t)\|_{\gamma_-,2} + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}|||\partial_t h|||_{\gamma_-,2} + |||wh|||_{\gamma_-,\infty}\\ &\quad + \frac{1}{\varepsilon^{2+\frac{3}{2m}}}\|\nu z\|_2 + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\|v\cdot\nabla_x z\|_2 + \|wz\|_\infty + \frac{1}{\varepsilon^{2+\frac{3}{2m}}}\|S(0)\|_2.\end{aligned} \tag{5.162}$$

Absorbing $o(1)\sup_{[0,t]}\|g(t)\|_{\gamma_+,\infty}$ into the left-hand side, we have

$$\sup_{[0,t]}\|g(t)\|_{\gamma_+,\infty}\lesssim\delta|||g|||_{\infty}+o(1)\sup_{[0,t]}\|g(t)\|_{\infty}+E. \tag{5.163}$$

On the other hand, taking the supremum over $[0,t]\times\Omega\times\mathbb{R}^3$ in (5.159), we have

$$\begin{aligned}\sup_{[0,t]}\|g(t)\|_{\infty}\lesssim{}&\delta|||g|||_{\infty}+\frac{1}{\varepsilon^{\frac{3}{2m}}\delta^{\frac{3}{2m}}}\sup_{[0,t]}\|\mathbb{P}[f(t)]\|_{2m}+\frac{1}{\varepsilon^{\frac{3}{2}}\delta^{\frac{3}{2}}}\sup_{[0,t]}\|(\mathbb{I}-\mathbb{P})[f(t)]\|_{2}\\&+\|wz\|_{\infty}+|||wh|||_{\gamma_-,\infty}+\left|\left|\left|\nu^{-1}wS\right|\right|\right|_{\infty}.\end{aligned} \tag{5.164}$$

Based on Theorem 5.8, we obtain

$$\sup_{[0,t]}\|g(t)\|_{\infty}\lesssim\delta|||g|||_{\infty}+o(1)\Big(\sup_{[0,t]}\|g(t)\|_{\gamma_+,\infty}+\sup_{[0,t]}\|g(t)\|_{\infty}\Big)+E. \tag{5.165}$$

Absorbing $\delta|||g|||_{\infty}$ and $o(1)\sup_{[0,t]}\|g(t)\|_{\infty}$ into the left-hand side, we have

$$\sup_{[0,t]}\|g(t)\|_{\infty}\lesssim o(1)\sup_{[0,t]}\|g(t)\|_{\gamma_+,\infty}+E. \tag{5.166}$$

Inserting (5.163) into (5.166), and absorbing $\delta|||g|||_{\infty}$ and $o(1)|||g|||_{\infty}$ into the left-hand side, we get

$$\sup_{[0,t]}\|g(t)\|_{\infty}\lesssim E. \tag{5.167}$$

Then (5.163) implies

$$\sup_{[0,t]}\|g(t)\|_{\gamma_+,\infty}\lesssim E. \tag{5.168}$$

In summary, we have

$$\begin{aligned}|||g|||_{\infty}+|||g|||_{\gamma_+,\infty}\lesssim{}&\frac{1}{\varepsilon^{2+\frac{3}{2m}}}\|\mathbb{P}[S(t)]\|_{\frac{2m}{2m-1}}\\&+\frac{1}{\varepsilon^{1+\frac{3}{2m}}}\left\|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[S(t)]\right\|_{2}+\frac{1}{\varepsilon^{2+\frac{3}{2m}}}|||\mathbb{P}[\partial_t S]|||_{2}+\frac{1}{\varepsilon^{1+\frac{3}{2m}}}\left|\left|\left|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[\partial_t S]\right|\right|\right|_{2}\\&+\left|\left|\left|\nu^{-1}wS\right|\right|\right|_{\infty}+\frac{1}{\varepsilon^{\frac{3}{2m}}}\|h(t)\|_{\gamma_-,\frac{4m}{3}}+\frac{1}{\varepsilon^{1+\frac{3}{2m}}}\|h(t)\|_{\gamma_-,2}+\frac{1}{\varepsilon^{1+\frac{3}{2m}}}|||\partial_t h|||_{\gamma_-,2}\\&+|||wh|||_{\gamma_-,\infty}+\frac{1}{\varepsilon^{2+\frac{3}{2m}}}\|\nu z\|_{2}+\frac{1}{\varepsilon^{1+\frac{3}{2m}}}\|v\cdot\nabla_x z\|_{2}+\|wz\|_{\infty}+\frac{1}{\varepsilon^{2+\frac{3}{2m}}}\|S(0)\|_{2}.\end{aligned} \tag{5.169}$$

Then our result naturally follows. ∎

Remark 5.4. In the above proof, we use the traces $\|g(t)\|_{\infty}$, $\|g(t)\|_{\gamma_+,\infty}$, and $|||g|||_{\gamma_+,\infty}$ interchangeably with $|||g|||_{\infty}$ to perform absorbing argument. Roughly speaking, we track the solution using a mild formulation, so it is always continuous along the characteristics, which covers the whole domain $\mathbb{R}_+\times\Omega\times\mathbb{R}^3$, so $|||g|||_{\gamma_+,\infty}$ will control all the rest. To be more precise, it actually relies on Ukai's trace theorem in [40], which says that for transport operator $\partial_t+v\cdot\nabla_x$, such traces are always well-defined and controllable.

Remark 5.5 (exponential decay). Define $\tilde{f} = \mathrm{e}^{K_0 t} f$. Then $\tilde{f}$ satisfies

$$
\left\{
\begin{aligned}
&\varepsilon^2 \partial_t \tilde{f} + \varepsilon v \cdot \nabla_x \tilde{f} + \mathcal{L}[\tilde{f}] = \varepsilon^2 K_0 \tilde{f} + \mathrm{e}^{K_0 t} S(t,x,v) \ \text{ in } \ \mathbb{R}_+ \times \Omega \times \mathbb{R}^3,\\
&\tilde{f}(0,x,v) = z(x,v) \ \text{ in } \ \Omega \times \mathbb{R}^3,\\
&\tilde{f}(t,x_0,v) = \mathcal{P}[\tilde{f}](t,x_0,v) + \mathrm{e}^{K_0 t} h(t,x_0,v) \ \text{ on } \ \mathbb{R}_+ \times \gamma_-,
\end{aligned}
\right. \tag{5.170}
$$

where

$$
\mathcal{P}[\tilde{f}](t,x_0,v) = \mu^{\frac{1}{2}}(v) \int_{\mathfrak{u}\cdot n(x_0)>0} \mu^{\frac{1}{2}}(\mathfrak{u}) \tilde{f}(t,x_0,\mathfrak{u}) \left|\mathfrak{u} \cdot n(x_0)\right| \mathrm{d}\mathfrak{u}. \tag{5.171}
$$

The extra term is $\varepsilon^2 K_0 \tilde{f}$. Thanks to ε^2, based on L^2 and L^{2m} energy estimates in Lemma 5.5 and Theorem 5.8, for K_0 small, we can absorb this term into the left-hand side. Therefore, we can recover all estimates as in Theorem 5.13.

6. Hydrodynamic limit.

6.1. Nonlinear estimates.

LEMMA 6.1. *The nonlinear term* Γ *defined in* (1.22) *satisfies* $\Gamma[f,g] \in \mathcal{N}^{\perp}$. *Also, for* $0 \leq \varrho < \frac{1}{4}$ *and* $\vartheta \geq 0$,

$$
\left\|\Gamma[f,g]\right\|_2 \lesssim \Big(\sup_{x\in\Omega} \left|\nu g(x)\right|_2\Big) \left\|\nu f\right\|_2, \tag{6.1}
$$

$$
\left\|\nu^{-1}\Gamma[f,g]\right\|_{\infty,\vartheta,\varrho} \lesssim \left\|f\right\|_{\infty,\vartheta,\varrho} \left\|g\right\|_{\infty,\vartheta,\varrho}. \tag{6.2}
$$

Proof. See [46, Lemma 6.1] for the proof. ∎

6.2. Perturbed remainder estimates. We consider the perturbed evolutionary Boltzmann equation

$$
\left\{
\begin{aligned}
&\varepsilon^2 \partial_t f + \varepsilon v \cdot \nabla_x f + \mathcal{L}[f] = \Gamma[f,g] + \varepsilon^3 \Gamma[f,f] + S(t,x,v) \ \text{ in } \ \mathbb{R}_+ \times \Omega \times \mathbb{R}^3,\\
&f(0,x,v) = z(x,v) \ \text{ in } \ \Omega \times \mathbb{R}^3,\\
&f(t,x_0,v) = \mathcal{P}[f](t,x_0,v) + (\mu_b^{\varepsilon} - \mu)\mu^{-1}\mathcal{P}[f] + h(t,x_0,v)\\
&\quad \text{for } \ x_0 \in \partial\Omega \ \text{ and } \ v \cdot n < 0.
\end{aligned}
\right. \tag{6.3}
$$

Assume that a priori

$$
|||g|||_{\infty,\vartheta,\varrho} + |||\partial_t g|||_{\infty,\vartheta,\varrho} + \left|\left|\left|\varepsilon^3 f\right|\right|\right|_{\infty,\vartheta,\varrho} = o(1)\varepsilon. \tag{6.4}
$$

THEOREM 6.2. *Assume* (5.3) *and* (5.4) *hold. The solution* $f(t,x,v)$ *to the equation* (6.3) *satisfies*

$$
\begin{aligned}
&\frac{1}{\varepsilon^{\frac{1}{2}}} \left\|(1-\mathcal{P})[f(t)]\right\|_{\gamma_+,2} + \frac{1}{\varepsilon} \left\|(\mathbb{I}-\mathbb{P})[f(t)]\right\|_{\nu} + \left\|\mathbb{P}[f(t)]\right\|_{2m}\\
&\left\|f(t)\right\|_2 + \frac{1}{\varepsilon^{\frac{1}{2}}} |||(1-\mathcal{P})[f]|||_{\gamma_+,2} + \frac{1}{\varepsilon} |||(\mathbb{I}-\mathbb{P})[f]|||_{\nu} + |||\mathbb{P}[f]|||_2\\
&\quad + \frac{1}{\varepsilon^{\frac{1}{2}}} |||(1-\mathcal{P})[\partial_t f]|||_{\gamma_+,2} + \frac{1}{\varepsilon} |||(\mathbb{I}-\mathbb{P})[\partial_t f]|||_{\nu} + |||\mathbb{P}[\partial_t f]|||_2
\end{aligned} \tag{6.5}
$$

$$
\begin{aligned}
&\lesssim o(1)\varepsilon^{\frac{3}{2m}}\Big(\|f(t)\|_{\gamma_+,\infty}+\|f(t)\|_\infty\Big)+\frac{1}{\varepsilon^2}\|\mathbb{P}[S(t)]\|_{\frac{2m}{2m-1}}+\frac{1}{\varepsilon}\Big\|\nu^{-\frac12}(\mathbb{I}-\mathbb{P})[S(t)]\Big\|_2\\
&\quad+\frac{1}{\varepsilon^2}|||\mathbb{P}[S]|||_2+\frac{1}{\varepsilon}\Big|\Big|\Big|\nu^{-\frac12}(\mathbb{I}-\mathbb{P})[S]\Big|\Big|\Big|_2+\frac{1}{\varepsilon^2}|||\mathbb{P}[\partial_tS]|||_2+\frac{1}{\varepsilon}\Big|\Big|\Big|\nu^{-\frac12}(\mathbb{I}-\mathbb{P})[\partial_tS]\Big|\Big|\Big|_2\\
&\quad+\|h(t)\|_{\gamma_-,\frac{4m}{3}}+\frac{1}{\varepsilon}\|h(t)\|_{\gamma_-,2}+\frac{1}{\varepsilon}|||h|||_{\gamma_-,2}+\frac{1}{\varepsilon}|||\partial_th|||_{\gamma_-,2}+\frac{1}{\varepsilon^2}\|\nu z\|_2\\
&\quad+\frac{1}{\varepsilon}\|v\cdot\nabla_xz\|_2+\frac{1}{\varepsilon^2}\|S(0)\|_2.
\end{aligned}
$$

Proof. Since the perturbed term $\Gamma[f,g],\Gamma[f,f]\in\mathcal{N}^\perp$, we apply Theorem 5.8 to (6.3) to obtain

(6.6)

$$
\begin{aligned}
&\frac{1}{\varepsilon^{\frac12}}\|(1-\mathcal{P})[f(t)]\|_{\gamma_+,2}+\frac{1}{\varepsilon}\|(\mathbb{I}-\mathbb{P})[f(t)]\|_\nu+\|\mathbb{P}[f(t)]\|_{2m}\\
&\quad+\frac{1}{\varepsilon^{\frac12}}|||(1-\mathcal{P})[\partial_tf]|||_{\gamma_+,2}+\frac{1}{\varepsilon}|||(\mathbb{I}-\mathbb{P})[\partial_tf]|||_\nu+|||\mathbb{P}[\partial_tf]|||_2\\
&\lesssim o(1)\varepsilon^{\frac{3}{2m}}\Big(\|f(t)\|_{\gamma_+,\infty}+\|f(t)\|_\infty\Big)+\frac{1}{\varepsilon^2}\|\mathbb{P}[S(t)]\|_{\frac{2m}{2m-1}}\\
&\quad+\frac{1}{\varepsilon}\Big\|\nu^{-\frac12}(\mathbb{I}-\mathbb{P})[S(t)]\Big\|_2+\frac{1}{\varepsilon^2}|||\mathbb{P}[\partial_tS]|||_2+\frac{1}{\varepsilon}\Big|\Big|\Big|\nu^{-\frac12}(\mathbb{I}-\mathbb{P})[\partial_tS]\Big|\Big|\Big|_2+\|h(t)\|_{\gamma_-,\frac{4m}{3}}\\
&\quad+\frac{1}{\varepsilon}\|h(t)\|_{\gamma_-,2}+\frac{1}{\varepsilon}|||\partial_th|||_{\gamma_-,2}+\frac{1}{\varepsilon^2}\|\nu z\|_2+\frac{1}{\varepsilon}\|v\cdot\nabla_xz\|_2+\frac{1}{\varepsilon^2}\|S(0)\|_2\\
&\quad+\frac{1}{\varepsilon}\Big\|\nu^{-\frac12}\Gamma[f,g](t)\Big\|_2+\frac{1}{\varepsilon}\Big|\Big|\Big|\nu^{-\frac12}\partial_t\Gamma[f,g]\Big|\Big|\Big|_2\\
&\quad+\frac{1}{\varepsilon}\Big\|\varepsilon^3\nu^{-\frac12}\Gamma[f,f](t)\Big\|_2+\frac{1}{\varepsilon}\Big|\Big|\Big|\varepsilon^3\nu^{-\frac12}\partial_t\Gamma[f,f]\Big|\Big|\Big|_2\\
&\quad+\Big\|(\mu_b^\varepsilon-\mu)\mu^{-1}\mathcal{P}[f(t)]\Big\|_{\gamma_-,\frac{4m}{3}}+\frac{1}{\varepsilon}\Big\|(\mu_b^\varepsilon-\mu)\mu^{-1}\mathcal{P}[f(t)]\Big\|_{\gamma_-,2}\\
&\quad+\frac{1}{\varepsilon}\Big|\Big|\Big|(\mu_b^\varepsilon-\mu)\mu^{-1}\mathcal{P}[\partial_tf]\Big|\Big|\Big|_{\gamma_-,2}+\frac{1}{\varepsilon^2}\|\Gamma[f,g](0)\|_2+\frac{1}{\varepsilon^2}\Big\|\varepsilon^3\Gamma[f,f](0)\Big\|_2.
\end{aligned}
$$

Also, based on Lemma 5.5, we have the L^2 estimate

(6.7)

$$
\begin{aligned}
&\|f(t)\|_2+\frac{1}{\varepsilon^{\frac12}}|||(1-\mathcal{P})[f]|||_{\gamma_+,2}+\frac{1}{\varepsilon}|||(\mathbb{I}-\mathbb{P})[f]|||_\nu+|||\mathbb{P}[f]|||_2\\
&\quad\lesssim\frac{1}{\varepsilon^2}|||\mathbb{P}[S]|||_2+\frac{1}{\varepsilon}\Big|\Big|\Big|\nu^{-\frac12}(\mathbb{I}-\mathbb{P})[S]\Big|\Big|\Big|_2+\frac{1}{\varepsilon}|||h|||_{\gamma_-,2}+\|z\|_2\\
&\qquad+\frac{1}{\varepsilon}\Big|\Big|\Big|\nu^{-\frac12}\Gamma[f,g]\Big|\Big|\Big|_2+\frac{1}{\varepsilon}\Big|\Big|\Big|\varepsilon^3\nu^{-\frac12}\Gamma[f,f]\Big|\Big|\Big|_2++\frac{1}{\varepsilon}\Big|\Big|\Big|(\mu_b^\varepsilon-\mu)\mu^{-1}\mathcal{P}[f]\Big|\Big|\Big|_{\gamma_-,2}.
\end{aligned}
$$

Step 1: Bulk perturbation terms.
Using Lemma 6.1 and (6.4), we have

$$
(6.8)\quad \frac{1}{\varepsilon}\Big\|\nu^{-\frac12}\Gamma[f,g](t)\Big\|_2\lesssim o(1)\Big\|\nu^{\frac12}f(t)\Big\|_2\lesssim o(1)\|\mathbb{P}[f(t)]\|_\nu+o(1)\|(\mathbb{I}-\mathbb{P})[f(t)]\|_\nu.
$$

Note that direct computation reveals that

$$\left\|\mathbb{P}[f(t)]\right\|_{2m} \gtrsim \left\|\mathbb{P}[f(t)]\right\|_{\nu}, \tag{6.9}$$

so inserting (6.8) into (6.6), we can absorb $o(1)\left\|\mathbb{P}[f(t)]\right\|_{\nu}$ and $o(1)\left\|(\mathbb{I}-\mathbb{P})[f(t)]\right\|_{\nu}$ into the left-hand side (LHS). On the other hand, Using Lemma 6.1 and (6.4), we have

$$\frac{1}{\varepsilon}\left\|\left|\nu^{-\frac{1}{2}}\partial_t\Gamma[f,g]\right|\right\|_2 \lesssim o(1)\left\|\left|\nu^{\frac{1}{2}}f\right|\right\|_2 + o(1)\left\|\left|\nu^{\frac{1}{2}}\partial_t f\right|\right\|_2. \tag{6.10}$$

Then $o(1)\left\|\nu^{\frac{1}{2}}f\right\|_2$ can be handled by L^2 estimates and $o(1)\left\|\nu^{\frac{1}{2}}\partial_t f\right\|_2$ can be absorbed into the LHS. Similarly,

$$\frac{1}{\varepsilon}\left\|\varepsilon^3\nu^{-\frac{1}{2}}\Gamma[f,f](t)\right\|_2 \lesssim \frac{1}{\varepsilon}\left\|\varepsilon^3 f(t)\right\|_{\infty,\vartheta,\varrho}\left\|\nu^{\frac{1}{2}}f(t)\right\|_2 \lesssim o(1)\left\|\nu^{\frac{1}{2}}f(t)\right\|_2, \tag{6.11}$$

$$\frac{1}{\varepsilon}\left\|\left|\varepsilon^3\nu^{-\frac{1}{2}}\partial_t\Gamma[f,f]\right|\right\|_2 \lesssim \frac{1}{\varepsilon}\left\|\left|\varepsilon^3 f\right|\right\|_{\infty,\vartheta,\varrho}\left\|\left|\nu^{\frac{1}{2}}\partial_t f\right|\right\|_2 \lesssim o(1)\left\|\left|\nu^{\frac{1}{2}}\partial_t f\right|\right\|_2. \tag{6.12}$$

Both of them can be absorbed into the LHS of (6.6). A similar argument justifies the absorbing in (6.7)

Step 2: Boundary perturbation terms.
On the other hand, due to (1.10), we know

$$\left\|(\mu_b^\varepsilon-\mu)\mu^{-1}\mathcal{P}[f(t)]\right\|_{\gamma_-,\frac{4m}{3}} \lesssim o(1)\varepsilon\left\|f(t)\right\|_{\gamma_+,\infty}, \tag{6.13}$$

which can be combined with the corresponding term on the right-hand side of (6.6). Also,

$$\frac{1}{\varepsilon}\left\|(\mu_b^\varepsilon-\mu)\mu^{-1}\mathcal{P}[f(t)]\right\|_{\gamma_-,2} \lesssim o(1)\left\|\mathcal{P}[f(t)]\right\|_{\gamma_-,2}, \tag{6.14}$$

$$\frac{1}{\varepsilon}\left\|\left|(\mu_b^\varepsilon-\mu)\mu^{-1}\mathcal{P}[\partial_t f]\right|\right\|_{\gamma_-,2} \lesssim o(1)\left\|\left|\mathcal{P}[\partial_t f]\right|\right\|_{\gamma_-,2}. \tag{6.15}$$

Note that both of them involve $\mathcal{P}[f]$, which has been controlled by the proof of Theorem 5.5 (Step 2). Hence, adding (6.7) to (6.6) and absorbing all new terms into the LHS, we can close the proof. ∎

THEOREM 6.3. *Assume* (5.3) *and* (5.4) *hold. The solution* $f(t,x,v)$ *to* (6.3) *satisfies for* $\vartheta \geq 0$ *and* $0 \leq \varrho < \frac{1}{4}$,

(6.16)

$$\begin{aligned}
&\left\|\left|f\right|\right\|_{\infty,\vartheta,\varrho} + \left\|\left|f\right|\right\|_{\gamma_+,\infty,\varrho,\vartheta} \\
&\lesssim \frac{1}{\varepsilon^{2+\frac{3}{2m}}}\left\|\mathbb{P}[S(t)]\right\|_{\frac{2m}{2m-1}} + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\left\|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[S(t)]\right\|_2 + \frac{1}{\varepsilon^{2+\frac{3}{2m}}}\left\|\left|\mathbb{P}[S]\right|\right\|_2 \\
&\quad + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\left\|\left|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[S]\right|\right\|_2 + \frac{1}{\varepsilon^{2+\frac{3}{2m}}}\left\|\left|\mathbb{P}[\partial_t S]\right|\right\|_2 + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\left\|\left|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[\partial_t S]\right|\right\|_2 \\
&\quad + \left\|\left|\nu^{-1}S\right|\right\|_{\infty,\vartheta,\varrho} + \frac{1}{\varepsilon^{\frac{3}{2m}}}\left\|h(t)\right\|_{\gamma_-,\frac{4m}{3}} + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\left\|h(t)\right\|_{\gamma_-,2} + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\left\|\left|h\right|\right\|_{\gamma_-,2} \\
&\quad + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\left\|\left|\partial_t h\right|\right\|_{\gamma_-,2} + \left\|\left|h\right|\right\|_{\gamma_-,\infty,\varrho,\vartheta} + \frac{1}{\varepsilon^{2+\frac{3}{2m}}}\left\|\nu z\right\|_2 + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\left\|v\cdot\nabla_x z\right\|_2 \\
&\quad + \left\|z\right\|_{\infty,\vartheta,\varrho} + \frac{1}{\varepsilon^{2+\frac{3}{2m}}}\left\|S(0)\right\|_2.
\end{aligned}$$

Proof. Since we already have bounds for f in L^{2m} as in Theorem 6.2, following the proof of Theorem 5.13, we obtain

(6.17)
$$
\begin{aligned}
&|||f|||_{\infty,\vartheta,\varrho} + |||f|||_{\gamma_+,\infty,\varrho,\vartheta} \\
&\quad\lesssim \frac{1}{\varepsilon^{2+\frac{3}{2m}}}\|\mathbb{P}[S(t)]\|_{\frac{2m}{2m-1}} + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\left\|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[S(t)]\right\|_2 + \frac{1}{\varepsilon^{2+\frac{3}{2m}}}|||\mathbb{P}[S]|||_2 \\
&\qquad + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\left|\left|\left|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[S]\right|\right|\right|_2 + \frac{1}{\varepsilon^{2+\frac{3}{2m}}}|||\mathbb{P}[\partial_t S]|||_2 + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\left|\left|\left|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[\partial_t S]\right|\right|\right|_2 \\
&\qquad + \left|\left|\left|\nu^{-1}S\right|\right|\right|_{\infty,\vartheta,\varrho} + \frac{1}{\varepsilon^{\frac{3}{2m}}}\|h(t)\|_{\gamma_-,\frac{4m}{3}} + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\|h(t)\|_{\gamma_-,2} + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}|||h|||_{\gamma_-,2} \\
&\qquad + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}|||\partial_t h|||_{\gamma_-,2} + |||h|||_{\gamma_-,\infty,\varrho,\vartheta} + \frac{1}{\varepsilon^{2+\frac{3}{2m}}}\|\nu z\|_2 + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\|v\cdot\nabla_x z\|_2 \\
&\qquad + \|z\|_{\infty,\vartheta,\varrho} + \frac{1}{\varepsilon^{2+\frac{3}{2m}}}\|S(0)\|_2 + \left|\left|\left|\nu^{-1}\Gamma[f,g]\right|\right|\right|_{\infty,\vartheta,\varrho} + \left|\left|\left|\varepsilon^3\nu^{-1}\Gamma[f,f]\right|\right|\right|_{\infty,\vartheta,\varrho} \\
&\qquad + \left\|(\mu_b^\varepsilon-\mu)\mu^{-1}\mathcal{P}[f]\right\|_{\gamma_-,\infty,\varrho,\vartheta}.
\end{aligned}
$$

Using Lemma 6.1 and (6.4), we have

$$
\left|\left|\left|\nu^{-1}\Gamma[f,g]\right|\right|\right|_{\infty,\vartheta,\varrho} \lesssim |||f|||_{\infty,\vartheta,\varrho}|||g|||_{\infty,\vartheta,\varrho} \lesssim o(1)|||f|||_{\infty,\vartheta,\varrho}, \tag{6.18}
$$

$$
\left|\left|\left|\varepsilon^3\nu^{-1}\Gamma[f,f]\right|\right|\right|_{\infty,\vartheta,\varrho} \lesssim |||f|||_{\infty,\vartheta,\varrho}\left|\left|\left|\varepsilon^3 f\right|\right|\right|_{\infty,\vartheta,\varrho} \lesssim o(1)|||f|||_{\infty,\vartheta,\varrho}. \tag{6.19}
$$

Inserting (6.18) into (6.17), we can absorb $o(1)\|f\|_{\infty,\vartheta,\varrho}$ into the LHS. Also, using (1.10), we have

$$
\left|\left|\left|(\mu_b^\varepsilon-\mu)\mu^{-1}\mathcal{P}[f]\right|\right|\right|_{\gamma_-,\infty,\varrho,\vartheta} \lesssim o(1)|||f|||_{\gamma_+,\infty,\varrho,\vartheta}. \tag{6.20}
$$

Inserting (6.20) into (6.17) and absorbing $o(1)|||f|||_{\gamma_+,\infty,\varrho,\vartheta}$ into the LHS, we obtain the desired result. ∎

6.3. Analysis of asymptotic expansion.

6.3.1. Analysis of initial layers. We first prove a theorem about the well-posedness and decay of the initial layer equation.

THEOREM 6.4. *For equation*

$$
\begin{cases}
\partial_\tau g + \mathcal{L}[g] = S(\tau,v) \quad \text{in} \quad \mathbb{R}_+\times\mathbb{R}^3, \\
g(0,v) = z(v)
\end{cases} \tag{6.21}
$$

with

$$
|z|_{\infty,\vartheta,\varrho} \lesssim 1, \quad \left\|e^{K_0 t}S\right\|_{\infty,\vartheta,\varrho} \lesssim 1, \tag{6.22}
$$

there exists a unique solution $g(\tau,\mathfrak{v})$ and a function $g_\infty\in\mathcal{N}$ satisfying

$$
|g_\infty| \lesssim 1, \quad \left\|e^{K_0\tau}(g-g_\infty)\right\|_{\infty,\vartheta,\varrho} \lesssim 1. \tag{6.23}
$$

Proof. This is very similar to the analysis of the ε-Milne problem with geometric correction as [46, section 3], but much simpler. We decompose $g = r + q$, where $r\in\mathcal{N}^\perp$ and $q = \sum_{k=0}^{4} q_k(\tau)\varphi_k(v)\in\mathcal{N}$. Then using the same $L^2 - L^\infty$ estimates as [46, section 3], we can get the desired result. ∎

With this theorem in hand, based on the analysis in section 2.6, we know $\mathcal{F}_1 = 0$ and $\mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$ are well-defined.

THEOREM 6.5. *For $K_0 > 0$ sufficiently small, the initial layer satisfies*

$$\left\|\mathrm{e}^{K_0\tau}\mathcal{F}_2(x)\right\|_{\infty,\vartheta,\varrho} \lesssim 1, \quad \left\|\mathrm{e}^{K_0\tau}\mathcal{F}_3(x)\right\|_{\infty,\vartheta,\varrho} \lesssim 1, \quad \left\|\mathrm{e}^{K_0\tau}\mathcal{F}_4(x)\right\|_{\infty,\vartheta,\varrho} \lesssim 1. \tag{6.24}$$

In particular, since $\partial_t = \varepsilon^{-2}\partial_\tau$, we have the time derivative estimate.

THEOREM 6.6. *For $K_0 > 0$ sufficiently small, the initial layer satisfies*

$$\left\|\mathrm{e}^{K_0\sigma}\frac{\partial\mathcal{F}_2(x)}{\partial t}\right\|_{\infty,\vartheta,\varrho} \lesssim \varepsilon^{-2}, \quad \left\|\mathrm{e}^{K_0\sigma}\frac{\partial\mathcal{F}_3(x)}{\partial t}\right\|_{\infty,\vartheta,\varrho} \lesssim \varepsilon^{-2}, \quad \left\|\mathrm{e}^{K_0\sigma}\frac{\partial\mathcal{F}_4(x)}{\partial t}\right\|_{\infty,\vartheta,\varrho} \lesssim \varepsilon^{-2}. \tag{6.25}$$

Note that due to rescaling $\tau = \frac{t}{\varepsilon^2}$, the bound for $\partial_t\mathcal{F}_k$ is much worse than $\mathcal{F}_k$. This is the main reason that we have to expand the initial layer to more orders than interior solutions and boundary layers. Also, this is why we have to enforce the compatibility condition (1.17) and let $\mathcal{F}_1$ vanish.

The space derivative version follows the same fashion.

THEOREM 6.7. *For $K_0 > 0$ sufficiently small, the initial layer satisfies*

$$\left\|\mathrm{e}^{K_0\tau}\nabla_x\mathcal{F}_2(x)\right\|_{\infty,\vartheta,\varrho} \lesssim 1, \quad \left\|\mathrm{e}^{K_0\tau}\nabla_x\mathcal{F}_3(x)\right\|_{\infty,\vartheta,\varrho} \lesssim 1, \quad \left\|\mathrm{e}^{K_0\tau}\nabla_x\mathcal{F}_4(x)\right\|_{\infty,\vartheta,\varrho} \lesssim 1. \tag{6.26}$$

The above estimates do not involve the spatial integral. Obviously, the x integral estimates also hold.

6.3.2. Analysis of boundary layers. Based on the analysis in sections 2.6 and 3, we know $\mathscr{F}_1 = 0$ and $\mathscr{F}_2$ is well-defined.

THEOREM 6.8. *For $K_0 > 0$ sufficiently small, the boundary layer $\mathscr{F}_2$ satisfies*

$$\left\|\mathrm{e}^{K_0\eta}\mathscr{F}_2(t)\right\|_{\infty,\vartheta,\varrho} \lesssim 1, \tag{6.27}$$

and

$$\begin{aligned}
&\left\|\mathrm{e}^{K_0\eta}v_\eta\frac{\partial\mathscr{F}_2(t)}{\partial\eta}\right\|_{\infty,\vartheta,\varrho} + \left\|\mathrm{e}^{K_0\eta}\frac{\partial\mathscr{F}_2(t)}{\partial\iota_1}\right\|_{\infty,\vartheta,\varrho} + \left\|\mathrm{e}^{K_0\eta}\frac{\partial\mathscr{F}_2(t)}{\partial\iota_2}\right\|_{\infty,\vartheta,\varrho} \lesssim \left|\ln(\varepsilon)\right|^8,\\
&\left\|\mathrm{e}^{K_0\eta}\nu\frac{\partial\mathscr{F}_2(t)}{\partial v_\eta}\right\|_{\infty,\vartheta,\varrho} + \left\|\mathrm{e}^{K_0\eta}\nu\frac{\partial\mathscr{F}_2(t)}{\partial v_\phi}\right\|_{\infty,\vartheta,\varrho} + \left\|\mathrm{e}^{K_0\eta}\nu\frac{\partial\mathscr{F}_2(t)}{\partial v_\psi}\right\|_{\infty,\vartheta,\varrho} \lesssim \left|\ln(\varepsilon)\right|^8.
\end{aligned} \tag{6.28}$$

However, the tricky part is the estimate of $\mathscr{F}_3$, which essentially satisfies a stationary linearized Boltzmann equation

$$\begin{cases}
\varepsilon v\cdot\nabla_x\mathscr{F}_3(t) + \mathcal{L}[\mathscr{F}_3(t)] = Z(t) \ \text{ in } \ \tilde{\Omega}\times\mathbb{R}^3,\\
\mathscr{F}_3(t)(x_0,v) = \mathcal{P}[\mathscr{F}_3(t)](x_0,v) + b(t) \ \text{ for } \ x_0\in\partial\Omega \ \text{ and } \ v\cdot n<0,
\end{cases} \tag{6.29}$$

where

$$
\begin{aligned}
Z :=\ & 2\Gamma[\mathscr{F}_1, \mathscr{F}_2] + 2\Gamma[F_1, \mathscr{F}_2] + 2\Gamma[F_2, \mathscr{F}_1] \\
& + \frac{1}{P_1P_2}\left(\frac{\partial_{11}r\cdot\partial_2 r}{P_1(\varepsilon\kappa_1\eta-1)}v_\phi v_\psi + \frac{\partial_{12}r\cdot\partial_2 r}{P_2(\varepsilon\kappa_2\eta-1)}v_\psi^2\right)\frac{\partial\mathscr{F}_2}{\partial v_\phi} \\
& + \frac{1}{P_1P_2}\left(\frac{\partial_{22}r\cdot\partial_1 r}{P_2(\varepsilon\kappa_2\eta-1)}v_\phi v_\psi + \frac{\partial_{12}r\cdot\partial_1 r}{P_1(\varepsilon\kappa_1\eta-1)}v_\phi^2\right)\frac{\partial\mathscr{F}_2}{\partial v_\psi} \\
& + \frac{v_\phi}{P_1(\varepsilon\kappa_1\eta-1)}\frac{\partial\mathscr{F}_2}{\partial\iota_1} + \frac{v_\psi}{P_2(\varepsilon\kappa_2\eta-1)}\frac{\partial\mathscr{F}_2}{\partial\iota_2},
\end{aligned}
\tag{6.30}
$$

and

$$
\begin{aligned}
b :=\ & \varepsilon^{-2}\Big(\mu_b^\varepsilon - \mu - \varepsilon\mu^{\frac12}\mu_1\Big)\mu^{-1}\mathcal{P}[F_1+\mathscr{F}_1] + \varepsilon^{-1}\Big(\mu_b^\varepsilon-\mu\Big)\mu^{-1}\mathcal{P}[F_2+\mathscr{F}_2] \\
& + \varepsilon^{-3}\mu^{-\frac12}\Big(\mu_b^\varepsilon - \mu - \varepsilon\mu^{\frac12}\mu_1 - \varepsilon^2\mu^{\frac12}\mu_2\Big) - \Big((B_3+C_3) - \mathcal{P}[B_3+C_3]\Big).
\end{aligned}
\tag{6.31}
$$

Based on stationary L^{2m} estimates in Remark 4.1, we obtain

$$
\begin{aligned}
& \frac{1}{\varepsilon^{\frac12}}\left\|(1-\mathcal{P})[\mathscr{F}_3(t)]\right\|_{\gamma_+,2} + \frac{1}{\varepsilon}\left\|(\mathbb{I}-\mathbb{P})[\mathscr{F}_3(t)]\right\|_\nu + \left\|\mathbb{P}[\mathscr{F}_3(t)]\right\|_{2m} \\
&\quad \lesssim \frac{1}{\varepsilon^2}\left\|\mathbb{P}[Z(t)]\right\|_{\frac{2m}{2m-1}} + \frac{1}{\varepsilon}\left\|\nu^{-\frac12}(\mathbb{I}-\mathbb{P})[Z(t)]\right\|_2 + \left\|\nu^{-1}Z(t)\right\|_{\infty,\vartheta,\varrho} + \left\|b(t)\right\|_{\gamma_-,\frac{4m}{3}} \\
&\qquad + \frac{1}{\varepsilon}\left\|b(t)\right\|_{\gamma_-,2} + \left\|b(t)\right\|_{\gamma_-,\infty,\varrho,\vartheta} \\
&\quad \lesssim \frac{1}{\varepsilon^{1+\frac{1}{2m}}}\left|\ln(\varepsilon)\right|^8,
\end{aligned}
\tag{6.32}
$$

where we strongly rely on the rescaling $\eta = \frac{\mathfrak{N}}{\varepsilon}$ and the exponential decay of Z in η. Then using the stationary L^∞ estimates in Theorem 4.6, we have

$$
\begin{aligned}
& \left\|\mathscr{F}_3(t)\right\|_{\infty,\vartheta,\varrho} + \left\|\mathscr{F}_3(t)\right\|_{\gamma_+,\infty,\varrho,\vartheta} \\
&\quad \lesssim \frac{1}{\varepsilon^{2+\frac{3}{2m}}}\left\|\mathbb{P}[Z(t)]\right\|_{\frac{2m}{2m-1}} + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\left\|\nu^{-\frac12}(\mathbb{I}-\mathbb{P})[Z(t)]\right\|_2 + \left\|\nu^{-1}Z(t)\right\|_{\infty,\vartheta,\varrho} \\
&\qquad + \frac{1}{\varepsilon^{\frac{3}{2m}}}\left\|b(t)\right\|_{\gamma_-,\frac{4m}{3}} + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\left\|b(t)\right\|_{\gamma_-,2} + \left\|b(t)\right\|_{\gamma_-,\infty,\varrho,\vartheta} \\
&\quad \lesssim \frac{1}{\varepsilon^{1+\frac{2}{m}}}\left|\ln(\varepsilon)\right|^8.
\end{aligned}
\tag{6.33}
$$

Note that we lose the decay of $\mathscr{F}_3$ in η.

The above is only the instantaneous version. The corresponding accumulative version for both $\mathscr{F}_k$ and $\partial_t\mathscr{F}_k$ also hold when taking time decay into consideration.

6.3.3. Analysis of interior solutions. Based on the analysis in matching procedure, we know F_k are well-defined satisfying corresponding fluid equations.

THEOREM 6.9. *For $K_0 > 0$ sufficiently small, the interior solution satisfies*

$$
\begin{aligned}
& \left|\left|\left|\langle v\rangle^\vartheta \mathrm{e}^{\varrho|v|^2}F_1\right|\right|\right|_{L_t^\infty H_x^3 L_v^\infty} \lesssim 1, \quad \left|\left|\left|\langle v\rangle^\vartheta \mathrm{e}^{\varrho|v|^2}F_2\right|\right|\right|_{L_t^\infty H_x^3 L_v^\infty} \lesssim 1, \\
& \left|\left|\left|\langle v\rangle^\vartheta \mathrm{e}^{\varrho|v|^2}F_3\right|\right|\right|_{L_t^\infty H_x^3 L_v^\infty} \lesssim 1.
\end{aligned}
\tag{6.34}
$$

6.3.4. Analysis of initial-boundary interaction. The compatibility condition (1.17) implies that at the corner points $(0, x_0, v)$, (1.21) is naturally satisfied. Also, we have the simplified expansion at these points:

- By our construction in section 2.6, $\mathscr{F}_1 = 0$ and $\mathcal{F}_1 = 0$. Also,

$$F_1(0, x_0, v) = A_1(t, x_0, v) + B_1(t, x_0, v) + C_1(t, x_0, v) = \rho_{0,1}(x_0) \tag{6.35}$$

with

$$A_1(t, x_0, v) = \rho_{0,1}(x_0)\mu^{\frac{1}{2}}(v), \quad B_1(t, x_0, v) = 0, \quad C_1(t, x_0, v) = 0. \tag{6.36}$$

In the end, we know

$$F_1(0, x_0, v) = A_1(t, x_0, v) = \rho_{0,1}(x_0)\mu^{\frac{1}{2}}(v). \tag{6.37}$$

- By our construction in section 2.6, at (t, x_0, v), $\mathscr{F}_2$ and $\mathcal{F}_2$ satisfy trivial equations with zero source term and zero data, so $\mathscr{F}_2(0, x, v) = 0$ and $\mathcal{F}_2(t, x_0, v) = 0$. Also,

$$F_2(0, x_0, v) = A_2(t, x_0, v) + B_2(t, x_0, v) + C_2(t, x_0, v) = \rho_{0,2}(x_0) \tag{6.38}$$

with

$$A_2(t, x_0, v) = \rho_{0,2}(x_0)\mu^{\frac{1}{2}}(v), \quad B_2(t, x_0, v) = 0, \quad C_2(t, x_0, v) = 0. \tag{6.39}$$

Here the space derivative $\nabla_x f_{0,1}(x_0, v) = 0$ plays a key role. In the end, we know

$$F_2(0, x_0, v) = A_2(t, x_0, v) = \rho_{0,2}(x_0)\mu^{\frac{1}{2}}(v). \tag{6.40}$$

- Based on our construction in section 2.6, we know

$$F_3(0, x_0, v) = A_3(t, x_0, v) + B_3(t, x_0, v) + C_3(t, x_0, v). \tag{6.41}$$

In particular, have

$$B_3(t, x_0, v) = 0, \quad C_3(t, x_0, v) = 0. \tag{6.42}$$

Here the space derivative $\nabla_x f_{0,1}(x_0, v) = \nabla_x f_{0,2}(x_0, v) = 0$ and $\nabla_x^2 f_{0,1}(x_0, v) = 0$ play a key role. Also, these space derivatives accompanied with $\partial_t \mu_1 (t, x_0, v) = 0$ yield $v \cdot \nabla_x \mathcal{F}_2 = 0$. Hence, we know $\mathscr{F}_3$ and $\mathcal{F}_3$ satisfy trivial equations with zero source term and zero data, so $\mathscr{F}_3(0, x, v) = 0$ and $\mathcal{F}_3(t, x_0, v) = 0$. In the end, we know

$$F_3(0, x_0, v) = A_3(t, x_0, v) = \rho_{0,3}(x_0)\mu^{\frac{1}{2}}(v). \tag{6.43}$$

- In summary, we have shown that at the corner point $(0, x_0, v)$, both the initial layer and boundary layer are zero up to third order.

6.4. Proof of the main theorem. Now we turn to the proof of the main result, Theorem 1.1. The asymptotic analysis already reveals that the construction of the interior solution, initial layer, and boundary layer is valid. Here, we focus on the remainder estimates. We divide the proof into several steps:

Step 1: Remainder definitions.

Define the remainder as

$$\varepsilon^3 R = f^\varepsilon - Q - \mathscr{Q} - \mathcal{Q}, \tag{6.44}$$

where

$$Q := \sum_{k=1}^{3} \varepsilon^k F_k, \quad \mathscr{Q} := \sum_{k=1}^{3} \varepsilon^k \mathscr{F}_k, \quad \mathcal{Q} := \sum_{k=1}^{4} \varepsilon^k \mathcal{F}_k. \tag{6.45}$$

In other words, we have

$$f^\varepsilon = Q + \mathscr{Q} + \mathcal{Q} + \varepsilon^3 R. \tag{6.46}$$

We write $\mathscr{L}$ to denote the linearized Boltzmann operator:

$$\mathscr{L}[f] = \varepsilon^2 \partial_t f + \varepsilon v \cdot \nabla_x f + \mathcal{L}[f]. \tag{6.47}$$

In studying initial layer in section 2.2, we utilize the equivalent form:

$$\mathscr{L}[f] = \partial_\tau f + \varepsilon v \cdot \nabla_x u + \mathcal{L}[f]. \tag{6.48}$$

In studying boundary layer in section 2.3, we use another equivalent form:

$$\begin{aligned}
\mathscr{L}[f] = {} & \varepsilon^2 \partial_t f + v_\eta \frac{\partial f}{\partial \eta} - \frac{\varepsilon}{R_1 - \varepsilon\eta}\left(v_\phi^2 \frac{\partial f}{\partial v_\eta} - v_\eta v_\phi \frac{\partial f}{\partial v_\phi}\right) \\
& - \frac{\varepsilon}{R_2 - \varepsilon\eta}\left(v_\psi^2 \frac{\partial f}{\partial v_\eta} - v_\eta v_\psi \frac{\partial f}{\partial v_\psi}\right) \\
& - \frac{\varepsilon}{P_1 P_2}\left(\frac{\partial_{11} r \cdot \partial_2 r}{P_1(\varepsilon\kappa_1\eta - 1)} v_\phi v_\psi + \frac{\partial_{12} r \cdot \partial_2 r}{P_2(\varepsilon\kappa_2\eta - 1)} v_\psi^2 \right) \frac{\partial f}{\partial v_\phi} \\
& - \frac{\varepsilon}{P_1 P_2}\left(\frac{\partial_{22} r \cdot \partial_1 r}{P_2(\varepsilon\kappa_2\eta - 1)} v_\phi v_\psi + \frac{\partial_{12} r \cdot \partial_1 r}{P_1(\varepsilon\kappa_1\eta - 1)} v_\phi^2 \right) \frac{\partial f}{\partial v_\psi} \\
& - \varepsilon\left(\frac{v_\phi}{P_1(\varepsilon\kappa_1\eta - 1)} \frac{\partial f}{\partial \iota_1} + \frac{v_\psi}{P_2(\varepsilon\kappa_2\eta - 1)} \frac{\partial f}{\partial \iota_2}\right) + \mathcal{L}[f].
\end{aligned}$$

Step 2: Representation of $\mathscr{L}[R]$.

Equation (1.21) is actually

$$\mathscr{L}[f^\varepsilon] = \Gamma[f^\varepsilon, f^\varepsilon], \tag{6.49}$$

which means

$$\mathscr{L}[Q + \mathscr{Q} + \mathcal{Q} + \varepsilon^3 R] = \Gamma[Q + \mathscr{Q} + \mathcal{Q} + \varepsilon^3 R, Q + \mathscr{Q} + \mathcal{Q} + \varepsilon^3 R]. \tag{6.50}$$

Note that the right-hand side of (6.50), i.e., the nonlinear term can be decomposed as

$$\begin{aligned}
\Gamma[Q + \mathscr{Q} + \mathcal{Q} + \varepsilon^3 R, Q + \mathscr{Q} + \mathcal{Q} + \varepsilon^3 R] = {} & \varepsilon^6 \Gamma[R, R] + 2\varepsilon^3 \Gamma[R, Q + \mathscr{Q} + \mathcal{Q}] \\
& + \Gamma[Q + \mathscr{Q} + \mathcal{Q}, Q + \mathscr{Q} + \mathcal{Q}].
\end{aligned} \tag{6.51}$$

Then we turn to the LHS of (6.50). The interior contribution is

$$\begin{aligned}\mathscr{L}[Q] &= \varepsilon^2\partial_t\Big(\varepsilon F_1+\varepsilon^2F_2+\varepsilon^3F_3\Big)+\varepsilon v\cdot\nabla_x\Big(\varepsilon F_1+\varepsilon^2F_2+\varepsilon^3F_3\Big)\\ &\quad+\mathcal{L}[\varepsilon F_1+\varepsilon^2F_2+\varepsilon^3F_3]\\ &=\varepsilon^4v\cdot\nabla_xF_3+\varepsilon^4\partial_tF_2+\varepsilon^5\partial_tF_3+\varepsilon^2\Gamma[F_1,F_1]+2\varepsilon^3\Gamma[F_1,F_2].\end{aligned}\tag{6.52}$$

On the other hand, we consider the boundary layer contribution. Since $\mathscr{F}_1=0$, $\mathscr{F}_2$ and $\mathscr{F}_3$ terms are all included in boundary layer construction except the time derivatives, we compute

$$\mathscr{L}[\mathscr{Q}]=\varepsilon^4\partial_t\mathscr{F}_2+\varepsilon^5\partial_t\mathscr{F}_3+2\varepsilon^3\Gamma[F_1,\mathscr{F}_2].\tag{6.53}$$

Also, since $\mathcal{F}_1=0$, the initial layer contribution

$$\mathscr{L}[\mathcal{Q}]=\varepsilon^5v\cdot\nabla_x\mathcal{F}_4+2\varepsilon^3\Gamma[F_1,\mathcal{F}_2]+2\varepsilon^4\Gamma[\mathcal{F}_2,\mathcal{F}_2]+2\varepsilon^4\Gamma[F_2,\mathcal{F}_2]+2\varepsilon^4\Gamma[F_1,\mathcal{F}_3].\tag{6.54}$$

Therefore, inserting (6.51), (6.52), (6.53), and (6.54) into (6.50), we have

$$\mathscr{L}[R]=\varepsilon^3\Gamma[R,R]+2\Gamma[R,Q+\mathscr{Q}+\mathcal{Q}]+S_1+S_2,\tag{6.55}$$

where

$$S_1=-\varepsilon v\cdot\nabla_xF_3-\varepsilon\partial_tF_2-\varepsilon^2\partial_tF_3-\varepsilon\partial_t\mathscr{F}_2-\varepsilon^2\partial_t\mathscr{F}_3-\varepsilon^2v\cdot\nabla_x\mathcal{F}_4,\tag{6.56}$$

$$\begin{aligned}S_2&=\varepsilon\bigg(2\Gamma[\mathscr{F}_2,\mathcal{F}_2]+2\Gamma[F_1,F_3]+2\Gamma[F_2,F_2]+2\Gamma[F_1,\mathscr{F}_3]\bigg)\\ &\quad+\text{higher-order }\Gamma\text{ terms up to }\varepsilon^4.\end{aligned}\tag{6.57}$$

Step 3: Representation of $R-\mathcal{P}[R]$ and $R(0)$.

The boundary condition of (1.21) is essentially

$$f^\varepsilon=\mu_b^\varepsilon\mu^{-1}\mathcal{P}[f^\varepsilon]+\mu^{-\frac12}(\mu_b^\varepsilon-\mu),\tag{6.58}$$

which means

$$Q+\mathscr{Q}+\varepsilon^3R=\mathcal{P}[Q+\mathscr{Q}+\varepsilon^3R]+(\mu_b^\varepsilon-\mu)\mu^{-1}\mathcal{P}[Q+\mathscr{Q}+\varepsilon^3R]+\mu^{-\frac12}(\mu_b^\varepsilon-\mu).\tag{6.59}$$

Based on the boundary condition expansion in section 2.6, we have

$$R-\mathcal{P}[R]=H[R]+h,\tag{6.60}$$

where

$$H[R](t,x_0,v)=(\mu_b^\varepsilon-\mu)\mu^{-1}\mathcal{P}[R]\tag{6.61}$$

and

$$h=-\varepsilon\mathcal{F}_4.\tag{6.62}$$

In other words, the only contribution is from the initial layer $\mathcal{F}_4$ at the corner point. On the other hand, for initial data,

$$R(0) = z = \varepsilon \mathcal{F}_4(0). \tag{6.63}$$

In other words, the only contribution is from the initial data of initial layer $\mathcal{F}_4$.

Step 4: Remainder estimate.

Equation (6.55), initial condition (6.63), and boundary condition (6.60) form a system that fits into (6.3):

(6.64)

$$\begin{cases} \varepsilon^2 \partial_t R + \varepsilon v \cdot \nabla_x R + \mathcal{L}[R] = \Gamma[R, 2(Q + \mathscr{Q} + \mathcal{Q}) + \varepsilon^3 R] + S_1(t,x,v) + S_2(t,x,v) \\ \quad \text{in } \mathbb{R}_+ \times \Omega \times \mathbb{R}^3, \\ R(0,x,v) = z(x,v) \text{ in } \Omega \times \mathbb{R}^3, \\ R(t,x_0,v) = \mathcal{P}[R](t,x_0,v) + H[R](t,x_0,v) + h(t,x_0,v) \text{ for } x_0 \in \partial\Omega \\ \quad \text{and } v \cdot n < 0. \end{cases}$$

Hence, by Theorem 6.3, we have

(6.65)

$$\begin{aligned} &\left\|\left|R\right|\right\|_{\infty,\vartheta,\varrho} + \left\|\left|R\right|\right\|_{\gamma_+,\infty,\varrho,\vartheta} \\ &\quad\lesssim \frac{1}{\varepsilon^{2+\frac{3}{2m}}} \|\mathbb{P}[S_1(t)]\|_{\frac{2m}{2m-1}} + \frac{1}{\varepsilon^{1+\frac{3}{2m}}} \left\|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[S_1(t)]\right\|_2 + \frac{1}{\varepsilon^{2+\frac{3}{2m}}} \left\|\left|\mathbb{P}[S_1]\right|\right\|_2 \\ &\qquad + \frac{1}{\varepsilon^{1+\frac{3}{2m}}} \left\|\left|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[S_1]\right|\right\|_2 + \frac{1}{\varepsilon^{2+\frac{3}{2m}}} \left\|\left|\mathbb{P}[\partial_t S_1]\right|\right\|_2 + \frac{1}{\varepsilon^{1+\frac{3}{2m}}} \left\|\left|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[\partial_t S_1]\right|\right\|_2 \\ &\qquad + \left\|\left|\nu^{-1} S\right|\right\|_{\infty,\vartheta,\varrho} + \frac{1}{\varepsilon^{2+\frac{3}{2m}}} \|S_1(0)\|_2 + \frac{1}{\varepsilon^{2+\frac{3}{2m}}} \|\mathbb{P}[S_2(t)]\|_{\frac{2m}{2m-1}} \\ &\qquad + \frac{1}{\varepsilon^{1+\frac{3}{2m}}} \left\|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[S_2(t)]\right\|_2 + \frac{1}{\varepsilon^{2+\frac{3}{2m}}} \left\|\left|\mathbb{P}[S_2]\right|\right\|_2 + \frac{1}{\varepsilon^{1+\frac{3}{2m}}} \left\|\left|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[S_2]\right|\right\|_2 \\ &\qquad + \frac{1}{\varepsilon^{2+\frac{3}{2m}}} \left\|\left|\mathbb{P}[\partial_t S_2]\right|\right\|_2 + \frac{1}{\varepsilon^{1+\frac{3}{2m}}} \left\|\left|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[\partial_t S_2]\right|\right\|_2 + \left\|\left|\nu^{-1} S\right|\right\|_{\infty,\vartheta,\varrho} \\ &\qquad + \frac{1}{\varepsilon^{2+\frac{3}{2m}}} \|S_2(0)\|_2 + \frac{1}{\varepsilon^{\frac{3}{2m}}} \|h(t)\|_{\gamma_-,\frac{4m}{3}} + \frac{1}{\varepsilon^{1+\frac{3}{2m}}} \|h(t)\|_{\gamma_-,2} + \frac{1}{\varepsilon^{1+\frac{3}{2m}}} \left\|\left|h\right|\right\|_{\gamma_-,2} \\ &\qquad + \frac{1}{\varepsilon^{1+\frac{3}{2m}}} \left\|\left|\partial_t h\right|\right\|_{\gamma_-,2} + \left\|\left|h\right|\right\|_{\gamma_-,\infty,\varrho,\vartheta} + \frac{1}{\varepsilon^{2+\frac{3}{2m}}} \|\nu z\|_2 \\ &\qquad + \frac{1}{\varepsilon^{1+\frac{3}{2m}}} \|v \cdot \nabla_x z\|_2 + \|z\|_{\infty,\vartheta,\varrho}. \end{aligned}$$

Step 5: Estimate of S_1.

Using results in section 6.3.3, for the interior contribution $S_{IS} := -\varepsilon v \cdot \nabla_x F_3 - \varepsilon \partial_t F_2 - \varepsilon^2 \partial_t F_3$,

$$\begin{aligned} &\|S_{IS}(t)\|_{\frac{2m}{2m-1}} + \left\|\nu^{-\frac{1}{2}} S_{IS}(t)\right\|_2 + \left\|\left|S_{IS}\right|\right\|_2 + \left\|\left|\nu^{-\frac{1}{2}} S_{IS}\right|\right\|_2 \\ &\quad + \left\|\left|\partial_t S_{IS}\right|\right\|_2 + \left\|\left|\nu^{-\frac{1}{2}} \partial_t S_{IS}\right|\right\|_2 + \left\|\left|\nu^{-1} S_{IS}\right|\right\|_{\infty,\vartheta,\varrho} + \|S_{IS}(0)\|_2 \lesssim \varepsilon. \end{aligned} \tag{6.66}$$

Using results in section 6.3.2, for the boundary layer contribution $S_{BL} := -\varepsilon\partial_t\mathscr{F}_2 - \varepsilon^2\partial_t\mathscr{F}_3$, note that $\|g(t)\|_{L^p} \lesssim \|g(t)\|_{L^{2m}}$ for $1 \leq p \leq 2m$:

(6.67)
$$\|\mathbb{P}[S_{BL}](t)\|_{\frac{2m}{2m-1}} \lesssim \varepsilon^{1-\frac{1}{2m}}\left|\ln(\varepsilon)\right|^8, \quad \left\|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[S_{BL}](t)]\right\|_2 \lesssim \varepsilon^{2-\frac{1}{2m}}\left|\ln(\varepsilon)\right|^8,$$
$$|||\mathbb{P}[S_{BL}]|||_2 \lesssim \varepsilon^{1-\frac{1}{2m}}\left|\ln(\varepsilon)\right|^8, \quad \left|\left|\left|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[S_{BL}]\right|\right|\right|_2 \lesssim \varepsilon^{2-\frac{1}{2m}}\left|\ln(\varepsilon)\right|^8,$$
$$|||\mathbb{P}[\partial_t S_{BL}]|||_2 \lesssim \varepsilon^{1-\frac{1}{2m}}\left|\ln(\varepsilon)\right|^8, \quad \left|\left|\left|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[\partial_t S_{BL}]\right|\right|\right|_2 \lesssim \varepsilon^{2-\frac{1}{2m}}\left|\ln(\varepsilon)\right|^8,$$

(6.68)
$$\left|\left|\left|\nu^{-1}S_{BL}\right|\right|\right|_{\infty,\vartheta,\varrho} \lesssim \varepsilon^{1-\frac{1}{m}}\left|\ln(\varepsilon)\right|^8, \quad \|S_{BL}(0)\|_2 \lesssim \varepsilon^{1-\frac{1}{2m}}\left|\ln(\varepsilon)\right|^8.$$

Using results in section 6.3.1, for the initial layer contribution $S_{IL} := -\varepsilon^2 v\cdot\nabla_x\mathcal{F}_4$, note the rescaling $\tau = \frac{t}{\varepsilon^2}$:

$$\|\mathbb{P}[S_{IL}](t)\|_{\frac{2m}{2m-1}} \lesssim \varepsilon^2, \quad \left\|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[S_{IL}](t)]\right\|_2 \lesssim \varepsilon^2, \tag{6.69}$$
$$|||\mathbb{P}[S_{IL}]|||_2 \lesssim \varepsilon^3, \quad \left|\left|\left|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[S_{IL}]\right|\right|\right|_2 \lesssim \varepsilon^3,$$
$$|||\mathbb{P}[\partial_t S_{IL}]|||_2 \lesssim \varepsilon, \quad \left|\left|\left|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[\partial_t S_{IL}]\right|\right|\right|_2 \lesssim \varepsilon,$$
$$\left|\left|\left|\nu^{-1}S_{IL}\right|\right|\right|_{\infty,\vartheta,\varrho} \lesssim \varepsilon^2, \quad \|S_{IL}(0)\|_2 \lesssim \varepsilon^2. \tag{6.70}$$

Hence, we have

$$\|\mathbb{P}[S_1](t)\|_{\frac{2m}{2m-1}} \lesssim \varepsilon^{1-\frac{1}{2m}}\left|\ln(\varepsilon)\right|^8, \quad \left\|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[S_1](t)]\right\|_2 \lesssim \varepsilon^{2-\frac{1}{2m}}\left|\ln(\varepsilon)\right|^8, \tag{6.71}$$
$$|||\mathbb{P}[S_1]|||_2 \lesssim \varepsilon^{1-\frac{1}{2m}}\left|\ln(\varepsilon)\right|^8, \quad \left|\left|\left|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[S_1]\right|\right|\right|_2 \lesssim \varepsilon^{2-\frac{1}{2m}}\left|\ln(\varepsilon)\right|^8,$$
$$|||\mathbb{P}[\partial_t S_1]|||_2 \lesssim \varepsilon^{1-\frac{1}{2m}}\left|\ln(\varepsilon)\right|^8, \quad \left|\left|\left|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[\partial_t S_1]\right|\right|\right|_2 \lesssim \varepsilon^{2-\frac{1}{2m}}\left|\ln(\varepsilon)\right|^8,$$
$$\left|\left|\left|\nu^{-1}S_1\right|\right|\right|_{\infty,\vartheta,\varrho} \lesssim \varepsilon^{1-\frac{1}{m}}\left|\ln(\varepsilon)\right|^8, \quad \|S_1(0)\|_2 \lesssim \varepsilon^{1-\frac{1}{2m}}\left|\ln(\varepsilon)\right|^8. \tag{6.72}$$

Step 6: Estimate of S_2.

It suffices to consider the leading-order term $2\varepsilon\Gamma[\mathscr{F}_2, \mathcal{F}_2]$ which contains the most dangerous initial layer $\mathcal{F}_2$. Note that the time derivative estimate is the worst one. Using nonlinear estimates in Lemma 6.1 and rescaling $\eta = \frac{\mu}{\varepsilon}$ and $\tau = \frac{t}{\varepsilon^2}$, we have

$$\|\mathbb{P}[S_2](t)\|_{\frac{2m}{2m-1}} = 0, \quad \left\|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[S_2](t)]\right\|_2 \lesssim \varepsilon^{\frac{3}{2}}, \tag{6.73}$$
$$|||\mathbb{P}[S_2]|||_2 = 0, \quad \left|\left|\left|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[S_2]\right|\right|\right|_2 \lesssim \varepsilon^{\frac{5}{2}},$$
$$|||\mathbb{P}[\partial_t S_2]|||_2 = 0, \quad \left|\left|\left|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})[\partial_t S_2]\right|\right|\right|_2 \lesssim \varepsilon^{\frac{1}{2}},$$
$$\left|\left|\left|\nu^{-1}S_2\right|\right|\right|_{\infty,\vartheta,\varrho} \lesssim \varepsilon, \quad \|S_2(0)\|_2 \lesssim \varepsilon^{\frac{3}{2}}. \tag{6.74}$$

Step 7: Estimate of h and z.

For boundary data $h = -\varepsilon\mathcal{F}_4$, we have

$$\|h(t)\|_{\gamma_-,\frac{4m}{3}} \lesssim \varepsilon, \quad \|h(t)\|_{\gamma_-,2} \lesssim \varepsilon, \quad |||h|||_{\gamma_-,2} \lesssim \varepsilon^2, \tag{6.75}$$
$$|||\partial_t h|||_{\gamma_-,2} \lesssim 1, \quad |||h|||_{\gamma_-,\infty,\varrho,\vartheta} \lesssim \varepsilon.$$

For initial data $z = -\varepsilon\mathcal{F}_4(0)$, we have

$$\|\nu z\|_2 \lesssim \varepsilon, \quad \|v\cdot\nabla_x z\|_2 \lesssim \varepsilon, \quad \|z\|_{\infty,\vartheta,\varrho} \lesssim \varepsilon. \tag{6.76}$$

Step 8: Synthesis.
Summarizing all of the above, we have

(6.77)

$$\begin{aligned}
&|||R|||_{\infty,\vartheta,\varrho} + |||R|||_{\gamma_+,\infty,\varrho,\vartheta} \\
&\lesssim \frac{1}{\varepsilon^{2+\frac{3}{2m}}}\left(\varepsilon^{1-\frac{1}{2m}}|\ln(\varepsilon)|^8\right) + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\left(\varepsilon^{2-\frac{1}{2m}}|\ln(\varepsilon)|^8\right) + \frac{1}{\varepsilon^{2+\frac{3}{2m}}}\left(\varepsilon^{1-\frac{1}{2m}}|\ln(\varepsilon)|^8\right) \\
&\quad + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\left(\varepsilon^{2-\frac{1}{2m}}|\ln(\varepsilon)|^8\right) + \frac{1}{\varepsilon^{2+\frac{3}{2m}}}\left(\varepsilon^{1-\frac{1}{2m}}|\ln(\varepsilon)|^8\right) \\
&\quad + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\left(\varepsilon^{2-\frac{1}{2m}}|\ln(\varepsilon)|^8\right) + \left(\varepsilon^{1-\frac{1}{m}}|\ln(\varepsilon)|^8\right) + \frac{1}{\varepsilon^{2+\frac{3}{2m}}}\left(\varepsilon^{1-\frac{1}{2m}}|\ln(\varepsilon)|^8\right) \\
&\quad + \frac{1}{\varepsilon^{2+\frac{3}{2m}}}\Big(0\Big) + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\left(\varepsilon^{\frac{3}{2}}\right) + \frac{1}{\varepsilon^{2+\frac{3}{2m}}}\Big(0\Big) + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\left(\varepsilon^{\frac{5}{2}}\right) \\
&\quad + \frac{1}{\varepsilon^{2+\frac{3}{2m}}}\Big(0\Big) + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\left(\varepsilon^{\frac{1}{2}}\right) + \Big(\varepsilon\Big) + \frac{1}{\varepsilon^{2+\frac{3}{2m}}}\left(\varepsilon^{\frac{3}{2}}\right) \\
&\quad + \frac{1}{\varepsilon^{\frac{3}{2m}}}\Big(\varepsilon\Big) + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\Big(\varepsilon\Big) + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\left(\varepsilon^2\right) + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\Big(1\Big) + \Big(\varepsilon\Big) \\
&\quad + \frac{1}{\varepsilon^{2+\frac{3}{2m}}}\Big(\varepsilon\Big) + \frac{1}{\varepsilon^{1+\frac{3}{2m}}}\Big(\varepsilon\Big) + \Big(\varepsilon\Big) \\
&\lesssim \frac{1}{\varepsilon^{1+\frac{2}{m}}}|\ln(\varepsilon)|^8.
\end{aligned}$$

We have shown

$$\frac{1}{\varepsilon^3}\left\|f^\varepsilon - \sum_{k=1}^{3}\varepsilon^k F_k - \sum_{k=1}^{3}\varepsilon^k \mathscr{F}_k - \sum_{k=1}^{4}\varepsilon^k \mathcal{F}_k\right\|_{\infty,\vartheta,\varrho} \lesssim \varepsilon^{-1-\frac{2}{m}}|\ln(\varepsilon)|^8. \tag{6.78}$$

Therefore, we know

$$\|f^\varepsilon - \varepsilon F_1 - \varepsilon\mathscr{F}_1 - \mathcal{F}_1\|_{\infty,\vartheta,\varrho} \lesssim \varepsilon^{2-\frac{2}{m}}|\ln(\varepsilon)|^8. \tag{6.79}$$

Since $\mathscr{F}_1 = \mathcal{F}_1 = 0$, then we naturally have for $F = F_1$,

$$\|f^\varepsilon - \varepsilon F\|_{\infty,\vartheta,\varrho} \lesssim \varepsilon^{2-\frac{2}{m}}|\ln(\varepsilon)|^8. \tag{6.80}$$

Here $\frac{3}{2} < m < 3$, so we may further bound

$$\|f^\varepsilon - \varepsilon F\|_{\infty,\vartheta,\varrho} \lesssim C(\delta)\varepsilon^{\frac{4}{3}-\delta} \tag{6.81}$$

for any $0 < \delta << 1$. The exponential decay in time can be justified in a similar fashion using Remark 5.5. The positivity of F follows from a similar argument as in [11, section 3.8].

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