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ABSTRACT

Low-frequency hydromagnetic turbulence is thought to play an important role in charged particle energization in space and astrophysical environments. For understanding large-scale turbulence in magnetized plasmas, low-frequency electromagnetic turbulence has been widely investigated within the theoretical framework of incompressible magnetohydrodynamic (MHD) theory. Among the existing works is the weak turbulence formalism of incompressible MHD turbulence. The present paper revisits the existing formalism under the assumption of zero residual energy. Under the strict assumption of turbulence taking place in a two-dimensional plane, which can be interpolated to a three-dimensional situation with azimuthal symmetry, the well-known steady-state turbulent spectrum of $k_\perp^{-2}$ is recovered, where $k_\perp$ denotes the wave number perpendicular to the ambient magnetic field.

I. INTRODUCTION

Low-frequency hydromagnetic turbulence in plasmas, or electrically conducting ionized gas immersed in ambient magnetic field, is widely investigated within the framework of magnetohydrodynamics (MHD) in the literature.[1–4] Many MHD turbulence theories are built upon the classic phenomenological fluid turbulence theory pioneered by Kolmogorov.[5] Similar phenomenological theories for MHD turbulence were subsequently developed in Refs. 6–9. Extended discussions along similar lines can also be found in later works.[10–12] Important developments in the more quantitative theory of MHD turbulence include those of Ref. 13, where the strong turbulence regime of MHD is modeled. Various renormalization group approaches to MHD turbulence are also found in the literature,[14,15] which is a concept well developed in the context of neutral fluid turbulence.[16–19] Among the important milestones in this regard is the formulation of the weak MHD turbulence theory for anisotropic turbulence in plasmas with a strong ambient magnetic field.[20–23] The above references rectified an earlier theory[24] and demonstrated that the three-wave resonant interaction characterizes the weak MHD turbulence. An excellent and systematic review of all these and other developments until the early 2000s can be found in Ref. 2, for instance. Numerous reviews with focus on applications to space and heliospheric physics are also available in the literature, but herewith, we only cite a few, for example, Refs. 25–28, and a recent review of MHD turbulence in the magnetosphere.[4] We should also note that direct numerical simulations of MHD turbulence are also widely available in the literature, which are too numerous to give a complete account of, but some representative works widely cited in the literature include Refs. 29–37. More recent selective works include Refs. 38–40, to name just a few.

Of particular interest to the present authors is the issue of weak MHD turbulence.[20–24] Because of the well-defined linear physics provided by the shear Alfvén wave dynamics for strongly magnetized plasmas, it seems that the weak turbulence approximation for MHD turbulence should be quite valid, which is in contrast to the problem of neutral fluid turbulence where the linear behavior is practically absent. Because of this, we paid attention to this problem and found that the existing formalism pioneered in the above references parallels that of the Langmuir/ion–sound wave weak turbulence problem, which is another area that is well developed in kinetic plasma theory, e.g., see Ref. 41. Upon a careful survey of the MHD weak turbulence theory, we deemed that an alternative formulation of the same problem within a framework that is often employed in the Langmuir/ion–sound wave weak turbulence theory might be a useful contribution to the subject matter. This is the motivation for the present paper.

A major finding in the existing MHD weak turbulence theory, according to Refs. 22 and 23, is that the weak anisotropic MHD turbulence energy spectrum in the asymptotically steady state should be characterized by an inverse power-law spectrum, $k_\perp^{-2}$, where $k_\perp$ represents the wave number perpendicular to the ambient magnetic field. With the alternative formulation of the present paper, we independently derive this result. However, we found an interesting feature associated with the weak MHD spectrum. That is, while the $k_\perp^{-2}$

spectrum appears to be generally supported by our numerical computation of the resultant weak MHD turbulence equation, we also found that other spectra are theoretically possible. Such a finding will also be discussed in the present paper.

The organization of the present paper is as follows: In Sec. II, we formulate the weak MHD turbulence theory in a format that closely resembles those of existing works, namely, those of Refs. 20–24, particularly those of Refs. 22 and 23. However, the formulation will follow a style that is reminiscent of the plasma kinetic weak turbulence theory found in the literature.[41–46,48] Then, Sec. III expounds on the implications of the weak MHD turbulence theory, which includes discussions on energy conservation, the anisotropic nature of turbulent spectra, and the quasi-steady-state spectrum as well as numerical demonstrations thereof. Finally, Sec. IV summarizes the present findings.

II. DERIVATION OF WAVE KINETIC EQUATIONS FOR WEAK MHD TURBULENCE

We start from the equations of incompressible magnetohydrodynamic (MHD) theory.[1,3] Assuming constant density, $\rho = \text{const}$, and thus, $\nabla \cdot \mathbf{u} = 0$, $\mathbf{u}$ being the fluid velocity, we have

$$\begin{aligned}\frac{d\mathbf{u}}{dt} &= -\nabla P_* + \frac{\mathbf{B}\cdot\nabla\mathbf{B}}{4\pi\rho} + \nu\nabla^2\mathbf{u},\\ \frac{d\mathbf{B}}{dt} &= (\mathbf{B}\cdot\nabla)\mathbf{u} + \nu\nabla^2\mathbf{B},\end{aligned} \tag{1}$$

where $\mathbf{B}$ is the magnetic field, which of course, satisfies $\nabla\cdot\mathbf{B} = 0$; the material derivative is defined by $d/dt = \partial/\partial t + \mathbf{u}\cdot\nabla$; $P_* = P/\rho + B^2/(8\pi\rho)$ represents the total pressure; and we have assumed that the velocity-space dissipation rate and the magnetic resistivity are identical. We separate the magnetic field into an ambient component and fluctuations, $\mathbf{B} = \mathbf{B}_0 + \delta\mathbf{B}$. We then define the Alfvén velocity, $\mathbf{c}_A = \mathbf{B}_0/(4\pi\rho)^{1/2}$, and the Elsässer fields, $\mathbf{z}^\pm = \mathbf{u}\pm\mathbf{b}$, where $\mathbf{b} = \delta\mathbf{B}/(4\pi\rho)^{1/2}$. Then, it is a well-known and straightforward exercise to rewrite the MHD equation (1) as

$$\begin{aligned}\frac{\partial\mathbf{z}^\alpha}{\partial t} - \alpha(\mathbf{c}_A\cdot\nabla)\mathbf{z}^\alpha + (\mathbf{z}^{-\alpha}\cdot\nabla)\mathbf{z}^\alpha + \nabla P_* - \nu\nabla^2\mathbf{z}^\alpha &= 0,\\ \nabla\cdot\mathbf{z}^\alpha &= 0,\end{aligned} \tag{2}$$

where $\alpha = \pm$ denotes the two signs of Elsässer fields.

In what follows, we will rewrite Eq. (2) in spectral representation. In doing so, we keep in mind that the spectral amplitudes vary slowly in time. That is, we treat the spectral amplitudes as an adiabatic function of time, while the physical quantities also oscillate in fast time scale of Alfvén waves.[41,43,46,47] This is a shortcut method to implement the multiple (or two)-time scale analysis into the problem at hand. We also treat the angular frequency ω to possess an infinitesimal positive imaginary part, which is a consequence of the causality relationship. In short, we express the physical quantities as $\mathbf{z}^\alpha(\mathbf{r},t) = \sum_{\mathbf{k},\omega}\mathbf{z}^\alpha_{\mathbf{k},\omega}(t)e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}$ well as $P_*(\mathbf{r},t) = \sum_{\mathbf{k},\omega}P_{\mathbf{k},\omega}(t)e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}$, together with the inverse transformation, $\mathbf{z}^\alpha_{\mathbf{k},\omega}(t) = (2\pi)^{-4}\int d\mathbf{r}\int dt\,\mathbf{z}^\alpha(\mathbf{r},t)e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega t}$ and $P_{\mathbf{k},\omega}(t) = (2\pi)^{-4}\int d\mathbf{r}\int dt\,P_*(\mathbf{r},t)e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega t}$. Here, the slow time dependence of the spectral amplitudes $\mathbf{z}^\alpha_{\mathbf{k},\omega}(t)$ and $P_{\mathbf{k},\omega}(t)$ is meant to be adiabatic. Implementing this spectral transformation to Eq. (2), we obtain[20,22]

$$\begin{aligned}0 &= \left(i\frac{\partial}{\partial t} + \omega + \alpha\mathbf{k}\cdot\mathbf{c}_A + ik^2\nu\right)\mathbf{z}^\alpha_{\mathbf{k},\omega} - \mathbf{k}P_{\mathbf{k},\omega}\\ &\quad -\frac{1}{2}\sum_{\mathbf{k}',\omega'}\left[(\mathbf{k}-\mathbf{k}')\cdot\mathbf{z}^{-\alpha}_{\mathbf{k}',\omega'}\mathbf{z}^\alpha_{\mathbf{k}-\mathbf{k}',\omega-\omega'} + (\mathbf{k}'\leftrightarrow\mathbf{k}-\mathbf{k}')\right],\\ &\qquad \mathbf{k}\cdot\mathbf{z}^\alpha_{\mathbf{k},\omega} = 0,\end{aligned} \tag{3}$$

where the symbol $(\mathbf{k}'\leftrightarrow\mathbf{k}-\mathbf{k}')$ denotes the repetition of the first term within the square bracket except $\mathbf{k}'$ and $\mathbf{k}-\mathbf{k}'$ are to be interchanged, and $i\partial/\partial t$ acts upon the adiabatic time variation of the spectral amplitude $\mathbf{z}^\alpha_{\mathbf{k},\omega}(t)$. In what follows, we absorb the slow time derivative to the angular frequency, $\omega + i\partial/\partial t \to \omega$, and reintroduce $i\partial/\partial t$ later at an appropriate stage. Note that the nonlinear term in Eq. (3) is expressed by writing the integral over $\mathbf{k}'$ and ω' in a symmetric form. This shortcut two-time scale treatise is adopted in the theory of kinetic weak plasma turbulence,[41,43,46,47] whereas in standard MHD (and fluid) turbulence theories the spectral transformation is applied only for the spatial coordinates.[1,2,19,22,23] The advantage of this method is that the basic equations turn into a set of algebraic equations, which facilitates subsequent manipulations.

By taking the dot product $\mathbf{k}\cdot$ of Eq. (3) and making use of the incompressibility condition, $\mathbf{k}\cdot\mathbf{z}^\alpha_{\mathbf{k},\omega} = 0$ one may obtain $P_{\mathbf{k},\omega}$, which is inserted back to Eq. (3). The result is

$$D_\alpha(\mathbf{k},\omega)z^{\alpha i}_{\mathbf{k},\omega} = \frac{1}{2}\sum_{\mathbf{k}',\omega'}P_{ij}(\mathbf{k})k_k\left[z^{\alpha j}_{\mathbf{k}-\mathbf{k}',\omega-\omega'}z^{-\alpha k}_{\mathbf{k}',\omega'} + (\mathbf{k}'\leftrightarrow\mathbf{k}-\mathbf{k}')\right], \tag{4}$$

where we have made use of $\mathbf{k}'\cdot\mathbf{z}^{-\alpha}_{\mathbf{k}',\omega'} = 0$ and $(\mathbf{k}-\mathbf{k}')\cdot\mathbf{z}^{-\alpha}_{\mathbf{k}-\mathbf{k}',\omega-\omega'} = 0$, and where the linear response function $D_\alpha(\mathbf{k},\omega)$ and the projection operator $P_{ij}(\mathbf{k})$ are defined, respectively, by

$$\begin{aligned}D_\alpha(\mathbf{k},\omega) &= \omega + \alpha k_\parallel c_A + ik^2\nu,\\ P_{ij}(\mathbf{k}) &= \delta_{ij} - \frac{k_ik_j}{k^2}.\end{aligned} \tag{5}$$

Here, we should keep in mind that the angular frequency ω is implicitly assumed to have an infinitesimal positive imaginary part, $\omega + i0$. In order to proceed, we take the dot product $z^{\alpha i}_{-\mathbf{k},-\omega}$ and take the ensemble average of Eq. (4),

$$\begin{aligned}&D_\alpha(\mathbf{k},\omega)\langle z^{\alpha i}_{\mathbf{k},\omega}z^{\alpha i}_{-\mathbf{k},-\omega}\rangle\\ &= \frac{1}{2}\sum_{\mathbf{k}',\omega'}P_{ij}(\mathbf{k})k_k\left[\langle z^{\alpha j}_{\mathbf{k}-\mathbf{k}',\omega-\omega'}z^{-\alpha k}_{\mathbf{k}',\omega'}z^{\alpha i}_{-\mathbf{k},-\omega}\rangle + (\mathbf{k}'\leftrightarrow\mathbf{k}-\mathbf{k}')\right].\end{aligned} \tag{6}$$

As is apparent, Eq. (6) is not closed since the two-body correlation function $\langle z^{\alpha i}_{\mathbf{k},\omega}z^{\alpha i}_{-\mathbf{k},-\omega}\rangle$ is coupled to the three-body correlations, $\langle z^{\alpha j}_{\mathbf{k}-\mathbf{k}',\omega-\omega'}z^{-\alpha k}_{\mathbf{k}',\omega'}z^{\alpha i}_{-\mathbf{k},-\omega}\rangle$ and $\langle z^{\alpha j}_{\mathbf{k}',\omega'}z^{-\alpha k}_{\mathbf{k}-\mathbf{k}',\omega-\omega'}z^{\alpha i}_{-\mathbf{k},-\omega}\rangle$, and so on, *ad infinitum*. Consequently, one must close the infinite hierarchy of correlations.

Note that for homogeneous and stationary turbulence, we have $\langle z^{\alpha i}_{\mathbf{k},\omega}z^{\alpha i}_{-\mathbf{k},-\omega}\rangle = \langle\mathbf{z}^\alpha_{\mathbf{k},\omega}\cdot\mathbf{z}^\alpha_{-\mathbf{k},-\omega}\rangle = \langle z^2_\alpha\rangle_{\mathbf{k},\omega}$, which is related to the spectral energy density of the Elsässer fields. Note also that $\langle z^{\alpha i}_{\mathbf{k},\omega}z^{\alpha i}_{-\mathbf{k},-\omega}\rangle = \langle u^2\rangle_{\mathbf{k},\omega} + \langle b^2\rangle_{\mathbf{k},\omega} + 2\alpha\,\mathrm{Re}\langle\mathbf{u}_{\mathbf{k},\omega}\cdot\mathbf{b}^*_{\mathbf{k},\omega}\rangle$. Here, $\langle u^2\rangle_{\mathbf{k},\omega} + \langle b^2\rangle_{\mathbf{k},\omega}$ is the total (particle plus field) energy, the quantity $\langle\mathbf{u}_{\mathbf{k},\omega}\cdot\mathbf{b}^*_{\mathbf{k},\omega}\rangle$ is associated with the alignment between the flow field vector and the fluctuating magnetic field vector. If $\langle\mathbf{u}_{\mathbf{k},\omega}\cdot\mathbf{b}^*_{\mathbf{k},\omega}\rangle = 0$, then the Elsässer fields are symmetric, $\langle z^2_+\rangle_{\mathbf{k},\omega} = \langle z^2_-\rangle_{\mathbf{k},\omega}$, and the turbulence is said to be balanced. Note that in general, the dot product of Elsässer fields of opposite signs is not necessarily zero, $\langle \mathbf{z}^{\pm}_{\mathbf{k},\omega}\cdot\mathbf{z}^{\mp}_{-\mathbf{k},-\omega}\rangle = \langle u^2\rangle_{\mathbf{k},\omega} - \langle b^2\rangle_{\mathbf{k},\omega} \mp 2i\mathrm{Im}\langle \mathbf{u}_{\mathbf{k},\omega}\cdot\mathbf{b}^*_{\mathbf{k},\omega}\rangle$. The quantity $\langle u^2\rangle_{\mathbf{k},\omega} - \langle b^2\rangle_{\mathbf{k},\omega}$ is known as the residual energy, the difference between the flow kinetic energy and the field energy. In general, the residual energy is generally finite, as some simulations indicate, and as Ref. 49 point out. For non-zero residual energy, in addition to the ensemble-averaged nonlinear equation for $\langle \mathbf{z}^{\alpha}_{\mathbf{k},\omega}\cdot\mathbf{z}^{\alpha}_{-\mathbf{k},-\omega}\rangle$ specified by Eq. (6), one must also construct the separate equation for $\langle \mathbf{z}^{\alpha}_{\mathbf{k},\omega}\cdot\mathbf{z}^{-\alpha}_{-\mathbf{k},-\omega}\rangle$. Such a formalism will lead to the coupling of energy density for Elsässer field, $\langle \mathbf{z}^{\alpha}_{\mathbf{k},\omega}\cdot\mathbf{z}^{\alpha}_{-\mathbf{k},-\omega}\rangle$, and the residual energy field $\langle \mathbf{z}^{\alpha}_{\mathbf{k},\omega}\cdot\mathbf{z}^{-\alpha}_{-\mathbf{k},-\omega}\rangle$. However, in the present discussion, we are concerned with a relatively simple and ideal case of zero residual energy, $\langle \mathbf{z}^{\alpha}_{\mathbf{k},\omega}\cdot\mathbf{z}^{-\alpha}_{-\mathbf{k},-\omega}\rangle = 0$, which Ref. 22 also assumed. This simplifies the analysis in that the ensuing formalism will only involve nonlinear interactions between the Elsässer fields of opposite signs, $\langle \mathbf{z}^{+}_{\mathbf{k},\omega}\cdot\mathbf{z}^{+}_{-\mathbf{k},-\omega}\rangle$ and $\langle \mathbf{z}^{-}_{\mathbf{k},\omega}\cdot\mathbf{z}^{-}_{-\mathbf{k},-\omega}\rangle$.

In order to achieve closure, one needs to obtain the three-body cumulants $\langle z^{\alpha j}_{\mathbf{k}-\mathbf{k}',\omega-\omega'} z^{-\alpha k}_{\mathbf{k}',\omega'} z^{\alpha i}_{-\mathbf{k},-\omega}\rangle$ and $\langle z^{\alpha j}_{\mathbf{k}',\omega'} z^{-\alpha k}_{\mathbf{k}-\mathbf{k}',\omega-\omega'} z^{\alpha i}_{-\mathbf{k},-\omega}\rangle$, which is done in the customary manner in standard weak turbulence theory. We first write the solution to (3) iteratively, $z^{\alpha i}_{\mathbf{k},\omega} = z^{\alpha(0)i}_{\mathbf{k},\omega} + z^{\alpha(1)i}_{\mathbf{k},\omega} + \cdots$, where $z^{\alpha(0)i}_{\mathbf{k},\omega}$ satisfies the linear equation, $D_\alpha(\mathbf{k},\omega) z^{\alpha(0)i}_{\mathbf{k},\omega} = 0$. This means that odd moments of the zeroth order solution are all zero, including the third order moment, $\langle z^{\alpha(0)i}_{\mathbf{k},\omega} z^{\beta(0)j}_{\mathbf{k}',\omega'} z^{\alpha(0)k}_{\mathbf{k}'',\omega''}\rangle = 0$. Since $D_\alpha(\mathbf{k},\omega) z^{\alpha(0)i}_{\mathbf{k},\omega} = 0$, the nonlinear correction is obtained by

$$z^{\alpha(1)i}_{\mathbf{k},\omega} = \frac{1}{2}\frac{1}{D_\alpha(\mathbf{k},\omega)}\sum_{\mathbf{k}',\omega'} P_{ij}(\mathbf{k}) k_k \left[z^{\alpha(0)j}_{\mathbf{k}-\mathbf{k}',\omega-\omega'} z^{-\alpha(0)k}_{\mathbf{k}',\omega'} + (\mathbf{k}\leftrightarrow\mathbf{k}-\mathbf{k}')\right]. \tag{7}$$

With this solution, we write the three-body cumulants of relevance as

$$\begin{aligned}\langle z^{\alpha j}_{\mathbf{k}-\mathbf{k}',\omega-\omega'} z^{-\alpha k}_{\mathbf{k}',\omega'} z^{\alpha i}_{-\mathbf{k},-\omega}\rangle &= \langle z^{\alpha(1)j}_{\mathbf{k}-\mathbf{k}',\omega-\omega'} z^{-\alpha(0)k}_{\mathbf{k}',\omega'} z^{\alpha(0)i}_{-\mathbf{k},-\omega}\rangle \\ &\quad + \langle z^{\alpha(0)j}_{\mathbf{k}-\mathbf{k}',\omega-\omega'} z^{-\alpha(1)k}_{\mathbf{k}',\omega'} z^{\alpha(0)i}_{-\mathbf{k},-\omega}\rangle \\ &\quad + \langle z^{\alpha(0)j}_{\mathbf{k}-\mathbf{k}',\omega-\omega'} z^{-\alpha(0)k}_{\mathbf{k}',\omega'} z^{\alpha(1)i}_{-\mathbf{k},-\omega}\rangle.\end{aligned} \tag{8}$$

By making use of the short-hand notation, $q = (\mathbf{k},\omega)$, and by iteration we may construct the three-body cumulants of relevance upon coupling Eqs. (7) and (8),

$$\begin{aligned}&\langle z^{\alpha j}_{\mathbf{k}-\mathbf{k}',\omega-\omega'} z^{-\alpha k}_{\mathbf{k}',\omega'} z^{\alpha i}_{-\mathbf{k},-\omega}\rangle \\ &= \frac{1}{2D_\alpha(q-q')}\sum_{q''} P_{jl}(\mathbf{k}-\mathbf{k}')(\mathbf{k}-\mathbf{k}')_m \\ &\quad\times\left(\langle z^{\alpha l}_{q-q'-q''} z^{-\alpha m}_{q''} z^{-\alpha k}_{q'} z^{\alpha i}_{-q}\rangle + \langle z^{\alpha l}_{q''} z^{-\alpha m}_{q-q'-q''} z^{-\alpha k}_{q'} z^{\alpha i}_{-q}\rangle\right) \\ &\quad + \frac{1}{2D_{-\alpha}(q')}\sum_{q''} P_{kl}(\mathbf{k}') k'_m \left(\langle z^{-\alpha l}_{q'-q''} z^{\alpha m}_{q''} z^{\alpha j}_{q-q'} z^{\alpha i}_{-q}\rangle\right. \\ &\quad \left.+ \langle z^{-\alpha l}_{q''} z^{\alpha m}_{q'-q''} z^{\alpha j}_{q-q'} z^{\alpha i}_{-q}\rangle\right) - \frac{1}{2D_\alpha(-q)}\sum_{q''} P_{il}(\mathbf{k}) k_m \\ &\quad\times\left(\langle z^{\alpha l}_{-q-q''} z^{-\alpha m}_{q''} z^{\alpha j}_{q-q'} z^{-\alpha k}_{q'}\rangle + \langle z^{\alpha l}_{q''} z^{-\alpha m}_{-q-q''} z^{\alpha j}_{q-q'} z^{-\alpha k}_{q'}\rangle\right),\end{aligned} \tag{9}$$

where we have ignored the superscript (0) after everything is said and done.

At this point, it is apparent that the three-body cumulants depend on four-body cumulants, $\langle z^{\alpha i}_q z^{\beta j}_{q'} z^{\gamma k}_{q''} z^{\delta l}_{q'''}\rangle$. In order to truncate this infinite chain of hierarchy, we write the four-body cumulant into products of two-body correlation functions while ignoring the irreducible four-body correlation function, $\langle z^{\alpha i}_q z^{\beta j}_{q'} z^{\gamma k}_{q''} z^{\delta l}_{q'''}\rangle = \langle z^{\alpha i}_q z^{\beta j}_{q'}\rangle\langle z^{\gamma k}_{q''} z^{\delta l}_{q'''}\rangle + \langle z^{\alpha i}_q z^{\gamma k}_{q''}\rangle\langle z^{\beta j}_{q'} z^{\delta l}_{q'''}\rangle + \langle z^{\alpha i}_q z^{\delta l}_{q'''}\rangle\langle z^{\beta j}_{q'} z^{\gamma k}_{q''}\rangle$. We further make use of $\langle z^{\alpha i}_q z^{\beta j}_{q'}\rangle = \delta(q+q')\langle z^{\alpha i}_q z^{\beta j}_{-q'}\rangle$ and the fact that $P_{jl}(\mathbf{k}-\mathbf{k}')\delta(q-q') = 0$, $P_{kl}(\mathbf{k}')\delta(q') = 0$, $P_{il}(-\mathbf{k})\delta(-q) = 0$, and $P_{kl}(\mathbf{k}-\mathbf{k}')\delta(q-q') = 0$. Then, we have

$$\begin{aligned}&\langle z^{\alpha j}_{\mathbf{k}-\mathbf{k}',\omega-\omega'} z^{-\alpha k}_{\mathbf{k}',\omega'} z^{\alpha i}_{-\mathbf{k},-\omega}\rangle \\ &= \frac{P_{jl}(\mathbf{k}-\mathbf{k}')}{D_\alpha(q-q')}(\mathbf{k}-\mathbf{k}')_m\left(\langle z^{\alpha l}_q z^{\alpha i}_{-q}\rangle\langle z^{-\alpha m}_{-q'} z^{-\alpha k}_{q'}\rangle + \langle z^{\alpha l}_{-q'} z^{-\alpha k}_{q'}\rangle\langle z^{-\alpha m}_q z^{\alpha i}_{-q}\rangle\right) \\ &\quad + \frac{P_{kl}(\mathbf{k}')}{D_{-\alpha}(q')} k'_m \left(\langle z^{-\alpha l}_{-q+q'} z^{\alpha j}_{q-q'}\rangle\langle z^{\alpha m}_q z^{\alpha i}_{-q}\rangle + \langle z^{-\alpha l}_q z^{\alpha i}_{-q}\rangle\langle z^{\alpha m}_{-q+q'} z^{\alpha j}_{q-q'}\rangle\right) \\ &\quad - \frac{P_{ilm}(\mathbf{k})}{D_\alpha(-q)} k_m \left(\langle z^{\alpha l}_{-q+q'} z^{\alpha j}_{q-q'}\rangle\langle z^{-\alpha m}_{-q'} z^{-\alpha k}_{q'}\rangle + \langle z^{\alpha l}_{-q'} z^{-\alpha k}_{q'}\rangle\langle z^{-\alpha m}_{-q+q'} z^{\alpha j}_{q-q'}\rangle\right).\end{aligned} \tag{10}$$

This method of constructing the three-body cumulant based upon the iterative solution (7) rather than taking the direct triple product of Eq. (2)—see, e.g., Refs. 1, 2, 19, 22, and 23—is again a standard practice adopted in the kinetic weak plasma turbulence formalism.[41,43,46] Obviously, this method bypasses the actual evolution equation for the triple correlation function and hence is convenient for achieving the closure of the hierarchy of correlations.

Inserting this into Eq. (6), making use of the incompressibility condition $k_i z^{\alpha i}_q = 0$ associated with various terms, as well as the property $\langle z^{\alpha i}_q z^{\alpha j}_q\rangle = \delta_{ij}\langle z^2_\alpha\rangle_q$, we arrive at

$$\begin{aligned}0 &= D_\alpha(q)\langle z^2_\alpha\rangle_q - \frac{1}{2}\sum_{q'} k^2 P_{ij}(\mathbf{k})\left(\frac{P_{ij}(\mathbf{k}')\langle z^2_{-\alpha}\rangle_{q-q'}\langle z^2_\alpha\rangle_q}{D_\alpha(q')}\right. \\ &\quad \left.+ \frac{P_{ij}(\mathbf{k})\langle z^2_\alpha\rangle_{q-q'}\langle z^2_{-\alpha}\rangle_{q'}}{D^*_\alpha(q)} + (\mathbf{k}\leftrightarrow\mathbf{k}-\mathbf{k}')\right),\end{aligned} \tag{11}$$

where we have written down the terms with interchange of dummy arguments $(\mathbf{k}'\leftrightarrow\mathbf{k}-\mathbf{k}')$ specifically, and we have made use of $k_j P_{ij}(\mathbf{k}) = 0$. Here, we have made use of the fact that ω is considered to have an infinitesimal positive imaginary part, which ensures causality. Thus, we have $D_\alpha(q) = D_\alpha(\mathbf{k},\omega) = \omega + \alpha k_\parallel c_A + ik^2\nu + i0$, which means that we may write $D_\alpha(-q) = -\omega - \alpha k_\parallel c_A + ik^2\nu + i0 = -D^*_\alpha(q)$. In arriving at Eq. (11), we have made use of the fact that the cross product $\langle \mathbf{z}^\alpha\cdot\mathbf{z}^{-\alpha}\rangle_q$ associated with the residual energy is zero.

At this point, we reintroduce the slow time dependence on $D_\alpha(q)$ via $D_\alpha(q) = D_\alpha(\mathbf{k},\omega)\to D_\alpha(\mathbf{k},\omega + i\partial/\partial t)$, which modifies the first term of Eq. (11),

$$\begin{aligned}D_\alpha(q)\langle z^2_\alpha\rangle_q &\to D_\alpha(q)\langle z^2_\alpha\rangle_q + \frac{i}{2}\frac{\partial D_\alpha(q)}{\partial\omega}\frac{\partial\langle z^2_\alpha\rangle_q}{\partial t} \\ &= D_\alpha(q)\langle z^2_\alpha\rangle_q + \frac{i}{2}\frac{\partial\langle z^2_\alpha\rangle_q}{\partial t},\end{aligned} \tag{12}$$

where we have made use of the fact that $\partial D_\alpha(q)/\partial\omega = 1$, by virtue of the definition $D_\alpha(q) = \omega + \alpha k_\parallel c_A + ik^2\nu$. We may split $D_\alpha(q)$ into real and imaginary parts, $D_\alpha(q) = \mathrm{Re}\, D_\alpha(q) + i\,\mathrm{Im}\, D_\alpha(q)$, where $\mathrm{Re}\, D_\alpha(q) = \omega + \alpha k_\parallel c_A$ and $\mathrm{Im}\, D_\alpha(q) = k^2\nu$. This modifies the equations at hand,

$$\begin{aligned} 0 &= \mathrm{Re}\, D_\alpha(q)\langle z_\alpha^2\rangle_q + ik^2\nu\langle z_\alpha^2\rangle_q + \frac{i}{2}\frac{\partial\langle z_\alpha^2\rangle_q}{\partial t} \\ &\quad -\frac{1}{2}\sum_{q'} k^2\left[\left(1+\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2k'^2}\right)\frac{\langle z_{-\alpha}^2\rangle_{q-q'}\langle z_\alpha^2\rangle_q}{D_\alpha(q')}\right. \\ &\quad \left. +\frac{2\langle z_\alpha^2\rangle_{q-q'}\langle z_{-\alpha}^2\rangle_{q'}}{D_\alpha^*(q)} + (\mathbf{k}\leftrightarrow\mathbf{k}-\mathbf{k}')\right], \end{aligned} \tag{13}$$

where we have made use of the definition (5) for the projection operator, $P_{ij}(\mathbf{k})$, in order to express the final equation in explicit notation. This procedure of first absorbing the slow-time derivative into the angular frequency and reintroducing it later is the essence of the short-cut two-time scale analysis, which is adopted in the kinetic weak turbulence theory.[41,43,46]

The real part of Eq. (13), while ignoring nonlinear terms, namely, $\mathrm{Re}\, D_\alpha(q) = 0$, leads to the Alfvén wave dispersion relation, $\omega = -\alpha k_\parallel c_A$, so that we may write the energy density for Elsässer fields in terms of the intensities,

$$\begin{aligned} \langle z_\alpha^2\rangle_q &= I_\mathbf{k}^\alpha\,\delta(\omega+\alpha\omega_\mathbf{k}), \\ \omega_\mathbf{k} &= k_\parallel c_A. \end{aligned} \tag{14}$$

We next consider the imaginary part of Eq. (13),

$$\begin{aligned} 0 &= \left(\frac{\partial I_\mathbf{k}^\alpha}{\partial t} + 2k^2\nu I_\mathbf{k}^\alpha\right)\delta(\omega+\alpha\omega_\mathbf{k}) \\ &\quad -\mathrm{Im}\sum_{q'}\left\{k^2\left(1+\frac{[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')]^2}{k^2(\mathbf{k}-\mathbf{k}')^2}\right)\frac{I_{\mathbf{k}'}^{-\alpha}I_\mathbf{k}^\alpha}{D_\alpha(q-q')}\right. \\ &\quad \times\delta(\omega+\alpha\omega_\mathbf{k})\delta(\omega'-\alpha\omega_{\mathbf{k}'}) \\ &\quad +\frac{2k^2I_{\mathbf{k}'}^{-\alpha}I_{\mathbf{k}-\mathbf{k}'}^\alpha}{D_\alpha^*(q)}\delta(\omega-\omega'+\alpha\omega_{\mathbf{k}-\mathbf{k}'})\delta(\omega'-\alpha\omega_{\mathbf{k}'}) \\ &\quad +k^2\left(1+\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2k'^2}\right)\frac{I_{\mathbf{k}-\mathbf{k}'}^{-\alpha}I_\mathbf{k}^\alpha}{D_\alpha(q')}\delta(\omega-\omega'-\alpha\omega_{\mathbf{k}-\mathbf{k}'})\delta(\omega+\alpha\omega_\mathbf{k}) \\ &\quad \left.+\frac{2k^2I_{\mathbf{k}'}^\alpha I_{\mathbf{k}-\mathbf{k}'}^{-\alpha}}{D_\alpha^*(q)}\delta(\omega-\omega'-\alpha\omega_{\mathbf{k}-\mathbf{k}'})\delta(\omega'+\alpha\omega_{\mathbf{k}'})\right\}. \end{aligned} \tag{15}$$

We evaluate the inverse dielectric functions by ignoring the principal part contributions,

$$\begin{aligned} \frac{1}{D_\alpha(q-q')} &= \frac{1}{(\omega-\omega'+\alpha\omega_{\mathbf{k}-\mathbf{k}'}+i0)\,\dfrac{\partial D_\alpha(q-q')}{\partial(\omega-\omega')}} \\ &= -i\pi\,\delta(\omega-\omega'+\alpha\omega_{\mathbf{k}-\mathbf{k}'}), \\ \frac{1}{D_\alpha(q')} &= \frac{1}{(\omega'+\alpha\omega_{\mathbf{k}'}+i0)\,\dfrac{\partial D_\alpha(q')}{\partial\omega'}} = -i\pi\,\delta(\omega'+\alpha\omega_{\mathbf{k}'}), \\ \frac{1}{D_\alpha^*(q)} &= \frac{1}{(\omega+\alpha\omega_\mathbf{k}+i0)\,\dfrac{\partial D_\alpha^*(q)}{\partial\omega}} = i\pi\,\delta(\omega+\alpha\omega_\mathbf{k}), \end{aligned} \tag{16}$$

where we have again made use of the fact that $\partial D_\alpha(q)/\partial\omega = 1$. This leads to the wave kinetic equation for the intensities $I_\mathbf{k}^\alpha$ associated with the Elsässer fields,

$$\begin{aligned} \frac{\partial I_\mathbf{k}^\alpha}{\partial t} &= -2k^2\nu I_\mathbf{k}^\alpha - \pi k^2\int d\mathbf{k}'\left\{\left[\left(1+\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2k'^2}\right)I_\mathbf{k}^\alpha - 2I_{\mathbf{k}'}^\alpha\right]I_{\mathbf{k}-\mathbf{k}'}^{-\alpha}\right. \\ &\quad \left.\times\,\delta(\omega_\mathbf{k}-\omega_{\mathbf{k}'}+\omega_{\mathbf{k}-\mathbf{k}'}) + (\mathbf{k}\leftrightarrow\mathbf{k}-\mathbf{k}')\right\}, \end{aligned} \tag{17}$$

where we now resort back to the long-hand notation. This form of wave kinetic equation is essentially the same as those found in the literature, e.g., Refs. 22 and 23.

III. IMPLICATIONS OF WEAK MHD TURBULENCE THEORY

The first implication of weak MHD turbulence theory relates to the energy conservation theorem. It can be shown that Eq. (17) satisfies the conservation of total energy associated with the Elsässer fields in the absence of dissipation. In order to show this, it is useful to multiply Eq. (17) with $\omega = \alpha\omega_\mathbf{k}$, sum over both signs of $\alpha = \pm$, and integrate over $\mathbf{k}$. Note that $\hbar\omega$ represents the quantum mechanical energy, although the present discussion is purely classical. Nevertheless, this shows that the quantity considered below represents the time rate of change in the total energy (TE) of Elsässer fields:

$$\begin{aligned} \frac{d}{dt}(\mathrm{TE}) &= \sum_{\alpha=\pm1}\frac{\partial}{\partial t}\int d\mathbf{k}\,\alpha\omega_\mathbf{k}I_\mathbf{k}^\alpha, \\ &= -2\int d\mathbf{k}\,k^2\nu\sum_{\alpha=\pm1}\alpha\omega_\mathbf{k}I_\mathbf{k}^\alpha - \pi k^2\int d\mathbf{k}\int d\mathbf{k}'\sum_{\alpha=\pm1} \\ &\quad \times\left\{\left[\left(1+\frac{[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')]^2}{k^2(\mathbf{k}-\mathbf{k}')^2}\right)I_\mathbf{k}^\alpha - 2I_{\mathbf{k}-\mathbf{k}'}^\alpha\right]\right. \\ &\quad \times I_{\mathbf{k}'}^{-\alpha}\alpha\omega_\mathbf{k}\delta(\omega_\mathbf{k}+\omega_{\mathbf{k}'}-\omega_{\mathbf{k}-\mathbf{k}'}) \\ &\quad \left.+\left[\left(1+\left(\frac{\mathbf{k}\cdot\mathbf{k}'}{k^2k'^2}\right)^2\right)I_\mathbf{k}^\alpha - 2I_{\mathbf{k}'}^\alpha\right]I_{\mathbf{k}-\mathbf{k}'}^{-\alpha}\alpha\omega_\mathbf{k}\delta(\omega_\mathbf{k}-\omega_{\mathbf{k}'}+\omega_{\mathbf{k}-\mathbf{k}'}).\right. \end{aligned} \tag{18}$$

We pay attention to the nonlinear term, which is expressed concretely as

$$\begin{aligned} \mathrm{NL} = -\pi k^2\int d\mathbf{k}\int d\mathbf{k}'\Bigg\{&\left(1+\frac{[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')]^2}{k^2(\mathbf{k}-\mathbf{k}')^2}\right) \\ &\times I_\mathbf{k}^+I_{\mathbf{k}'}^-\omega_\mathbf{k}\,\delta(\omega_\mathbf{k}+\omega_{\mathbf{k}'}-\omega_{\mathbf{k}-\mathbf{k}'}) && \text{(A)} \\ &+\left(1+\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2k'^2}\right)I_\mathbf{k}^+I_{\mathbf{k}-\mathbf{k}'}^-\omega_\mathbf{k}\,\delta(\omega_\mathbf{k}-\omega_{\mathbf{k}'}+\omega_{\mathbf{k}-\mathbf{k}'}), && \text{(A}'\text{)} \\ &-\left(1+\frac{[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')]^2}{k^2(\mathbf{k}-\mathbf{k}')^2}\right)I_\mathbf{k}^-I_{\mathbf{k}'}^+\omega_\mathbf{k}\,\delta(\omega_\mathbf{k}+\omega_{\mathbf{k}'}-\omega_{\mathbf{k}-\mathbf{k}'}) && \text{(B)} \\ &-\left(1+\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2k'^2}\right)I_\mathbf{k}^-I_{\mathbf{k}-\mathbf{k}'}^+\omega_\mathbf{k}\,\delta(\omega_\mathbf{k}-\omega_{\mathbf{k}'}+\omega_{\mathbf{k}-\mathbf{k}'}) && \text{(B}'\text{)} \\ &-2I_{\mathbf{k}-\mathbf{k}'}^+I_{\mathbf{k}'}^-\omega_\mathbf{k}\,\delta(\omega_\mathbf{k}+\omega_{\mathbf{k}'}-\omega_{\mathbf{k}-\mathbf{k}'}) && \text{(C)} \\ &-2I_{\mathbf{k}'}^+I_{\mathbf{k}-\mathbf{k}'}^-\omega_\mathbf{k}\,\delta(\omega_\mathbf{k}-\omega_{\mathbf{k}'}+\omega_{\mathbf{k}-\mathbf{k}'}) && \text{(C}'\text{)} \\ &+2I_{\mathbf{k}'}^+I_{\mathbf{k}-\mathbf{k}'}^-\omega_\mathbf{k}\,\delta(\omega_\mathbf{k}+\omega_{\mathbf{k}'}-\omega_{\mathbf{k}-\mathbf{k}'}) && \text{(D)} \\ &+2I_{\mathbf{k}-\mathbf{k}'}^+I_{\mathbf{k}'}^-\omega_\mathbf{k}\,\delta(\omega_\mathbf{k}-\omega_{\mathbf{k}'}+\omega_{\mathbf{k}-\mathbf{k}'})\Bigg\}, && \text{(D}'\text{)} \end{aligned}$$

where we have indicated four similar pairs of terms with (A), (A′), (B), (B′), (C), (C′), (D), and (D′). After trivial changes of dummy variables, we can show that each pair cancels out to zero. Specifically, for the terms (A′) and (B′), we first interchange $\mathbf{k}'$ and $\mathbf{k}-\mathbf{k}'$, and subsequently change the signs $\mathbf{k}\to-\mathbf{k}$ and $\mathbf{k}'\to-\mathbf{k}'$ (and make use of the symmetry properties $I^\alpha_{-\mathbf{k}}=I^\alpha_{\mathbf{k}}$ and $\omega_{-\mathbf{k}}=-\omega_{\mathbf{k}}$). Then, one can easily show that these terms exactly cancel out (A) and (B) terms, respectively. For the terms (C′) and (D′), we simply make the change of signs, $\mathbf{k}\to-\mathbf{k}$ and $\mathbf{k}'\to-\mathbf{k}'$, plus invoke appropriate symmetry properties to cancel out the other two respective terms, (C) and (D). In short, we have

$$\sum_{\alpha=\pm1}\frac{\partial}{\partial t}\int d\mathbf{k}\,\alpha\omega_{\mathbf{k}}I^\alpha_{\mathbf{k}}=-2\int d\mathbf{k}\,k^2\nu\sum_{\alpha=\pm1}\alpha\omega_{\mathbf{k}}I^\alpha_{\mathbf{k}}. \tag{19}$$

If we ignore the dissipation term on the right-hand side, then we have the wave energy conservation theorem. That is, the exchange of wave momentum and energy via three-wave resonance does not affect the total energy content. It only leads to the redistribution of wave momentum and energy, that is, the cascade of turbulence. This is, of course, to be expected, since the three-wave resonance that leads to the turbulent cascade does not alter the energy content until the cascade reaches the scale, $\lambda\propto\nu^{-1/2}$, the so-called dissipation range, at which point the turbulent energy is dissipated by viscosity or magnetic resistivity.

The second implication of the weak MHD turbulence theory is the inherent anisotropic nature of the turbulence spectrum.[22,23] That is, the cascade in directions transverse to the ambient magnetic field, or equivalently, perpendicular to the Alfvén wave propagation direction, is the dominant process. To see this, let us consider the three-wave resonance conditions, which are given by $\omega_{\mathbf{k}}+\omega_{\mathbf{k}'}-\omega_{\mathbf{k}-\mathbf{k}'}=2k'_\parallel c_A$ and $\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}+\omega_{\mathbf{k}-\mathbf{k}'}=2(k_\parallel-k'_\parallel)c_A$. This allows one to write the wave kinetic equation (17) as

$$\frac{\partial I^\alpha_{\mathbf{k}}}{\partial t}=-2k^2\nu I^\alpha_{\mathbf{k}}-\frac{\pi k^2}{2c_A}\int d\mathbf{k}'\left\{\left[\left(1+\frac{[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')]^2}{k^2(\mathbf{k}-\mathbf{k}')^2}\right)I^\alpha_{\mathbf{k}}-2I^\alpha_{\mathbf{k}-\mathbf{k}'}\right]\right.$$
$$\left.\times I^{-\alpha}_{\mathbf{k}'}\delta\left(k'_\parallel\right)+\left[\left(1+\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2k'^2}\right)I^\alpha_{\mathbf{k}}-2I^\alpha_{\mathbf{k}'}\right]I^{-\alpha}_{\mathbf{k}-\mathbf{k}'}\delta\left(k_\parallel-k'_\parallel\right)\right\}. \tag{20}$$

This result shows that the cascade process primarily affects the perpendicular wave vector, since the $\mathbf{k}'$ integration along the parallel direction is suppressed by the delta function conditions, $\delta(k'_\parallel)$ and $\delta(k'_\parallel-k_\parallel)$.

To see this more clearly, let us assume a Cartesian coordinate system where both $\mathbf{k}$ and $\mathbf{k}'$ vectors lie in the xz plane. In such a situation, since the y axis is a trivial coordinate, the problem effectively reduces to a two-dimensional situation where physical quantities are implicitly assumed as translationally invariant along the y axis. Such an effective two-dimensional problem, however, can be translated into a genuine three-dimensional result if we consider a cylindrical, or azimuthal, symmetry associated with physical quantities. Such a consideration will be relevant later, when we discuss the steady-state spectrum of the weak MHD turbulence. For now, let us assume that $\mathbf{k}=\hat{\mathbf{x}}k_x+\mathbf{z}k_z$ and $\mathbf{k}'=\hat{\mathbf{x}}k'_x+\mathbf{z}k'_z$. Then, Eq. (20) can be written as

$$\begin{aligned}\frac{\partial I_\alpha(k_x,k_z)}{\partial t}&=-2k^2\nu I_\alpha(k_x,k_z)\\&\quad-\frac{\pi k^2}{2c_A}\int_{-\infty}^{\infty}dk'_x\left\{\left[\left(1+\frac{[k_x(k_x-k'_x)+k_z^2]^2}{k^2[(k_x-k'_x)^2+k_z^2]}\right)I_\alpha(k_x,k_z)\right.\right.\\&\quad\left.-2I_\alpha(k_x-k'_x,k_z)\right]I_{-\alpha}(k'_x,0)\\&\quad+\left[\left(1+\frac{(k_xk'_x+k_z^2)^2}{k^2(k'^2_x+k_z^2)}\right)I_\alpha(k_x,k_z)-2I_\alpha(k'_x,k_z)\right]\\&\quad\left.\times I_{-\alpha}(k_x-k'_x,0)\right\}.\end{aligned} \tag{21}$$

From this, it is clear that dynamical processes affect k_x while changes in intensity along k_z come about only indirectly. As a consequence, we may consider the dynamics along k_x only by taking $k_z=0$ in Eq. (21),

$$\begin{aligned}\frac{\partial W_\alpha(k_x)}{\partial t}&=-2k_\perp^2\nu W_\alpha(k_x)\\&\quad-\frac{\pi k_x^2}{2c_A}\int_{-\infty}^{\infty}dk'_x\{[W_\alpha(k_x)-W_\alpha(k_x-k'_x)]W_{-\alpha}(k'_x)\\&\quad+[W_\alpha(k_x)-W_\alpha(k'_x)]W_{-\alpha}(k_x-k'_x)\},\end{aligned} \tag{22}$$

where $W_\alpha(k_x)=I_\alpha(k_x,0)$. This equation describes the perpendicular cascade of highly anisotropic weak MHD turbulence along the perpendicular direction.

A question that naturally arises relates to the steady-state spectrum, which points to the third implication of the present problem. Let us assume a steady state and ignore dissipation in Eq. (22). Let us also assume that in the steady state the weak MHD turbulence is in balance, $W_+(k_x)=W_-(k_x)=W(k)$, that is, $\mathbf{u}_{\mathbf{k}}\cdot\mathbf{b}_{\mathbf{k}}=0$, and for the sake of notational simplicity, we suppress the subscript x, and write $k=k_x$. Then, we have

$$0=\int_{-\infty}^{\infty}dk'\{[W(k')+W(k-k')]W(k)-2W(k')W(k-k')\}. \tag{23}$$

Obviously, the constant $W(k)=W_0$ satisfies the above. Of course, $W(k)=W_0$ over an entire spectral range may lead to divergence when integrated over an infinite range of k, but constant $W(k)$ may be realized over a finite space. Note that an inverse power-law spectrum k^{-s} also leads to divergence at $k=0$, but such a distribution can be realized over a finite k domain. Consequently, one cannot blindly apply the power-law spectrum over an entire domain. Nevertheless, suppose that the spectrum is given by a power law,

$$W(k)\propto\frac{1}{k^s}. \tag{24}$$

Then, we have

$$\begin{aligned}0&=\frac{2}{k^s}\int_{-\infty}^{\infty}\frac{dx}{x^s}-2\int_{-\infty}^{\infty}\frac{dx}{x^s(k-x)^s},\\&=\frac{2[1+(-1)^s]}{k^s}\int_0^\infty\frac{dx}{x^s}-2\int_{-\infty}^{k}\frac{dx}{x^s(k-x)^s}-2(-1)^s\int_k^\infty\frac{dx}{x^s(x-k)^s},\\&=\frac{2[1+(-1)^s]}{k^s}\int_0^\infty\frac{dx}{x^s}-2(-1)^s\int_{-k}^{\infty}x^{-s}(x+k)^{-s}\,dx-2(-1)^s\\&\quad\times\int_k^\infty x^{-s}(x-k)^{-s}\,dx.\end{aligned} \tag{25}$$

Upon making use of the following integral identity:

$$\int_u^\infty x^{-s}(x-u)^{-s}\,dx = u^{1-2s}B(2s-1,1-s) = u^{1-2s}\,\frac{\Gamma(2s-1)\Gamma(1-s)}{\Gamma(s)}, \tag{26}$$

we obtain

$$0 = \frac{2\left[1+(-1)^s\right]}{k^s}\int_0^\infty \frac{dx}{x^s}. \tag{27}$$

From this, it is clear that all odd integer s values satisfy the equation. Note that the second and third terms on the right-hand sides of second and third lines in Eq. (25) exactly cancel each other out. This means that the steady-state weak MHD turbulence spectrum along the perpendicular direction is not unique such that multiple solutions are possible including $W(k_x) = \text{const}$ and $W(k_x) \propto k_x^{-1}$, k_x^{-3}, k_x^{-5}. Note that even more divergent spectra of the type k_x^{2n+1}, $n = 0, 1, 2, 3, \ldots$ are also possible in a mathematical sense, although for all practical purposes, power-law solutions with positive spectral indices are not of particular physical significance. Of these possible solutions, we are particularly interested in $W(k_x) \propto k_x^{-1}$ since this solution is related to the steady-state spectrum discussed in the literature, namely, that the perpendicular spectrum of weak MHD turbulence is $\propto k_\perp^{-2}$.[22] In order to recover such a relationship, we invoke the conservation relationship between the Cartesian representation in the two-dimensional sheet discussed thus far and three-dimensional turbulence with cylindrical (or azimuthal) symmetry,

$$\int_{-\infty}^\infty W(k_x)\,dk_x = 2\pi\int_0^\infty W(k_\perp)\,k_\perp dk_\perp. \tag{28}$$

Upon substituting $W(k_x) = Ck_x^{-1}$ on the left-hand side, changing the dummy integral variable from k_x to $k_\perp$, and extending the perpendicular $k_\perp$ integral on the right-hand side to negative range of $k_\perp$ under the assumption that the intensity is invariant under the change of sign of $k_\perp$ (which is justified *a posteriori*), we have

$$C\int_{-\infty}^\infty k_\perp^{-1}\,dk_\perp = \pi\int_{-\infty}^\infty W(k_\perp)\,k_\perp dk_\perp. \tag{29}$$

A direct comparison of both sides of the equality leads to the desired result,

$$W(k_\perp) \propto \frac{1}{k_\perp^2}. \tag{30}$$

We have also solved the evolution equation (22) by numerical means. MHD has no characteristic scale, but suppose that we normalize the equation into a dimensionless form by introducing an arbitrary scale length, L, and normalize the wave numbers with respect to L. We also consider the dimensionless time with respect to L and Alfvén speed c_A. The dissipation rate is also normalized into a dimensionless form. Finally, the Elsässer wave energy density is also written in dimensionless form,

$$x = k_\perp L, \quad T = \frac{c_A t}{L}, \quad \nu \to \frac{\nu}{Lc_A}, \quad W_\alpha(x) \to \frac{\pi W_\alpha(k_x)}{2L^2 c_A^2}. \tag{31}$$

Then, we have

$$\begin{aligned}\frac{\partial W_\alpha(x)}{\partial T} &= -2x^2\nu W_\alpha(x) + 4x^2\Bigg(\int_0^x ds W_{-\alpha}(s)W_\alpha(x-s) \\ &\quad + \int_0^\infty ds W_{-\alpha}(s)W_\alpha(x+s) + \int_x^\infty ds W_{-\alpha}(s)W_\alpha(s-x) \\ &\quad - 4W_\alpha(x)\int_0^\infty ds W_{-\alpha}(s)\Bigg),\end{aligned} \tag{32}$$

where we have rewritten the s integrals as integrals over positive s only by making use of the symmetry $W_\alpha(-s) = W_\alpha(s)$.

In Fig. 1, we showcase a sample result where we initiated the computation with an identical Gaussian spectral profile for both $W(x) = W_+(x) = W_-(x) = W_0\exp[-(x-x_0)^2/\Delta]$, where $W_0 = 0.1$, $x_0 = 0.2$, and $\Delta = 0.005$. We have considered a range of normalized perpendicular wave numbers, $0 < x = k_x L < 1$, and have solved the evolution equation (32) up to $T = 500$. The initial profile is plotted with dots, while the snapshots at different times, $T = 100$, 200, 300, 400, and 500, are plotted by means of colored curves, with corresponding time intervals indicated with a color scheme. As one can see, the initial Gaussian profile gradually evolves into a quasi-power-law spectral profile characterized by k_x^{-1} by the forward cascade process, which translates to $k_\perp^{-2}$, as noted above, in agreement with Ref. 22. We have also considered normalized dissipation rate of $\nu = 10^{-4}$, but the numerical solution is hardly affected even if we set ν equal to zero. Note that the quasi-exponential turnover feature near maximum $x_{\max} = k_x L = 1$ is simply the result of the boundary condition rather than dissipation by finite resistivity. We find that this result is rather interesting. Despite the prediction that any of the spectral forms $W(k_x) = \text{const}$ and/or $W(k_x) = k_x^{2n+1}$, $n = 0, \pm1, \pm2, \pm3, \ldots$, can be realized, which follows from the strict mathematical analysis of steady-state equation, the numerical solution shown in Fig. 1 nonetheless corresponds to k_x^{-1} spectrum over a finite range, which implies $k_\perp^{-2}$ spectral shape in 3D geometry with cylindrical (or equivalently, azimuthal) symmetry.

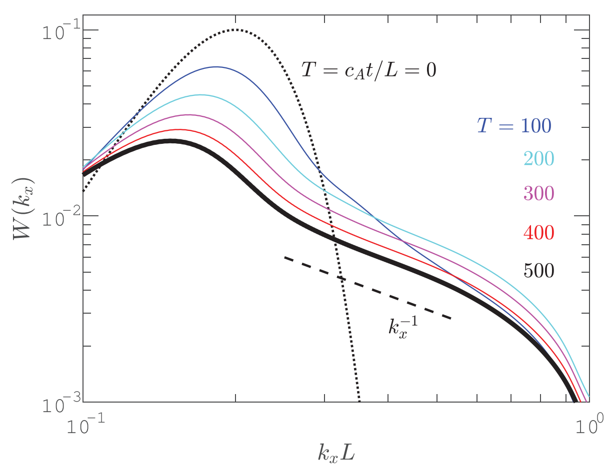

FIG. 1. Initially Gaussian profile $W(x) = W_+(x) = W_-(x) = W_0\exp[-(x-x_0)^2/\Delta]$, where $W_0 = 0.1$, $x_0 = 0.2$, and $\Delta = 0.005$, evolves into a power-law spectral profile characterized by k_x^{-1} by forward cascade, which translates to $k_\perp^{-2}$, in agreement with Galtier *et al.*[22]

IV. SUMMARY AND CONCLUSIONS

The study of magnetohydrodynamic (MHD) turbulence is important due to its varied applications including its role not only in heating the solar corona and acceleration of the wind but also in other astrophysical and laboratory environments.[1–4,25–28] In the present paper, we reformulated the weak MHD turbulence theory[22,23] under a slightly different approach often employed in the Langmuir/ion–sound wave kinetic weak turbulence theory.[41,43,46] We have confirmed that the weak anisotropic MHD turbulence energy spectrum in the asymptotically steady state behaves as $k_\perp^{-2}$, where $k_\perp$ represents the wave number perpendicular to the ambient magnetic field, in agreement with Refs. 22 and 23. This is rather interesting since such a quasi-steady-state solution, according to purely mathematical reasoning, is not a unique solution and that other spectra are theoretically possible. Nevertheless, our numerical computation demonstrated that the $k_\perp^{-2}$ (or in a two-dimensional sheet-like geometry, k_x^{-1}) spectrum is indeed reproduced as a quasi-steady-state solution.

An important point of the present work is that the standard methodologies widely employed in the MHD (and fluid) turbulence problem and the kinetic plasma turbulence situations are mutually equivalent such that these methodologies and approaches may be employed interchangeably for other more complex situations. Nonetheless, we find that the standard method involved in the kinetic plasma turbulence theory, which involves the shortcut two-time scale analysis and the iterative solution-based construction of the three-body correlation function, offers the possibility of a more convenient pathway to construct a similar analysis for more complex problems.

Before we close, we mention that the weak MHD turbulence theory as discussed in this paper, or for that matter, any other turbulence theory based on fluid equations, lacks the crucial element of wave–particle interaction process. Instead, MHD or fluid turbulence theories only deal with the process of spectral transfer of wave energy. In order to encapsulate the wave–particle aspect of the problem, one needs to move beyond the fluid models of turbulence and incorporate kinetic effects. Some discussions along such a line of approach may represent the issue of absorption and emission of low-frequency fluctuations by particles. For instance, Refs. 50–52 investigated the issue of spontaneous thermal emission of Alfvénic fluctuations by means of kinetic theory, which could, in principle, be coupled to the wave kinetic equation for Alfvénic turbulence once the MHD weak turbulence theory is reformulated within the framework of plasma kinetic theory. Such a task is, however, beyond the scope of the present paper.

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DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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