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Anomalous Proton Velocity Diffusion by Quasi-monochromatic Kinetic Alfvén Waves

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Abstract

The anomalous diffusion of resonant protons in parallel and perpendicular velocity space by kinetic Alfvén waves is discussed. The velocity diffusion coefficient is calculated by employing an autocorrelation function for proton trajectories. It is found that for protons resonant with the waves, the perpendicular diffusion coefficient decays away for a sufficiently long time, but parallel diffusion monotonically increases in time until it saturates at a certain level. This result indicates that a portion of resonant protons can undergo anomalous diffusion along the background magnetic field even if the intensity of the kinetic Alfvén wave is sufficiently low. The present findings imply that under suitable conditions, astrophysical charged-particle acceleration can take place in the parallel direction.

Unified Astronomy Thesaurus concepts: Solar magnetic fields (1503)

1. Introduction

The charged-particle acceleration is a long standing problem (Axford 1965; Burlaga 1967; Bell 1978a, 1978b). Among the proposed mechanisms is diffusive shock acceleration of protons (see, e.g., (Jokipii 1986), which may be operative at the termination shock of the solar wind. This is the accepted model for the origin of anomalous cosmic rays. Protons and electrons may also be accelerated in the inner heliospheric environment, such as in association with solar flares or coronal mass ejections, and the related shock waves thereof. In the study of charged-particle heating, which is intimately related to the solar wind acceleration and coronal heating problem, the role of kinetic Alfvén waves is emphasized (see, e.g., Vásconez et al. 2015). This research topic is of contemporary significance thanks to the inner heliospheric space missions, such as the Parker Solar Probe and the Solar Orbiter. The interaction between waves and charged particles is of fundamental importance in plasma physics. It is well known that charged particles can be accelerated to high energies via a number of wave–particle interaction processes including the customary resonant wave–particle interaction (Karimabadi et al. 1992; Miller et al. 1997), and nonstandard acceleration processes such as the resonance overlap in a single large-amplitude quasi-monochromatic spectrum of waves (Karney 1978; Karimabadi et al. 1990), for instance. The stochastic acceleration of protons by kinetic Alfvén waves in the solar wind, may be an important process, which has been studied by many authors (see, e.g., Voitenko & Goossens 2004; Chandran et al. 2010; Hoppock et al. 2018; Choi et al. 2019). The stochastic acceleration can be defined as a process for which a particle can gain energy via nonresonant interaction with waves. During impulsive solar flares large amounts of energies are released in the form of energetic electrons or protons with energies up to tens of keV for electrons and ∼MeV for ions. Among the acceleration processes is the stochastic energization of the charged particles via interaction with background waves, including obliquely propagating kinetic Alfvén waves (Miller 1991; Smith & Brecht 1993; Lee & Völk 1973; Miller et al. 1997; Karimabadi et al. 1992). An important point is that most charged-particle acceleration mechanisms involve energization in directions perpendicular to the ambient magnetic field. Quasi-isotropic energy distribution of charged-particle emerge only as a result of slow pitch-angle diffusion.

In this paper, we investigate the resonant interaction of protons with quasi-monochromatic kinetic Alfvén waves, but instead of the classical approach such as linear or quasilinear theoretical methods, we employ the perturbative Hamiltonian method combined with the autocorrelation function scheme in order to compute the resonant wave–particle diffusion coefficient. By employing such a tool, a potentially significant aspect of the resonant wave–particle interaction that had been overlooked in the past is unveiled. That is, as will be discussed subsequently, we found that a relatively large energy gain along the parallel direction with respect to the ambient magnetic field may be achieved via an anomalous diffusion process. In such a process, finite Larmor radius effects play a key role, which is absent in the traditional linear or quasilinear picture of the resonant wave–particle interaction process. As will be shown, the anomalous diffusion takes place only along the parallel direction, but is not associated with perpendicular diffusion. Along the parallel direction, resonant protons experience a constant force in the frame moving with the wave, which remains finite even when particle motions are averaged over random phases of the waves, but in the perpendicular direction, such an anomalous effect is diminished by the gyromotion of the protons. This finding thus implies that under suitable conditions, charged particles may experience direct and efficient parallel energization, which has not been discussed in the literature.

It is important to note that the main focus of the present paper relates to the charged-particle energization in perpendicular or parallel momentum (or velocity) space, which is related to, but distinct from, the charged-particle transport in real (or configuration) space. Energetic charged-particle transport in magnetized plasmas in real space, such as the cosmic-ray transport across the interplanetary or interstellar medium, is of great importance, and had been studied for several decades.

Customarily, the spatial transport is discussed in conjunction with the momentum transport by means of quasilinear theory under the gyro-kinetic transformation of variables (Schlickeiser 2002), but alternatively, momentum and spatial transport theory can be constructed on the basis of the autocorrelation function scheme (Matthaeus et al. 1990; Bieber et al. 1994; Shalchi 2009, 2020; Shalchi & Weinhorst 2009). The present discussion also adopts the autocorrelation function methodology, but our main focus is on charged-particle transport and diffusion in momentum (or velocity) space.

The organization of the present paper is as follows: In Section 2 the proton trajectory in the presence of a finite spectrum of a kinetic Alfvén wave is calculated by employing the perturbative Hamiltonian equation of motion introduced by Choi et al. (2019). Then Section 3 calculates the diffusion coefficients of protons in velocity space by making use of the autocorrelation function, wherewith we discuss the anomalous parallel diffusion of the protons. Finally, Section 4 concludes the paper, and discusses possible applications of the present findings in the context of heliospheric and astrophysical environments.

2. Motion of Charged Protons in the Presence of the Kinetic Alfvén Wave

The present investigation is a direct extension of our recent work on proton acceleration by the monochromatic kinetic Alfvén wave (Choi et al. 2019). Following Choi et al. (2019), the wave magnetic field and the associated vector potential in the case of a monochromatic kinetic Alfvén wave are expressed by

$$\begin{aligned} \boldsymbol{B}_1(\mathrm{x}, t) &= \hat{x} B_{1x} \cos(k_z z + k_y y - \omega t), \\ \boldsymbol{A}_1(\mathrm{x}, t) &= |\mathrm{A}_1|(\hat{y} - k_z k_y \rho^2 \hat{z}) \sin \\ &\quad \times (k_z z + k_y y - \omega t), \end{aligned} \tag{1}$$

where $B_{1x} = -\mathrm{A}_1 k_z (1 + k_y^2 \rho^2)$, $|\mathrm{A}_1|$ is the amplitude of perturbed vector potential, and $\rho = \sqrt{2T_i/m_i}$ is the proton gyroradius, T_i and m_i being the proton temperature and mass, respectively. We now generalize the situation by considering a broadband kinetic Alfvén wave. We thus express the vector potential as follows:

$$\boldsymbol{A}_1(\mathrm{x}, t) = \int d^3\mathrm{k} \int d\omega \boldsymbol{A}_1(\mathrm{k}, \omega) e^{i\mathrm{k}\cdot\mathrm{x} - i\omega t}. \tag{2}$$

Upon assuming that the perturbation spectrum is composed of the kinetic Alfvén wave only, we express $\mathrm{A}_1(\mathrm{k}, \omega) = \mathrm{A}_1(\mathrm{k})\delta[\omega - \omega_0(\mathrm{k})]$, where $\omega_0(\mathrm{k}) = k_z V_A \sqrt{1 + k_\perp^2 \rho^2}$ is the dispersion relation for kinetic Alfvén waves and $V_A = B_0/\sqrt{4\pi n_0 m_i}$ is the Alfvén speed, B_0 and n_0 being the ambient magnetic field intensity and ambient plasma density, respectively. Note the symmetry property, $\omega_0(-\boldsymbol{k}) = -\omega_0(\boldsymbol{k})$. Making use of these the vector potential (2) can be re-expressed as

$$\begin{aligned} \boldsymbol{A}_1(\boldsymbol{x}, t) = &\int_{-\infty}^{\infty} dk_z \int_{-\infty}^{\infty} dk_y \\ &\times (\hat{y} - k_z k_y \rho^2 \hat{z}) \mathrm{A}_1(k_z, k_y) e^{i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega_0(\boldsymbol{k})t}, \end{aligned} \tag{3}$$

where we have assumed, without loss of generality, that the ambient magnetic field is directed along the z-axis and that $\boldsymbol{k}$ lies in the yz plane, so that we may write $\mathrm{k} \cdot \mathrm{x} = k_y y + k_z z$. Note that we alternatively may write $k_y = k_\perp$ and $k_z = k_\parallel$, where $\perp$ and $\parallel$ refer to directions with respect to the ambient magnetic field vector. Following Choi et al. (2019), it is possible to obtain the relationship between the perturbed vector potentials along versus across the magnetic field,

$$\frac{A_z}{A_y} = \frac{E_{1z}}{E_{1y}} = -k_z k_y \rho^2. \tag{4}$$

The force balance equation for protons is

$$\dot{\mathrm{v}} = \frac{e}{m}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}), \tag{5}$$

where $\boldsymbol{E} = -\nabla\Phi - \partial\boldsymbol{A}/\partial t$ and $\boldsymbol{B} = \nabla \times \mathrm{A}$. For the perturbed field, we make use of a spectrum of kinetic Alfvén waves. Then it is straightforward to show that the equation of motion, namely, $\dot{v}_x(t)$, $\dot{v}_y(t)$, and $\dot{v}_z(t)$, are given by

$$\dot{v}_{1x} = \Omega_0 v_{1y}, \tag{6}$$

$$\begin{aligned} \dot{v}_{1y} = -\Omega_0 v_{1x} + i\frac{e}{m}\int_{-\infty}^{\infty} dk_z \int_{-\infty}^{\infty} dk_\perp [\omega_0 - v_{z0} k_z \\ \times (1 + k_\perp^2 \rho^2)] \mathrm{A}_1(k_z, k_\perp) e^{i(k_z z + k_\perp y - \omega_0 t)}, \end{aligned} \tag{7}$$

$$\begin{aligned} \dot{v}_{1z} = i\frac{e}{m}\int_{-\infty}^{\infty} dk_z \int_{-\infty}^{\infty} dk_\perp [v_0 \sin(\Omega_0 t + \phi_0) k_z \\ \times (1 + k_\perp^2 \rho^2) - k_z k_\perp \rho^2 \omega_0] \\ \times \mathrm{A}_1(k_z, k_\perp) \sum_{n=-\infty}^{\infty} J_n(k_\perp \rho) e^{i\lambda_n t} e^{i\xi_n}, \end{aligned} \tag{8}$$

which can readily be integrated in time in a trivial manner, and thus obtain $v_x(t)$, $v_y(t)$, and $v_z(t)$ explicitly. For the present purpose, however, it is more advantageous to leave the solution in terms of time derivatives. In (6), (7), and (8), $J_n(k_\perp \rho)$ is the Bessel function of the first kind of order n, $\Omega_0 = eB_0/m$ is the proton gyro frequency and

$$\begin{aligned} \lambda_n &= k_z v_{z0} - \omega_0 + n\Omega_0, \\ \xi_n &= k_z z_0 + k_\perp y_0 + n\phi_0, \end{aligned} \tag{9}$$

where ϕ_0 is initial phase of the gyrating particle and z_0, y_0 is initial position of the particle.

In the present investigation, we extend our recent work (Choi et al. 2019) by employing a narrow band, quasi-monochromatic, Alfvén wave spectrum. In the Earth's magnetosphere whistler mode chorus or ion-cyclotron waves of a narrow band are often observed, which are believed to play an important role for diffusion of resonantly interacting electrons or protons (Lyons 1974; Hikishima et al. 2009, 2010). In the literature the energy distribution of such waves is modeled as a Gaussian function in wave frequency and wavenumber (Lyons 1974). In a similar vein, we deal with the diffusion of protons by kinetic Alfvén waves with a Gaussian spectral distribution in k_z and $k_\perp$ having a peak wave intensity located at central perpendicular and parallel wavenumbers, $k_{\perp 0}$ and k_{z0}, respectively:

$$A_1(k_z, k_\perp) = \frac{-iA_1}{4\pi\delta k_z \delta k_\perp} \times \left[\exp\left(-\frac{(k_z - k_{z0})^2}{2(\delta k_z)^2} - \frac{(k_\perp - k_{\perp 0})^2}{2(\delta k_\perp)^2}\right) - \exp\left(-\frac{(k_z + k_{z0})^2}{2(\delta k_z)^2} - \frac{(k_\perp + k_{\perp 0})^2}{2(\delta k_\perp)^2}\right)\right], \quad (10)$$

where δk_z and $\delta k_\perp$ represent the widths of the spectrum along parallel and perpendicular directions, respectively. Note that Equation (10) satisfies the symmetry property, $A_1(k_z, k_{\perp 0}) = A_1^*(-k_z, -k_{\perp 0})$. In the limit of $\delta k_z \to 0$ and $\delta k_\perp \to 0$ one can easily see that the above spectrum reduces to a pair of counter-streaming monochromatic kinetic Alfvén waves in oblique directions with respect to the ambient magnetic field. While Equation (10) is quite general, for situations characterized by $\delta k_\perp \ll k_{\perp 0}$ and $\delta k_z \ll k_{z0}$, the problem reduces to quasi-monochromatic kinetic Alfvén waves. As we shall see later, while the formulation is general, we focus on quasi-monochromatic situations characterized by $\delta k_\perp \ll k_{\perp 0}$ and $\delta k_z \ll k_{z0}$. It should be noted that our choice of Gaussian spectrum is a highly idealized model. We adopt such a model because a narrow Gaussian spectrum is a direct generalization of the monochromatic model considered earlier by us (Choi et al. 2019) so that a direct comparison can be made. For a realistic turbulence spectrum in the heliosphere, of course, our model is not appropriate. For a more realistic model spectrum Matthaeus et al. (1990) consider a slab model, Shalchi & Weinhorst (2009) adopt a 2D model spectrum that includes inertial and energy range, to name just a couple. A recent review by Shalchi (2020) discusses a number of realistic turbulence spectra in the heliospheric environment.

3. Diffusion Coefficients

It is possible to derive the charged-particle transport equation that includes momentum space diffusion as well as spatial (or configurational space) diffusion, starting from the standard Vlasov equation by transforming the variables to gyro centered phase space, as shown by Schlickeiser (2002) and Shalchi (2009). Transformation of the phase space variables between real phase space and gyro-phase becomes very complicated especially when the equation becomes nonlinear (Frieman & Liu 1982). According to modern gyro-kinetic theory using Lie-transform perturbation methods, its relation to real physical variable is mathematically clear, but physically subtle (for example, exact particle density and gyro-phase center density; Brizard & Hahm 2007). However, when we consider a homogeneous magnetic field or very local transport event, we may follow the above-referenced works in order to derive the charged-particle transport equation that describes momentum and spatial diffusion. The purpose of the present paper is, however, restricted to the issue of momentum space diffusion only.

We calculate the velocity diffusion coefficients for protons by making use of the Taylor–Green–Kubo (TGK) formalism (Taylor 1922; Green 1951; Kubo 1957), which in the standard form, applies to the spatial diffusion. The TGK spatial diffusion coefficient is defined as

$$D_{ij} = \left.\frac{\langle \Delta r_i(t) \Delta r_j(t)\rangle}{2t}\right|_{t\to\infty} = \lim_{t\to\infty} \frac{1}{2t}\int_0^t dt' \int_{t'-t}^{t'} d\tau C_{ij}(\tau), \quad (11)$$

where

$$C_{ij}(\tau) = \langle v_i(\tau) v_j(0)\rangle \quad (12)$$

is the velocity autocorrelation function, with the ensemble average denoted by the bracket. Note that the TGK spatial diffusion coefficient pertains to the charged-particle transport in configuration space (Bieber et al. 1994; Matthaeus et al. 2003; Shalchi & Weinhorst 2009; Shalchi 2010, 2020). In a straightforward extension of this formalism to velocity space, the corresponding TGK velocity space diffusion coefficient is given (Smith & Kaufman 1978; Shalchi 2009, 2011) as

$$D_{ij} = \frac{\langle \Delta v_i(t) \Delta v_j(t)\rangle}{2} = \frac{1}{2}\int_0^t dt' \int_{t'-t}^{t'} d\tau C_{ij}(\tau), \quad (13)$$

where

$$C_{ij}(\tau) = \langle \dot{v}_i(t+\tau) \dot{v}_j(t)\rangle_{z_0, y_0, \phi_0}. \quad (14)$$

Note that now the autocorrelation function involves the acceleration, or the force exerted on each particle.

3.1. Diffusion along the Magnetic Field

Making use of Equation (8), the zz component of the correlation function along the magnetic field is given as follows:

$$C_z(\tau) = \frac{e^2}{m^2}\int_{-\infty}^{\infty} dk_z \int_{-\infty}^{\infty} dk_\perp \left[k_z^2 v_0^2 (1 + k_\perp^2 \rho^2)^2 C_1(\tau) + k_z^2 k_\perp^2 \rho^4 \omega_0^2 \sum_{n=-\infty}^{\infty} J_n^2(k_\perp \rho) e^{i\lambda_n \tau}\right] |A_1(k_z, k_\perp)|^2 \, e^{i(k_z v_{z0} - \omega_0)\tau}, \quad (15)$$

where

$$C_1(\tau) = \frac{1}{2}\sum_{n=-\infty}^{\infty} J_n(k_\perp \rho) \times [J'_{n+1}(k_\perp \rho) e^{i\lambda_{n+1}\tau} - J'_{n-1}(k_\perp \rho) e^{i\lambda_{n-1}\tau}]. \quad (16)$$

Defining

$$C_{z,k_{z0},k_{\perp 0}} = \left(\frac{A_1}{4\pi\delta k_z \delta k_\perp}\frac{e}{m}\right)^2 \int_{-\infty}^{\infty} dk_z \int_{-\infty}^{\infty} dk_\perp \times \left[k_z^2 v_0^2 (1 + k_\perp^2 \rho^2)^2 C_1(\tau) + k_z^2 k_\perp^2 \rho^4 \omega_0^2 \sum_{n=-\infty}^{\infty} J_n^2(k_\perp \rho) e^{i\lambda_n \tau}\right] \times \exp\left(-\frac{(k_z - k_{z0})^2}{2(\delta k_z)^2} - \frac{(k_\perp - k_{\perp 0})^2}{2(\delta k_\perp)^2} + i(k_z v_{z0} - \omega_0)\tau\right), \quad (17)$$

and assuming $k_{z0}/\delta k_z \gg 1$ and $k_{\perp 0}/\delta k_\perp \gg 1$, i.e., quasi-monochromatic kinetic Alfvén wave situation, we obtain

$$C_z(\tau) \approx C_{z,k_{z0},k_{\perp 0}} + C_{z,-k_{z0},-k_{\perp 0}} = C_{z,k_{z0},k_{\perp 0}} + C_{z,-k_{z0},k_{\perp 0}} = 2\,\mathrm{Re}[C_{z,k_{z0},k_{\perp 0}}], \tag{18}$$

where we have invoked the symmetry relation, $C_{z,k_{z0},k_{\perp 0}} = C_{z,k_{z0},-k_{\perp 0}}$. Making use of Equations (17) and (18) we obtain

$$C_z(\tau) = \frac{|A_1|^2}{4\pi\delta k_z\delta k_\perp}\left(\frac{e}{m}\right)^2\int_{-\infty}^{\infty}dk_z\int_{-\infty}^{\infty}dk_\perp \sum_{n=-\infty}^{\infty}[C_2(k_\perp, k_z)\cos\lambda_{n+1}\tau + C_3(k_\perp, k_z)\cos\lambda_{n-1}\tau + C_4(k_\perp, k_z)\cos\lambda_n\tau], \tag{19}$$

where

$$\begin{pmatrix} C_2(k_\perp, k_z)\\ C_3(k_\perp, k_z)\\ C_4(k_\perp, k_z)\end{pmatrix} = \begin{pmatrix} \frac{1}{2}k_z^2 v_0^2(1 + k_\perp^2\rho^2)^2 J_n(k_\perp\rho)J'_{n+1}(k_\perp\rho)\\ -\frac{1}{2}k_z^2 v_0^2(1 + k_\perp^2\rho^2)^2 J_n(k_\perp\rho)J'_{n-1}(k_\perp\rho)\\ k_z^2 k_\perp^2\rho^4\omega_0^2 J_n^2(k_\perp\rho)\end{pmatrix} \times \exp\left(-\frac{(k_z - k_{z0})^2}{2(\delta k_z)^2} - \frac{(k_\perp - k_{\perp 0})^2}{2(\delta k_\perp)^2}\right). \tag{20}$$

The velocity dispersion $\langle(\Delta v_z)^2\rangle$ follows upon making use of the autocorrelation function for parallel acceleration:

$$\langle(\Delta v_z)^2\rangle = 2\int_0^t d\tau(t-\tau)C_z(\tau) = \frac{|A_1|^2}{2\pi\delta k_z\delta k_\perp}\left(\frac{e}{m}\right)^2\int_{-\infty}^{\infty}dk_z\int_{-\infty}^{\infty}dk_\perp \times\sum_{n=-\infty}^{\infty}[C_2(k_\perp, k_z) + C_3(k_\perp, k_z) + C_4(k_\perp, k_z)]\frac{1-\cos\lambda_n t}{\lambda_n^2}. \tag{21}$$

This leads to the parallel diffusion coefficient for quasi-monochromatic kinetic Alfvén waves,

$$D_{z,b}(t) = \frac{\langle(\Delta v_z)^2\rangle}{2t}. \tag{22}$$

For the sake of simplicity, let us assume a single wavenumber $k_{\perp 0}$ along the perpendicular direction by setting $\delta k_\perp \to 0$. That is, we consider a monochromatic spectrum along the perpendicular wavenumber, but consider a finite spectral width along parallel wavenumber. Then we find that

$$D_{z,b}(t) = \frac{1}{2t}\frac{|A_1|^2}{8\pi}\left(\frac{e}{m}\right)^2\left\{v_0^2(1 + k_{\perp 0}^2\rho^2)^2\sum_{n=-\infty}^{\infty}\times\left(\frac{2nJ_n(k_{\perp 0}\rho)}{k_{\perp 0}\rho}\right)^2 G_1(n, k_{z0}, v_{z0}) + 4k_{\perp 0}^2\rho^4 V_A^2\left(1 + \frac{k_{\perp 0}^2\rho^2}{2}\right)\sum_{n=-\infty}^{\infty}J_n^2(k_{\perp 0}\rho)\,G_2(n, k_{z0}, v_{z0})\right\}, \tag{23}$$

where

$$\begin{pmatrix} G_1(n, k_{z0}, v_{z0})\\ G_2(n, k_{z0}, v_{z0})\end{pmatrix} = \frac{1}{\delta k_z}\int_{-\infty}^{\infty}dk_z\begin{pmatrix} k_z^2\\ k_z^4\end{pmatrix} \times\exp\left(-\frac{(k_z - k_{z0})^2}{2(\delta k_z)^2}\right)\frac{1-\cos\lambda_n t}{\lambda_n}. \tag{24}$$

From 9, we may express the parallel wavenumber as

$$k_z = \frac{\lambda_n + n\Omega_0}{v_{z0} - V_A\sqrt{1 + k_{\perp 0}^2\rho^2/2}}. \tag{25}$$

By trivially changing the integral variable we obtain

$$G_1(n, k_{z0}, v_{z0}) = \frac{1}{(\delta k_z)v_{z0}'^3}\int_{-\infty}^{\infty}d\lambda_n \times(\lambda_n + n\Omega_0)^2\frac{1-\cos\lambda_n t}{\lambda_n}\exp\left(-\frac{(\lambda_n + n\Omega_0 - k_{z0}v'_{z0})^2}{2(\delta k_z)^2 v_{z0}'^2}\right), \tag{26}$$

where

$$v'_{z0} = v_{z0} - V_A\sqrt{1 + \frac{k_{\perp 0}^2\rho^2}{2}}. \tag{27}$$

If we restrict ourselves to protons whose parallel velocity v_{z0} satisfies the resonant condition,

$$n_0\Omega_0 - k_{z0}v'_{z0} = n_0\Omega_0 - k_{z0}v_{z0} - \omega_0(k_{z0}, k_{\perp 0})k_{z0} = 0, \tag{28}$$

then, we have

$$G_1(n, k_{z0}, v_{z0}) \sim \int_{-\infty}^{\infty}d\lambda_n\frac{1-\cos\lambda_n t}{\lambda_n}\exp\left(-\frac{[\lambda_n + (n - n_0)\Omega_0]^2}{2(\delta k_z)^2 v_{z0}'^2}\right). \tag{29}$$

Owing to the factor $(1 - \cos\lambda_n t)/\lambda_n$, one may easily see that the dominant contribution to the integrand comes from $\lambda_n \sim 0$. On the other hand, for a sufficiently narrow k_z spectrum with $\delta k_z \ll 0$, we can also see that the dominant contribution comes from $n = n_0$. In short, we may approximate

$$G_1(n, k_{z0}, v_{z0}) \approx G_1(n_0, k_{z0}, v_{z0})\delta_{n,n_0}, \tag{30}$$

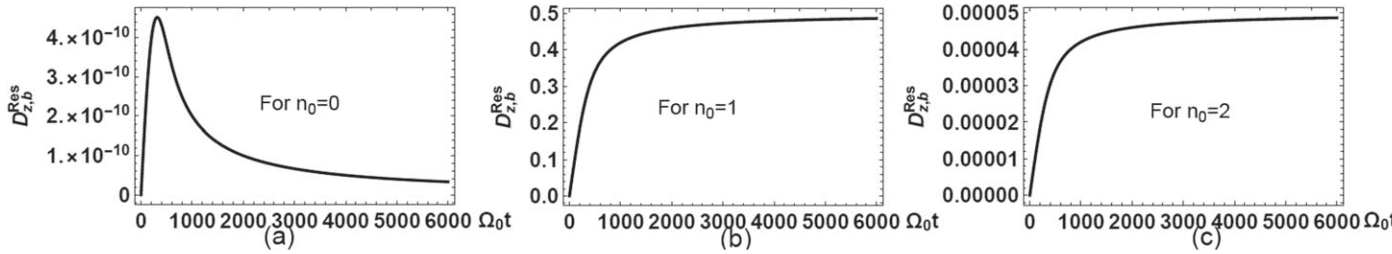

Figure 1. The parallel diffusion coefficient $D_{z,b}^{\rm Res}(t)$, for (a) Landau resonance $n_0 = 0$, (b) cyclotron resonance $n_0 = 1$, and (c) harmonic cyclotron resonance $n_0 = 2$, versus time.

where δ_{n,n_0} is the Kronecker delta. Note that $n_0 = 0$ corresponds to the Landau resonance, while $n_0 = 1$ and $n_0 = 2$ corresponds to first and second cyclotron harmonic resonance. Consequently, the resonant part to G_1 can be integrated exactly to yield

$$G_1^{\rm Res}(n_0, k_{z0}, v_{z0}) = \frac{1}{(\delta k_z) v_{z0}'^3}\left\{n_0^2 \Omega_0^2 \,\pi t\ \mathrm{erf}\left(\frac{t}{2\sqrt{A}}\right) + (1 + 2An_0^2\Omega_0^2)\sqrt{\frac{\pi}{A}}\left[\exp\left(-\frac{t^2}{4A}\right) - 1\right]\right\}, \tag{31}$$

where

$$A = \frac{1}{2(\delta k_z)^2 v_{z0}'^2}, \tag{32}$$

and the superscript designates the fact that k_{z0} and v_{z0} satisfy the resonance condition (28). In a similar manner, we may also obtain G_2 for resonant protons,

$$G_2^{\rm Res}(n_0, k_{z0}, v_{z0}) = \frac{1}{(\delta k_z) v_{z0}'^5}\left\{\frac{\sqrt{\pi}}{4A^{5/2}} \times \left[2A + (t^2 - 1)\exp\left(-\frac{t^2}{4A}\right)\right] + n_0^4 \Omega_0^4\, \pi t\ \mathrm{erf}\left(\frac{t}{2\sqrt{A}}\right) + 2n_0^2\Omega_0^2 \times (3 + An_0^2\Omega_0^2)\sqrt{\frac{\pi}{A}}\left[\exp\left(-\frac{t^2}{4A}\right) - 1\right]\right\}. \tag{33}$$

Combing Equations (23), (31), and (33), we arrive at the diffusion coefficient for resonant protons as

$$D_{z,b}^{\rm Res}(t) = \frac{1}{2t}\frac{|\mathrm{A}_1|^2}{4\sqrt{2\pi}}\left(\frac{e}{m}\right)^2 \left\{v_0^2 (1 + k_{\perp 0}^2\rho^2)^2 \times \left(\frac{2n_0 J_{n_0}(k_{\perp 0}\rho)}{k_{\perp 0}\rho}\right)^2 G_1^{\rm Res}(n_0, k_{z0}, v_{z0}) + 4k_{\perp 0}^2 \rho^4 V_A^2 \left(1 + \frac{k_{\perp 0}^2\rho^2}{2}\right) J_{n_0}^2 \times (k_{\perp 0}\rho) G_2^{\rm Res}(n_0, k_{z0}, v_{z0})\right\}. \tag{34}$$

Figure 1 plots the parallel diffusion coefficient for protons at various resonant harmonic mode number n_0, versus normalized time. For numerical purposes we chose normalized input parameters corresponding to $e\mathrm{A}_1/m = 0.1$, $V_A/c = 0.1$, $\rho\Omega_0/c = 0.1$, $v_{z0}/c = 0.05$, $k_{\perp 0} = 0.1$, and $\delta k_z = 0.1$, with k_{z0} determined according to Equation (28). In Figure 1, panel (a) shows the case of Landau resonance with $n_0 = 0$, panel (b) displays the cyclotron resonance with $n_0 = 1$, and panel (c) depicts the second harmonic cyclotron resonance with $n_0 = 2$. Figure 1(a) shows that the diffusion coefficient for Landau resonance peaks in an early time period but subsequently decreases with the behavior $\propto\mathcal{O}(t^{-1})$, which is characteristic of normal diffusion. Note that the normal diffusion brought on by Landau resonance is rather insignificant, the magnitude being only on the order of $\sim\mathcal{O}(10^{-10})$. In contrast to the Landau resonance, the cyclotron resonance leads to anomalous diffusion, as shown in Figure 1(b) and Figure 1(c). The cyclotron-resonant diffusion coefficients do not exhibit $\propto\mathcal{O}(t^{-1})$ temporal behavior, but instead, they asymptotically approach constant values for $t \to \infty$. This is representative of anomalous diffusion. Note that the anomalous diffusion coefficient is higher in magnitude over that of Landau resonance by a factor of $\mathcal{O}(10^6)$ to $\mathcal{O}(10^8)$. Also, it is seen that the fundamental harmonic resonance leads to higher magnitude when compared with the harmonic cyclotron resonance.

For the case of monochromatic waves, we also obtained a similar anomalous behavior associated with the diffusion coefficient. In the limit of the monochromatic $\delta k_z \to 0$ case, we have already obtained a diffusion coefficient in the z direction (Choi et al. 2019) as

$$D_z^{\rm mono}(t) = \frac{1}{8t}\frac{e^2 \mathrm{A}_1^2 v_0^2}{m^2 k_{z0}^2}(1 + k_{\perp 0}^2\rho^2)^2 \sum_n [J_n]^2 \times (W_{n+1} + W_{n-1}) + \left(\frac{e\mathrm{A}_1}{mk_{z0}} k_{\perp 0}\omega\rho^2\right)^2 \sum_n [J_n]^2 W_n, \tag{35}$$

where

$$W_n = \frac{[1 - \cos(\lambda_n t)]}{\lambda_n^2}.$$

At the resonance $\lambda_n = 0$, we obtain $W_n = t^2/2$. Thus, for sufficiently long t, the diffusion coefficient behaves as $D_z^{\rm mono}(t) \sim t$, which grows linearly in time, which of course is an anomalous behavior associated with diffusion. The purpose of Choi et al. (2019), however, was to discuss stochastic diffusion and acceleration of charged protons, for which the focus was on nonresonant wave–particle interaction. Consequently, in Choi et al. (2019) we considered protons far from resonance. In contrast, the diffusion coefficient of protons who are resonant with the quasi-monochromatic wave approaches constant magnitude asymptotically. Note that each quasi-monochromatic wave within the spectrum has a random phase and resonant protons also acquire the same phase. The

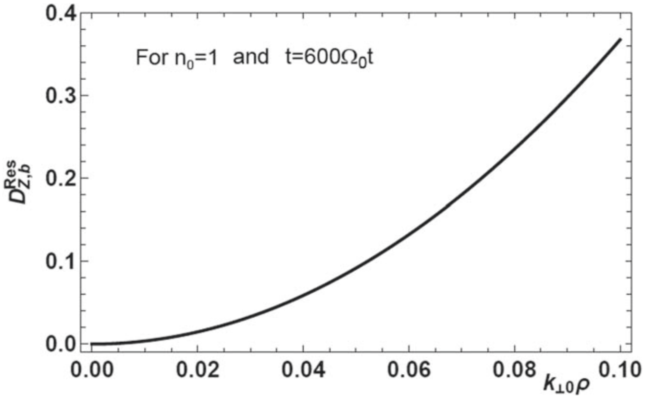

Figure 2. The parallel diffusion coefficient $D^{\mathrm{Res}}_{z,b}(k_{\perp 0}\rho)$ for $n_0 = 1$ versus normalized gyroradius, computed for dimensionless time $600\Omega_0 t$, for which the diffusion coefficient has attained the asymptotically steady state value.

ensemble average process for resonant protons phase mixes such motion and the overall diffusion coefficient becomes constant for a large time as

$$D^{\mathrm{Res}}_{z,b}(\infty) = \frac{|A_1|^2\Omega_0^2}{4(\delta k_z)v'_{z0}}\left[\frac{v_0^2}{v'^2_{z0}}(1 + k^2_{\perp 0}\rho^2)^2[n_0 J'_{n_0}]^2 + \frac{k^2_{\perp 0}\rho^4\Omega_0^2 V_A^2}{v'^4_{z0}}\left(1 + \frac{k^2_{\perp 0}\rho^2}{2}\right)[n_0^2 J_{n_0}]^2\right]. \tag{36}$$

This shows that even for a narrow spectrum of δk_z in the z direction with monochromatic wavenumber in perpendicular direction, the existence of broad spectrum changes the characteristics of diffusion in a fundamental way.

In Figure 2 we plot the parallel diffusion coefficient versus normalized gyroradius, for $n_0 = 1$, and for $600\Omega_0 t$, for which the diffusion coefficient is at near saturation magnitude. One can see that in the limit of infinitesimal gyroradius, the diffusion coefficient tends to zero, while for finite gyroradius, the saturated diffusion coefficient becomes finite and increases monotonically as a function of $k_{\perp 0}\rho$. This indicates that the finite Larmor radius effect leads to enhanced diffusion of protons along the parallel direction. Since $D^{\mathrm{Res}}_{z,b}(k_{\perp 0}\rho)$ is inversely proportional to δk_z, in the limit of monochromatic waves, the diffusion coefficient diverges while for the quasi-monochromatic spectrum, they converge to finite values.

3.2. Diffusion across the Magnetic Field

The proton gyromotion becomes important for perpendicular diffusion, and as shown by Smith & Kaufman (1978), the so-called "apparent diffusion" arises where diffusion coefficient decrease as $O(t^{-1})$. The diffusion coefficient is given by

$$D^{\mathrm{Res}}_{\perp,b} = \frac{1}{2}\int_0^t dt'\int_{t'-t}^{t'} d\tau[C_{xx}(\tau) + C_{yy}(\tau)], \tag{37}$$

where

$$C_{xx}(\tau) = \langle \dot{v}_x(t+\tau)\dot{v}_x(t)\rangle_{z_0,y_0,\phi_0},$$
$$C_{yy}(\tau) = \langle \dot{v}_y(t+\tau)\dot{v}_y(t)\rangle_{z_0,y_0,\phi_0}. \tag{38}$$

We may obtain specific expressions for perpendicular forces—for detailed intermediate steps, see the Appendix. The result is

$$\dot{v}_x(\xi) = \frac{dv_x(\xi)}{d\xi} = v_0\Omega_0\sin(\Omega_0\xi + \phi'_0) + \sum_n R_n(\xi)[e^{-i(\beta_n\xi+\phi_n)}D_+(\xi) - \text{c.c.}],$$
$$\dot{v}_y(\xi) = \frac{dv_y(\xi)}{d\xi} = v_0\Omega_0\cos(\Omega_0\xi + \phi'_0) + \sum_n R_n(\xi)[e^{-i(\beta_n\xi+\phi_n)}D_-(\xi) - \text{c.c.}], \tag{39}$$

where

$$\xi = t - \xi_0,\quad \xi_0 = -z_0(v_{z0} - V_{As})^{-1},$$
$$V_{As} = V_A\sqrt{1 + k^2_{\perp 0}\rho^2},$$
$$\beta_n = k_{z0}v_{z0} - \omega_0 + n\Omega_0,$$
$$(\delta\omega)^2 = \frac{(\delta k_z)^2}{2}(v_{z0} - V_{As})^2,$$
$$\phi_n = n\phi_0 + k_{\perp 0}y_0 - n\Omega_0 z_0(v_{z0} - V_{As})^{-1},$$
$$\phi'_0 = \Omega_0\xi_0 + \phi_0. \tag{40}$$

Defining the Dawson-F function as D_F, we have

$$R_n(\xi) = \frac{iE_n(\xi)e^{-(\delta\omega)^2\xi^2}}{4(\delta\omega)},$$
$$D_+(\xi) = D_F\left(\frac{\Omega_0 - \beta_n}{2\delta\omega} + i\delta\omega\xi\right) + D_F\left(\frac{\Omega_0 + \beta_n}{2\delta\omega} - i\delta\omega\xi\right),$$
$$D_-(\xi) = D_F\left(\frac{\Omega_0 - \beta_n}{2\delta\omega} + i\delta\omega\xi\right) - D_F\left(\frac{\Omega_0 + \beta_n}{2\delta\omega} - i\delta\omega\xi\right),$$
$$D_F(Z) = \frac{\sqrt{\pi}}{2i}e^{-Z^2}\operatorname{erf}(iZ) = \frac{\sqrt{\pi}}{2}e^{-Z^2}\operatorname{erfi}(Z), \tag{41}$$

Here we used the property that $\operatorname{erfi}(Z) = -iZ\operatorname{erf}(iZ)$, where erf is the error function, and erfi is the imaginary error function.

$$\langle e^{-i\phi_n - i\phi_{n'}}\rangle_{y_0,\phi_0} = 0,$$
$$\langle e^{-i\phi_n + i\phi_{n'}}\rangle_{y_0,\phi_0} = \delta_{n,n'}, \tag{42}$$

and the correlation functions $C_x(\tau)$ and $C_y(\tau)$ can be obtained as follows:

$$\langle \dot{v}_{1x}(0)\dot{v}_{1x}(\tau)\rangle_{\phi_0,y_0} = -\sum_n R_n(0)R_n(\tau) \times[e^{-i\beta_n\tau}D_+(0)D_+(\tau) + \text{c.c.}],$$
$$\langle \dot{v}_{1y}(0)\dot{v}_{1y}(\tau)\rangle_{\phi_0,y_0} = -\sum_n R_n(0)R_n(\tau) \times[e^{-i\beta_n\tau}D_-(0)D_-(\tau) + \text{c.c.}]. \tag{43}$$

We may calculate $e^{-i\beta_n\tau}D_+(0)D_+(\tau)$ and $e^{-i\beta_n\tau}D_-(0)D_-(\tau)$. Take, for instance,

$$e^{-i\beta_n\tau}D_-(\tau) = \frac{i\sqrt{\pi}}{2}e^{(\delta\omega)^2\tau^2} \times \left[\mathrm{erf}\left((\delta\omega)\tau - i(n_0 - n + 1)\frac{\Omega_0}{2(\delta\omega)}\right)\exp \times \left(-i\Omega_0\tau - \frac{(n_0 - n + 1)^2\Omega_0^2}{4(\delta\omega)^2}\right) + \mathrm{erf}\left((\delta\omega)\tau + i(n - n_0 + 1)\frac{\Omega_0}{2(\delta\omega)}\right)\exp \times \left(i\Omega_0\tau - \frac{(n - n_0 + 1)^2\Omega_0^2}{(\delta\omega)^2}\right)\right]. \tag{44}$$

Since $\Omega_0/\delta\omega \gg 1$, this quantity is dominant only when $n_0 - n + 1 = 0$ or $n - n_0 + 1 = 0$. This approximation gives

$$e^{-i\beta_n\tau}D_-(\tau) \approx \frac{i\sqrt{\pi}}{2}e^{(\delta\omega)^2\tau^2}\,\mathrm{erf}[(\delta\omega)\tau] \times [e^{-i\Omega_0\tau}\delta_{n,n_0+1} + e^{i\Omega_0\tau}\delta_{n,n_0-1}], \tag{45}$$

which leads to

$$e^{-i\beta_n\tau}D_-(\tau)D_-(0) + \mathrm{c.c.} = \sqrt{\pi}\ e^{(\delta\omega)^2\tau^2}\,\mathrm{erf} \times [(\delta\omega)\tau]\sin(\Omega_0\tau)D_F\left(\frac{\Omega_0}{\delta\omega}\right)[\delta_{n,n_0-1} + \delta_{n,n_0+1}], \tag{46}$$

where we have made use of $\beta_{n_0+1} = -\Omega_0$, from which we have $D_-(0)|_{n=n_0\ +\ 1} = D_F[\Omega_0/(\delta\omega)]$. We may implement similar procedures for $e^{-i\beta_n\tau}D_+(0)D_+(\tau)$ as well.

At this point, we assume that v_{z0} satisfies a specific resonance condition,

$$k_{z0}v_{z0} - \omega_0 + n_0\Omega_0 = 0. \tag{47}$$

Then, for such a value of v_{z0}, we have

$$\beta_n = (n - n_0)\Omega_0, \tag{48}$$

where the resonance occurs at $n = n_0$. Taking the zeroth-order gyromotion into account as well, we arrive at

$$C_y(\tau) = \frac{v_0^2\Omega_0^2}{2}\cos(\Omega_0\tau) + \frac{\pi^{3/2}}{2^6(\delta\omega)^2}\frac{e^2A_1^2}{m^2}k_{z0}^2V_{Am}^2\Omega_0^2D_F\left(\frac{\Omega_0}{\delta\omega}\right) \times (J_{n_0-1}^2 + J_{n_0+1}^2)\,\mathrm{erf}[(\delta\omega)\tau]\sin(\Omega_0\tau). \tag{49}$$

It can be shown that $C_x(\tau)$ is the same as $C_y(\tau)$. The derivation is omitted. As a consequence, for a sufficiently long time, the diffusion coefficient in the perpendicular direction is given by

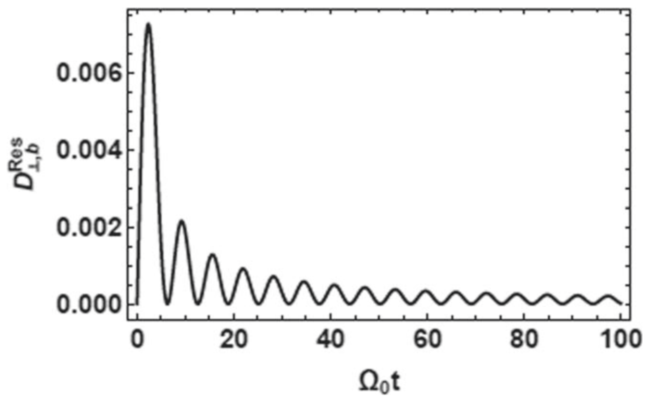

Figure 3. The perpendicular diffusion coefficient $D_{\perp,b}^{\mathrm{Res}}(t)$.

$$D_{\perp,b}^{Res}(t) = \frac{1}{t}\int_0^t d\tau(t - \tau)C_\perp(\tau) = \frac{v_0^2[1 - \cos(\Omega_0 t)]}{t} + \frac{\pi^{3/2}}{2^5(\delta\omega)^2}\frac{e^2A_1^2}{m^2}k_{z0}^2V_{Am}^2\Omega_0^2D_F\left(\frac{\Omega_0}{\delta\omega}\right)(J_{n_0-1}^2 + J_{n_0+1}^2) \times \frac{1}{\Omega_0^2 t}\left[\left(2 + \frac{\Omega_0^2}{(\delta\omega)^2}\right)\frac{1}{\sqrt{\pi}}D_F\left(\frac{\Omega_0}{2(\delta\omega)}\right) - \frac{\Omega_0}{\sqrt{\pi}(\delta\omega)} + \Omega_0 t\exp\left(-\frac{\Omega_0^2}{4(\delta\omega)^2}\right) - \sin(\Omega_0 t)\right]. \tag{50}$$

In this equation, one can see that the first term represents an "apparent" diffusion by gyromotion (Smith & Kaufman 1978), while the second term is anomalous diffusion by quasi-monochromatic kinetic Alfvén wave. In arriving at Equation (50), we have assumed that a single perpendicular wavenumber dominates the $k_\perp$ integral and $\delta k_z \ll k_{z0}$ or $\Omega_0/\delta\omega \gg 1$ as was done for the parallel diffusion. This indicates that the Dawson-F function with a large argument leads to

$$D_F\left(\frac{\Omega}{\delta\omega}\right) \ll 1. \tag{51}$$

Owing to this term in the perpendicular direction, the anomalous or second term of Equation (50) is negligibly small compared to first term. Comparing a such feature with the parallel diffusion coefficient in (36), one may readily see that in perpendicular direction, the anomalous term is negligible. One may conclude that for protons near resonance, anomalous diffusion across the magnetic field can be ignored.

In Figure 3 we show the perpendicular diffusion coefficient of resonant protons $D_{\perp,b}^{\mathrm{Res}}$ given by Equation (50) for the time t. Upon comparison with Figure 1, which shows the parallel diffusion coefficient, it is shown that the perpendicular diffusion coefficient is generally higher in magnitude. But this diffusion is due to the simple gyromotion of the proton and anomalous diffusion of resonant protons does not show up due to its small magnitude.

4. Summary and Conclusions

In the present paper, we have calculated the velocity diffusion coefficients for protons in both parallel and perpendicular directions, assuming the protons are immersed in an ambient magnetic field plus quasi-monochromatic kinetic Alfvén wave field. It is found that for the protons moving coherently with the waves, they exhibit an unexpected behavior of anomalous diffusion in the parallel direction. While normal diffusion approaches zero for a sufficiently long time, the anomalous part of the parallel diffusion coefficient saturates to a finite level asymptotically. The anomalous parallel diffusion has a close connection to perpendicular proton motion in that the saturated parallel diffusion coefficient depends on the finite gyroradius. This is evident from the fact that the fastest anomalous parallel diffusion occurs for protons resonant with the $n_0 = 1$ cyclotron mode (Figure 1), but is absent for the $n_0 = 0$ Landau resonant case, for which no perpendicular motion is included. This verifies the interpretation that the anomalous parallel diffusion is intimately related to the perpendicular proton motion. In the customary linear or quasilinear diffusion process, the particles are assumed to execute the unperturbed orbit, hence, finite gyroradius terms induced by the waves, namely, terms associated with $J_n(k_\perp \rho)$, are precluded.

The physical origin of the anomalous parallel diffusion can be understood from the force Equation (8). In the vicinity of the resonance $\lambda_n \approx 0$ or $\lambda_{n\pm1} \approx 0$, one can see that a proton in parallel direction experiences a constant force, which is independent of time. This in turn leads to the asymptotically constant terms in the autocorrelation $C_z(\tau)$, as can be easily seen from 19 and 20. Consequently, the diffusion coefficient $D_{z,b}$ increases in time until is asymptotically approaches the saturation level (Figure 2). A similar anomalous term is also present in the perpendicular diffusion coefficient, but as shown by Equation (50), it is associated with a small factor $D_F(\Omega_0/\delta\omega)$, which for narrow spectrum, becomes vanishingly small as $\delta\omega/\Omega_0$ approaches zero. Thus, the anomalous diffusion in the perpendicular direction is negligibly small.

The present work is based on a Gaussian form kinetic Alfvén wave model intended to directly extend our previous work based on a monochromatic kinetic Alfvén wave (Choi et al. 2019). We find the following differences and new features. In the monochromatic case, we cannot calculate the diffusion coefficients for resonant particles since the resonant particles suffer from a constant force by the wave. But when the wave becomes quasi-monochromatic, different wavenumbers imply different forces with random phase to the particles such that it leads to anomalous diffusion. Another important fact is that for resonant particles, the diffusion coefficient across the magnetic field is an order of magnitude smaller compared to the parallel diffusion. In other words, the main anomalous diffusion occurs along the magnetic field rather than across the magnetic field. This was not addressed in Choi et al. (2019). The potential importance of the present findings in the context of astrophysics is as follows. In most charged-particle acceleration mechanisms, the energization takes place in directions perpendicular to the ambient magnetic field vector. For instance, shock-related energization processes involve predominant perpendicular acceleration. Standard resonant and stochastic heating by kinetic Alfvén waves discussed in the context of coronal heating and solar wind acceleration also involve predominant perpendicular heating. Quasi-isotropic energy distribution of energetic particles come about as a result of slow pitch-angle diffusion. In the absence of large-scale parallel electric field, direct acceleration along the ambient magnetic field vector is generally not available. The present finding that under certain conditions, charged particles may undergo anomalous parallel diffusion implies a new parallel energization/heating mechanism. Further investigations of the ramifications of the present findings, including demonstrations by numerical simulations is called for, but such a task is beyond the scope of the present work.

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Appendix
Intermediate Steps

From the force balance Equation (7), we have

$$\frac{\partial A_{1y}}{\partial t} = \int_{-\infty}^{\infty} dk_z \int_{-\infty}^{\infty} dk_\perp (-i\omega_0) \mathrm{A}_1(k_z, k_\perp) e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_0 t)} \quad \text{(A1)}$$

and

$$B_{1x} = \int_{-\infty}^{\infty} dk_z \int_{-\infty}^{\infty} dk_\perp (-ik_z) (1 + k_\perp^2 \rho^2) \mathrm{A}_1(k_z, k_\perp) e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_0 t)}. \quad \text{(A2)}$$

Thus we obtain

$$\frac{e}{m} v_{z0} B_{1x} - \frac{e}{m}\frac{\partial A_{1y}}{\partial t} = \frac{ie}{m}\int_{-\infty}^{\infty} dk_z \int_{-\infty}^{\infty} dk_\perp \eta (k_z, k_\perp) \mathrm{A}_1(k_z, k_\perp) e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_0 t)}, \quad \text{(A3)}$$

where

$$\eta(k_z, k_\perp) = \omega_0 - k_z v_{z0}(1 + k_\perp^2 \rho^2) \equiv k_z V_{Am}. \quad \text{(A4)}$$

Here, we define modified Alfvén speed,

$$V_{Am} = V_A(1 + k_\perp^2 \rho^2)^{1/2} - v_{z0}(1 + k_\perp^2 \rho^2). \quad \text{(A5)}$$

Returning to the original force balance equation, we have

$$\dot{v}_{1y}(t) = -\Omega_0 v_{1x} + \frac{ie}{m}\int_{-\infty}^{\infty} dk_z \int_{-\infty}^{\infty} \eta \times (k_z, k_\perp) \mathrm{A}_1(k_z, k_\perp) e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_0 t)} dk_\perp, \quad \text{(A6)}$$

In order to carry out the integral in Equation (A6), we substitute Equation (10) in (A6), which results in the following expressions for the integral of relevance:

$$
\begin{aligned}
G(t) &= \frac{ie}{m}\int_{-\infty}^{\infty} dk_z \int_{-\infty}^{\infty} \eta(k_z, k_\perp) \mathrm{A}_1(k_z, k_\perp) e^{i(\mathrm{k}\cdot\mathrm{x}-\omega_0 t)} dk_\perp \\
&= \frac{e}{4m}\frac{\mathrm{A}_1}{\sqrt{\pi}(\delta k_z)}\int_{-\infty}^{\infty} dk_z \int_{-\infty}^{\infty} \frac{dk_\perp}{\delta k_\perp}\eta \\
&\quad\times (k_z, k_\perp) e^{i\mathrm{k}\cdot\mathrm{x}(t) - i\omega_0 t}\sum_{+,-}(\pm)\exp \\
&\quad\times\left(-\frac{(k_z \mp k_{z0})^2}{2(\delta k_z)^2} - \frac{(k_\perp \mp k_{\perp 0})^2}{2(\delta k_\perp)^2}\right) \\
&= \frac{e}{m}\frac{\mathrm{A}_1 V_{Am}}{2\sqrt{2}(\delta k_z)}\int_{-\infty}^{\infty} dk_z k_z e^{ik_z z(t) - i\omega_0 t} \\
&\quad\times \sum_{+,-}(\pm) e^{\pm} ik_{\perp 0} y(t) \exp\left(-\frac{(k_z \mp k_{z0})^2}{2(\delta k_z)^2}\right) \\
&= \frac{\sqrt{\pi} e \mathrm{A}_1 V_{Am}}{2m}\exp\left(-\frac{(\delta k_z)^2 (z - V_{As} t)^2}{2}\right) \\
&\quad\times [k_{z0}\cos(\mathrm{k}\cdot\mathrm{x} - \omega_0 t) - (\delta k_z)^2 \\
&\quad\times (z - V_{As} t)\sin(\mathrm{k}\cdot\mathrm{x} - \omega_0 t)],
\end{aligned}
\tag{A7}
$$

where $V_{As} = V_A(1 + k_{\perp 0}^2 \rho^2)^{1/2}$. For finite t, we may approximately obtain $G(t)$ by

$$
\begin{aligned}
G(t) &= \frac{\sqrt{\pi}}{2}\frac{e\mathrm{A}_1}{m}\exp\left(-\frac{(\delta k_z)^2 (z - V_{As} t)^2}{2}\right) \\
&\quad\times k_{z0} V_{Am}\cos(\mathrm{k}\cdot\mathrm{x} - \omega_0 t).
\end{aligned}
\tag{A8}
$$

Upon making use of the Bessel function property, $e^{iz\cos\theta} = \sum_n i^n J_n(z) e^{\mathrm{i}n\theta}$ (known as the Jacobi–Anger expansion) and defining $y(t) = \rho\sin(\Omega_0 t + \phi_0) + y_0$, Equation (A8) may be rewritten as

$$
\begin{aligned}
G(t) &= \frac{\sqrt{\pi}}{2}\frac{e\mathrm{A}_1}{m} k_{z0} V_{Am}\exp\left(-\frac{(\delta k_{z0})^2 (z - V_{As} t)^2}{2}\right) \\
&\quad\times \sum_n J_n(k_{\perp 0}\rho)\cos[n\phi_0 + k_{\perp 0} y_0 \\
&\quad + k_{z0} z_0 + (k_{z0} v_{z0} - \omega_0 + n\Omega_0) t].
\end{aligned}
\tag{A9}
$$

Upon defining $t = \xi + \xi_0$, where $\xi_0 = -z_0/(v_{z0} - V_{As})$, the function $G(\xi)$ expressed in terms of transformed time variable ξ is given by its derivative,

$$
\begin{aligned}
F(\xi) \equiv \frac{dG(\xi)}{d\xi} &= -\frac{\sqrt{\pi}}{2}\frac{e\mathrm{A}_1}{m} k_{z0} V_{Am} e^{-\delta\omega^2\xi^2} \\
&\quad\times \sum_n J_n(k_{\perp 0}\rho)\beta_n \sin(\beta_n\xi + \phi_n),
\end{aligned}
\tag{A10}
$$

where $\beta_n = k_{z0} v_{z0} - \omega_0 + n\Omega_0$, $\phi_n = n\phi_0 + k_{\perp 0} y_0 - n\Omega_0 z_0/(v_{z0} - V_{As})$, and $\delta\omega^2 = \delta k_z^2 (v_{z0} - V_{As})^2/2$. Returning to the force balance equation, we have

$$
\ddot{v}_y + \Omega_0^2 v_y = \sum_n E_n e^{-\delta\omega^2\xi^2}\sin(\beta_n\xi + \phi_n) = F(\xi),
\tag{A11}
$$

where

$$
E_n(\xi) = -\frac{\sqrt{\pi}}{2}\frac{e\mathrm{A}_1}{m} k_{z0} V_{Am} J_n(k_{\perp 0}\rho)\beta_n.
\tag{A12}
$$

The solution to Equation (A11) is represented as

$$
v_y(\xi) = v_{0y}(\xi) + \sum_n v_{1yn}(\xi),
\tag{A13}
$$

where the particular solution v_{1yn} is given by

$$
\begin{aligned}
v_{1yn}(\xi) &= \frac{iE_n e^{-\xi^2\delta\omega^2}}{4(\delta\omega)\Omega_0}\Bigg\{ e^{-i(\beta_n\xi + \phi_n)} \\
&\quad \Bigg[D_F\left(\frac{\Omega_0 + \beta_n}{2(\delta\omega)} - i(\delta\omega)\xi\right) \\
&\quad + D_F\left(\frac{\Omega_0 - \beta_n}{2(\delta\omega)} + i(\delta\omega)\xi\right)\Bigg] - c.c.\Bigg\}.
\end{aligned}
\tag{A14}
$$

Here, $D_F(Z)$ is defined as

$$
\begin{aligned}
D_F(Z) &= \frac{1}{2}\int_0^\infty e^{-\frac{t^2}{4}}\sin(Zt) dt \\
&= \frac{\sqrt{\pi}}{2i} e^{-Z^2}\,\mathrm{erf}(iZ),
\end{aligned}
\tag{A15}
$$

and we have made use of the property $D_F(Z^*) = D_F^*(Z)$. Note that $D_F(Z)$ is also known as the Dawson function (or Dawson's integral).

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