

An Area-Depth Symmetric q, t -Catalan Polynomial

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Abstract

We define two symmetric q, t -Catalan polynomials in terms of the area and depth statistic and in terms of the dinv and depth statistics. We prove symmetry using an involution on plane trees. The same involution proves symmetry of the Tutte polynomials. We also provide a combinatorial proof of a remark by Garsia et al. regarding parking functions and the number of connected graphs on a fixed number of vertices.

Mathematics Subject Classifications: 05A19, 05E10, 05C05, 05C30

1 Introduction

The q, t -Catalan functions were first introduced in connection with Macdonald polynomials and Garsia–Haiman’s theory of diagonal harmonics [10] as certain rational functions in q and t . They can be obtained as the bigraded Hilbert series of the alternating component of a certain module of diagonal harmonics, whose dimension is equal to the Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$. In terms of symmetric functions, they can be expressed using of the nabla operator and the elementary symmetric functions e_n as

$$C_n(q, t) = \langle \nabla e_n, e_n \rangle.$$

The combinatorics of the q, t -Catalan polynomials was developed in various papers [11, 14, 15]. In particular, Haglund [14] gave a combinatorial formula as a sum over all Dyck paths graded by the area and bounce statistics (see (5)). Shortly thereafter, Haiman announced a different combinatorial formula using the area and dinv statistics (see (3)). The zeta map [1, 15] relates these two combinatorial formulas. One of the main open problems related to the q, t -Catalan polynomials $C_n(q, t)$ is a combinatorial proof of its symmetry in q and t .

In this paper, we introduce two different q, t -analogues of the Catalan numbers C_n , which are symmetric in q and t . We also get a new formula for the original q, t -Catalan polynomials (see Corollary 35).

The first polynomial $F_n(q, t)$ (see (8)) is the sum over all Dyck paths graded by area and depth. There are several maps from Dyck paths to plane trees, see for example Definitions 7, 9 and 11 from [22, 15, 6] below. The intuition for the depth statistics is that it is the sum over the depths of the various vertices in the plane tree. (This is related to a particular labelling of the vertices in a plane tree as given in Definition 20.) The symmetry in q and t is proved by defining a duality on plane trees, which switches the area and depth sequence. This duality turns out to be a composition of the maps in Definitions 7 and 9. We prove that on Dyck paths, the corresponding involution is equal to a recursively defined involution introduced by Deutsch [9]. In particular, this gives an alternative proof of the symmetry of the Tutte polynomial for the Catalan matroid [5]. The polynomials $F_n(q, t)$ satisfy a recursion that relates them to q, t -Catalan polynomials defined in terms of increasing/decreasing factorizations [18, Section 5] and to Hurwitz graphs [4].

The second polynomial $G_n(q, t)$ (see (9)) is defined in terms of the dinv and dinv of depth statistics denoted ddinv . The dinv statistics can be formulated using the area sequence, so using the depth sequence instead yields the dinv of depth statistics. This polynomial is also symmetric in q and t .

We also address a remark in [12] stating that the sum of parking functions graded by two to the area is equal to the number of connected graphs on a fixed number of vertices.

The paper is organized as follows. In Section 2 we review the definitions associated with the q, t -Catalan polynomials $C_n(q, t)$ and define the polynomials $F_n(q, t)$ and $G_n(q, t)$. In particular, the definition of depth and ddinv is given. Furthermore, we review several maps from Dyck paths to plane trees. Our main results are stated in Section 3. In particular, a recursion for $F_n(q, t)$ is proved as well as symmetry of $F_n(q, t)$ and $G_n(q, t)$ using an involution ω that interchanges area and depth. The paper concludes with some further results on parking functions.

2 Background and Definitions

In Section 2.1, we review Dyck paths and their various statistics. In Section 2.2, we define new statistics and related polynomials. In Section 2.3 we give background knowledge on plane trees and their various connections to Dyck paths. We conclude in Section 2.4 with the definition and some results on parking functions and labelled trees.

2.1 Dyck Paths

A *Dyck path* of semilength n is a lattice path with vertices in $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ from $(0, 0)$ to (n, n) consisting of North $(1, 0)$ and East $(0, 1)$ steps that never passes below the line $y = x$. Let the set of all Dyck paths with semilength n be denoted by D_n . It is well known that D_n is enumerated by the n -th *Catalan number* $C_n = \frac{1}{n+1} \binom{2n}{n}$.

Given $\pi \in D_n$, let the *area sequence* of π be the vector $(a_1(\pi), a_2(\pi), \dots, a_n(\pi))$, where $a_i(\pi)$ is the number of full unit squares in the i -th row completely between π and the diagonal $y = x$. Let

$$\text{area}(\pi) = \sum_{i=1}^n a_i(\pi), \quad (1)$$

that is, the total number of squares between the path π and the diagonal. Note that a Dyck path is uniquely determined by its area sequence. Additionally, a vector $(a_1, a_2, \dots, a_n) \in \mathbb{Z}_{\geq 0}^n$ is an area sequence of some Dyck path in D_n if and only if $a_1 = 0$ and $0 \leq a_i \leq a_{i-1} + 1$ for $2 \leq i \leq n$.

Using the area sequence of a Dyck path π , we can define another statistic on Dyck paths as follows

$$\text{dinv}(\pi) = |\{(i, j) \mid i < j, a_i(\pi) = a_j(\pi)\} \cup \{(i, j) \mid i < j, a_i(\pi) = a_j(\pi) + 1\}|. \quad (2)$$

The q, t -Catalan polynomial is defined as

$$C_n(q, t) = \sum_{\pi \in D_n} q^{\text{area}(\pi)} t^{\text{dinv}(\pi)}. \quad (3)$$

The polynomial $C_n(q, t)$ is symmetric in q and t , that is, $C_n(q, t) = C_n(t, q)$ (see for example [15]). It is an open question to find a combinatorial proof of its symmetry.

To define the bounce statistic of $\pi \in D_n$, we first must construct the bounce path $\mathcal{B}(\pi)$ by the following algorithm:

- (1) Start at the point $(0, 0)$.
- (2) Continue North until the start of an East step of π is met.
- (3) Continue East until the diagonal $y = x$ is met.
- (4) If the bounce path has reached the point (n, n) , then stop. Otherwise go back to step (2).

Let $(0, 0) = (b_0, b_0), (b_1, b_1), \dots, (b_k, b_k) = (n, n)$ be the points on the diagonal that $\mathcal{B}(\pi)$ touches. Then *bounce* is defined as

$$\text{bounce}(\pi) = \sum_{i=1}^{k-1} n - b_i. \quad (4)$$

Proposition 1. [15] *We have*

$$C_n(q, t) = \sum_{\pi \in D_n} q^{\text{area}(\pi)} t^{\text{bounce}(\pi)}. \quad (5)$$

There exists a bijection $\zeta: D_n \rightarrow D_n$ on Dyck paths, called the *zeta map*, which has the property that for $\pi \in D_n$

$$\begin{aligned} \text{area}(\pi) &= \text{bounce}(\zeta(\pi)), \\ \text{dinv}(\pi) &= \text{area}(\zeta(\pi)). \end{aligned}$$

This proves that (3) and (5) are equal. The inverse of the zeta map first appeared connection with nilpotent ideals in certain Borel subalgebras of $\mathfrak{sl}(n)$ [1]. For its connections with the combinatorics of q, t -Catalan polynomials, see [15]. The zeta map was further studied and generalized in [2, 7, 24, 8]. For the definition of the zeta map, see [15, Theorem 3.15]. In Proposition 13 below, we state another formulation of the zeta map in terms of plane trees (which can also serve as the definition).

2.2 Depth polynomials

Let $\pi \in D_n$. We produce a labelling for π column-by-column using the following algorithm:

- (1) In the leftmost column, label all cells directly to the right of a North step with a 0.
- (2) In the i -th column from the left, locate the bottommost cell c in the column that is directly right of a North step; note that such a cell may not exist. From c travel Southwest diagonally until a cell c' that is already labelled is reached. Let ℓ be the labelling of c' . Label all cells directly to the right of a North step in the i -th column with an $\ell + 1$.

Define this to be the *depth labelling* of π . The *depth sequence* $(d_1(\pi), d_2(\pi), \dots, d_n(\pi))$ of π can be obtained by reading the entries of the depth labelling of π in the following manner:

- (1) Let v be the empty vector. Let c be the cell directly right of the first North step of π .
- (2) Append the label of c to the end of v . If the length of v is n , then stop and let

$$(d_1(\pi), d_2(\pi), \dots, d_n(\pi)) = v.$$

- (3) Otherwise, travel Northeast diagonally from c until a cell that is labelled is reached. If this cell exists and has not been seen before, then redefine c to be this cell. If no such cell exists or the cell was already visited before by the algorithm, then consider the set of all cells that have been visited already but have a labelled cell directly above them that has not been visited. Out of this set choose the rightmost one and let c be the cell directly above this cell. Go back to step (2).

Remark 2. Note that in the above definition, the rightmost cell of all visited cells with a labelled cell directly above is also the cell in this set with the largest label. Namely, look at the lowest cell in the same column as c , which is labelled. All cells that were already visited but have a labelled cell directly above them are to the left of this cell on the same diagonal or lower. By the construction of the labels, these cells all have strictly smaller labels.

Define the *depth* statistic as follows

$$\mathbf{depth}(\pi) = \sum_{i=1}^n d_i(\pi). \quad (6)$$

Similar to how **dinv** was defined in terms of the area sequence in (2), we can associate a “dinv” type statistic called **ddinv** to the depth sequence of a Dyck path. Formally,

$$\mathbf{ddinv}(\pi) = |\{(i, j) \mid i < j, d_i(\pi) = d_j(\pi)\} \cup \{(i, j) \mid i < j, d_i(\pi) = d_j(\pi) + 1\}|. \quad (7)$$

Example 3. In Figure 1, a Dyck path $\pi \in D_9$ with its depth labelling is shown. The depth sequence is $(0, 1, 1, 2, 0, 1, 2, 2, 0)$. Hence the depth is $\mathbf{depth}(\pi) = 9$. Finally

$\{(1, 5), (1, 9), (5, 9), (2, 3), (2, 6), (3, 6), (4, 7), (4, 8), (7, 8), (2, 5), (2, 9), (3, 5), (3, 9), (6, 9), (4, 6)\}$

are pairs contributing to the **ddinv** statistic in (7), hence $\mathbf{ddinv}(\pi) = 15$.

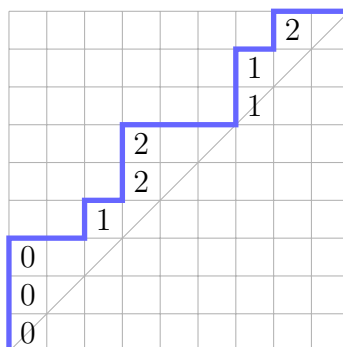


Figure 1: Example of a Dyck path $\pi \in D_9$ with its depth labelling.

Next we define two q, t -Catalan polynomials using the just introduced statistics:

$$F_n(q, t) = \sum_{\pi \in D_n} q^{\mathbf{area}(\pi)} t^{\mathbf{depth}(\pi)} \quad (8)$$

and

$$G_n(q, t) = \sum_{\pi \in D_n} q^{\mathbf{dinv}(\pi)} t^{\mathbf{ddinv}(\pi)}. \quad (9)$$

We will prove various properties of these polynomials in Section 3, including that they are symmetric in q and t .

Example 4. We list the first few polynomials:

n	$C_n(q, t)$	$F_n(q, t)$	$G_n(q, t)$
1	1	1	1
2	$q + t$	$q + t$	$q + t$
3	$q^3 + q^2t + qt^2 + t^3 + qt$	$q^3 + q^2t + qt^2 + t^3 + qt$	$q^2t^2 + q^3 + t^3 + 2qt$
4	$q^6 + q^5t + q^4t^2 + q^3t^3 + q^2t^4 + qt^5 + t^6$ $+q^4t + q^3t^2 + q^2t^3 + qt^4$ $+q^3t + q^2t^2 + qt^3$	$q^6 + q^5t + q^4t^2 + 2q^3t^3 + q^2t^4 + qt^5 + t^6$ $+q^4t + qt^4$ $+q^3t + 2q^2t^2 + qt^3$	$q^5t^2 + q^4t^3 + q^3t^4 + q^2t^5$ $+q^6 + q^4t^2 + q^2t^4 + t^6$ $+2q^3t + 2qt^3 + q^2t + qt^2$

Remark 5. Note that $C_n(1, 1) = F_n(1, 1) = G_n(1, 1) = C_n$ are all equal to the n -th Catalan number. The difference $F_n(q, t) - C_n(q, t)$ can be written as $(1 - t)(1 - q)M_n(q, t)$. Evaluating $M_n(1, 1)$ yields the sequence 0, 0, 0, 1, 14, 124, 888, 5615, 32714, $\dots$, which curiously is the 5-th number after each 1 in the Riordan array, see [17]. Both $(G_n - C_n)/((q - 1)(t - 1))$ and $(G_n - F_n)/((q - 1)(t - 1))$ are also conjectured to have positive coefficients. At $q = t = 1$, the corresponding sequences are 0, 0, 0, 1, 11, 83, 530, 3071, 16997, 86778, 436084, $\dots$ and 0, 0, 0, 1, 10, 69, 406, 2183, 11082, 54064, 256204, $\dots$, which do not seem to appear in [17].

2.3 Plane Trees

In this paper, all rooted trees are drawn with the root on top and its descendants below. The principal subtrees of a rooted tree T are the rooted trees obtained by removing the root of T and considering the children of the root of T to be the new roots of their respective trees.

Definition 6. A *plane tree* is a rooted tree, which either consists only of the root vertex r or it consists recursively of the root r and its principal subtrees $(T_1, \dots, T_k)$ which themselves are plane trees. Note that the subtrees are linearly ordered. Let the set of all plane trees on $n + 1$ vertices be denoted by $\mathcal{T}_{n+1}$.

Note that $\mathcal{T}_{n+1}$ is also enumerated by the n -th Catalan number C_n . This can be shown by a bijection between D_n and $\mathcal{T}_{n+1}$. Here, we discuss three such bijections that will be useful to us. The first bijection can, for example, be found in [22, Page 10].

Definition 7. Let the *Stanley map* $\sigma: D_n \rightarrow \mathcal{T}_{n+1}$ be defined as follows:

- (1) Consider the Dyck path π as a string $\pi_1\pi_2\cdots\pi_{2n}$ of length $2n$ in the alphabet $\{N, E\}$ corresponding to the North and East steps of π .
- (2) Start at the root node. Label this as vertex v .
- (3) For $1 \leq i \leq 2n$, if $\pi_i = N$ then add a child to the right of all preexisting children of v . Label this new child as v . If $\pi_i = E$, set v to be the parent of v .

Example 8. The Dyck path of Figure 1 corresponds to the plane tree in Figure 2A under σ .

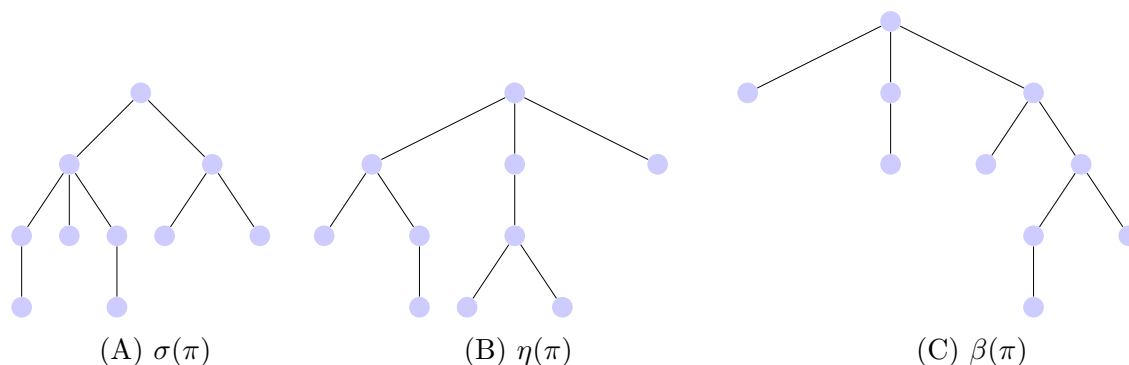


Figure 2: Plane trees corresponding to the Dyck path π of Figure 1 under σ, η , and β , respectively.

The next bijection is the restriction of a bijection between parking functions and labelled trees to Dyck paths. The bijection on parking functions can, for example, be found in [16] and [15, Chapter 5].

Definition 9. Let the *Haglund–Loehr map* $\eta: D_n \rightarrow \mathcal{T}_{n+1}$ be defined as follows:

- (1) For each cell in the first column that lies directly right of a North step attach a child to the root vertex. Associate the rightmost child to the topmost cell in the first column, the second rightmost child to the second topmost cell in the first column, and so on such that the leftmost child is associated with the bottommost cell in the first column.
- (2) To determine the children of any other vertex v , travel on the Northeast diagonal from its associated cell under π until it reaches a cell directly to the right of a North step. If this cell exists and is the bottommost cell in its column that is directly right of a North step, then attach k children to v , where k is the number of cells in this column that lie directly right of a North step. For each of these new vertices, associate them to the appropriate cell as laid out above.

Example 10. The Dyck path of Figure 1 corresponds to the plane tree in Figure 2B under η .

The last map we mention can be found in [6].

Definition 11. Let the *Benchekroun–Moszkowski map* $\beta: D_n \rightarrow \mathcal{T}_{n+1}$ be defined as follows:

- (1) Consider the Dyck path π as a string $\pi_1\pi_2\cdots\pi_{2n}$ of length $2n$ in the alphabet $\{N, E\}$ corresponding to the North and East steps of π . Append $\pi_0 = E$ to the front of the string.
- (2) For each vertex, we attach one of two states: “Checked” or “Not Checked”. Start with just the root vertex in the “Not Checked” state.

- (3) Recursively consider π_i for $i = 0, 1, \dots, 2n$. If $\pi_i = E$, then find the set of all closest vertices to the root in the “Not Checked” state. Out of these vertices choose the leftmost vertex and label this vertex as v . Let k be the number of consecutive North steps directly following π_i . Append k children to v all in “Not Checked” state. Change the state of vertex v to “Checked”. If $\pi_i = N$, then perform no action on the graph.

Example 12. The Dyck path of Figure 1 corresponds to the plane tree in Figure 2C under β .

It turns out that σ and β can be used to obtain the zeta map.

Proposition 13. [6] Let $\pi \in D_n$. Then $\zeta(\pi) = \beta^{-1} \circ \sigma(\pi)$.

2.4 Labelled trees and parking functions

A *labelled tree* on n vertices is a tree T with vertex set $\{0, 1, \dots, n-1\}$, where the vertex labelled 0 is considered to be the root of the tree. We also use the convention that in a drawing of a labelled tree any vertex v sits above its children and the labels of its children increase from left to right. Let $\mathcal{L}_n$ represent the set of all labelled trees on n vertices. The cardinality of $\mathcal{L}_n$ is known to be n^{n-2} .

A *coinversion* of $T \in \mathcal{L}_n$ is an ordered pair (i, j) such that j is a descendant of i and $0 < i < j$. Denote the number of coinversions of a labelled tree T by $\text{coinv}(T)$. Gessel and Wang [13] proved combinatorially that

$$\sum_{G \in \mathcal{C}_n} q^{e(G)} = q^{n-1} \sum_{T \in \mathcal{L}_n} (1+q)^{\text{coinv}(T)}, \quad (10)$$

where $\mathcal{C}_n$ is the set of all *labelled connected graphs* with vertex set $\{0, \dots, n-1\}$ and $e(G)$ is the number of edges in G . Evaluating at $q = 1$ gives the surprising result that

$$|\mathcal{C}_n| = \sum_{T \in \mathcal{L}_n} 2^{\text{coinv}(T)}. \quad (11)$$

Remark 14. Note that Gessel and Wang [13] studied tree inversions instead of coinversions, but these statistics can be seen to be jointly equidistributed on labelled trees by relabelling vertex i by $n+1-i$ for $i \neq 0$ which was observed by Irving and Rattan [18].

A *parking function* P on n cars is equivalent to a Dyck path $\pi \in D_n$, where the numbers 1 through n are placed directly right of the North steps of π such that each number appears exactly once and the numbers in each column are strictly decreasing. We refer to the labels $\{1, \dots, n\}$ in the parking function P as cars. Denote the set of all parking functions on n cars by $\mathcal{P}_n$. The area of parking function P is taken to be the area of its corresponding Dyck path. The cardinality of $\mathcal{P}_n$ is known to be $(n+1)^{n-1}$ which implies that there exists a bijection with labelled trees on $n+1$ vertices. We review the bijection discovered in [16].

Definition 15. Let the *Haglund–Loehr map* $\lambda: \mathcal{P}_n \rightarrow \mathcal{L}_{n+1}$ be defined as follows:

- (1) Start with the root vertex labelled 0. For each car labelled i in the first column of the parking function, attach a vertex labelled i to the root 0.
- (2) To determine the children of any other vertex v , travel Northeast from its associated car until it reaches another car. If this car exists and is the bottommost car in its column, attach a child labelled i to v for each car i in the column.

Observe that restricting λ to the parking functions containing car i in row i recovers η of Definition 9 (by ordering siblings in increasing order and then disregarding the labels on the tree).

Haglund and Loehr [16] also defined a function $\tilde{d}_i(T)$ on the vertices $0 \leq i \leq n$ for $T \in \mathcal{L}_{n+1}$ such that $\tilde{d}_0(T) = 0$ and $\tilde{d}_j(T) = \tilde{d}_i(T) + k - 1$, where vertex j is the k -th smallest/leftmost child of vertex i . For any $P \in \mathcal{P}_n$, we have

$$\text{area}(P) = \sum_{i=0}^n \tilde{d}_i(\lambda(P)). \quad (12)$$

3 Results

In Section 3.1, we prove a recursion for the polynomials $F_n(q, t)$. In Section 3.2, we introduce the notation of a dual plane tree using various reading words. We use this to prove in Section 3.3 that $F_n(q, t)$ and $G_n(q, t)$ are symmetric in q and t . This also gives an expression of the usual Catalan polynomials in terms of the depth and dinv of depth statistics. In Section 3.4, we relate the involution that interchanges depth and area used to prove the symmetry in Section 3.3 to an involution by Deutsch [9]; this yields an easy proof of the symmetry of the Tutte polynomials of the Catalan matroid [5]. In Section 3.5, we consider the setup of parking functions and address a remark in [12].

3.1 Recursion for $F_n(q, t)$

We begin by giving a recursion for $F_n(q, t)$.

Proposition 16. *We have $F_0(q, t) = 1$ and for any $n \geq 1$*

$$F_n(q, t) = \sum_{k=1}^n q^{k-1} t^{n-k} F_{k-1}(q, t) F_{n-k}(q, t).$$

Proof. Let

$$D_n(k) = \{\pi \in D_n \mid \pi \text{ first touches the diagonal at } (k, k)\}.$$

Let $f: D_n(k) \rightarrow D_{k-1} \times D_{n-k}$ be the classical bijection sending

$$\pi = \pi_1 \pi_2 \cdots \pi_{2n} \quad \mapsto \quad (\pi_2 \cdots \pi_{2k-1}, \pi_{2k+1} \cdots \pi_{2n}).$$

Let $f_1(\pi)$ and $f_2(\pi)$ be the first and second component of $f(\pi)$, respectively. Note that appending a North step to the beginning and an East step at the end of a Dyck path of semilength m increases the area by m . As π is obtained by concatenating N , $f_1(\pi)$, E , and $f_2(\pi)$, we have $q^{\text{area}(\pi)} = q^{k-1} q^{\text{area}(f_1(\pi))} q^{\text{area}(f_2(\pi))}$. Now consider the depth labelling of π . Observe that the labellings of all North steps after π_{2k+1} can be uniquely determined by the labelling to the right of π_{2k+1} . Since the labelling to the right of the first North step is 0 and (k, k) is the first time π touches the diagonal, we have that the labelling to the right π_{2k+1} is 1. However, looking at the corresponding depth labelling in $f_2(\pi)$, this value is a zero. Thus, to get from the depth labelling of $f_2(\pi)$ to the that of $\pi_{2k+1} \cdots \pi_{2n}$ in π , we must add 1 to each of the $n - k$ labels. Additionally, from the definition of the depth labelling, we see that the portion of π from $(0, 1)$ to $(k - 1, k)$ corresponding to $f_1(\pi)$ has the same depth labelling as $f_1(\pi)$. This gives us that $t^{\text{depth}(\pi)} = t^{n-k} t^{\text{depth}(f_1(\pi))} t^{\text{depth}(f_2(\pi))}$. Therefore,

$$\sum_{\pi \in D_n(k)} q^{\text{area}(\pi)} t^{\text{depth}(\pi)} = q^{k-1} t^{n-k} F_{k-1}(q, t) F_{n-k}(q, t). \quad (13)$$

Summing over k from 1 to n gives the desired result. $\square$

The recursion in Proposition 16 relates the polynomials $F_n(q, t)$ to the q, t -Catalan polynomials in [18, Section 5] in terms of increasing/decreasing factorizations and to Hurwitz graphs [4] since they satisfy the same recurrence. Note that in [4] the authors defined a statistics **bmaj** on Dyck paths, which corresponds to our depth statistics. However, **depth** and **bmaj** are defined in different ways. In particular, the depth sequence is a refinement of depth, which will be used in subsequent sections to define a duality.

3.2 Dual plane trees

We define two labellings of plane trees and an associated reading word to each labelling.

Definition 17. The *labelling* A of a plane tree T , denoted by T_A , is defined recursively by the following algorithm:

- (1) Label the root as 0.
- (2) For any other vertex v , let m be the labelling of its parent w . Label v as $m + k - 1$, where v is the k -th leftmost child of w .

Definition 18. Let T be a plane tree with $n+1$ vertices. The *reading word* of T_A , denoted by $\text{read}_A(T)$, is given by the following algorithm:

- (1) Start by setting $\text{read}_A(T)$ to be an empty vector. Append the labels of the children of the root in increasing order.
- (2) If the length of $\text{read}_A(T)$ equals n , then output $\text{read}_A(T)$. Otherwise, consider the set of vertices whose labels have already been added to $\text{read}_A(T)$ but whose children's labels have not been added. Find the vertex in this set with the largest label and at least one child. Call this vertex v . Append the labels of all the children of v in increasing order.

Note that the definition of the reading word in Definition 18 is well-defined. To show this, it suffices to explain why no two vertices with the same label will be considered by the definition at the same step. Let v and w be any two vertices that have the same label. If one is an ancestor of the other, then they would not be considered at the same point anywhere in the algorithm. Otherwise, consider the closest common ancestor of v and w and label it x . Let v' (resp. w') be the child of x on the path from v (resp. w) to x . As the label of w' is strictly larger than that of v' , w will be considered before v' and thus before v in the algorithm.

Example 19. The labelling T_A of the tree T in Figure 2A is given in Figure 3A. The corresponding reading word is $\text{read}_A(T) = (0, 1, 1, 2, 0, 1, 2, 2, 0)$.

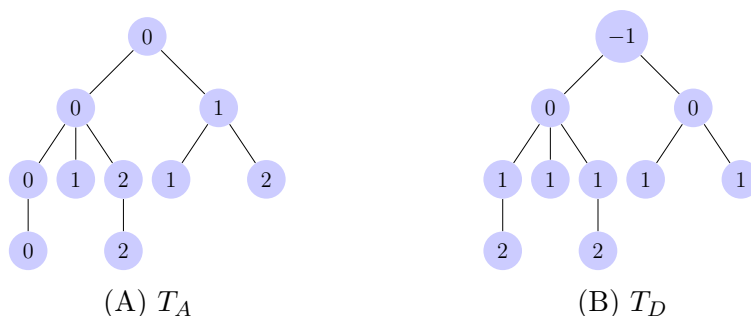


Figure 3: Plane tree labellings T_A and T_D of the plane tree in Figure 2A.

Definition 20. The *labelling D* of a plane tree T , denoted by T_D , is defined by labelling a vertex v by the number of edges in the path from v to the root minus one.

Remark 21. Note that the map $\lambda: \mathcal{P}_n \rightarrow \mathcal{L}_{n+1}$ of Definition 15 on parking functions with car i in row i (or equivalently map η) sends the coinversions of labelled trees in the codomain to the labelling D defined in Definition 20.

Definition 22. Let T be a plane tree with $n + 1$ vertices. The *reading word* of T_D , denoted by $\text{read}_D(T)$, is defined by the following algorithm:

- (1) Start by setting $\text{read}_D(T)$ to be an empty vector. Append the label of the root.
- (2) If the length of $\text{read}_D(T)$ equals $n + 1$, then remove the label corresponding to the root from $\text{read}_D(T)$ and output $\text{read}_D(T)$. Otherwise consider the set of all vertices whose vertices have already been added to $\text{read}_D(T)$ but have at least one child whose label has not been added. Find the vertex in this set with the largest label and call the vertex v . Attach to $\text{read}_D(T)$ the label of the leftmost child of v that has not already been added.

This definition is also well-defined as vertices with the same labels will never be considered at the same time.

Example 23. The labelling T_D of the tree T in Figure 2A is given in Figure 3B. The corresponding reading word is $\text{read}_D(T) = (0, 1, 2, 1, 1, 2, 0, 1, 1)$.

Definition 24. Let T be a plane tree. Let the k -th child of a vertex v be the k -th leftmost child of v . We define the *dual plane tree* of T , denoted by T^{dual} , by the following algorithm:

- (1) **Initialization:** Set T^{dual} to be a single vertex u which we label as the root of T^{dual} . If the root of T has a child, then add a child to u of T^{dual} . Set this to be the 1-st child of u and associate this child with the 1-st child of the root in T .
- (2) **Determining if a non-root vertex v in T^{dual} has a child:** Look at the associated vertex v' of v in the original plane tree T . If v' has a sibling to its right, then attach a child to v which will be the 1-st child of v . Associate the child of v in T^{dual} with the sibling directly right of v' in T . If v' has no sibling to its right, then v has no children.
- (3) **Determining if a vertex v (including the root) in T^{dual} has a k -th child for $k > 1$:** Let w be the $(k - 1)$ -th child of v . Look at the associated vertex w' of w in T . If w' has a child, then attach a k -th child to v . Associate the k -th child of v to the 1-st child of w' . If w' has no children, then v has no k -th child.

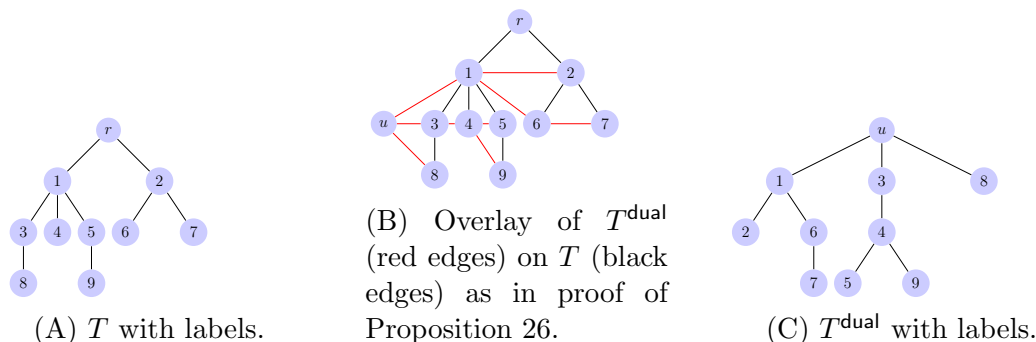


Figure 4: Construction of the dual plane tree T^{dual} of the plane T in Figure 2A.

Example 25. The dual plane tree T^{dual} of the plane tree T in Figure 2A is given in Figure 4C. Observe, by comparing with Figure 2, that in this example $T^{\text{dual}} = \eta \circ \sigma^{-1}(T)$. This will be proved in general in Corollary 33.

It is easy to see that $T^{\text{dual}} \in \mathcal{T}_{n+1}$ by observing that every non-root node of T is paired with a non-root node of T^{dual} , there are no loops in T^{dual} , and the children of every vertex are given a proper ordering. To show that the term dual plane tree is not a misnomer, we also prove that this operation is an involution.

Proposition 26. Let T be a plane tree. Then $(T^{\text{dual}})^{\text{dual}} = T$.

Proof. Draw the plane tree T in the canonical way with every vertex sitting above all of its descendants and the order of its children increasing from left to right. Next place the root of T^{dual} to the left of all vertices in T and draw the plane tree T^{dual} on top of T such that any vertex in T^{dual} is drawn on top of its corresponding vertex in T . Under this configuration all vertices in T^{dual} sit to the left of their descendants, and the order of their children increase from top to bottom. Since a vertex v and its corresponding vertex v' lie on top of each other in the specified configuration, we will abuse notation and refer to both as vertex v . Interchanging the position of the two trees (i.e. flipping the plane along the perpendicular bisector of the two root nodes), we clearly see that for a vertex v in T its first child corresponds to the sibling on the right of v in T^{dual} and its k -th child corresponds to the first sibling of the $(k - 1)$ -th child of v for $k > 1$. Thus, $(T^{\text{dual}})^{\text{dual}} = T$. $\square$

The two reading words are related under the dual map on plane trees.

Proposition 27. *Let T be a plane tree. Then*

$$\text{read}_D(T^{\text{dual}}) = \text{read}_A(T) \quad \text{and} \quad \text{read}_A(T^{\text{dual}}) = \text{read}_D(T).$$

Proof. It suffices to prove that $\text{read}_D(T) = \text{read}_A(T^{\text{dual}})$ since then this implies that $\text{read}_D(T^{\text{dual}}) = \text{read}_A((T^{\text{dual}})^{\text{dual}})$ which equals $\text{read}_A(T)$ by Proposition 26.

Let $\text{read}_D(T) = (r_1, r_2, \dots, r_n)$ and $\text{read}_A(T^{\text{dual}}) = (s'_1, s'_2, \dots, s'_n)$. Let v_i be the vertex in T that has label r_i . Similarly, let w_i be the vertex in T^{dual} that has label s'_i . We will prove by induction that $(r_1, r_2, \dots, r_k) = (s'_1, s'_2, \dots, s'_k)$ and that w_k corresponds to v_k under **dual** for $1 \leq k \leq n$. We have that both w_1 and v_1 are the leftmost child of their respective root nodes and the labelling of each is equal to zero. By the definition of T^{dual} , we have w_1 corresponds to v_1 . Assume that $(r_1, r_2, \dots, r_k) = (s'_1, s'_2, \dots, s'_k)$ and that w_k corresponds to v_k . If v_{k+1} is a child of v_k , then $r_{k+1} = r_k + 1$. Note that by definition of T^{dual} , w_k must have a sibling to its right. This implies that w_{k+1} is the sibling directly right of w_k and w_{k+1} corresponds to v_{k+1} . We have $r_{k+1} = r_k + 1 = s'_k + 1 = s'_{k+1}$. If v_{k+1} is not a child of v_k , then v_{k+1} is the leftmost unvisited child of y , where $y = v_i$ for some $1 \leq i < k$ and y has the largest label out of all parents with unvisited children. Note as v_k does not have any children, w_k has no siblings to its right. Thus, to find w_{k+1} we look for the leftmost child of the vertex x , where $x = w_\ell$ for some $1 \leq \ell \leq k$ and x has the largest label out of all parents with unvisited children. The condition that x has unvisited children in T^{dual} implies that the parent of its corresponding vertex $x' = v_\ell$ in T has an unvisited child. Thus the parent of x' either is y or has label smaller than y . If it has a label smaller than y then by the definition of T^{dual} and our inductive hypothesis, there exists v_j with $1 \leq j < k$ that has unvisited children and label strictly greater than x which is a contradiction. Therefore x' is the rightmost visited child of y and the leftmost child of x corresponds to the sibling to the right of x' . This implies that w_{k+1} corresponds with v_{k+1} and $w_{k+1} = w_\ell = v_\ell = v_{k+1}$. $\square$

3.3 Symmetry of $F_n(q, t)$ and $G_n(q, t)$

In this section, we prove the symmetry of the polynomials $F_n(q, t)$ and $G_n(q, t)$. We do so by defining an involution on Dyck paths using the Stanley and Haglund–Loehr maps σ and η , which switches the area and depth statistics. We begin by relating the area and depth sequences under the Stanley and Haglund–Loehr maps using the two reading words above. Recall that $a_i(\pi)$ and $d_i(\pi)$ are defined in Sections 2.1 and 2.2.

Proposition 28. *Let $\pi \in D_n$. Then*

$$\begin{aligned}\text{read}_D(\sigma(\pi)) &= (a_1(\pi), a_2(\pi), \dots, a_n(\pi)), \\ \text{read}_A(\sigma(\pi)) &= (d_1(\pi), d_2(\pi), \dots, d_n(\pi)).\end{aligned}$$

Proof. Let $(r_1, r_2, \dots, r_n) = \text{read}_D(\sigma(\pi))$. We use induction on $1 \leq k \leq n$ to prove that

$$(r_1, r_2, \dots, r_k) = (a_1(\pi), a_2(\pi), \dots, a_k(\pi))$$

and the k -th vertex (excluding the root) added in the creation of $\sigma(\pi)$ corresponds to the vertex with label r_k . Observe that r_1 corresponds to the label of the leftmost child of the root node. Note that this is the first node added in $\sigma(\pi)$. Thus, $r_1 = 0 = a_1(\pi)$. Assume that $(r_1, r_2, \dots, r_k) = (a_1(\pi), a_2(\pi), \dots, a_k(\pi))$ and r_k is the label of the k -th vertex v_k added in the creation of $\sigma(\pi)$ excluding the root. If the $(k+1)$ -th vertex v_{k+1} added to $\sigma(\pi)$ is a child of v_k , then in the Dyck path $a_{k+1}(\pi) = a_k(\pi) + 1$. Since the label of v_k was added last to $(r_1, \dots, r_k)$, we know that in the previous step the parent of v_k had the largest label out of all parents containing a child whose label was not already appended to the reading word. As v_k has a larger label than its parent and contains a child v_{k+1} , r_{k+1} is the label of the leftmost available child of v_k which would coincide with v_{k+1} . We have the label of v_{k+1} is one more than v_k giving us $r_{k+1} = r_k + 1 = a_k(\pi) + 1 = a_{k+1}(\pi)$. Now assume that v_{k+1} is not a child of v_k . In the Dyck path, this corresponds to a block of East steps after the k -th North step. Let ℓ denote the size of this block of East steps. We see that $a_{k+1}(\pi) = a_k(\pi) + \ell - 1$. In the tree, this corresponds to going ℓ vertices towards the root along the path from v_k to the root and attaching a new vertex v_{k+1} to this vertex w . Note that this implies that v_k and all vertices strictly between v_k and w do not have any additional children that have not already been added. This implies that w has the largest label of all vertices that contain a child whose label has not been appended to the reading word. Thus, r_{k+1} corresponds to the label of v_{k+1} which is one more than the label of w . Thus, $r_{k+1} = r_k - \ell + 1 = a_k(\pi) - \ell + 1 = a_{k+1}(\pi)$. By induction, we obtain $\text{read}_D(\sigma(\pi)) = (a_1(\pi), a_2(\pi), \dots, a_n(\pi))$.

Let $(s_1, s_2, \dots, s_n) = \text{read}_A(\sigma(\pi))$. Similar to the previous paragraph, we use induction on $1 \leq k \leq n$ to prove that

$$(s_1, s_2, \dots, s_k) = (d_1(\pi), d_2(\pi), \dots, d_k(\pi))$$

and the North step corresponding to $d_k(\pi)$ created the vertex v corresponding to the label s_k in $\sigma(\pi)$. We have that $d_1(\pi) = 0$ corresponds to the first North step which created the

leftmost child of the root node. Note that $s_1 = 0$ and also corresponds to the leftmost vertex of the root node. Assume that $(s_1, s_2, \dots, s_k) = (d_1(\pi), d_2(\pi), \dots, d_k(\pi))$ and the North step corresponding to $d_k(\pi)$ in the Dyck path created the vertex v_k corresponding to the label s_k in $\sigma(\pi)$. If the vertex v_{k+1} corresponding to s_{k+1} is a sibling of v_k then $s_{k+1} = s_k + 1$. By the previous paragraph, siblings correspond to North steps on the same diagonal. Note that no other North step can lie between the diagonal connecting the North step N_k of v_k and the North step N_{k+1} of v_{k+1} (keep in mind that N_k does not mean the k -th North step of π). Also, N_{k+1} needs to be the bottommost North step in its column, otherwise v_k and v_{k+1} would not be siblings in $\sigma(\pi)$. Since the depth label $d_k(\pi)$ corresponds to N_k , we have that $d_{k+1}(\pi)$ is the labelling of N_{k+1} . Thus, $d_{k+1}(\pi) = d_k(\pi) + 1 = s_k + 1 = s_{k+1}$. Assume that the vertex v_{k+1} corresponding to s_{k+1} is not a sibling of v_k . This implies that v_{k+1} is the leftmost child of the vertex w with the largest labelling in $(s_1, s_2, \dots, s_k)$ whose children's labels have not been added yet. Looking at the North step N_k corresponding to d_k , we have that the first North step reached by traveling northeast from N_k is not in the bottom of its column. Thus to find the North step corresponding to $d_{k+1}(\pi)$, we must find the largest labeled cell visited by $(d_1(\pi), d_2(\pi), \dots, d_k(\pi))$ that has a labelled cell directly above which has not been visited. Note that having a labeled cell directly above corresponds to having a child. Thus the North step corresponding to $d_{k+1}(\pi)$ is the same as the North step corresponding to v_{k+1} and is one cell directly above the North step corresponding to w . Note that the labelling of w is s_i and the labelling of its corresponding North step is $d_i(\pi)$ for some $1 \leq i \leq n$. As v_{k+1} is the leftmost child of w , we have $s_{k+1} = s_i$. Similarly, as $d_{k+1}(\pi)$ lies in the same column as $d_i(\pi)$, we have $d_{k+1}(\pi) = d_i(\pi)$. By induction $d_i(\pi) = s_i$, implying $d_{k+1}(\pi) = s_{k+1}$. By induction we obtain $\text{read}_A(\sigma(\pi)) = (d_1(\pi), d_2(\pi), \dots, d_n(\pi))$. $\square$

Proposition 29. *Let $\pi \in D_n$. Then*

$$\begin{aligned}\text{read}_A(\eta(\pi)) &= (a_1(\pi), a_2(\pi), \dots, a_n(\pi)), \\ \text{read}_D(\eta(\pi)) &= (d_1(\pi), d_2(\pi), \dots, d_n(\pi)).\end{aligned}$$

Proof. Let x be the parking function obtained by labelling the North step in the i -th row by i . Then [16] have showed the first equality.

We prove the second equality by induction. Let $(r_1, r_2, \dots, r_n) = \text{read}_D(\eta(\pi))$. We prove that $(r_1, r_2, \dots, r_k) = (d_1(\pi), d_2(\pi), \dots, d_k(\pi))$ for $1 \leq k \leq n$ and the North step corresponding to $d_k(\pi)$ created the vertex v corresponding to the label r_k in $\eta(\pi)$. We have that $d_1(\pi) = 0$ and it lies to the right of the first North step. The first North step under the map η creates the leftmost child of the root which is precisely the vertex whose label is $r_1 = 0$. Assume that $(r_1, r_2, \dots, r_k) = (d_1(\pi), d_2(\pi), \dots, d_k(\pi))$ and the North step corresponding to $d_k(\pi)$ created the vertex v_k whose label is r_k . Let v_{k+1} be the vertex whose label is r_{k+1} . Also define N_k and N_{k+1} to be the North steps that created v_k and v_{k+1} , respectively. Assume that the vertex v_{k+1} is a child of v_k . As v_{k+1} is a child of v_k , we obtain $r_{k+1} = r_k + 1$. By the definition of read_D , we have that v_{k+1} is the leftmost child of v_k . This implies that their A label is the same. Since $\text{read}_A(\eta(\pi)) = (a_1(\pi), a_2(\pi), \dots, a_n(\pi))$, we have the North steps that created v_k and v_{k+1}

under η lie on the same diagonal. By the definition of η , we have that N_{k+1} must be at the bottom of its column and no other North step lies between the N_k and N_{k+1} . Thus $d_{k+1}(\pi)$ is the depth labelling of N_{k+1} which satisfies $d_{k+1}(\pi) = d_k(\pi) + 1 = r_k + 1$. Assume now that v_{k+1} is not a child of v_k which implies by the definition of read_D that v_k does not have any children. Consider the subset S' of $S = \{v_1, v_2, \dots, v_k\}$ containing all vertices with a child that is not also in S . Let w be the vertex in S' with the largest label. We have that v_{k+1} is the leftmost child of w that is not in S . As v_k does not have a child, the first North step attained by traveling Northeast from N_k is not at the bottom of its column or does not exist. Thus to find N_{k+1} , we must find the largest labeled cell visited by $(d_1(\pi), d_2(\pi), \dots, d_k(\pi))$ that has a labelled cell directly above which has not been visited. Note that having two North steps consecutively corresponds to them being siblings under η . Additionally, observe that the vertex in S with the largest label out of vertices in S containing a sibling not in S is a child of w . Thus v_{k+1} and the node created by N_{k+1} are the same. All the children of w have the same D labelling, and depth labelings in the same column of π are equal. Paired with the inductive hypothesis, this implies $r_{k+1} = d_{k+1}$. $\square$

We are now ready to show that combining the Stanley and Haglund–Loehr maps gives an involution that interchanges area and depth.

Proposition 30. *Let $\omega = \sigma^{-1} \circ \eta: D_n \rightarrow D_n$. Then ω is an involution which interchanges the depth and area sequence.*

Proof. By Propositions 28 and 29 we have that

$$\begin{aligned} (d_1(\omega(\pi)), d_2(\omega(\pi)), \dots, d_n(\omega(\pi))) &= (a_1(\pi), a_2(\pi), \dots, a_n(\pi)), \\ (a_1(\omega(\pi)), a_2(\omega(\pi)), \dots, a_n(\omega(\pi))) &= (d_1(\pi), d_2(\pi), \dots, d_n(\pi)). \end{aligned}$$

Additionally, we have

$$(a_1(\omega^2(\pi)), a_2(\omega^2(\pi)), \dots, a_n(\omega^2(\pi))) = (d_1(\omega(\pi)), d_2(\omega(\pi)), \dots, d_n(\omega(\pi)))$$

implying $(a_1(\omega^2(\pi)), a_2(\omega^2(\pi)), \dots, a_n(\omega^2(\pi))) = (a_1(\pi), a_2(\pi), \dots, a_n(\pi))$. Since the area sequence uniquely determines a Dyck path, we have that ω is an involution. $\square$

Example 31. Consider the Dyck path π in Figure 1 with area and depth sequences (see also Example 3)

$$a(\pi) = (0, 1, 2, 1, 1, 2, 0, 1, 1) \quad \text{and} \quad d(\pi) = (0, 1, 1, 2, 0, 1, 2, 2, 0).$$

Then $\omega(\pi)$ is given in Figure 5 and it is easy to check that $a(\omega(\pi)) = d(\pi)$ and $d(\omega(\pi)) = a(\pi)$.

Corollary 32. *Let $\pi \in D_n$. Then $\omega(\pi) = \sigma^{-1}((\sigma(\pi))^{\text{dual}}) = \eta^{-1}((\eta(\pi))^{\text{dual}})$.*

Proof. By Proposition 30, it suffices to prove that the area sequences of $\sigma^{-1}((\sigma(\pi))^{\text{dual}})$ and $\eta^{-1}((\eta(\pi))^{\text{dual}})$ are equal to the depth sequence of π . Using Propositions 27, 28, and 29, we observe that this is indeed the case. $\square$

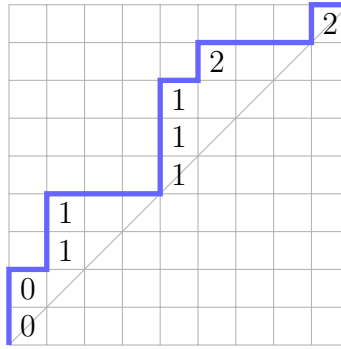


Figure 5: $w(\pi)$ with π as in Figure 1 with depth labelling.

Corollary 33. *Let $T \in \mathcal{T}_{n+1}$. Then $T^{\text{dual}} = \eta \circ \sigma^{-1}(T)$.*

Proof. This follows directly from Proposition 30 and Corollary 32. $\square$

Finally, we are ready to prove the symmetry of $F_n(q, t)$ and $G_n(q, t)$.

Theorem 34. *We have*

$$F_n(q, t) = F_n(t, q) \quad \text{and} \quad G_n(q, t) = G_n(t, q).$$

Proof. By Proposition 30, ω is a bijection on D_n that interchanges the area and depth sequence of a Dyck path. As **area** and **depth** are defined as the sum of their respective sequences, we have that ω interchanges **area** and **depth**, thereby proving symmetry of $F_n(q, t)$.

By (2) and (7), the definitions of **dinv** and **ddinv** are identical except with the area and depth sequence interchanged. Since by Proposition 30 the involution ω interchanges the area and depth sequences, ω also interchanges **dinv** and **ddinv**. Thus, $G_n(q, t)$ is symmetric in q and t . $\square$

From a similar argument, we obtain the following corollary.

Corollary 35. *We have*

$$C_n(q, t) = \sum_{\pi \in D_n} q^{\text{depth}(\pi)} t^{\text{ddinv}(\pi)}.$$

3.4 The Deutsch involution and ω

We now define an involution $(\cdot)'$ on Dyck paths first introduced by Deutsch in [9].

Definition 36. We define $(\cdot)': D_n \rightarrow D_n$ recursively as follows:

- (1) $\varepsilon' = \varepsilon$, where ε is the empty Dyck path.
- (2) For $\pi \in D_n$ and $n \geq 1$, write $\pi = N\alpha E\beta$, where α and β are Dyck paths. Note that α, β are allowed to be empty. Then define $\pi' = N\beta' E\alpha'$.

The map $\omega = \sigma^{-1} \circ \eta$ gives an explicit description of Deutsch's recursive operator as we first observed using FindStat [20].

Proposition 37. *Let $\pi \in D_n$. Then $\omega(\pi) = \pi'$.*

Proof. By Proposition 30, it suffices to prove that

$$(d_1(\pi), d_2(\pi), \dots, d_n(\pi)) = (a_1(\pi'), a_2(\pi'), \dots, a_n(\pi')).$$

We proceed by induction on n . We have that both the area and depth sequence of ε are $\emptyset$. Assume that

$$(d_1(\pi), d_2(\pi), \dots, d_j(\pi)) = (a_1(\pi'), a_2(\pi'), \dots, a_j(\pi'))$$

for all $\pi \in D_j$, where $0 \leq j \leq n$. Let $\pi \in D_{n+1}$ and let α and β be Dyck paths such that $\pi = N\alpha E\beta$. Let $k-1$ be the semilength of α . We have that (k, k) is the first time the path π touches the diagonal after $(0, 0)$. From the definition of the depth labelling and the argument in the proof of Proposition 16, we have $(d_1(\pi), d_2(\pi), \dots, d_{n+1}(\pi)) = (0, d_1(\beta) + 1, d_2(\beta), \dots, d_{n+1-k}(\beta), d_1(\alpha), d_2(\alpha), \dots, d_{k-1}(\alpha))$. From the definition of the area sequence and $(\cdot)'$, we have that

$$(a_1(\pi'), a_2(\pi'), \dots, a_{n+1}(\pi')) = (0, a_1(\beta') + 1, a_2(\beta') + 1, \dots, a_{n+1-k}(\beta') + 1, a_1(\alpha'), a_2(\alpha'), \dots, a_{k-1}(\alpha')).$$

Note that α and β have semilength strictly less than $n+1$. Hence by induction

$$(d_1(\beta), \dots, d_{n+1-k}(\beta)) = (a_1(\beta'), \dots, a_{n+1-k}(\beta'))$$

and

$$(d_1(\alpha), d_2(\alpha), \dots, d_{k-1}(\alpha)) = (a_1(\alpha'), a_2(\alpha'), \dots, a_{k-1}(\alpha')).$$

Thus,

$$(d_1(\pi), d_2(\pi), \dots, d_{n+1}(\pi)) = (a_1(\pi'), a_2(\pi'), \dots, a_{n+1}(\pi')).$$

□

Using Corollary 32 and Proposition 37, we find a relation between the $(\cdot)^{\text{dual}}$ operator defined on plane trees and the one defined on Dyck paths.

Corollary 38. *The following diagram commutes:*

$$\begin{array}{ccc} D_n & \xleftarrow{(\cdot)'} & D_n \\ \sigma \text{ or } \eta \downarrow & & \downarrow \sigma \text{ or } \eta \\ \mathcal{T}_{n+1} & \xleftarrow{(\cdot)^{\text{dual}}} & \mathcal{T}_{n+1}. \end{array}$$

Deutsch proved [9] that the operator $(\cdot)'$ interchanges the initial rise (IR) of a Dyck path (the number of North steps before the first East step) with its number of returns (RET) (the number of times the Dyck path touches the diagonal excluding the point $(0, 0)$). We see that the initial rise and the number of returns of a Dyck path correspond

to the length of the leftmost path from the root to a leaf and the number of children of the root, respectively, under σ (and vice versa under η). This gives an alternate explanation of the symmetry of the *Tutte polynomial*

$$T_{\text{Cat}_n}(q, t) = \sum_{\pi \in D_n} q^{\text{IR}(\pi)} t^{\text{RET}(\pi)}$$

associated with the Catalan matroid Cat_n defined in [5].

Stump [23] proved that the coefficient of $q^a t^b$ of $T_{\text{Cat}_n}(q, t)$ only depends on the sum $a + b$ using a map given by Speyer [21]. This map τ fixes Dyck paths π , where $\text{RET}(\pi) = 1$ and sends Dyck paths $\pi = N\alpha_1 E N\alpha_2 E N\alpha_3 E \cdots N\alpha_k E$ to $NN\alpha_1 E \alpha_2 E N\alpha_3 E \cdots N\alpha_k E$, where $\text{RET}(\pi) = k > 1$ and α_i is a Dyck path that is possibly empty. Speyer's map has a nice relation with ω as follows.

Proposition 39. *Let $\pi \in D_n$. Then $\tau^{-1} \circ \omega(\pi) = \omega \circ \tau(\pi)$.*

Proof. If $\text{RET}(\pi) = 1$, then $\tau(\pi) = \pi$ and $\omega \circ \tau(\pi) = \omega(\pi)$. As ω interchanges initial rises and the number of returns, we have $\text{IR}(\omega(\pi)) = 1$. This implies that $\tau^{-1} \circ \omega(\pi) = \omega(\pi)$. Thus, we have $\tau^{-1} \circ \omega(\pi) = \omega \circ \tau(\pi)$.

If $\text{RET}(\pi) = k > 1$, let $\pi = N\alpha_1 E N\alpha_2 E N\alpha_3 E \cdots N\alpha_k E$, where α_i is a possibly empty Dyck path. We show $\omega(\pi) = \tau \circ \omega \circ \tau(\pi)$. From Definition 36 and Proposition 37

$$\omega(\pi) = N(N\alpha_2 E N\alpha_3 E \cdots N\alpha_k E)' E \alpha_1'.$$

On the other hand,

$$\begin{aligned} \tau(\pi) &= NN\alpha_1 E \alpha_2 E N\alpha_3 E \cdots N\alpha_k E, \\ \omega \circ \tau(\pi) &= N(N\alpha_3 E \cdots N\alpha_k E)' E (N\alpha_1 E \alpha_2)' \\ &= N(N\alpha_3 E \cdots N\alpha_k E)' E N\alpha_2' E \alpha_1', \\ \tau \circ \omega \circ \tau(\pi) &= NN(N\alpha_3 E \cdots N\alpha_k E)' E \alpha_2' E \alpha_1' \\ &= N(N\alpha_2 E N\alpha_3 E \cdots N\alpha_k E)' E \alpha_1'. \end{aligned}$$

Hence, $\omega(\pi) = \tau \circ \omega \circ \tau(\pi)$. □

3.5 Parking Functions

Kreweras [19] essentially proved recursively

$$\sum_{T \in \mathcal{L}_{n+1}} q^{\text{coinv}(T)} = \sum_{\pi \in \mathcal{P}_n} q^{\text{area}(\pi)}. \quad (14)$$

Combining this with (10), one obtains the formula

$$q^n \sum_{\pi \in \mathcal{P}_n} (1 + q)^{\text{area}(\pi)} = \sum_{G \in \mathcal{C}_{n+1}} q^{e(G)}, \quad (15)$$

which was also observed in [3]. Following in Gessel and Wang's footsteps [13], we provide a combinatorial proof of this formula.

We start by defining an algorithm that produces a specific spanning tree from a labelled connected graph. Recall from Section 2.4 that $\mathcal{L}_n$ is the set of all labelled trees on n vertices and $\mathcal{C}_n$ is the set of all labelled connected graphs with vertex set $\{0, \dots, n-1\}$.

Definition 40. Let $\mathcal{S}: \mathcal{C}_n \rightarrow \mathcal{L}_n$ be given by the following algorithm:

- (1) Start with all vertices of $G \in \mathcal{C}_n$ in the “Not Seen” state.
- (2) Visit vertex 0 and set its state to “Seen”. Visit all vertices v adjacent to 0 in increasing label order including the edges from vertex 0 to v .
- (3) If all vertices of G are in the “Seen” state, then return the subgraph of G comprised of all vertices and edges that were visited. Otherwise, find the vertex v that was visited last and is in the “Not Seen” state. Set v to “Seen”. Visit all “Not Seen” vertices w adjacent to v that have not been visited already including the edge between v and w , where vertices with smaller labels are visited first.

Clearly, $\mathcal{S}(G)$ is connected and acyclic for any $G \in \mathcal{C}_n$ implying that $\mathcal{S}$ is well defined. For $T \in \mathcal{L}_n$, let $\mathcal{G}_\mathcal{S}(T)$ denote the set of all labelled connected graphs G satisfying $\mathcal{S}(G) = T$.

Example 41. Consider the labelled connected graph G in Figure 6A. Its spanning tree $\mathcal{S}(G)$ is given in Figure 6B.

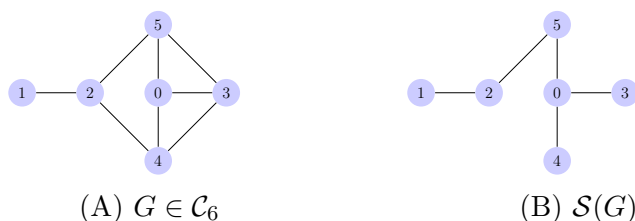


Figure 6: Connected labelled graph G and its spanning tree $\mathcal{S}(G)$.

We will now associate a set of labelled connected graphs to a labelled tree by adding certain edges to the tree. Let $T \in \mathcal{L}_n$. To each vertex i in T , we associate a set of edges $\mathcal{E}_T(i)$ that are not in T as follows. Let Q be the unique path from i to the root node 0 in T . We let

$$\mathcal{E}_T(i) = \{\{i, j\} \mid j \text{ is a sibling of some vertex } k \in Q \text{ and } j < k\}.$$

Define $\mathcal{E}_T = \bigcup_{i=0}^n \mathcal{E}_T(i)$ and let $\mathcal{G}_\mathcal{E}(T)$ denote the set of all connected graphs obtained by adding some subset of edges from $\mathcal{E}_T$ to T .

Example 42. Let T be the labelled tree in Figure 6B. Then

$$\mathcal{E}_T = \{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{5, 3\}, \{5, 4\}, \{4, 3\}\}.$$

Proposition 43. Let $T \in \mathcal{L}_n$. Then $\mathcal{G}_S(T) = \mathcal{G}_E(T)$.

Proof. Let $G \in \mathcal{G}_S(T)$. This implies that $G = T \sqcup S$, where S is a set of edges not in T . Assume $S \not\subseteq \mathcal{E}_T$ and let $e = \{v, w\} \in S - \mathcal{E}_T$, where v was “Seen” before w in the construction of $\mathcal{S}(G)$. Observe that in the step, where v is marked as “Seen”, all visited vertices that are “Not Seen” are endpoints of edges in $\mathcal{E}_T$. Hence, vertex w has not been visited when v was marked as “Seen”. This implies that v is a parent of w in T which contradicts e being an edge not in T . Therefore, $\mathcal{G}_S(T) \subseteq \mathcal{G}_E(T)$.

Let $G = T \sqcup S$, where $S \subseteq \mathcal{E}_T$ and assume $\mathcal{S}(G) \neq T$. Let edge $e = \{v, w\} \in S$ be the first edge used in $\mathcal{S}(G)$ that is not present in T . Assume v is marked “Seen” before vertex w in the construction of $\mathcal{S}(G)$. Recall that w is a smaller sibling of a vertex on the path from v to the root in T . This implies that w has already been visited when v is marked as “Seen”, and thus the edge e cannot have been used in the construction of $\mathcal{S}(G)$. Therefore, $\mathcal{G}_E(T) \subseteq \mathcal{G}_S(T)$ and $\mathcal{G}_S(T) = \mathcal{G}_E(T)$. $\square$

Observe that the following relation holds between the number of associated edges of a vertex to the statistic defined after Definition 15.

Lemma 44. Let i be a vertex in $T \in \mathcal{L}_n$. Then $|\mathcal{E}_T(i)| = \tilde{d}_i(T)$.

Proof. We induct on the distance of vertex i to the root, where distance is defined as the length of the path between the two vertices. The only vertex that is distance 0 from the root is the root itself. We clearly have $\mathcal{E}_T(0) = \emptyset$ and $\tilde{d}_0(T) = 0$. Assume that $|\mathcal{E}_T(i)| = \tilde{d}_i(T)$ for all vertices i that are distance m from the root. Let j be a vertex that is distance $m + 1$ from the root and let Q be the unique path from j to the root. By assumption, $|\mathcal{E}_T(i)| = \tilde{d}_i(T)$, where i is the parent of j . Let S be the set of all siblings of j that are smaller than j . Observe that $\mathcal{E}_T(j) = \mathcal{E}_T(i) \cup S$ and $\tilde{d}_j(T) = \tilde{d}_i(T) + k - 1$ where j is the k -th smallest child of i . Thus, $|\mathcal{E}_T(j)| = \tilde{d}_j(T)$. $\square$

We now prove (14) combinatorially.

Theorem 45. We have

$$q^n \sum_{\pi \in \mathcal{P}_n} (1 + q)^{\text{area}(\pi)} = \sum_{G \in \mathcal{C}_{n+1}} q^{e(G)}.$$

Proof. By Proposition 43 and using the fact that a tree on $n + 1$ vertices has n edges, we observe

$$\sum_{G \in \mathcal{C}_{n+1}} q^{e(G)} = \sum_{T \in \mathcal{L}_{n+1}} q^{e(T)} (1 + q)^{|\mathcal{E}_T|} = \sum_{T \in \mathcal{L}_{n+1}} q^n (1 + q)^{|\mathcal{E}_T|}. \quad (16)$$

Using Lemma 44, Definition 15, and (12), we have

$$\sum_{T \in \mathcal{L}_{n+1}} q^n (1 + q)^{|\mathcal{E}_T|} = \sum_{T \in \mathcal{L}_{n+1}} q^n (1 + q)^{\sum_{i=0}^n \tilde{d}_i(T)} = \sum_{\pi \in \mathcal{P}_n} q^n (1 + q)^{\text{area}(\pi)}. \quad (17)$$

Combining the equations above, we obtain the desired result. $\square$

Substituting $q = 1$ into Theorem 45 gives the following result which provides an explicit proof of a remark found in [12, Section 3].

Corollary 46. *The following identity holds*

$$\sum_{\pi \in \mathcal{P}_n} 2^{\text{area}(\pi)} = |\mathcal{C}_{n+1}|.$$

From Theorem 45 and (10), we obtain a new proof of the fact that `area` and `coinv` are equidistributed over labelled trees/parking functions [18].

Corollary 47. *The following identity holds*

$$\sum_{T \in \mathcal{L}_n} q^{\text{area}(\lambda^{-1}(T))} = \sum_{T \in \mathcal{L}_n} q^{\text{coinv}(T)}.$$

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