Discrete Mathematics 345 (2022) 112902

Contents lists available at ScienceDirect

Discrete Mathematics

www.elsevier.com/locate/disc

An improvement to the Hilton-Zhao vertex-splitting conjecture

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ARTICLE INFO

Article history:
Received 30 June 2021
Received in revised form 6 January 2022
Accepted 18 March 2022
Available online xxxx

Keywords:
Overfull graph
Multifan
Kierstead path
Vertex-splitting

ABSTRACT

For a simple graph G, denote by n, $\Delta(G)$, and $\chi'(G)$ its order, maximum degree, and chromatic index, respectively. A graph G is *edge-chromatic critical* if $\chi'(G) = \Delta(G) + 1$ and $\chi'(H) < \chi'(G)$ for every proper subgraph H of G. Let G be an n-vertex connected regular class 1 graph, and let G^* be obtained from G by splitting one vertex of G into two vertices. Hilton and Zhao in 1997 conjectured that G^* must be edge-chromatic critical if $\Delta(G) > n/3$, and they verified this when $\Delta(G) \ge \frac{n}{2}(\sqrt{7} - 1) \approx 0.82n$. In this paper, we prove it for $\Delta(G) \ge 0.75n$.

1. Introduction

We consider only simple graphs. Let G be a graph. Denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively. An *edge k-coloring* of G is a mapping φ from $E(G)$ to the set of integers $[1, k] := \{1, \dots, k\}$, called *colors*, such that no two adjacent edges receive the same color with respect to φ. The *chromatic index* of G, denoted $\chi'(G)$, is defined to be the smallest integer k so that G has an edge k-coloring. We denote by $\mathcal{C}^k(G)$ the set of all edge k-colorings of G. In 1960's, Vizing [10] showed that every simple graph G has chromatic index either $\Delta(G)$ or $\Delta(G) + 1$. If $\chi'(G) = \Delta(G)$, then G is said to be of *class 1*; otherwise, it is said to be of *class 2*. Holyer [4] showed that it is NP-complete to determine whether an arbitrary graph is of class 1. However, by Vizing's Theorem and the fact that $\chi'(G) \ge |E(G)|/\lfloor |V(G)|/2 \rfloor$, a graph G must be class 2 if $|E(G)| > \Delta(G)\lfloor |V(G)|/2 \rfloor$. Such graphs are called *overfull*. Easily implied by its definition, overfull graphs are of odd order.

We call G *edge-chromatic critical* or *Δ-critical* if $\chi'(G) = \Delta(G)+1$ and $\chi'(H) < \Delta(G) + 1$ for every proper subgraph H of G. For example, odd cycles and the graph obtained from the Petersen graph by deleting one vertex are edge-chromatic critical. We study sufficient conditions for a class 2 graph to be edge-chromatic critical. A *vertex-splitting* in G at a vertex v gives a new graph G^* obtained by replacing v with two new adjacent vertices v_1 and v_2 and partitioning the neighborhood $N_G(v)$ into two nonempty subsets that, respectively, serve as the set of neighbors of v_1 and v_2 from $V(G)$ in G^*. We say G^* is obtained from G by a vertex-splitting. Vertex-splitting was called the "Mőbius-type gluing technique" in [1] and [7]. It is easy to see that a regular class 1 graph has even order, and that every graph obtained from a regular graph of even order by a vertex-splitting is overfull. Hilton and Zhao [3] in 1997 proposed the following conjecture.

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1 This author was supported in part by NSF grant DMS-1855716.

https://doi.org/10.1016/j.disc.2022.112902

Conjecture 1 *(Vertex-splitting conjecture). Let G be an n-vertex connected class 1 Δ-regular graph with $\Delta > \frac{n}{3}$. If G^* is obtained from G by a vertex-splitting, then G^* is Δ-critical.*

Since the graph G^* above is overfull and so is class 2, the difficulty of the vertex-splitting conjecture lies in checking that every edge of G^* is critical, i.e., its deletion decreases the chromatic index of G^*. Hilton and Zhao [3] in the same paper verified the conjecture for graphs G with $\Delta(G) \geq \frac{n}{2}(\sqrt{7}-1) \approx 0.82n$. Song [5] in 2002 showed that the conjecture is true for a special class of graphs G with $\Delta(G) \geq \frac{n}{2}$. Except this result, to our best knowledge, we are not aware of any other progress on the conjecture. In this paper, we verify the conjecture for graphs G with $\Delta(G) \geq 0.75n$ as follows.

Theorem 1. *Let n and Δ be positive integers such that $\Delta \geq \frac{3(n-1)}{4}$. If G is obtained from an $(n-1)$-vertex Δ-regular class 1 graph by a vertex-splitt, then G is Δ-critical.*

Note that the $(n-1)$-vertex Δ-regular class 1 in Theorem 1 is connected by the lower bound on Δ. The remainder of this paper is organized as follows. In Section 2, we introduce some definitions and preliminary results. In Section 3, we prove Theorem 1. In the last Section, we prove one newly developed adjacency lemma.

2. Definitions and preliminary results

Let G be a graph. For $e \in E(G)$, $G-e$ denotes the graph obtained from G by deleting the edge e. The symbol Δ is reserved for $\Delta(G)$, the maximum degree of G throughout this paper. A *k-vertex* in G is a vertex of degree k in G, and a *k-neighbor* of a vertex v is a neighbor of v that is a k-vertex in G. For $u, v \in V(G)$, we use $\text{dist}_G(u,v)$ to denote the distance between u and v, which is the length of a shortest path connecting u and v in G. For $S \subseteq V(G)$, define $\text{dist}_G(u,S) = \min_{v\in S} \text{dist}_G(u,v)$.

An edge $e \in E(G)$ is a *critical edge* of G if $\chi'(G-e) < \chi'(G)$. If G is connected, class 2, and every edge of G is critical, then clearly G is Δ-critical. Edge-chromatic critical graphs have more structure than general class 2 graphs. For example, Vizing's Adjacency Lemma (VAL) from 1965 [10] is a useful tool that reveals certain structure at a vertex by assuming the criticality of an edge.

Lemma 2 *(Vizing's Adjacency Lemma (VAL), [10]). Let G be a class 2 graph with maximum degree Δ. If $e = xy$ is a critical edge of G, then x has at least $\Delta - d_G(y) + 1$ Δ-neighbors in $V(G) \setminus \{y\}$.*

Let G be a graph and $\varphi \in \mathcal{C}^k(G-e)$ for some edge $e \in E(G)$ and some integer $k \geq 0$. For any $v \in V(G)$, the set of colors *present* at v is $\varphi(v) = \{\varphi(f) : f \text{ is an edge incident to } v\}$, and the set of colors *missing* at v is $\overline{\varphi}(v) = [1,k] \setminus \varphi(v)$. For a vertex set $X \subseteq V(G)$, define $\overline{\varphi}(X) = \bigcup_{v\in X} \overline{\varphi}(v)$. We call X *elementary* with respect to φ or simply *φ-elementary* if $\overline{\varphi}(u) \cap \overline{\varphi}(v) = \emptyset$ for every two distinct vertices $u, v \in X$.

For two distinct colors $\alpha, \beta \in [1,k]$, let H be the spanning subgraph of G with $E(H) = \{f \in E(G) : \varphi(f) \in \{\alpha,\beta\}\}$. Each component of H is either an even cycle or a path, which is called an *(α,β)-chain* of G with respect to φ. If we interchange the colors α and β on an (α,β)-chain C of G, we get a new edge k-coloring of G, and we write

$$\varphi' = \varphi/C.$$

This operation is called a *Kempe change*.

Let $x, y \in V(G)$, and $\alpha, \beta \in [1,k]$ be two distinct colors. If x and y are contained in the same (α,β)-chain of G with respect to φ, we say x and y are *(α,β)-linked* with respect to φ. For a vertex-edge sequence S, we use $V(S)$ to denote the set of all vertices contained in the sequence.

2.1. Multifan and Kierstead path

The fan argument was introduced by Vizing [8,9] in his classical results on the upper bounds of chromatic indices for simple graphs and multigraphs. We will use multifan, a generalized version of Vizing fan, given by Stiebitz et al. [6], in our proof.

Let G be a graph, $e = rs_1 \in E(G)$ and $\varphi \in \mathcal{C}^k(G-e)$ for some integer $k \geq 0$. A *multifan* centered at r with respect to e and φ is a sequence $F_\varphi(r, s_1 : s_p) := (r, rs_1, s_1, rs_2, s_2, \ldots, rs_p, s_p)$ with $p \geq 1$ consisting of distinct vertices $r, s_1, s_2, \ldots, s_p$ and distinct edges $rs_1, rs_2, \ldots, rs_p$ satisfying the following condition:

(F1) For every edge rs_i with $i \in [2,p]$, there exists $j \in [1, i-1]$ such that $\varphi(rs_i) \in \overline{\varphi}(s_j)$.

We will simply denote a multifan $F_\varphi(r, s_1 : s_p)$ by F if φ and the vertices and edges in $F_\varphi(r, s_1 : s_p)$ are clear. The following result regarding a multifan can be found in [6, Theorem 2.1].

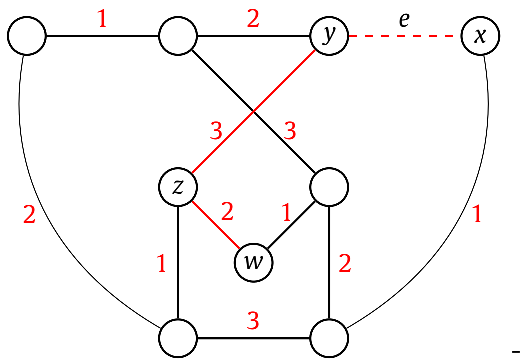

Fig. 1. A Kierstead path with non-elementary vertex set in a 3-coloring of $P^* - xy$.

Lemma 3. *Let G be a class 2 graph and $F_\varphi(r, s_1 : s_p)$ be a multifan with respect to a critical edge $e = rs_1$ and a coloring $\varphi \in \mathcal{C}^\Delta(G-e)$. Then the following statements hold.*

(a) *$V(F)$ is φ-elementary.*
(b) *Let $\alpha \in \overline{\varphi}(r)$. Then for every $i \in [1, p]$ and $\beta \in \overline{\varphi}(s_i)$, r and s_i are (α, β)-linked with respect to φ.*

Let G be a graph, $e = v_0v_1 \in E(G)$, and $\varphi \in \mathcal{C}^k(G-e)$ for some integer $k \geq 0$. A *Kierstead path* with respect to e and φ is a sequence $K = (v_0, v_0v_1, v_1, v_1v_2, v_2, \ldots, v_{p-1}, v_{p-1}v_p, v_p)$ with $p \geq 1$ consisting of distinct vertices $v_0, v_1, \ldots, v_p$ and distinct edges $v_0v_1, v_1v_2, \ldots, v_{p-1}v_p$ satisfying the following condition:

(K1) For every edge v_iv_{i+1} with $i \in [1, p-1]$, there exists $j \in [0, i-1]$ such that $\varphi(v_iv_{i+1}) \in \overline{\varphi}(v_j)$.

Clearly a Kierstead path with at most 3 vertices is a multifan. We consider Kierstead paths with 4 vertices. The result below was proved in Theorem 3.3 from [6].

Lemma 4. *Let G be a class 2 graph, $e = v_0v_1 \in E(G)$ be a critical edge, and $\varphi \in \mathcal{C}^\Delta(G-e)$. If $K = (v_0, v_0v_1, v_1, v_1v_2, v_2, v_2v_3, v_3)$ is a Kierstead path with respect to e and φ, then the following statements hold.*

(a) *If $\min\{d_G(v_1), d_G(v_2)\} < \Delta$, then $V(K)$ is φ-elementary.*
(b) $|\overline{\varphi}(v_3) \cap (\overline{\varphi}(v_0) \cup \overline{\varphi}(v_1))| \leq 1$.

3. Proof of Theorem 1

The proof of Theorem 1 is mainly an application of a new adjacency lemma, Lemma 5 below. The truth of the vertex-splitting conjecture could be proved when $\Delta \geq \frac{n}{2}$ if the vertex set of every Kierstead path on four vertices is elementary. Unfortunately, this is not true. A counterexample has been found using the graph P^* obtained from the Petersen graph by deleting one vertex, see Fig. 1: let $\varphi \in \mathcal{C}^3(P^* - xy)$ be the given coloring. Then $K = (x, xy, y, yz, z, zw, w)$ is a Kierstead path as $\varphi(yz) = 3 \in \overline{\varphi}(x)$ and $\varphi(zw) = 2 \in \overline{\varphi}(x)$, and $V(K)$ is not φ-elementary since $3 \in \overline{\varphi}(x) \cap \overline{\varphi}(w)$.

We define a *short-kite* to be a 6-vertex graph consisting of a 4-cycle *abuca* and two additional edges ux and uy. The new adjacency lemma below is an attempt to reveal some elementary properties of a Kierstead path on four vertices by incorporating some additional structure to the path.

Lemma 5. *Let G be a class 2 graph, $H \subseteq G$ be a short-kite with $V(H) = \{a, b, c, u, x, y\}$, and let $\varphi \in \mathcal{C}^\Delta(G - ab)$. Suppose*

$$K = (a, ab, b, bu, u, ux, x) \quad \text{and} \quad K^* = (b, ab, a, ac, c, cu, u, uy, y)$$

are two Kierstead paths with respect to ab and φ. If $\overline{\varphi}(x) \cup \overline{\varphi}(y) \subseteq \overline{\varphi}(a) \cup \overline{\varphi}(b)$, then $\max\{d_G(x), d_G(y)\} = \Delta$.

The proof of Lemma 5 will be given in the last section. Since all vertices not missing a given color α are covered by the matching that consists of all edges colored by α in G, we have the Parity Lemma below, which has appeared in many papers, for example, see [2, Lemma 2.1].

Lemma 6 *(Parity Lemma). Let G be an n-vertex multigraph and $\varphi \in \mathcal{C}^k(G)$ for some integer $k \geq \Delta(G)$. Then for any color $\alpha \in [1, \Delta]$,* $|\{v \in V(G) : \alpha \in \overline{\varphi}(v)\}| \equiv n \pmod 2$.

Let G be a graph and $u, v \in V(G)$ be adjacent. We call (u, v) a *full-deficiency pair* of G if $d(u) + d(v) = \Delta(G) + 2$. Note that if we split a vertex x of a Δ-regular graph into u and v, then we obtain a full-deficiency pair (u, v). If G is Δ-critical, then a full-deficiency pair (u, v) of G is called a *saturating pair* of $G - uv$ in [1].

Lemma 7. *If G is an n-vertex class 2 graph with a full-deficiency pair (a, b) such that ab is a critical edge of G, then G satisfies the following properties.*

(i) *For every $x \in (N_G(a) \cup N_G(b)) \setminus \{a, b\}$, $d_G(x) = \Delta$;*
(ii) *For every $x \in V(G) \setminus \{a, b\}$, if $\text{dist}_G(x, \{a, b\}) = 2$, then $d_G(x) \geq \Delta - 1$. If also $d_G(a) < \Delta$ and $d_G(b) < \Delta$, then $d_G(x) = \Delta$;*
(iii) *For every $x \in V(G) \setminus \{a, b\}$, if $d_G(x) \geq n - |N_G(a) \cup N_G(b)|$, then $d_G(x) \geq \Delta - 1$. If also $d_G(a) < \Delta$ and $d_G(b) < \Delta$, then $d_G(x) = \Delta$.*

Proof. We let $\varphi \in \mathcal{C}^\Delta(G - ab)$ and $F = (b, ba, a)$ be a multifan with respect to ab and φ. By Lemma 3(a),

$$|\overline{\varphi}(V(F))| = |\overline{\varphi}(a)| + |\overline{\varphi}(b)| = (\Delta - (d_G(a) - 1)) + (\Delta - (d_G(b) - 1)) = \Delta. \tag{1}$$

By Lemma 3, for every $\varphi' \in \mathcal{C}^\Delta(G - ab)$, $\{a, b\}$ is φ'-elementary and for every $i \in \overline{\varphi}'(a)$ and $j \in \overline{\varphi}'(b)$, a and b are (i, j)-linked with respect to φ'. We will use this fact very often.

Since $\overline{\varphi}(a) \cap \overline{\varphi}(b) = \emptyset$ and $\overline{\varphi}(a) \cup \overline{\varphi}(b) = [1, \Delta]$, it follows that $\varphi(a) = \overline{\varphi}(b)$. Thus for any $x \in N_G(a) \setminus \{b\}$, (a, ab, b, ax, x) is a multifan with respect to ab and φ and so $\{a, b, x\}$ is φ-elementary by Lemma 3(a). It follows from (1) that $d_G(x) = \Delta$. Symmetrically, for each $x \in N_G(b) \setminus \{a\}$, $d_G(x) = \Delta$. This proves (i).

For (ii), let $x \in V(G) \setminus \{a, b\}$ such that $\text{dist}_G(x, \{a, b\}) = 2$. We assume that $\text{dist}_G(x, b) = 2$ and let $u \in (N_G(b)) \setminus \{a\}) \cap N_G(x)$. Then by (1), $K = (a, ab, b, bu, u, ux, x)$ is a Kierstead path with respect to ab and φ. By (1) and Lemma 4(b), it follows that $d_G(x) \geq \Delta - 1$. If $d_G(a) < \Delta$ and $d_G(b) < \Delta$, then $V(K)$ is φ-elementary by Lemma 4(a). Since $\overline{\varphi}(a) \cup \overline{\varphi}(b) = [1, \Delta]$ by (1), it follows that $d_G(x) = \Delta$.

For (iii), let $x \in V(G) \setminus \{a, b\}$ such that $d_G(x) \geq n - |N_G(a) \cup N_G(b)|$. The degree condition on x implies that $N_G(x) \cap (N_G(a) \cup N_G(b)) \neq \emptyset$, which implies that $\text{dist}_G(x, \{a, b\}) \leq 2$. But $\text{dist}_G(x, \{a, b\}) \neq 0$ since $x \notin \{a, b\}$ by hypothesis, and if $\text{dist}_G(x, \{a, b\}) = 1$ or 2 then the result follows from (i) or (ii) respectively. □

Corollary 8. *Let G be an n-vertex class 2 graph with a full-deficiency pair (a, b) such that ab is a critical edge of G. If $\Delta \geq \frac{3(n-1)}{4}$, then there exists at most one vertex $x \in V(G) \setminus \{a, b\}$ such that $d_G(x) = \Delta - 1$.*

Proof. Assume to the contrary that there exist distinct $x, y \in V(G) \setminus \{a, b\}$ such that $d_G(x) = d_G(y) = \Delta - 1$. By Lemma 7(i), $x, y \notin (N_G(a) \cup N_G(b)) \setminus \{a, b\}$. Since a and b are adjacent in G and $d_G(a) + d_G(b) = \Delta + 2$, it follows that $d_G(x) + |N_G(a) \cup N_G(b)| \geq \Delta - 1 + (\frac{1}{2}\Delta + 2) = \frac{3}{2}\Delta + 1 \geq \frac{9}{8}(n-1) + 1 \geq n$. Hence by Lemma 7(iii), we may assume that one of a and b, w.l.o.g., b, has degree Δ. Thus $d_G(a) = 2$. Let c be the other neighbor of a in G. As ab is a critical edge of G and $d_G(a) = 2$ and $d_G(b) = \Delta$, Lemma 2(VAL) implies that $d_G(c) = \Delta$. Thus (a, c) is a full-deficiency pair of G as well. Similarly, we may assume $x, y \notin N_G(c)$.

Since $d_G(b) = d_G(c) = \Delta$ and $x, y \notin N_G(b) \cup N_G(c)$, we get $|N_G(b) \cap N_G(c)| \geq 2\Delta - (n-2) \geq \frac{3(n-1)}{2} - (n-2) > \frac{n}{2}$. Similarly, $|N_G(x) \cap N_G(y)| \geq 2(\Delta - 1) - (n-2) > \frac{n}{2} - 2$. Since $b, c \notin \big(N_G(b) \cap N_G(c)\big) \cup \big(N_G(x) \cap N_G(y)\big)$, we get $|N_G(b) \cap N_G(c) \cap N_G(x) \cap N_G(y)| > \frac{n}{2} + (\frac{n}{2} - 2) - (n-2) = 0$. Let $u \in N_G(b) \cap N_G(c) \cap N_G(x) \cap N_G(y)$, H be the short-kite with $V(H) = \{a, b, c, u, x, y\}$, and $\varphi \in \mathcal{C}^\Delta(G - ab)$. As $\{a, b\}$ is φ-elementary, $|\overline{\varphi}(a) \cup \overline{\varphi}(b)| = |\overline{\varphi}(a)| + |\overline{\varphi}(b)| = \Delta$ and so $\overline{\varphi}(a) \cup \overline{\varphi}(b) = [1, \Delta]$. Thus $K = (a, ab, b, bu, u, ux, x)$ and $K^* = (b, ab, a, ac, c, cu, u, uy)$ are two Kierstead paths with respect to ab and φ, and $\overline{\varphi}(x) \cup \overline{\varphi}(y) \subseteq \overline{\varphi}(a) \cup \overline{\varphi}(b)$. However, $d_G(x) = d_G(y) = \Delta - 1$, contradicting Lemma 5. □

Theorem 1. *Let n and Δ be positive integers such that $\Delta \geq \frac{3(n-1)}{4}$. If G is obtained from an $(n-1)$-vertex Δ-regular class 1 graph by a vertex-splitting, then G is Δ-critical.*

Proof. Since $\Delta \geq \frac{3(n-1)}{4}$, any Δ-regular graph of order $(n-1)$ is connected. Thus G is obtained from a connected graph by vertex-splitting and so is also connected. Since G is overfull, it is class 2. Therefore, to show that G is Δ-critical, we only need to show that every edge of G is critical. Suppose to the contrary that there exists $xy \in E(G)$ such that xy is not a critical edge of G. Let $G^* = G - xy$. Then $\chi'(G^*) = \Delta + 1$.

Let ab be the edge of G whose contraction gives an $(n-1)$-vertex Δ-regular class 1 graph. Then (a, b) is a full-deficiency pair in G such that ab is a critical edge, which implies $ab \neq xy$, and every vertex in $V(G) \setminus \{a, b\}$ has degree Δ. Since any Δ-coloring of $G - ab$ gives a Δ-coloring of $G^* - ab$, ab is also a critical edge of G^*. If $\{a, b\} \cap \{x, y\} = \emptyset$, then (a, b) is a full-deficiency pair in G^* and $d_{G^*}(x) = d_{G^*}(y) = \Delta - 1$, which contradicts Corollary 8. Thus $\{a, b\} \cap \{x, y\} \neq \emptyset$, say $a = x$. Then we have $d_{G^*}(a) + d_{G^*}(b) = \Delta + 1$. However, Lemma 2(VAL) implies that $d_{G^*}(a) + d_{G^*}(b) \geq \Delta + 2$, a contradiction. □

4. Proof of Lemma 5

We start with some notation. Let G be a graph and $\varphi \in \mathcal{C}^k(G-e)$ for some edge $e \in E(G)$ and some integer $k \geq 0$. For all the concepts below, when we use them later on, if we skip φ, we mean the concept is defined with respect to the current edge coloring.

Let $x, y \in V(G)$, and $\alpha, \beta \in [1,k]$ be three colors. Let P be an (α,β)-chain of G with respect to φ that contains both x and y. If P is a path, denote by $P_{[x,y]}(\alpha,\beta,\varphi)$ the subchain of P that has endvertices x and y. By *swapping colors* along $P_{[x,y]}(\alpha,\beta,\varphi)$, we mean exchanging the two colors α and β on the path $P_{[x,y]}(\alpha,\beta,\varphi)$.

Define $P_x(\alpha,\beta,\varphi)$ to be the (α,β)-chain of G that contains x; it is a path or a cycle. If P_x is a path with x as an endvertex and $u, v \in P_x$ and u lies between x and v on P_x, then we say that P_x *meets* u *before* v. Suppose the current color of an edge uv of G is α, the notation $uv: \alpha \to \beta$ means to recolor the edge uv using the color β. If $|\overline{\varphi}(x)| = 1$, we will also use $\overline{\varphi}(x)$ to denote the color that is missing at x.

Lemma 5. *Let G be a class 2 graph, $H \subseteq G$ be a short-kite with $V(H) = \{a,b,c,u,x,y\}$, and let $\varphi \in \mathcal{C}^\Delta(G-ab)$. Suppose*

$$K = (a, ab, b, bu, u, ux, x) \quad \text{and} \quad K^* = (b, ab, a, ac, c, cu, u, uy, y)$$

are two Kierstead paths with respect to ab and φ. If $\overline{\varphi}(x) \cup \overline{\varphi}(y) \subseteq \overline{\varphi}(a) \cup \overline{\varphi}(b)$, then $\max\{d_G(x), d_G(y)\} = \Delta$.

Proof. Assume to the contrary that $\max\{d_G(x), d_G(y)\} \leq \Delta - 1$. Since $\emptyset \neq \overline{\varphi}(x) \subseteq \overline{\varphi}(a) \cup \overline{\varphi}(b)$, $V(K)$ is not φ-elementary. It follows from Lemma 4(a) that $d_G(b) = d_G(u) = \Delta$, and from Lemma 4(b) that $|\overline{\varphi}(x)| \leq 1$, so that $d_G(x) = \Delta - 1$.

Let $\overline{\varphi}(b) = \{1\}$. Then $\varphi(ac) = 1$. Let φ' be the coloring obtained from φ by uncoloring ac and coloring ab with color 1, and let $K' = (a, ac, c, cu, u, uy, y)$. Note that $\overline{\varphi}'(a) = \overline{\varphi}(a)$ and $\overline{\varphi}'(c) = \overline{\varphi}(c) \cup \overline{\varphi}(b)$. Since K^* is a Kierstead path with respect to ab and φ, it follows that K' is a Kierstead path with respect to ac and φ'. Now the argument of the previous paragraph applied to φ' and K' shows that $d_G(c) = \Delta$ and $d_G(y) = \Delta - 1$.

Since (a, ab, b) is a multifan with respect to ab and φ, a and b are $(1,\alpha)$-linked for any $\alpha \in \overline{\varphi}(a)$ by Lemma 3(b). We will need the following observation. Let α be a color and v a vertex such that $\alpha \in \overline{\varphi}(a)$, $v \notin \{a,b\}$, and $\{1,\alpha\} \cap \overline{\varphi}(v) \neq \emptyset$. Then a and b are $(1,\alpha)$-linked through c, and so $P_v(1,\alpha)$ does not pass through a, b, or c. Thus a $(1,\alpha)$-swap at v gives a coloring that satisfies all the properties of φ; in particular, K and K^* are still Kierstead paths.

We wish to choose φ so that $\varphi(uy) = \overline{\varphi}(b) = 1$. Let $\overline{\varphi}(y) = \alpha$ and $\varphi(uy) = \beta$. Assume $\beta \neq 1$. Then $\beta \in \overline{\varphi}(a)$, since K^* is a Kierstead path and $\overline{\varphi}(c) = \emptyset$. If $\alpha \neq 1$ then $\alpha \in \overline{\varphi}(a)$, since $\overline{\varphi}(y) \subseteq \overline{\varphi}(a) \cup \overline{\varphi}(b)$; thus a $(1,\alpha)$-swap at y makes $\overline{\varphi}(y) = \overline{\varphi}(b) = 1$. So we may assume that $\overline{\varphi}(y) = 1$ anyway. Now a $(1,\beta)$-swap at y makes $\varphi(uy) = \overline{\varphi}(b) = 1$. So we may assume that $\varphi(uy) = 1$ anyway. We consider now two cases.

Case 1. $\overline{\varphi}(x) = \overline{\varphi}(y)$.

Let $\varphi(ux) = \gamma$ and $\overline{\varphi}(x) = \overline{\varphi}(y) = \eta$. As $\varphi(uy) = 1$, we have $\overline{\varphi}(b) = 1 \notin \{\gamma, \eta\}$. As K is a Kierstead path and $\overline{\varphi}(x) \subseteq \overline{\varphi}(a) \cup \overline{\varphi}(b)$, $\gamma, \eta \in \overline{\varphi}(a)$. Denote by $P_u^*(1,\gamma)$ the maximal subpath of the chain $P_u(1,\gamma)$ that starts at u along uy and does not include the edge ux.

Claim 1a. *We may assume that $P_u^*(1,\gamma)$ ends at x, or at some vertex $z \in V(G) \setminus \{a,b,c,u,x,y\}$, or passing c ends at a.*

Proof. Since (a, ab, b) is a multifan with respect to ab and φ, we have $P_a(1,\gamma) = P_b(1,\gamma)$. If $u \notin P_a(1,\gamma)$, then the $(1,\gamma)$-chain containing u is a cycle or a path with endvertices contained in $V(G) \setminus \{a,b,c,u,x,y\}$. Thus $P_u^*(1,\gamma)$ ends at x or at some $z \in V(G) \setminus \{a,b,c,u,x,y\}$. Hence we assume $u \in P_a(1,\gamma)$. As a consequence, $P_u^*(1,\gamma)$ ends at either b or a. If $P_x(1,\gamma)$ ends at b, we color ab by 1, uncolor ac, and exchange the vertex labels b and c. This gives an edge Δ-coloring of $G - ab$ such that $P_u^*(1,\gamma)$ ends at a. Thus, if $u \in P_a(1,\gamma)$, we may always assume that $P_u^*(1,\gamma)$ ends at a. □

Let $\varphi(bu) = \delta$. Again, $\delta \in \overline{\varphi}(a)$. Clearly, $\delta \neq 1, \gamma$. Since a and b are $(1,\delta)$-linked with respect to φ, $P_b(1,\delta)$ cannot end at y, and so $\eta \neq \delta$. Thus $1, \gamma, \delta$ and η are pairwise distinct.

Claim 1b. *It holds that $ub \in P_y(\eta,\delta)$ and $P_y(\eta,\delta)$ meets u before b.*

Proof. Let φ^* be obtained from φ by coloring ab by δ and uncoloring bu. Note that $\overline{\varphi}^*(b) = 1, \overline{\varphi}^*(u) = \delta$ and $\varphi^*(uy) = 1$. Thus $F^* = (u, ub, b, uy, y)$ is a multifan with respect to bu and φ^*. Thus u and y are (η,δ)-linked by Lemma 3(b). By uncoloring ab and coloring bu by δ, we get back the original coloring φ. Therefore, under the coloring φ, $u \in P_y(\eta,\delta)$ and $P_y(\eta,\delta)$ meets u before b. □

We apply the following operations based on φ starting with the current coloring of H. We first do $ux: \gamma \to \eta$ and let φ_1 be the resulting coloring, see Fig. 2(a). Note that $\gamma \in \overline{\varphi}_1(u)$ and both ux and another edge incident with u are colored

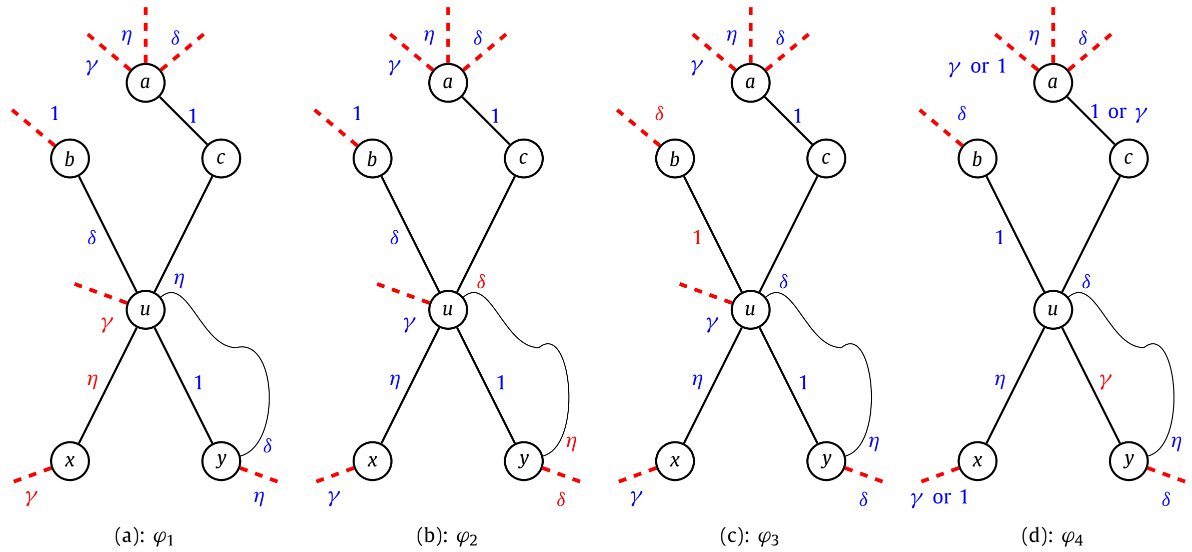

Fig. 2. Recoloring operations on H in Case 1.

by η under φ_1. As $P_y(\eta,\delta,\varphi)=P_y(\eta,\delta,\varphi_1)$, by Claim 1b, we still have $ub\in P_y(\eta,\delta,\varphi_1)$ and $P_y(\eta,\delta,\varphi_1)$ meets u before b. We then do a (δ,η)-swap on $P_{[u,y]}(\eta,\delta,\varphi_1)$ and let the resulting coloring be φ_2, see Fig. 2(b). Now, under φ_2, both ub and another edge incident with u are colored by δ. Next we do $ub:\delta\to 1$ and let the resulting coloring be φ_3, see Fig. 2(c). Under φ_3, both ub and uy are colored by 1. As $\eta,\delta,1,\gamma$ are pairwise distinct, and $ub\notin P^*_u(1,\gamma,\varphi)$, we still have $P^*_u(1,\gamma,\varphi_3)=P^*_u(1,\gamma,\varphi)$. Thus by Claim 1a, $P^*_u(1,\gamma,\varphi_3)$ ends at x, or at some vertex $z\in V(G)\setminus\{a,b,c,u,x,y\}$, or passing c ends at a. Now we do a $(1,\gamma)$-swap on $P^*_u(1,\gamma,\varphi_3)$ and let the resulting coloring be φ_4, see Fig. 2(d). As $\gamma\in\overline{\varphi}_3(u)$ and $b\notin P^*_u(1,\gamma,\varphi_3)$, now under φ_4, all the edges incident with u are colored by distinct colors. Thus φ_4 is an edge Δ-coloring. As $\delta\in\overline{\varphi}_4(a)\cap\overline{\varphi}_4(b)$, coloring the edge ab by the color δ gives an edge Δ-coloring of G, contradicting the assumption that $\chi'(G)=\Delta+1$.

Case 2. $\overline{\varphi}(x)\neq\overline{\varphi}(y)$.

Let

$$\varphi(bu)=\alpha,\quad \varphi(ux)=\beta,\quad \overline{\varphi}(x)=\tau,\quad \text{and}\quad \overline{\varphi}(y)=\gamma.$$

As $\varphi(uy)=\overline{\varphi}(b)=1$, $1\notin\{\alpha,\beta,\gamma\}$, and clearly, $\alpha\neq\beta$. Since a and b are $(1,\alpha)$-linked, $P_b(1,\alpha)$ cannot end at y, and so $\gamma\neq\alpha$. We will prove $\beta\neq\gamma$ later, when we need it. Since K is a Kierstead path, $\alpha,\beta\in\overline{\varphi}(a)$. Since $\overline{\varphi}(y)\subseteq\overline{\varphi}(a)\cup\overline{\varphi}(b)$, $\gamma\in\overline{\varphi}(a)$. Thus $\alpha,\beta,\gamma\in\overline{\varphi}(a)$ and $\tau\in\overline{\varphi}(a)\cup\{1\}$.

Claim 2. *We may assume that $\overline{\varphi}(x)=\tau=1$. (See Fig. 3(a).)*

Proof. Suppose $\tau\neq 1$; then $\tau\in\overline{\varphi}(a)$. Thus a and b are $(1,\tau)$-linked. If $uy\notin P_x(1,\tau)$, we simply do a $(1,\tau)$-swap at x. So assume $uy\in P_x(1,\tau)$, which implies $uy\notin P_b(1,\tau)$.

We do five swaps in succession, none of which change the color of bu, ux, or uy. We first do a $(1,\tau)$-swap at b; now $\overline{\varphi}(b)=\varphi(ac)=\tau$, and a and b are $(1,\tau)$-linked, (α,τ)-linked and (γ,τ)-linked through c. Next we do an (α,τ)-swap at x, so that there is no edge of color α at x. Then we do two swaps at b, a (γ,τ)-swap at b followed by a $(1,\gamma)$-swap; now $\overline{\varphi}(b)=\varphi(ac)=\varphi(uy)=1$, and a and b are lined by a $(1,\alpha)$-chain $ac\ldots yub$ which cannot pass though x. Finally, a $(1,\alpha)$-swap at x gives the desired coloring. □

Note that b is linked to a by a $(1,\beta)$-chain that does not pass through u, since $1\in\overline{\varphi}(x)$. So we now do a $(1,\beta)$-swap at b, color ab by α, recolor bu by β, and uncolor ux. See Fig. 3(b). Note that $F^*=(u,ux,x,uy,y)$ is now a multifan. The following facts now hold:

(1) $\varphi(bu)=\beta$, $\varphi(uy)=1$, $\overline{\varphi}(u)=\alpha$, $\overline{\varphi}(x)=\{1,\beta\}$, $\overline{\varphi}(y)=\gamma$.
(2) u and x are $(1,\alpha)$-linked and (α,β)-linked (clearly). Also u and y are (γ,α)-linked by Lemma 3(b) applied to F^*.

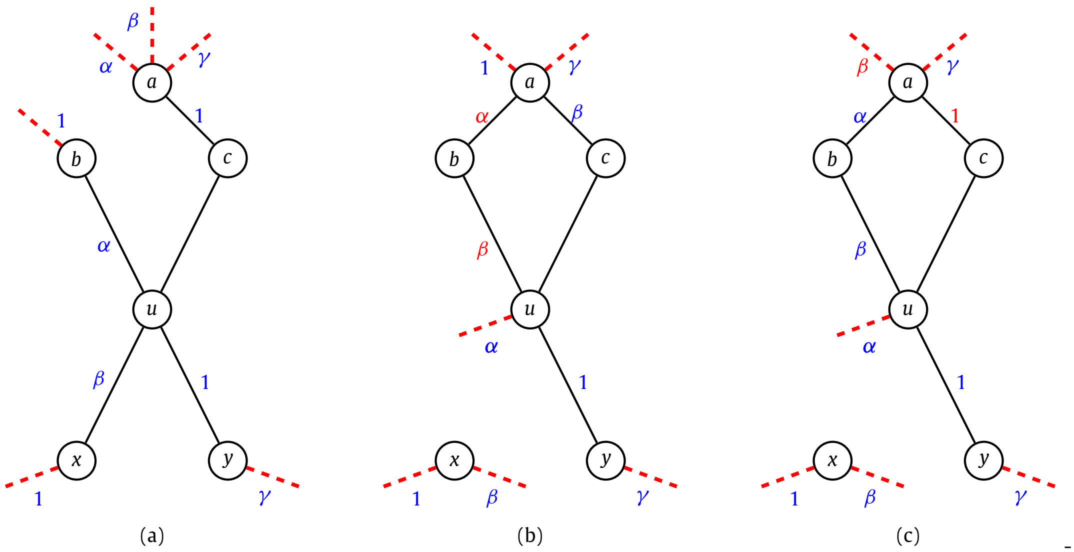

Fig. 3. Colors on the edges of H in Case 2.

Since $V(F^*)$ is φ-elementary by Lemma 3(a), we have $\beta \neq \gamma$.

We will do four swaps at a in succession, through which (1) and (2) hold: for each step, the truth of (2) before the swap implies that (1) still holds after the swap; but (1) $\Rightarrow$ (2). The four swaps at a, are, in order, an (α, γ), an (α, β)-swap, a $(1, \alpha)$-swap, and another (α, γ)-swap. See Fig. 3(c) for the resulting coloring after these four swaps. The effects of these swaps on $\varphi(ab)$, $\varphi(ac)$ and two known elements of $\overline{\varphi}(a)$ are as follows.

	$\varphi(ab)$	$\varphi(ac)$	$\in \overline{\varphi}(a)$
Before the first swap:	α	β	$1, \gamma$
After the (α, γ)-swap:	γ	β	$1, \alpha$
After the (α, β)-swap:	γ	α	$1, \beta$
After the $(1, \alpha)$-swap:	γ	1	α, β
After the (α, γ)-swap:	α	1	β, γ

Now $P_u(\alpha, \beta) = uba$, which contradicts the assertion in (2) that u and x are (α, β)-linked. This contradiction completes the proof of Lemma 5. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The authors are very grateful to the two anonymous referees for their careful reading and valuable comments, which greatly improved this paper.

References

[1] S. Bonvicini, A. Vietri, A Möbius-type gluing technique for obtaining edge-critical graphs, Ars Math. Contemp. 19 (2) (2020) 209–229.
[2] S. Grünewald, E. Steffen, Independent sets and 2-factors in edge-chromatic-critical graphs, J. Graph Theory 45 (2) (2004) 113–118.
[3] A.J.W. Hilton, C. Zhao, Vertex-splitting and chromatic index critical graphs, in: Second International Colloquium on Graphs and Optimization, Leukerbad, 1994, vol. 76, 1997, pp. 205–211.
[4] I. Holyer, The NP-completeness of edge-coloring, SIAM J. Comput. 10 (4) (1981) 718–720.
[5] Z. Song, A further extension of Yap's construction for Δ-critical graphs, Discrete Math. 243 (1–3) (2002) 283–290.
[6] M. Stiebitz, D. Scheide, B. Toft, L.M. Favrholdt, Vizing's theorem and Goldberg's conjecture, in: Graph Edge Coloring, in: Wiley Series in Discrete Mathematics and Optimization., John Wiley & Sons, Inc., Hoboken, NJ, 2012, With a preface by Stiebitz and Toft.

[7] A. Vietri, An analogy between edge colourings and differentiable manifolds, with a new perspective on 3-critical graphs, Graphs Comb. 31 (6) (2015) 2425–2435.
[8] V.G. Vizing, On an estimate of the chromatic class of a p-graph, Diskretn. Anal. 3 (1964) 25–30.
[9] V.G. Vizing, The chromatic class of a multigraph, Kibernetika 1965 (3) (1965) 29–39.
[10] V.G. Vizing, Critical graphs with given chromatic class, Diskretn. Anal. 5 (1965) 9–17.