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Overfullness of critical class 2 graphs with a small core degree

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A R T I C L E I N F O

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Let G be a simple graph, and let n, $\Delta(G)$ and $\chi'(G)$ be the order, the maximum degree and the chromatic index of G, respectively. We call G *overfull* if $|E(G)|/\lfloor n/2\rfloor > \Delta(G)$, and *critical* if $\chi'(H) < \chi'(G)$ for every proper subgraph H of G. Clearly, if G is overfull then $\chi'(G) = \Delta(G) + 1$ by Vizing's Theorem. The *core* of G, denoted by G_Δ, is the subgraph of G induced by all its maximum degree vertices. Hilton and Zhao conjectured that for any critical class 2 graph G with $\Delta(G) \geq 4$, if the maximum degree of G_Δ is at most two, then G is overfull, which in turn gives $\Delta(G) > n/2 + 1$. We show that for any critical class 2 graph G, if the minimum degree of G_Δ is at most two and $\Delta(G) > n/2+1$, then G is overfull.
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1. Introduction

We will mainly adopt the notation from the book [13]. Graphs in this paper are simple, i.e., finite, undirected, without loops or multiple edges. Let G be a graph. A *k-edge-*

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coloring of G is a map φ: $E(G) \to \{1, 2, \dots, k\}$ that assigns to every edge e of G a color $\varphi(e) \in \{1, 2, \dots, k\}$ such that no two adjacent edges receive the same color. Denote by $\mathcal{C}^k(G)$ the set of all k-edge-colorings of G. The *chromatic index* $\chi'(G)$ is the least integer $k \geq 0$ such that $\mathcal{C}^k(G) \neq \emptyset$. Denote by $\Delta(G)$ the maximum degree of G. In 1960's, Vizing [16] and, independently, Gupta [6] proved that $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. This leads to a natural classification of graphs. Following Fiorini and Wilson [4], we say a graph G is of *class 1* if $\chi'(G) = \Delta(G)$ and of *class 2* if $\chi'(G) = \Delta(G) + 1$. Holyer [9] showed that it is NP-complete to determine whether an arbitrary graph is of class 1.

A graph G is *critical* if $\chi'(H) < \chi'(G)$ for every proper subgraph H of G. In investigating the classification problem, critical graphs are of particular interest. Critical graphs of class 2 have rather more structures than arbitrary graphs of class 2, and it follows from Vizing's Theorem that every graph of class 2 contains a critical graph of class 2 with the same maximum degree as a subgraph. In this paper, we call a critical class 2 graph Δ-*critical* if $\Delta(G) = \Delta$.

Since every matching of G has at most $\lfloor |V(G)|/2 \rfloor$ edges, $\chi'(G) \geq |E(G)|/\lfloor |V(G)|/2 \rfloor$. A graph G is *overfull* if $|E(G)|/\lfloor |V(G)|/2 \rfloor > \Delta(G)$. Clearly, if G is overfull then $\chi'(G) = \Delta(G) + 1$ by Vizing's Theorem, and so G is of class 2. Applying Edmonds' matching polytope theorem, Seymour [12] showed that whether a graph G contains an overfull subgraph of maximum degree $\Delta(G)$ can be determined in polynomial time. A number of outstanding conjectures listed in *Twenty Pretty Edge Coloring Conjectures* in [13] lie in deciding when a Δ-critical graph is overfull.

The *core* of a graph G, denoted by G_Δ, is the subgraph induced by all its maximum degree vertices. Vizing [16] proved that if G_Δ has at most two vertices then G is class 1. Fournier [5] generalized Vizing's result by showing that if G_Δ is acyclic then G is class 1. Thus a necessary condition for a graph to be class 2 is to have a core that contains cycles. A long-standing conjecture of Hilton and Zhao [7] claims that for a connected graph G with $\Delta \geq 4$, if the maximum degree of G_Δ is at most two, then G is overfull. We [1], along with Guangming Jing, recently confirmed this conjecture, which in turn implies $\Delta(G) > n/2 + 1$, where $n = |V(G)|$ is the order of G. In this paper, by imposing a condition on the maximum degree of G, we relax the condition $\Delta(G_\Delta) \leq 2$, and show a result analogous to the Hilton-Zhao Conjecture as follows.

Theorem 1.1. *Let G be a Δ-critical graph of order n. If $\delta(G_\Delta) \leq 2$ and $\Delta(G) > n/2 + 1$, then G is overfull.*

By the proof of the Hilton-Zhao Conjecture [1], for $\Delta \geq 4$, the connected class 2 graphs with maximum degree Δ and $\Delta(G_\Delta) \leq 2$ are Δ-critical with $\Delta(G) > n/2 + 1$. Thus, implicitly, Theorem 1.1 is much stronger than the Hilton-Zhao Conjecture, but we don't have a direct proof for that. A graph G is said to be *just overfull* if $|E(G)| = \Delta(G)\lfloor \frac{1}{2}|V(G)| \rfloor + 1$. We hope that the new edge coloring techniques we introduced in our proof may shed some light on attacking the Just Overfull Conjecture – Conjecture 4.23 (page 72) in [13].

Conjecture 1.2. *Let G be a Δ-critical graph of order n. If $\Delta(G) \geq n/2$, then G is just overfull.*

Chetwynd and Hilton in 1986 [2,3] made a much stronger conjecture, commonly referred to as *the Overfull Conjecture* that for a Δ-critical graph of order n, if $\Delta(G) > n/3$ then G is overfull. Except some very special results [3,8,11], the Overfull Conjecture seems untouchable with current edge coloring techniques.

Let G be a graph and $H \subseteq G$ be a subgraph. For $v \in V(G)$, $N(v)$ is the set of neighbors of v in G and $d(v) = |N(v)|$ is the degree of v in G. Let $N_H(v) = N(v) \cap V(H)$ and $d_H(v) = |N_H(v)|$. More generally, for a subset $S \subseteq V(G)$, let $N_H(S) = \cup_{v \in S} N_H(v)$ be the neighborhood of S in G that is contained in $V(H)$. For two vertices u and v, we write $u \sim v$ if they are adjacent, and write $u \nsim v$ if otherwise. For a nonnegative integer k, a *k-vertex* is a vertex of degree k. We denote by V_k and $N_k(v)$ the set of all k-vertices, repetitively, in $V(G)$ and $N(v)$. Let $N[v] = N(v) \cup \{v\}$ and $N_k[v] = N_k(v) \cup \{v\}$. For convenience, for any nonnegative integers p and q, let $[p, q] = \{i \in \mathbb{Z} : p \leq i \leq q\}$.

A vertex v of a graph G is called *light* if it is adjacent to at most two $\Delta(G)$-vertices, i.e., $d_{G_\Delta}(v) \leq 2$. An edge e of G is *critical* if $\chi'(G-e) < \chi'(G)$. Clearly, if G is Δ-critical then every edge of G is critical. In a Δ-critical graph, for a light vertex, what can we say about its neighbors? The following lemma reveals some of their properties.

Lemma 1.3 *(Vizing's Adjacency Lemma (VAL)). Let G be a class 2 graph with maximum degree Δ. If $e = xy$ is a critical edge of G, then x is adjacent to at least $\Delta - d(y) + 1$ Δ-vertices from $V(G) \setminus \{y\}$.*

Let G be a Δ-critical graph and r be a light vertex of G. We claim $d(s) \geq \Delta - 1$ for every $s \in N(r)$. Otherwise, by VAL, r is adjacent to at least $\Delta - d(s) + 1 \geq 3$ vertices of degree Δ, giving a contradiction. Consequently, we have $d(s) = \Delta - 1$ or $d(s) = \Delta$. As $\Delta \geq 3$ and r is light, we have $N(r) = N_\Delta(r) \cup N_{\Delta-1}(r)$. We also see that r must be adjacent to exactly two Δ-vertices if r is light. These facts will be frequently used throughout this paper.

Theorem 1.1 is a consequence of the following three technical results.

Theorem 1.4. *Let G be a class 2 graph with maximum degree Δ, $r \in V_\Delta$ be light, and $s \in N_{\Delta-1}(r)$. If rs is a critical edge of G, then all vertices in $N(s) \setminus N(r)$ are Δ-vertices.*

Theorem 1.5. *Let G be a class 2 graph with maximum degree Δ, $r \in V(G)$ be a light Δ-vertex and $s \in N_{\Delta-1}(r)$ such that rs is a critical edge. For every $x \in V(G) \setminus N[r]$, if $d(x) \leq \Delta - 3$, then $N(x) \cap N(s) \subseteq N(r) \setminus N_\Delta(r)$.*

Theorem 1.6. *Let G be a Δ-critical graph of order n. If $\Delta > n/2 + 1$ and $\delta(G_\Delta) \leq 2$, then n is odd.*

Proof of Theorem 1.1. Let G be a Δ-critical graph of order n such that $\delta(G_\Delta) \le 2$ and $\Delta > n/2 + 1$. By Theorem 1.6, n is odd. Let r be a light Δ-vertex of G. Since $|N_{\Delta-1}(r)| = \Delta - 2$, we have $2|E(G)| \le n\Delta - (\Delta - 2)$. Thus to show $2|E(G)| \ge (n-1)\Delta + 2$ (i.e., G is overfull), we only need to show that all vertices in $V(G) \setminus N_{\Delta-1}(r)$ are Δ-vertices.

Assume to the contrary that there exists $x \in V(G) \setminus N_{\Delta-1}(r)$ such that $d(x) \le \Delta - 1$. Since every vertex in $N[r] \setminus N_{\Delta-1}(r)$ is a Δ-vertex, we have $x \notin N[r]$. Since n is odd, $\Delta > n/2 + 1$ implies $\Delta \ge (n+1)/2 + 1$. We first suppose that $d(x) \ge \Delta - 2$, i.e., $|N(x)| \ge (n-1)/2$. Since $|N_{\Delta-1}(r)| = \Delta - 2 \ge (n-1)/2$ and $r \notin N(x)$, we conclude that $N(x) \cap N_{\Delta-1}(r) \ne \emptyset$. Let $s \in N(x) \cap N_{\Delta-1}(r)$. Since G is Δ-critical, rs is a critical edge of G. Applying Theorem 1.4, we get $d(x) = \Delta$, a contradiction. Thus $d(x) \le \Delta - 3$. Since G is Δ-critical, x has a neighbor u with degree Δ. As $\Delta \ge (n+1)/2 + 1$ and $|N_{\Delta-1}(r)| = \Delta - 2$, we find a vertex $s \in N(u) \cap N_{\Delta-1}(r)$. Thus $u \in N(x) \cap N(s)$. Since $d(u) = \Delta$ and $d(x) \le \Delta - 3$, $u \notin N(r) \setminus N_\Delta(r)$. Again, rs is a critical edge of G as G is Δ-critical. Applying the contrapositive statement of Theorem 1.5, we get $d(x) \ge \Delta - 2$, which gives a contradiction. □

Theorems 1.4 to 1.6 study some structural properties of vertices outside the neighborhood of a light vertex. The study of structural properties of vertices beyond a given neighborhood plays a key role in our proof, and we believe that the technique may be useful on tackling other edge coloring problems involving overfull properties.

2. Preliminaries

This section is divided into three subsections. We first give some basic notation and terminologies, then define a slightly modified and specific Vizing fan centering at a light vertex, and finally we investigate some properties of a Δ-edge-coloring around a light vertex.

2.1. Basic notation and terminologies

Let G be a graph with maximum degree Δ, and let $e \in E(G)$ and $\varphi \in \mathcal{C}^\Delta(G-e)$. When we apply some definitions later, we may drop the phrase "w.r.t. φ" or surpass the coloring symbol φ whenever the coloring φ is clearly understood.

For a vertex $v \in V(G)$, define the two color sets

$$\varphi(v) = \{\varphi(f) : f \ne e \text{ is incident to } v\} \quad \text{and} \quad \overline{\varphi}(v) = [1, \Delta] \setminus \varphi(v).$$

We call $\varphi(v)$ the set of colors *present* at v and $\overline{\varphi}(v)$ the set of colors *missing* at v. If $|\overline{\varphi}(v)| = 1$, we will also use $\overline{\varphi}(v)$ to denote the color missing at v.

For a vertex set $X \subseteq V(G)$, define $\overline{\varphi}(X) = \bigcup_{v \in X} \overline{\varphi}(v)$ to be the set of missing colors of X. The set X is *elementary* w.r.t. φ or simply φ-*elementary* if $\overline{\varphi}(u) \cap \overline{\varphi}(v) = \emptyset$ for any two distinct vertices $u, v \in X$.

For a color α, the edge set $E_\alpha = \{f \in E(G) \mid \varphi(f) = \alpha\}$ is called a *color class*. Clearly, E_α is a *matching* of G (possibly empty). For two distinct colors α, β, the subgraph of G induced by $E_\alpha \cup E_\beta$ is a union of disjoint paths and even cycles, which are referred to as (α, β)-*chains* of G w.r.t. φ. These chains are also called *Kempe-chain*. For $x, y \in V(G)$, if x and y are contained in the same (α, β)-chain with respect to φ, we say x and y are (α, β)-*linked*. Otherwise, they are (α, β)-*unlinked*.

For a vertex v, let $C_v(\alpha, \beta, \varphi)$ denote the unique (α, β)-chain containing v. If $C_v(\alpha, \beta, \varphi)$ is a path, we just write it as $P_v(\alpha, \beta, \varphi)$ and simply as $P_v(\alpha, \beta)$ is φ is understood. The notation $P_v(\alpha, \beta, \varphi)$ is commonly used when we know $|\overline{\varphi}(v) \cap \{\alpha, \beta\}| = 1$. If we interchange the colors α and β on an (α, β)-chain C of G, we briefly say that the new coloring is obtained from φ by an (α, β)-*swap* on C, and we write it as φ/C. This operation is called a *Kempe change*. If $C = uv$ is just an edge, the notation $uv : \alpha \to \beta$ means to recolor the edge uv that is colored by α using the color β.

Suppose that α, β, γ are three colors such that $\alpha \in \overline{\varphi}(x)$ and $\beta, \gamma \in \varphi(x)$. An $(\alpha, \beta) - (\beta, \gamma)$ *swap at* x consists of two operations: first swaps colors on $P_x(\alpha, \beta, \varphi)$ to get a new coloring φ', and then swaps colors on $P_x(\beta, \gamma, \varphi')$. When $\beta = \alpha$, an (α, α)-swap is just a vacuous recoloring operation.

For a given path P, a vertex u and an edge uv, we write $u \in P$ and $uv \in P$ for $u \in V(P)$ and $uv \in E(P)$, respectively. Suppose $x \in P$. For two vertices $u, v \in P_x(\alpha, \beta, \varphi)$, if u lies between x and v, then we say that $P_x(\alpha, \beta, \varphi)$ *meets* u *before* v.

2.2. Modified Vizing fans and Kierstead paths

The fan argument was introduced by Vizing [14,15] in his classic results on the upper bounds of chromatic indices. We will use multifans, a generalized version of Vizing fans, given by Stiebitz et al. [13], in our proof. To simplify the arguments, we will not include maximum degree vertices in our fans except the center vertex.

Definition 2.1. Let G be a graph with maximum degree Δ. For an edge $e = rs_1 \in E(G)$ and a coloring $\varphi \in \mathcal{C}^\Delta(G - e)$, a *multifan* centered at r w.r.t. e and φ is a sequence $F_\varphi(r, s_1 : s_p) = (r, rs_1, s_1, rs_2, s_2, \ldots, rs_p, s_p)$ with $p \geq 1$ consisting of distinct vertices $r, s_1, s_2, \ldots, s_p$ and edges $rs_1, rs_2, \ldots, rs_p$ satisfying the following condition:

(F1) For every edge rs_i with $i \in [2, p]$, there exists $j \in [1, i-1]$ such that $\varphi(rs_i) \in \overline{\varphi}(s_j)$,

and none of $s_1, \ldots, s_p$ is a Δ-vertex.

We will simply denote a multifan $F_\varphi(r, s_1 : s_p)$ by F if we do not need to emphasize the center r, and the non-center starting and ending vertices. We also notice that if $F_\varphi(r, s_1 : s_p)$ is a multifan, then for any integer $p^* \in [1, p]$, $F_\varphi(r, s_1 : s_{p^*})$ is also a multifan. The following result regarding a multifan can be found in [13, Theorem 2.1].

Lemma 2.2. *Let G be a class 2 graph, $e = rs_1$ be a critical edge and $\varphi \in \mathcal{C}^\Delta(G-e)$. If $F_\varphi(r, s_1 : s_p)$ is a multifan w.r.t. e and φ, then $V(F)$ is φ-elementary.*

Suppose that $e = rs_1$ is a critical edge of a class 2 graph G and $F_\varphi(r, s_1 : s_p)$ is a multifan w.r.t. e and a coloring $\varphi \in \mathcal{C}^\Delta(G-e)$. Given a color $\alpha \in \overline{\varphi}(s_1)$, we call a vertex s_ℓ with $\ell \in [2, p]$ an *α-inducing vertex* if there exists a subsequence $(s_{\ell_1}, s_{\ell_2}, \dots, s_{\ell_k})$ terminated at $s_{\ell_k} = s_\ell$ such that $\varphi(rs_{\ell_1}) = \alpha \in \overline{\varphi}(s_1)$ and for each $i \in [2, k]$, $\varphi(rs_{\ell_i}) \in \overline{\varphi}(s_{\ell_{i-1}})$. We also call the above sequence an *α-inducing sequence*, and a color $\beta \in \overline{\varphi}(s_\ell)$ an *α-inducing color* or a color *induced by α*. For convention, α itself is also called an α-inducing color. Since $V(F)$ is elementary, every color in $\overline{\varphi}(V(F)\setminus\{r\})$ is induced by a color in $\overline{\varphi}(s_1)$.

As a consequence of Lemma 2.2, we have the following linkage properties of vertices in a multifan.

Lemma 2.3. *Let G be a class 2 graph, $e = rs_1$ be a critical edge and $\varphi \in \mathcal{C}^\Delta(G-e)$. Then, for every multifan $F_\varphi(r, s_1 : s_p)$, the following three statements hold.*

(a) For any color $\gamma \in \overline{\varphi}(r)$ and any color $\delta \in \overline{\varphi}(s_i)$ with $i \in [1, p]$, vertices r and s_i are (γ, δ)-linked w.r.t. φ.
(b) For $i, j \in [1, p]$, if two colors $\delta \in \overline{\varphi}(s_i)$ and $\lambda \in \overline{\varphi}(s_j)$ are induced by two different colors in $\overline{\varphi}(s_1)$, then the corresponding vertices s_i and s_j are (δ, λ)-linked.
(c) For $i, j \in [1, p]$, suppose two colors $\delta \in \overline{\varphi}(s_i)$ and $\lambda \in \overline{\varphi}(s_j)$ are induced by the same color in $\overline{\varphi}(s_1)$. If s_i and s_j are not (δ, λ)-linked and $j > i$, then $r \in P_{s_j}(\delta, \lambda, \varphi)$.

The proof of Lemma 2.3(a) can be found in [13, Theorem 2.1], and the proof of Lemma 2.3(b) and (c) can be found in [1, Lemma 3.2]. All the three proofs go by contradiction and argue in the similar way. Suppose the desired linkage does not exist. Then we will be able to find a Kempe-chain starting at a vertex of F, containing no edges of F, and ending at a vertex outside of $V(F)$. Swapping the two colors on the Kempe-chain gives a new edge coloring φ_1. A subsequence of F is still a multifan with respect to φ_1 but its vertex set is not φ_1-elementary, which contradicts Lemma 2.2.

Let G be a class 2 graph, $r \in V(G)$ be a light vertex, $rs_1 \in E(G)$ be a critical edge and $\varphi \in \mathcal{C}^\Delta(G - rs_1)$. Let $F_\varphi(r, s_1 : s_p)$ be a multifan w.r.t. rs_1 and φ. By VAL, except two Δ-vertices, all other neighbors of r are $(\Delta-1)$-vertices. In particular, $d(s_i) = \Delta - 1$ for all $i \in [1, p]$. Hence, $|\overline{\varphi}(s_1)| = 2$ and $|\overline{\varphi}(s_i)| = 1$ for each $i \in [2, p]$. Assume without loss of generality $\overline{\varphi}(s_1) = \{2, \Delta\}$. Then, all 2-inducing vertices form a 2-inducing sequence and all Δ-inducing vertices form a Δ-inducing sequence. By relabeling if necessary, we assume $(s_2, \dots, s_\alpha)$ is a 2-inducing sequence and $(s_{\alpha+1}, \dots, s_p)$ is a Δ-inducing sequence for some $\alpha \in [1, p]$, where we define $(s_2, \dots, s_\alpha)$ to be the empty sequence if $\alpha < 2$. We call a multifan *typical* at a light vertex r, denoted by $F_\varphi(r, s_1 : s_\alpha : s_\beta)$, if $1 \in \overline{\varphi}(r)$, $\overline{\varphi}(s_1) = \{2, \Delta\}$ and either $|V(F)| = 2$ or $|V(F)| \geq 3$ with the following two conditions.

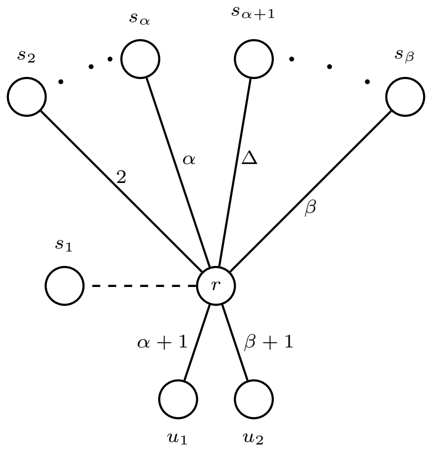

Fig. 1. A typical multifan $F_\varphi(r, s_1 : s_\alpha : s_\beta)$ at a light vertex r, where $\overline{\varphi}(r) = 1$ and $\overline{\varphi}(s_1) = \{2, \Delta\}$.

(1) $(s_2, \ldots, s_\alpha)$ is a 2-inducing sequence and $(s_{\alpha+1}, \ldots, s_\beta)$ is a Δ-inducing sequence of F.
(2) For each $i \in [2, \beta]$, $\varphi(rs_i) = i$ and $\overline{\varphi}(s_i) = i + 1$ except when $i = \alpha + 1 \in [3, \beta]$. In this case, $\varphi(rs_{\alpha+1}) = \Delta$ and $\overline{\varphi}(s_{\alpha+1}) = \alpha + 2$.

A *typical multifan* at a light vertex r is depicted in Fig. 1.

By relabeling vertices and colors if necessary, every multifan centered at a light vertex r is corresponding to a typical multifan at r on the same vertex set. Thus in this paper, we assume all multifans at r are typical.

We close this subsection with Kierstead paths, which were introduced by Kierstead [10] in his work on edge colorings of multigraphs.

Definition 2.4. Let G be a graph, $e = v_0v_1 \in E(G)$, and $\varphi \in \mathcal{C}^\Delta(G - e)$. A *Kierstead path* w.r.t. e and φ is a sequence $K = (v_0, v_0v_1, v_1, v_1v_2, v_2, \ldots, v_{p-1}, v_{p-1}v_p, v_p)$ with $p \geq 1$ consisting of distinct vertices $v_0, v_1, \ldots, v_p$ and edges $v_0v_1, v_1v_2, \ldots, v_{p-1}v_p$ satisfying the following condition:

(K1) For every edge $v_{i-1}v_i$ with $i \in [2, p]$, there exists $j \in [0, i-2]$ such that $\varphi(v_{i-1}v_i) \in \overline{\varphi}(v_j)$.

Clearly a Kierstead path with at most three vertices is a multifan. So we consider Kierstead paths with four vertices and restrict on its simple graph version. The following lemma was proved in Theorem 3.3 from [13].

Lemma 2.5. *Let G be a class 2 graph, $e = v_0v_1 \in E(G)$ be a critical edge, and $K = (v_0, v_0v_1, v_1, v_1v_2, v_2, v_2v_3, v_3)$ be a Kierstead path w.r.t. e and a coloring $\varphi \in \mathcal{C}^\Delta(G - e)$. If $\min\{d_G(v_1), d_G(v_2)\} < \Delta$, then $V(K)$ is φ-elementary.*

Let G be a class 2 graph of maximum degree Δ, e be a critical edge and $\varphi \in \mathcal{C}^\Delta(G-e)$. Let T be a sequence of vertices and edges of G. We denote by $V(T)$ and $E(T)$ the set of vertices and the set of edges that are contained in T, respectively. For simplicity, we write $\overline{\varphi}(T)$ for $\overline{\varphi}(V(T))$. If $V(T)$ is φ-elementary, then for a color $\tau \in \overline{\varphi}(T)$, we denote by $\overline{\varphi}_T^{-1}(\tau)$ the unique vertex in $V(T)$ at which τ is missing. A coloring $\varphi' \in \mathcal{C}^\Delta(G-e)$ is called *T-stable* w.r.t. φ if $\overline{\varphi}'(x) = \overline{\varphi}(x)$ for every vertex $x \in V(T)$ and $\varphi'(f) = \varphi(f)$ for every edge $f \in E(T)$. Clearly, φ is T-stable w.r.t. itself.

Let $F = F_\varphi(r, s_1 : s_\alpha : s_\beta)$ be a typical multifan w.r.t. $e = rs_1$ and $\varphi \in \mathcal{C}^\Delta(G - rs_1)$. By the definition above, if φ' is F-stable, then F is also a typical multifan w.r.t. e and φ'. Let $\gamma, \delta \in [1, \Delta]$ be two colors and P be a (γ, δ)-path. If $E(P) \cap E(F) = \emptyset$ and neither endvertices of P is in $V(F)$, then Kempe change φ/P gives an F-stable coloring. Applying Lemma 2.3, we have the following results on stable coloring, which will be used heavily in our proofs.

Lemma 2.6. *Let G be a class 2 graph and $F = F_\varphi(r, s_1 : s_\alpha : s_\beta)$ be a typical multifan w.r.t. a light vertex r, critical edge rs_1, and a coloring $\varphi \in \mathcal{C}^\Delta(G - rs_1)$. For any color $\gamma \in \overline{\varphi}(F)$ and $x \notin V(F)$, the following statements hold.*

- *the Kempe change $\varphi/P_x(1, \gamma, \varphi)$ gives an F-stable coloring provided $\overline{\varphi}(x) \cap \{1, \gamma\} \neq \emptyset$.*
- *if γ is 2-inducing, then the Kempe change $\varphi/P_x(\gamma, \Delta, \varphi)$ gives an F-stable coloring provided $r \notin P_x(\gamma, \Delta, \varphi)$ and $\overline{\varphi}(x) \cap \{\gamma, \Delta\} \neq \emptyset$; and*
- *if γ is Δ-inducing, then the Kempe change $\varphi/P_x(2, \gamma, \varphi)$ gives an F-stable coloring provided $r \notin P_x(\gamma, 2, \varphi)$ and $\overline{\varphi}(x) \cap \{\gamma, 2\} \neq \emptyset$.*

2.3. τ-sequence, rotation, and shifting

Throughout this subsection, we assume that G is a class 2 graph, $r \in V(G)$ is a light vertex, $e = rs_1 \in E(G)$ is a critical edge of G and $\varphi \in \mathcal{C}^\Delta(G - e)$. We also assume that $N_\Delta(r) = \{u_1, u_2\}$ and $N_{\Delta-1}(r) = \{s_1, \ldots, s_q\}$, where $q = d(r) - 2$. Furthermore, we assume that $F = F_\varphi(r, s_1 : s_\alpha : s_\beta)$ is a typical multifan at r. Since $|\overline{\varphi}(s_i)| = 1$ for each $i \in [2, q]$, for notation convenience, we also use $\overline{\varphi}(s_i)$ to denote the color that is missing at s_i.

We call F a *maximum multifan at r* if $|V(F)|$ is maximum over all colorings in $\mathcal{C}^\Delta(G-e)$ and all multifans centered at r. Clearly, if F is maximum, then colors $\alpha+1$ and $\beta+1$ are assigned to edges ru_1 and ru_2, respectively, i.e., $\alpha+1, \beta+1 \notin \{\varphi(rs_{\beta+1}), \ldots, \varphi(rs_q)\}$ (see Fig. 1).

Definition 2.7. For a color $\tau \notin \overline{\varphi}(F)$, a *$\tau$-sequence* is a sequence of distinct vertices $(v_1, v_2, \ldots, v_t)$ with $v_i \in \{s_{\beta+1}, \ldots, s_q\}$ such that $\varphi(rv_1) = \tau$, and the following three conditions are satisfied.

(i) $\{v_1, \ldots, v_{t-1}\}$ is elementary and $\overline{\varphi}(v_i) \notin \overline{\varphi}(F)$ for each $i \in [1, t-1]$;

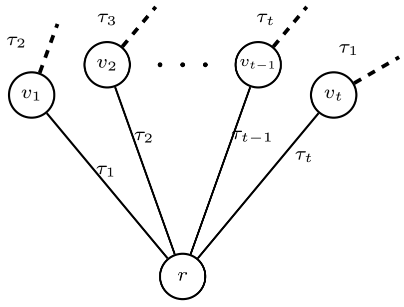

Fig. 2. A rotation in the neighborhood of r.

(ii) $\varphi(rv_i) = \overline{\varphi}(v_{i-1})$ for each $i \in [2, t]$; and

(iii) There are three possibilities for $\overline{\varphi}(v_t)$: (A) $\overline{\varphi}(v_t) = \tau$, (B) $\overline{\varphi}(v_t) \in \overline{\varphi}(F)$, or (C) $\overline{\varphi}(v_t) = \overline{\varphi}(v_i)$ for some $i \in [1, t-1]$. Accordingly, we name the τ-sequence type A, type B, and type C, respectively, where a type A sequence is also called a *rotation*.

An example of a rotation is given in Fig. 2, where $\tau_i = \overline{\varphi}(v_{i-1})$ for each $i \in [2, t]$.

Lemma 2.8. *If F is maximum, then for any color $\tau \notin \overline{\varphi}(F)$, there is a unique τ-sequence.*

Proof. Since $\tau \notin \overline{\varphi}(r)$, there is a vertex $s \in N(r)$ such that $\varphi(rs) = \tau$. Since F is maximum, we have $\alpha+1, \beta+1 \in \{\varphi(ru_1), \varphi(ru_2)\}$, and so $s \notin \{u_1, u_2\} =: N_\Delta(r)$. Since $\varphi(rs_i) \in \overline{\varphi}(F)$ for all $i \in [2, \beta]$, $s = v_1$ for some $v_1 \in \{s_{\beta+1}, \ldots, s_q\}$, where we recall $q = d(r) - 2$.

Starting with a singleton sequence (v_1), let $(v_1, \ldots, v_{t-1})$ be a longest sequence of vertices in $N(r) \setminus V(F)$ satisfying the following two conditions:

(i) $\{v_1, \ldots, v_{t-1}\}$ is elementary and $\overline{\varphi}(v_i) \notin \overline{\varphi}(F)$ for each $i \in [1, t-1]$; and

(ii) $\varphi(rv_i) = \overline{\varphi}(v_{i-1})$ for each $i \in [2, t-1]$.

Let v_t be a vertex in $N(r)$ such that $\varphi(rv_t) = \overline{\varphi}(v_{t-1})$. Since $\overline{\varphi}(v_{t-1}) \notin \overline{\varphi}(F)$, $v_t \in \{s_{\beta+1}, \ldots, s_q\}$. If $v_t = v_1$, then $(v_1, \ldots, v_{t-1})$ is a τ-sequence of type A. Thus we assume that $v_t \neq v_1$, i.e., $\overline{\varphi}(v_{t-1}) \neq \tau$. Since $\overline{\varphi}(v_{t-1}) \neq \overline{\varphi}(v_i)$ for all $i \in [1, t-2]$, $v_t \notin \{v_2, \ldots, v_{t-1}\}$. Hence $v_t \neq v_i$ for each $i \in [1, t-1]$. By the maximality of $(v_1, \ldots, v_{t-1})$, $\overline{\varphi}(v_t)$ can only have three possibilities, (A), (B) or (C), as listed in condition (iii) of Definition 2.7.

Moreover, since each $|\overline{\varphi}(s_i)| = 1$ for all $i \in [\beta+1, q]$, the sequence above is unique. □

Lemma 2.9. *If F is maximum, then for any color $\tau \notin \overline{\varphi}(F)$, $r \in P_{s_1}(\tau, \Delta)$ and $r \in P_{s_1}(2, \tau)$.*

Proof. We only show $r \in P_{s_1}(\tau, \Delta)$ since the proof for the other case is symmetric. Suppose to the contrary that $r \notin P_{s_1}(\tau, \Delta)$. Let $\varphi(rv_1) = \tau$ for $v_1 \in \{s_{\beta+1}, \ldots, s_q\}$ and $(v_1, \ldots, v_t)$ be the τ-sequence by Lemma 2.8. Let $\varphi' = \varphi/C_r(\tau, \Delta)$. Notice that under the coloring φ', $\varphi'(rv_1) = \Delta$ and $\varphi'(rs_{\alpha+1}) = \tau$, and the color on each edge from $E(F) \setminus \{rs_{\alpha+1}\}$ and the missing color on each vertex of F are the same as the corresponding colors under φ. Hence, $F' = (r, rs_1, s_1, \ldots, rs_\alpha, s_\alpha, rv_1, v_1, \ldots, rv_t, v_t)$ is a multifan w.r.t. e and φ'.

We consider three cases according to the type of this τ-sequence: type A, type B, or type C with respect to the coloring φ. We note that if $\overline{\varphi}'(v_t) = \tau$, then, due to $\varphi'(rs_{\alpha+1}) = \tau$, F' can be extended to a larger multifan:

$$F^* = (r, rs_1, s_1, \ldots, rs_\alpha, s_\alpha, rv_1, v_1, \ldots, rv_t, v_t, rs_{\alpha+1}, s_{\alpha+1}, \ldots, rs_\beta, s_\beta)$$

which is also larger than F, giving a contradiction to the maximality of F. We will use F' and F^* to lead a contradiction in our proof.

Type A: In this case $\overline{\varphi}(v_t) = \tau$. If $C_r(\tau, \Delta, \varphi) \neq P_{v_t}(\tau, \Delta, \varphi)$, then $\overline{\varphi}'(v_t) = \tau$, and so F^* is a multifan w.r.t. e and φ', giving a contradiction. Thus $C_r(\tau, \Delta, \varphi) = P_{v_t}(\tau, \Delta, \varphi)$, which in turn gives $\overline{\varphi}'(v_t) = \Delta$. In this case, $\Delta \in \overline{\varphi}'(s_1) \cap \overline{\varphi}'(v_t)$, and so F' is not elementary, giving a contradiction.

Type B: In this case $\overline{\varphi}(v_t)$, denoted by γ, is in $\overline{\varphi}(F)$. If $\gamma \in \overline{\varphi}(\{r, s_1, \ldots, s_\alpha\})$ and $\gamma \neq \Delta$, then F' is not elementary, giving a contradiction. Thus, we have either $\gamma = \Delta$ or $\gamma \in \overline{\varphi}(\{s_{\alpha+1}, \ldots, s_\beta\})$.

Assume first that $\gamma \neq \Delta$. Let $\gamma = \overline{\varphi}(s_j)$ for some $j \in [\alpha + 1, \beta]$. Since $P_r(1, \gamma, \varphi) = P_{s_j}(1, \gamma, \varphi)$ and $1, \gamma \notin \{\tau, \Delta\}$, we still have $P_r(1, \gamma, \varphi') = P_r(1, \gamma, \varphi) = P_{s_j}(1, \gamma, \varphi')$. Let $\varphi'' = \varphi'/P_{v_t}(1, \gamma, \varphi')$. Under φ'', F' is also a multifan. However, color $1 \in \overline{\varphi}''(r) \cap \overline{\varphi}''(v_t)$, giving a contradiction to $V(F')$ being elementary. Thus $\gamma = \Delta$. If $C_r(\tau, \Delta, \varphi) = P_{v_t}(\tau, \Delta, \varphi)$, then $\overline{\varphi}'(v_t) = \tau$, which in turn shows that F^* is a multifan w.r.t. e and φ', a contradiction. Thus $C_r(\tau, \Delta, \varphi) \neq P_{v_t}(\tau, \Delta, \varphi)$. So $\varphi'(v_t) = \Delta$, which in turn shows that F' is not elementary since Δ is also in $\overline{\varphi}'(s_1)$, a contradiction.

Type C: Suppose $\overline{\varphi}(v_t) = \overline{\varphi}(v_{i-1}) = \tau_i$ for some $i \in [2, t]$ and some $\tau_i \in [1, \Delta] \setminus \overline{\varphi}(F)$. Note that one of v_{i-1} and v_t is $(1, \tau_i)$-unlinked with r. By doing a $(1, \tau_i)$-swap at a vertex in $\{v_{i-1}, v_t\}$ that is $(1, \tau_i)$-unlinked with r, we convert this case to the Type B case. □

Note that under the condition of Lemma 2.9, if P is a $(2, \tau)$- or (τ, Δ)-chain disjoint from $P_{s_1}(2, \tau)$ or $P_{s_1}(\tau, \Delta)$, then we also have $r \notin P$, and so the Kempe change φ/P gives an F-stable coloring.

Definition 2.10. Let $h, \ell \in [1, q]$. The *shifting from s_h to s_ℓ* is a recoloring operation $rs_i :\ \varphi(rs_i) \to \overline{\varphi}(s_i)$ for all $i \in [h, \ell]$, i.e., replacing the current color on the edge rs_i with the missing color at s_i for all $i \in [h, \ell]$.

We apply shiftings when the sequence $(s_h, \dots, s_\ell)$ forms a rotation or is a type B τ-sequence, where $\tau = \varphi(rs_h)$, such that $\overline{\varphi}(s_\ell) = 1$. Since $1 \in \overline{\varphi}(r)$, we obtain another Δ-edge coloring in both cases. We do not know whether a shifting can be achieved through a sequence of Kempe changes. So, in this paper, "Kempe changes" do not include "shifting". In the proof, we sometimes use the following weaker version of "stable" coloring.

Definition 2.11. A coloring $\varphi' \in \mathcal{C}^\Delta(G - rs_1)$ is $V(F-r)$-*stable* (w.r.t. F and φ) if $V(F)$ is the vertex set of a multifan $F_{\varphi'}$ at r w.r.t. rs_1 and φ', $\overline{\varphi}'(s_1) = \overline{\varphi}(s_1) = \{2, \Delta\}$, and $\overline{\varphi}'(V(F_{\varphi'}) \setminus \{r\}) = \overline{\varphi}(V(F) \setminus \{r\})$. Moreover, a $V(F-r)$-stable coloring φ' is called $V(F)$-*stable* if $\overline{\varphi}'(r) = \overline{\varphi}(r)$.

Lemma 2.12. *For any color $\gamma \in \overline{\varphi}(F)$ and a vertex $x \in V(G) \setminus V(F)$, the following two statements hold.*

- *if γ is 2-inducing, then the Kempe change $\varphi/P_x(\gamma, \Delta, \varphi)$ gives a $V(F)$-stable coloring provided $\overline{\varphi}(x) \cap \{\gamma, \Delta\} \neq \emptyset$, and*
- *if γ is Δ-inducing, then the Kempe change $\varphi/P_x(2, \gamma, \varphi)$ gives a $V(F)$-stable coloring provided $\overline{\varphi}(x) \cap \{\gamma, 2\} \neq \emptyset$.*

Proof. By symmetry, we only prove the first statement. If $r \notin P_x(\gamma, \Delta, \varphi)$, we are done by Lemma 2.6. Assume $r \in P_x(\gamma, \Delta, \varphi)$. Since $\overline{\varphi}(x) \cap \{\gamma, \Delta\} \neq \emptyset$, $P_x(\gamma, \Delta, \varphi)$ is disjoint from $P_{s_1}(\gamma, \Delta, \varphi) = P_{\overline{\varphi}_F^{-1}(\gamma)}(\gamma, \Delta, \varphi)$. Let $\varphi' = \varphi/P_x(\gamma, \Delta, \varphi)$. Note that $\varphi(rs_{\alpha+1}) = \Delta$. Let $s_i = \overline{\varphi}_F^{-1}(\gamma)$ for some $i \in [1, \alpha]$. We have $\varphi'(rs_{\alpha+1}) = \gamma$ and $\varphi'(rs_i) = \Delta$. So,

$$F' = (r, rs_1, s_1, rs_2, \dots, rs_{i-1}, s_{i-1}, rs_{\alpha+1}, s_{\alpha+1}, \dots, s_\beta, rs_i, \dots, s_\alpha)$$

is a multifan w.r.t. rs_1 and φ'. Clearly, $\overline{\varphi}'(s_1) = \overline{\varphi}(s_1) = \{2, \Delta\}$ and $\overline{\varphi}'(V(F')) = \overline{\varphi}(V(F))$. Hence, φ' is $V(F)$-stable. □

Let $\tau \in [1, \Delta] \setminus \overline{\varphi}(F)$ and $(v_1, v_2, \dots, v_t)$ be the τ-sequence at r. If the τ-sequence is of type A, the shifting of this sequence yields a coloring in $\mathcal{C}^\Delta(G - e)$, which is F-stable. We call such an operation an *A-shifting*. If the τ-sequence is of type B and satisfies $\overline{\varphi}(v_t) = 1$, the shifting of this sequence yields a coloring $\varphi' \in \mathcal{C}^\Delta(G - e)$ with $\overline{\varphi}'(r) = \tau$, which is $V(F-r)$-stable. We call such an operation a *B-shifting*.

Let P be a $(\tau, *)$-chain with endvertices x and y, where $*$ represents any color from $[1, \Delta] \setminus \{\tau\}$. Suppose that $rv_1 \in E(P)$ and $x, y \notin \{v_1, \dots, v_t\}$. If either the A-shifting or the B-shifting is eligible, we do it and obtain a new coloring φ'. Notice that $\overline{\varphi}'(v_1) = \tau$. So, either $P_x(\tau, *, \varphi') = P_{v_1}(\tau, *, \varphi')$ or $P_y(\tau, *, \varphi') = P_{v_1}(\tau, *, \varphi')$ but not both. Consequently, x and y are $(\tau, *)$-unlinked w.r.t. coloring φ'. We will use this "unlink" technique in the following lemma. In the proofs, we may need to preserve some colors at a vertex, which leads to the following definition.

Definition 2.13. Given a set S of colors, a coloring φ' is called *S-avoiding* (w.r.t. φ) if every Kempe change applied in obtaining φ' from coloring φ does not involve any color from S.

In the following lemma, whenever $P_x(\tau, \Delta)$ or $P_x(2, \tau)$ is used, it implicitly implies that one of the two colors from τ and Δ or from 2 and τ is missing at x.

Lemma 2.14. *Suppose F is a maximum multifan, $N[r] \neq V(G)$ and $\overline{\varphi}(F) \neq [1, \Delta]$. For any vertex $x \in V(G) \setminus N[r]$ and any color $\tau \in [1, \Delta] \setminus \overline{\varphi}(F)$ with $\overline{\varphi}(x) \cap \{\tau, \Delta\} \neq \emptyset$, the following statements hold.*

(i) If $\tau \in \overline{\varphi}(x)$, then there is an F-stable coloring $\varphi' \in \mathcal{C}^\Delta(G - rs_1)$ such that $1 \in \overline{\varphi}'(x)$.
(ii) If $\tau \in \overline{\varphi}(x)$, then there exists an F-stable and $\{\Delta\}$-avoiding coloring φ' such that $1 \in \overline{\varphi}'(x)$ unless the τ-sequence $(v_1, \ldots, v_t)$ with respect to φ is of type B and $\overline{\varphi}(v_t) = \Delta$.
(iii) Suppose $r \in P_{s_1}(\tau, \Delta, \varphi') = P_x(\tau, \Delta, \varphi')$ for every F-stable and $\{\tau, \Delta\}$-avoiding coloring φ'. Then the τ-sequence $(v_1, \ldots, v_t)$ with respect to φ is of type B with $\overline{\varphi}(v_t) = \Delta$.
(iv) Suppose $r \in P_{s_1}(\tau, \Delta, \varphi') = P_x(\tau, \Delta, \varphi')$ for every F-stable and $\{2, \tau, \Delta\}$-avoiding coloring φ'. Then the τ-sequence $(v_1, \ldots, v_t)$ with respect to φ is of type B with $\overline{\varphi}'(v_t) \in \{2, \Delta\}$.
(v) Suppose $r \in P_{s_1}(\tau, \Delta, \varphi') = P_x(\tau, \Delta, \varphi')$ and $r \in P_{s_1}(2, \tau, \varphi') = P_x(2, \tau, \varphi')$ for every $V(F)$-stable coloring φ' with $1, \tau \in \overline{\varphi}'(x)$. Then the τ-sequence $(v_1, \ldots, v_t)$ with respect to φ is of type B with $\overline{\varphi}(v_t) = 1$.
(vi) Suppose $r \in P_{s_1}(\tau, \Delta, \varphi') = P_x(\tau, \Delta, \varphi')$ for every $V(F)$-stable and $\{1, \tau, \Delta\}$-avoiding coloring φ'. Then we can modify φ into a $V(F)$-stable and $\{1, \tau, \Delta\}$-avoiding coloring φ^* such that the τ-sequence $(v_1, \ldots, v_t)$ with respect to φ^* is of type B in which $\overline{\varphi}^*(v_t)$ is in $\{1, \Delta\}$ or is 2-inducing.
(vii) Suppose $r \in P_{s_1}(2, \tau, \varphi') = P_x(2, \tau, \varphi')$ for every $V(F)$-stable and $\{1, \tau, \Delta\}$-avoiding coloring φ'. Then we can modify φ into a $V(F)$-stable and $\{1, \tau, \Delta\}$-avoiding coloring φ^* such that the τ-sequence $(v_1, \ldots, v_t)$ with respect to φ^* is of type B in which $\overline{\varphi}^*(v_t)$ is in $\{1, \Delta\}$ or is 2-inducing.

Proof. Let $(v_1, \ldots, v_t)$ be the τ-sequence with respect to φ. We will apply either an A-shifting or a B-shifting on $(v_1, \ldots, v_t)$ to cutoff the linkage between either x and r or x and s_1 by the remark prior to Definition 2.13. We show the statements by considering the type of the τ-sequence $(v_1, \ldots, v_t)$ one by one.

Assume first that $(v_1, \ldots, v_t)$ is of type A. We prove (i) and (ii) together. For (i), we may assume that x and r are $(1, \tau)$-linked for every F-stable φ', since otherwise we just apply a $(1, \tau)$-swap at x to get a desired coloring. For (ii), we may assume that x and r are $(1, \tau)$-linked for every F-stable and $\{\Delta\}$-avoiding φ', since otherwise we just apply a $(1, \tau)$-swap at x to get a desired coloring. Thus v_t and r are $(1, \tau)$-unlinked and so we apply a $(1,\tau)$-swap at v_t, and then apply a B-shifting from v_1 to v_t to obtain a coloring φ'. Let φ'' be obtained from φ' by renaming τ as 1 and vice versa. Then φ'' is F-stable and it is a desired coloring for both (i) and (ii). For each of (iii) to (vii), by applying the A-shifting on $(v_1,\dots,v_t)$, we cutoff the linkage between x and s_1 and so obtain a contradiction to the assumptions of the statements.

Assume now that $(v_1,\dots,v_t)$ is of type B. Let $\overline{\varphi}(v_t)=\gamma\in\overline{\varphi}(F)$. Recall that r and $\overline{\varphi}_F^{-1}(\gamma)$ are $(1,\gamma)$-linked by Lemma 2.3(a). For (i), we simply apply a $(1,\gamma)$-swap at v_t and then apply a shifting from v_1 to v_t. Then by renaming τ as 1 and vice versa, we obtain an F-stable coloring φ' such that color 1 is missing at x. For (ii), we repeat exactly the same argument as for (i) unless $\gamma=\Delta$. For Statements (iii) and (iv), we suppose by contradiction that the corresponding conclusion does not hold. We simply apply a $(1,\gamma)$-swap at v_t, apply a shifting from v_1 to v_t, and then rename τ as 1 and vice versa. Denote the resulting coloring by φ'. For (iii), φ' is F-stable and $\{\tau,\Delta\}$-avoiding but $P_{s_1}(\tau,\Delta,\varphi')\neq P_x(\tau,\Delta,\varphi')$, a contradiction. For (iv), φ' is F-stable and $\{2,\tau,\Delta\}$-avoiding but $P_{s_1}(\tau,\Delta,\varphi')\neq P_x(\tau,\Delta,\varphi')$, a contradiction. We show (v) now. We may assume that $\gamma\neq 1$ and γ is 2-inducing by the symmetry between 2 and Δ. Since s_1 and $\overline{\varphi}_F^{-1}(\gamma)$ are (γ,Δ)-linked by Lemma 2.3(b), we first apply a (γ,Δ)-swap at v_t. The resulting coloring φ' is $V(F)$-stable and $1,\tau\in\overline{\varphi}'(x)$, so we still have $r\in P_{s_1}(\tau,\Delta,\varphi')=P_x(\tau,\Delta,\varphi')$. Now we apply a (τ,Δ)-swap at v_t to get a new coloring φ''. The coloring φ'' is still $V(F)$-stable satisfying $1,\tau\in\overline{\varphi}''(x)$. However, the sequence $(v_1,\dots,v_t)$ is of type A with respect to φ'', and so we can reach a contradiction as in the first case. For (vi) and (vii), we are done if γ is 1 or Δ or 2-inducing. If γ is Δ-inducing, we apply a $(2,\gamma)$-swap at v_t. The resulting coloring is $V(F)$-stable by Lemma 2.12 and is $\{1,\tau,\Delta\}$-avoiding. Now the missing color of v_t is a 2-inducing color, as desired.

Assume finally that $(v_1,\dots,v_t)$ is of type C. That is, $\overline{\varphi}(v_t)=\overline{\varphi}(v_{i-1})=\tau_i$ for some $i\in[2,t]$ and some $\tau_i\in[1,\Delta]\setminus\overline{\varphi}(F)$. We show that under the assumption of each statement, we can reduce this sequence into a type B τ-sequence with respect to an F-stable coloring. For each of (i) to (iv), since one of v_{i-1} and v_t is $(1,\tau_i)$-unlinked with r, we apply a $(1,\tau_i)$ swap at a vertex in $\{v_{i-1},v_t\}$ that is $(1,\tau_i)$-unlinked with r, resulting in a type B τ-sequence $(v_1,\dots,v_{i-1})$ or $(v_1,\dots,v_t)$ such that the color 1 is missing at the last vertex of the sequence. For (i) and (ii), we can find a desired coloring as in type B case; and for (iii) and (iv), we obtain a contradiction as in type B case. For (v), since $\tau_i\in[1,\Delta]\setminus\overline{\varphi}(F)$, by Lemma 2.9, $r\in P_{s_1}(\tau_i,\Delta)$. Since one of v_{i-1} and v_t is (τ_i,Δ)-unlinked with r, we apply a (τ_i,Δ)-swap at a vertex in $\{v_{i-1},v_t\}$ that is (τ_i,Δ)-unlinked with r, resulting in a type B τ-sequence $(v_1,\dots,v_{i-1})$ or $(v_1,\dots,v_t)$ such that the color Δ is missing at the last vertex of the sequence. Then we can obtain a contradiction as in the type B case. For (vi) and (vii), since $\tau_i\in[1,\Delta]\setminus\overline{\varphi}(F)$, by Lemma 2.9, $r\in P_{s_1}(2,\tau_i)$. Since one of v_{i-1} and v_t is $(2,\tau_i)$-unlinked with r, we apply a $(2,\tau_i)$-swap at a vertex in $\{v_{i-1},v_t\}$ that is $(2,\tau_i)$-unlinked with r, resulting in a type B τ-sequence $(v_1,\dots,v_{i-1})$ or $(v_1,\dots,v_t)$. $\square$

3. Proof of Theorem 1.4

Theorem 1.4. *Let G be a class 2 graph with maximum degree Δ, $r \in V_\Delta$ be light, and $s \in N_{\Delta-1}(r)$. If rs is a critical edge of G, then all vertices in $N(s) \setminus N(r)$ are Δ-vertices.*

Proof. Assume to the contrary that there exists $x \in N(s) \setminus N(r)$ with $d(x) < \Delta$. Clearly, $x \neq r$. Denote s by s_1. Let $\varphi \in \mathcal{C}^\Delta(G - rs_1)$ and assume the corresponding multifan F w.r.t. rs_1 is maximum and typical.

We claim that there is an F-stable coloring such that color 1 is missing at x. To see this, let $\tau \in \overline{\varphi}(x)$. If $\tau \in \overline{\varphi}(F)$, then $\overline{\varphi}_F^{-1}(\tau)$ and r are $(1, \tau)$-linked by Lemma 2.3(a). So, $P_x(1, \tau)$ does not contain any edge of F and does not end at any vertex in F. Hence $\varphi/P_x(1, \tau)$ is F-stable such that color 1 is missing at x. We assume that $\tau \notin \overline{\varphi}(F)$. By Lemma 2.14(i), there is an F-stable coloring such that color 1 is missing at x. So the claim is true and we may assume $1 \in \overline{\varphi}(x)$.

Let $\varphi(s_1x) = \tau$. If $\tau \in \overline{\varphi}(F)$, we may assume it is 2-inducing. Since $\overline{\varphi}_F^{-1}(\tau)$ and r are $(1, \tau)$-linked by Lemma 2.3(a), we apply a $(1, \tau)$-swap at x and get an F-stable coloring. We then apply a (τ, Δ)-swap at x and get a new coloring φ'. Since τ is 2-inducing, it follows that s_1 and $\overline{\varphi}_F^{-1}(\tau)$ are (τ, Δ)-linked, and Δ is still missing at s_1. We see that $F^* = (r, rs_1, s_1, s_1x, x)$ is a multifan w.r.t. φ'. However, we have $\Delta \in \overline{\varphi}'(s_1) \cap \overline{\varphi}'(x)$, contradicting $V(F^*)$ being elementary. Thus we assume that $\tau \notin \overline{\varphi}(F)$. We apply a $(1, \Delta)$-swap at x and get an F-stable coloring φ'. Then $P_{s_1}(\tau, \Delta, \varphi') = s_1x$ does not contain vertex r, showing a contradiction to Lemma 2.9. $\square$

4. Proof of Theorem 1.5

In this section, we let G be a class 2 graph with maximum degree Δ, $rs_1 \in E(G)$ be a critical edge with r being a light vertex and $s_1 \in N_{\Delta-1}(r)$, and let $x \in V(G) \setminus N[r]$ with $d(x) \leq \Delta - 2$.

Theorem 1.5. *If $d(r) = \Delta$ and $d(x) \leq \Delta - 3$, then $N(x) \cap N(s_1) \subseteq N(r) \setminus N_\Delta(r)$.*

The proof of Theorem 1.5 is based on the following three lemmas whose proofs will be given in the following three subsections, respectively. Let $\varphi \in \mathcal{C}^\Delta(G - rs_1)$ and F be a typical multifan w.r.t. rs_1 and φ. We additionally assume that F is a maximum multifan w.r.t. edge rs_1.

Lemma 4.1. *Suppose that r is a Δ-vertex and $u \in N(s_1) \cap N(x)$ with $u \notin N(r) \setminus N_\Delta(r)$. Then there is no $V(F)$-stable coloring φ_1 such that $\varphi_1(ux) = \Delta$ and $\{1, 2\} \subseteq \overline{\varphi}_1(x)$ or $\varphi_1(ux) = 2$ and $\{1, \Delta\} \subseteq \overline{\varphi}_1(x)$.*

Lemma 4.2. *Suppose that r is a Δ-vertex and $u \in N(s_1) \cap N(x)$ with $u \notin N(r) \setminus N_\Delta(r)$. Then there is no $V(F)$-stable coloring φ_1 such that $\varphi_1(s_1u) = 1$ and $\{2, \Delta\} \subseteq \overline{\varphi}_1(x)$.*

By relaxing the condition $d(r) = \Delta$ to $d(r) \geq \Delta - 1$, we have the following results.

Lemma 4.3. *Under the assumption $d(r) \geq \Delta - 1$, the following statements hold.*

(i) Assume $d_G(x) \leq \Delta - 3$. If there is a $V(F)$-stable coloring φ such that $1 \in \overline{\varphi}(x)$, then there is a $V(F)$-stable coloring φ_1 such that $\{2, \Delta\} \subseteq \overline{\varphi}_1(x)$.

(ii) If there is a $V(F)$-stable coloring φ such that $\{2, \Delta\} \subseteq \overline{\varphi}(x)$, then there is a $V(F)$-stable coloring φ_1 such that $\varphi_1(ux) \in \{2, \Delta\}$ and $1 \in \overline{\varphi}_1(x)$. Furthermore, we can choose φ_1 such that if $\varphi_1(s_1u)$ is 2-inducing, then $\varphi_1(ux) = \Delta$; and if $\varphi_1(s_1u)$ is Δ-inducing, then $\varphi_1(ux) = 2$.

(iii) If there is a $V(F)$-stable coloring φ such that $\varphi(ux) \in \{2, \Delta\}$ and $1 \in \overline{\varphi}(x)$, where $\varphi(ux) = \Delta$ if $\varphi(s_1u)$ is 2-inducing, and $\varphi(ux) = 2$ if $\varphi(s_1u)$ is Δ-inducing, then there is a $V(F)$-stable coloring φ_1 such that $\varphi_1(s_1u) = 1$ and $\{2, \Delta\} \cap \overline{\varphi}_1(x) \neq \emptyset$.

(iv) Assume $d_G(x) \leq \Delta - 3$. If there is a $V(F)$-stable coloring φ such that $\varphi(s_1u) = 1$ and $\{2, \Delta\} \cap \overline{\varphi}(x) \neq \emptyset$, then there is a $V(F)$-stable coloring φ_1 such that $\varphi_1(s_1u) = 1$ and $\{2, \Delta\} \subseteq \overline{\varphi}_1(x)$.

Lemma 4.3 describes a process of modifying φ into a $V(F)$-stable coloring φ_1 such that $\varphi_1(s_1u) = 1$ and $\{2, \Delta\} \subseteq \overline{\varphi}_1(x)$, which in turn gives a contradiction to Lemma 4.2. We list the processes as separate statements as some of them will be applied independently in the last section also.

Proof of Theorem 1.5. Let $N_{\Delta-1}(r) = \{s_1, s_2, \ldots, s_{\Delta-2}\}$. Suppose to the contrary that there is a vertex $u \in V(G)$ such that $u \notin N(r) \setminus N_\Delta(r)$ and u is adjacent to both x and s_1. Then $u \notin \{r, s_1, \ldots, s_{\Delta-2}\}$ since $x \notin N(r)$. Following the notation given at the beginning of this section, we let $\varphi \in \mathcal{C}^\Delta(G - rs_1)$, and F be a typical multifan w.r.t. rs_1 and φ. We also assume that F is a maximum multifan w.r.t. rs_1.

We claim that there exists a $V(F)$-stable coloring φ' such that $1 \in \overline{\varphi}'(x)$. Let $\tau \in \overline{\varphi}(x)$. If $\tau \in \overline{\varphi}(F)$, then by Lemma 2.6, the Kempe change $\varphi/P_x(1, \tau, \varphi)$ gives an F-stable coloring φ'. Clearly, $1 \in \overline{\varphi}'(x)$. Thus, we assume that $\tau \in [1, \Delta] \setminus \overline{\varphi}(F)$. By Lemma 2.14(i), there is an F-stable coloring φ' such that $1 \in \overline{\varphi}'(x)$. Now applying Lemma 4.3(i)-(iv), we can modify φ' into a $V(F)$-stable coloring φ_1 such that $\varphi_1(s_1u) = 1$ and $\{2, \Delta\} \subseteq \overline{\varphi}_1(x)$. This gives a contradiction to Lemma 4.2. □

4.1. Proof of Lemma 4.1

By symmetry, we only prove the first part of the conclusion. Suppose to the contrary that there is a $V(F)$-stable coloring φ_1 such that $\varphi_1(ux) = \Delta$ and $\{1, 2\} \subseteq \overline{\varphi}_1(x)$. Notice that $1 \in \overline{\varphi}_1(r) \cap \overline{\varphi}_1(x)$ and $2 \in \overline{\varphi}_1(s_1) \cap \overline{\varphi}_1(x)$. Let $\tau = \varphi_1(s_1u)$.

Consider first that $\tau \in \overline{\varphi}_1(F)$. If $\tau = 1$, then $\varphi_1(s_1u) \in \overline{\varphi}_1(r)$ and $\varphi_1(ux) = \Delta \in \overline{\varphi}_1(s_1)$, and so $K = (r, rs_1, s_1, s_1u, u, ux, x)$ is a Kierstead path. Since $d(s_1) = \Delta - 1$, $V(K)$ is elementary by Lemma 2.5, showing a contradiction to $2 \in \overline{\varphi}_1(x) \cap \overline{\varphi}_1(s_1)$. So,

$\tau \neq 1$. We claim that τ is Δ-inducing. Suppose to the contrary that τ is 2-inducing. We let $\psi = \varphi_1/P_x(1,\tau,\varphi_1)$. By Lemma 2.6, ψ is F-stable. If $s_1u \notin P_x(1,\tau,\varphi_1)$, then $\psi(s_1u) = \tau$, and so $P_{s_1}(\tau,\Delta,\psi) = s_1ux$, contradicting that s_1 and $\overline{\psi}_F^{-1}(\tau)$ are (τ,Δ)-linked (Lemma 2.3(b)). If $s_1u \in P_x(1,\gamma,\varphi_1)$, then $\psi(s_1u) = 1 \in \overline{\psi}(r)$ and $\psi(ux) = \Delta \in \overline{\psi}(s_1)$, so under ψ, $K = (r, rs_1, s_1, s_1u, u, ux, x)$ is a Kierstead path with $d(s_1) = \Delta - 1 < \Delta$, but 2 is missing at both s_1 and x, showing a contradiction to $V(K)$ being elementary (Lemma 2.5). Thus τ is Δ-inducing. We apply $(\Delta,1)-(1,2)$-swaps at x and get an F-stable coloring φ' (Lemma 2.6). Notice that $\varphi'(ux) = 2$ and $\{1,\Delta\} \subseteq \overline{\varphi}'(x)$. This gives back to the previous case by the symmetry between 2 and Δ, which leads to a contradiction. Thus, $\tau \in [1,\Delta] \setminus \overline{\varphi}_1(F)$.

Since F is a maximum multifan, by Lemma 2.8 there is a unique τ-sequence $(v_1,\ldots,v_t)$. We claim that $s_1u \in P_x(1,\tau,\varphi_1) = P_r(1,\tau,\varphi_1)$. Otherwise, let $\varphi' = \varphi_1/P_x(1,\tau,\varphi_1)$. Clearly, $\tau \in \overline{\varphi}'(x)$. If $s_1u \notin P_x(1,\tau,\varphi_1)$, then $P_{s_1}(\tau,\Delta,\varphi') = s_1ux$. In this case, if $P_x(1,\tau,\varphi_1)$ did not end at r, then φ' is F-stable, which in turn gives $r \in P_{s_1}(\tau,\Delta,\varphi') = s_1ux$ by Lemma 2.9, a contradiction; if $P_x(1,\tau,\varphi_1)$ ended at r, then φ' is $V(F-r)$-stable and $\overline{\varphi}'(r) = \tau$, which in turn gives $P_{s_1}(\tau,\Delta,\varphi') = s_1ux$, which should contain r and end at r by Lemma 2.3(a), giving a contradiction. Then we assume that $P_x(1,\tau,\varphi_1)$ contains edge s_1u and does not end at r. In this case, $\varphi'(s_1u) = 1 \in \overline{\varphi}'(r)$ and $\varphi'(ux) = \varphi_1(ux) = \Delta \in \overline{\varphi}'(s_1)$, and so $K' = (r, rs_1, s_1, s_1u, u, ux, x)$ is a Kierstead path. But, $2 \in \overline{\varphi}'(s_1) \cap \overline{\varphi}'(x)$ shows that $V(K')$ is not elementary, a contradiction.

We consider below the τ-sequence $(v_1,\ldots,v_t)$ according to its type, but deal with the situation in the following claim first.

Claim 4.1. *There does not exist a $V(F)$-stable coloring φ' with $\varphi'(s_1u) = \tau$, $\varphi'(ux) = \Delta$, $2 \in \overline{\varphi}'(x)$, and the τ-sequence w.r.t. φ' is of type B with $\overline{\varphi}'(v_t) = 1$.*

Proof. Suppose to the contrary that there is such a $V(F)$-stable coloring. We also assume that under coloring φ', the τ-sequence is also $(v_1,\ldots,v_t)$. We do the B-shifting from v_1 to v_t to get a new coloring φ^*. Note that φ^* is a $V(F-r)$-stable coloring, and $\varphi^*(s_1u) = \tau = \overline{\varphi}^*(r)$ and $\varphi^*(ux) = \Delta \in \overline{\varphi}^*(s_1)$, which in turn shows that $K = (r, rs_1, s_1, s_1u, u, ux, x)$ is a Kierstead path. But, $2 \in \overline{\varphi}^*(s_1) \cap \overline{\varphi}^*(x)$ shows that $V(K)$ is not elementary, a contradiction. □

If the τ-sequence is of type A, i.e., $\overline{\varphi}_1(v_t) = \tau$, we apply a $(1,\tau)$-swap at v_t to get a coloring φ'. Since $s_1u \in P_x(1,\tau,\varphi_1) = P_r(1,\tau,\varphi_1)$, φ' is F-stable. We also notice that $\varphi'(ux) = \varphi_1(ux) = \Delta$ and $2 \in \overline{\varphi}'(x)$, which gives a contradiction to Claim 4.1.

Suppose that the τ-sequence is of type C, more specifically, $\overline{\varphi}_1(v_t) = \overline{\varphi}_1(v_{i-1}) = \tau_i$ for some $i \in [2,t]$. Since one of v_{i-1} and v_t is $(1,\tau_i)$-unlinked with r, we apply a $(1,\tau_i)$-swap at a vertex in $\{v_{i-1}, v_t\}$ that is $(1,\tau_i)$-unlinked with r to get an F-stable coloring φ'. Clearly, $1 \in \overline{\varphi}'(v_{i-1})$ or $1 \in \overline{\varphi}'(v_t)$. In either case, the resulting τ-sequence is of type B with color 1 missing at the last vertex, which gives a contradiction to Claim 4.1.

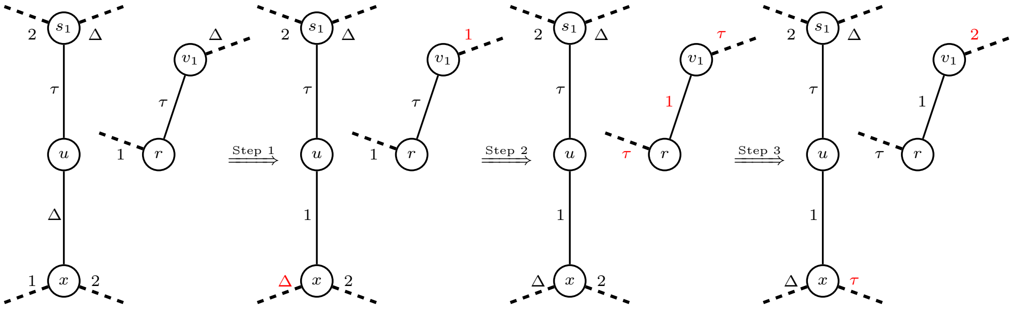

Fig. 3. Three steps of Kempe changes.

Suppose now that the τ-sequence is of type B and let $\overline{\varphi}_1(v_t) = \gamma$ for some $\gamma \in \overline{\varphi}_1(F)$. By Claim 4.1, $\gamma \neq 1$. If $\gamma \neq \Delta$, we first apply a $(1,\gamma)$-swap at v_t and get an F-stable coloring φ'. Note that $1 \notin \overline{\varphi}'(x)$ may occur. Under coloring φ', the τ-sequence is of type B and $1 \in \overline{\varphi}'(v_t)$, giving a contradiction to Claim 4.1. Thus, $\gamma = \Delta$. We consider two cases regarding whether $t = 1$.

Case 1. $t = 1$.

We first do three Kempe changes as below. Step 1: $(1,\Delta)$-swap(s) at both v_1 and x (s_1 and r are $(1,\Delta)$-linked); Step 2: a $(1,\tau)$-swap at v_1 (only changes the color on the edge rv_1); and Step 3: $(2,\tau)$-swap(s) at both x and v_1 (s_1 and r are $(2,\tau)$-linked). See Fig. 3 for this sequence of changes.

Note that Step 1 gives an F-stable coloring, Step 2 gives a $V(F-r)$-stable coloring, and Step 3 gives a stable coloring w.r.t. the new multifan obtained in Step 2.

We then color rs_1 by τ and uncolor s_1u to give a coloring φ', which is followed by 5 Kempe changes as follows. Step 1: ux: $1 \to \tau$; Step 2: $(1,2)$-swap(s) at both x and v_1 (s_1 and u are $(1,2)$-linked); Step 3: $(1,\Delta)$-swap(s) at both x and v_1 (s_1 and u are $(1,\Delta)$-linked); Step 4: a $(1,\tau)$-swap on the $(1,\tau)$-chain containing s_1r; Step 5: $(1,\Delta)$-swap(s) at both x and v_1 (s_1 and u are $(1,\Delta)$-linked). Since every recoloring is a Kempe change, the final coloring is in $\mathcal{C}^\Delta(G-s_1u)$. See Fig. 4 for this sequence of changes.

Under the current coloring, we have $P_{s_1}(1,2) = s_1rv_1$. On the other hand, since s_1u is uncolored and 1 and 2 are missing at u and s_1 respectively, $P_{s_1}(1,2) = P_u(1,2)$, giving a contradiction.

Case 2. $t \geq 2$.

Let $\varphi_1(rv_t) = \overline{\varphi}_1(v_{t-1}) = \tau_t$. As the τ-sequence $(v_1,\ldots,v_t)$ is of type B, we have $\tau_t \neq \tau$. We may assume that v_{t-1} and r are $(1,\tau_t)$-linked. Otherwise, the $(1,\tau_t)$-swap at v_{t-1} gives an F-stable coloring that contradicts Claim 4.1. We apply a $(1,\tau_t)$-swap at x and get an F-stable coloring. By Lemma 2.9, $r \in P_{s_1}(\tau_t,\Delta)$. As $\varphi_1(rv_t) = \tau_t$ and $\overline{\varphi}_1(v_t) = \Delta$, we then have $r \in P_{s_1}(\tau_t,\Delta) = P_{v_t}(\tau_t,\Delta)$. We thus apply (τ_t,Δ)-swaps at both x and v_{t-1} and get an F-stable coloring φ'. Note that $\varphi'(ux) = \tau_t$ and $\Delta \in \overline{\varphi}'(x) \cap \overline{\varphi}'(v_{t-1}) \cap \overline{\varphi}'(v_t)$.

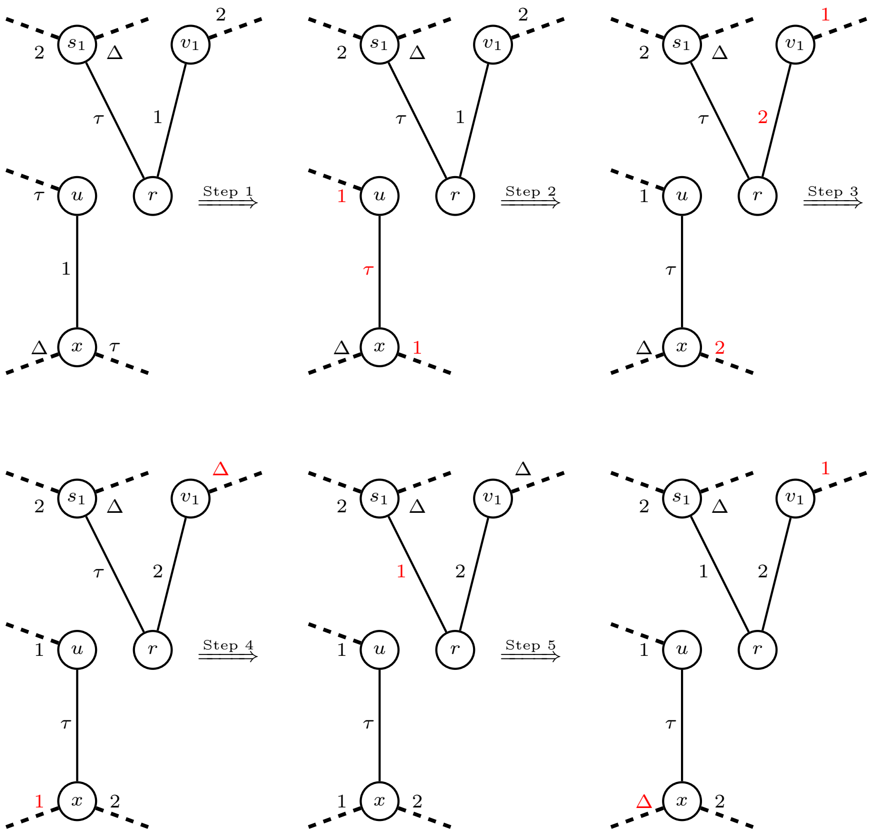

Fig. 4. Five steps of Kempe changes.

By Lemma 2.9, $r \in P_{s_1}(\tau, \Delta)$. We claim that $P_{s_1}(\tau, \Delta, \varphi') = P_x(\tau, \Delta, \varphi')$. Suppose to the contrary that $P_{s_1}(\tau, \Delta, \varphi') \neq P_x(\tau, \Delta, \varphi')$. If $P_x(\tau, \Delta, \varphi') \neq P_{v_t}(\tau, \Delta, \varphi')$, we do the following sequence of five Kempe changes: the (τ, Δ)-swap at x, the $(1, \Delta)$-swap at v_t (s_1 and r are $(1, \Delta)$-linked), the $(1, \tau_t)$-swap on the $(1, \tau)$-chain containing ux, the $(2, \Delta)$-swap at x, and the $(1, \Delta)$-swap at x. Except the Kempe change that the $(2, \Delta)$-swap at x may possibly change the colors on two edges of F, all other changes are F-stable. Thus the final resulting coloring is $V(F)$-stable. Under the current coloring, $P_{s_1}(\tau, \Delta) = s_1ux$ that does not contain vertex r, giving a contradiction to Lemma 2.9.

Under the assumption that $P_{s_1}(\tau, \Delta, \varphi') \neq P_x(\tau, \Delta, \varphi')$, by the argument above, we assume then that $P_x(\tau, \Delta, \varphi') = P_{v_t}(\tau, \Delta, \varphi')$. We do the (τ, Δ)-swap at x that is also the (τ, Δ)-swap at v_t to get an F-stable coloring. Note that Δ is no longer missing at x unless τ is also previously missing at x. Since s_1 and r are $(1, \Delta)$-linked, we apply a $(1, \Delta)$-swap at v_{t-1} to get an F-stable coloring, and apply a shifting from v_1 to v_{t-1}, which give a $V(F - r)$-stable coloring. Denote the corresponding new multifan by F^*. Since s_1 and r are (τ, Δ)-linked, we apply a (τ, Δ)-swap at both x and v_t to get an F^*-stable coloring such that Δ is missing at x. Since $r \in P_{s_1}(\tau_t, \Delta) = P_{v_t}(\tau_t, \Delta)$ by Lemma 2.9, we do the (Δ, τ_t)-swap at x, which does not affect the multifan. Denote the resulting coloring by φ^*. Since $\varphi^*(s_1 u) = \tau = \overline{\varphi}^*(r)$ and $\varphi^*(ux) = \Delta \in \overline{\varphi}^*(s_1)$, $(r, rs_1, s_1, s_1 u, u, ux, x)$ is a Kierstead path. But, $2 \in \overline{\varphi}^*(s_1) \cap \overline{\varphi}^*(x)$, giving a contradiction. Therefore $r \in P_{s_1}(\tau, \Delta, \varphi') = P_x(\tau, \Delta, \varphi')$.

Recall that $\varphi'(s_1 u) = \tau$, $\varphi'(ux) = \tau_t$, $\Delta \in \overline{\varphi}'(x) \cap \overline{\varphi}'(v_{t-1}) \cap \overline{\varphi}'(v_t)$, and $2 \in \overline{\varphi}'(x)$. We apply a (τ, Δ)-swap at both v_{t-1} and v_t. Under the new coloring, τ is missing at both v_{t-1} and v_t. We may assume that v_t and r are $(1, \tau)$-unlinked by doing the A-shifting from v_1 to v_{t-1} if necessary. Thus we apply a $(1, \tau)$-swap at v_t. Denote the new coloring by φ^*. If $\varphi^*(s_1 u) = \tau$, we apply a $(1, \tau_t)$-swap on the $(1, \tau_t)$-chain containing ux and then apply a $(1, \Delta)$-swap at x. This gives a type A τ-sequence $(v_1, \dots, v_{t-1})$, which we have dealt with previously. Thus $\varphi^*(s_1 u) = 1$. We apply a $(1, \tau_t)$-swap on the $(1, \tau_t)$-chain containing ux and then apply a $(1, \Delta)$-swap at both x and v_t. This leads back to Case 1 with v_t in the place of v_1 and τ_t in the place of τ. $\square$

4.2. Proof of Lemma 4.2

Let $u \in N(s_1) \cap N(x)$ with $u \notin N(r) \setminus N_\Delta(r)$. Suppose to the contrary that there is a $V(F)$-stable coloring φ_1 such that $\varphi_1(s_1 u) = 1$ and $\{2, \Delta\} \subseteq \overline{\varphi}_1(x)$. Note that $u \neq r$, as every neighbor of r has degree at least $\Delta - 1$ in G while $d(x) \leq \Delta - 3$. Thus $u \notin N[r] \setminus N_\Delta(r)$. Let $\varphi_1(ux) = \tau$. Clearly, $\tau \neq 1$.

Since F is a maximum multifan, $r \in P_{s_1}(\tau, \Delta)$ and $r \in P_{s_1}(2, \tau)$ by Lemma 2.9. We claim that $P_{s_1}(\tau, \Delta) = P_x(\tau, \Delta)$ and $P_{s_1}(2, \tau) = P_x(2, \tau)$. Otherwise, say $P_x(\tau, \Delta)$ and $P_{s_1}(\tau, \Delta)$ are disjoint. We apply a (τ, Δ)-swap at x and get an F-stable coloring φ'. Since $\varphi'(s_1 u) = \varphi_1(s_1 u) = 1 \in \overline{\varphi}'(r)$ and $\varphi'(ux) = \Delta \in \overline{\varphi}'(s_1)$, $K = (r, rs_1, s_1, s_1 u, u, ux, x)$ is a Kierstead path and $2 \in \overline{\varphi}'(s_1) \cap \overline{\varphi}'(x)$, contradicting $V(K)$ being elementary (Lemma 2.5).

We claim that $\tau \notin \overline{\varphi}_1(F)$. Otherwise, $P_{s_1}(\tau, \Delta) = P_{\overline{\varphi_1}_F^{-1}(\tau)}(\tau, \Delta)$ if τ is 2-inducing and $P_{s_1}(2, \tau) = P_{\overline{\varphi_1}_F^{-1}(\tau)}(2, \tau)$ if τ is Δ-inducing. In either case, we get a contradiction to the previous claim. Since the multifan F is maximum, there is a unique τ-sequence $(v_1, \dots, v_t)$ by Lemma 2.8. Since $r \in P_{s_1}(2, \tau) = P_x(2, \tau)$ and $r \in P_{s_1}(\tau, \Delta) = P_x(\tau, \Delta)$, $rv_1 \in P_x(2, \tau)$ and $rv_1 \in P_x(\tau, \Delta)$.

If the τ-sequence is of type A, we do the A-shifting and get an F-stable coloring φ', and under this coloring $P_x(\Delta, \tau, \varphi') \neq P_{s_1}(\Delta, \tau, \varphi')$. But, $\overline{\varphi}'(x) = \overline{\varphi}_1(x) \supseteq \{2, \Delta\}$ and $\varphi'(s_1 u) = \varphi_1(s_1 u) = 1$, giving a contradiction.

Suppose then that the τ-sequence is of type B: $\overline{\varphi}_1(v_t) = \gamma$ for some $\gamma \in \overline{\varphi}_1(F)$. If $\gamma = 1$, we do the B-shifting and get a $V(F - r)$-stable coloring. Note that F is also a multifan w.r.t. the new coloring and that τ and Δ are missing at r and s_1, respectively. We then apply a (τ, Δ)-swap at x and get an F-stable coloring φ' w.r.t. the previous coloring. Note that F is a multifan w.r.t. rs_1 and φ', $\overline{\varphi}'(r) = \tau$, $\{2, \tau\} \subseteq \overline{\varphi}'(x)$ and $\varphi'(ux) = \Delta$, showing a contradiction to Lemma 4.1. So, $\gamma \neq 1$, say γ is 2-inducing. Let $\varphi' = \varphi / P_{v_t}(1, \gamma, \varphi)$. If $s_1 u \notin P_{v_t}(1, \gamma, \varphi)$, the argument turns back to $\overline{\varphi}_1(v_t) = \gamma = 1$

case, which we just settled. Thus we assume $\varphi'(s_1u) = \gamma$. We apply a shifting from v_1 to v_t. Then as s_1 and r are (τ, Δ)-linked, we apply a (τ, Δ)-swap at x. Now up to exchanging the role of 1 and τ, we have a $V(F)$-stable coloring φ'' such that $\overline{\varphi}''(r) = 1$, $\varphi''(ux) = \Delta$ and $\{1, 2\} \subseteq \overline{\varphi}''(x)$. This again gives a contradiction to Lemma 4.1.

Thus the τ-sequence is of type C: $\overline{\varphi}_1(v_t) = \overline{\varphi}_1(v_{i-1}) = \tau_i$ for some $i \in [2, t]$ and some $\tau_i \in [1, \Delta] \setminus \overline{\varphi}(F)$. We first apply a $(1, \Delta)$-swap at x. One of v_{i-1} and v_t is (τ_i, Δ)-unlinked with s_1. We may assume that v_t and s_1 are (τ_i, Δ)-unlinked (the proof for the other case is similar). By Lemma 2.9, we have $r \notin P_{v_t}(\tau_i, \Delta)$. We first apply a (τ_i, Δ)-swap at v_t and then a $(1, \Delta)$-swap at both x and v_t. This converts the problem back to the type B τ-sequence case. $\square$

4.3. Proof of Lemma 4.3

For (i), we assume that $1 \in \overline{\varphi}(x)$. Let $\tau \in \overline{\varphi}(x) \setminus \{1\}$. If $\tau \in \overline{\varphi}(F)$, we apply a $(1, 2)$-swap at x, and then apply $(\tau, 1) - (1, \Delta)$-swaps at x to get a desired coloring φ_1. Thus $\tau \in [1, \Delta] \setminus \overline{\varphi}(F)$. By Lemma 2.9, for any $V(F)$-stable coloring φ' such that $1, \tau \in \overline{\varphi}'(x)$, we have $r \in P_{s_1}(2, \tau, \varphi')$ and $r \in P_{s_1}(\tau, \Delta, \varphi')$. We may further assume that $r \in P_{s_1}(2, \tau, \varphi') = P_x(2, \tau, \varphi')$ and $r \in P_{s_1}(\tau, \Delta, \varphi') = P_x(\tau, \Delta, \varphi')$. For otherwise, say there is a $V(F)$-stable coloring φ' such that $1, \tau \in \overline{\varphi}'(x)$ and $P_{s_1}(2, \tau, \varphi') \neq P_x(2, \tau, \varphi')$, then under φ', we apply a $(2, \tau)$-swap at x and then a $(1, \Delta)$-swap at x in getting a desired coloring φ_1. Applying Lemma 2.14(v), we know that the τ-sequence $(v_1, \dots, v_t)$ is of type B with $\overline{\varphi}(v_t) = 1$. Let $\lambda \in \overline{\varphi}(x) \setminus \{1, \tau\}$. Following a same argument as above, we may assume that $\lambda \in [1, \Delta] \setminus \overline{\varphi}(F)$, and $r \in P_{s_1}(2, \lambda, \varphi') = P_x(2, \lambda, \varphi')$ and $r \in P_{s_1}(\lambda, \Delta, \varphi') = P_x(\lambda, \Delta, \varphi')$ for any $V(F)$-stable coloring φ' such that $1, \lambda \in \overline{\varphi}'(x)$. Applying Lemma 2.14(v) again, we know that the λ-sequence $(w_1, \dots, w_k)$ is of type B with $\overline{\varphi}(w_k) = 1$.

If the two sequences are disjoint, then x is $(1, 2)$-linked with at most one of v_t and w_k. Assume, without loss of generality, that x and v_t are $(1, 2)$-unlinked. We apply a $(1, 2)$-swap at x and then apply a shifting from v_1 to v_t. Now τ is missing at r and r and s_1 are (τ, Δ)-linked. We apply a (τ, Δ)-swap at x. This gives a desired coloring φ_1 up to exchanging the role of 1 and τ. Therefore, the τ-sequence and the λ-sequence intersect. Assume that $v_i = w_j$ is the first common vertex of the two sequences. Then, the two sequences are identical after this vertex.

If both i, j are at least two, then $\overline{\varphi}(v_{i-1}) = \overline{\varphi}(w_{j-1})$, name it γ. By the definition of τ-sequence, $\gamma \in [1, \Delta] \setminus \overline{\varphi}(F)$. Since F is maximum, $r \in P_{s_1}(\gamma, \Delta)$ by Lemma 2.9. One of v_{i-1} and w_{j-1}, say v_{i-1}, is not on $P_{s_1}(\gamma, \Delta)$. We apply a (γ, Δ)-swap at v_{i-1} and get an F-stable coloring φ'. But, the τ-sequence ends with a vertex missing color Δ rather than 1 but the color 1 and τ are still missing at x, giving a contradiction to Lemma 2.14(v). Assume then that one of i and j is 1. Assume, without loss of generality, that $j = 1$, i.e., $\lambda = \overline{\varphi}(v_{i-1})$ and $w_1, \dots, w_k$ is the same as $v_i, \dots, v_t$. We first apply a $(1, 2)$-swap at both x and v_t. One of x and v_{i-1} is $(1, \lambda)$-unlinked with r. If x and r are $(1, \lambda)$-unlinked, we apply $(\lambda, 1) - (1, \Delta)$-swaps at x to get a desired coloring. If v_{i-1} and

r are $(1,\lambda)$-unlinked, we apply a $(1,\lambda)$-swap at v_{i-1} and then apply a shifting from v_1 to v_{i-1}. Next, we apply a (τ,Δ)-swap at x and get a $V(F-r)$-coloring φ'. Switching colors 1 and τ for the entire graph, we get a $V(F)$-stable coloring φ_1 with $\{2,\Delta\} \subseteq \overline{\varphi}_1(x)$.

For (ii), let $\varphi(ux) = \tau$. If $\tau \in \overline{\varphi}(F)$, we may assume that either $\tau = 1$ or τ is 2-inducing. We apply $(\Delta,\tau)-(\tau,1)$-swaps at x. This gives a $V(F)$-stable coloring φ_1 such that $\varphi_1(ux) = \Delta$ and $\{1,2\} \subseteq \overline{\varphi}_1(x)$, showing a contradiction to Lemma 4.1. Thus $\tau \in [1,\Delta] \setminus \overline{\varphi}(F)$. If there is an F-stable and $\{2,\tau,\Delta\}$-avoiding coloring φ' such that $r \notin P_x(\tau,\Delta,\varphi')$, then as $r \in P_{s_1}(\tau,\Delta,\varphi')$ by Lemma 2.9, we apply a (τ,Δ)-swap at x to get a $V(F)$-stable coloring φ'. If $\varphi'(s_1u)$ is not Δ-inducing, then we apply a $(1,2)$-swap at x to get a desired coloring φ_1. If $\varphi'(s_1u)$ is Δ-inducing, we then apply $(2,\Delta)-(\Delta,1)$-swaps at x to get a desired coloring φ_1. Thus $r \in P_x(\tau,\Delta,\varphi')$ for every F-stable and $\{2,\tau,\Delta\}$-avoiding coloring φ', and thus $r \in P_{s_1}(\tau,\Delta,\varphi') = P_x(\tau,\Delta,\varphi')$ by Lemma 2.9. Applying Lemma 2.14(iv), the τ-sequence $(v_1,\ldots,v_t)$ is of type B with $\overline{\varphi}(v_t) \in \{2,\Delta\}$. Let $\varphi(s_1u) = \gamma$. Note that $\gamma \notin \{2,\tau,\Delta\}$. By symmetry, we may assume that either γ is 2-inducing or $\gamma \in [1,\Delta] \setminus (\overline{\varphi}(F) \setminus \{1\})$. We may assume that $\overline{\varphi}(v_t) = 2$. As otherwise, if $\overline{\varphi}(v_t) = \Delta$, we apply a $(1,\Delta)$-swap at both x and v_t and then apply a $(1,2)$-swap at v_t and a $(1,\Delta)$-swap at x, which converts back to the case when $\overline{\varphi}(v_t) = 2$. We apply a $(1,2)$-swap at v_t and then apply a shifting from v_1 to v_t. Next, we apply a (τ,Δ)-swap at x. Since $\gamma \notin \{2,\tau,\Delta\}$ and s_1 and r are both $(1,2)$- and $(1,\Delta)$-linked, the color on s_1u is still γ. Up to exchanging the role of 1 and τ, we get a desired coloring φ_1.

For (iii), by symmetry, we let $\varphi(ux) = \Delta$ and $1 \in \overline{\varphi}(x)$ and show that there is a $V(F)$-stable coloring φ_1 such that $\varphi_1(s_1u) = 1$ and $\{2,\Delta\} \cap \overline{\varphi}_1(x) \neq \emptyset$. Let $\varphi(s_1u) = \tau$. Assume first that $\tau \in \overline{\varphi}(F)$. Clearly, $\tau \neq 1$. As otherwise, $K = (r, rs_1, s_1, s_1u, u, ux, x)$ is a Kierstead path, but $1 \in \overline{\varphi}(s_1) \cap \overline{\varphi}(x)$, showing a contradiction to Lemma 2.5. Since $\varphi(ux) = \Delta$, the assumption of (iii) implies that τ is 2-inducing. We apply a $(1,\tau)$-swap at x. Denote the new coloring by φ^*. If $\varphi^*(s_1u) = \tau$, then $P_{s_1}(\tau,\Delta) = s_1ux$, contradicting Lemma 2.3(b). Thus, $\varphi^*(s_1u) = 1$. As s_1 and $\overline{\varphi_1}_F^{-1}(\tau)$ are (τ,Δ)-linked, we apply a (τ,Δ)-swap at x, which gives a desired coloring φ_1.

Thus $\tau \in [1,\Delta] \setminus \overline{\varphi}(F)$. We first apply a $(1,2)$-swap at x and still denote the resulting coloring by φ. We have $\varphi(ux) = \Delta$ and $2 \in \overline{\varphi}(x)$. Let $(v_1,\ldots,v_t)$ be the τ-sequence guaranteed by Lemma 2.8. For any $V(F)$-stable and $\{2,\tau,\Delta\}$-avoiding coloring φ', as the multifan corresponding to F under φ' is still maximum, by Lemma 2.9, we have $r \in P_{s_1}(2,\tau,\varphi')$. Thus it must be the case that $r \in P_{s_1}(2,\tau,\varphi') = P_x(2,\tau,\varphi')$. (As if x and s_1 were $(2,\tau)$-unlinked with respect to φ', we apply a $(2,\tau)$-swap at x to get a coloring φ''. Then $P_{s_1}(\tau,\Delta,\varphi'') = s_1ux$, which does not contain r, showing a contradiction to Lemma 2.9.) By Lemma 2.14(iv) (the symmetric version with the roles of 2 and Δ exchanged), the τ-sequence $(v_1,\ldots,v_t)$ is of type B and $\overline{\varphi}(v_t) \in \{2,\Delta\}$. If $\overline{\varphi}(v_t) = 2$, since $r \in P_{s_1}(2,\tau) = P_x(2,\tau)$, we apply a $(2,\tau)$-swap at v_t and then apply a shifting from v_1 to v_t. This gives a type A τ-sequence. We then apply a shifting from v_1 to v_t. Denote the new coloring by φ^*. Then $P_{s_1}(2,\tau,\varphi^*) \neq P_x(2,\tau,\varphi^*)$. Still $r \in P_{s_1}(2,\tau,\varphi^*)$ by Lemma 2.9. We apply a $(2,\tau)$-swap at x to get φ^{**}. Then $P_{s_1}(\tau,\Delta,\varphi^{**}) = s_1ux$, showing a contradiction to Lemma 2.9. If $\overline{\varphi}(v_t) = \Delta$, since s_1 and r are $(1,\Delta)$-linked, we apply a $(1,\Delta)$-swap at v_t. Then we apply a shifting from v_1 to v_t and swap the colors 1 and τ in the entire graph. Denote the new coloring by φ_1 (note that $\varphi_1(ux)$ could be Δ or τ and has to be τ, as otherwise $K = (r, rs_1, s_1, s_1u, u, ux, x)$ is a Kierstead path with $2 \in \overline{\varphi}_1(s_1) \cap \overline{\varphi}_1(x)$). We have $\varphi_1(s_1u) = 1$ and $2 \in \overline{\varphi}_1(x)$, and so φ_1 is a desired coloring.

For (iv), by symmetry, we let $\varphi(s_1u) = 1$ and $2 \in \overline{\varphi}(x)$ and show that there is a $V(F)$-stable coloring φ_1 such that $\varphi_1(s_1u) = 1$ and $\{2,\Delta\} \subseteq \overline{\varphi}_1(x)$. Let $\tau \in \overline{\varphi}(x) \setminus \{2\}$. If $\tau = 1$ or is 2-inducing, we simply apply a (τ,Δ)-swap at x. Thus we may assume that τ is Δ-inducing. We apply $(2,1) - (1,\Delta)$-swaps at x and then apply a $(2,\tau)$-swap at x to get a desired $V(F)$-stable coloring φ_1.

Thus $\tau \in [1,\Delta] \setminus \overline{\varphi}(F)$. We first apply a $(1,2)$-swap at x to get an F-stable coloring φ^*. Since s_1 and r are $(1,2)$-linked, we still have $\varphi^*(s_1u) = 1$. Now $\{1,\tau\} \subseteq \overline{\varphi}^*(x)$. By Lemma 2.9, $r \in P_{s_1}(2,\tau,\varphi')$ and $r \in P_{s_1}(\tau,\Delta,\varphi')$ for every $V(F)$-stable coloring φ'. Thus we may assume that $P_x(\tau,\Delta,\varphi') = P_{s_1}(\tau,\Delta,\varphi')$ and $P_x(2,\tau,\varphi') = P_{s_1}(2,\tau,\varphi')$ for every $V(F)$-stable coloring φ' with $\{1,\tau\} \subseteq \overline{\varphi}'(x)$. As otherwise, we can simply apply either a $(2,\tau)$-swap and then a $(1,\Delta)$-swap at x or a (τ,Δ)-swap and then a $(1,2)$-swap at x to get a desired coloring φ_1. By Lemma 2.14(v), the τ-sequence $(v_1,\ldots,v_t)$ is of type B such that $\overline{\varphi}^*(v_t) = 1$. Since $d_G(x) \le \Delta - 3$, we let $\lambda \in \overline{\varphi}^*(x) \setminus \{1,\tau\}$. Using the same arguments as above, we may assume that $\lambda \in [1,\Delta] \setminus \overline{\varphi}^*(F)$ and that the λ-sequence $(w_1,\ldots,w_k)$ is of type B such that $\overline{\varphi}^*(w_k) = 1$.

If the two sequences are disjoint, then x is $(1,2)$-linked with at most one of v_t and w_k. Assume, without loss of generality, that x and v_t are $(1,2)$-unlinked. We apply a $(1,2)$-swap at v_t. Denote the new coloring by φ^{**}. The coloring φ^{**} is $V(F)$-stable with $\{1,\tau\} \subseteq \overline{\varphi}^{**}(x)$. Furthermore, we may still assume that $r \in P_{s_1}(\tau,\Delta,\varphi^{**}) = P_x(\tau,\Delta,\varphi^{**})$ and $r \in P_{s_1}(2,\tau,\varphi^{**}) = P_x(2,\tau,\varphi^{**})$. However, the τ-sequence $(v_1,\ldots,v_t)$ is of type B such that $\overline{\varphi}^*(v_t) = 2$ now, showing a contradiction to Lemma 2.14(v). Therefore, the τ-sequence and the λ-sequence intersect. Assume that $v_i = w_j$ is the first common vertex of the two sequences. Then, the two sequences are identical after this vertex.

If both i,j are at least two, then $\overline{\varphi}^*(v_{i-1}) = \overline{\varphi}^*(w_{j-1})$, name it γ. By the definition of τ-sequence, $\gamma \in [1,\Delta] \setminus \overline{\varphi}^*(F)$. Since F is maximum, $r \in P_{s_1}(\gamma,\Delta)$ by Lemma 2.9. One of v_{i-1} and w_{j-1}, say v_{i-1}, is not on $P_{s_1}(\gamma,\Delta)$. We apply a (γ,Δ)-swap at v_{i-1} and get a coloring φ^{**}. The condition of Lemma 2.14(v) is satisfied by φ^{**}, but the current τ-sequence $(v_1,\ldots,v_{i-1})$ ends with a vertex missing color Δ rather than 1, giving a contradiction to Lemma 2.14(v). Therefore one of i and j is 1. Assume, without loss of generality, that $j = 1$, i.e., $\lambda = \overline{\varphi}^*(v_{i-1})$ and $w_1,\ldots,w_k$ is the same as $v_i,\ldots,v_t$. We first apply a $(1,2)$-swap at both x and v_t. Since $r \in P_{s_1}(\lambda,\Delta,\varphi^*) = P_x(\lambda,\Delta,\varphi^*)$, we apply a (λ,Δ)-swap at v_{i-1} to get a new coloring φ^{**}. Again, the condition of Lemma 2.14(v) is satisfied by φ^{**}, but the current τ-sequence $(v_1,\ldots,v_{i-1})$ ends at a vertex missing color Δ rather than 1, giving a contradiction to Lemma 2.14(v). $\square$

5. Proof of Theorem 1.6

We introduce some new concepts in order to prove Theorem 1.6.

5.1. Pseudo-fan

Let G be a class 2 graph and rs_1 be a critical edge. A *pseudo-fan* (P-fan) at r w.r.t. rs_1 and a coloring $\varphi \in \mathcal{C}^\Delta(G - rs_1)$ is a sequence

$$S = S_\varphi(r, s_1 : s_t : s_p) = (r, rs_1, s_1, rs_2, s_2, \ldots, rs_t, s_t, rs_{t+1}, s_{t+1}, \ldots, s_{p-1}, rs_p, s_p)$$

such that all $s_1, \ldots, s_p$ are distinct vertices in $N_{\Delta-1}(r)$ and the following conditions hold:

(P1) $(r, rs_1, s_1, rs_2, s_2, \ldots, rs_t, s_t)$, denoted by $F_\varphi(r, s_1 : s_t)$, is a maximum multifan at r.

(P2) The vertex set $V(S)$ is φ'-elementary for every F-stable φ' w.r.t. φ.

Clearly every maximum multifan is a P-fan, and if S is a P-fan w.r.t. φ and $F = F_\varphi(r, s_1 : s_t)$, then by the definition above, S is also a P-fan w.r.t. every F-stable coloring φ'. The result below is a modification of Lemma 3.6 from [1].

Lemma 5.1. *Let G be a class 2 graph with maximum degree Δ, $r \in V_\Delta$ be light, and $S = S_\varphi(r, s_1 : s_p : s_q)$ be a P-fan w.r.t. rs_1 and a coloring $\varphi \in \mathcal{C}^\Delta(G - rs_1)$. Then the following two statements hold, where $F = F_\varphi(r, s_1 : s_p)$.*

(a) For every $v_1 \in V(S) \setminus V(F)$, the $\varphi(rv_1)$-sequence $(v_1, \ldots, v_t)$ is a rotation at r, and v_i and r are $(1, \overline{\varphi}(v_i))$-linked for each $i \in [1, t]$.

(b) For any i, j with $i \in [1, p]$ and $j \in [p + 1, q]$ and colors $\gamma \in \overline{\varphi}(s_i)$ and $\delta \in \overline{\varphi}(s_j)$, $r \in P_{s_i}(\gamma, \delta) = P_{s_j}(\gamma, \delta)$. Moreover, if $\varphi(rz) = \gamma$ for some $z \in N(r)$, then $P_{s_i}(\gamma, \delta)$ meets z before r.

Proof. By relabeling colors and vertices, we assume F is typical. Let $F = F_\varphi(r, s_1 : s_\alpha : s_\beta)$ be a typical multifan, where $\beta = p$.

For Statement (a), we let $v_1 \in V(S) \setminus V(F)$, and let $\varphi(rv_1) = \tau$. Since F is maximum, by Lemma 2.8, we let $(v_1, \ldots, v_t)$ be the τ-sequence at r. We show first that the sequence is a rotation or a type A sequence. We may assume the sequence is of type B or C. If $(v_1, \ldots, v_t)$ is of type B, i.e., $\overline{\varphi}(v_t) = \gamma \in \overline{\varphi}(F)$, since $\overline{\varphi}_F^{-1}(\gamma)$ and r are $(1, \gamma)$-linked, we apply a $(1, \gamma)$-swap at v_t to get φ'. Then we apply the B-shifting from v_1 to v_t and exchange the role of 1 and τ in the entire graph. This results in an F-stable coloring, yet $V(S)$ is not elementary, contradicting (P2) of the definition of a P-fan. If $(v_1, \ldots, v_t)$ is of type C, i.e., $\overline{\varphi}(v_t) = \overline{\varphi}(v_{i-1}) = \tau_i$ for some $i \in [2, t]$ and some $\tau_i \in [1, \Delta] \setminus \overline{\varphi}(F)$, since one of v_{i-1} and v_t is $(1, \tau_i)$-unlinked with r, we apply a $(1, \tau_i)$-swap at a vertex in $\{v_{i-1}, v_t\}$ that is $(1, \tau_i)$-unlinked with r. This gives an F-stable coloring such that the corresponding τ-sequence is of type B, converting the problem to the previous case. Thus the τ-sequence $(v_1, \ldots, v_t)$ is a rotation. Moreover, v_i and r are $(1, \overline{\varphi}(v_i))$-linked for each $i \in [1, t]$. As otherwise, a $(1, \overline{\varphi}(v_i))$-swap at v_i would give rise to a type B τ-sequence, contradicting what was proved above. The proof of Statement (a) is now complete.

By Statement (a), we let $(v_1, \ldots, v_t)$ be the rotation containing s_j, where $v_1 = s_j$. For the first part of Statement (b), suppose to the contrary that $r \in P_{s_i}(\gamma, \delta) = P_{v_1}(\gamma, \delta)$ does not hold. Assume without loss of generality that $i \in [1, \alpha]$. Then we have the following three cases: $r \notin P_{s_i}(\gamma, \delta)$ and $r \notin P_{v_1}(\gamma, \delta)$; $r \notin P_{s_i}(\gamma, \delta)$ and $r \in P_{v_1}(\gamma, \delta)$; and $r \in P_{s_i}(\gamma, \delta)$ and $r \notin P_{v_1}(\gamma, \delta)$.

Suppose first that $r \notin P_{s_i}(\gamma, \delta)$ and $r \notin P_{v_1}(\gamma, \delta)$. Then let $\varphi' = \varphi/Q$, where Q is the (γ, δ)-chain containing r. Note that φ' and φ agree on every edge incident to r except two edges rv_2 and rz where z is the vertex in $N(r)$ such that $\varphi(rz) = \gamma$. Since $r \notin P_{s_i}(\gamma, \delta)$, $r \notin P_{v_1}(\gamma, \delta)$ and $V(S)$ is φ-elementary, $\overline{\varphi}'(s_i) = \overline{\varphi}(s_i)$ for all $s_i \in V(S)$. Thus under the new coloring φ', $F^* = (r, rs_1, s_1, \ldots, s_i, rv_2, v_2, \ldots, rv_t, v_t, rv_1, v_1, rs_{i+1}, s_{i+1}, \ldots, s_\beta)$ is a multifan. This is because, if $i < \alpha$, then $\overline{\varphi}'(s_i) = \gamma = \varphi'(rv_2)$ and $\overline{\varphi}'(v_1) = \delta = \varphi'(rs_{i+1})$; and if $i = \alpha$, then $\varphi'(s_{i+1}) = \Delta \in \overline{\varphi}'(s_1)$. As $|V(F)| < |V(F^*)|$, we obtain a contradiction to the maximality assumption of F. Suppose then that $r \notin P_{s_i}(\gamma, \delta)$ and $r \in P_{v_1}(\gamma, \delta)$. Then let $\varphi' = \varphi/P_{v_1}(\gamma, \delta)$. Similar to the case above, one can easily check that $F^* = (r, rs_1, s_1, \ldots, s_i, rv_2, v_2, \ldots, rv_t, v_t, rv_1, v_1)$ is a multifan. Since $\overline{\varphi}'(s_i) = \overline{\varphi}'(v_1) = \gamma$, we obtain a contradiction to Lemma 2.2 that $V(F^*)$ is φ'-elementary. Suppose lastly that $r \in P_{s_i}(\gamma, \delta)$ and $r \notin P_{v_1}(\gamma, \delta)$. Then let $\varphi' = \varphi/P_{v_1}(\gamma, \delta)$. Note that φ' is F-stable w.r.t. φ, thus by the definition of a P-fan, $V(S)$ is φ'-elementary. But $\overline{\varphi}'(s_i) = \overline{\varphi}'(v_1) = \gamma$, a contradiction. This completes the proof of the first part of Statement (b).

For the second part of Statement (b), assume to the contrary that $P_{s_i}(\gamma, \delta)$ meets r before z. Then $P_{s_i}(\gamma, \delta)$ meets v_2 before r. Let φ' be obtained from φ by shifting from v_1 to v_t. Then $r \notin P_{s_i}(\delta, \gamma, \varphi')$, showing a contradiction to the first part of Statement (b). □

5.2. Two structural lemmas

Lemma 5.2. *Let G be a class 2 graph with maximum degree $\Delta \geq 3$, $r \in V_\Delta$ be light, and rs_1 be a critical edge. If $S = S(r, s_1 : s_p : s_q)$ is a P-fan w.r.t. rs_1 and a coloring $\varphi \in \mathcal{C}^\Delta(G - rs_1)$, then for any $x \in N(V(S)) \setminus N[r]$, $d(x) \neq \Delta - 1$.*

Proof. Suppose to the contrary that there is a degree $(\Delta - 1)$ vertex $x \notin N[r]$ and a vertex $s^* \in S$ such that $x \sim s^*$. Let $F = F(r, s_1 : s_\alpha : s_\beta)$ be the maximum multifan contained in S. We further assume that F is typical. Since rs_1 is a critical edge of G, every edge of F is a critical edge of G. Thus by Theorem 1.4, $s^* \in V(S) \setminus V(F)$.

We may first assume $1 \in \overline{\varphi}(x)$. To see this, let $\tau \in \overline{\varphi}(x)$. If $\tau \in \overline{\varphi}(F)$, since $\overline{\varphi}_F^{-1}(\tau)$ and r are $(1, \tau)$-linked by Lemma 2.3(a), we simply apply a $(1, \tau)$-swap at x. Thus we assume that $\tau \in [1, \Delta] \setminus \overline{\varphi}(F)$. By Lemma 2.14(i), there is an F-stable coloring such that 1 is missing at x. We then apply a $(1, \Delta)$-swap at x, still call the resulting coloring φ. We now have $\Delta \in \overline{\varphi}(x)$.

We claim that there is a $V(F)$-stable coloring, still call it φ, such that $\varphi(s^*x) \in \{2, \Delta\}$. Let $\varphi(s^*x) = \tau$. Assume first that $\tau \in \overline{\varphi}(F)$. If τ is not Δ-inducing, we simply apply a (τ, Δ)-swap at x. Otherwise, we do $(\Delta, 1) - (1, 2) - (2, \tau)$-swaps at x, and get a desired $V(F)$-stable coloring. Thus, we may assume $\tau \in [1, \Delta] \setminus \overline{\varphi}(F)$. For every $V(F)$-stable and

$\{\tau, \Delta\}$-avoiding coloring φ', since F is maximum, $r \in P_{s_1}(\tau, \Delta, \varphi')$ (by Lemma 2.9). We claim $P_x(\tau, \Delta, \varphi') = P_{s_1}(\tau, \Delta, \varphi')$. Otherwise, a (τ, Δ)-swap at x gives a desired coloring. Applying Lemma 2.14(iii), the τ-sequence $(v_1, \dots, v_t)$ is of type B and $\overline{\varphi}(v_t) = \Delta$. Since $r \in P_{s_1}(\tau, \Delta, \varphi) = P_x(\tau, \Delta, \varphi)$, we apply a (τ, Δ)-swap at v_t to get an F-stable coloring, and then do the A-shifting from v_1 to v_t. Under the new coloring, $P_{s_1}(\tau, \Delta) \neq P_x(\tau, \Delta)$. Since still $r \in P_{s_1}(\tau, \Delta)$ by Lemma 2.9, we apply a (τ, Δ)-swap at x to get a desired $V(F)$-stable coloring. So we may assume $\varphi(s^*x) = \Delta$.

We then show that there is a $V(F)$-stable coloring, still call it φ, such that $\varphi(s^*x) = \Delta$ and $1 \in \overline{\varphi}(x)$. Let $\tau \in \overline{\varphi}(x)$. If $\tau \in \overline{\varphi}(V(S))$, by Lemma 2.3(a) and Lemma 5.1(a), we simply apply a $(1, \tau)$-swap at x. Thus $\tau \in [1, \Delta] \setminus \overline{\varphi}(S)$. We may further assume that there is no F-stable and $\{\Delta\}$-avoiding coloring φ' such that $1 \in \overline{\varphi}'(x)$. In particular, we have $P_x(1, \tau, \varphi) = P_r(1, \tau, \varphi)$. By Lemma 2.14(ii), the τ-sequence $(v_1, \dots, v_t)$ at r is of type B such that $\overline{\varphi}(v_t) = \Delta$. Let $\overline{\varphi}(s^*) = \delta$. As $V(S)$ is elementary and $\Delta \in \overline{\varphi}(s_1)$, we have $v_t \notin V(S)$, and so $s^* \neq v_t$. We also note that $\delta \neq \tau$. Otherwise, by Lemma 5.1(a), $P_{s^*}(1, \tau, \varphi) = P_r(1, \tau, \varphi)$, which gives a contradiction to $P_x(1, \tau, \varphi) = P_r(1, \tau, \varphi)$. By Lemma 5.1(b), $r \in P_{s_1}(\delta, \Delta, \varphi) = P_{s^*}(\delta, \Delta, \varphi)$. We apply a (δ, Δ)-swap at v_t and get an F-stable coloring φ^* with δ missing at v_t. Applying Lemma 5.1(a) to $s^* \in V(S)$, we get $P_r(1, \delta, \varphi^*) = P_{s^*}(1, \delta, \varphi^*)$. We apply a $(1, \delta)$-swap at v_t. Note that by Lemma 5.1(a), the $\varphi(rs^*)$-sequence containing s^* at r is a rotation, thus $s^* \notin \{v_1, \dots, v_t\}$. We apply the B-shifting from v_1 to v_t followed by switching color 1 and τ for the entire graph, which results in a desired $V(F)$-stable coloring.

Hence, we may assume that $\varphi(s^*x) = \Delta$, $1 \in \overline{\varphi}(x)$, and $\overline{\varphi}(s^*) = \delta$. By Lemma 5.1(a) that $P_r(1, \delta, \varphi) = P_{s^*}(1, \delta, \varphi)$, we apply a $(1, \delta)$-swap at x. Under the new coloring, $P_{s^*}(\delta, \Delta) = s_1x$, showing a contradiction to the fact that s^* and s_1 are (δ, Δ)-linked (Lemma 5.1(b)). $\square$

Lemma 5.3. *Let G be a class 2 graph with maximum degree $\Delta \geq 3$, $r \in V_{\Delta-1}$ be light, and F be a multifan at r w.r.t. edge rs_1 and a coloring $\varphi \in \mathcal{C}^\Delta(G - rs_1)$. If F is maximum, then $\overline{\varphi}(r) \nsubseteq \overline{\varphi}(x)$ for any $x \in V(G) \setminus N[r]$ with $(N(x) \cap N(s_1)) \setminus N_{\Delta-1}[r] \neq \emptyset$.*

Proof. Suppose to the contrary that there exists a vertex $x \in V(G) \setminus N[r]$ such that $(N(x) \cap N(s_1)) \setminus N_{\Delta-1}[r] \neq \emptyset$ and $\overline{\varphi}(r) \subseteq \overline{\varphi}(x)$. Let $u \in (N(x) \cap N(s_1)) \setminus N_{\Delta-1}[r]$, $\overline{\varphi}(r) = \{1, \Delta - 1\}$ and $\overline{\varphi}(s_1) = \{2, \Delta\}$. So, $\{1, \Delta - 1\} \subseteq \overline{\varphi}(x)$. Our goal is to modify φ in getting a $V(F)$-stable coloring φ' such that $K = (r, rs_1, s_1, s_1u, u, ux, x)$ is a Kierstead path but $\overline{\varphi}'(x) \cap (\overline{\varphi}'(s_1) \cup \overline{\varphi}'(r)) \neq \emptyset$, in achieving a contradiction to Lemma 2.5. Since r is light, we may assume that $F = F(r, s_1 : s_\alpha : s_\beta)$ is typical.

By applying $(1, 2)$- and $(\Delta - 1, \Delta)$-swaps at x when it is necessary, we may assume that $2, \Delta \in \overline{\varphi}(x)$. Applying Lemma 4.3(ii) and then Lemma 4.3(iii), we may assume that there is a $V(F)$-stable coloring, still denoted by φ, such that $\varphi(s_1u) = 1$ and $\Delta \in \overline{\varphi}(x)$. We show next that there is a $V(F)$-stable coloring, still denoted by φ, such that $\varphi(s_1u) = 1$ and $\varphi(ux) = \Delta$.

Let $\varphi(ux) = \tau$. Suppose first that $\tau \in \overline{\varphi}(F)$. If τ is not Δ-inducing, we apply a (τ, Δ)-swap at x in getting a desired $V(F)$-stable coloring. If τ is Δ-inducing, we apply $(\Delta, 1) - (1, 2) - (2, \tau)$-swaps at x in getting a desired $V(F)$-stable coloring. Suppose then that $\tau \in [1, \Delta] \setminus \overline{\varphi}(F)$. We claim that for every $V(F)$-stable and $\{1, \tau, \Delta\}$-avoiding coloring φ', $P_x(\tau, \Delta, \varphi') = P_{s_1}(\tau, \Delta, \varphi')$. Otherwise, since F is maximum, $r \in P_{s_1}(\tau, \Delta, \varphi')$ by Lemma 2.9. Then the (τ, Δ)-swap at x gives a $V(F)$-stable coloring φ^* such that $\varphi^*(ux) = \Delta$ and $\varphi^*(s_1u) = 1$, which is what we want. By Lemma 2.14(vi), we may assume that the τ-sequence $(v_1, \dots, v_t)$ is of type B such that $\overline{\varphi}(v_t) \in \{1, \Delta\}$ or is 2-inducing. If $\overline{\varphi}(v_t) = 1$, we apply a $(1, 2)$-swap at v_t, so the color missing at v_t is 2-inducing. Thus we only need to consider two cases: either $\overline{\varphi}(v_t)$ is 2-inducing or $\overline{\varphi}(v_t) = \Delta$. If $\overline{\varphi}(v_t)$ is 2-inducing, let $\overline{\varphi}(v_t) = \gamma$ for some $\gamma \in \overline{\varphi}(F)$, we apply a $(\gamma, \Delta - 1)$-swap at v_t, where $\Delta - 1$ is another color missing at r. Then we apply the B-shifting from v_1 to v_t and get a $V(F - r)$-stable coloring φ'. In particular, we have $\tau \in \overline{\varphi}'(r)$. Since $\varphi'(s_1u) = 1 \in \overline{\varphi}'(r)$ and $\varphi'(ux) = \tau \in \overline{\varphi}'(r)$, $K = (r, rs_1, s_1, s_1u, u, ux, x)$ is a Kierstead path. But Δ is missing at both s_1 and x, achieving a contradiction to Lemma 2.5. Thus $\overline{\varphi}(v_t) = \Delta$. Since $r \in P_{s_1}(\tau, \Delta, \varphi) = P_x(\tau, \Delta, \varphi)$, we apply a (τ, Δ)-swap at v_t, resulting in a type A τ-sequence. Then the A-shifting from v_1 to v_t gives a $V(F)$-stable coloring φ' such that x and s_1 are (τ, Δ)-unlinked. Since still $r \in P_{s_1}(\tau, \Delta, \varphi')$ by Lemma 2.9, we apply a (τ, Δ)-swap at x in getting a desired coloring.

Therefore we assume that $\varphi(s_1u) = 1$ and $\varphi(ux) = \Delta$. Since $\varphi(s_1u) = 1 \in \overline{\varphi}(r)$ and $\varphi(ux) = \Delta \in \overline{\varphi}(s_1)$, $K = (r, rs_1, s_1, s_1u, u, ux, x)$ is a Kierstead path. We next show that there is a $V(F)$-stable coloring φ' keeping the Kierstead path but $\overline{\varphi}'(x) \cap (\overline{\varphi}'(s_1) \cup \overline{\varphi}'(r)) \neq \emptyset$, which gives a contradiction to Lemma 2.5.

Let $\tau \in \overline{\varphi}(x)$. If $\tau \in \overline{\varphi}(F)$, we simply apply a $(\tau, \Delta - 1)$-swap at x to get a contradiction. Thus, $\tau \in [1, \Delta] \setminus \overline{\varphi}(F)$. We claim that for any $V(F)$-stable and $\{1, \tau, \Delta\}$-avoiding coloring φ', $P_x(2, \tau, \varphi') = P_{s_1}(2, \tau, \varphi')$. Otherwise, since F is maximum, by Lemma 2.9 we have $r \in P_{s_1}(2, \tau, \varphi')$. Then the $(2, \tau)$-swap at x gives a $V(F)$-stable coloring that maintains the Kierstead path, but 2 is missing at both x and s_1, a contradiction. By Lemma 2.14(vii), we may assume that the τ-sequence $(v_1, \dots, v_t)$ is of type B such that $\overline{\varphi}(v_t) \in \{1, \Delta\}$ or is 2-inducing. If $\overline{\varphi}(v_t) = 1$, we apply a $(1, 2)$-swap at v_t. Thus we only need to consider two cases where $\overline{\varphi}(v_t) \neq 1$. If $\overline{\varphi}(v_t)$ is 2-inducing, let $\overline{\varphi}(v_t) = \gamma$ for some $\gamma \in \overline{\varphi}(F)$, we apply a $(\gamma, \Delta - 1)$-swap at v_t and then apply the B-shifting from v_1 to v_t. Now $K = (r, rs_1, s_1, s_1u, u, ux, x)$ is a Kierstead path but τ is missing at both r and x, achieving a contradiction to Lemma 2.5. Thus $\overline{\varphi}(v_t) = \Delta$. Now applying $(\Delta, 1) - (1, 2) - (2, \Delta - 1)$-swaps at v_t and then the B-shifting from v_1 to v_t gives the same contradiction as right before. □

5.3. Proof of Theorem 1.6

Since all vertices not missing a given color α are saturated by the matching that consists of all edges colored by α in G, we have the following result.

Lemma 5.4 *(Parity Lemma). Let G be an n-vertex graph and $\varphi \in \mathcal{C}^\Delta(G)$. Then for any color $\alpha \in [1, \Delta]$, $|\{v \in V(G) : \alpha \in \overline{\varphi}(v)\}| \equiv n \pmod 2$.*

Theorem 2.5. *Let G be a Δ-critical graph with n vertices. If G has a light Δ-vertex and $\Delta > n/2 + 1$, then n is odd.*

Proof. Let r be a light Δ-vertex of G. Recall that $N(r) = N_\Delta(r) \cup N_{\Delta-1}(r)$. We prove first that $d(x) = \Delta$ for every $x \in V(G) \setminus N[r]$. Assume to the contrary that there exists $x \in V(G) \setminus N[r]$ with $d(x) \le \Delta - 1$. If $d(x) \ge \Delta - 2 \ge (n-1)/2$, since $d(r) = \Delta \ge (n+3)/2$, we get $|N(r) \cap N(x)| \ge d(r) + d(x) - |N(x) \cup N(r)| \ge (n+1) - (n-2) = 3$. Since $|N_\Delta(r)| = 2$, there exists $s \in N_{\Delta-1}(r)$ such that $x \sim s$. Since G is Δ-critical, rs is a critical edge of G. But this gives a contradiction to Theorem 1.4. Thus $d(x) \le \Delta - 3$. Then for any $u \in N_\Delta(x)$, there exists $s \in N_{\Delta-1}(r)$ such that $u \sim s$. Since every neighbor of r from $N(r) \setminus N_\Delta(r)$ has degree $\Delta - 1$ and $d(u) = \Delta$, we have $u \notin N(r) \setminus N_\Delta(r)$. Again, using that rs is a critical edge of G, we obtain a contradiction to Theorem 1.5.

Assume to the contrary that n is even. We first claim that $|N(s) \cap N_{\Delta-1}(r)| \le \frac{\Delta-4}{2}$ for any $s \in N_{\Delta-1}(r)$. Let $s \in N_{\Delta-1}(r)$, $\varphi \in \mathcal{C}^\Delta(G - rs)$, and $X \subseteq N_{\Delta-1}[r]$ be a largest φ-elementary set that contains r and s. By the Parity Lemma, every color from $\overline{\varphi}(X)$ is missing at another vertex from $V(G) \setminus X$. Since all vertices in $V(G) \setminus N_{\Delta-1}(r)$ are of maximum degree, we have $|N_{\Delta-1}(r) \setminus X| \ge |\overline{\varphi}(N_{\Delta-1}(r) \setminus X)| \ge |\overline{\varphi}(X)| = |X| + 1$. On the other hand, we have $|N_{\Delta-1}(r) \setminus X| + |X \setminus \{r\}| = \Delta - 2$. Combining the two formulas above, we get $|X| \le \frac{\Delta-2}{2}$. Thus $|\overline{\varphi}(X)| = |X| + 1 \le \frac{\Delta}{2}$. Since φ is an edge coloring, in $G - rs$, all colors on edges incident with r are distinct and distinct from the color missing at r. Therefore, there are at most $\frac{\Delta-2}{2}$ edges rs' with $s' \in N(r)$ such that $\varphi(rs')$ is missing at a vertex of G. Those edges include ru_1, ru_2 for $u_1, u_1 \in N_\Delta(r)$, as $\varphi(ru_1)$ and $\varphi(ru_2)$ are missed at vertices from a maximum multifan at r with respect to rs and φ. Let $Y = \{x \in N(r) : \varphi(rx)$ presents at every vertex of $G\}$. Then $Y \subseteq N_{\Delta-1}(r)$ and $|Y| \ge \Delta - 1 - \frac{\Delta-2}{2} \ge \frac{\Delta}{2}$. Now to show $|N(s) \cap N_{\Delta-1}(r)| \le \frac{\Delta-4}{2}$, it suffices to show that $N(s) \cap Y = \emptyset$. For otherwise, if there exists $x \in N(s) \cap Y$, let G_1 be obtained from $G - rs$ by deleting all the edges colored by $\varphi(rx)$. Then G_1 is still a class 2 graph and r is a light $\Delta(G_1)$-vertex in G_1, and $\varphi \in \mathcal{C}^{\Delta-1}(G_1 - rs)$. However, we have $d_{G_1}(x) = \Delta(G_1) - 1$ but $x \in N_{G_1}(s) \setminus N_{G_1}(r)$, contradicting Theorem 1.4.

Let $N_{\Delta-1}(r) = \{s_1, \ldots, s_{\Delta-2}\}$, $\varphi \in \mathcal{C}^\Delta(G - rs_1)$, and let X be a largest φ-elementary set that contains r and s_1 such that $X \subseteq N_{\Delta-1}[r]$. By the same argument as above, we have $|X| \le \frac{\Delta-4}{2}$. Since $|N_{\Delta-1}(r)| = \Delta - 2$, there exists a vertex $x \in N_{\Delta-1}(r)$ such that the color $\tau = \varphi(rx)$ is presented at every vertex of G. Let G_1 be obtained from G by deleting all the edges colored by τ. Then G_1 is still a class 2 graph such that r is a light maximum degree vertex, and $\varphi \in \mathcal{C}^{\Delta-1}(G_1 - rs_1)$. As $\Delta(G_1) = \Delta - 1 \ge n/2 + 1$, there exists $s^* \in N_{G_1}(r)$ with $d_{G_1}(s^*) = \Delta(G_1) - 1$ such that $x \sim s^*$ in G_1. Note that G_1 is still a class 2 graph, and φ, being restricted on G_1, is a $\Delta(G_1)$-coloring of G_1. Let $F_\varphi(r, s_1 : s_\alpha : s_\beta)$ be a maximum typical multifan at r and S be a maximum P-fan containing F. If $s^* \in V(S)$, then we obtain a contradiction to Lemma 5.2. Thus

$s^* \notin V(S)$. Since $V(S)$ is a largest P-fan containing F, there is a $V(F)$-stable coloring φ such that $V(S) \cup \{s^*\}$ is not φ-elementary. Since $V(S)$ is φ-elementary by the definition of S, $\overline{\varphi}(s^*) \in \overline{\varphi}(S)$. As for every $\gamma \in \overline{\varphi}(S) \setminus \overline{\varphi}(r)$, $\overline{\varphi}_F^{-1}(\gamma)$ and r are $(1, \gamma)$-linked by Lemma 2.3(a) and Lemma 5.1(a), we apply a $(1, \overline{\varphi}(s^*))$-swap at s^*. Let $\varphi(rs^*) = \delta$.

If rs^* is a critical edge of G_1, then we already reach a contradiction to Theorem 1.4. Thus, rs^* is not a critical edge of G_1. We let $G_2 = G_1 - rs^*$. Note that G_2 is still a class 2 graph with $r \in V_{\Delta(G_2)-1}$ being a light vertex. The coloring φ, being restricted on G_2, is a $\Delta(G_2)$-coloring of G_2, and $F_\varphi(r, s_1 : s_\alpha : s_\beta)$ is still a maximum typical multifan at r. By the choice of φ before, we have $\overline{\varphi}(r) = \overline{\varphi}(s^*) = \{1, \delta\}$.

Since s_1 is adjacent in G_2 to at most $\frac{\Delta-4}{2}$ vertices from $\{s_1, \ldots, s_{\Delta-2}\}$, and $d_{G_2}(s_1) \geq \Delta - 2$, s_1 is adjacent in G_2 to at least $\Delta/2 - 1$ vertices from $V(G) \setminus \{r, s_1, \ldots, s_{\Delta-2}\}$. Similarly, $d_{G_2}(s^*) = \Delta(G_2) - 2 = \Delta - 3$, s^* is adjacent in G_2 to at most $\frac{\Delta-4}{2}$ vertices from $\{s_1, \ldots, s_{\Delta-2}\}$, and $s^* \nsim r$, it follows that s^* is adjacent in G_2 to at least $\Delta/2 - 1$ vertices from $V(G) \setminus \{r, s_1, \ldots, s_{\Delta-2}\}$. Since $\Delta \geq n/2 + 2$, $|V(G) \setminus \{r, s_1, \ldots, s_{\Delta-2}\}| \leq n/2 - 1$. As $2(\Delta/2 - 1) \geq n/2$, there exists $u \in (N_{G_2}(s_1) \cap N_{G_2}(s^*)) \setminus \{r, s_1, \ldots, s_{\Delta-2}\}$. Since $\overline{\varphi}(r) = \overline{\varphi}(s^*) = \{1, \delta\}$, we obtain a contradiction to Lemma 5.3. □

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