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# Two-fluid approach to weak plasma turbulence

**Peter H Yoon**

IPST, University of Maryland, College Park, MD 20742-2431, United States of America

E-mail: yoonp@umd.edu

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**Abstract**

Weakly turbulent processes that take place in plasmas are customarily formulated in terms of kinetic theory. However, owing to an inherent complexity associated with the problem, thus far the theory is fully developed largely for unmagnetized plasmas. In the present paper it is shown that a warm two fluid theory can successfully be employed in order to partially formulate the weak turbulence theory in spatially uniform plasma. Specifically, it is shown that the nonlinear wave-wave interaction, or decay processes, can be reproduced by the two-fluid formalism. The present finding shows that the same approach can in principle be extended to magnetized plasmas, which is a subject of future work.

Keywords: fluid, plasma, weak turbulence, nonlinear

## 1. Introduction

The weak turbulence theory in plasma physics is a well established research tool that may be employed in numerous studies of nonlinear plasma processes. It was developed by early pioneers of modern plasma physics [1–11], and its further development continues to this day [12–19]. Recently, the present author published a monograph in which a systematic exposition of the subject is presented in a pedagogic manner [20].

One of the most successful applications of the weak turbulence theory relates to the electrostatic turbulence involving Langmuir and ion acoustic waves and the ensuing wave-particle interaction that leads to the formation of suprathermal electron distribution function [21, 22]. Electron and ion velocity distribution functions measured in space since the early days of space research have consistently demonstrated that the space plasma is in a state of non-thermal quasi-equilibrium characterized by quasi inverse power-law velocity tail population [23–26]. Such a feature is again verified by more recent satellite missions [27, 28]. In order to fit the observation, an empirical model known as the kappa distribution was put forth in [23]. Later, however, the conceptual basis for the kappa distribution in the context of non extensive thermo-statical theory [29–31] became available. The kappa model of non thermal distribution enjoys an alternative theoretical justification, which is based upon the plasma weak turbulence theory. It was shown by the present author that the asymptotically steady state Langmuir turbulence leads to the formation of electron kappa distribution [21, 22].

Another useful application of the weak turbulence theory is on the radio emission by partial conversion of electrostatic turbulence to transverse electromagnetic wave, which is known as the plasma emission [32]. The plasma emission is a fundamental mechanism that is responsible for the solar radio bursts, known as the types II and III radio bursts [33]. Type III radio bursts, in particular, are radio emission that emanates from the solar active region. During the solar flare, energetic electron beam emerges from the active region, which gradually propagates out into the interplanetary space. As the electron beam interacts with the background plasma, the beam-plasma instability sets in, and Langmuir waves are excited. The partial conversion of Langmuir waves through nonlinear wave-wave interaction leads to the said plasma emission [34].

The standard weak turbulence theory has successfully addressed the plasma emission problem [35–41]. Recently, complete equations of electromagnetic weak turbulence theory are solved in order to quantitatively investigate the plasma emission process [42], and its validity was confirmed by

$2+1/2$ dimensional (two dimensional space and three dimensional velocity) particle-in-cell (PIC) simulation [43]. While PIC simulations of plasma emission process have been carried out [44–54], a quantitative comparison between the PIC simulation and weak turbulence theory has not been attempted until [43].

However, one of the key ingredients in the standard method is that the existence of background quasi static magnetic field is ignored. In the interplanetary space, especially near Earth orbit, the solar wind magnetic field is quite weak so that the assumption of unmagnetized plasma may be adequate. However, near the solar source, the influence of strong magnetic field associated with the solar active region cannot always be ignored [55]. The recent Parker Solar Probe (PSP) observation of type III bursts also poses some open questions regarding the polarization characteristics [56]. Note that without the ambient magnetic field, the radio emissions should bear no signatures of circular polarity, but the authors report such polarizations. Ma *et al* [57] analyzed the PSP observation by surveying the low frequency cutoffs associated with the type III bursts, and speculate that the cyclotron maser emission may also be effective in the radiation process. Of course, the cyclotron emission requires the presence of ambient magnetic field. All these call for the generalization of standard weak turbulence theory to include the effects of ambient magnetic field. Such a task, which has been attempted sporadically [15–18, 58–69], is by no means complete. The reason is because of the extreme complexity, which is inherent to the theoretical development. A brief overview may help set the stage for the present work.

Among the early works, Tsytovich *et al* [58] derived the formal equations of weak turbulence theory for magnetized plasmas, which involve the nonlinear response tensor expressed in terms of multiple Bessel function series. After making a series of simplifications, they applied the formalism to the problem of two plasmons (of frequency $\omega\sim\omega_{pe}$, where $\omega_{pe}=\sqrt{4\pi n_0e^2/m_e}$ is the plasma frequency, $n_0$, $e$, and $m_e$ being the ambient density, unit charge, and electron mass, respectively) interacting with a lower-hybrid wave, $\omega\sim|\Omega_e\Omega_i|^{1/2}$. Here, $\Omega_e=-eB_0/m_ec$ and $\Omega_i=eB_0/m_ic$ denote the electron and proton cyclotron frequency, respectively, $B_0$, $m_i$, and $c$ being the ambient magnetic field intensity, proton mass, and the speed of light *in vacuo*. In the same vein, Melrose and Sy [59] adopted basically the same methodology as pioneered in [58], but the authors applied the formalism to Thomson scattering in magnetized plasmas. Along this line of research, Pustovalov and Silin [60] also undertook the task of formulating weak turbulence theory for magnetized plasmas within the framework of Vlasov kinetic theory. The general expressions for nonlinear susceptibilities, derived in the above references, is rather formidable so that relatively few actual applications are made, a recent example being [70].

In a separate development, Porkolab and coauthors, e.g. [61–63], etc discussed the weak turbulence theory with the effects of $B$ field included, but they were interested in nonlinear interaction of electrostatic Bernstein waves in hot plasmas. The related work by [64] also pertains to interactions of electrostatic cyclotron waves in magnetized plasmas.

An important development was initiated in [65], which formulated the problem with the warm fluid approach. Instead of the highly complex kinetic formalism, the method in [65] is far simpler, and in the cold-plasma limit, the formalism becomes equivalent to those of [58, 59], and also that of [60], if the same cold-plasma limit is taken in their more formal kinetic theory. However, the issue of whether the warm fluid formalism is also equivalent to the approximate kinetic formalism had not been addressed in the literature. The full kinetic version of nonlinear susceptibility (in both unmagnetized and magnetized plasmas) is impractical so that suitable approximations are customarily taken. As will be discussed shortly, the present paper will demonstrate that the approximate kinetic version of nonlinear susceptibility, already amply discussed in the literature [1–10, 71] can be derived from the warm fluid weak turbulence theory for unmagnetized plasmas. This demonstration makes the prospect of formulating a similar weak turbulence theory for magnetized plasmas based upon the warm fluid theory also feasible. It should be noted that the cold-plasma formalism [58–60, 65] is not entirely useful since finite temperature effects cannot always be ignored. This is because in the cold-plasma limit, the ion-sound or magneto-ion acoustic wave, which mediates the three-wave decay process, does not even exist. Moreover, judging from the experience in unmagnetized plasma theory, some approximate limiting forms of kinetic nonlinear response function usually end up having an inverse proportionality to the temperature, $\propto 1/T$, so that for such cases, the cold-plasma limit ($T\to 0$) is simply meaningless. Thus, one needs to capture certain thermal effects via adopting warm-fluid formalism.

Returning to the overview, Trakhtengerts [65] heuristically discussed two plasmons interacting with low-frequency magneto-ion-acoustic wave. Later, [71–74] addressed the issue of second-harmonic plasma emission within the framework of cold magnetized plasma theory. However, the cold-plasma theory of plasma emission in magnetized plasmas is not entirely compatible with that of unmagnetized plasma emission theory. For instance, in the cold magnetized plasma theory, ion-acoustic mode ($S$) does not exist, and Langmuir mode reduces to the upper-hybrid (or more accurately, $Z$ mode) wave with no thermal effects.

Meanwhile, the present author [66–68] formally derived (and numerically solved) the complete kinetic equations of weak turbulence theory for magnetized plasmas, but only for limiting situations of either exactly parallel or perpendicular propagation. The focus on these papers was either on low-frequency ion-cyclotron turbulence or nonlinear interaction of upper-hybrid (or $Z$ mode) waves with $X$ mode radiation. On a different development, [15–18, 75, 76] took a different approach by adopting the drift kinetic equation as a basis to analyze the whistler/lower-hybrid turbulence problem. In this approach, cyclotron waves and their harmonics are ignored, which makes the analysis tractable. In yet another development, incompressible magnetohydrodynamic (MHD) turbulence problem has also been analyzed within the standard framework and methodology commonly adopted in the theory of weak turbulence [77–79]. An area that was overlooked in this overview is that of (strong) MHD turbulence, for which perturbative expansion in some smallness parameter may not be suitable. The full MHD turbulence is often treated with full numerical simulations or by means of phenomenological approach, and it is a mature subject area [80, 81]. Also largely left out of the overview is the recently rapidly developing area of dealing with large scale plasma turbulence with hybrid [82], gyro-kinetic [83, 84], or even full Vlasov/PIC simulations [85–87].

As briefly reviewed, a number of important milestones have been achieved on the weak turbulence theory of magnetized plasmas throughout the long history of plasma physics, and advances in the problem of weak turbulence in magnetized plasma have been in diverse directions. However, thus far, no comprehensive theory emerged that can be considered as equivalent to that of unmagnetized plasmas and no quantitative numerical analysis of the sort discussed in [42] is available, let alone, direct comparison with simulations, as in [43]. There are, however, reasons to believe that some fundamental progresses can be made if one employs the warm fluid theory instead of the much more difficult kinetic theory. We make note of the fact that the main nonlinear mechanism for conversion of electrostatic waves into radiation involves nonlinear wave-wave decay interaction, which involves only waves but no particles. This points to the possibility of avoiding the arduous task of employing the kinetic theoretical method in the attempt to generalize the standard weak turbulence theory to include the effects of ambient magnetic field, but instead to resort to the much simpler fluid theoretical framework for such an effort. Before we carry out such a task, however, it is useful to first demonstrate that the standard weak turbulence theory for unmagnetized plasma can be partially reconstructed within the fluid theoretical paradigm as a proof of concept.

The purpose of the present paper is thus, to revisit the problem of weak plasma turbulence from the perspective of two-fluid theory. Our aim is to show that the nonlinear three wave interaction process can be reproduced by the two-fluid approach. In the subsequent analysis, we begin with the relatively simple electrostatic formalism, and then we move on to the more general electromagnetic treatment. The inclusion of ambient magnetic field effects based upon the methodology outlined in the present paper is the subject of future work.

## 2. Two-fluid formulation of weakly turbulent plasma processes

### 2.1. Electrostatic approximation

Let us first consider the simple case of electrostatic approximation. We start from electrostatic two-fluid equations in unmagnetized plasmas:

$$\begin{aligned}
&\frac{\partial n_a}{\partial t}+\nabla\cdot(n_a\mathbf{v}_a)=0,\\
&m_a n_a\left(\frac{\partial}{\partial t}+\mathbf{v}_a\cdot\nabla\right)\mathbf{v}_a+\nabla P_a+e_a n_a\mathbf{E}=0,\\
&\nabla\cdot\mathbf{E}=4\pi\sum_a e_a n_a,\\
&\frac{\partial\mathbf{E}}{\partial t}=4\pi\sum_a e_a n_a\mathbf{v}_a,
\end{aligned}\tag{1}$$

where $a=i,e$ denotes charged particle species ($i$ for protons and $e$ for electrons); $n_a$, $\mathbf{v}_a$, $P_a$, and $\mathbf{E}$ denotes fluid density, velocity, pressures, and electric field; and $e_a$ and $m_a$ stand for charge and mass for charged particle species labeled $a$. Let us assume isotropic pressure $P_a=n_aT_a$, and separate quantities in terms of averages and fluctuations: $n_a=n_0+\delta n_a$, $\mathbf{v}_a=\delta\mathbf{v}_a$, and $\mathbf{E}=\delta\mathbf{E}$. Here, we have assumed that there is no net flow associated with the plasma fluid and that the system is free of large scale electric field. In the present analysis we adopt the weak turbulence ordering, which amounts to the assumption that the fluctuating quantities are treated as small perturbations to the average quantities. Specifically, it is assumed that $|\delta n_a|\ll n_0$. Making use of the charge neutrality $\sum_a e_a=0$, we arrive at:

$$\begin{aligned}
&\frac{\partial\,\delta n_a}{\partial t}+n_0\nabla\cdot\delta\mathbf{v}_a+\nabla\cdot(\delta n_a\,\delta\mathbf{v}_a)=0,\\
&\left(\frac{\partial}{\partial t}+\delta\mathbf{v}_a\cdot\nabla\right)\delta\mathbf{v}_a-\frac{e_a}{m_a}\,\delta\mathbf{E}+\frac{T_a}{m_a n_0}\nabla\delta n_a\\
&\quad-\frac{T_a}{m_a n_0^2}\,\delta n_a\nabla\delta n_a=0,\\
&\nabla\cdot\delta\mathbf{E}=\sum_a 4\pi e_a\,\delta n_a,\\
&\frac{\partial\,\delta\mathbf{E}}{\partial t}=-\sum_a 4\pi e_a n_0\,\delta\mathbf{v}_a-\sum_a 4\pi e_a\,\delta n_a\,\delta\mathbf{v}_a.
\end{aligned}\tag{2}$$

In the momentum equation above we have expanded the inverse of density, $1/n_a=1/(n_0+\delta n_a)\approx n_0^{-1}(1-\delta n_a/n_0)$. The perturbed electric field vector can be expressed in terms of the electrostatic potential, $\delta\mathbf{E}=-\nabla\delta\phi$. Note that the last two equations can be used interchangeably, as they are equivalent. Spectral representation of this set of equations are:

$$\begin{aligned}
\frac{\delta n^a_{\mathbf{k},\omega}}{n_0}&=\frac{k}{\omega}\,\delta v^a_{\mathbf{k},\omega}+\frac{1}{\omega}\sum_{\omega',\mathbf{k}'}\frac{\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|}\frac{\delta n^a_{\mathbf{k}',\omega'}}{n_0}\,\delta v^a_{\mathbf{k}-\mathbf{k}',\omega-\omega'},\\
\delta v^a_{\mathbf{k},\omega}&=\frac{e_a}{m_a\omega}k\,\delta\phi_{\mathbf{k},\omega}+\frac{kv_a^2}{\omega}\frac{\delta n^a_{\mathbf{k},\omega}}{n_0}+\frac{1}{\omega}\sum_{\omega',\mathbf{k}'}\\
&\quad\times\frac{[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')][\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')]}{kk'|\mathbf{k}-\mathbf{k}'|}\,\delta v^a_{\mathbf{k}',\omega'}\,\delta v^a_{\mathbf{k}-\mathbf{k}',\omega-\omega'},\\
\delta\phi_{\mathbf{k},\omega}&=\sum_a\frac{4\pi e_a n_0}{k\omega}\left(\delta v^a_{\mathbf{k},\omega}+\sum_{\omega',\mathbf{k}'}\frac{\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')}{k|\mathbf{k}-\mathbf{k}'|}\frac{\delta n^a_{\mathbf{k}',\omega'}}{n_0}\right.\\
&\quad\left.\times\,\delta v^a_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\right).
\end{aligned}\tag{3}$$

Here, we have expressed the velocity perturbation as $\delta\mathbf{v}^a_{\mathbf{k},\omega} = (\mathbf{k}/k)\,\delta v^a_{\mathbf{k}}$, and have introduced the fluid thermal speed, $v_a = (T_a/m_a)^{\frac{1}{2}}$. Note that the angular frequency has an implicit slow-time derivative, $\omega \to \omega + i(\partial/\partial t)$, where time-derivative operator $i(\partial/\partial t)$ is to be regarded as a small correction. We initially disregard the time derivative in the definition for angular frequency $\omega$ so that the ensuing analysis can be treated as algebraic methodology. However, we reintroduce $i(\partial/\partial t)$ at an appropriate later stage in the theoretical development [3, 88].

We write the particle fluid quantities as in perturbation series, with each term proportional to the electric field amplitude, $\delta n^a_{\mathbf{k},\omega} = \delta n^{a(1)}_{\mathbf{k},\omega} + \delta n^{a(2)}_{\mathbf{k},\omega} + \cdots$ and $\delta v^a_{\mathbf{k},\omega} = \delta v^{a(1)}_{\mathbf{k},\omega} + \delta v^{a(2)}_{\mathbf{k},\omega} + \cdots$, where $\delta n^{a(1)}_{\mathbf{k},\omega}, \delta v^{a(1)}_{\mathbf{k},\omega} \propto \mathscr{O}(\delta\phi_{\mathbf{k},\omega})$, $\delta n^{a(2)}_{\mathbf{k},\omega}, \delta v^{a(2)}_{\mathbf{k},\omega} \propto \mathscr{O}[(\delta\phi_{\mathbf{k},\omega})^2]$, etc. Making use of the above we may solve equation (3) by iterative means in terms of the field strength,

$$
\begin{aligned}
\delta n^{a(1)}_{\mathbf{k},\omega} &= \frac{e_a n_0}{m_a}\frac{k^2}{\omega^2 - k^2 v_a^2}\,\delta\phi_{\mathbf{k},\omega},\\
\delta v^{a(1)}_{\mathbf{k},\omega} &= \frac{e_a}{m_a}\frac{\omega k}{\omega^2 - k^2 v_a^2}\,\delta\phi_{\mathbf{k},\omega},\\
\delta n^{a(2)}_{\mathbf{k},\omega} &= \frac{e_a^2 n_0}{2m_a^2}\frac{\omega^2}{\omega^2 - k^2 v_a^2}\sum_{\omega',\mathbf{k}'}\\
&\quad\times\frac{\omega'(\omega-\omega')}{\omega(\omega'^2 - k'^2 v_a^2)[(\omega-\omega')^2 - (\mathbf{k}-\mathbf{k}')^2 v_a^2]}\\
&\quad\times\left(\frac{k^2}{\omega}\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}') + \frac{k'^2}{\omega'}\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')\right.\\
&\qquad\left. + \frac{|\mathbf{k}-\mathbf{k}'|^2}{\omega-\omega'}\mathbf{k}\cdot\mathbf{k}'\right)\delta\phi_{\mathbf{k}',\omega'}\,\delta\phi_{\mathbf{k}-\mathbf{k}',\omega-\omega'},\\
\delta v^{a(2)}_{\mathbf{k},\omega} &= \frac{e_a^2}{m_a^2}\frac{\omega^2}{\omega^2 - k^2 v_a^2}\sum_{\omega',\mathbf{k}'}\\
&\quad\times\frac{k'[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')]\omega'(\omega-\omega')}{\omega(\omega'^2 - k'^2 v_a^2)[(\omega-\omega')^2 - (\mathbf{k}-\mathbf{k}')^2 v_a^2]}\\
&\quad\times\left(\frac{\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')}{kk'} + \frac{kk' v_a^2}{\omega\omega'}\right)\delta\phi_{\mathbf{k}',\omega'}\,\delta\phi_{\mathbf{k}-\mathbf{k}',\omega-\omega'}.
\end{aligned}
\tag{4}
$$

Making use of the perturbative solutions the wave equation can be expressed as follows:

$$
\begin{aligned}
0 &= k\epsilon(\mathbf{k},\omega)\,\delta\phi_{\mathbf{k},\omega} - i\sum_{\omega',\mathbf{k}'} k'|\mathbf{k}-\mathbf{k}'|\\
&\quad\times\chi(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')\,\delta\phi_{\mathbf{k}',\omega'}\,\delta\phi_{\mathbf{k}-\mathbf{k}',\omega-\omega'},
\end{aligned}
\tag{5}
$$

where linear and nonlinear susceptibilities are given by:

$$
\begin{aligned}
\epsilon(\mathbf{k},\omega) &= 1 + \sum_a \chi_a(\mathbf{k},\omega),\\
\chi_a(\mathbf{k},\omega) &= -\frac{\omega_{pa}^2}{\omega^2 - k^2 v_a^2},\\
\chi(\mathbf{k}_1,\omega_2|\mathbf{k}_1,\omega_2) &= \sum_a \chi_a(\mathbf{k}_1,\omega_2|\mathbf{k}_1,\omega_2),\\
\chi_a(\mathbf{k}_1,\omega_2|\mathbf{k}_1,\omega_2) &= \frac{-ie_a}{2m_a}\frac{\omega_{pa}^2\,\omega_1\,\omega_2\,\omega}{(\omega_1^2 - k_1^2 v_a^2)(\omega_2^2 - k_2^2 v_a^2)(\omega^2 - k^2 v_a^2)}\\
&\quad\times\left(\frac{k_1}{\omega_1}\frac{\mathbf{k}_2\cdot\mathbf{k}}{k_2 k} + \frac{k_2}{\omega_2}\frac{\mathbf{k}_1\cdot\mathbf{k}}{k_1 k} + \frac{k}{\omega}\frac{\mathbf{k}_1\cdot\mathbf{k}_2}{k_1 k_2}\right).
\end{aligned}
\tag{6}
$$

In the nonlinear susceptibility, $\omega$ and $\mathbf{k}$ are short-hand notations for $\omega = \omega_1 + \omega_2$ and $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$. In equation (6) $\omega_{pa} = (4\pi n_0 e^2/m_a)^{1/2}$ is the plasma oscillation frequency defined for species $a$.

It is interesting to compare the fluid versions of plasma susceptibilities against the kinetic versions [10, 12–14, 20]:

$$
\begin{aligned}
\chi_a(\mathbf{k},\omega) &= \sum_a \omega_{pa}^2 \int d\mathbf{v}\,\frac{1}{\omega - \mathbf{k}\cdot\mathbf{v}}\,\mathbf{k}\cdot\frac{\partial F_a(\mathbf{v})}{\partial\mathbf{v}},\\
\chi_a(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2) &= \frac{-i}{2}\frac{e_a}{m_a}\frac{\omega_{pa}^2}{k_1 k_2 k}\int d\mathbf{v}\,\frac{1}{\omega - \mathbf{k}\cdot\mathbf{v}}\frac{\partial}{\partial\mathbf{v}}\\
&\quad\cdot\left(\frac{\mathbf{k}_1\mathbf{k}_2}{\omega_2 - \mathbf{k}_2\cdot\mathbf{v}} + \frac{\mathbf{k}_2\mathbf{k}_1}{\omega_1 - \mathbf{k}_1\cdot\mathbf{v}}\right)\cdot\frac{\partial F_a(\mathbf{v})}{\partial\mathbf{v}}.
\end{aligned}
\tag{7}
$$

Here, $F_a(\mathbf{v})$ represents the particle velocity distribution function, normalized according to $\int d\mathbf{v}F_a(\mathbf{v}) = 1$, and $\omega$ is assumed to have a finite positive imaginary part, $\omega \to \omega + i0$. In kinetic theory, thermal speed is defined with a factor 2 higher than that of the fluid theory, $v_a^2 = 2T_a/m_a$. As discussed in [10, 12–14, 20], linear and nonlinear susceptibilities in kinetic theory enjoy the following approximate properties:

$$
\begin{aligned}
\chi_a(\mathbf{k},\omega) &= \mathrm{Re}\,\chi_a(\mathbf{k},\omega) + i\,\mathrm{Im}\,\chi_a(\mathbf{k},\omega),\\
\mathrm{Re}\,\chi_a(\mathbf{k},\omega) &= \begin{cases} -(\omega_{pa}^2/\omega^2)\left[1 + \frac{3}{2}(k^2 v_a^2/\omega^2)\right] & \mathbf{k}\cdot\mathbf{v} \ll \omega\\ 2\omega_{pa}^2/(k^2 v_a^2) & \mathbf{k}\cdot\mathbf{v} \gg \omega\end{cases},\\
\mathrm{Im}\,\chi_a(\mathbf{k},\omega) &= -\frac{\pi\omega_{pa}^2}{k^2}\int d\mathbf{v}\,\mathbf{k}\cdot\frac{\partial F_a}{\partial\mathbf{v}}\,\delta(\omega - \mathbf{k}\cdot\mathbf{v}),
\end{aligned}
\tag{8}
$$

and

$$
\begin{aligned}
\chi_a(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2) &= \frac{-i}{2}\frac{e_a}{m_a}\frac{\omega_{pa}^2}{\omega_1\omega_2\omega}\left(\frac{k_1}{\omega_1}\frac{\mathbf{k}_2\cdot\mathbf{k}}{k_2 k} + \frac{k_2}{\omega_2}\frac{\mathbf{k}_1\cdot\mathbf{k}}{k_1 k}\right.\\
&\qquad\left. + \frac{k}{\omega}\frac{\mathbf{k}_1\cdot\mathbf{k}_2}{k_1 k_2}\right),\\
&\qquad(\omega_1 \gg k_1 v_a,\ \omega_2 \gg k_2 v_a,\ \omega \gg k v_a),\\
&= \frac{ie_a}{T_a}\frac{\omega_{pa}^2}{\omega_2\omega}\frac{\mathbf{k}_2\cdot\mathbf{k}}{k_1 k_2 k},\ (\omega_1 \ll k_1 v_a,\ \omega_2 \gg k_2 v_a,\\
&\qquad\omega \gg k v_a),\\
&= \frac{ie_a}{T_a}\frac{\omega_{pa}^2}{\omega_1\omega}\frac{\mathbf{k}_1\cdot\mathbf{k}}{k_1 k_2 k},\ (\omega_1 \gg k_1 v_a,\ \omega_2 \ll k_2 v_a,\\
&\qquad\omega \gg k v_a),\\
&= \frac{ie_a}{T_a}\frac{\omega_{pa}^2}{\omega_1\omega_2}\frac{\mathbf{k}_1\cdot\mathbf{k}_2}{k_1 k_2 k},\ (\omega_1 \gg k_1 v_a,\ \omega_2 \gg k_2 v_a,\\
&\qquad\omega \ll k v_a).
\end{aligned}
\tag{9}
$$

Note that the fluid nonlinear susceptibility (6) enjoys the exact same limiting forms as the kinetic nonlinear susceptibility (9). This shows that the present two-fluid formulation does indeed partially correspond to that of kinetic theory. Note also that the fluid linear susceptibility is slightly different in limiting form when compared to that of kinetic formalism. Specifically, under the two-fluid treatment, the linear susceptibility has the following form:

$$\chi_a(\mathbf{k},\omega) = \begin{cases} -(\omega_{pa}^2/\omega^2)\left[1+(k^2v_a^2/\omega^2)\right] & \mathbf{k}\cdot\mathbf{v} \ll \omega \\ \omega_{pe}^2/(k^2v_a^2) & \mathbf{k}\cdot\mathbf{v} \gg \omega \end{cases}, \tag{10}$$

which differs slightly from equation (8) in that not only the mathematical forms of the limiting expressions are not identical but also that the linear dielectric susceptibility has no imaginary part in the fluid limit. This means that in the two-fluid formulation the linear wave property of small amplitude perturbation will be distinct from that of kinetic theory. Moreover, as the principal parts of the inverse linear dielectric constant is intimately associated with the induced scattering processes while the residue contributions from the same inverse linear dielectric constant is related to the decay processes, and since the two-fluid formalism does not have imaginary parts of the linear dielectric constant, the induced scattering is absent in the two-fluid theory of weak turbulence.

Let us proceed with the rest of the formulation. We multiply $\delta\phi^*_{\mathbf{k},\omega} = \delta\phi_{-\mathbf{k},-\omega}$ to (5) and take the ensemble average to arrive at:

$$\begin{aligned} 0 = k\epsilon(\mathbf{k},\omega)\left\langle\delta\phi^2\right\rangle_{\mathbf{k},\omega} - i\int d\omega'\int d\mathbf{k}'\,k'|\mathbf{k}-\mathbf{k}'| \\ \times\,\chi(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')\langle\delta\phi_{\mathbf{k}',\omega'}\,\delta\phi_{\mathbf{k}-\mathbf{k}',\omega-\omega'} \\ \times\,\delta\phi_{-\mathbf{k},-\omega}\rangle. \end{aligned} \tag{11}$$

The subsequent formulation involves the computation of third-body cumulant, $\langle\delta\phi_{\mathbf{k}',\omega'}\,\delta\phi_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\,\delta\phi_{-\mathbf{k},-\omega}\rangle$, which can be done following the customary closure scheme, as explained fully in the author's monograph [20] as well as in other literature [10, 12–14]. In this scheme we take each of $\delta\phi_{\mathbf{k}',\omega'}$, $\delta\phi_{\mathbf{k}-\mathbf{k}',\omega-\omega'}$, and $\delta\phi_{-\mathbf{k},-\omega}$ and write as sums of leading solution that satisfies the linear dispersion relation plus nonlinear correction. The nonlinear correction is computed from the wave equation (5). This procedure leads to the four-body cumulant $\langle\delta\phi_{\mathbf{k}_1,\omega_1}\,\delta\phi_{\mathbf{k}_2,\omega_2}\,\delta\phi_{\mathbf{k}_3,\omega_3}\,\delta\phi_{\mathbf{k}_4,\omega_4}\rangle$, which is expressed as products of two-body cumulants while ignoring irreducible four-body correlation under the assumption of the so-called Bogoliubov's hierarchy of correlations. These methodologies are exactly the same whether one is concerned with kinetic formalism or fluid-theoretical paradigm. Consequently, we do not explicitly rehash the derivation that involves these steps. Again, all such processes are explained in detail in the present author's recent monograph [20] and in other works [10, 12–14]. Of the various terms that results from following the above explained steps, we subsequently ignore those terms that lead to $\epsilon(0,0)$ in the denominator, since $\epsilon(0,0)\to\infty$ such that the inverse $1/\epsilon(0,0)\to 0$. We write down the final result,

$$\begin{aligned} &\langle\delta\phi_{\mathbf{k}',\omega'}\,\delta\phi_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\,\delta\phi_{-\mathbf{k},-\omega}\rangle \\ &= \frac{2ik|\mathbf{k}-\mathbf{k}'|\chi(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')}{k'\epsilon(\mathbf{k}',\omega')} \\ &\times\left\langle\delta\phi^2\right\rangle_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\left\langle\delta\phi^2\right\rangle_{\mathbf{k},\omega} \\ &+\frac{2ikk'\chi(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')}{|\mathbf{k}-\mathbf{k}'|\epsilon(\mathbf{k}-\mathbf{k}',\omega-\omega')}\left\langle\delta\phi^2\right\rangle_{\mathbf{k}',\omega'}\left\langle\delta\phi^2\right\rangle_{\mathbf{k},\omega} \\ &-\frac{2ik'|\mathbf{k}-\mathbf{k}'|\chi^*(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')}{k\epsilon^*(\mathbf{k},\omega)} \\ &\times\left\langle\delta\phi^2\right\rangle_{\mathbf{k}',\omega'}\left\langle\delta\phi^2\right\rangle_{\mathbf{k}-\mathbf{k}',\omega-\omega'}. \end{aligned} \tag{12}$$

Inserting (12) to (11), making use of $k^2\langle\delta\phi\rangle_{\mathbf{k},\omega} = \left\langle\delta E^2\right\rangle_{\mathbf{k},\omega}$, and reinstating the slow time derivative to the leading linear response term, $\epsilon(\mathbf{k},\omega)\left\langle\delta E^2\right\rangle_{\mathbf{k},\omega}\to \epsilon(\mathbf{k},\omega)\left\langle\delta E^2\right\rangle_{\mathbf{k},\omega}+(i/2)[\partial\epsilon(\mathbf{k},\omega)/\partial\omega](\partial/\partial t)\left\langle\delta E^2\right\rangle_{\mathbf{k},\omega}$, we obtain the formal wave kinetic equation,

$$\begin{aligned} 0 = \epsilon(\mathbf{k},\omega)\left\langle\delta E^2\right\rangle_{\mathbf{k},\omega} + \frac{i}{2}\frac{\partial\epsilon(\mathbf{k},\omega)}{2}\partial\omega\frac{\partial}{\partial t}\left\langle\delta E^2\right\rangle_{\mathbf{k},\omega} + 2\int d\omega' \\ \times\int d\mathbf{k}'\left(\frac{\{\chi(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')\}^2}{\epsilon(\mathbf{k}',\omega')}\left\langle\delta E^2\right\rangle_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\right. \\ \times\left\langle\delta E^2\right\rangle_{\mathbf{k},\omega} \\ +\frac{\{\chi(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')\}^2}{\epsilon(\mathbf{k}-\mathbf{k}',\omega-\omega')}\left\langle\delta E^2\right\rangle_{\mathbf{k}',\omega'} \\ \times\left\langle\delta E^2\right\rangle_{\mathbf{k},\omega} - \frac{|\chi(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')|^2}{\epsilon^*(\mathbf{k},\omega)} \\ \left.\times\left\langle\delta E^2\right\rangle_{\mathbf{k}',\omega'}\left\langle\delta E^2\right\rangle_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\right). \end{aligned} \tag{13}$$

Here, we have resorted back to the electric field fluctuation spectrum. This result is formally identical to the kinetic version of the same equation [10, 12–14, 20], except that the definitions of linear and nonlinear dielectric susceptibilities are defined differently.

As in the standard method, by considering the linear response of equation (13) and setting equal to zero, $\epsilon(\mathbf{k},\omega)\left\langle\delta E^2\right\rangle_{\mathbf{k},\omega}=0$, we obtain the linear eigenmode and eigenvalue. For high frequency Langmuir mode, satisfying $\omega^2\gg k^2v_e^2$ and $\omega^2\gg k^2v_i^2$, we may ignore ion response and approximate the electron response via (10). For low frequency ion sound mode, which satisfy $\omega^2\gg k^2v_i^2$ and $\omega^2\ll k^2v_e^2$, we retain both electron and ion responses, but we adopt opposite limiting forms by virtue of (10). This leads to the fluid version of the Langmuir and ion sound wave dispersion relations, $\omega=\omega^L_{\mathbf{k}}$ and $\omega=\omega^S_{\mathbf{k}}$, where $\omega^L_{\mathbf{k}}=\omega_{pe}\left[1+\frac{1}{2}(k^2v_e^2/\omega_{pe}^2)\right] = \omega_{pe}\left(1+\frac{1}{2}k^2\lambda_D^2\right)$ and $\omega^S_{\mathbf{k}}=kc_s(1+T_i/T_e)^{\frac{1}{2}}(1+k^2\lambda_D^2)^{-\frac{1}{2}}\approx kc_S$, where $\omega_{pe}=(4\pi n_0e^2/m_e)^{\frac{1}{2}}$ is the electron plasma oscillation frequency, $\lambda_D=T_e^{\frac{1}{2}}/(4\pi n_0e^2)^{-\frac{1}{2}}$ is the Debye length, and $c_s=(T_e/m_i)^{\frac{1}{2}}$ is the ion sound speed. As a consequence of linear dispersion relation, we may write the electric field spectrum as $\left\langle\delta E^2\right\rangle_{\mathbf{k},\omega}=\sum_{\alpha=L,S}\sum_{\sigma=\pm1}I^{\sigma\alpha}_{\mathbf{k}}\,\delta(\omega-\sigma\omega^{\alpha}_{\mathbf{k}})$.

Implementing the above prescription and coupling with the imaginary part of (13) and making use of the inverse of linear dielectric response function, we obtain:

$$
\begin{aligned}
\frac{1}{\epsilon(\mathbf{k}',\sigma\omega_{\mathbf{k}}^{\alpha}-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma})} &= \mathscr{P}\frac{1}{\epsilon(\mathbf{k}',\sigma\omega_{\mathbf{k}}^{\alpha}-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma})} \\
&\quad -\sum_{\beta}\sum_{\sigma'=\pm1} \\
&\quad \times\frac{i\pi\,\delta(\sigma\omega_{\mathbf{k}}^{\alpha}-\sigma'\omega_{\mathbf{k}'}^{\beta}-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma})}{\partial\epsilon(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^{\beta})/\partial(\sigma'\omega_{\mathbf{k}'}^{\beta})}, \\
\frac{1}{\epsilon(\mathbf{k}-\mathbf{k}',\sigma\omega_{\mathbf{k}}^{\alpha}-\sigma'\omega_{\mathbf{k}'}^{\beta})} &= \mathscr{P}\frac{1}{\epsilon(\mathbf{k}-\mathbf{k}',\sigma\omega_{\mathbf{k}}^{\alpha}-\sigma'\omega_{\mathbf{k}'}^{\beta})} \\
&\quad -\sum_{\gamma}\sum_{\sigma''=\pm1} \\
&\quad \times\frac{i\pi\,\delta(\sigma\omega_{\mathbf{k}}^{\alpha}-\sigma'\omega_{\mathbf{k}'}^{\beta}-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma})}{\partial\epsilon(\mathbf{k}-\mathbf{k}',\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma})/\partial(\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma})}, \\
\frac{1}{\epsilon^{*}(\mathbf{k},\sigma'\omega_{\mathbf{k}'}^{\beta}+\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma})} &= \mathscr{P}\frac{1}{\epsilon^{*}(\mathbf{k},\sigma'\omega_{\mathbf{k}'}^{\beta}+\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma})} \\
&\quad +\sum_{\alpha}\sum_{\sigma=\pm1} \\
&\quad \times\frac{i\pi\,\delta(\sigma'\omega_{\mathbf{k}'}^{\beta}+\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma}-\sigma\omega_{\mathbf{k}}^{\alpha})}{\partial\epsilon(\mathbf{k},\sigma\omega_{\mathbf{k}}^{\alpha})\partial(\sigma\omega_{\mathbf{k}}^{\alpha})}, \quad (14)
\end{aligned}
$$

which are again the same in the present two-fluid versus the customary and more complete kinetic approaches. However, the main difference is that in the two-fluid approach, the linear dielectric constant does not have an imaginary part, which implies that Landau damping associated with Langmuir and ion-sound waves is absent in the present two-fluid formalism. As a result, the principal parts of (14) do not have any contribution to the imaginary part of the nonlinear wave kinetic equation (13). Consequently, the resulting wave kinetic equation does not have linear wave-particle resonance term, i.e. induced emission, nor does it have nonlinear wave-particle resonance term, or induced scattering term,

$$
\begin{aligned}
\frac{\partial I_{\mathbf{k}}^{\sigma\alpha}}{\partial t} &= \frac{4\pi}{\partial\epsilon(\mathbf{k},\sigma\omega_{\mathbf{k}}^{\alpha})/\partial(\sigma\omega_{\mathbf{k}}^{\alpha})}\sum_{\beta,\gamma}\sum_{\sigma',\sigma''=\pm1}\int d\mathbf{k}' \\
&\quad \times|\chi(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^{\beta}|\mathbf{k}-\mathbf{k}',\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma})|^{2} \\
&\quad \times\left(\frac{I_{\mathbf{k}'}^{\sigma'\beta}I_{\mathbf{k}-\mathbf{k}'}^{\sigma''\gamma}}{\partial\epsilon(\mathbf{k},\sigma\omega_{\mathbf{k}}^{\alpha})\partial(\sigma\omega_{\mathbf{k}}^{\alpha})}-\frac{I_{\mathbf{k}}^{\sigma\alpha}I_{\mathbf{k}-\mathbf{k}'}^{\sigma''\gamma}}{\partial\epsilon(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^{\beta})/\partial(\sigma'\omega_{\mathbf{k}'}^{\beta})}\right. \\
&\quad \left.-\frac{I_{\mathbf{k}}^{\sigma\alpha}I_{\mathbf{k}'}^{\sigma'\beta}}{\partial\epsilon(\mathbf{k}-\mathbf{k}',\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma})/\partial(\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma})}\right) \\
&\quad \times\delta(\sigma\omega_{\mathbf{k}}^{\alpha}-\sigma'\omega_{\mathbf{k}'}^{\beta}-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma}). \quad (15)
\end{aligned}
$$

This form of wave kinetic equation is identical to the nonlinear equation depicting the three wave decay term. Since this equation does not have linear term, or induced emission term, it cannot describe Landau damping. Also, since the two-fluid approach does not have the concept of particle kinetic equation, the above equation cannot describe the damping of the wave via particles absorbing wave energy in the first place. Equation (15) does not allow for the description of induced scattering either, which is sometimes known as the nonlinear Landau damping. However, it is adequate to describe the nonlinear interaction among three waves. Before we move on to electromagnetic formalism, we note that one may further manipulate equation (15) by invoking the approximate properties of nonlinear susceptibility (9), which is the same both in two-fluid and kinetic formalisms, as already pointed out. The result, which is identical to the three-wave decay interaction equation, already discussed in the standard literature, e.g. [10, 12–14, 20], is shown here for the purpose of completeness,

$$
\begin{aligned}
\frac{\partial I_{\mathbf{k}}^{\sigma L}}{\partial t} &= \sigma\omega_{\mathbf{k}}^{L}\sum_{\sigma',\sigma''=\pm1}\int d\mathbf{k}'\,\frac{\pi}{2}\frac{e^{2}}{T_{e}^{2}}\frac{\mu_{\mathbf{k}-\mathbf{k}'}(\mathbf{k}\cdot\mathbf{k}')^{2}}{k^{2}k'^{2}|\mathbf{k}-\mathbf{k}'|^{2}} \\
&\quad \times\left[\sigma\omega_{\mathbf{k}}^{L}I_{\mathbf{k}'}^{\sigma'L}\frac{I_{\mathbf{k}-\mathbf{k}'}^{\sigma''S}}{\mu_{\mathbf{k}-\mathbf{k}'}}\right. \\
&\quad \left.-\left(\sigma'\omega_{\mathbf{k}'}^{L}\frac{I_{\mathbf{k}-\mathbf{k}'}^{\sigma''S}}{\mu_{\mathbf{k}-\mathbf{k}'}}+\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L}I_{\mathbf{k}'}^{\sigma'L}\right)I_{\mathbf{k}}^{\sigma L}\right] \\
&\quad \times\delta(\sigma\omega_{\mathbf{k}}^{L}-\sigma'\omega_{\mathbf{k}'}^{L}-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{S}), \\
\frac{\partial}{\partial t}\frac{I_{\mathbf{k}}^{\sigma S}}{\mu_{\mathbf{k}}} &= \sigma\omega_{\mathbf{k}}^{L}\sum_{\sigma',\sigma''=\pm1}\int d\mathbf{k}'\,\frac{\pi}{4}\frac{e^{2}}{T_{e}^{2}}\frac{\mu_{\mathbf{k}}[\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')]^{2}}{k^{2}k'^{2}|\mathbf{k}-\mathbf{k}'|^{2}} \\
&\quad \times\left[\sigma\omega_{\mathbf{k}}^{L}I_{\mathbf{k}'}^{\sigma'L}I_{\mathbf{k}-\mathbf{k}'}^{\sigma''L}\right. \\
&\quad \left.-\left(\sigma'\omega_{\mathbf{k}'}^{L}I_{\mathbf{k}-\mathbf{k}'}^{\sigma''L}+\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L}I_{\mathbf{k}'}^{\sigma'L}\right)\frac{I_{\mathbf{k}}^{\sigma S}}{\mu_{\mathbf{k}}}\right] \\
&\quad \times\delta(\sigma\omega_{\mathbf{k}}^{S}-\sigma'\omega_{\mathbf{k}'}^{L}-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L}), \\
\mu_{\mathbf{k}} &= \frac{k^{3}v_{e}^{3}}{\omega_{pe}^{3}}\left(\frac{m_{e}}{m_{i}}\right)^{1/2}. \quad (16)
\end{aligned}
$$

Here, $\alpha = L,S$ denotes Langmuir and ion sound wave. To reiterate, this form of nonlinear equation is identical to that one encounters in kinetic theory of weak turbulence theory, except that the linear and nonlinear wave-particle resonant interaction terms are missing. Having said that, however, we should caution the readers that the subtle difference is how the linear dispersion relations for Langmuir ($L$) and ($S$) are defined. Whereas in the present two-fluid formalism, they are defined by $\omega_{\mathbf{k}}^{L} = \omega_{pe}\left(1+\frac{1}{2}k^{2}\lambda_{D}^{2}\right)$ and $\omega_{\mathbf{k}}^{S} = kc_{s}(1+T_{i}/T_{e})^{\frac{1}{2}}(1+k^{2}\lambda_{D}^{2})^{-\frac{1}{2}}$, the kinetic theoretical definitions are $\omega_{\mathbf{k}}^{L} = \omega_{pe}\left(1+\frac{3}{2}k^{2}\lambda_{D}^{2}\right)$ and $\omega_{\mathbf{k}}^{S} = kc_{s}(1+3T_{i}/T_{e})^{\frac{1}{2}}(1+k^{2}\lambda_{D}^{2})^{-\frac{1}{2}}$.

### 2.2. *Electromagnetic formalism*

We next move on to the more general electromagnetic problem, yet without the constant magnetic field. The two-fluid equations are thus generalized to:

$$
\begin{aligned}
&\frac{\partial n_{a}}{\partial t}+\nabla\cdot(n_{a}\mathbf{v}_{a}) = 0, \\
&\left(\frac{\partial}{\partial t}+\mathbf{v}_{a}\cdot\nabla\right)\mathbf{v}_{a}+\frac{\nabla(n_{a}T_{a})}{m_{a}n_{a}}-\frac{e_{a}}{m_{a}}\left(\mathbf{E}+\frac{\mathbf{v}_{a}}{c}\times\mathbf{B}\right) = 0, \\
&\nabla\times\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} = 0, \\
&\nabla\times\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} = \sum_{a}\frac{4\pi e_{a}n_{a}\mathbf{v}_{a}}{c}. \quad (17)
\end{aligned}
$$

Again separating quantities in terms of averages and fluctuations, we obtain the spectral form of nonlinear equations,

$$
\begin{aligned}
\frac{\delta n^a_{\mathbf{k},\omega}}{n_0} &= \frac{\mathbf{k}\cdot\delta\mathbf{v}^a_{\mathbf{k},\omega}}{\omega} + \frac{1}{\omega}\sum_{\mathbf{k}',\omega'}\frac{\delta n^a_{\mathbf{k}',\omega'}}{n_0}\mathbf{k}\cdot\delta\mathbf{v}^a_{\mathbf{k}-\mathbf{k}',\omega-\omega},\\
\delta\mathbf{v}^a_{\mathbf{k},\omega} &= \frac{ie_a}{m_a\omega}\delta\mathbf{E}_{\mathbf{k},\omega} + \frac{\mathbf{k}v_{Ta}^2}{\omega}\frac{\delta n^a_{\mathbf{k},\omega}}{n_0} + \frac{1}{\omega}\sum_{\mathbf{k}',\omega'}\\
&\quad\times[(\mathbf{k}-\mathbf{k}')\cdot\delta\mathbf{v}^a_{\mathbf{k}',\omega'}]\,\delta\mathbf{v}^a_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\\
&\quad+\sum_{\mathbf{k}',\omega'}\frac{ie_a}{m_a\omega(\omega-\omega')}(\delta\mathbf{v}^a_{\mathbf{k}',\omega'}\cdot\delta\mathbf{E}_{\mathbf{k}-\mathbf{k}',\omega-\omega'})\\
&\quad\times(\mathbf{k}-\mathbf{k}')\\
&\quad-\sum_{\mathbf{k}',\omega'}\frac{ie_a}{m_a\omega(\omega-\omega')}[(\mathbf{k}-\mathbf{k}')\cdot\delta\mathbf{v}^a_{\mathbf{k}',\omega'}]\\
&\quad\times\delta\mathbf{E}_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\\
&\quad-\sum_{\mathbf{k}',\omega'}(\mathbf{k}-\mathbf{k}')\frac{v_{T_a}^2}{\omega}\frac{\delta n^a_{\mathbf{k}',\omega'}}{n_0}\frac{\delta n^a_{\mathbf{k}-\mathbf{k}',\omega-\omega'}}{n_0},\\
0 &= \left(1-\frac{c^2k^2}{\omega^2}\right)\delta\mathbf{E}_{\mathbf{k},\omega} + \frac{c^2}{\omega^2}(\mathbf{k}\cdot\delta\mathbf{E}_{\mathbf{k},\omega})\mathbf{k} + i\sum_a\\
&\quad\times\frac{4\pi e_a n_0}{\omega}\delta\mathbf{v}^a_{\mathbf{k},\omega} + i\sum_a\frac{4\pi e_a n_0}{\omega}\sum_{\mathbf{k}',\omega'}\frac{\delta n^a_{\mathbf{k}',\omega'}}{n_0}\\
&\quad\times\delta\mathbf{v}^a_{\mathbf{k}-\mathbf{k}',\omega-\omega'}.
\end{aligned}
\tag{18}
$$

We again implement the iterative solution method by decomposing the fluid particle quantities by first- and second order density and velocities. The first order solution can be obtained immediately,

$$
\begin{aligned}
\frac{\delta n^{a(1)}_{\mathbf{k},\omega}}{n_0} &= \frac{ie_a}{m_a}\frac{k_j}{\omega^2-k^2v_a^2}\delta E^j_{\mathbf{k},\omega},\\
(\delta\mathbf{v}^{a(1)}_{\mathbf{k},\omega})_i &= \frac{ie_a}{m_a}\frac{\omega^2\delta_{ij}-v_a^2k^2\delta_{ij}+v_a^2k_ik_j}{\omega(\omega^2-k^2v_a^2)}\delta E^j_{\mathbf{k},\omega},
\end{aligned}
\tag{19}
$$

where repeated indices represent the dot product, i.e. the Einstein convention, and $\delta_{ij}$ denote the Kronecker delta, $\delta=0$ if $i\neq j$ and 1 if $i=j$. In writing the second order equations, we reshuffle and combine certain terms. The result is:

$$
\begin{aligned}
\frac{\delta n^{a(2)}_{\mathbf{k},\omega}}{n_0} &= \frac{1}{\omega^2-k^2v_a^2}\sum_{\mathbf{k}',\omega'}\\
&\quad\times\Bigg((\mathbf{k}\cdot\delta\mathbf{v}^{a(1)}_{\mathbf{k}-\mathbf{k}',\omega-\omega'})[(\mathbf{k}-\mathbf{k}')\cdot\delta\mathbf{v}^{a(1)}_{\mathbf{k}',\omega'}]\\
&\quad+\frac{ie_a}{m_a}\frac{\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')}{\omega-\omega'}(\delta\mathbf{v}^{a(1)}_{\mathbf{k}',\omega'}\cdot\delta\mathbf{E}_{\mathbf{k}-\mathbf{k}',\omega-\omega'})\\
&\quad-\frac{ie_a}{m_a}\frac{(\mathbf{k}-\mathbf{k}')\cdot\delta\mathbf{v}^{a(1)}_{\mathbf{k}',\omega'}}{\omega-\omega'}(\mathbf{k}\cdot\delta\mathbf{E}_{\mathbf{k}-\mathbf{k}',\omega-\omega'})\Bigg)\\
&\quad-\sum_{\mathbf{k}',\omega'}\Bigg(\frac{\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')v_a^2}{\omega^2-k^2v_a^2}\frac{\delta n^{a(1)}_{\mathbf{k}',\omega'}}{n_0}\frac{\delta n^{a(1)}_{\mathbf{k}-\mathbf{k}',\omega-\omega'}}{n_0}\\
&\quad-\frac{\delta n^{a(1)}_{\mathbf{k}',\omega'}}{n_0}\frac{\mathbf{k}\cdot\delta\mathbf{v}^{a(1)}_{\mathbf{k}-\mathbf{k}',\omega-\omega'}}{\omega}\Bigg),\\
\delta\mathbf{v}^{a(2)}_{\mathbf{k},\omega} &= \frac{\mathbf{k}v_a^2}{\omega}\frac{1}{\omega^2-k^2v_a^2}\\
&\quad\times\Bigg(\sum_{\mathbf{k}',\omega'}(\mathbf{k}\cdot\delta\mathbf{v}^{a(1)}_{\mathbf{k}-\mathbf{k}',\omega-\omega'})[(\mathbf{k}-\mathbf{k}')\cdot\delta\mathbf{v}^{a(1)}_{\mathbf{k}',\omega'}]\\
&\quad+\frac{ie_a}{m_a}\frac{\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')}{\omega-\omega'}(\delta\mathbf{v}^{a(1)}_{\mathbf{k}',\omega'}\cdot\delta\mathbf{E}_{\mathbf{k}-\mathbf{k}',\omega-\omega'})\\
&\quad-\frac{ie_a}{m_a}\sum_{\mathbf{k}',\omega'}\frac{(\mathbf{k}-\mathbf{k}')\cdot\delta\mathbf{v}^{a(1)}_{\mathbf{k}',\omega'}}{\omega-\omega'}(\mathbf{k}\cdot\delta\mathbf{E}_{\mathbf{k}-\mathbf{k}',\omega-\omega'})\\
&\quad-\sum_{\mathbf{k}',\omega'}[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')v_a^2]\frac{\delta n^{a(1)}_{\mathbf{k}',\omega'}}{n_0}\frac{\delta n^{a(1)}_{\mathbf{k}-\mathbf{k}',\omega-\omega'}}{n_0}\Bigg)\\
&\quad+\sum_{\mathbf{k}',\omega'}\Bigg(\frac{\mathbf{k}v_a^2}{\omega}\frac{\delta n^{a(1)}_{\mathbf{k}',\omega'}}{n_0}\frac{\mathbf{k}\cdot\delta\mathbf{v}^{a(1)}_{\mathbf{k}-\mathbf{k}',\omega-\omega'}}{\omega}\\
&\quad+\frac{(\mathbf{k}-\mathbf{k}')\cdot\delta\mathbf{v}^{a(1)}_{\mathbf{k}',\omega'}}{\omega}\delta\mathbf{v}^{a(1)}_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\\
&\quad+\frac{ie_a}{m_a\omega}\frac{\mathbf{k}-\mathbf{k}'}{\omega-\omega'}(\delta\mathbf{v}^{a(1)}_{\mathbf{k}',\omega'}\cdot\delta\mathbf{E}_{\mathbf{k}-\mathbf{k}',\omega-\omega'})\\
&\quad-\frac{ie_a}{m_a\omega}\frac{(\mathbf{k}-\mathbf{k}')\cdot\delta\mathbf{v}^{a(1)}_{\mathbf{k}',\omega'}}{\omega-\omega'}\delta\mathbf{E}_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\\
&\quad-\frac{(\mathbf{k}-\mathbf{k}')v_a^2}{\omega}\frac{\delta n^{a(1)}_{\mathbf{k}',\omega'}}{n_0}\frac{\delta n^{a(1)}_{\mathbf{k}-\mathbf{k}',\omega-\omega'}}{n_0}\Bigg).
\end{aligned}
\tag{20}
$$

Once we solve for the fluid particle quantities via iterative means, then we couple the fluid quantities with the wave equation,

$$
\begin{aligned}
0 &= \left(1-\frac{c^2k^2}{\omega^2}\right)\delta\mathbf{E}_{\mathbf{k},\omega} + \frac{c^2}{\omega^2}\mathbf{k}(\mathbf{k}\cdot\delta\mathbf{E}_{\mathbf{k},\omega}) + i\sum_a\frac{4\pi e_a n_0}{\omega}\\
&\quad\times\delta\mathbf{v}^{a(1)}_{\mathbf{k},\omega} + i\sum_a\frac{4\pi e_a n_0}{\omega}\delta\mathbf{v}^{a(2)}_{\mathbf{k},\omega} + i\sum_a\frac{4\pi e_a n_0}{\omega}\sum_{\mathbf{k}',\omega'}\\
&\quad\times\frac{\delta n^{a(1)}_{\mathbf{k}',\omega'}}{n_0}\delta\mathbf{v}^{a(1)}_{\mathbf{k}-\mathbf{k}',\omega-\omega'}.
\end{aligned}
\tag{21}
$$

An immediately obvious fact is that the second order density perturbation is not needed in the final wave equation. This facilitates the computation.

For the sake of notational simplicity, let us omit $\delta$ in front of perturbed quantities and make use of the shorthand notation $K=(\mathbf{k},\omega)$. Making use of the first order solution the second order velocity perturbation is thus expressed in shorthand notation by:

$$
\begin{aligned}
(\mathbf{v}_K^{a(2)})_i = & -\frac{e_a^2}{m_a^2\omega}\sum_{K'}\left(\frac{(\omega'^2-k'^2v_a^2)(\mathbf{k}-\mathbf{k}')_j+v_a^2\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')k_j'}{\omega'(\omega-\omega')(\omega'^2-k'^2v_a^2)[(\omega-\omega')^2-(\mathbf{k}-\mathbf{k}')^2v_a^2]}\right.\\
& \times\{[(\omega-\omega')^2-(\mathbf{k}-\mathbf{k}')^2v_a^2]\delta_{ik}+v_a^2(\mathbf{k}-\mathbf{k}')_i(\mathbf{k}-\mathbf{k}')_k\}+\frac{(\mathbf{k}-\mathbf{k}')_i[(\omega'^2-k'^2v_a^2)\delta_{jk}+v_a^2k_j'k_k']}{\omega'(\omega-\omega')(\omega'^2-k'^2v_a^2)}\\
& -\frac{\delta_{ik}[(\omega'^2-k'^2v_a^2)(\mathbf{k}-\mathbf{k}')_j+v_a^2\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')k_j']}{\omega'(\omega-\omega')(\omega'^2-k'^2v_a^2)}\\
& +\frac{k_i[(\omega'^2-k'^2v_a^2)(\mathbf{k}-\mathbf{k}')_j+v_a^2\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')k_j']}{(\omega^2-k^2v_a^2)\omega'(\omega-\omega')(\omega'^2-k'^2v_a^2)[(\omega-\omega')^2-(\mathbf{k}-\mathbf{k}')^2v_a^2]}\\
& \times\{[(\omega-\omega')^2-(\mathbf{k}-\mathbf{k}')^2v_a^2]k_k+v_a^2\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')(\mathbf{k}-\mathbf{k}')_k\}v_a^2\\
& +\frac{\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')[(\omega'^2-k'^2v_a^2)\delta_{jk}k_i+v_a^2k_ik_j'k_k']v_a^2}{(\omega^2-k^2v_a^2)\omega'(\omega-\omega')(\omega'^2-k'^2v_a^2)}\\
& -\frac{k_ik_k[(\omega'^2-k'^2v_a^2)(\mathbf{k}-\mathbf{k}')_j+v_a^2\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')k_j']v_a^2}{(\omega^2-k^2v_a^2)\omega'(\omega-\omega')(\omega'^2-k'^2v_a^2)}\\
& -\frac{\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')k_ik_j'(\mathbf{k}-\mathbf{k}')_kv_a^4}{(\omega^2-k^2v_a^2)(\omega'^2-k'^2v_a^2)[(\omega-\omega')^2-(\mathbf{k}-\mathbf{k}')^2v_a^2]}\\
& +\frac{k_ik_j'\{[(\omega-\omega')^2-(\mathbf{k}-\mathbf{k}')^2v_a^2]k_k+v_a^2\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')(\mathbf{k}-\mathbf{k}')_k\}v_a^2}{\omega(\omega-\omega')(\omega'^2-k'^2v_a^2)[(\omega-\omega')^2-(\mathbf{k}-\mathbf{k}')^2v_a^2]}\\
& \left.-\frac{(\mathbf{k}-\mathbf{k}')_ik_j'(\mathbf{k}-\mathbf{k}')_kv_a^2}{(\omega'^2-k'^2v_a^2)[(\omega-\omega')^2-(\mathbf{k}-\mathbf{k}')^2v_a^2]}\right)E_{K'}^jE_{K-K'}^k.
\end{aligned}
\tag{22}
$$

As is apparent, even with the fluid approach, which is far simpler than the kinetic theory, the second order velocity perturbation is still quite complex. This is mostly owing to the fluid thermal correction. However, the second order perturbation is essentially a next-order correction to the linear perturbation. As such, it is not necessary to retain the full thermal correction term. We thus focus only on the leading terms, that is, we ignore terms with explicit $v_a^2$ dependence in the overall numerator, but only retain thermal correction in the denominator. Then it can be shown that the final result, which is shown with long-hand notation again, is quite compactly given by:

$$
\begin{aligned}
(\delta\mathbf{v}_{\mathbf{k},\omega}^{a(2)})_i = & -\frac{e_a^2}{m_a^2}\sum_{\mathbf{k}',\omega'}\frac{\omega'(\mathbf{k}-\mathbf{k}')_i\delta_{jk}}{\omega(\omega-\omega')(\omega'^2-k'^2v_a^2)}\\
& \times\delta E_{\mathbf{k}',\omega'}^j\delta E_{\mathbf{k}-\mathbf{k}',\omega-\omega'}^k.
\end{aligned}
\tag{23}
$$

Inserting the iterative solution to the wave equation, and rewriting the nonlinear term in symmetric fashion with respect to dummy integral variables, we have:

$$
\begin{aligned}
0 = & \left[\delta_{ij}-\sum_a\frac{\omega_{pa}^2}{\omega^2}\frac{\omega^2\delta_{ij}-v_a^2k^2\delta_{ij}+v_a^2k_ik_j}{\omega^2-k^2v_a^2}\right.\\
& \left.\times-\frac{c^2k^2}{\omega^2}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\right]\delta E_{\mathbf{k},\omega}^j+\sum_{\mathbf{k}',\omega'}\sum_a\frac{-ie_a}{2m_a}\frac{\omega_{pa}^2}{\omega^2}\\
& \times\left(\frac{\omega'(\mathbf{k}-\mathbf{k}')_i\delta_{jk}}{(\omega-\omega')(\omega'^2-k'^2v_a^2)}\right.\\
& +\frac{\omega(\omega-\omega')k_j'\delta_{ik}}{(\omega'^2-k'^2v_a^2)[(\omega-\omega')^2-(\mathbf{k}-\mathbf{k}')^2v_a^2]}\\
& +\frac{(\omega-\omega')k'_i\delta_{jk}}{\omega'[(\omega-\omega')^2-(\mathbf{k}-\mathbf{k}')^2v_a^2]}\\
& \left.+\frac{\omega\omega'(\mathbf{k}-\mathbf{k}')_k\delta_{ij}}{(\omega'^2-k'^2v_a^2)[(\omega-\omega')^2-(\mathbf{k}-\mathbf{k}')^2v_a^2]}\right)\\
& \times\delta E_{\mathbf{k}',\omega'}^j\delta E_{\mathbf{k}-\mathbf{k}',\omega-\omega'}^k.
\end{aligned}
\tag{24}
$$

Rearranging terms and ignoring thermal correction in the numerator, the above reduces to:

$$
\begin{aligned}
0 = & \Lambda_{ij}(\mathbf{k},\omega)\,\delta E_{\mathbf{k},\omega}^j+\sum_{\mathbf{k}',\omega'}\chi_{ijk}(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')\\
& \times\delta E_{\mathbf{k}',\omega'}^j\delta E_{\mathbf{k}-\mathbf{k}',\omega-\omega'}^k,
\end{aligned}
$$

$$\Lambda_{ij}(\mathbf{k},\omega)=\epsilon_{ij}(\mathbf{k},\omega)-\frac{c^2k^2}{\omega^2}\left(\delta_{ij}-\frac{k_ikj}{k^2}\right),$$
$$\epsilon_{ij}(\mathbf{k},\omega)=\delta_{ij}+\sum_a\chi^a_{ij}(\mathbf{k},\omega),$$
$$\chi^a_{ij}(\mathbf{k},\omega)=-\sum_a\frac{\omega_{pa}^2}{\omega^2}\delta_{ij}-\sum_a\frac{\omega_{pa}^2}{\omega^2}\frac{k^2v_a^2}{\omega^2-k^2v_a^2}\frac{k_ik_j}{k^2},$$
$$\chi_{ijk}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2)=\sum_a\frac{-ie_a}{2m_a}\times\frac{\omega_{pa}^2\omega_1\omega_2\omega}{(\omega_1^2-k_1^2v_a^2)(\omega_2^2-k_2^2v_a^2)(\omega^2-k^2v_a^2)}\times\left(\frac{k_i\delta_{jk}}{\omega}+\frac{k_{1j}\delta_{ik}}{\omega_1}+\frac{k_{2k}\delta_{ij}}{\omega_2}\right). \tag{25}$$

The rest of the analysis is equivalent to the kinetic formalism, namely, we take the product of the wave equation with $\delta E_i(\mathbf{k}',\omega')$, and take the ensemble average. We also obtain the three body cumulant in the same manner as explained already. The result is:

$$\begin{aligned}0=&\Lambda_{ij}(\mathbf{k},\omega)\,\langle\delta E_i\,\delta E_j\rangle_{\mathbf{k},\omega}-2\sum_{\mathbf{k}',\omega'}\chi_{ijk}(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')\\&\times\Big\{\Big[\Lambda^{-1}_{jl}(\mathbf{k}',\omega')\chi_{lmn}(-\mathbf{k}+\mathbf{k}',-\omega+\omega'|\mathbf{k},\omega)\\&\quad\times\langle\delta E_k\,\delta E_m\rangle_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\\&\quad+\Lambda^{-1}_{kl}(\mathbf{k}-\mathbf{k}',\omega-\omega')\chi_{lmn}(-\mathbf{k}',-\omega'|\mathbf{k},\omega)\\&\quad\times\langle\delta E_j\,\delta E_m\rangle_{\mathbf{k}',\omega'}\Big]\langle\delta E_i\,\delta E_n\rangle_{\mathbf{k},\omega}\\&\quad+\Lambda^{-1}_{il}(-\mathbf{k},-\omega)\chi_{lmn}(-\mathbf{k}',-\omega'|-\mathbf{k}+\mathbf{k}',-\omega+\omega')\\&\quad\times\langle\delta E_j\,\delta E_m\rangle_{\mathbf{k}',\omega'}\langle\delta E_k\,\delta E_n\rangle_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\Big\}.\end{aligned} \tag{26}$$

This is the nonlinear spectral balance equation based upon which further manipulation will lead to the nonlinear wave kinetic equation, to be discussed next. The first step toward the derivation is to introduce the slow time dependence to $\Lambda_{ij}(\mathbf{k},\omega)$ via $\Lambda_{ij}(\mathbf{k},\omega)\to\Lambda_{ij}(\mathbf{k},\omega+i\partial/\partial t)\approx\Lambda_{ij}(\mathbf{k},\omega)+(i/2)(\partial\Lambda_{ij}(\mathbf{k},\omega)/\partial\omega)(\partial/\partial t)$. We then invoke the symmetry relations $\Lambda_{ij}(-\mathbf{k},-\omega)=\Lambda^*_{ij}(\mathbf{k},\omega)$, $\chi_{ijk}(-\mathbf{k}_1,-\omega_1|-\mathbf{k}_2,-\omega_2)=\chi^*_{ijk}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2)$, $\chi_{ijk}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2)=\chi_{ikj}(\mathbf{k}_2,\omega_2|\mathbf{k}_1,\omega_1)$, and $\chi_{ijk}(\mathbf{k}_1+\mathbf{k}_2,\omega_1+\omega_2|-\mathbf{k}_2,-\omega_2)=-\chi_{jik}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2)$. Further, we make use of the specific diagonal expressions for the linear dielectric susceptibility tensor and its inverse,

$$\begin{aligned}\Lambda_{ij}(\mathbf{k},\omega)&=\frac{k_i\,k_j}{k^2}\,\epsilon_\parallel(\mathbf{k},\omega)+\left(\delta_{ij}-\frac{k_i\,k_j}{k^2}\right)\Lambda_\perp(\mathbf{k},\omega),\\\Lambda^{-1}_{ij}(\mathbf{k},\omega)&=\frac{k_i\,k_j}{k^2}\frac{1}{\epsilon_\parallel(\mathbf{k},\omega)}+\left(\delta_{ij}-\frac{k_i\,k_j}{k^2}\right)\frac{1}{\Lambda_\perp(\mathbf{k},\omega)},\\\Lambda_\perp(\mathbf{k},\omega)&=\epsilon_\perp(\mathbf{k},\omega)-\frac{c^2k^2}{\omega^2},\\\epsilon_\parallel(\mathbf{k},\omega)&=1-\sum_a\frac{\omega_{pa}^2}{\omega^2-k^2v_a^2},\\\epsilon_\perp(\mathbf{k},\omega)&=1-\sum_a\frac{\omega_{pa}^2}{\omega^2}.\end{aligned} \tag{27}$$

Finally, the spectral wave energy density tensor is decomposed into longitudinal and transverse parts in diagonal form,

$$\langle E_iE_j\rangle_{\mathbf{k},\omega}=\frac{k_i\,k_j}{k^2}\left\langle E_\parallel^2\right\rangle_{\mathbf{k},\omega}+\frac{1}{2}\left(\delta_{ij}-\frac{k_i\,k_j}{k^2}\right)\left\langle E_\perp^2\right\rangle_{\mathbf{k},\omega}. \tag{28}$$

Implementing all of the above processes to nonlinear wave equation (26) leads to quite a lengthy expression, which is fully written down in the author's monograph so that readers may independently check its accuracy [20]. The full expression in the monograph, however, includes terms that arise as a result of formulating the problem from the framework of kinetic theory. In the present fluid formalism, we have only partial terms that relate to the fluid treatment. Nevertheless these partial terms are sufficient for the description of electromagnetic decay interaction. For the sake of completeness, we show the intermediate result,

$$\begin{aligned}0=&\frac{i}{2}\frac{\partial\epsilon_\parallel(\mathbf{k},\omega)}{\partial\omega}\frac{\partial\langle E_\parallel^2\rangle_{\mathbf{k},\omega}}{\partial t}+\frac{i}{2}\frac{\partial\Lambda_\perp(\mathbf{k},\omega)}{\partial\omega}\frac{\partial\langle E_\perp^2\rangle_{\mathbf{k},\omega}}{\partial t}+\langle E_\parallel^2\rangle_{\mathbf{k},\omega}\epsilon_\parallel(\mathbf{k},\omega)+\langle E_\perp^2\rangle_{\mathbf{k},\omega}\Lambda_\perp(\mathbf{k},\omega)+2\int d\mathbf{k}'\int d\omega'\\&\times\Bigg\{\Bigg[\frac{k_j'k_l'}{k'^2}\frac{\chi_{ijk}(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')\chi^{(2)}_{nlm}(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')}{\epsilon_\parallel(\mathbf{k}',\omega')}\\&+\left(\delta_{jl}-\frac{k_j'k_l'}{k'^2}\right)\frac{\chi_{ijk}(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')\chi_{nlm}(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')}{\Lambda_\perp(\mathbf{k}',\omega')}\Bigg]\\&\times\Bigg[\frac{k_ik_n}{k^2}\frac{(\mathbf{k}-\mathbf{k}')_k(\mathbf{k}-\mathbf{k}')_m}{|\mathbf{k}-\mathbf{k}'|^2}\langle E_\parallel^2\rangle_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\langle E_\parallel^2\rangle_{\mathbf{k},\omega}+\frac{1}{2}\frac{k_ik_n}{k^2}\left(\delta_{km}-\frac{(\mathbf{k}-\mathbf{k}')_k(\mathbf{k}-\mathbf{k}')_m}{|\mathbf{k}-\mathbf{k}'|^2}\right)\\&\times\langle E_\perp^2\rangle_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\langle E_\parallel^2\rangle_{\mathbf{k},\omega}+\frac{1}{2}\left(\delta_{in}-\frac{k_ik_n}{k^2}\right)\frac{(\mathbf{k}-\mathbf{k}')_k(\mathbf{k}-\mathbf{k}')_m}{|\mathbf{k}-\mathbf{k}'|^2}\langle E_\parallel^2\rangle_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\langle E_\perp^2\rangle_{\mathbf{k},\omega}\\&+\frac{1}{4}\left(\delta_{in}-\frac{k_ik_n}{k^2}\right)\left(\delta_{km}-\frac{(\mathbf{k}-\mathbf{k}')_k(\mathbf{k}-\mathbf{k}')_m}{|\mathbf{k}-\mathbf{k}'|^2}\right)\langle E_\perp^2\rangle_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\langle E_\perp^2\rangle_{\mathbf{k},\omega}\Bigg]\\&+\Bigg[\frac{(\mathbf{k}-\mathbf{k}')_k(\mathbf{k}-\mathbf{k}')_l}{|\mathbf{k}-\mathbf{k}'|^2}\frac{\chi_{ijk}(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')\chi_{nml}(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')}{\epsilon_\parallel(\mathbf{k}-\mathbf{k}',\omega-\omega')}\end{aligned}$$

$$+\left(\delta_{kl}-\frac{(\mathbf{k}-\mathbf{k}')_k(\mathbf{k}-\mathbf{k}')_l}{|\mathbf{k}-\mathbf{k}'|^2}\right)\frac{\chi^{(2)}_{ijk}(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')\,\chi^{(2)}_{nml}(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')}{\Lambda_\perp(\mathbf{k}-\mathbf{k}',\omega-\omega')}\Bigg]$$
$$\times\left[\frac{k_ik_n}{k^2}\frac{k_j'k_m'}{k'^2}\langle E_\parallel^2\rangle_{\mathbf{k}',\omega'}\langle E_\parallel^2\rangle_{\mathbf{k},\omega}+\frac{1}{2}\frac{k_ik_n}{k^2}\left(\delta_{jm}-\frac{k_j'k_m'}{k'^2}\right)\langle E_\perp^2\rangle_{\mathbf{k}',\omega'}\langle E_\parallel^2\rangle_{\mathbf{k},\omega}\right.$$
$$\left.+\frac{1}{2}\left(\delta_{in}-\frac{k_ik_n}{k^2}\right)\frac{k_j'k_m'}{k'^2}\langle E_\parallel^2\rangle_{\mathbf{k}',\omega'}\langle E_\perp^2\rangle_{\mathbf{k},\omega}+\frac{1}{4}\left(\delta_{in}-\frac{k_ik_n}{k^2}\right)\left(\delta_{jm}-\frac{k_j'k_m'}{k'^2}\right)\langle E_\perp^2\rangle_{\mathbf{k}',\omega'}\langle E_\perp^2\rangle_{\mathbf{k},\omega}\right]$$
$$-\left[\frac{k_ik_l}{k^2}\frac{1}{\epsilon_\parallel^*(\mathbf{k},\omega)}\chi_{ijk}(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')\,\chi^*_{lmn}(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')\right.$$
$$\left.+\left(\delta_{il}-\frac{k_ik_l}{k^2}\right)\frac{1}{\Lambda_\perp^*(\mathbf{k},\omega)}\chi_{ijk}(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')\,\chi^*_{lmn}(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')\right]$$
$$\times\left[\frac{k_j'k_m'}{k'^2}\frac{(\mathbf{k}-\mathbf{k}')_k(\mathbf{k}-\mathbf{k}')_n}{|\mathbf{k}-\mathbf{k}'|^2}\langle E_\parallel^2\rangle_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\langle E_\parallel^2\rangle_{\mathbf{k}',\omega'}\right.$$
$$+\frac{1}{2}\frac{k_j'k_m'}{k'^2}\left(\delta_{kn}-\frac{(\mathbf{k}-\mathbf{k}')_k(\mathbf{k}-\mathbf{k}')_n}{|\mathbf{k}-\mathbf{k}'|^2}\right)\langle E_\parallel^2\rangle_{\mathbf{k}',\omega'}\langle E_\perp^2\rangle_{\mathbf{k}-\mathbf{k}',\omega-\omega'}$$
$$+\frac{1}{2}\left(\delta_{jm}-\frac{k_j'k_m'}{k'^2}\right)\frac{(\mathbf{k}-\mathbf{k}')_k(\mathbf{k}-\mathbf{k}')_n}{|\mathbf{k}-\mathbf{k}'|^2}\langle E_\perp^2\rangle_{\mathbf{k}',\omega'}\langle E_\parallel^2\rangle_{\mathbf{k}-\mathbf{k}',\omega-\omega'}$$
$$\left.\left.+\frac{1}{4}\left(\delta_{jm}-\frac{k_j'k_m'}{k'^2}\right)\left(\delta_{kn}-\frac{(\mathbf{k}-\mathbf{k}')_k(\mathbf{k}-\mathbf{k}')_n}{|\mathbf{k}-\mathbf{k}'|^2}\right)\langle E_\perp^2\rangle_{\mathbf{k}',\omega'}\langle E_\perp^2\rangle_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\right]\right\}. \tag{29}$$

This equation can further be analyzed for specific eigenmodes, that is, the two longitudinal electrostatic modes, Langmuir and ion sound waves, and the transverse electromagnetic mode.

Electrostatic linear eigenmodes have already been discussed in the previous subsection. They satisfy the longitudinal wave dispersion relation, $\epsilon_\parallel(\mathbf{k},\omega_\mathbf{k}^\alpha)=0$. We already discussed that $\alpha=L$ and $S$ correspond to Langmuir and ion-sound modes. The transverse electromagnetic ($T$) mode satisfies the transverse wave dispersion relation. For $T$ mode the ion response can be ignored and we have $\Lambda_\perp(\mathbf{k},\omega_\mathbf{k}^T)=1-\omega_{pe}^2/(\omega_\mathbf{k}^T)^2-c^2k^2/(\omega_\mathbf{k}^T)^2$, from which we easily obtain $\omega_\mathbf{k}^T=(\omega_{pe}^2+c^2k^2)^{\frac{1}{2}}$. As with the longitudinal mode, the transverse mode spectrum can be represented by $\langle E_\perp^2\rangle_{\mathbf{k},\omega}=\sum_{\sigma=\pm1}I_\mathbf{k}^{\sigma T}\,\delta(\omega-\sigma\omega_\mathbf{k}^T)$. Making use of the short-hand notations, $\epsilon_\parallel'(\mathbf{k},\omega)=\partial\epsilon_\parallel(\mathbf{k},\omega)/\partial\omega$ and $\Lambda_\perp'(\mathbf{k},\omega)=\partial\Lambda_\perp(\mathbf{k},\omega)/\partial\omega$, and making note of the fact that inverses of linear dielectric response function makes no contributions to the wave kinetic equation from their principal parts, we can show, after some tedious but otherwise straightforward exercise (note that [20] fully spells out the intermediate steps) that the wave equation (29) decouples into an equation for longitudinal mode and an equation for transverse mode,

$$0=\epsilon_\parallel'(\mathbf{k},\sigma\omega_\mathbf{k}^\alpha)\frac{\partial I_\mathbf{k}^{\sigma\alpha}}{\partial t}+4\pi\int d\mathbf{k}'$$
$$\times\left\{\sum_{\beta,\gamma}\sum_{\sigma',\sigma''}\frac{k_ik_n}{k^2}\frac{k_j'k_m'}{k'^2}\frac{(\mathbf{k}-\mathbf{k}')_k(\mathbf{k}-\mathbf{k}')_l}{|\mathbf{k}-\mathbf{k}'|^2}\right.$$
$$\times\chi_{ijk}(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^\beta|\mathbf{k}-\mathbf{k}',\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^\gamma)\,\chi^*_{nml}$$
$$\times(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^\beta|\mathbf{k}-\mathbf{k}',\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^\gamma)$$
$$\times\left[\frac{I_{\mathbf{k}-\mathbf{k}'}^{\sigma''\gamma}I_\mathbf{k}^{\sigma\alpha}}{\epsilon_\parallel'(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^\beta)}+\frac{I_{\mathbf{k}'}^{\sigma'\beta}I_\mathbf{k}^{\sigma\alpha}}{\epsilon_\parallel'(\mathbf{k}-\mathbf{k}',\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^\gamma)}\right.$$
$$\left.-\frac{I_{\mathbf{k}-\mathbf{k}'}^{\sigma''\gamma}I_{\mathbf{k}'}^{\sigma'\beta}}{\epsilon_\parallel'(\mathbf{k},\sigma\omega_\mathbf{k}^\alpha)}\right]\delta(\sigma\omega_\mathbf{k}^\alpha-\sigma'\omega_{\mathbf{k}'}^\beta,-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^\gamma)$$
$$+\sum_\beta\sum_{\sigma',\sigma''}\frac{k_ik_n}{k^2}\frac{k_j'k_m'}{k'^2}\left(\delta_{kl}-\frac{(\mathbf{k}-\mathbf{k}')_k(\mathbf{k}-\mathbf{k}')_l}{|\mathbf{k}-\mathbf{k}'|^2}\right)$$
$$\times\chi_{ijk}(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^\beta|\mathbf{k}-\mathbf{k}',\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^T)\,\chi^*_{nml}$$
$$\times(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^\beta|\mathbf{k}-\mathbf{k}',\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^T)$$
$$\times\left[\frac{I_{\mathbf{k}-\mathbf{k}'}^{\sigma''T}I_\mathbf{k}^{\sigma\alpha}}{\epsilon_\parallel'(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^\beta)}+\frac{2I_{\mathbf{k}'}^{\sigma'\beta}I_\mathbf{k}^{\sigma\alpha}}{\Lambda_\perp'(\mathbf{k}-\mathbf{k}',\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^T)}\right.$$
$$\left.-\frac{I_{\mathbf{k}'}^{\sigma'\beta}I_{\mathbf{k}-\mathbf{k}'}^{\sigma''T}}{\epsilon_\parallel'(\mathbf{k},\sigma\omega_\mathbf{k}^\alpha)}\right]\delta(\sigma\omega_\mathbf{k}^\alpha-\sigma'\omega_{\mathbf{k}'}^\beta,-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^T)+\sum_{\sigma',\sigma''}$$
$$\times\frac{1}{4}\frac{k_ik_n}{k^2}\left(\delta_{jm}-\frac{k_j'k_m'}{k'^2}\right)\left(\delta_{kl}-\frac{(\mathbf{k}-\mathbf{k}')_k(\mathbf{k}-\mathbf{k}')_l}{|\mathbf{k}-\mathbf{k}'|^2}\right)$$
$$\times\chi_{ijk}(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^T|\mathbf{k}-\mathbf{k}',\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^T)\,\chi^*_{nml}$$
$$\times(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^T|\mathbf{k}-\mathbf{k}',\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^T)$$
$$\times\left[\frac{2I_{\mathbf{k}-\mathbf{k}'}^{\sigma''T}I_\mathbf{k}^{\sigma\alpha}}{\Lambda_\perp'(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^T)}+\frac{2I_{\mathbf{k}'}^{\sigma'T}I_\mathbf{k}^{\sigma\alpha}}{\Lambda_\perp'(\mathbf{k}-\mathbf{k}',\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^T)}\right.$$
$$\left.\left.-\frac{I_{\mathbf{k}'}^{\sigma'T}I_{\mathbf{k}-\mathbf{k}'}^{\sigma''T}}{\epsilon_\parallel'(\mathbf{k},\sigma\omega_\mathbf{k}^\alpha)}\right]\delta(\sigma\omega_\mathbf{k}^\alpha-\sigma'\omega_{\mathbf{k}'}^T,-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^T)\right\}, \tag{30}$$

where

$$\epsilon'_{\parallel}(\mathbf{k},\sigma\omega_{\mathbf{k}}^{L}) = \frac{2}{\sigma\omega_{\mathbf{k}}^{L}}, \quad \epsilon'_{\parallel}(\mathbf{k},\sigma\omega_{\mathbf{k}}^{S}) = \frac{2}{\sigma\omega_{\mathbf{k}}^{S}}, \quad \Lambda'_{\perp}(\mathbf{k},\sigma\omega_{\mathbf{k}}^{T}) = \frac{2}{\sigma\omega_{\mathbf{k}}^{T}}. \tag{31}$$

When compared with the electrostatic formalism—see equation (15), we now have additional decay processes involving $T$ mode. When compared with kinetic formalism [20], the present two fluid theory does not have wave-particle interactions so that induced emission and induced scattering terms are missing—as we already noted in the previous subsection—but the decay processes are reproduced exactly in the formal sense. The difference lies in the definition of susceptibilities, but the formal result is identical to the kinetic counterpart (minus the particle effects).

The electromagnetic formalism not only modifies the longitudinal wave kinetic equation, but as noted, it leads to the wave kinetic equation for $T$ mode. Following the prescribed and systematic methods, the formal wave kinetic equation (29) leads to the formal $T$ mode wave kinetic equation, which results in the following:

$$\begin{aligned}
0 = {} & \Lambda'_{\perp}(\mathbf{k},\sigma\omega_{\mathbf{k}}^{T})\frac{\partial I_{\mathbf{k}}^{\sigma T}}{\partial t} + 4\pi\int d\mathbf{k}' \\
& \times\Bigg\{\sum_{\beta,\gamma}\sum_{\sigma',\sigma''}\frac{1}{2}\left(\delta_{in}-\frac{k_i k_n}{k^2}\right)\frac{k'_j k'_m}{k'^2}\frac{(\mathbf{k}-\mathbf{k}')_k(\mathbf{k}-\mathbf{k}')_l}{|\mathbf{k}-\mathbf{k}'|^2} \\
& \times\chi_{ijk}(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^{\beta}|\mathbf{k}-\mathbf{k}',\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma})\chi^{*}_{nml} \\
& \times(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^{\beta}|\mathbf{k}-\mathbf{k}',\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma}) \\
& \times\left[\frac{I_{\mathbf{k}-\mathbf{k}'}^{\sigma''\gamma}I_{\mathbf{k}}^{\sigma T}}{\epsilon'_{\parallel}(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^{\beta})}+\frac{I_{\mathbf{k}'}^{\sigma'\beta}I_{\mathbf{k}}^{\sigma T}}{\epsilon'_{\parallel}(\mathbf{k}-\mathbf{k}',\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma})}\right. \\
& \quad\left.\times-\frac{2I_{\mathbf{k}-\mathbf{k}'}^{\sigma''\gamma}I_{\mathbf{k}'}^{\sigma'\beta}}{\Lambda'_{\perp}(\mathbf{k},\sigma\omega_{\mathbf{k}}^{T})}\right]\delta(\sigma\omega_{\mathbf{k}}^{T}-\sigma'\omega_{\mathbf{k}'}^{\beta},-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma}) \\
& +\sum_{\gamma}\sum_{\sigma',\sigma''}\frac{1}{2}\left(\delta_{in}-\frac{k_i k_n}{k^2}\right)\left(\delta_{jm}-\frac{k'_j k'_m}{k'^2}\right) \\
& \times\frac{(\mathbf{k}-\mathbf{k}')_k(\mathbf{k}-\mathbf{k}')_l}{|\mathbf{k}-\mathbf{k}'|^2} \\
& \times\chi_{ijk}(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^{T}|\mathbf{k}-\mathbf{k}',\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma})\chi^{*}_{nml} \\
& \times(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^{T}|\mathbf{k}-\mathbf{k}',\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma}) \\
& \times\left[\frac{2I_{\mathbf{k}-\mathbf{k}'}^{\sigma''\gamma}I_{\mathbf{k}}^{\sigma T}}{\Lambda'_{\perp}(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^{T})}+\frac{I_{\mathbf{k}'}^{\sigma'T}I_{\mathbf{k}}^{\sigma T}}{\epsilon'_{\parallel}(\mathbf{k}-\mathbf{k}',\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma})}\right. \\
& \quad\left.\times-\frac{2I_{\mathbf{k}-\mathbf{k}'}^{\sigma''\gamma}I_{\mathbf{k}'}^{\sigma'T}}{\Lambda'_{\perp}(\mathbf{k},\sigma\omega_{\mathbf{k}}^{T})}\right]\delta(\sigma\omega_{\mathbf{k}}^{T}-\sigma'\omega_{\mathbf{k}'}^{T},-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma})\Bigg\}.
\end{aligned} \tag{32}$$

As with the electrostatic case, formal equations (30) and (32) can be further simplified by taking advantage of approximate forms of nonlinear susceptibility. For electromagnetic case, the two-fluid version of nonlinear susceptibility can be approximated under various limiting conditions as follows:

$$\begin{aligned}
\chi_{ijk}^{a}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2) = {} & \frac{-ie_a}{2m_a}\frac{\omega_{pa}^2}{\omega_1\omega_2(\omega_1+\omega_2)} \\
& \times\left(\frac{k_{1j}\delta_{ik}}{\omega_1}+\frac{k_{2k}\delta_{ij}}{\omega_2}+\frac{k_i\delta_{jk}}{\omega}\right), \\
& (\omega_1\gg k_1 v_a,\quad \omega_2\gg k_2 v_a,\quad \omega\gg k v_a), \\
= {} & \frac{ie_a}{2T_a}\frac{\omega_{pa}^2}{\omega_2\omega}\frac{k_{1j}}{k_1^2}\delta_{ik}, \\
& (\omega_1\ll k_1 v_a,\quad \omega_2\gg k_2 v_a,\quad \omega\gg k v_a), \\
= {} & \frac{ie_a}{2T_a}\frac{\omega_{pa}^2}{\omega_1\omega}\frac{k_{2k}}{k_2^2}\delta_{ij}, \\
& (\omega_1\gg k_1 v_a,\quad \omega_2\ll k_2 v_a,\quad \omega\gg k v_a), \\
= {} & \frac{ie_a}{2T_a}\frac{\omega_{pa}^2}{\omega_1\omega_2}\frac{k_i}{k^2}\delta_{jk}, \\
& (\omega_1\gg k_1 v_a,\quad \omega_2\gg k_2 v_a,\quad \omega\ll k v_a),
\end{aligned} \tag{33}$$

where $\mathbf{k}=\mathbf{k}_1+\mathbf{k}_2$ and $\omega=\omega_1+\omega_2$. This leads to the explicit representation of the wave kinetic equations for $L$, $S$, and $T$, which are equivalent to the wave kinetic equations for the same modes derived under the framework of kinetic theory, except that the present two-fluid formalism only reproduces terms that depict various three wave decay processes:

$$\begin{aligned}
\frac{\partial I_{\mathbf{k}}^{\sigma L}}{\partial t} = {} & -\pi(\sigma\omega_{\mathbf{k}}^{L})\sum_{\sigma',\sigma''}\int d\mathbf{k}' \\
& \times\Bigg\{\frac{e^2}{4T_e^2}\frac{\mu_{\mathbf{k}'}[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')]^2}{k^2k'^2|\mathbf{k}-\mathbf{k}'|^2} \\
& \quad\times\left(\sigma'\omega_{\mathbf{k}'}^{L}I_{\mathbf{k}-\mathbf{k}'}^{\sigma''L}I_{\mathbf{k}}^{\sigma L}+\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L}\frac{I_{\mathbf{k}'}^{\sigma'S}}{\mu_{\mathbf{k}'}}I_{\mathbf{k}}^{\sigma L}-\sigma\omega_{\mathbf{k}}^{L}\right. \\
& \quad\quad\left.\times I_{\mathbf{k}-\mathbf{k}'}^{\sigma''L}\frac{I_{\mathbf{k}'}^{\sigma'S}}{\mu_{\mathbf{k}'}}\right)\delta(\sigma\omega_{\mathbf{k}}^{L}-\sigma'\omega_{\mathbf{k}'}^{S},-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L}) \\
& +\frac{e^2}{4T_e^2}\frac{\mu_{\mathbf{k}-\mathbf{k}'}(\mathbf{k}\cdot\mathbf{k}')^2}{k^2k'^2|\mathbf{k}-\mathbf{k}'|^2} \\
& \quad\times\left(\sigma'\omega_{\mathbf{k}'}^{L}\frac{I_{\mathbf{k}-\mathbf{k}'}^{\sigma''S}}{\mu_{\mathbf{k}-\mathbf{k}'}}I_{\mathbf{k}}^{\sigma L}+\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L}I_{\mathbf{k}'}^{\sigma'L}I_{\mathbf{k}}^{\sigma L}-\sigma\omega_{\mathbf{k}}^{L}\right. \\
& \quad\quad\left.\times\frac{I_{\mathbf{k}-\mathbf{k}'}^{\sigma''S}}{\mu_{\mathbf{k}-\mathbf{k}'}}I_{\mathbf{k}'}^{\sigma'L}\right)\delta(\sigma\omega_{\mathbf{k}}^{L}-\sigma'\omega_{\mathbf{k}'}^{L},-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{S}) \\
& +\frac{e^2}{2T_e^2}\frac{\mu_{\mathbf{k}'}(\mathbf{k}\times\mathbf{k}')^2}{k^2k'^2|\mathbf{k}-\mathbf{k}'|^2} \\
& \quad\times\left(\sigma'\omega_{\mathbf{k}'}^{L}\frac{I_{\mathbf{k}-\mathbf{k}'}^{\sigma''T}}{2}I_{\mathbf{k}}^{\sigma L}+\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{T}\frac{I_{\mathbf{k}'}^{\sigma'S}}{\mu_{\mathbf{k}'}}I_{\mathbf{k}}^{\sigma L}-\sigma\omega_{\mathbf{k}}^{L}\right. \\
& \quad\quad\left.\times\frac{I_{\mathbf{k}'}^{\sigma'S}}{\mu_{\mathbf{k}'}}\frac{I_{\mathbf{k}-\mathbf{k}'}^{\sigma''T}}{2}\right)\delta(\sigma\omega_{\mathbf{k}}^{L}-\sigma'\omega_{\mathbf{k}'}^{S},-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{T}) \\
& +\frac{e^2}{8\,m_e^2\,\omega_{pe}^2}\frac{(\mathbf{k}\times\mathbf{k}')^2}{k^2k'^2|\mathbf{k}-\mathbf{k}'|^2}\left(\frac{k^2}{\sigma\omega_{\mathbf{k}}^{L}}+\frac{k'^2}{\sigma'\omega_{\mathbf{k}'}^{L}}\right)^2 \\
& \quad\times\left(\sigma'\omega_{\mathbf{k}'}^{L}\frac{I_{\mathbf{k}-\mathbf{k}'}^{\sigma''T}}{2}I_{\mathbf{k}}^{\sigma L}+\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{T}I_{\mathbf{k}'}^{\sigma'L}I_{\mathbf{k}}^{\sigma L}-\sigma\omega_{\mathbf{k}}^{L}\right.
\end{aligned}$$

$$\times I_{\mathbf{k}'}^{\sigma' L} \frac{I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' T}}{2} \Bigg) \delta(\sigma\omega_{\mathbf{k}}^L - \sigma'\omega_{\mathbf{k}'}^L - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^T)$$
$$+ \frac{e^2}{4m_e^2} \frac{k^2 \{k'^2 |\mathbf{k}-\mathbf{k}'|^2 + [\mathbf{k}' \cdot (\mathbf{k}-\mathbf{k}')]^2\}}{k'^2 |\mathbf{k}-\mathbf{k}'|^2 (\omega_{\mathbf{k}'}^T)^2 (\omega_{\mathbf{k}-\mathbf{k}'}^T)^2}$$
$$\times \Bigg( \sigma'\omega_{\mathbf{k}'}^T \frac{I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' T}}{2} I_{\mathbf{k}}^{\sigma L} + \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^T \frac{I_{\mathbf{k}'}^{\sigma' T}}{2} I_{\mathbf{k}}^{\sigma L} - \sigma\omega_{\mathbf{k}}^L$$
$$\times \frac{I_{\mathbf{k}'}^{\sigma' T}}{2} \frac{I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' T}}{2} \Bigg) \delta(\sigma\omega_{\mathbf{k}}^L - \sigma'\omega_{\mathbf{k}'}^T - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^T) \Bigg\}, \quad (34)$$

$$\frac{\partial}{\partial t} \frac{I_{\mathbf{k}}^{\sigma S}}{\mu_{\mathbf{k}}} = -\pi\sigma\omega_{\mathbf{k}}^L \sum_{\sigma',\sigma''} \int d\mathbf{k}'$$
$$\times \Bigg\{ \frac{e^2}{4T_e^2} \frac{\mu_{\mathbf{k}} [\mathbf{k}' \cdot (\mathbf{k}-\mathbf{k}')]^2}{k^2 k'^2 |\mathbf{k}-\mathbf{k}'|^2}$$
$$\times \Bigg( \sigma'\omega_{\mathbf{k}'}^L I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L} \frac{I_{\mathbf{k}}^{\sigma S}}{\mu_{\mathbf{k}}} + \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L I_{\mathbf{k}'}^{\sigma' L} \frac{I_{\mathbf{k}}^{\sigma S}}{\mu_{\mathbf{k}}} - \sigma\omega_{\mathbf{k}}^L$$
$$\times I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L} I_{\mathbf{k}'}^{\sigma' L} \Bigg) \delta(\sigma\omega_{\mathbf{k}}^S - \sigma'\omega_{\mathbf{k}'}^L - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L)$$
$$+ \frac{e^2}{2T_e^2} \frac{\mu_{\mathbf{k}} (\mathbf{k}\times\mathbf{k}')^2}{k^2 k'^2 |\mathbf{k}-\mathbf{k}'|^2}$$
$$\times \Bigg( \sigma'\omega_{\mathbf{k}'}^L \frac{I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' T}}{2} \frac{I_{\mathbf{k}}^{\sigma S}}{\mu_{\mathbf{k}}} + \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^T I_{\mathbf{k}'}^{\sigma' L} \frac{I_{\mathbf{k}}^{\sigma S}}{\mu_{\mathbf{k}}} - \sigma\omega_{\mathbf{k}}^L$$
$$\times I_{\mathbf{k}'}^{\sigma' L} \frac{I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' T}}{2} \Bigg) \delta(\sigma\omega_{\mathbf{k}}^S - \sigma'\omega_{\mathbf{k}'}^L - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^T) \Bigg\}, \quad (35)$$

$$\frac{\partial}{\partial t} \frac{I_{\mathbf{k}}^{\sigma T}}{2} = -\pi\sigma\omega_{\mathbf{k}}^T \sum_{\sigma',\sigma''} \int d\mathbf{k}'$$
$$\times \Bigg\{ \frac{e^2}{32\, m_e^2\, \omega_{pe}^2} \frac{(\mathbf{k}\times\mathbf{k}')^2}{k^2 k'^2 |\mathbf{k}-\mathbf{k}'|^2} \left( \frac{k'^2}{\sigma'\omega_{\mathbf{k}'}^L} - \frac{|\mathbf{k}-\mathbf{k}'|^2}{\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L} \right)^2$$
$$\times \Bigg( \sigma'\omega_{\mathbf{k}'}^L I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L} \frac{I_{\mathbf{k}}^{\sigma T}}{2} + \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L I_{\mathbf{k}'}^{\sigma' L} \frac{I_{\mathbf{k}}^{\sigma T}}{2} - \sigma\omega_{\mathbf{k}}^T$$
$$\times I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L} I_{\mathbf{k}'}^{\sigma' L} \Bigg) \delta(\sigma\omega_{\mathbf{k}}^T - \sigma'\omega_{\mathbf{k}'}^L - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L)$$
$$+ \frac{e^2}{8T_e^2} \frac{\mu_{\mathbf{k}'} (\mathbf{k}\times\mathbf{k}')^2}{k^2 k'^2 |\mathbf{k}-\mathbf{k}'|^2}$$
$$\times \Bigg( \sigma'\omega_{\mathbf{k}'}^L I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L} \frac{I_{\mathbf{k}}^{\sigma T}}{2} + \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L \frac{I_{\mathbf{k}'}^{\sigma' S}}{\mu_{\mathbf{k}'}} \frac{I_{\mathbf{k}}^{\sigma T}}{2} - \sigma\omega_{\mathbf{k}}^T$$
$$\times I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L} \frac{I_{\mathbf{k}'}^{\sigma' S}}{\mu_{\mathbf{k}'}} \Bigg) \delta(\sigma\omega_{\mathbf{k}}^T - \sigma'\omega_{\mathbf{k}'}^S - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L)$$
$$+ \frac{e^2}{8T_e^2} \frac{\mu_{\mathbf{k}-\mathbf{k}'} (\mathbf{k}\times\mathbf{k}')^2}{k^2 k'^2 |\mathbf{k}-\mathbf{k}'|^2}$$
$$\times \Bigg( \sigma'\omega_{\mathbf{k}'}^L \frac{I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S}}{\mu_{\mathbf{k}-\mathbf{k}'}} \frac{I_{\mathbf{k}}^{\sigma T}}{2} + \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L I_{\mathbf{k}'}^{\sigma' L} \frac{I_{\mathbf{k}}^{\sigma T}}{2} - \sigma\omega_{\mathbf{k}}^T$$
$$\times \frac{I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S}}{\mu_{\mathbf{k}-\mathbf{k}'}} I_{\mathbf{k}'}^{\sigma' L} \Bigg) \delta(\sigma\omega_{\mathbf{k}}^T - \sigma'\omega_{\mathbf{k}'}^L - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^S)$$
$$+ \frac{1}{4} \frac{e^2}{m_e^2} \frac{|\mathbf{k}-\mathbf{k}'|^2}{(\sigma\omega_{\mathbf{k}}^T)^2 (\sigma'\omega_{\mathbf{k}'}^T)^2} \left( 1 + \frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k'^4} \right)$$
$$\times \Bigg( \sigma'\omega_{\mathbf{k}'}^T I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L} \frac{I_{\mathbf{k}}^{\sigma T}}{2} + \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L \frac{I_{\mathbf{k}'}^{\sigma' T}}{2} \frac{I_{\mathbf{k}}^{\sigma T}}{2} - \sigma\omega_{\mathbf{k}}^T$$
$$\times I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L} \frac{I_{\mathbf{k}'}^{\sigma' T}}{2} \Bigg) \delta(\sigma\omega_{\mathbf{k}}^T - \sigma'\omega_{\mathbf{k}'}^T - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L) \Bigg\}. \quad (36)$$

This completes the derivation wave kinetic equations that depict nonlinear wave-wave interaction processes starting from the two-fluid framework. The final result is consistent with that of kinetic theory, which shows that it is possible to formulate the weak turbulence theory for more complex problems such as that of magnetized plasmas.

## 3. Summary and discussion

The present paper shows that the two-fluid formalism can successfully reproduce the nonlinear wave kinetic equation that involves decay processes. This lays the foundation for similar approaches that may be taken for magnetized plasmas. The significance of the present work is in such a context. That is, the purpose of the present author's research is to build upon the experience of developing the weak turbulence theory for unmagnetized plasma on the basis of two-fluid theory, and then proceed to formulate the weak turbulence theory for magnetized plasmas. In doing so, however, while the presence of ambient magnetic field will greatly complicate the problem, fortunately, the situation there might simplify the matter in another way.

For unmagnetized plasmas the high frequency waves, Langmuir ($L$) and transverse radiation ($T$) cannot undergo wave-wave interaction unless there exists a low frequency ion sound ($S$) wave. The ion sound (or ion acoustic) wave does not exist in cold plasma, since it is a thermal plasma mode. This means that the weak turbulence theory, whether based upon kinetic theory or two-fluid theory, cannot entirely ignore thermal corrections. This generally makes the analysis quite cumbersome. For instance, in the present paper, we encountered the extremely complex form of the second-order velocity perturbation that includes thermal effects—see equation (22)—albeit, still simpler than the kinetic theoretical counterpart.

However, for magnetized plasmas, the low frequency wave naturally exists in the form of magnetosonic/whistler mode, even if one ignores the proton response and thermal effects. This means that one may employ the standard magnetoionic theory of magnetized plasma waves within the context of the weak turbulence theory. Recall that the magnetoionic theory is a cold plasma theory of waves in magnetized plasma in which ion response is ignored. In magnetoionic theory, fast extraordinary ($X$) and ordinary ($O$) modes constitute the radiation, while high frequency plasma oscillation-upper hybrid mode known as the $Z$ (or slow extraordinary) mode exists. The low frequency magnetosonic/whistler, or $W$ mode, is a low frequency mode, which may participate in three wave interaction with any two of the three high frequency modes. This points to the possibility that the extension of the present two-fluid equation based weak turbulence theory may be employed for magnetized plasmas in a relatively straightforward manner since we may ignore proton dynamics and thermal effects at the outset. Indeed, such a possibility has already been entertained in the literature, but only in qualitative terms [89–91], or by means of numerical simulation [92, 93]. A quantitative weak turbulence theory that may deal with such an interaction among magnetoionic modes is still lacking.

Note that for both magnetized and unmagnetized plasmas, the plasma emission at the second harmonic does not involve low frequency mode at all. For unmagnetized plasmas, the merging of two Langmuir waves into a transverse EM radiation at twice the plasma frequency occurs, while for magnetized plasmas, the merging of two $Z$ modes may lead to the radiation emission at approximately twice the upper-hybrid frequency in the form of either $X$ or $O$ mode [59, 74]—see also, [94]. In short, the comprehensive weak turbulence theory that is capable of quantitative numerical analysis, and that naturally lends itself to comparative analysis again PIC simulation is still lacking, which is the long term research goal of the present author. The present paper, which partially reformulates the weak turbulence theory for unmagnetized plasmas from the perspective of warm fluid theory represents a proof of concept, and it is a start for the future research.

## Data availability statement

No new data were created or analysed in this study.

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## ORCID iD

Peter H Yoon https://orcid.org/0000-0001-8134-3790

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