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ABSTRACT

The properties of the collective subluminal electrostatic fluctuations in isotropic plasmas are investigated using the covariant kinetic theory of linear fluctuations based on the correct momentum–velocity relation. The covariant theory correctly accounts for the differences in subluminal and superluminal fluctuations in contrast to the non-covariant theory. The general formalism developed here is valid in unmagnetized plasmas and in magnetized plasmas for wavevectors of electrostatic waves parallel to the direction of the uniform magnetic field. Of particular interest are potential differences between the covariant and the non-covariant approach and the consequences of these differences in modifying observational predictions. For thermal particle distributions of protons and electrons with nonrelativistic equal temperatures, the covariant and non-covariant theories yield exactly the same dispersion function and relation for weakly damped electrostatic waves. Also, the quasi-equilibrium wavenumber spectrum of collective thermal electrostatic noise agrees in both theories apart from the important wavenumber restriction $|k| > k_c = \omega_{p,e}/c$. While the non-covariant analysis also yields eigenmode fluctuations at small wavenumbers with superluminal phase speeds, the correct covariant analysis indicates that subluminal electrostatic fluctuations are only generated at wavenumbers $|k| > k_c$ by spontaneous emission of the plasma particles. As a consequence, the nonrelativistic thermal electrostatic noise wavenumber spectrum is limited to the wavenumber range $\omega_{p,e} \le |k| \le k_{\max}$. Within a linear fluctuation theory, superluminal electrostatic noise cannot be generated.

I. INTRODUCTION

During the last few years, tremendous progress with the analysis of small-amplitude fluctuations in space plasmas has been achieved (for a detailed review see the monograph[1]). The system of the linearized and Fourier-Laplace transformed Klimontovich and Maxwell equations provides general expressions for the electromagnetic fluctuation spectra (electric and magnetic field, charge and current densities) from uncorrelated plasma particles in unmagnetized[2] and magnetized[3–5] plasmas for arbitrary complex frequencies $\omega = \omega_R + \imath\Gamma$.

Using the correct momentum–velocity relation from the special theory of relativity,

$$\vec{p} = m_a E \vec{v},\ E = \sqrt{1 + (p^2/m_a^2 c^2)}, \tag{1}$$

between particle momentum $\vec{p}$ and particle velocity $\vec{v}$ with the particle Lorentz factor E, the theory accounts for the difference in subluminal [with phase velocities (ω_R/k) less than the speed of light] and superluminal [with phase velocities (ω_R/k) greater than the speed of light] fluctuations. Hereafter, we refer to this approach as the *covariant* approach as we use the correct velocity–momentum relation from the special theory of relativity.

This difference is missing when nonrelativistic kinetic equations for the particles, with $E = 1$ in Eq. (1), are used—hereafter referred to as the *non-covariant* approach—as in existing studies of electrostatic fluctuations in the literature (see Refs. 6 and 7 and references therein). It is the purpose of the present manuscript to investigate the properties of subluminal electrostatic eigenmode fluctuations with the correct momentum–velocity relation. In particular, we are interested in establishing any differences between the covariant and the non-covariant approach, treated separately in Appendix, and how these differences quantitatively modify observational predictions.

The analysis presented here will be useful for further studies of this series addressing the generation of electrostatic noise in the interplanetary medium by velocity anisotropic strahl plasma distributions injected into an isotropic thermal background plasma. The recent observations from the Helios and Parker Solar Probe missions[8,9] are showing that the parallel component of the electron strahl temperature is independent of radial distance from the Sun at values of about 100 eV. Even in the absence of additional mechanisms of electron acceleration, such high temperature population will exhibit a small fraction of electrons moving at ~5 percent of the speed of light. Although the signal of electrostatic fluctuations observed by the Parker Solar Probe so far does not appear to be crucially affected by relativistic effects,[10] the well-established theoretical results[11,12] claim that a very small fraction of fast suprathermal electrons could notably modify the observed power spectra in the vicinity of the Langmuir resonance and be potentially visible during future Parker Solar Probe encounters. It is important to emphasize that even for nonrelativistic particle velocities, a relativistic dispersion theory is necessary as the phase speeds of the electrostatic fluctuations can reach the speed of light at small wavenumbers.

The organization of this manuscript is as follows: in Sec. II, we start with the general covariant expressions for the form factor and the dispersion function for adopted gyrotropic particle distribution functions which determine the fluctuation spectrum of electrostatic waves. We then simplify these general expressions for isotropic distribution functions and the limit of weakly damped fluctuations. In Sec. III, we adopt thermal isotropic distribution functions to derive the dispersion functions and the weakly damped thermal electrostatic noise (TEN) for arbitrary values of the temperature. The general electrostatic eigenmode is investigated in Sec. IV by calculating the spontaneous emission coefficient of electrostatic fluctuations as input quantity for the kinetic equation for the intensity of collective electrostatic eigenmodes. Then for thermal plasmas, the weakly damped thermal electrostatic noise from the covariant fluctuation theory is determined again for arbitrary temperature values. In Secs. V and VI, we apply our results to the case of a thermal electron–proton plasma of equal nonrelativistic temperature and compare them with the corresponding non-covariant analysis treated in Appendix. We conclude with Sec. VII.

II. BASIC EQUATIONS

A. Electrostatic waves

The electric fluctuation spectrum of longitudinal electrostatic waves in unmagnetized plasma, and in magnetized plasmas with wave vectors parallel to the direction of the ordered magnetic field, is given by[2,13,14]

$$\langle \delta E^2\rangle(k,\omega)=\frac{K_L(k,z)}{|\Lambda_L(k,\omega)|^2}, \tag{2}$$

with the electrostatic form factor

$$K_L(k,z)=-\sum_a \frac{\omega_{p,a}^2 m_a}{2\pi^2|\omega|^2}\Im\left[\int_{-\infty}^{\infty}dp_\parallel\int_0^\infty dp_\perp \frac{p_\perp v_\parallel^2 F_a(p_\parallel,p_\perp)}{\omega-kv_\parallel}\right], \tag{3}$$

in terms of the plasma frequency $\omega_{p,a}^2=4\pi e^2 n_0/m_a$, $k=|\vec{k}|$, and the longitudinal dispersion function,

$$\Lambda_L(k,\omega)=1+\sum_a \frac{2\pi\omega_{p,a}^2}{\omega}\int_{-\infty}^{\infty}dp_\parallel p_\parallel\int_0^\infty dp_\perp \frac{p_\perp}{E(\omega-kv_\parallel)}\frac{\partial F_a}{\partial p_\parallel}, \tag{4}$$

holding both for any gyrotropic plasma particle distribution functions $F_a(p_\parallel,p_\perp)$ and arbitrary complex frequency $\omega=\omega_R+\imath\Gamma$ with $\omega_R=\Re\omega$ and $\Gamma=\Im\omega$.

The two-dimensional momentum integrals in Eqs. (3) and (4) cannot be done separately as the integrand contains the plasma particle Lorentz factor $E=\sqrt{1+((p_\parallel^2+p_\perp^2)/m_a^2c^2)}$ in $v_\parallel=p_\parallel/(m_aE)$. In order to decouple the two integrals, it is convenient[15] to transform to the new momentum variables of integration,

$$y=p_\parallel/(m_ac),\quad E=\sqrt{1+\frac{p_\parallel^2+p_\perp^2}{m_a^2c^2}}, \tag{5}$$

implying

$$\begin{aligned}\frac{\partial F_a(p_\parallel,p_\perp)}{\partial p_\parallel}&=(m_ac)^{-1}\left[\frac{\partial F_a(E,y)}{\partial y}+\frac{y}{E}\frac{\partial F_a(E,y)}{\partial E}\right],\\ \frac{\partial F_a(p_\parallel,p_\perp)}{\partial p_\perp}&=(m_ac)^{-1}\frac{\sqrt{E^2-1-y^2}}{E}\frac{\partial F_a(E,y)}{\partial E}.\end{aligned} \tag{6}$$

The electrostatic form factor (3) and the longitudinal dispersion function (4) then become

$$\begin{aligned}K_L(k,z)=&\frac{1}{2\pi^2|z|^2(kc)^3}\sum_a \omega_{p,a}^2 m_ac^2(m_ac)^3\\ &\times\int_1^\infty dE\,\Im\left[\int_{-\sqrt{E^2-1}}^{\sqrt{E^2-1}}dy\frac{y^2F_a(E,y)}{y-Ez}\right]\end{aligned} \tag{7}$$

and

$$\begin{aligned}\Lambda_L(k,\omega)=&1-\frac{2\pi}{\omega kc}\sum_a\omega_{p,a}^2(m_ac)^3\\ &\times\int_1^\infty dE\,E\int_{-\sqrt{E^2-1}}^{\sqrt{E^2-1}}dy\frac{y}{y-\frac{E\omega}{kc}}\left[\frac{\partial F_a(E,y)}{\partial y}+\frac{y}{E}\frac{\partial F_a(E,y)}{\partial E}\right]\\ =&1-\frac{2\pi}{zk^2c^2}\sum_a\omega_{p,a}^2(m_ac)^{3\prime}\\ &\times\int_1^\infty dE\,E\int_{-\sqrt{E^2-1}}^{\sqrt{E^2-1}}dy\frac{y}{y-Ez}\left[\frac{\partial F_a(E,y)}{\partial y}+\frac{y}{E}\frac{\partial F_a(E,y)}{\partial E}\right]\end{aligned} \tag{8}$$

in terms of the complex phase speed

$$z=R+\imath I=\frac{\omega}{kc},\quad R=\frac{\omega_R}{kc},\quad I=\frac{\Gamma}{kc}. \tag{9}$$

B. Isotropic distribution functions

An important simplification of the analysis results if only isotropic particle distribution functions $F_a(E)$ with $\partial F_a(E)/\partial y = 0$ are considered. In this case, Eqs. (7) and (8) reduce to

$$K_L(k,z) = \sum_a \frac{\omega_{p,a}^2 m_a c^2 (m_a c)^3}{2\pi^2 |z|^2 (kc)^3} \int_1^\infty dE F_a(E) \Im[\Psi(E,z)] \tag{10}$$

and

$$\Lambda_L(k,z) = 1 - \frac{2\pi}{zk^2c^2} \sum_a \omega_{p,a}^2 (m_a c)^3 \int_1^\infty dE \frac{\partial F_a(E)}{\partial E} \Psi(E,z) \tag{11}$$

with the same integral

$$\begin{aligned}\Psi(E,z) &= \int_{-\sqrt{E^2-1}}^{\sqrt{E^2-1}} dy \frac{y^2}{y-Ez} = \int_{-\sqrt{E^2-1}}^{+\sqrt{E^2-1}} dy \frac{y^2 - E^2z^2 + E^2z^2}{y-Ez} \\ &= 2E\sqrt{E^2-1}z + E^2z^2 \int_{-\sqrt{E^2-1}}^{+\sqrt{E^2-1}} \frac{dy}{y-Ez} \\ &= 2E\sqrt{E^2-1}z - E^2z^2 J(E,z)\end{aligned} \tag{12}$$

involving the complex logarithmic function

$$J(E,z) = J(E,R,I) = -\int_{-\sqrt{1-E^{-2}}}^{\sqrt{1-E^{-2}}} \frac{dt}{t-z} = -\int_{-\sqrt{1-E^{-2}}}^{\sqrt{1-E^{-2}}} \frac{dt}{t-R-\imath I}. \tag{13}$$

Accounting for the correct analytical continuation in the negative complex frequency plane $I < 0$ provides[16]

$$\begin{aligned}J(E,R,I) = &\frac{1}{2}\ln \frac{\left(\sqrt{1-E^{-2}}+R\right)^2 + I^2}{\left(\sqrt{1-E^{-2}}-R\right)^2 + I^2} \\ &-\imath\Bigg(\arctan\frac{\sqrt{1-E^{-2}}-R}{I} + \arctan\frac{\sqrt{1-E^{-2}}+R}{I} \\ &+\pi\sigma\Theta[1-R]\Theta\left[E-(1-R^2)^{-1/2}\right]\Bigg),\end{aligned} \tag{14}$$

with the Heaviside step function Θ and $\sigma = 0,1,2$ for $I > 0, = 0, < 0$.

The integral (12) yields with

$$\Im[\Psi(E,z)] = 2E\sqrt{E^2-1}\Im[z] - E^2\Im\left[z^2 J(E,z)\right], \tag{15}$$

for the form factor (10)

$$\begin{aligned}K_L(k,z) = &\frac{\omega_{p,e}^2 m_e c^2}{\pi^2 |z|^2 (kc)^3} \sum_a (m_a c)^3 \Bigg(\Im[z] \int_1^\infty dE F_a(E) E\sqrt{E^2-1} \\ &-\frac{1}{2}\int_1^\infty dE F_a(E) E^2 \Im\left[z^2 J(E,z)\right]\Bigg),\end{aligned} \tag{16}$$

where we use $\omega_{p,a}^2 m_a = \omega_{p,e}^2 m_e$. Likewise, the isotropic dispersion function (11) becomes

$$\begin{aligned}\Lambda_L(k,z) = 1 - &\frac{2\pi}{k^2c^2} \sum_a \omega_{p,a}^2 (m_a c)^3 \Bigg[2\int_1^\infty dE\, E\sqrt{E^2-1} \frac{\partial F_a(E)}{\partial E} \\ &-z\int_1^\infty dE\, E^2 \frac{\partial F_a(E)}{\partial E} J(E,z)\Bigg].\end{aligned} \tag{17}$$

C. Weakly damped fluctuations

For weakly damped fluctuations with $|I| \ll |R|$, which needs to be checked *a posteriori* (done here in Sec. VI D), the complex logarithmic function (14) is approximated by its limit $I \to 0^-$ leading to

$$J(E,R) = J_0(E,R) - \imath\pi\Theta(E - E_c(R))\Theta(1-R^2), \tag{18}$$

with its real part

$$J_0(E,R) = \ln\left|\frac{\sqrt{1-E^{-2}}+R}{\sqrt{1-E^{-2}}-R}\right| = \frac{1}{2}\ln\frac{\left(R+\sqrt{1-E^{-2}}\right)^2}{\left(R-\sqrt{1-E^{-2}}\right)^2} \tag{19}$$

and

$$E_c(R) = \frac{1}{\sqrt{1-R^2}}. \tag{20}$$

We note that the imaginary part in the function (18) only occurs for subluminal ($R^2 < 1$) fluctuations.

The form factor (16) for weakly damped fluctuations then is given by

$$\begin{aligned}K_L(k,R) &\simeq K_L(k,R,I=0) \\ &= \frac{\omega_{p,e}^2 m_e c^2 \Theta(1-R^2)}{2\pi (kc)^3} \sum_a (m_a c)^3 \int_{\frac{1}{\sqrt{1-R^2}}}^\infty dE F_a(E) E^2,\end{aligned} \tag{21}$$

whereas the dispersion relation (17) becomes

$$\begin{aligned}\Lambda_L(k,R) &\simeq \Lambda_L(k,R,I=0) \\ &= 1 - \frac{2\pi}{k^2c^2}\sum_a \omega_{p,a}^2 (m_a c)^3 \\ &\times \Bigg[2\int_1^\infty dE\, E\sqrt{E^2-1}\frac{\partial F_a(E)}{\partial E} \\ &-R\int_1^\infty dE\, E^2 \frac{\partial F_a(E)}{\partial E} J_0(E,R) + \imath\pi R\Theta(1-R^2) \\ &\times \int_{\frac{1}{\sqrt{1-R^2}}}^\infty dE\, E^2 \frac{\partial F_a(E)}{\partial E}\Bigg].\end{aligned} \tag{22}$$

The eigenmode dispersion relation of weakly damped electrostatic oscillations is then given by

$$\begin{aligned}&\Re\Lambda_L(k,R,I=0) = \Lambda(k,R) = 0, \\ \Lambda(k,R) = 1 - &\frac{2\pi}{k^2c^2}\sum_a \omega_{p,a}^2 (m_a c)^3 \\ &\times\left[2\int_1^\infty dE\, E\sqrt{E^2-1}\frac{\partial F_a(E)}{\partial E} - R\int_1^\infty dE\, E^2\frac{\partial F_a(E)}{\partial E} J_0(E,R)\right],\end{aligned} \tag{23}$$

whereas the damping rate is obtained by the standard Taylor expansion[17] of the dispersion function near $I = 0$ as

$$\begin{aligned}I(k) &= -\frac{\Im\Lambda_L(k,R,I=0)}{\frac{\partial\Re\Lambda_L(k,R,I=0)}{\partial R}} \\ &= \frac{2\pi^2 R\Theta(1-R^2)}{k^2c^2 \frac{\partial\Lambda(k,R)}{\partial R}} \sum_a \omega_{p,a}^2 (m_a c)^3 \int_{\frac{1}{\sqrt{1-R^2}}}^\infty dE\, E^2 \frac{\partial F_a(E)}{\partial E}.\end{aligned} \tag{24}$$

Both the form factor (21) and the damping rate (24) vanish for superluminal fluctuations.

III. THERMAL ISOTROPIC DISTRIBUTION FUNCTION

For the thermal distribution function

$$F_a(E) = \frac{\mu_a e^{-\mu_a E}}{4\pi(m_a c)^3 K_2(\mu_a)} \tag{25}$$

with

$$\mu_a = \frac{m_a c^2}{k_B T_a} \tag{26}$$

and K_2 denoting the modified Bessel function of order 2, we notice that

$$\frac{\partial F_a(E)}{\partial E} = -\mu_a F_a(E). \tag{27}$$

The isotropic form factor (16) then is given by

$$K_L(k,z) = \frac{\omega_{p,e}^2 m_e c^2}{4\pi^3 |z|^2 (kc)^3} \sum_a \frac{\mu_a}{K_2(\mu_a)} \left(\Im[z] \int_1^\infty dE E\sqrt{E^2-1} e^{-\mu_a E} - \frac{1}{2}\int_1^\infty dE\, E^2 e^{-\mu_a E} \Im\left[z^2 J(E.z)\right] \right). \tag{38}$$

With the integral

$$\int_1^\infty dE\, E\sqrt{E^2-1} e^{-\mu E} = -\frac{\partial}{\partial \mu}\int_1^\infty dE\, \sqrt{E^2-1} e^{-\mu E} = -\frac{\partial}{\partial \mu}\left[\frac{K_1(\mu)}{\mu}\right] = \frac{K_2(\mu)}{\mu}, \tag{29}$$

the thermal form factor (28) reduces to

$$\begin{aligned} K_L(k,z) &= \frac{\omega_{p,e}^2 m_e c^2}{4\pi^3 |z|^2 (kc)^3} \left(\Im[z] - \sum_a \frac{\mu_a}{2K_2(\mu_a)} \times \int_1^\infty dE E^2 e^{-\mu_a E} \Im\left[z^2 J(E.z)\right] \right) \\ &= \frac{\omega_{p,e}^2 m_e c^2}{4\pi^3 |z|^2 (kc)^3} \left(\Im[z] + \sum_a \frac{\mu_a}{2K_2(\mu_a)} \times \int_1^\infty dE\, \frac{\partial w_2(\mu_a,E)}{\partial E} \Im\left[z^2 J(E.z)\right] \right), \end{aligned} \tag{30}$$

where we introduce the function

$$w_2(\mu_a, E) = \int_E^\infty dx\, x^2 e^{-\mu_a x} = e^{-\mu_a E}\left[\frac{E^2}{\mu_a} + \frac{2E}{\mu_a^2} + \frac{2}{\mu_a^3}\right], \tag{31}$$

which obviously obeys

$$\frac{\partial w_2(\mu_a,E)}{\partial E} = -E^2 e^{-\mu_a E}. \tag{32}$$

Likewise, with Eqs. (25) and (27) the dispersion function (17) reduces to

$$\Lambda_L(k,z) = 1 + \frac{1}{k^2c^2}\sum_a \frac{\omega_{p,a}^2 \mu_a^2}{K_2(\mu_a)} \left[\int_1^\infty dE\, E\sqrt{E^2-1} e^{-\mu_a E} - \frac{z}{2}\int_1^\infty dE\, E^2 e^{-\mu_a E} J(E,z) \right]. \tag{33}$$

With the integral (29) and the function (31) we obtain for the thermal dispersion function (33)

$$\begin{aligned} \Lambda_L(k,z) &= 1 + \sum_a \frac{\omega_{p,a}^2 \mu_a}{k^2c^2} L(z,\mu_a), \\ L(z,\mu_a) &= 1 - \frac{\mu_a z}{2K_2(\mu_a)} \int_1^\infty dE E^2 e^{-\mu_a E} J(E,z) \\ &= 1 + \frac{\mu_a z}{2K_2(\mu_a)} \int_1^\infty dE\, \frac{\partial w_2(\mu_a,E)}{\partial E} J(E,z). \end{aligned} \tag{34}$$

A. Weakly damped fluctuations

For weakly damped fluctuations with $|I| \ll |R|$ the thermal electrostatic form factor (30) reduces to

$$\begin{aligned} K_L(k,R) &\simeq K_L(k,R,I=0) \\ &= \frac{\omega_{p,e}^2 m_e c^2}{(2\pi)^3 R^2 (kc)^3} \sum_a \frac{\mu_a}{K_2(\mu_a)} \int_1^\infty dE\, \frac{\partial w_2(\mu_a,E)}{\partial E} \Im\left[R^2 J(E.R)\right] \\ &= -\frac{\pi \omega_{p,e}^2 m_e c^2 \Theta(1-R^2)}{(2\pi)^3 (kc)^3} \sum_a \frac{\mu_a}{K_2(\mu_a)} \int_{E_c(R)}^\infty dE\, \frac{\partial w_2(\mu_a,E)}{\partial E} \\ &= \frac{\omega_{p,e}^2 m_e c^2 \Theta(1-R^2)}{8\pi^2 (kc)^3} \sum_a \frac{\mu_a w_2\left(\mu_a, \frac{1}{\sqrt{1-R^2}}\right)}{K_2(\mu_a)}, \end{aligned} \tag{35}$$

where we used the imaginary part of the function (18).

Likewise, inserting the function (18) the thermal dispersion function (34) becomes

$$\begin{aligned} \Lambda_L(k,R,I=0) &= \Lambda(k,R) + \imath \frac{\pi R \Theta(1-R^2)}{2k^2c^2} \sum_a \frac{\omega_{p,a}^2 \mu_a^2 w_2(\mu_a,E_c)}{K_2(\mu_a)}, \\ \Lambda(k,R) &= \Re\Lambda_L(k,R,I=0) = 1 + \sum_a \frac{\omega_{p,a}^2 \mu_a}{k^2c^2} L_0(R,\mu_a), \end{aligned} \tag{36}$$

with

$$L_0(R,\mu_a) = 1 + \frac{\mu_a R}{2K_2(\mu_a)} \int_1^\infty dE\, \frac{\partial w_2(\mu_a,E)}{\partial E} J_0(E,R). \tag{37}$$

The eigenmode dispersion relation of weakly damped electrostatic oscillations is given by

$$\Re\Lambda_L(k,R,I=0) = \Lambda(k,R) = 0, \tag{38}$$

whereas the associated damping rate is

$$I(k) = -\frac{\Im\Lambda(k,R,I=0)}{\frac{\partial \Re\Lambda(k,R,I=0)}{\partial R}} = -\frac{\pi R \Theta(1-R^2)}{2k^2c^2\left[\frac{\partial \Lambda(k,R)}{\partial R}\right]} \sum_a \frac{\omega_{p,a}^2 \mu_a^2 w_2(\mu_a,E_c)}{K_2(\mu_a)}. \tag{39}$$

B. Weakly damped thermal electrostatic noise

Consequently, according to Eq. (2) the weakly damped thermal electrostatic noise (TEN) from covariant fluctuation theory then is given by

$$\begin{aligned}\langle\delta E^2\rangle(k,R) &= \frac{K_L(k,R)}{|\Lambda(k,R)|^2} = \frac{\omega_{p,e}^2 m_e c^2\Theta(1-R^2)}{8\pi^2(kc)^3|\Lambda(k,R)|^2} \\ &\times \sum_a \frac{\mu_a w_2\left(\mu_a, \frac{1}{\sqrt{1-R^2}}\right)}{K_2(\mu_a)} \\ &= \frac{\omega_{p,e}^2 m_e c^2\Theta(1-R^2)}{8\pi^2(kc)^3|\Lambda(k,R)|^2}\sum_a \frac{e^{-\frac{\mu_a}{\sqrt{1-R^2}}}}{K_2(\mu_a)} \\ &\times \left[\frac{1}{1-R^2} + \frac{2}{\mu_a\sqrt{1-R^2}} + \frac{2}{\mu_a^2}\right], \end{aligned} \tag{40}$$

where we inserted Eqs. (35) and (31). The form (40) is particularly useful for non-collective fluctuations when the denominator $\Lambda(k,R)$ does not become close to zero. However, most of the fluctuation intensities occur for collective modes, defined by $\Lambda(k,R)=0$, where the earlier approach by Yoon *et al.*[18] has to be taken.

IV. ELECTROSTATIC EIGENMODE

According to Eq. (38), the electrostatic eigenmode obeys the dispersion relation

$$\Lambda(k,R) = \Re\Lambda_L(k,R) = 0, \tag{41}$$

providing $R=R(k)$ or $\omega_R = \omega_R(k)$. The fluctuating electric field is of the form

$$(\delta E)_{k,\omega} = (\delta E)_k \delta(\omega - \omega(k)), \tag{42}$$

yielding for the electrostatic wave intensity

$$\begin{aligned} S(k,t) &= \int d\omega \int d\omega' \,\langle(\delta E)_{k,\omega}(\delta E)^*_{k,\omega'}\rangle \exp\left[-\imath(\omega-\omega')t\right] \\ &= \langle\delta E^2\rangle_k e^{2\Gamma(k)t}, \end{aligned} \tag{43}$$

so that as a consequence of induced emission (for positive $\Gamma(k)$) or induced absorption [for negative $\Gamma(k)$]

$$\left[\frac{\partial S(k,t)}{\partial t}\right]_{\rm ind} = 2\Gamma(k)S(k,t). \tag{44}$$

A. Spontaneous emission coefficient

To account for spontaneous wave emission, we rewrite Eq. (2) as in Refs. 4 and 18

$$\Lambda\left(k, \omega + \frac{\imath}{2}\frac{\partial}{\partial t}\right)\langle\delta E^2\rangle(k,R) = \frac{K_L(k,R)}{\Lambda^*(k,\omega)}, \tag{45}$$

where we have introduced the slow adiabatic time derivative $\partial/\partial t$ on the left-hand side.[14] Performing the Taylor expansion to first order of the left-hand side and of the denominator of the right-hand side of the last equation yields

$$\begin{aligned} &\Lambda(k,\omega(k))\langle\delta E^2\rangle(k,R) + \frac{\imath}{2}\frac{\partial\Lambda(k,\omega)}{\partial\omega}\frac{\partial}{\partial t}\langle\delta E^2\rangle(k,R) \\ &= \frac{K_L(k,R)}{\left[\Lambda(k,\omega(k)) + \frac{\partial\Lambda(k,\omega)}{\partial\omega}(\omega-\omega(k))\right]^*}. \end{aligned} \tag{46}$$

For eigenmodes according to Eq. (41), we use $\Lambda(k,\omega_R(k)) = \Lambda^*(k,\omega_R(k)) = 0$ leaving with Eqs. (42)–(43)

$$\frac{\imath}{2}\frac{\partial\Lambda(k,\omega)}{\partial\omega_R}\frac{\partial S(k,t)}{\partial t}\delta(\omega-\omega_R(k)) = \frac{K_L(k,R)}{\left[\frac{\partial\Lambda(k,\omega)}{\partial\omega}(\omega-\omega_R(k))\right]^*} \tag{47}$$

or

$$\frac{\partial S(k,t)}{\partial t}\delta(\omega-\omega_R(k)) = \frac{-2\imath K_L(k,R)}{\left|\frac{\partial\Lambda(k,\omega)}{\partial\omega}\right|^2(\omega-\omega_R(k))^*}. \tag{48}$$

Integrating Eq. (48) over all frequencies provides with the residuum theorem for the spontaneous emission coefficient of thermal electrostatic waves,

$$\alpha(k,t) = \left[\frac{\partial S(k,t)}{\partial t}\right]_{\rm spont} = \frac{4\pi K_L(k,R(k))}{\left|\frac{\partial\Lambda(k,\omega_R)}{\partial\omega_R}\right|^2} = \frac{4\pi k^2c^2 K_L(k,R(k))}{\left|\frac{\partial\Lambda(k,R)}{\partial R}\right|^2}. \tag{49}$$

B. Kinetic equation for the intensity of collective electrostatic eigenmodes

Combining spontaneous emission [from Eq. (49)] and induced emission/absorption [from Eq. (44)], the kinetic equation for the time evolution of electrostatic waves reads

$$\begin{aligned}\frac{\partial S(k,t)}{\partial t} &= \left[\frac{\partial S(k,t)}{\partial t}\right]_{\rm spont} + \left[\frac{\partial S(k,t)}{\partial t}\right]_{\rm ind} \\ &= \alpha(k,t) + 2\Gamma(k,t)S(k,t).\end{aligned} \tag{50}$$

In general, both the spontaneous emission coefficient $\alpha(k,t)$ and the growth rate $\Gamma(k,t)$ can depend on time.

For steady growth rates $\Gamma(k)$ and steady spontaneous emission coefficients $\alpha(k)$, Eq. (50) provides the solution

$$S(k,t) = S_0(k)e^{2\Gamma(k)(t-t_0)} + \frac{\alpha(k)}{2\Gamma(k)}\left[e^{2\Gamma(k)(t-t_0)} - 1\right], \tag{51}$$

using the initial condition $S(k,t_0) = S_0(k)$.

In isotropic unmagnetized plasmas (and in magnetized plasmas with parallel wave vectors), the imaginary part of the eigenmode frequency $\Gamma(k) < 0$ is always negative, so that no growing electrostatic eigenmode exists.[21] In nonrelativistic plasma kinetic theory this result is known as Newcomb-Gardner theorem.[19,20] In this case, the solution (51) for large times $t - t_0 \gg |2\Gamma(k)|^{-1}$ approaches the quasi-equilibrium spectrum,

$$S(k, t-t_0 \gg |2\Gamma(k)|^{-1}) \simeq S_\infty(k) = -\frac{\alpha(k)}{2\Gamma(k)} = -\frac{\alpha(k)}{2kcI(k)}$$
$$= -\frac{2\pi kcK_L(k, R(k))}{I(k)|\frac{\partial\Lambda(k,R)}{\partial R}|^2}$$
$$= \frac{2\pi kcK_L(k, R(k))}{\Im\Lambda_L(k, R)\frac{\partial\Lambda^*(k,R)}{\partial R}}, \tag{52}$$

where we inserted the spontaneous emission coefficient (49) and (24).

C. Thermal plasmas

In the case of thermal plasmas, we use Eqs. (35), (36), and (39) in Eq. (52) to obtain for the quasi-equilibrium spectrum of collective TEN,

$$S_\infty(k) = \frac{\omega_{p,e}^2 m_e c^2 \Theta(1-R^2)}{2\pi^2 R\frac{\partial\Lambda^*(k,R)}{\partial R}} \frac{\sum_a \frac{\mu_a w_2\left(\mu_a, \frac{1}{\sqrt{1-R^2}}\right)}{K_2(\mu_a)}}{\sum_a \frac{\omega_{p,a}^2\mu_a^2 w_2\left(\mu_a, \frac{1}{\sqrt{1-R^2}}\right)}{K_2(\mu_a)}}. \tag{53}$$

D. Lower wavenumber limit

The thermal form factor $K_L(k, R(k))$ given in Eq. (35), determining the spontaneous emission coefficient (49) and the quasi-equilibrium spectrum of collective TEN (53), has non-zero values only for subluminal fluctuations. It has been shown before[22] that subluminal solutions to the thermal electrostatic dispersion relation exist only for wavenumbers $|k| > k_c$ with the critical wavenumber

$$k_c(\mu_a) = \left[\sum_a \frac{\omega_{p,a}^2}{c^2}\frac{2K_0(\mu_a)+\mu_a K_1(\mu_a)}{\mu_a K_2(\mu_a)}\right]^{1/2}$$
$$\simeq \begin{cases} \sqrt{\sum_a \frac{\omega_{p,a}^2}{c^2}\left(1+\frac{1}{2\mu_a}\right)} & \text{for } \mu_a \gg 1, \\ \sqrt{\sum_a \frac{\omega_{p,a}^2\mu_a}{c^2}\left(\ln\frac{2}{\mu_a} - 0.0772\right)} & \text{for } \mu_a \ll 1. \end{cases} \tag{54}$$

Consequently, the longitudinal collective fluctuations from the thermal electrostatic eigenmode can only be generated by the spontaneous emission of thermal plasma particles at wavenumbers greater than k_c. Mathematically, the subluminality condition $\Theta(1-R^2)$ corresponds to $\Theta(|k|-k_c)$, so that the quasi-equilibrium spectrum (53) of collective TEN reads

$$S_\infty(k) = \frac{\omega_{p,e}^2 m_e c^2 \Theta(|k|-k_c)}{2\pi^2 R\frac{\partial\Lambda^*(k,R)}{\partial R}} \frac{\sum_a \frac{\mu_a w_2\left(\mu_a, \frac{1}{\sqrt{1-R^2}}\right)}{K_2(\mu_a)}}{\sum_a \frac{\omega_{p,a}^2\mu_a^2 w_2\left(\mu_a, \frac{1}{\sqrt{1-R^2}}\right)}{K_2(\mu_a)}}. \tag{55}$$

V. NONRELATIVISTIC ELECTRON–PROTON PLASMA

As application, we consider a thermal electron–proton plasma of equal nonrelativistic temperature $T_e = T_p = T_0$, so that $\mu_p = \mu_e/\xi = 1836\mu_e \gg 1$ with the electron–proton mass ratio $\xi = m_e/m_p = 1/1836$ and $\mu_e = m_e c^2/(k_B T_0) \gg 1$. Then the function (31) is well approximated as

$$w_2(\mu_a \gg 1, E_c) \simeq \frac{e^{-\frac{\mu_a}{\sqrt{1-R^2}}}}{\mu_a(1-R^2)}, \tag{56}$$

so that the ratio in Eq. (55) becomes

$$\frac{\sum_a \frac{\mu_a w_2\left(\mu_a, \frac{1}{\sqrt{1-R^2}}\right)}{K_2(\mu_a)}}{\sum_a \frac{\omega_{p,a}^2\mu_a^2 w_2\left(\mu_a, \frac{1}{\sqrt{1-R^2}}\right)}{K_2(\mu_a)}} \simeq \frac{1}{\omega_{p,e}^2\mu_e}. \tag{57}$$

With $k_c(\mu_a \gg 1) \simeq \omega_{p,e}/c$ from Eq. (54), the quasi-equilibrium fluctuation spectrum (55) for collective weakly damped fluctuations then becomes

$$S_\infty(k) \simeq \frac{m_e c^2\Theta(|k|-\omega_{p,e}/c)}{2\pi^2 R\mu_e\frac{\partial\Lambda^*(k,R)}{\partial R}} = \frac{k_B T_0\Theta(|k|-\omega_{p,e}/c)}{2\pi^2 R(k)\frac{\partial\Lambda(k,R)}{\partial R}}. \tag{58}$$

It remains to calculate the real part of the phase speed as a function of wavenumber $R(\kappa)$ from solving the dispersion relation (41) as well as $\partial\Lambda^*(k,R)/\partial R = \partial\Lambda(k,R)/\partial R$.

In their Sec. IV, Touil *et al.*[23] have demonstrated that for nonrelativistic plasma temperatures the longitudinal thermal dispersion function (34) is well approximated to the well-known textbook expression,

$$\Lambda_L(k, z) \simeq 1 - \sum_a \frac{\omega_{p,a}^2}{k^2c^2\beta_a^2} Z'\left(\frac{z}{\beta_a}\right), \tag{59}$$

in terms of the derivative of the Fried–Conte plasma dispersion function[24]

$$Z(x) = \pi^{-1/2}\int_{-\infty}^{\infty} dt\frac{e^{-t^2}}{t-x}, \tag{60}$$

and the normalized (in units of c) thermal velocity

$$\beta_a = \sqrt{\frac{2}{\mu_a}} = \sqrt{\frac{2k_B T_a}{m_a c^2}}, \quad \beta_p = \sqrt{\frac{m_e}{m_p}}\beta_e = \sqrt{\xi}\beta_e, \tag{61}$$

in an equal temperature plasma. We note that $\omega_{p,p}^2/\beta_p^2 = \omega_{p,e}^2/\beta_e^2$, so that Eq. (59) reads

$$\Lambda_L(k, z) \simeq 1 - \frac{\omega_{p,e}^2}{k^2c^2\beta_e^2}\sum_a Z'\left(\frac{z}{\beta_a}\right)$$
$$= 1 - \frac{1}{\beta_e^2\kappa^2}\left[Z'\left(\frac{z}{\beta_e}\right) + Z'\left(\frac{z}{\sqrt{\xi}\beta_e}\right)\right], \tag{62}$$

in terms of the normalized wavenumber

$$\kappa = \frac{kc}{\omega_{p,e}}. \tag{63}$$

The normalization (63) implies for the real part of the phase speed $R = f/\kappa$ with the normalized frequency $f = \omega_R/\omega_{p,e}$. In terms of the normalized wavenumber (63), the quasi-equilibrium fluctuation spectrum (58) for collective weakly damped fluctuations from covariant fluctuation theory reads

$$S_{\infty}(\kappa) = \frac{k_B T_0 \Theta(|\kappa|-1)}{2\pi^2 R(\kappa)\dfrac{\partial \Lambda(\kappa,R)}{\partial R}}. \tag{64}$$

For weakly damped fluctuations, Eq. (62) provides

$$\Lambda(\kappa,R) = \Re\Lambda(\kappa,R,I=0)$$
$$= 1 - \frac{1}{\beta_e^2\kappa^2}\left[\Re Z'\left(\frac{R}{\beta_e}\right) + \Re Z'\left(\frac{R}{\sqrt{\xi}\beta_e}\right)\right]. \tag{65}$$

Consequently,

$$\frac{\partial \Lambda(\kappa,R)}{\partial R} = -\frac{1}{\kappa^2\beta_e^3}\left[\Re Z''\left(\frac{R}{\beta_e}\right) + \frac{1}{\sqrt{\xi}}\Re Z''\left(\frac{R}{\sqrt{\xi}\beta_e}\right)\right]. \tag{66}$$

With the nonrelativistic temperature limits ($\mu_a \gg 1$) of the functions (31) and (A4), the damping rate $I(\kappa)$ from Eq. (39) becomes

$$I(\kappa) \simeq -\sqrt{\frac{\pi}{2}}\frac{R\mu_e^{3/2}\Theta(|\kappa|-1)}{(1-R^2)\kappa^2\left[\dfrac{\partial\Lambda(k,R)}{\partial R}\right]}\exp\left[-\mu_e\left(\frac{1}{\sqrt{1-R^2}}-1\right)\right]$$
$$= -\frac{2\sqrt{\pi}R\Theta(|\kappa|-1)}{(1-R^2)\beta_e^3\kappa^2\left[\dfrac{\partial\Lambda(k,R)}{\partial R}\right]}\exp\left[-\frac{2}{\beta_e^2}\left(\frac{1}{\sqrt{1-R^2}}-1\right)\right], \tag{67}$$

where we inserted $\mu_e = 2/\beta_e^2$.

A. Comparison with non-covariant analysis

Until here, all results are derived by using the correct covariant approach. In Appendix, we derive the corresponding results from the non-covariant approach based on the strictly nonrelativistic analysis. In comparison, we note:

(1). As Eq. (A16) agrees exactly with Eq. (62), we find that for nonrelativistic thermal plasmas covariant and non-covariant theories yield exactly the same dispersion function and dispersion relation for weakly damped electrostatic waves.

(2). Concerning the quasi-equilibrium spectrum of collective TEN, we note that the non-covariant spectrum (A25) agrees with the covariant spectrum (64) apart from the important wavenumber restriction $\Theta(|\kappa|-1)$. Whereas the non-covariant analysis yields eigenmode fluctuations also at small wavenumbers with superluminal phase speeds, the correct covariant analysis indicates that subluminal electrostatic fluctuations are only generated at wavenumbers $|\kappa| > 1$, corresponding to wavenumbers $|k| > k_c = \omega_{p,e}/c$ in a nonrelativistic thermal electron–proton plasma.

(3). Concerning the damping rates, we note that the non-covariant rate from Eq. (A22), listed here again as

$$I(\kappa) = -\frac{2\pi^{1/2} R e^{-\frac{R^2}{\beta_e^2}}}{\beta_e^3\kappa^2\dfrac{\partial\Re\Lambda(\kappa,R)}{\partial R}}, \tag{68}$$

agrees rather well with the covariant damping rate (67) apart from the important wavenumber restriction $\Theta(|\kappa|-1)$. Only for values of $|R|$ close to unity, the covariant damping rate differs from the non-covariant rate.

B. Solution of the nonrelativistic dispersion relation

As noted, the covariant and non-covariant approaches lead to the same nonrelativistic dispersion relation $\Lambda(\kappa,R)=0$, which according to Eq. (62) yields

$$(\beta_e\kappa)^2 = (k\lambda_D)^2 = \Re Z'\left(\frac{R}{\beta_e}\right) + \Re Z'\left(\frac{R}{\sqrt{\xi}\beta_e}\right), \tag{69}$$

where we introduce the Debye length

$$\lambda_D = \frac{v_{\mathrm{th,e}}}{\omega_{p,e}} = \frac{\beta_e c}{\omega_{p,e}} = \sqrt{\frac{k_B T_e}{4\pi e^2 n_e}} = 9.8\sqrt{\frac{T_e/\mathrm{K}}{n_e/\mathrm{cm}^{-3}}}\ \mathrm{cm}. \tag{70}$$

In terms of the variable

$$x = \frac{R}{\beta_e} = \frac{\omega_R}{k v_{\mathrm{th,e}}}, \tag{71}$$

the dispersion relation (69) can be written in the form

$$(k\lambda_D)^2 = (\beta_e\kappa)^2 = \Re Z'(x) + \Re Z'(43x). \tag{72}$$

With the numerically calculated Fried–Conte plasma dispersion function, we plot the dispersion relation (72) in Fig. 1. As the wavenumber k and the normalized phase speed x are real, solutions of the dispersion relation (72) are only possible if the right-hand side of Eq. (72) is positive. According to Fig. 1, this occurs for $x > 0.924$, corresponding to $|R| > 0.924\beta_e$ or

$$|\omega_R| > 0.924|k|v_{\mathrm{th,e}}. \tag{73}$$

Moreover, Fig. 1 indicates that the right-hand side of Eq. (72) is always smaller than 0.56. Consequently, collective longitudinal waves in an

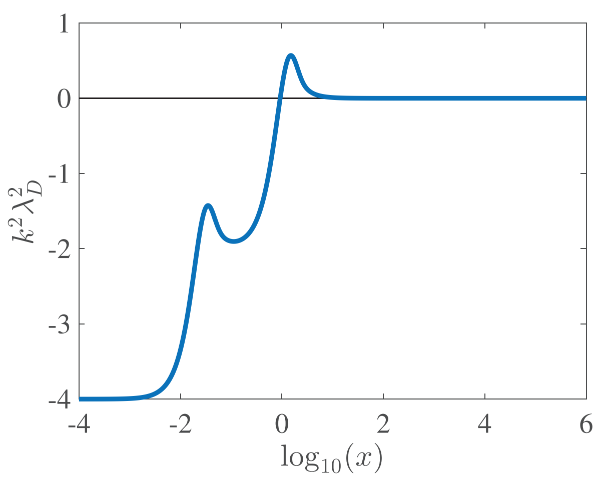

FIG. 1. Numerically solved dispersion relation (72) of longitudinal waves in an equal-temperature thermal electron–proton plasma with nonrelativistic temperatures.

equal-temperature thermal nonrelativistic electron–proton plasma only exist at wavenumbers between

$$0 \leq |k\lambda_D| = |\kappa\beta_e| \leq \sqrt{0.56} = 0.75, \tag{74}$$

indicating the existence of the maximum wavenumber

$$\kappa_{\max} = \frac{0.75}{\beta_e}, \quad k_{\max} = \frac{0.75}{\lambda_e}. \tag{75}$$

We emphasize that the lower limit on $x > 0.928$ and the upper wavenumber limit $\kappa \leq \kappa_{\max}$ result from the property of the real part of the Fried–Conte plasma dispersion function $\Re Z'(x)$. We are not aware that the existence of these limits has been realized before.

As solutions to the dispersion relation are only possible for $x > 0.924$, we need an accurate polynomial approximation of the real part of the Fried–Conte plasma dispersion function for these values of x, in order to investigate analytically the dispersion relation (72). These approximations are discussed next.

C. Fried–Conte plasma dispersion function

The Fried–Conte plasma dispersion function obeys the relation

$$1 + xZ(x) = -2Z'(x). \tag{76}$$

For the real part of the function $Z'(x)$, one finds in the literature[25] the polynomial approximations,

$$\Re Z'(x) \simeq \begin{cases} -2(1-2x^2) & \text{for } |x| \leq 1.03, \\ \frac{1}{x^2}\left(1 + \frac{3}{2x^2}\right) & \text{for } |x| \geq 1.03. \end{cases} \tag{77}$$

In Fig. 2, the numerically calculated and the standard polynomial approximations of the real part of $\Re Z'(x)$ are shown. While the polynomial approximations are very accurate at very small and very large values of the real argument x, they are too inaccurate for values of x near unity.

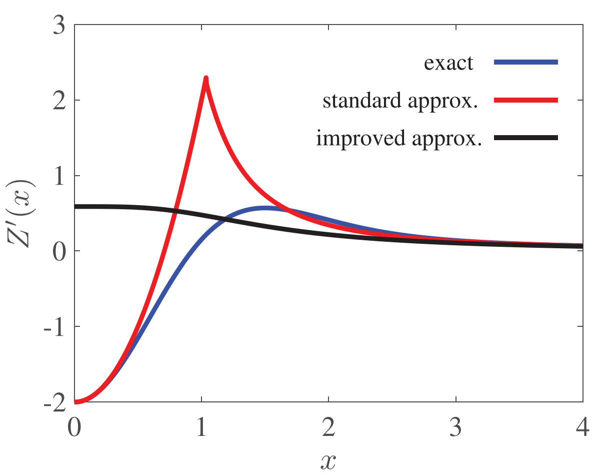

FIG. 2. Comparison of the numerically calculated (blue curve) real part of $\Re Z'(x)$ with the standard polynomial approximations (77) (red curves) and the improved polynomial approximations (78) (black curve) for real arguments x. As only values of $x > 0.924$ are of interest the improved approximation is significantly better than the standard approximation.

As we need good approximations for values of $x > 0.924$, we use instead of Eq. (77) the improved approximation,

$$\Re Z'(x) \simeq \frac{3 + 2x^2}{5.1 + 3.4x^2 + 2x^4}, \tag{78}$$

shown as black curve in Fig. 2. This approximation is indeed more accurate for values of $x > 0.924$. The improved approximation (78) implies

$$\Re Z''(x) \simeq -\frac{8x^3(x^2+3)}{(5.1 + 3.4x^2 + 2x^4)^2}. \tag{79}$$

VI. RESULTS

A. Subluminal dispersion relation

With the improved approximation (78), we obtain for the dispersion relation (72),

$$\begin{aligned}(\kappa\beta_e)^2 &= \frac{3 + 2x^2}{5.1 + 3.4x^2 + 2x^4} + \frac{3 + 2(43x)^2}{5.1 + 3.4(43x)^2 + 2(43x)^4} \\ &\simeq \frac{3 + 2x^2}{5.1 + 3.4x^2 + 2x^4} + \frac{\xi}{2x^2} \simeq \frac{3 + 2x^2}{5.1 + 3.4x^2 + 2x^4},\end{aligned} \tag{80}$$

where due to the smallness of $\xi = 1/1836$, we ignore the proton contribution completely. Setting

$$y = 2x^2, \ \alpha = (\kappa\beta_e)^2 \in [\beta_e^2, 0.56], \tag{81}$$

where the lower limit reflects the lower limit $|\kappa| = 1$ for subluminal electrostatic waves established in Sec. V A, the dispersion relation (80) reads

$$\alpha = \frac{2(3+y)}{y^2 + 3.4y + 10.2}, \tag{82}$$

with the solution

$$y = 2x^2 = \frac{1 - 1.7\alpha}{\alpha}\left[1 + \sqrt{1 + \frac{6\alpha}{1 - 1.7\alpha}}\right] \tag{83}$$

or

$$x^2 = \frac{R^2}{\beta_e^2} = \frac{1 - 1.7\alpha}{2\alpha}\left[1 + \sqrt{1 + \frac{6\alpha}{1 - 1.7\alpha}}\right]. \tag{84}$$

For small values of $\alpha \ll 1$, the solution (84) is well approximated by

$$x^2 \simeq \frac{1}{\alpha}\left[1 - 0.2\alpha - \frac{9}{4}\alpha^2\right] \tag{85}$$

or

$$R^2 \simeq \frac{1}{\kappa^2}\left[1 - 0.2(\beta_e\kappa)^2 - 2.25(\beta_e\kappa)^4\right]. \tag{86}$$

With the solution (85) of the dispersion relation inserted, we obtain for the earlier noted [see Eq. (73)] normalized phase speed restriction $x > 0.924$ the requirement

$$1-0.2\alpha-\frac{9}{4}\alpha^2>0.924^2\alpha=0.854\alpha \tag{87}$$

or

$$\alpha<0.472, \tag{88}$$

which is slightly stronger than the earlier found limit (81) that $\alpha<0.56$. Consequently, the subluminal solution (84) holds in the wavenumber range $1\le\kappa<0.687\beta_e^{-1}$. It corresponds to

$$f^2=\frac{\omega_R^2}{\omega_{p,e}^2}=R^2\kappa^2=\alpha x^2\simeq\frac{1-1.7\alpha}{2}\left[1+\sqrt{1+\frac{6\alpha}{1-1.7\alpha}}\right], \tag{89}$$

while the approximation (86) for small values of $\alpha\ll 1$ provides the well-known non-covariant dispersion relation,

$$f^2\simeq 1-0.2(\beta_e\kappa)^2-2.25(\beta_e\kappa)^4, \tag{90}$$

which apart from the first term differs significantly from the standard non-covariant dispersion relation,

$$f^2=1+\frac{3\beta_e^2\kappa^2}{2}. \tag{91}$$

In Fig. 3, we compare the improved dispersion relation (89) of subluminal electrostatic waves $f(\kappa)$ with its approximation (90). We find very good agreement of the approximation (88) with the Eq. (87) even at wavenumbers close to the maximum wavenumbers $\kappa_{\max}=0.687\beta_e^{-1}$. Both curves show a frequency decrease to the value $|f|=0.631$ for the maximum value $\beta_e\kappa_{\max}=0.687$.

B. Fluctuation spectrum

With the variable (71), we rewrite Eq. (66) as

$$\frac{\partial\Lambda(\kappa,R)}{\partial R}=-\frac{1}{\kappa^2\beta_e^3}\left[\Re Z''(x)+43\Re Z''(43x)\right]. \tag{92}$$

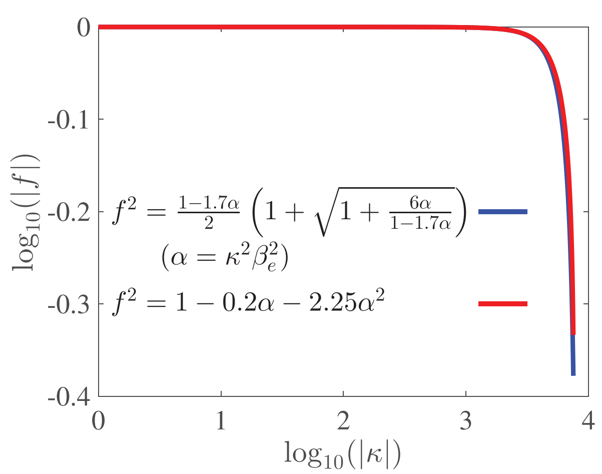

FIG. 3. Dispersion relation of subluminal electrostatic waves $f(\kappa)$ in an equal-temperature thermal electron–proton plasma with nonrelativistic temperatures ($\beta_e=10^{-4}$) calculated from Eq. (89) (blue curve) and its approximation (90) (red curve) with the improved polynomial approximation of the Fried–Conte plasma dispersion function.

The approximation (80) of the dispersion relation provides

$$5.1+3.4x^2+2x^4\simeq\frac{3+2x^2}{(\kappa\beta_e)^2}, \tag{93}$$

implying with Eq. (79)

$$\begin{aligned}\Re Z''(x)&\simeq-\frac{8x^3(x^2+3)(\kappa\beta_e)^4}{(3+2x^2)^2},\\ 43\Re Z''(43x)&\simeq-\frac{2x}{\xi}(\kappa\beta_e)^4=-3672x(\kappa\beta_e)^4.\end{aligned} \tag{94}$$

For values of $x>0.924$, the second contribution from protons in Eq. (94) is much greater than the first contribution from electrons. Consequently, Eq. (92) reduces to

$$\frac{\partial\Lambda(\kappa,R)}{\partial R}\simeq\frac{\Re Z''\left(\frac{x}{\sqrt{\xi}}\right)}{\sqrt{\xi}\kappa^2\beta_e^3}=\frac{43\Re Z''(43x)}{\kappa^2\beta_e^3}=3672x\beta_e\kappa^2=\frac{2}{\xi}R\kappa^2. \tag{95}$$

The quasi-equilibrium fluctuation spectrum (64) then becomes

$$\begin{aligned}S_\infty(\kappa)&=\frac{k_BT_0\Theta(|\kappa|-1)\Theta(\kappa_{\max}-|\kappa|)}{1836(2\pi)^2Rx\beta_e\kappa^2}\\ &=\frac{\xi k_BT_0\Theta(|\kappa|-1)\Theta(\kappa_{\max}-|\kappa|)}{(2\pi)^2\beta_e^2\kappa^2x^2}.\end{aligned} \tag{96}$$

Inserting the dispersion relation (85) then provides

$$S_\infty(\kappa)=\frac{\xi k_BT_0\Theta(|\kappa|-1)\Theta(0.687\beta_e^{-1}-|\kappa|)}{(2\pi)^2\left[1-0.2(\beta_e\kappa)^2-2.25(\beta_e\kappa)^4\right]}, \tag{97}$$

which is shown in Fig. 4. The fluctuation spectrum is basically flat and constant with a slight increase at large normalized wavenumbers.

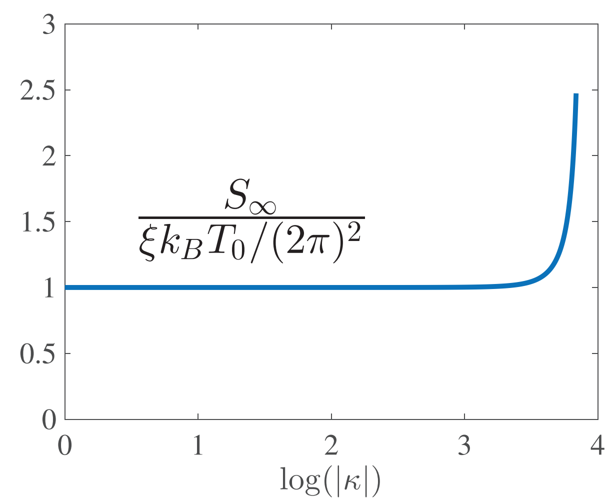

FIG. 4. Fluctuation wavenumber spectrum (97) of subluminal electrostatic waves in an equal-temperature thermal electron–proton plasma with nonrelativistic temperatures calculated for $\beta_e=10^{-4}$.

C. Total collective electrostatic electric fields

Integrating the spectra (97) over all wavenumbers, we obtain for the total collective longitudinal electric field in thermal nonrelativistic electron–proton plasmas,

$$\langle(\delta E)^2\rangle_{\rm tot} = \int d^3k S_\infty(k) = 4\pi\int_0^\infty dk\, k^2 S_\infty(k)$$
$$= 4\pi\left(\frac{\omega_{p,e}}{c}\right)^3\int_{\kappa_{\rm min}}^\infty d\kappa\,\kappa^2 S_\infty(\kappa). \quad (98)$$

The non-covariant theory has no lower limit $\kappa_{\rm min,non-cov} = 0$, whereas the covariant theory provides $\kappa_{\rm min,cov} = 1$. This difference in the lower limit of the covariant and non-covariant fluctuation theories applied to nonrelativistic thermal electron–proton plasmas, however, is not significant as we show next. We readily obtain

$$\langle(\delta E)^2\rangle_{\rm tot} = \frac{\xi k_B T_0}{\pi}\left(\frac{\omega_{p,e}}{\beta_e c}\right)^3\int_{\beta_e\kappa_{\rm min}}^{0.687} dy\,\frac{y^2}{1-0.2y^2-2.25y^4}$$
$$= \frac{\xi m_e c^2}{2\pi\beta_e}\left(\frac{\omega_{p,e}}{c}\right)^3 J(\beta_e\kappa_{\rm min}), \quad (99)$$

where we inserted $k_B T_0 = m_e c^2\beta_e^2/2$ and introduced the integral

$$J(Y) = \int_Y^{0.687} dy\,\frac{y^2}{1-0.2y^2-2.25y^4}$$
$$\simeq \int_Y^{0.687} dy\, y^2(1+0.2y^2)$$
$$= \left[\frac{y^3}{3}\left(1+\frac{3y^2}{25}\right)\right]_Y^{0.687} = 0.114 - \frac{Y^3}{3}\left(1+\frac{3Y^2}{25}\right). \quad (100)$$

The non-covariant lower limit then provides $J_{\rm noncov} = 0.114$ whereas the covariant lower limit with $\kappa_{\rm min,cov} = 1$ yields $J_{\rm cov} \simeq 0.114 - (\beta_e^3/3)$. The difference between the two is negligibly small as β_e^3 is of the order 10^{-11}. Using the non-covariant value $J_{\rm noncov} = 0.114$ then provides for the total collective longitudinal electric field (99)

$$\langle(\delta E)^2\rangle_{\rm tot} = \frac{0.114\xi m_e c^2}{2\pi\beta_e}\left(\frac{\omega_{p,e}}{c}\right)^3 = \frac{5.4\cdot 10^{-29} n_e^{3/2}}{(\beta_e/10^{-4})}\,\frac{\rm erg}{\rm cm^3}. \quad (101)$$

D. Damping rate and *a posteriori* consistency check

Equation (95) also enters the damping rates (67) and (68), providing

$$I(\kappa)_{\rm cov} = -\frac{\sqrt{\pi}\xi\Theta(|\kappa|-1)}{(1-R^2)(\beta_e\kappa)^4}\exp\left[-\frac{2}{\beta_e^2}\left(\frac{1}{\sqrt{1-R^2}}-1\right)\right] \quad (102)$$

and

$$I(\kappa)_{\rm non-cov} = -\frac{\pi^{1/2}\xi e^{-\frac{R^2}{\beta_e^2}}}{(\beta_e\kappa)^4}, \quad (103)$$

with the same R^2 taken from the dispersion relation (86), i.e.,

$$R^2 \simeq \frac{1}{\kappa^2}\left[1-0.2(\beta_e\kappa)^2 - 2.25(\beta_e\kappa)^4\right], \quad (104)$$

taken from the dispersion relation (84) and shown in Fig. 5. As can be seen, the values of the phase speed $|R|$ are considerably smaller than unity, so that the covariant and non-covariant damping rates divided by kc given in Eqs. (102) and (103), respectively, are practically equal and indistinguishable so that in Fig. 5 we only show one of them.

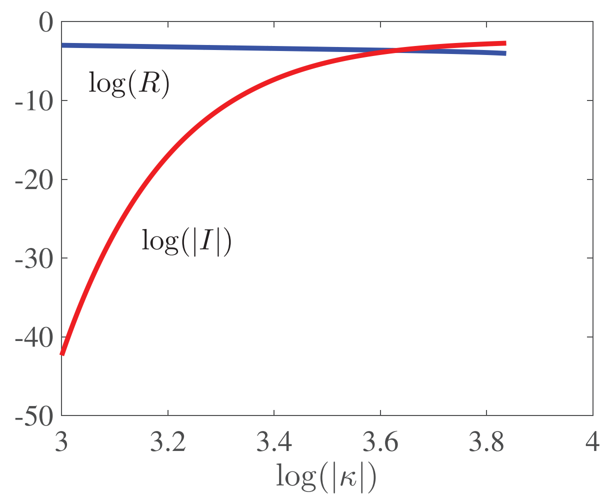

FIG. 5. Phase speed (blue curve) $|R|$ from Eq. (104) and damping rate $I(k) = -\Gamma/kc$ (red curve) of longitudinal waves as a function of the normalized wavenumber κ in an equal-temperature electron–proton plasma with nonrelativistic temperatures $\beta_e = 10^{-4}$. The intersection of the two curves occurs at $\kappa_c \simeq 4700$.

Moreover, the weak-damping condition is only fulfilled if $|I| \le |R|$, corresponding with Eqs. (103) and (104) to wavenumbers $|\kappa| \le \kappa_c$ where

$$\frac{\pi^{1/2}\xi e^{-\frac{1}{(\beta_e\kappa_c)^2}}}{(\beta_e\kappa_c)^4} \le \frac{1}{\kappa_c}, \quad (105)$$

yielding

$$\kappa_c \simeq 4.7\cdot 10^3(\beta_e/10^{-4})^{-1}, \quad (106)$$

in excellent agreement with Fig. 5. Only below the corresponding critical wavenumber $k_c = 0.47\omega_{p,e}/(c\beta_e)$ cm^{-1}, which is slightly smaller than the limits inferred before, our results for weakly damped longitudinal waves are valid.

VII. SUMMARY AND CONCLUSIONS

The properties of the collective subluminal electrostatic fluctuations in isotropic plasmas are investigated using the covariant kinetic theory of linear fluctuations based on the correct momentum–velocity relation. The covariant theory correctly accounts for the differences in subluminal and superluminal fluctuations in contrast to the non-covariant theory based on $\vec{p} = m_a\vec{v}$. The presented general formalism is valid in unmagnetized plasmas and in magnetized plasmas for wavevectors of electrostatic waves parallel to the direction of the uniform magnetic field. Of particular interest are potential differences between the covariant and the non-covariant approach and the consequences of these differences in modifying observational predictions.

As a first application, we consider thermal particle distributions of protons and electrons with nonrelativistic equal temperatures. We demonstrated that in this limit, covariant and non-covariant theories yield exactly the same dispersion function and relation for weakly damped electrostatic waves. Concerning the quasi-equilibrium spectrum of collective thermal electrostatic noise, the calculated non-covariant wavenumber spectrum agrees with the covariant wavenumber spectrum apart from the important wavenumber restriction $|k| > k_c = \omega_{p,e}/c$. While the non-covariant analysis also yields eigenmode fluctuations at small wavenumbers with superluminal phase speeds, the correct covariant analysis indicates that subluminal electrostatic fluctuations are only generated at wavenumbers $|k| > k_c$ by spontaneous emission of the plasma particles. As a consequence, the nonrelativistic thermal electrostatic noise wavenumber spectrum is limited to the wavenumber range $\omega_{p,e} \le |k| \le k_{\max}$. Within a linear fluctuation theory, superluminal electrostatic noise cannot be generated. Of course, this is modified if nonlinear effects, such as nonlinear Landau damping,[27–29] scatter subluminal waves to the superluminal regime.

However, when calculating the total strength of the collective electrostatic electric fields, the lower wavenumber limit has a negligibly small effect as the noise wavenumber spectrum is flat and contains most of its power at large wavenumbers near $k_{\max}$. The value of the maximum wavenumber $k_{\max}$ is set either by the requirement of the validity of the weak-damping approximation, and/or by a newly proposed improved, i.e., more accurate than the standard one, polynomial approximation of the real part of the derivative of the Fried–Conte plasma dispersion function. The strongest of these constraints provides $k_{\max} = 4.7\omega_{p,e}/(\beta_e c)$ cm^{-1} in terms of the electron plasma frequency and the thermal electron velocity $\beta_{e}c$.

Our results can be used to determine the intensity level of electrostatic noise in the solar wind plasma, quiet phases of the interstellar medium, and the intracluster gas in clusters of galaxies. The plasma particle distribution functions in these astrophysical objects are well represented by thermal Maxwellians with nonrelativistic plasma temperatures. Also, the equal temperature assumption for electrons and protons is reasonably well fulfilled; otherwise, it is straightforward to generalize our results to the case of different electron–proton plasma temperatures. Another future application is possible and planned where we will contrast our analytical formula with dedicated PIC simulations as was done before[30,31] in a pure electron and electron–positron plasma, respectively.

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APPENDIX: NONRELATIVISTIC ANALYSIS

In the strictly nonrelativistic analysis with $E = 1$ and $\vec{p} = m_a\vec{v}$, we obtain for Eqs. (3) and (4) with $F_a(\vec{p}) = f_a(\vec{v})/m_a^3$,

$$K_L(k,\omega) = -\frac{\omega_{p,e}^2 m_e}{2\pi^2|\omega|^2}\sum_a \Im\left[\int_{-\infty}^{\infty} dv_\parallel \int_0^\infty dv_\perp \frac{v_\perp v_\parallel^2 f_a(v_\parallel, v_\perp)}{\omega - kv_\parallel}\right] \tag{A1}$$

and

$$\Lambda_L(k,\omega) - 1 = \sum_a \frac{2\pi\omega_{p,a}^2}{\omega}\int_{-\infty}^{\infty} dv_\parallel \int_0^\infty dv_\perp \frac{v_\parallel v_\perp}{\omega - kv_\parallel}\frac{\partial f_a}{\partial v_\parallel}, \tag{A2}$$

respectively. Equation (A2) agrees exactly with Eq. (9) of Schlickeiser.[26]

For nonrelativistic temperatures $\mu_a \gg 1$, the thermal distribution function (25) reduces to

$$f_a(v) = m_a^3 F_a(E) \simeq \left(\frac{\mu_a}{2\pi c^2}\right)^{3/2} e^{-\mu_a(E-1)}, \tag{A3}$$

where we approximated

$$K_2(\mu_a \gg 1) \simeq \sqrt{\frac{\pi}{2\mu_a}} e^{-\mu_a}. \tag{A4}$$

With

$$E - 1 = \sqrt{1 + \left(\frac{p}{m_a c}\right)^2} - 1 \simeq \frac{p^2}{2m_a^2c^2} \tag{A5}$$

at nonrelativistic momenta we obtain for the distribution function (A3) with $p = m_a v$

$$f_a(v) \simeq \left(\frac{m_a}{2\pi k_B T_a}\right)^{3/2} e^{-\frac{p^2}{2m_a k_b T_a}} = \frac{e^{-\frac{v_\parallel^2 + v_\perp^2}{v_{\mathrm{th,a}}^2}}}{\pi^{3/2} v_{\mathrm{th,a}}^3}, \tag{A6}$$

with the thermal velocity

$$v_{\mathrm{th,a}} = \sqrt{\frac{2k_B T_a}{m_a}}. \tag{A7}$$

With the distribution function (A6), the nonrelativistic thermal form factor (A1) becomes

$$\begin{aligned} K_L(k,\omega) &\simeq -\frac{\omega_{p,e}^2 m_e}{4\pi^{7/2}|\omega|^2}\sum_a \frac{1}{v_{\mathrm{th,a}}}\Im\left[\int_{-\infty}^{\infty} dv_\parallel \frac{v_\parallel^2 e^{-\frac{v_\parallel^2}{v_{\mathrm{th,a}}^2}}}{\omega - kv_\parallel}\right] \\ &= \frac{\omega_{p,e}^2 m_e}{4\pi^{7/2}|\omega|^2 k}\sum_a v_{\mathrm{th,a}}\Im\left[\int_{-\infty}^{\infty} dy \frac{y^2 e^{-y^2}}{y - \frac{\omega}{k v_{\mathrm{th,a}}}}\right], \end{aligned} \tag{A8}$$

where we used

$$\int_0^\infty dv_\perp v_\perp e^{-\frac{v_\perp^2}{v_{\mathrm{th,a}}^2}} = \frac{v_{\mathrm{th,a}}^2}{2}, \tag{A9}$$

and substituted $v_\parallel = v_{\mathrm{th,a}} y$. The remaining integral is given by

$$\int_{-\infty}^{\infty} dy \frac{y^2}{y-\frac{\omega}{kv_{\rm th,a}}} e^{-y^2}$$
$$= \int_{-\infty}^{\infty} dy \frac{y^2 - \frac{\omega^2}{k^2 v_{\rm th,a}^2} + \frac{\omega^2}{k^2 v_{\rm th,a}^2}}{y - \frac{\omega}{kv_{\rm th,a}}} e^{-y^2}$$
$$= \int_{-\infty}^{\infty} dy \left(y + \frac{\omega}{kv_{\rm th,a}}\right) e^{-y^2} + \frac{\omega^2}{k^2 v_{\rm th,a}^2} \int_{-\infty}^{\infty} dy \frac{e^{-y^2}}{y - \frac{\omega}{kv_{\rm th,a}}}$$
$$= \frac{\omega}{kv_{\rm th,a}} \pi^{1/2} \left[1 + \frac{\omega}{kv_{\rm th,a}} Z\left(\frac{\omega}{kv_{\rm th,a}}\right)\right], \tag{A10}$$

in terms of the Fried–Conte plasma dispersion function,

$$Z(x) = \pi^{-1/2} \int_{-\infty}^{\infty} dy \frac{e^{-y^2}}{y-x}. \tag{A11}$$

Consequently, the nonrelativistic thermal form factor (A8) becomes

$$K_L(k,\omega) = \frac{\omega_{p,e}^2 m_e}{4\pi^3 |\omega|^2 k^2} \sum_a \Im\left\{\omega\left[1 + \frac{\omega}{kv_{\rm th,a}} Z\left(\frac{\omega}{kv_{\rm th,a}}\right)\right]\right\}$$
$$= -\frac{\omega_{p,e}^2 m_e}{(2\pi)^3 |\omega|^2 k^2} \sum_a \Im\left[\omega Z'\left(\frac{\omega}{kv_{\rm th,a}}\right)\right], \tag{A12}$$

as the plasma dispersion function satisfies

$$Z'(x) = -2[1 + xZ(x)]. \tag{A13}$$

Likewise, with the distribution function (A6) the nonrelativistic thermal dispersion function (A2) becomes

$$\Lambda_L(k,\omega) - 1 = \sum_a \frac{\omega_{p,a}^2}{\pi^{1/2} v_{\rm th,a} \omega} \int_{-\infty}^{\infty} dv_\parallel \frac{v_\parallel}{\omega - kv_\parallel} \frac{\partial}{\partial v_\parallel} e^{-\frac{v_\parallel^2}{v_{\rm th,a}^2}}$$
$$= \frac{2}{\pi^{1/2}} \sum_a \frac{\omega_{p,a}^2}{kv_{\rm th,a}\omega} \int_{-\infty}^{\infty} dy \frac{y^2 e^{-y^2}}{y - \frac{\omega}{kv_{\rm th,a}}}, \tag{A14}$$

where we use the integral (A9). With the integral (A10), we obtain

$$\Lambda_L(k,\omega) = 1 + \sum_a \frac{2\omega_{p,a}^2}{k^2 v_{\rm th,a}^2} \left[1 + \frac{\omega}{kv_{\rm th,a}} Z\left(\frac{\omega}{kv_{\rm th,a}}\right)\right]$$
$$= 1 - \sum_a \frac{\omega_{p,a}^2}{k^2 v_{\rm th,a}^2} Z'\left(\frac{\omega}{kv_{\rm th,a}}\right). \tag{A15}$$

1. Electron–proton plasma

For an equal temperature $T_e = T_p = T_0$ electron–proton plasma, the dispersion relation (A15) reads

$$\Lambda_L(k,z) = 1 - \frac{1}{\beta_e^2 \kappa^2} \left[Z'\left(\frac{z}{\beta_e}\right) + Z'\left(\frac{z}{\xi^{1/2}\beta_e}\right)\right], \tag{A16}$$

in terms of the complex phase speed (9) and the normalized wavenumber (63). Likewise the thermal form factor (A12) is given by

$$K_L(k,z) = -\frac{\omega_{p,e}^2 m_e c^2}{(2\pi)^3 |z|^2 (kc)^3} \Im\left\{z\left[Z'\left(\frac{z}{\beta_e}\right) + Z'\left(\frac{z}{\xi^{1/2}\beta_e}\right)\right]\right\}. \tag{A17}$$

2. Weakly damped fluctuations

For weakly damped fluctuations, the eigenmode dispersion relation then is given by

$$0 = \Lambda(k,R) = \Re\Lambda_L(k,R,I=0)$$
$$= 1 - \frac{1}{\beta_e^2\kappa^2}\left[Z'\left(\frac{R}{\beta_e}\right) + Z'\left(\frac{R}{\xi^{1/2}\beta_e}\right)\right], \tag{A18}$$

leading to the same dispersion relation (72), i.e.,

$$(k\lambda_D)^2 = \Re Z'(x) + \Re Z'(43x), \tag{A19}$$

resulting from using the covariant plasma equations in the nonrelativistic thermal limit.

For the damping rate, we use again the first Eq. (39) providing

$$I(\kappa) = -\frac{\Im\Lambda_L(\kappa,R,I=0)}{\frac{\partial\Re_L\Lambda(\kappa,R,I=0)}{\partial R}} = \frac{\Im\left[Z'\left(\frac{R}{\beta_e}\right) + Z'\left(\frac{R}{\xi^{1/2}\beta_e}\right)\right]}{(\beta_e\kappa)^2 \frac{\partial\Re\Lambda(\kappa,R,I=0)}{\partial R}}. \tag{A20}$$

With

$$\Im Z(x) = \pi^{1/2} e^{-x^2}, \quad \Im Z'(x) = -2\pi^{1/2} x e^{-x^2}, \tag{A21}$$

for $\Im(x) = 0$ the damping rate (A20) becomes

$$I(\kappa) \simeq -\frac{2\pi^{1/2} R e^{-\frac{R^2}{\beta_e^2}}}{\beta_e^3 \kappa^2 \frac{\partial\Re\Lambda(\kappa,R,I=0)}{\partial R}}, \tag{A22}$$

which differs considerably from the damping rate (67) derived from using the correct plasma equations. Most noteworthy is that the rate (A22) has no wavenumber restriction, holding for all values of κ.

From Eqs. (A16) and (A17), we obtain

$$K_L(k,R,I=0) = -\frac{\omega_{p,e}^2 m_e c^2}{(2\pi)^3 R (kc)^3} \Im\left[\left(Z'\left(\frac{R}{\beta_e}\right) + Z'\left(\frac{R}{\xi^{1/2}\beta_e}\right)\right],$$
$$\Im\Lambda_L(k,R) = -\frac{\Im\left[Z'\left(\frac{R}{\beta_e}\right) + Z'\left(\frac{R}{\xi^{1/2}\beta_e}\right)\right]}{\beta_e^2\kappa^2}, \tag{A23}$$

respectively, so that the ratio

$$\frac{2\pi kc K_L(k,R,I=0)}{\Im\Lambda_L(k,R)} = \frac{\omega_{p,e}^2 m_e c^2 (\beta_e\kappa)^2}{(2\pi)^2 R (kc)^2} = \frac{m_e c^2 \beta_e^2}{(2\pi)^2 R}, \tag{A24}$$

where we inserted $kc = \omega_{p,e}\kappa$ from Eq. (63). Consequently, we obtain for the quasi-equilibrium fluctuation spectrum (54)

$$S_\infty(k) = \frac{2\pi kcK_L(k,R,I=0)}{\Im\Lambda_L(\kappa,R,I=0)\dfrac{\partial\Re\Lambda^*(\kappa,R,I=0)}{\partial R}} = \frac{m_ec^2\beta_e^2}{(2\pi)^2R\dfrac{\partial\Re\Lambda^*(\kappa,R,I=0)}{\partial R}} = \frac{k_BT_0}{2\pi^2R\dfrac{\partial\Re\Lambda^*(\kappa,R,I=0)}{\partial R}}, \tag{A25}$$

where we use $m_ec^2\beta_e^2 = 2k_BT_0$. Apart from the missing wavenumber restriction, the spectrum (A25) agrees exactly with the spectrum (58) derived with the covariant analysis.

Moreover, Eq. (A16) provides

$$\frac{\partial\Re\Lambda^*(\kappa,R,I=0)}{\partial R} = \frac{\partial\Re\Lambda(\kappa,R,I=0)}{\partial R} = -\frac{1}{\beta_e^3\kappa^2}\Re\left[Z''\left(\frac{R}{\beta_e}\right) + \xi^{-1/2}Z''\left(\frac{R}{\xi^{1/2}\beta_e}\right)\right], \tag{A26}$$

so that

$$S_\infty(\kappa) = -\frac{\omega_{p,e}^2 m_ec^2\beta_e^3\kappa^2}{(2\pi)^2R\Re\left[Z''\left(\frac{R}{\beta_e}\right) + \xi^{-1/2}Z''\left(\frac{R}{\xi^{1/2}\beta_e}\right)\right]}. \tag{A27}$$

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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