Journal of Combinatorial Theory, Series B 150 (2021) 177–194

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

Laminar tight cuts in matching covered graphs ☆

Guantao Chen [a], Xing Feng [b], Fuliang Lu [c], Cláudio L. Lucchesi [d], Lianzhu Zhang [e]

[a] *Department of Mathematics and Statistics, Georgia State University, Atlanta, GA 30303, United States of America*
[b] *Faculty of Science, Jiangxi University of Science and Technology, Ganzhou 341000, China*
[c] *School of Mathematics and Statistics, Minnan Normal University, Zhangzhou 363000, China*
[d] *Institute of Computing, UNICAMP, Campinas, Brazil*
[e] *School of Mathematical Science, Xiamen University, Xiamen 361005, China*

A R T I C L E   I N F O

Article history:
Received 21 September 2019
Available online 3 June 2021

Keywords:
Tight cut
ELP-cut
Perfect matching
Matching covered graph

A B S T R A C T

An edge cut C of a graph G is *tight* if $|C \cap M| = 1$ for every perfect matching M of G. Barrier cuts and 2-separation cuts are called *ELP-cuts*, which are two important types of tight cuts in matching covered graphs. Edmonds, Lovász and Pulleyblank proved that if a matching covered graph has a nontrivial tight cut, then it also has a nontrivial ELP-cut. Carvalho, Lucchesi, and Murty made a stronger conjecture: given any nontrivial tight cut C in a matching covered graph G, there exists a nontrivial ELP-cut D in G which does not cross C. We confirm the conjecture in this paper.

☆ Research Supported by NSF grant DMS-1855716, National Natural Science Foundation of China (Grant Nos. 12001250 and 11671186), The Program of Qingjiang Excellent Young Talents, Jiangxi University of Science and Technology, Natural Science Foundation of Fujian Province (2020J01795) and The Brazilian National Research Council (CNPq).

E-mail addresses: gchen@gsu.edu (G. Chen), fengxing_fm@163.com (X. Feng), flianglu@163.com (F. Lu), lucchesi@ic.unicamp.br (C.L. Lucchesi), zhanglz@xmu.edu.cn (L. Zhang).

https://doi.org/10.1016/j.jctb.2021.05.003
0095-8956/

1. Introduction

All graphs considered in this paper are finite and may contain multiple edges, but no loops. We will generally follow the notation and terminology used by Bondy and Murty in [1].

Let G be a graph with vertex set V and edge set E. For any $X \subseteq V$, let $N_G(X)$ be the set of vertices of $V - X$ which are adjacent to vertices of X and let $\overline{X} = V - X$ be the complement of X in V. The set of *coboundary edges* of X in G, denoted by $\partial_G(X)$, is the set of edges with exact one end in X and one end in $\overline{X}$. For notational simplicity, we denote $\partial_G(X)$ by $\partial(X)$ whenever G is understood. A *cut* is a coboundary edge set. We call X and $\overline{X}$ the *shores* of $\partial(X)$. A cut $\partial(X)$ is *trivial* if $|X| = 1$ or $|\overline{X}| = 1$.

Let $C := \partial(X)$ be a cut of G. We denote by $G/(X \to x)$ the graph obtained from G by contracting X to a single vertex x (and removing any resulting loops). The graphs $G/(X \to x)$ and $G/(\overline{X} \to \overline{x})$ are the two *C-contractions of* G. Let $H := G/(X \to x)$ and let e be an edge of $\partial_G(X)$. We shall denote by e, with a mild abuse of language, the corresponding edge in $\partial_H(x)$ and regard e as a label of the edge. Thus, we shall use the equality $\partial_G(X) = C = \partial_H(x)$.

Let $C := \partial(X)$ and $D := \partial(Y)$ be two cuts of G. We say that C and D *cross* if all the four sets $X \cap Y, \overline{X} \cap Y, X \cap \overline{Y}$ and $\overline{X} \cap \overline{Y}$ are nonempty, and are *laminar* otherwise. So, two cuts C and D are laminar if and only if one of the two shores of C is a subset of one of the shores of D. A collection of cuts is *laminar* if no two of its cuts cross.

A graph is called *matching covered* if it is connected, has at least one edge and each of its edges is contained in some perfect matching. Suppose that our graph G is matching covered. A cut C of G is *tight* if $|C \cap M| = 1$ for every perfect matching M of G. Clearly, every trivial cut is a tight cut. We call a matching covered graph which is free of nontrivial tight cuts a *brace* if it is bipartite, and a *brick* otherwise.

Let S be a set of vertices of G. A component of $G - S$ is *odd* if it consists of an odd number of vertices, is *even* if it consists of an even number of vertices, and is *trivial* if it consists of only one vertex. We denote by $o(G - S)$ the number of odd components of $G - S$.

We shall make use of the following known facts about matching covered graphs and tight cuts.

Theorem 1.1 *(Tutte [8]). A graph G has a perfect matching if and only if $o(G-S) \leq |S|$ for every subset S of $V(G)$.*

A *barrier* of G is a nonempty set B of vertices of G such that $o(G - B) = |B|$. Moreover, B is a *trivial barrier* if $|B| = 1$. The graph G is *bicritical* if $G - S$ has a perfect matching, for each pair S of vertices of G. If G is bicritical, then, by Tutte's theorem, every barrier of G is trivial.

Corollary 1.2. *Let G be a matching covered graph and let S be a subset of $V(G)$. Then, $o(G-S) \leq |S|$, with equality only if S is independent and $G-S$ has no even components.*

Proof. A matching covered graph has perfect matchings, hence the asserted inequality holds, by Theorem 1.1. Suppose that S is not independent, let $e := vw$ be an edge of G having both ends in S. As G is matching covered, it has a perfect matching, M, that contains the edge e. Let $T := S - \{v, w\}$. Then, $M - e$ is a perfect matching of $H := G - \{v, w\}$. By Theorem 1.1, $o(G - S) = o(H - T) \leq |T| = |S| - 2$. The inequality is thus strict in this case. Now suppose that $G - S$ has an even component, K. As G is connected, K has a vertex, x, which is adjacent to a vertex, y of S. Let $U := S + x$. Every odd component of $G - S$ is an odd component of $G - U$. In addition, $K - x$ has at least one odd component which is not a component of $G - S$. Moreover, the set U is not independent, hence $o(G - S) \leq o(G - U) - 1 \leq |U| - 3 = |S| - 2$. □

Proposition 1.3 *([4]). Let G be a matching covered graph and let $\partial(X)$ and $\partial(Y)$ be two tight cuts such that $|X \cap Y|$ is odd. Then, $\partial(X \cap Y)$ and $\partial(X \cup Y)$ are also tight in G. Furthermore, no edge connects $X \cap \overline{Y}$ to $\overline{X} \cap Y$.* □

Proposition 1.4 *([6]). Every matching covered graph on four or more vertices is 2-connected.* □

Proposition 1.5 *([5]). Let G be a matching covered graph, and let C be a tight cut of G. Then, both C-contractions are matching covered. Moreover, if G' is a C-contraction of G, then a tight cut of G' is also a tight cut of G. Conversely, if a tight cut of G is a cut of G', then it is also tight in G'.* □

Corollary 1.6. *Let G be a matching covered graph, and let $C = \partial(X)$ be a tight cut of G. Then, both shores X and $\overline{X}$ of C induce connected graphs.*

Proof. The C-contraction $G' := G/(X \to x)$ of G is matching covered, hence it is 2-connected, by Proposition 1.4. Thus, $G' - x$ is connected. In other words, $\overline{X}$ induces a connected subgraph of G. Likewise, X also induces a connected subgraph of G. □

If C is a tight cut of G, then both C-contractions of G are matching covered. A nontrivial tight cut may help us to reduce a matching covered graph to smaller matching covered graphs. We may apply to G a procedure, called a *tight cut decomposition* of G, which produces a list of bricks and braces. If G itself is a brick or a brace, then the list consists of just G. Otherwise, G has a nontrivial tight cut, C. Then, both C-contractions of G are matching covered. One may recursively apply the tight cut decomposition procedure to each C-contraction of G, and then combine the resulting lists to produce a tight cut decomposition of G itself.

Theorem 1.7 *(Lovász [5]). Any two applications of the tight cut decomposition procedure to G produce the same list of bricks and braces, up to multiple edges.* □

In particular, any two applications of the tight cut decomposition procedure yield the same number of bricks, which is called the *brick number* of G and denoted by $b(G)$.

1.1. ELP-cuts

There are two types of tight cuts that play a critical role in studying matching theory. Let G be a matching covered graph. By Tutte's theorem, each barrier B of G is an independent set, and all components of $G - B$ are odd components. A cut C is called a *barrier-cut* if there exists a barrier B and a component H of $G - B$ such that $C = \partial(V(H))$. Clearly, a barrier-cut is a tight cut.

A *2-separation* of G is a pair S of vertices of G such that $G - S$ is not connected and each of the components of $G - S$ is even. Let $\{u, v\}$ be a 2-separation of G, and let us divide the components of $G - \{u, v\}$ into two nonempty subgraphs G_1 and G_2. Each of the two cuts $C' := \partial(V(G_1) + u)$ and $C'' := \partial(V(G_1) + v)$ is a *2-separation cut* of G and the pair $\{C', C''\}$ is a *2-separation cut pair* of G.

Barrier-cuts and 2-separation cuts are particular types of tight cuts and are called *ELP-cuts*, named after Edmonds, Lovász, and Pulleyblank, who proved the following fundamental result.

Theorem 1.8 *(The ELP Theorem [4]). Every matching covered graph that has a nontrivial tight cut has a nontrivial barrier or a 2-separation.* □

Their proof is based on linear programming techniques. Szigeti [7] gave a purely graph theoretical proof. Carvalho, Lucchesi and Murty [3] provided an alternative proof.

1.2. Ultimate ELP-cuts

We now define a special type of tight cut decomposition procedure. We may apply to G a procedure, called an *ELP-cut decomposition* of G, which produces a list of bricks and braces. If G itself is a brick or a brace, then the list consists of just G. Otherwise, by the ELP Theorem, G has a nontrivial ELP-cut, C. Then, both C-contractions of G are matching covered. One may recursively apply the ELP-cut decomposition procedure to each C-contraction of G, and then combine the resulting lists to produce an ELP-cut decomposition of G itself. Each cut used in an ELP-cut decomposition procedure is said to be an *ultimate* ELP-cut. It is easy to see that the following conjecture implies that every nontrivial tight cut of G is an ultimate ELP-cut.

Conjecture 1.9. [Carvalho, Lucchesi and Murty [3]] *Let C be a nontrivial tight cut of a matching covered graph G. Then, G has a nontrivial ELP-cut which is laminar with C.*

In 2002, the three authors proved a particular and very important case of the Conjecture, in which G is a brick and e is an edge of G such that $G-e$ is matching covered and $b(G-e)=2$ [2]. In [3] they also proved the validity of the Conjecture for bicritical graphs.

In this paper we present a proof of a result, our Main Theorem, that implies Conjecture 1.9. To state the Main Theorem we need one more definition and a simple result.

Let G be a matching covered graph, let C be a tight cut of G and let S be a set of vertices of G which is either a barrier or a 2-separation. The set S is *C-sheltered* if S is a subset of a shore of C, and is *C-avoiding* if each ELP-cut associated with S is laminar with C. If S is C-sheltered, then, as each shore of C induces a connected subgraph of G (Corollary 1.6), one of the components of $G-S$ is a supergraph of H, where H the subgraph of G that induced by the shore, containing no vertices of S, of C. It follows that if S is C-sheltered, then S is C-avoiding. We record this result for later reference.

Proposition 1.10. *Let S be either a 2-separation or a barrier of a matching covered graph G and let C be a tight cut of G. If S is C-sheltered, then some cut associated with S has a shore that is a superset of a shore of C, say, X, and all the other cuts associated with S have a shore that is a subset of $\overline{X}$. Consequently, if S is C-sheltered, then it is C-avoiding.* □

We now state our result, which, in view of the proposition above, implies Conjecture 1.9.

Theorem 1.11 *(Main Theorem). Let C be a nontrivial tight cut of a matching covered graph G. Then, G has a C-sheltered nontrivial barrier or a 2-separation cut which is laminar with C (see Example 1.12).*

Example 1.12. Consider the graph depicted in Fig. 1. The tight cut C is laminar with the cut D, which is a 2-separation cut associated with the pair $\{u_1, u_2\}$. The cut C is also laminar with the 2-separation cut F, which is associated with the pair $\{1, b_3\}$. The barriers $\{b_1, b_2, b_3\}$ and $\{2, 3\}$ are C-sheltered, whereas the barrier $\{1, 2, 3\}$ is not C-avoiding.

The proof of the Main Theorem will be given in Section 3, after we present some necessary results in Section 2.

2. Ingredients

In this section, we prove three lemmas, which play a crucial role in the proof of the Main Theorem.

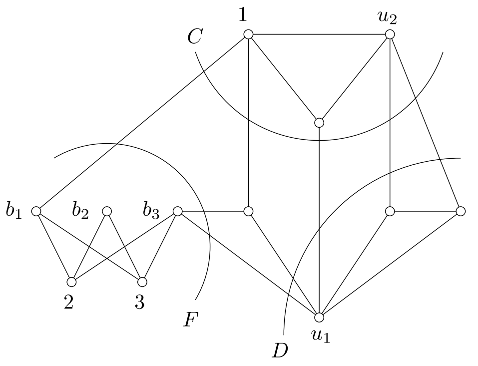

Fig. 1. The graph of Example 1.12.

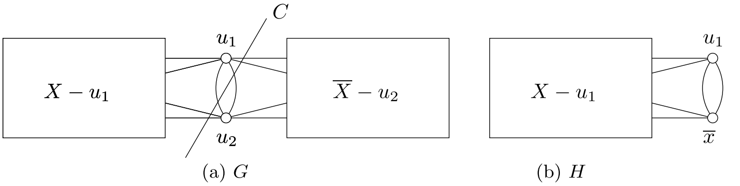

Fig. 2. Illustration for Lemma 2.1.

Lemma 2.1. *Let $C := \partial(X)$ be a 2-separation cut of a matching covered graph G, associated with a 2-separation $\{u_1, u_2\}$, where $u_1 \in X$ and $u_2 \in \overline{X}$. Let S_H denote either a 2-separation or a barrier of the C-contraction $H := G/(\overline{X} \to \overline{x})$ of G and let*

$$S := \begin{cases} S_H, & \text{if } \overline{x} \notin S_H, \\ (S_H - \overline{x}) + u_2, & \text{if } \overline{x} \in S_H. \end{cases}$$

Then, the following properties hold (see Fig. 2):

(i) *every component of $H - S_H$ that does not contain vertices in $\{u_1, \overline{x}\}$ is a component of $G - S$,*
(ii) *at most one component of $H - S_H$ contains vertices in $\{u_1, \overline{x}\}$, and*
(iii) *if S_H is a barrier of H, then S is a barrier of G and if S_H is a 2-separation of H, then S is a 2-separation of G.*

Proof. Let $\mathcal{H}$ denote the collection of components of $H - S_H$ and let $\mathcal{H}_0$ denote the collection of components of $H - S_H$ that contain at least one vertex in $\{u_1, \overline{x}\}$.

(i): Let $H[Y]$ be a component in $\mathcal{H} - \mathcal{H}_0$. (As indicated in Section 1, we shall make the abuse of language and identify edges in $\partial_G(\overline{X})$ with the corresponding edges in $\partial_H(\overline{x})$.) By the definition of $\mathcal{H}_0$, $Y \subseteq X - u_1$, hence $\partial_G(Y) = \partial_H(Y)$. For convenience, denote it by $\partial(Y)$. We shall now prove that $H[Y]$ is a component of $G-S$. It suffices to show that every edge of $\partial(Y)$ is incident with a vertex of S.

Let e be an edge of $\partial(Y)$. Then, e joins a vertex u in Y to a vertex v in S_H. If $v \neq \overline{x}$, then $v \in S$, by the definition of S. We may thus assume that $\overline{x} \in S_H$ and $v = \overline{x}$, which implies that $e \in C$. Every edge of C is incident in G with a vertex in $\{u_1, u_2\}$. Thus, either u_1 is an end of e in X or u_2 is an end of e in $\overline{X}$. As $Y \subseteq X - u_1$, the end of e in Y is not u_1. It follows that e is not incident with u_1, hence e is incident with u_2 in G. By the definition of S, u_2 is a vertex of S. Hence, we conclude that e is incident with a vertex of S in both alternatives. This conclusion holds for each edge $e \in \partial(Y)$.

(ii): The graph G is 2-connected (Proposition 1.4), hence the vertices u_1 and $\overline{x}$ are adjacent in the graph H. (ii) follows from the following stronger statement.

2.1.1. $|\mathcal{H}_0| \leq 1$, with equality if S_H is a barrier of H.

Proof. If neither u_1 nor $\overline{x}$ is in S_H, then, as u_1 and $\overline{x}$ are adjacent, it follows that $|\mathcal{H}_0| = 1$. Assume thus that at least one of u_1 and $\overline{x}$ is in S_H. In this case, the asserted inequality holds. Moreover, suppose that S_H is a barrier, then, by Corollary 1.2, S_H is independent. It follows that precisely one of the vertices in $\{u_1, \overline{x}\}$ is in S_H, hence the equality holds. □

(iii): By (i), all the components in $\mathcal{H} - \mathcal{H}_0$ are components of $G-S$. By (2.1.1), $|\mathcal{H}_0| \leq 1$, with equality if S_H is a barrier of H.

Consider first the case in which S_H is a barrier of H. Each of the $|S|-1$ (odd) components in $\mathcal{H} - \mathcal{H}_0$ is a component of $G-S$ by (i). By parity, $G-S$ has at least $|S|$ odd components. By Corollary 1.2, $G-S$ has precisely $|S|$ components, all of which are odd. Indeed, S is a barrier of G.

Finally, suppose that S_H is a 2-separation. Then, $H - S_H$ has only even components, one of which does not contain any vertex from $\{u_1, \overline{x}\}$, denoted by K. Thus, K is an even component of $G-S$ by (i). Moreover, K is a subgraph of $G[X-u_1]$, the subgraph of G induced by $X - u_1$, hence $G-S$ has two or more components. As K is even and G is matching covered, it follows that $G-S$ has only even components (Corollary 1.2). Indeed, S is a 2-separation of G. □

Let G be a matching covered graph, let $C := \partial(X)$ be a tight cut of G, let B denote a barrier of G and let $\mathcal{H}$ denote the set of components of $G-B$. For each shore Z of C, let $\mathcal{H}_Z$ be the set of those components $H \in \mathcal{H}$ such that $|V(H) \cap Z|$ is odd. As G is matching covered and B is a barrier of G, every component in $\mathcal{H}$ is odd. Thus, for each $H \in \mathcal{H}$, precisely one of $|V(H) \cap X|$ and $|V(H) \cap \overline{X}|$ is odd. Therefore, $|\mathcal{H}_X| + |\mathcal{H}_{\overline{X}}| = |\mathcal{H}| = |B|$. See Fig. 3.

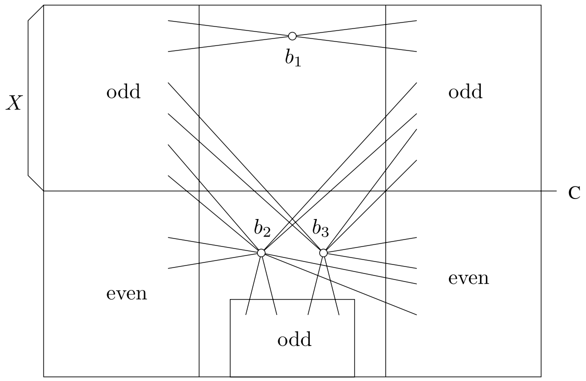

Fig. 3. A barrier $B := \{b_1, b_2, b_3\}$, where $|\mathcal{H}_X| = 2$ and $|\mathcal{H}_{\overline{X}}| = 1$.

Lemma 2.2. *Let G be a matching covered graph, let $C := \partial(X)$ be a tight cut of G, let B denote a barrier of G and let K be a component of $\mathcal{H}_X$ that contains a vertex adjacent to a vertex in $B \cap \overline{X}$. The following properties hold:*

(i) $|B \cap X| = |\mathcal{H}_X| - 1$ *and* $|B \cap \overline{X}| = |\mathcal{H}_{\overline{X}}| + 1$,
(ii) *each component in $\mathcal{H}_{\overline{X}}$ is a subgraph of $G[\overline{X}]$ and has no vertex adjacent to a vertex of $B \cap X$,*
(iii) *$B \cap \overline{X}$ is a (C-sheltered, possibly trivial) barrier of G, and*
(iv) *if C is nontrivial and B is C-avoiding and nontrivial, then $B \cap \overline{X}$ is a (C-sheltered) nontrivial barrier of G.*

Proof. The following simple statement is important in the proof of the lemma.

2.2.1. Let Z be a shore of C, let $H \in \mathcal{H}_Z$, let e be an edge in $\partial(V(H))$ incident with a vertex of $B \cap \overline{Z}$ and let M be a perfect matching of G that contains edge e. Then, the edge of $M \cap C$ has at least one end in H.

Proof. Let $e := vw$, $v \in V(H)$, $w \in B \cap \overline{Z}$. If $v \in Z$, then clearly e is the edge of $M \cap C$. Assume thus that $v \in \overline{Z}$. As $W := V(H) \cap \overline{Z}$ is even, it follows that $M \cap \partial(W)$ has another edge, f. The edge f cannot have an end in B, because H is matched by M to the end w of e. Thus, f has an end in $V(H) \cap Z$, hence f has both ends in H and is the edge of $M \cap C$. □

<u>(i)</u>: By hypothesis, K has a vertex, v, which is adjacent to a vertex, w, of $B \cap \overline{X}$. Let M be a perfect matching of G that contains edge vw. By (2.2.1), the edge of $M \cap C$, say, f, has at least one end in K. Clearly, either f has both ends in K or it is incident with a vertex of B. It follows that except for K, all the other components of $\mathcal{H}_X$ are matched by M to vertices of $B \cap X$ and every component of $\mathcal{H}_{\overline{X}}$ is matched by M to a vertex of $B \cap \overline{X}$. The asserted equality holds.

(ii): Assume, to the contrary, that $\mathcal{H}_{\overline{X}}$ has a component that contains a vertex adjacent to a vertex in $B \cap X$. From (i), with the roles of X and $\overline{X}$ interchanged, we deduce that

$$|B \cap \overline{X}| = |\mathcal{H}_{\overline{X}}| - 1 \quad \text{and} \quad |B \cap X| = |\mathcal{H}_X| + 1.$$

This is a contradiction to property (i). Let H be a component in $\mathcal{H}_{\overline{X}}$ and assume, to the contrary, that H has vertices in X. As H has an odd number of vertices in $\overline{X}$, it has an even number of vertices in X, hence $V(H) \cap X$ is a proper subset of X. The subgraph $G[X]$ of G is connected (Corollary 1.6), hence some vertex of $V(H) \cap X$ is adjacent to a vertex of $B \cap X$, a contradiction.

(iii): Let H be a component in $\mathcal{H}_{\overline{X}}$. From (ii), we deduce that H is a subgraph of $G[\overline{X}]$. Moreover, every edge of $\partial(V(H))$ is incident with a vertex of $B \cap \overline{X}$. We conclude that H is a component of $G - (B \cap \overline{X})$. This conclusion holds for each $H \in \mathcal{H}_{\overline{X}}$. From (i), we infer that the $|B \cap \overline{X}| - 1$ components in $\mathcal{H}_{\overline{X}}$ are (odd) components of $G - (B \cap \overline{X})$. By parity and Corollary 1.2, $B \cap \overline{X}$ is a barrier of G.

(iv): From (iii), we infer that $B \cap \overline{X}$ is a (C-sheltered) barrier of G. Suppose that C is nontrivial and B is C-avoiding and nontrivial. Assume, to the contrary, that $B \cap \overline{X}$ is trivial, let v denote the only vertex of $B \cap \overline{X}$. By (i), $\mathcal{H} = \mathcal{H}_X$.

By hypothesis, C is nontrivial, thus $\overline{X} - v$ is not empty. Consequently, some component of $G - B$, H, contains vertices of $\overline{X} - v$. Moreover, as $\mathcal{H} = \mathcal{H}_X$, the component H contains vertices in X. In sum, $V(H)$ contains vertices in both shores of C.

By hypothesis, B is nontrivial. As v is the only vertex of B in $\overline{X}$, it follows that B has vertices in X. Thus, $V(H)$ is not a superset of X. As v, a vertex of $\overline{X}$, is in B, it follows that $V(H)$ is not a superset of $\overline{X}$. Therefore, $V(H)$ is neither a subset nor a superset of any shore of C. Consequently, the cuts C and $\partial(V(H))$ cross, a contradiction to the hypothesis that B is C-avoiding. We conclude that $B \cap \overline{X}$ is nontrivial. □

Lemma 2.3. *Let $C := \partial(X)$ be a nontrivial tight cut of a matching covered graph G and let t be a vertex of $\overline{X}$. Suppose that the assertion of the Main Theorem holds for every graph having $|V(G)|$ or fewer vertices. Then, at least one of the following properties holds:*

(i) *the graph G has a 2-separation that does not contain the vertex t, and*
(ii) *G has a 2-separation, S, that contains the vertex t, associated with a cut $D := \partial(Y)$, such that $Y \subseteq \overline{X}$, and*
(iii) *the graph G has a C-sheltered nontrivial barrier.*

Proof. By induction on $|V(G)|$. Since G contains a nontrivial tight cut C, $|V(G)| \geq 6$. We first prove the following claim.

Claim 2.3.1. *If $|V(G)| = 6$, then C is an ELP-cut.*

Proof. Assume, to the contrary, that C is not an ELP-cut. Then, G has an ELP-cut $D := \partial(Y)$, by Theorem 1.8. The cuts C and D cross, as $|V(G)| = 6$. Adjust notation, by interchanging X with $\overline{X}$ if necessary, so that both $|X \cap Y|$ and $|\overline{X} \cap \overline{Y}|$ are odd. Then, both $X \cap Y$ and $\overline{X} \cap \overline{Y}$ contain exactly one vertex, say u and v respectively, since both $X \cap \overline{Y}$ and $\overline{X} \cap Y$ are non-empty sets and each of them contains an even number of vertices. No edge of G joins a vertex of $X \cap \overline{Y}$ to a vertex in $\overline{X} \cap Y$ (Proposition 1.3). Hence, C is a 2-separation cut of G associated with $\{u, v\}$, a contradiction. □

If $|V(G)| = 6$, then C is either a barrier cut or a 2-separation cut of G, by Claim 2.3.1. If C is a barrier cut, then alternative (iii) of the statement of the lemma holds. If C is a 2-separation cut, then either alternative (i) or alternative (ii) of the statement of the lemma holds.

Assume now, that $|V(G)| > 6$. By hypothesis, the assertion of the Main Theorem holds for G. If G has a C-sheltered nontrivial barrier, then alternative (iii) of the statement of the lemma holds. We may thus assume that G has a 2-separation T and its associated cut $F := \partial(Z)$, such that Z is a subset of a shore of C. If $t \notin T$, then the alternative (i) of the assertion holds. We may thus assume that $t \in T$. If $Z \subseteq \overline{X}$, then the alternative (ii) holds, with $Y := Z$ and $D := F$. If $Z = X$, then the alternative (ii) holds, with $Y := \overline{Z}$ and $D := F$. It now remains the case in which $Z \subset X$ and $t \in T$.

Let u be the vertex of $T - t$. One of u and t is in Z. As $t \notin X$, it follows that $u \in Z$. Let $Z' := (Z - u) + t$. The cuts F and $F' := \partial(Z')$ are members of a 2-separation cut pair of G. Let H be the F'-contraction $G/(Z' \to z')$ of G, let $X_H := (X - Z) + u$ and let $C_H := \partial(X_H)$. See Fig. 4.

We plan now to apply the induction hypothesis, with H, X_H, C_H and z' playing respectively the roles of G, X, C and t. The cuts C and F' cross, the intersection of the shores X and $\overline{Z'}$ is odd and equal to the shore X_H of C_H. The cut C_H is tight in G (Proposition 1.3). As C_H and F' are laminar, C_H is tight in H (Proposition 1.5). Moreover, as $Z \subset X$, it follows that C_H is a nontrivial tight cut of H. As $|V(H)| < |V(G)|$, we may infer, by hypothesis, that the assertion of the Main Theorem holds for every graph having $|V(H)|$ or fewer vertices. Moreover, $z' \in \overline{X_H}$. We now apply the induction hypothesis to H, X_H, C_H and z' playing respectively the roles of G, X, C and t. We consider separately the three possible cases.

Case 1. The graph H has a 2-separation which does not contain the vertex z'.

Let S be a 2-separation of H that does not contain the vertex z'. By Lemma 2.1, S is a 2-separation of G. Clearly, S does not contain the vertex t. The alternative (i) of the assertion holds.

Case 2. The graph H has a 2-separation S_H that contains the vertex z', associated with a 2-separation cut $D_H := \partial(Y_H)$, such that $Y_H \subseteq \overline{X_H}$.

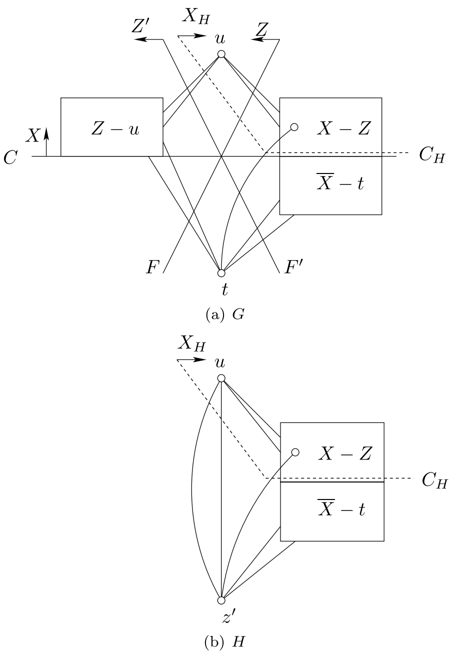

Fig. 4. The cut C_H in G and in H.

Let $W_H := Y_H - S_H$. As $z' \in S_H$, the vertex z' is not in W_H. As $Y_H \subseteq \overline{X_H}$, the vertex u is not in W_H. In sum, W_H and $\{u, z'\}$ are disjoint and $W_H \subset \overline{X}$. Let $S := (S_H - z') + t$ and let $Y := W_H + t = (Y_H - S_H) + t$. Thus, $Y \subseteq \overline{X}$. By Lemma 2.1, every (even) component of $H[W_H]$ is a component of $G - S$ and S is a 2-separation of G. Moreover, the vertex t is in S, the cut $D := \partial(Y)$ is a 2-separation cut of G associated with S and its shore Y is a subset of $\overline{X}$. The alternative (ii) of the assertion holds.

Case 3. The graph H has a C_H-sheltered nontrivial barrier.

Let S_H be a C_H-sheltered nontrivial barrier of H. We now apply Lemma 2.1. Let S be the set as defined in the statement of Lemma 2.1. Then, S is a (nontrivial) barrier of G. If S_H is a subset of X_H, then $S = S_H$ and S is a subset of X. If S_H is a subset of $\overline{X_H} - z'$, then $S = S_H$ and S is a subset of $\overline{X}$. Finally, if S_H is a subset of $\overline{X}$ and $z' \in S_H$, then S is equal to $(S_H - z') + t$ and is a subset of $\overline{X}$. In all alternatives, S is a C-sheltered nontrivial barrier of G, hence alternative (iii) of the assertion of the lemma holds.

The proof of the lemma is complete. □

3. Proof of the main theorem

Proof. Let G be a matching covered graph and let $C := \partial(X)$ be a nontrivial tight cut of $G := (V, E)$. If $|V| = 6$, then each nontrivial tight cut of G is an ELP-cut by Claim 2.3.1.

Clearly, the assertion holds. Thus, we may assume, as the induction hypothesis, that the assertion holds for every matching covered graph having fewer than $|V|$ vertices. We shall now prove that G has a 2-separation cut having a shore that is a subset of a shore of C or G has a C-sheltered nontrivial barrier.

As C is nontrivial and tight, it follows from Theorem 1.8 that G either has a 2-separation or a nontrivial barrier. We consider these possibilities separately.

Case 1. The graph G does not have 2-separations.

In this case, the graph G has a nontrivial barrier B, by Theorem 1.8. If B is C-avoiding, then, by Lemma 2.2(iv), one of $B \cap X$ and $B \cap \overline{X}$ is a nontrivial (C-sheltered) barrier of G. We may thus assume that $G - B$ has a component, $G[Y]$, such that $\partial(Y)$ and C cross. Let $H := G/(\overline{Y} \to \overline{y})$ and let $D := \partial(Y)$. Adjust notation, by interchanging X with $\overline{X}$ if necessary, so that $|X \cap Y|$ is odd.

Let $I := \partial(X \cap Y)$, let $U := \partial(\overline{X} \cap \overline{Y})$. The cuts I and U are both tight in G. Moreover, no edge of G joins a vertex of $\overline{X} \cap Y$ to a vertex in $X \cap \overline{Y}$ (Proposition 1.3). As $|X \cap Y|$ is odd, H has an odd number of vertices in X. Moreover, Y is neither a subset of X nor a superset of $\overline{X}$. The subgraph of G induced by $\overline{X}$ is connected. Thus, G has an edge joining a vertex of $\overline{X} \cap Y$ to a vertex in $B \cap \overline{X}$. Therefore, H has an odd number of vertices in X and has a vertex adjacent to a vertex in $B \cap \overline{X}$. By Lemma 2.2(iii), $B \cap \overline{X}$ is a (possibly trivial) C-sheltered barrier of G. If $B \cap \overline{X}$ is not a singleton, then the assertion of the theorem holds. We may thus assume that $|B \cap \overline{X}| = 1$. Let u be the only vertex of $B \cap \overline{X}$. By Lemma 2.2(i), every component of $G - B$ has an odd number of vertices in X.

Proposition 3.1. *The cut I is nontrivial.*

Proof. Assume, to the contrary, that $X \cap Y$ is a singleton, $\{v\}$. No edge of G joins a vertex of $X \cap \overline{Y}$ to a vertex of $\overline{X} \cap Y$. Thus, every edge of $\partial(\overline{X} \cap Y)$ is incident with a vertex in $\{u, v\}$. We conclude that $\{u, v\}$ is a 2-separation of G, a contradiction to the hypothesis that G is free of 2-separations. (In fact, the cut $\partial((\overline{X} \cap Y) + u)$ is a 2-separation cut of G associated with $\{u, v\}$.) □

We now apply Lemma 2.3, with H, $X \cap Y$, I and $\overline{y}$ playing, respectively, the roles of G, X, C and t. We then deduce that one of the following possibilities holds:

(i) the graph H has a 2-separation that does not contain the vertex $\overline{y}$, and
(ii) H has a 2-separation, S_H, that contains the vertex $\overline{y}$, associated with a cut $D_H := \partial(Y_H)$, such that $Y_H \subseteq (\overline{X} \cap Y) + \overline{y}$, and
(iii) the graph H has a nontrivial I-sheltered barrier, B_H.

We shall now eliminate the two first possibilities. Assume, to the contrary, that H has a 2-separation, S_1, that does not contain the vertex $\overline{y}$. One of the (even) components of

$H - S_1$, K_1, does not contain the vertex $\overline{y}$ and is a proper subgraph of H. In this case, K_1 is an even component of $G - S_1$. By Corollary 1.2, $G - S_1$ has no odd components, hence S_1 is a 2-separation of G. This is a contradiction to the hypothesis that G is free of 2-separations.

Assume, to the contrary, that H has a 2-separation, S_H, that contains the vertex $\overline{y}$, associated with the cut $D_H := \partial(Y_H)$, where $Y_H \subseteq (\overline{X} \cap Y) + \overline{y}$. Let v denote the vertex of $S_H - \overline{y}$ and let K be a component of $H[Y_H - S_H]$. Necessarily, K is even. The set $V(K)$ is a subset of $\overline{X} \cap Y$, hence $\partial_H(V(K)) = \partial_G(V(K))$. Let e be an edge of $\partial(V(K))$ that is not incident with v. Then, e is incident with $\overline{y}$ in H, hence, in G, e joins a vertex of $\overline{X} \cap Y$ to a vertex, w, of $\overline{Y}$. Necessarily $w = u$, hence K is an even component of $G - \{u, v\}$. By Corollary 1.2, $\{u, v\}$ is a 2-separation of G, again a contradiction to the hypothesis that G is free of 2-separations.

We deduce that H has a nontrivial I-sheltered barrier, B_H. If B_H is a subset of $X \cap Y$ or of $\overline{X} \cap Y$, then B_H is C-sheltered, and the assertion of the theorem holds. We may thus assume that B_H is a subset of $(\overline{X} \cap Y) + \overline{y}$ that contains the vertex $\overline{y}$. In this case, the set $B_G := (B_H - \overline{y}) \cup B$ is a barrier of G. One of the components of $H - B_H$, say, K, contains all the vertices of $X \cap Y$, by Proposition 1.10. As $\overline{y} \in B_H$, K is also a component of $G - B_G$ and $V(K) \cap X = X \cap Y$, hence K contains an odd number of vertices in X. Moreover, as H is 2-connected, K has vertices adjacent to at least one vertex of $B_H - \overline{y}$, which is a vertex of $B_G \cap \overline{X}$. Finally, $B_G \cap \overline{X}$ is nontrivial. By Lemma 2.2(iii), $B_G \cap \overline{X}$ is a nontrivial C-sheltered barrier of G.

The analysis of Case 1 is complete.

Case 2. The graph G has 2-separations.

If G has a C-sheltered 2-separation, then the assertion holds, by Proposition 1.10. We may thus assume that

$$\underline{\text{each 2-separation of } G \text{ contains a vertex in each shore of } C.}$$

Let S be a 2-separation of G and let K be a component of $G - S$. As K is even, it follows that $|X \cap V(K)| \equiv |\overline{X} \cap V(K)| \pmod 2$. We say that K is *balanced* if $|X \cap V(K)|$ is even and *unbalanced*, otherwise.

Proposition 3.2. *Let S be a 2-separation of G such that $G - S$ has an unbalanced component. Then, $G - S$ has precisely two components, both of which are unbalanced. Moreover, every edge of C has both ends in a component of $G - S$ (see Fig. 5).*

Proof. Suppose that $G - S$ has an unbalanced component, $K_1 := G[L_1]$. The shore X of C is odd and it contains precisely one vertex of S. Moreover, $|L_1 \cap X|$ is also odd. Thus, $|X - L_1 - S|$ is odd. It follows that $G - S$ has an unbalanced component, $K_2 := G[L_2]$, distinct from K_1.

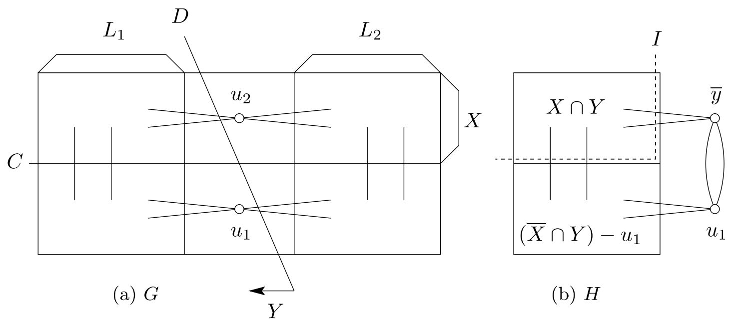

Fig. 5. The components $G[L_1]$ and $G[L_2]$ of $G - \{u_1, u_2\}$ are unbalanced.

Assume, to the contrary, that $G - S$ contains a component, $K_3 := G[L_3]$, not necessarily unbalanced, but distinct from both K_1 and K_2. Let e be an edge of $\partial(L_3)$ and let M be a perfect matching of G that contains edge e. Necessarily, e is incident with a vertex of S. As L_3 is even, M contains also an edge in $\partial(L_3)$ which is incident with the other vertex of S. For $i = 1, 2$, as $|X \cap L_i|$ is odd, the cut $\partial(X \cap L_i)$ contains an edge in M and that edge is not incident with a vertex of S, hence it is an edge having both ends in L_i, therefore it is an edge of C. We conclude that M contains more than one edge in C, a contradiction to the hypothesis that C is tight. Indeed, K_1 and K_2 are the only two components of $G - S$ and they are both unbalanced.

Let u_1 and u_2 be the two vertices of S. Let $Y := L_1 + u_1$ be the shore of a cut associated with S such that $|X \cap Y|$ is odd. We have assumed that S has one vertex in each shore of C, hence $u_1 \in \overline{X}$ and $u_2 \in X$. Let $D := \partial(Y)$ (see Fig. 5). The cuts C and D are tight and cross. No edge of G joins a vertex in $(\overline{X} \cap L_1) + u_1$ to a vertex in $(X \cap L_2) + u_2$ (Proposition 1.3). By symmetry, no edge of G joins a vertex in $(X \cap L_1) + u_2$ to a vertex in $(\overline{X} \cap L_2) + u_1$. Therefore, each edge of C has both ends in some component of $G - S$. □

Let S be a 2-separation of G. A component K of $G - S$ is *good* if K is balanced or each vertex of S is adjacent to two or more vertices of K. Let $\mathcal{S}$ be the collection of 2-separations of G. For each $S \in \mathcal{S}$, let $\mathcal{F}(S)$ be the set of good components of $G - S$. Let

$$\mathcal{F} := \bigcup_{S \in \mathcal{S}} \mathcal{F}(S).$$

Case 2.1. The collection $\mathcal{F}$ is empty.

Let S be a 2-separation of G. The hypothesis of the case implies that the components of $G - S$ are unbalanced. By Proposition 3.2, $G - S$ consists of precisely two components, $K_i := G[L_i]$, $i = 1, 2$. We have assumed that each shore of C contains a vertex of S. Let u_1 be the vertex of S in $\overline{X}$ and let u_2 be the vertex of S in X (Fig. 5).

The hypothesis of the case also implies that for $i = 1, 2$, one of u_1 and u_2 is adjacent only to one vertex of L_i. Adjust notation so that u_1 is adjacent to only one vertex of L_1, says v_1.

Proposition 3.3. *The vertex u_1 is adjacent only to one vertex of L_2.*

Proof. Suppose, to the contrary, that u_1 is adjacent to more than one vertex in L_2. By the hypothesis of the case, u_2 is adjacent only to one vertex in L_2, say, v_2. Let $T := \{v_1, v_2\}$. No edge of $G - T$ joins a vertex of $Z_1 := (L_1 - v_1) + u_2$ to a vertex of $Z_2 := (L_2 - v_2) + u_1$. Thus, $G - T$ has two or more components.

Let us now prove that each component of $G-T$ is even. If $G-T$ has an odd component, then, by parity, it has at least two odd components. In this case, by Corollary 1.2, $G-T$ has precisely two components, both odd. But if $G - T$ has only two components, then they are $G[Z_1]$ and $G[Z_2]$, both even. We conclude that each component of $G - T$ is even.

Let us now prove that each component of $G - T$ has an even number of vertices in each shore of C. If $G-T$ has only two components, then they are $G[Z_1]$ and $G[Z_2]$, and both have an even number of vertices in each shore of C. Alternatively, if $G - T$ has more than two components, then again each of these components has an even number of vertices in each shore of C, by Proposition 3.2.

As each component of $G-T$ is even, the pair T is a 2-separation of G. Each component of $G - T$ has an even number of vertices in each shore of C. We conclude that $\mathcal{F}$ is nonempty, a contradiction to the hypothesis of the case. □

In sum, for $i = 1, 2$, the vertex u_1 is adjacent to only one vertex of L_i, say, v_i. Clearly, $\{v_1, v_2\}$ is a C-sheltered nontrivial barrier of G.

Case 2.2. The collection $\mathcal{F}$ is nonempty.

Let K_1 be a minimal component in $\mathcal{F}$, let $L_1 := V(K_1)$ and let S be the associated 2-separation of G. Let u_1 and u_2 be two vertices of S. Let $Y := L_1 + u_1$ be the shore of a cut associated with S such that $|X \cap Y|$ is odd and let $D := \partial(Y)$. We have assumed that S has one vertex in each shore of C. If K_1 is unbalanced, then $u_1 \in \overline{X}$ and $u_2 \in X$ (see Fig. 5). Alternatively, if K_1 is balanced, then $u_1 \in X$ and $u_2 \in \overline{X}$ (see Fig. 6). Let $H := G/(\overline{Y} \to \overline{y})$ and let $I := \partial(X \cap Y)$. The cuts C and D cross. By Proposition 1.3, the cut I is tight and no edge joins a vertex of $\overline{X} \cap Y$ to a vertex of $X \cap \overline{Y}$.

Case 2.2.1. The cut I is trivial.

Suppose that I is trivial. If K_1 is unbalanced, then, as K_1 is good, it has two or more vertices adjacent to vertices of S, hence I is nontrivial, a contradiction to the hypothesis of the case. We deduce that K_1 is balanced. In this case, $X \cap Y = \{u_1\}$ and every edge of $\partial(\overline{X} \cap Y)$ is incident with a vertex in S. It follows that the shore $Y' := (\overline{X} \cap Y) + u_2$

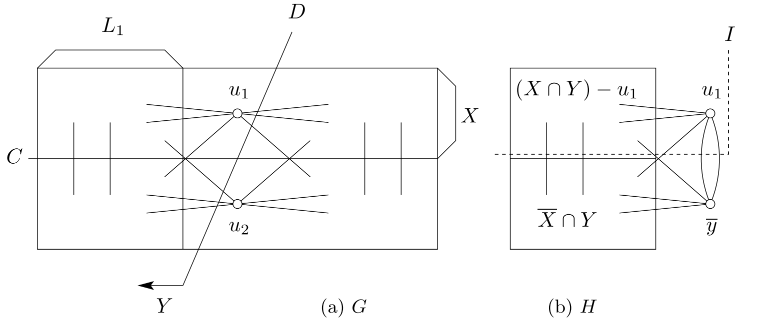

Fig. 6. The component $G[L_1]$ of $G - \{u_1, u_2\}$ is balanced.

of the associated cut $\partial(Y')$ is a subset of $\overline{X}$. The assertion of the theorem holds in this case. We may thus assume that

The cut I is nontrivial.

The cuts D and I are laminar, hence I is tight in H (Proposition 1.5). We now apply the induction hypothesis, with H and I playing respectively the roles of G and C.

Case 2.2.2. The graph H has a nontrivial I-sheltered barrier.

Let S_H be a nontrivial I-sheltered barrier of H. Let us now apply Lemma 2.1 to H, Y and $\overline{y}$ playing respectively the roles of G, X and $\overline{x}$. Let S be the set as defined in the statement of Lemma 2.1. By the item (iii) of Lemma 2.1, S is a (nontrivial) barrier of G. If $\overline{y} \notin S_H$, then S is equal to S_H and is a subset of $X \cap Y$ or of $\overline{X} \cap Y$, hence S is C-sheltered. We may thus assume that $\overline{y} \in S_H$, in which case $S = (S_H - \overline{y}) + u_2$ and S_H is a subset of $(\overline{X} \cap Y) + \overline{y}$. If K_1 is balanced, then $u_2 \in \overline{X}$ (Fig. 6), hence S is a subset of $\overline{X}$. We may thus assume that K_1 is unbalanced.

In H, some component of $H - S_H$, say, W, contains all the vertices of $X \cap Y$, by Proposition 1.10. By the 2-connectivity of H, W contains a vertex adjacent to a vertex of $S_H \cap \overline{X} \cap Y$. Moreover, by Lemma 2.1, W is a component of $G - S$. By Lemma 2.2(iii), the set $S - u_2$, which is equal to $S \cap \overline{X}$, is a barrier of G. If $S \cap \overline{X}$ is nontrivial, then the assertion of the theorem holds.

We may thus assume that $S \cap \overline{X}$ is trivial, let v be its only vertex. Let L be a component of $H - S_H$ distinct from W. Then, $V(L) \subset \overline{X} \cap Y$. The only vertex of $\overline{X} \cap Y$ adjacent to $\overline{y}$ is the vertex u_1, hence L contains u_1. We deduce that L and W are the only two components of $H - S_H$. As K_1 is unbalanced and good, u_1 is adjacent to two or more vertices of $\overline{X} \cap L_1$, hence L is nontrivial. The graph $L - u_1$ is a proper nonempty subgraph of G, hence the graph $G - \{u_1, v\}$ is not connected. By Corollary 1.2, $\{u_1, v\}$ is either a barrier or a 2-separation of G. As $\{u_1, v\}$ is C-sheltered, the assertion of the theorem holds, by Proposition 1.10. The analysis of Case 2.2.2 is complete.

Case 2.2.3. The graph H has a 2-separation cut which is laminar with I.

Let $D_H := \partial(Z_H)$ be a 2-separation cut associated with a 2-separation S_H, such that Z_H is a subset of a shore of I and the subgraph, induced by the vertex set $Z_H - S_H$, contains exactly one (even) component of $H - S_H$. Let us now apply Lemma 2.1 to H, Y and $\overline{y}$ playing respectively the roles of G, X and $\overline{x}$. Let S be the set as defined in the statement of Lemma 2.1. By the item (iii) of Lemma 2.1, S is a 2-separation of G. We have assumed that one of the vertices of S is in X, the other is in $\overline{X}$. Let w_1 be the vertex of S in X, and let w_2 be the vertex of S in $\overline{X}$.

Let $W_H := Z_H - S_H$. Then, W_H is a nonempty proper subset, with an even number of vertices, of a shore of I as Z_H is a subset of a shore of I. If $\overline{y} \notin W_H$ and $u_1 \notin W_H$, then the component induced by W_H is also a component of $G - S$, by the item (i) of Lemma 2.1. And since S meets both shores of C, it follows that one of the 2-separation cuts of G associated with S has a shore that is a subset of C. This implies that the assertion holds. If $\overline{y} \notin W_H$ and $u_1 \in W_H$, then, $\overline{y} \in S_H$ since u_1 and $\overline{y}$ are adjacent in H. So, the other vertex of S_H is w_1. If K_1 is balanced, then $W_H \subseteq X \cap Y$. As each edge of $\partial(W_H - u_1)$ is incident with a vertex of $\{u_1, w_1\}$ and W_H is a nonempty subset with an even number of vertices, $\{u_1, w_1\}$ is a C-sheltered barrier of G, the assertion of the theorem holds. If K_1 is unbalanced, then $W_H \subseteq \overline{X} \cap Y$. Since $G[\overline{X}]$ is connected, $G[\overline{X} \cap Y]$ is also connected. This implies that $W_H = \overline{X} \cap Y$. Thus, each component of $H - S_H$, distinct from $G[W_H]$, is a subgraph of $G[X \cap Y]$. Hence, the graph $G - \{u_2, w_1\}$ is not connected. By Corollary 1.2, $\{u_2, w_1\}$ is either a barrier or a 2-separation of G. As $\{u_2, w_1\}$ is C-sheltered, the assertion of the theorem holds, by Proposition 1.10. We may thus assume that $\overline{y} \in W_H$. Then, $S = S_H$. As W_H is a subset of a shore of I, it follows that W_H is a subset of the shore $(\overline{X} \cap Y) + \overline{y}$.

If K_1 is unbalanced, then, by the definition of K_1, the vertex u_2 is adjacent to at least two vertices of $X \cap Y$. Thus, $\overline{y}$ is adjacent to at least two vertices of $X \cap Y$ in H. Since $\overline{y} \in W_H$ and $W_H \subset (\overline{X} \cap Y) + \overline{y}$, both vertices of S_H are in $X \cap Y$, hence S is C-sheltered. It is a contradiction as $S = S_H$ and both X and $\overline{X}$ contains one of the vertices of S. So, we may thus assume that K_1 is balanced. The subgraph of G induced by $\overline{Y}$ is connected (Corollary 1.6). One of the components of $G - S$ is the graph $G[W]$, where $W = (W_H - \overline{y}) \cup \overline{Y}$. As W_H is even, then so too is W. The vertices u_1 and $\overline{y}$ are adjacent in H, therefore $u_1 \in S_H \cup W_H$, hence $u_1 \in S \cup W$. We conclude that every component of $G - S$ distinct from $G[W]$ is a proper subgraph of K_1. Moreover, the set $T_1 := W_H - \overline{y}$ is an odd subset of $\overline{X} \cap Y$, and $T_2 := \overline{X} \cap \overline{Y}$ is also odd. Clearly, $W \cap \overline{X} = T_1 \cup T_2$, hence $W \cap \overline{X}$ is even. That is, W is balanced. By Proposition 3.2, every component of $G - S$ is balanced. In particular, the components of $G - S$ distinct from W are proper subgraphs of K_1, a contradiction to the minimality of K_1.

The contradiction completes the proof of Case 2.2.3 and hence completes the proof of Case 2.2. So the proof of the main theorem is complete. □

Appendix A. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.jctb.2021.05.003.

References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory, Graduate Texts in Mathematics, Springer, New York, 2008.
[2] M.H. de Carvalho, C.L. Lucchesi, U.S.R. Murty, On a conjecture of Lovász concerning bricks: II. Bricks of finite characteristic, J. Comb. Theory, Ser. B 85 (1) (2002) 137–180.
[3] M.H. de Carvalho, C.L. Lucchesi, U.S.R. Murty, On tight cuts in matching covered graphs, J. Comb. 9 (2018) 163–184.
[4] J. Edmonds, L. Lovász, W.R. Pulleyblank, Brick decompositions and the matching rank of graphs, Combinatorica 2 (3) (1982) 247–274.
[5] L. Lovász, Matching structure and the matching lattice, J. Comb. Theory, Ser. B 43 (2) (1987) 187–222.
[6] L. Lovász, M.D. Plummer, Matching Theory, AMS Chelsea Publishing, Providence, RI, 2009.
[7] Z. Szigeti, Perfect matchings versus odd cuts, Combinatorica 22 (4) (2002) 575–589.
[8] W.T. Tutte, The factorization of linear graphs, J. Lond. Math. Soc. 22 (1947) 107–111.