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The conjunction of the linear arboricity conjecture and Lovász's path partition theorem

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ABSTRACT

A graph is a *linear forest* if each of its components is a path. Given a graph G with maximum degree $\Delta(G)$, motivated by the famous linear arboricity conjecture and Lovász's classic result on partitioning the edge set of a graph into paths, we call a partition $\mathcal{F} := F_1|\cdots|F_k$ of the edge set of G an *exact linear forest partition* if each F_i induces a linear forest, $k \le \lceil \frac{\Delta(G)+1}{2} \rceil$, and every vertex $v \in V(G)$ is on at most $\lceil \frac{d_G(v)+1}{2} \rceil$ non-trivial paths belonging to $\mathcal{F}$. In this paper, we prove the following two results.

- Every 2-degenerate graph has an exact linear forest partition, and so does every series-parallel graph, every outerplanar graph, and every subdivision of any graph provided each edge of the original graph is subdivided at least once.
- Let $p \in (0, 1)$ be a constant. If $G \sim G_{n,p}$, then a.a.s. G has an exact linear forest partition.

1. Introduction

We will mainly use the notation and terminologies from West [12]. Graphs in this paper are simple unless otherwise specified. Let G be a graph. We use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of G, respectively. The degree of vertex v in a graph G, written $d_G(v)$, is the number of edges incident to v in G. The maximum degree and minimum degree of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively.

A graph is a *linear forest* if each of its components is a path. The *linear arboricity* of a graph G, denoted by $la(G)$, is the least number of linear forests needed to partition the edge set of G. In 1981, Akiyama, Exoo, and Harary [1] made the following conjecture, commonly referred to as the *linear arboricity conjecture* (LAC).

Conjecture 1.1. *For every graph G, $la(G) \le \lceil \frac{\Delta(G)+1}{2} \rceil$.*

It is noteworthy that $la(G) \ge \lceil \frac{\Delta(G)}{2} \rceil$ for any graph G because the maximum degree of a path is at most 2. In addition, $la(G) \ge \lceil \frac{\Delta(G)+1}{2} \rceil$ for some graphs G; for example, regular graphs with even degree because for any linear forest partition

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$\mathcal{F}$ of $E(G)$, there is a vertex $v \in V(G)$ such that v is an end-vertex of a path belonging to $\mathcal{F}$. Recall that an edge-coloring of a graph is actually a partition of edges into matchings, and a matching can be viewed as a linear forest whose each component is an edge. Therefore, the above conjecture can be viewed as an analogue to Vizing's theorem. Alon [2] in 1988 showed that LAC is asymptotically correct as $\Delta \to \infty$. In the same paper, he also proved that LAC holds for graphs G with girth $\Omega(\Delta(G))$. The bound was subsequently improved by Alon and Spencer [3] in 1992, and by Ferber, Fox and Jain [7] in 2020. LAC was verified for planar graphs G with $\Delta(G) \neq 7$ by Wu [13] in 1991 and for planar graphs G with $\Delta(G) = 7$ by Wu and Wu [14] in 2008. McDiarmid and Reed [10] confirmed the conjecture for random regular graphs with fixed degree. Glock, Kühn and Osthus [8] showed that, for $p \in (0, 1)$, a.a.s. the random graph $G \sim G_{n,p}$ can be decomposed into $\lceil \frac{\Delta(G)}{2} \rceil$ linear forests.

Erdős asked what is the minimum number of paths into which the edge set of a connected graph of order n can be partitioned. Gallai conjectured that this number is $\lceil \frac{n}{2} \rceil$. (See [4,9].) Lovász [9] confirmed Gallai's conjecture for graphs with at most one even degree vertex. More specifically, Lovász proved the following result.

Theorem 1.2. *For any graph G, $E(G)$ can be partitioned into paths with the following two properties.*

- *For each odd degree vertex v, there is exactly one of these paths containing v as an end-vertex, and*
- *For each even degree vertex v, there are at most two of these paths containing v as an end-vertex.*

We call an edge set partition of a graph satisfying the above two properties a *Lovász's path partition*. Pyber [11] and Fan [6], as well as some others, improved Lovász's result. Inspired by LAC and Lovász's path partition theorem, we give the following definition.

Definition 1.3. For a graph G with maximum degree $\Delta(G)$, a partition $\mathcal{F} := F_1 | \cdots | F_k$ of $E(G)$ is called an *exact linear forest partition* of G if each F_i induces a linear forest, $k \le \lceil \frac{\Delta(G)+1}{2} \rceil$, and every vertex $v \in V(G)$ is on at most $\lceil \frac{d_G(v)+1}{2} \rceil$ non-trivial paths belonging to $\mathcal{F}$.

It is worth noting that if G has an exact linear forest partition $\mathcal{F}$, then it gives both LAC and a Lovász's path partition: Clearly, the exact linear forest partition of G is also a linear forest partition of G and $la(G) = k \le \lceil \frac{\Delta(G)+1}{2} \rceil$. Hence, LAC holds for G. For any vertex $v \in V(G)$, let p be the number of paths in $\mathcal{F}$ containing v and q be the number of paths in $\mathcal{F}$ containing v as an end-vertex. Since G has an exact linear forest partition $\mathcal{F}$, we have $p \le \lceil \frac{d_G(v)+1}{2} \rceil$. If $d_G(v)$ is odd, then $p \le \frac{d_G(v)+1}{2}$ and $q \ge 1$. Since $d_G(v) = q + 2(p-q) = 2p - q \le 2 \cdot \frac{d_G(v)+1}{2} - q = d_G(v) + 1 - q$, we have $q \le 1$, and so $q = 1$. Therefore, G has the first property in Theorem 1.2. If $d_G(v)$ is even, then $p \le \frac{d_G(v)+2}{2}$ and $q \ge 0$. Since $d_G(v) = q + 2(p-q) = 2p - q \le 2 \cdot \frac{d_G(v)+2}{2} - q = d_G(v) + 2 - q$, we have $q \le 2$. Hence, G has the second property in Theorem 1.2. Consequently, G has a Lovász's path partition.

A graph G is *k-degenerate* if every induced subgraph H has a vertex v with $d_H(v) \le k$.

Theorem 1.4. *If G is a 2-degenerate graph, then G has an exact linear forest partition.*

We believe that the above theorem can be extended to k-degenerate graph for small k, such as $k = 3, 4, 5$. For a given k, we guess that there is a positive number K such that every k-degenerate graph G with $\Delta(G) \ge K$ has an exact linear forest partition. With Yu, we [5] recently proved that LAC holds for k-degenerate graph G with $\Delta(G) \ge 2k^2 - 2k$.

A graph H is said to be a *minor* of a graph G if a copy of H can be obtained from G by deleting and/or contracting edges of G. Furthermore, a graph is called *series-parallel* if it has no K_4 as a minor. It is well-known that every graph with minimum degree $\delta \ge 3$ contains K_4 as a minor. Thus, every series-parallel graph is 2-degenerate, which in turn gives that every series-parallel graph has an exact linear forest partition.

A graph G is an *outerplanar graph* if and only if neither K_4 nor $K_{2,3}$ is a minor of G. Hence, every outerplanar graph is a series-parallel graph, and so each outerplanar graph has an exact linear forest partition.

An *H-subdivision* (or *subdivision of H*) is a graph obtained from H by replacing edges with pairwise internally disjoint paths. Clearly, given a graph G, if the vertices with degree at least three form an independent set, then G is 2-degenerate. Hence, each subdivision of a graph has an exact linear forest partition provided each edge of the original graph is subdivided at least once.

Fix $0 \le p \le 1$. Let $G_{n,p}$ denote a random graph on a set of n vertices such that each possible edge occurs independently with probability p. Given a probability space, a property is said to hold *asymptotically almost surely* (*a.a.s*) if over a sequence of sets, the probability converges to 1. The following result shows that almost all dense graphs have an exact linear forest partition.

Theorem 1.5. *Let $p \in (0, 1)$ be a constant. If $G \sim G_{n,p}$, then a.a.s. G has an exact linear forest partition.*

We make a rough guess that every graph might have an exact linear forest partition. In the remainder of this paper, we will give the proof of Theorem 1.4 in Section 2 and Theorem 1.5 in Section 3.

2. Theorem 1.4

A family $\mathcal{G}$ of graphs is said to be *monotonic decreasing* if $G \in \mathcal{G}$ and $H \subseteq G$, then $H \in \mathcal{G}$. Let $\mathcal{G}$ be a monotonically decreasing family of graphs. We call a graph $G \in \mathcal{G}$ a *minimal counterexample* to the exact linear forest partition if G does not have an exact linear forest partition, but every proper subgraph of G in $\mathcal{G}$ has one. With this, we state the following technical result, which gives Theorem 1.4.

Theorem 2.1. *Let $\mathcal{G}$ be a monotonic decreasing family of graphs. If $G \in \mathcal{G}$ is a minimal counterexample to the exact linear forest partition, then G is 2-connected and $\delta(G) \geq 3$.*

Proof. We first claim that G is connected. Suppose on the contrary that G is disconnected. Since G is the minimal counterexample, each of its components has an exact linear forest partition, which in turn gives a desired partition of $E(G)$, contradicting that G is a counterexample. The remainder of the proof is divided into two claims.

Claim 1. G is 2-connected.

Suppose the contrary: G has a cut-vertex v. We can assume $G := G_1 \cup G_2$ such that $V(G_1) \cap V(G_2) = \{v\}$, $E(G_1) \cap E(G_2) = \emptyset$ and $E(G) = E(G_1) \cup E(G_2)$. Since $\mathcal{G}$ is a monotonic decreasing family of graphs, we have $G_1, G_2 \in \mathcal{G}$. Since G is a minimal counterexample, for $1 \leq i \leq 2$, G_i has an exact linear forest partition $\mathcal{F}^i := F_1^i | \cdots | F_{k_i}^i$ such that $k_i \leq \lceil \frac{\Delta(G_i)+1}{2} \rceil$ and each vertex $u \in V(G_i)$ is on at most $\lceil \frac{d_{G_i}(u)+1}{2} \rceil$ paths belonging to $\mathcal{F}^i$. Clearly, $k_i \leq \lceil \frac{\Delta(G)+1}{2} \rceil$.

Let $d = d_G(v)$, $d_1 = d_{G_1}(v)$, and $d_2 = d_{G_2}(v)$. Clearly, $d = d_1 + d_2$. Let $k = \lceil \frac{\Delta(G)+1}{2} \rceil$. For $1 \leq i \leq 2$, let $\widetilde{\mathcal{F}}^i := \widetilde{F}_1^i | \cdots | \widetilde{F}_k^i$ be a new partition of $E(G_i)$ obtained from $\mathcal{F}^i$ by adding some empty sets and relabeling them if necessary such that $\widetilde{I}(v) \cup \widetilde{J}(v) \subseteq \{1, \ldots, \lceil \frac{d+1}{2} \rceil\}$ and $|\widetilde{I}(v) \cup \widetilde{J}(v)|$ is maximum, where $\widetilde{I}(v) = \{j : d_{\widetilde{F}_j^1}(v) > 0\}$ and $\widetilde{J}(v) = \{j : d_{\widetilde{F}_j^2}(v) > 0\}$. Clearly, $|\widetilde{I}(v)| \leq \lceil \frac{d_1+1}{2} \rceil \leq \lceil \frac{d+1}{2} \rceil$ and $|\widetilde{J}(v)| \leq \lceil \frac{d_2+1}{2} \rceil \leq \lceil \frac{d+1}{2} \rceil$. Let's consider the two cases as follows.

Case 1.1. Both d_1 and d_2 are odd.

In this case, $\lceil \frac{d_1+1}{2} \rceil + \lceil \frac{d_2+1}{2} \rceil = \lceil \frac{d+1}{2} \rceil$, and so $|\widetilde{I}(v)| + |\widetilde{J}(v)| \leq \lceil \frac{d_1+1}{2} \rceil + \lceil \frac{d_2+1}{2} \rceil = \lceil \frac{d+1}{2} \rceil$. Since $|\widetilde{I}(v) \cup \widetilde{J}(v)|$ is maximized, we have $|\widetilde{I}(v) \cap \widetilde{J}(v)| = 0$. Let $F_i = \widetilde{F}_i^1 \cup \widetilde{F}_i^2$ for all i such that $1 \leq i \leq k$. Since $|\widetilde{I}(v) \cap \widetilde{J}(v)| = 0$, if v is on a path in $\widetilde{F}_i^1$ then it is not on any path in $\widetilde{F}_i^2$, and vice versa, which in turn gives that each F_i is a linear forest. So $\mathcal{F} := F_1 | \cdots | F_k$ is a linear forest partition of $E(G)$. Obviously, $k \leq \lceil \frac{\Delta(G)+1}{2} \rceil$. Since $|\widetilde{I}(v) \cup \widetilde{J}(v)| \leq \lceil \frac{d+1}{2} \rceil$, vertex v is on at most $\lceil \frac{d+1}{2} \rceil = \lceil \frac{d_G(v)+1}{2} \rceil$ paths belonging to $\mathcal{F}$. For any vertex $x \in V(G)$ other than v, x is on at most $\lceil \frac{d_{G_i}(x)+1}{2} \rceil = \lceil \frac{d_G(x)+1}{2} \rceil$ paths belonging to $\mathcal{F}$. Therefore, $\mathcal{F}$ is an exact linear forest partition of $E(G)$, giving a contradiction.

Case 1.2. At least one of d_1 and d_2 is even.

In this case, $\lceil \frac{d_1+1}{2} \rceil + \lceil \frac{d_2+1}{2} \rceil = \lceil \frac{d+1}{2} \rceil + 1$, and so $|\widetilde{I}(v)| + |\widetilde{J}(v)| \leq \lceil \frac{d_1+1}{2} \rceil + \lceil \frac{d_2+1}{2} \rceil = \lceil \frac{d+1}{2} \rceil + 1$. Since $|\widetilde{I}(v) \cup \widetilde{J}(v)|$ is maximized, we have $|\widetilde{I}(v) \cap \widetilde{J}(v)| \leq 1$. Moreover, the equalities hold if and only if

$$|\widetilde{I}(v)| = \lceil \frac{d_1+1}{2} \rceil \text{ and } |\widetilde{J}(v)| = \lceil \frac{d_2+1}{2} \rceil. \tag{1}$$

If $|\widetilde{I}(v) \cap \widetilde{J}(v)| = 0$, similarly to the case 1, we then can construct an exact linear forest partition of $E(G)$. So we assume $|\widetilde{I}(v) \cap \widetilde{J}(v)| = 1$. In this case, we have (1). Let $\widetilde{I}_1(v) = \{i : d_{\widetilde{F}_i^1}(v) = 1\}$, $\widetilde{I}_2(v) = \{i : d_{\widetilde{F}_i^1}(v) = 2\}$, $\widetilde{J}_1(v) = \{i : d_{\widetilde{F}_i^2}(v) = 1\}$ and $\widetilde{J}_2(v) = \{i : d_{\widetilde{F}_i^2}(v) = 2\}$. We claim $\widetilde{I}_1(v) \neq \emptyset$, which is clearly true if d_1 is odd. If d_1 is even and $\widetilde{I}_1(v) = \emptyset$, then $\lceil \frac{d_1+1}{2} \rceil = |\widetilde{I}(v)| = |\widetilde{I}_2(v)| = \lceil \frac{d_1}{2} \rceil = \lceil \frac{d_1+1}{2} \rceil - 1$, giving a contradiction. Similarly, we have $\widetilde{J}_1(v) \neq \emptyset$. Relabeling the indices in $\widetilde{I}(v)$ and $\widetilde{J}(v)$ if necessary such that $\widetilde{I}_1(v) \cap \widetilde{J}_1(v) \neq \emptyset$, say $1 \in \widetilde{I}_1(v) \cap \widetilde{J}_1(v)$. We then have $\{1\} = \widetilde{I}_1(v) \cap \widetilde{J}_1(v) = \widetilde{I}(v) \cap \widetilde{J}(v)$ because of $|\widetilde{I}(v) \cap \widetilde{J}(v)| = 1$.

For $1 \leq i \leq 2$, let P_i be a path in $\widetilde{F}_1^i$ containing v. Since $1 \in \widetilde{I}_1(v)$, vertex v is an end-vertex of P_1. Similarly, vertex v is an end-vertex of P_2. Then, $P = P_1 \cup P_2$ is a path. Let $F_1 = (\widetilde{F}_1^1 \backslash \{P_1\}) \cup (\widetilde{F}_1^2 \backslash \{P_2\}) \cup \{P\}$ and $F_i = \widetilde{F}_i^1 \cup \widetilde{F}_i^2$ for $i \geq 2$. Thus $\mathcal{F} := F_1 | \cdots | F_k$ is a linear forest partition of $E(G)$ with $k \leq \lceil \frac{\Delta(G)+1}{2} \rceil$. Since $|\widetilde{I}(v) \cup \widetilde{J}(v)| = |\widetilde{I}(v)| + |\widetilde{J}(v)| - 1 = \lceil \frac{d+1}{2} \rceil = \lceil \frac{d_G+1}{2} \rceil$, vertex v is on $\lceil \frac{d_G+1}{2} \rceil$ paths belonging to $\mathcal{F}$, and for any vertex $x \in V(G)$ other than v, x is on at most $\lceil \frac{d_{G_i}(x)+1}{2} \rceil = \lceil \frac{d_G(x)+1}{2} \rceil$ paths belonging to $\mathcal{F}$. Hence, $\mathcal{F}$ is an exact linear forest partition of $E(G)$, giving a contradiction.

Claim 2. $\delta(G) \geq 3$.

Suppose the contrary: there exists a vertex $v \in V(G)$ such that $d_G(v) = 2$. Let u, w be the two neighbors of v and let $H = G - vw$. Since $\mathcal{G}$ is a monotonic decreasing family of graphs, we have $H \in \mathcal{G}$. Given the assumption that G is a minimal counterexample to the exact linear forest partition, $E(H)$ has an exact linear forest partition $\mathcal{F} := F_1 | \cdots | F_k$ such that $k \leq \lceil \frac{\Delta(H)+1}{2} \rceil \leq \lceil \frac{\Delta(G)+1}{2} \rceil$ and for each $x \in V(H)$, $|I(x)| \leq \lceil \frac{d_H(x)+1}{2} \rceil$, where $I(x) = \{i : d_{F_i}(x) > 0 \text{ and } 1 \leq i \leq k\}$. Adding some empty sets to the partition $\mathcal{F}$ if necessary such that $k = \lceil \frac{\Delta(G)+1}{2} \rceil$. Since $w \in V(H)$, we have $|I(w)| \leq \lceil \frac{d_H(w)+1}{2} \rceil = \lceil \frac{d_G(w)}{2} \rceil \leq \lceil \frac{d_G(w)+1}{2} \rceil$.

We first assume that $|I(w)| \leq \lceil \frac{d_G(w)+1}{2} \rceil - 1$. So $|I(w)| < k$. Without loss of generality, we assume that $1 \notin I(w)$, i.e., $d_{F_1}(w) = 0$. If $d_{F_1}(v) = 0$, then let $F_1^* = F_1 \cup \{vw\}$. If $d_{F_1}(v) = 1$, then there exists a path P belonging to F_1 such that v is an end-vertex of P. Let P^* be a path obtained by adding edge vw in P. Let $F_1^* = (F_1 \backslash \{P\}) \cup \{P^*\}$ and $F_i^* = F_i$ for

$i \in \{2, \ldots, k\}$. Since $d_{F_1}(w) = 0$, F_1^* is a linear forest. Therefore, $\mathcal{F}^* := F_1^*|\cdots|F_k^*$ is also a linear forest partition of $E(G)$ with $k \le \lceil \frac{\Delta(G)+1}{2} \rceil$. Moreover, vertex v is on at most $2 = \lceil \frac{d_G(v)+1}{2} \rceil$ paths belonging to $\mathcal{F}^*$, vertex w is on $|I(w)| + 1 \le (\lceil \frac{d_G(w)+1}{2} \rceil - 1) + 1 = \lceil \frac{d_G(w)+1}{2} \rceil$ paths belonging to $\mathcal{F}^*$, and any other vertex x is on at most $\lceil \frac{d_H(x)+1}{2} \rceil = \lceil \frac{d_G(x)+1}{2} \rceil$ paths belonging to $\mathcal{F}^*$. So, $\mathcal{F}^*$ is an exact linear forest partition of $E(G)$, giving a contradiction.

We now assume that $|I(w)| = \lceil \frac{d_G(w)+1}{2} \rceil$. Recall $|I(w)| \le \lceil \frac{d_H(w)+1}{2} \rceil = \lceil \frac{d_G(w)}{2} \rceil$. In this case, $d_G(w)$ is odd, i.e., $d_H(w)$ is even. Let $I_1(w) = \{i : d_{F_i}(w) = 1\}$ and $I_2(w) = \{i : d_{F_i}(w) = 2\}$. Clearly, $I(w) = I_1(w) \cup I_2(w)$.

We first claim $|I_1(w)| \ge 1$. Otherwise, we have $\lceil \frac{d_G(w)+1}{2} \rceil = |I(w)| = |I_2(w)| = \frac{d_H(w)}{2} = \frac{d_G(w)-1}{2}$, giving a contradiction. Further, we claim $|I_1(w)| \ge 2$. If $|I_1(w)| = 1$, then $d_H(w) = |I_1(w)| + 2|I_2(w)|$ is odd, giving a contradiction. So $|I_1(w)| \ge 2$ and $|I_1(w)|$ is even.

Since $|I_1(w)| \ge 2$ and $|I(v)| = d_H(v) = 1$, we assume without loss of generality that $I(v) = \{1\}$ and $2 \in I_1(w) \backslash I(v)$. Let P be a path in F_2 containing w as an end-vertex and $P^* = P \cup \{vw\}$ be a path obtained by adding the edge vw in P. Let $F_2^* = (F_2 \backslash \{P\}) \cup \{P^*\}$ and $F_i^* = F_i$ for each $i \ne 2$. Let $\mathcal{F}^* := F_1^*|\cdots|F_k^*$. Clearly, $\mathcal{F}^*$ is a linear forest partition of $E(G)$ with $k \le \lceil \frac{\Delta(G)+1}{2} \rceil$. Moreover, vertex v is on two paths belonging to $\mathcal{F}^*$, vertex w is on $\lceil \frac{d_G(w)+1}{2} \rceil$ paths belonging to $\mathcal{F}^*$, and any other vertex x is on $|I(x)| \le \lceil \frac{d_H(x)+1}{2} \rceil = \lceil \frac{d_G(x)+1}{2} \rceil$ paths belonging to $\mathcal{F}^*$. So, $\mathcal{F}^*$ is an exact linear forest partition of $E(G)$, giving a contradiction. □

3. Theorem 1.5

Glock, Kühn, and Osthus [8] proved the following optimal decomposition results for random graphs.

Theorem 3.1. *Let $p \in (0,1)$ be a constant and let $G \sim G_{n,p}$. Let $odd(G)$ be the number of odd degree vertices in G. The following a.a.s. hold:*

(i) *G can be decomposed into $\lceil \frac{\Delta(G)}{2} \rceil$ cycles and a matching of size $odd(G)/2$.*
(ii) *G can be decomposed into $\max\{odd(G)/2, \lceil \frac{\Delta(G)}{2} \rceil\}$ paths.*
(iii) *G can be decomposed into $\lceil \Delta(G)/2 \rceil$ linear forests.*

Our proof of Theorem 1.5 is inspired by the proof of Theorem 3.1. In the next subsection, we will state some additional notation and preliminary results given in [8].

3.1. Notation and preliminary results

Let G be a multigraph or digraph. Denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. Given $U \subseteq V(G)$, let $G - U$ be the graph obtained by deleting vertices in U from G, and $G[U]$ be the subgraph of G induced by U. If $F \subseteq E(G)$, then let $G \backslash F$ be the graph obtained by removing all edges in F from G, and $G \cup F$ denote the graph obtained by adding the edges in F to G.

Let $\varepsilon, p \in (0,1)$ and $e_G(S,T)$ be the number of edges in G between disjoint $S, T \subseteq V(G)$. A graph G of order n is said to be *lower-(p,ε)-regular* if we have

$$e_G(S,T) \ge (p - \varepsilon) \cdot |S| \cdot |T|$$

for all disjoint $S, T \subseteq V(G)$ with $|S|, |T| \ge \varepsilon n$.

Proposition 3.2. [Lemma 3.5 in [8]] *Let $\varepsilon, p \in (0,1)$ be constant. The following hold a.a.s. for the random graph $G \sim G_{n,p}$:*

(i) $\Delta(G) - \delta(G) \le 4\sqrt{n \log n}$,
(ii) *G is lower-(p,ε)-regular,*
(iii) *G has a unique vertex of maximum degree.*

Proposition 3.3. [Proposition 3.1. in [8]] *Let $0 < 1/n_0 \ll \varepsilon, p < 1$, and let G be a lower-(p,ε)-regular (di-)graph on $n \ge n_0$ vertices. Then the following hold:*

(i) *If G' is obtained from G by adding a new vertex w and arbitrary edges at w, then G' is lower-$(p, 2\varepsilon)$-regular.*
(ii) *Let H be a graph on $V(G)$ such that $\Delta(H) \le \eta n$. Let $\varepsilon' := \max\{2\varepsilon, 2\sqrt{\eta}\}$. Then $G \backslash H$ is lower-(p,ε')-regular.*
(iii) *If $U \subseteq V(G)$ has size at least βn, then $G[U]$ is lower-$(p, \varepsilon/\beta)$-regular.*

A graph G is *Eulerian* if and only if it is connected and its vertices all have even degree. A digraph G is *Eulerian* if it has a closed trial containing all edges of G.

Proposition 3.4. [Lemma 3.12 in [8]] *Let $0 < 1/n_0 \ll \varepsilon \ll p, \alpha < 1$. Suppose that G is an Eulerian graph on $n \geq n_0$ vertices. Assume further that G is lower-(p, ε)-regular and $\delta(G) \geq \alpha n$. Then there exists an orientation G' of G such that G' is Eulerian and lower-$(p/4, \varepsilon)$-regular.*

Let G be a (di-)graph and let $M = \{x_1y_1, \ldots, x_my_m\}$ be a matching in the complete graph on $V(G)$ such that $d_G(x_i) \leq d_G(y_i)$ for all $i \in \{1, \ldots, m\}$. We say that a subgraph $F \subseteq G$ is *consistent with M* if for all $i \in \{1, \ldots, m\}$, $x_i \in V(F)$ implies $y_i \in V(F)$.

Theorem 3.5. [Theorem 4.1 in [8]] *Let $0 < 1/n_0 \ll \eta, \varepsilon \ll p < 1$ be such $\varepsilon^2 \leq \eta$. Suppose that G is a lower-(p, ε)-regular digraph on $n \geq n_0$ vertices. Moreover, assume that G is Eulerian and $\Delta(G) - \delta(G) \leq \eta n$. Let M be any matching in the complete graph on $V(G)$. Then G can be decomposed into $\frac{\Delta(G)}{2}$ cycles which are consistent with M.*

Corollary 3.6. [Corollary 4.2 in [8]] *Let $0 < 1/n_0 \ll \eta, \varepsilon \ll p < 1$. Suppose that G is a lower-(p, ε)-regular graph on $n \geq n_0$ vertices. Moreover, assume that $\Delta(G) - \delta(G) \leq \eta n$ and that G is Eulerian. Let M be any matching in the complete graph on $V(G)$. Then G can be decomposed into $\frac{\Delta(G)}{2}$ cycles which are consistent with M.*

3.2. Proof of Theorem 1.5

We will prove the following result which by Proposition 3.2 in turn gives Theorem 1.5.

Theorem 3.7. *Let $0 < 1/n_0 \ll \eta, \varepsilon \ll p < 1$. Suppose that G is a lower-(p, ε)-regular graph on n vertices. If $n \geq n_0$ and $\Delta(G) - \delta(G) \leq \eta n$, then G has an exact linear forest partition.*

Proof. Denote by $W = \{x_1, y_1, x_2, y_2, \ldots, x_m, y_m\}$ the set of odd degree vertices in G. We assume without loss of generality that $d_G(x_i) \leq d_G(y_i)$ for $i \in \{1, \ldots, m\}$. The proof is divided into two cases according to whether $|W| \geq \Delta(G)$.

Case 1. $|W| \geq \Delta(G)$.

Let H be the graph obtained from G by the following two steps: (1). Add a new vertex z and edges between z and every vertex in $\lceil \frac{\Delta(G)}{2} \rceil$ pairs $\{x_i, y_i\}$ of W with $i \in \{1, \ldots, \lceil \frac{\Delta(G)}{2} \rceil\}$; (2). In the remaining $m - \lceil \frac{\Delta(G)}{2} \rceil$ pairs, let $M = \{x_iy_i \ : \ x_iy_i \in E(G)\}$ and $M^* = \{x_iy_i \ : \ x_iy_i \in E(G^c)\}$, where G^c is the complement of G. Remove M from the graph and add M^*.

Clearly, H is Eulerian, $\Delta(H) \leq \Delta(G) + 1$ and $\delta(H) \geq \delta(G) - 1$. Hence, $\Delta(H) - \delta(H) \leq \Delta(G) - \delta(G) + 2$, which in turn implies that H is lower-$(p, 4\varepsilon)$-regular by Proposition 3.3 (i) and (ii). By Corollary 3.6, we obtain a cycle partition $\mathcal{C} := C_1|\cdots|C_k$ of H with $k = \frac{\Delta(H)}{2}$, which are consistent with M.

Note that $d_H(z) = 2 \cdot \lceil \frac{\Delta(G)}{2} \rceil$. We claim that $d_H(z) = \Delta(H)$. For any vertex $v \in V(G)$, we have $d_H(v) \leq d_G(v) + 1 \leq \Delta(G) + 1$. If $\Delta(G)$ is odd, then $d_H(z) = \Delta(G) + 1$, and so $d_H(z) = \Delta(H)$. If $\Delta(G)$ is even, then $v \in V(G) \backslash W$ for any vertex $v \in V(G)$ with $d_G(v) = \Delta(G)$. So $d_H(v) = d_G(v) = \Delta(G) = d_H(z)$. Hence, $d_H(z) = \Delta(H)$. We now are going to get a linear forest partition of G in the following three steps.

(1) Remove z in H.

Since $d_H(z) = \Delta(H)$, we then have that z is contained in each of $\frac{\Delta(H)}{2} = \lceil \frac{\Delta(G)}{2} \rceil$ cycles of $\mathcal{C}$. Therefore, we obtain a path partition $\mathcal{P} := P_1|\cdots|P_k$ of $(G\backslash M) \cup M^*$, i.e., $H - z$, where $P_i = C_i - z$ and $k = \lceil \frac{\Delta(G)}{2} \rceil$. Clearly, every vertex in $\{x_1, y_1, \ldots, x_k, y_k\}$ becomes end-vertex of some path P_i in $\mathcal{P}$.

(2) Remove edges of M^* one by one from $(G\backslash M) \cup M^*$.

For each path P_i, if $E(P_i) \cap M^* \neq \emptyset$, then we let $F_i^* = P_i \backslash M^*$; if $E(P_i) \cap M^* = \emptyset$, then we let $F_i^* = P_i$. Obviously, F_i^* is a linear forest for $i \in \{1, \ldots, k\}$. Hence, we obtain a linear forest partition $\mathcal{F}^* := F_1^*|\cdots|F_k^*$ of $G\backslash M$. Clearly, every vertex in pair $\{x_j, y_j\}$ with $x_jy_j \in M^*$ for $j \in \{k+1, \ldots, m\}$ becomes an end-vertex of some path belonging to $\mathcal{F}^*$.

(3) Add edges of M one by one from $G\backslash M$.

For any edge $x_iy_i \in M$, since $x_iy_i \in E(G)$, we have $d_{G\backslash M}(y_i) = d_H(y_i) = d_G(y_i) - 1 < \Delta(G)$. Since $y_iz \notin E(H)$ and $y_i \notin V(M^*)$, operations in (1) and (2) do not affect y_i. Thus, y_i is not a leaf in any F_j^* for $j \in \{1, \ldots, k\}$. Therefore, there exists some $j \in \{1, \ldots, k\}$ such that $d_{F_j^*}(y_i) = 0$, and so $y_i \notin V(C_j)$. Since cycle C_j is consistent with M, we have $x_i \notin V(C_j)$, which in turn shows that $d_{F_j^*}(x_i) = 0$. Hence, $F_j^* \cup \{x_iy_i\}$ is also a linear forest. Let $F_j = F_j^* \cup \{x_iy_i\}$ and $F_\ell = F_\ell^*$ for $\ell \neq j$. In the same fashion, we add all edges of M to get G. Thus, we obtain a linear forest partition $\mathcal{F} := F_1|\cdots|F_k$ of G, where $k = \lceil \frac{\Delta(G)}{2} \rceil$. Moreover, only the vertices in pair $\{x_i, y_i\}$ with $x_iy_i \in M$ for $i \in \{k+1, \ldots, m\}$ become end-vertices of new paths (i.e., x_iy_i) belonging to F_j.

We now show that each vertex $v \in V(G)$ is on at most $\lceil \frac{d_G(v)+1}{2} \rceil$ nontrivial paths in $\mathcal{F}$. Note that only the vertices in W are the end-vertices of paths belonging to $\mathcal{F}$ and each vertex in W is an end-vertex of exactly one path in $\mathcal{F}$. Consequently, for any vertex $v \in V(G)$, if $v \in W$, then v is on exactly $1 + \frac{d_G(v)-1}{2} = \lceil \frac{d_G(v)+1}{2} \rceil$ nontrivial paths in $\mathcal{F}$; and if $v \in V(G)\backslash W$, then v is on exactly $\frac{d_G(v)}{2} < \lceil \frac{d_G(v)+1}{2} \rceil$ nontrivial paths in $\mathcal{F}$. Hence, for each vertex v, v is on at most $\lceil \frac{d_G(v)+1}{2} \rceil$ nontrivial paths in $\mathcal{F}$, which completes the proof of Case 1.

Case 2. $|W| < \Delta(G)$.

Let $M = \{x_iy_i \ : \ x_iy_i \in E(G)\}$ and let $M^* = \{x_iy_i \ : \ x_iy_i \in E(G^c)\}$. Let $H = (G \backslash M) \cup M^*$. Clearly, H is Eulerian and lower-$(p, 2\varepsilon)$-regular. Applying Proposition 3.4, we get an orientation D of H which is Eulerian and lower-$(p/4, 2\varepsilon)$-regular. Let D^* be the digraph obtained from D as follows: Add a new vertex z and connect it to all vertices of W by one edge according to the following rule. If $x_iy_i \in E(D)$, then remove it and add x_iz and zy_i. If $y_ix_i \in E(D)$, then remove it and add y_iz and zx_i. Else (i.e., $x_iy_i \in M$), add edge x_iy_i, y_iz and zx_i. Note that G is the underlying graph of $D^* - z$. Let's consider the following two cases.

Case 2.1. G has a unique vertex v_0 of maximum degree and $\Delta(G)$ is even.

Let $\ell = \frac{\Delta(G)-|W|}{2}$ and U be an ℓ-vertex set in $V(G) \backslash (W \cup \{v_0\})$. Let D^{**} be the digraph obtained from D^* by adding two edges which are oriented in opposite directions between z and each vertex of U. Clearly, D^{**} is Eulerian.

We claim $d_{D^{**}}(z) = \Delta(D^{**}) = \Delta(G)$. Let $u \in V(G) \backslash \{v_0\}$. If $u \in V(G) \backslash (W \cup \{v_0\})$, then $d_G(u) \leq \Delta(G) - 2$. Hence, $d_{D^{**}}(u) \leq d_G(u) + 2 \leq (\Delta(G) - 2) + 2 = \Delta(G)$. If $u \in W$, then $d_G(u) \leq \Delta(G) - 1$, which gives us that $d_{D^{**}}(u) = d_G(u) + 1 \leq (\Delta(G) - 1) + 1 = \Delta(G)$. Note that $d_{D^{**}}(z) = |W| + 2\ell = |W| + 2 \cdot \frac{\Delta(G)-|W|}{2} = \Delta(G)$. Therefore, $\Delta(D^{**}) = \Delta(G) = d_{D^{**}}(z)$.

By Proposition 3.3, D^{**} is lower-$(p/4, 8\varepsilon)$-regular. Applying Theorem 3.5, we can obtain a cycle partition $\mathcal{C} := C_1|\cdots|C_k$ of D^{**}, where $k = \frac{\Delta(D^{**})}{2} = \frac{\Delta(G)}{2}$. Since $d_{D^{**}}(z) = \Delta(D^{**}) = \Delta(G)$, we then get that z is contained in each of these $\frac{\Delta(G)}{2}$ cycles. Note that G is the underlying graph of $D^{**} - z = D^* - z$. After removing z from each cycle of $\mathcal{C}$, we get a disjoint path partition $\mathcal{P} := P_1|\cdots|P_k$ of G with $k = \frac{\Delta(G)}{2}$, where $P_i = C_i - z$. Let $F_i = P_i$ for $i \in \{1, \ldots, k\}$. So, we get a linear forest partition $\mathcal{F} := F_1|\cdots|F_k$ of $E(G)$.

We observe the following facts to show that each vertex $v \in V(G)$ is on at most $\lceil \frac{d_G(v)+1}{2} \rceil$ paths belonging to $\mathcal{F}$. Only the vertices in $W \cup U$ can be an end-vertex of a path belonging to $\mathcal{F}$. Moreover, each vertex in W is an end-vertex of exactly one of these paths, and each vertex in U (which has even degree) may be end-vertex of 0 or 2 of such paths. Therefore, for any vertex $v \in V(G)$, if $v \in W$, then v is on exactly $1 + \frac{d_G(v)-1}{2} = \lceil \frac{d_G(v)+1}{2} \rceil$ nontrivial paths belonging to $\mathcal{F}$; if $v \in U$, then v is on at most $2 + \frac{d_G(v)-2}{2} = \lceil \frac{d_G(v)+1}{2} \rceil$ nontrivial paths belonging to $\mathcal{F}$; if $v \in V(G) \backslash \{U, W\}$, then v is on exactly $\frac{d_G(v)}{2} < \lceil \frac{d_G(v)+1}{2} \rceil$ nontrivial paths belonging to $\mathcal{F}$. Hence, for each vertex $v \in V(G)$, v is on at most $\lceil \frac{d_G(v)+1}{2} \rceil$ nontrivial paths in $\mathcal{F}$, which completes the proof of Case 2.1.

Case 2.2. At least two $\Delta(G)$-vertices in G or $\Delta(G)$ is odd.

Let $\ell = \lceil \frac{\Delta(G)+1-|W|}{2} \rceil$ and U be an ℓ-vertex set in $V(G) \backslash W$. Let D^{**} be obtained from D by adding two edges with opposite orientations between z and each vertex in U. Clearly, D^{**} is Eulerian.

Note that $d_{D^{**}}(z) = |W| + 2\ell = |W| + 2 \cdot \lceil \frac{\Delta(G)+1-|W|}{2} \rceil = 2 \cdot \lceil \frac{\Delta(G)+1}{2} \rceil$. We claim that $d_{D^{**}}(z) = \Delta(D^{**})$. Let $u \in V(D^{**}) \backslash \{z\}$ and $S = \{v : d_G(v) = \Delta(G)\}$. Suppose $|S| \geq 1$ and $\Delta(G)$ is odd. If $u \in W$, then $d_G(u) \leq \Delta(G)$. And so $d_{D^{**}}(u) = d_G(u) + 1 \leq \Delta(G) + 1$. If $u \in V(G) \backslash W$, then $d_G(u) \leq \Delta(G) - 1$. Hence, $d_{D^{**}}(u) \leq d_G(u) + 2 \leq \Delta(G) + 1 = 2 \cdot \lceil \frac{\Delta(G)+1}{2} \rceil$. Suppose $|S| \geq 2$ and $\Delta(G)$ is even. If $u \in W$, then $d_G(u) \leq \Delta(G) - 1$, which in turn gives $d_{D^{**}}(u) = d_G(u) + 1 \leq \Delta(G)$. If $u \in V(G) \backslash W$, then $d_G(u) \leq \Delta(G)$. Hence, $d_{D^{**}}(u) \leq d_G(u) + 2 = \Delta(G) + 2 = 2 \cdot \lceil \frac{\Delta(G)+1}{2} \rceil$. Therefore, $\Delta(D^{**}) = 2 \cdot \lceil \frac{\Delta(G)+1}{2} \rceil = d_{D^{**}}(z)$.

By Proposition 3.3, D^{**} is lower-$(p/4, 8\varepsilon)$-regular. Applying Theorem 3.5, we can obtain a cycle partition $\mathcal{C} := C_1|\cdots|C_k$ of D^{**}, where $k = \frac{\Delta(D^{**})}{2} = \lceil \frac{\Delta(G)+1}{2} \rceil$. Since $d_{D^{**}}(z) = \Delta(D^{**}) = \Delta(G)$, we then get that z is contained in each of these $\lceil \frac{\Delta(G)+1}{2} \rceil$ cycles and G is the underlying graph of $D^{**} - z$. Similarly to Case 2.1., we get a linear forest partition $\mathcal{F} := F_1|\cdots|F_k$ of $E(G)$, where $F_i = C_i - z$ and $k = \lceil \frac{\Delta(G)+1}{2} \rceil$, and for each vertex $v \in V(G)$, v is on at most $\lceil \frac{d_G(v)+1}{2} \rceil$ nontrivial paths belonging to $\mathcal{F}$, which completes the proof of Case 2.2. □

Declaration of competing interest

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References

[1] Jin Akiyama, Geoffrey Exoo, Frank Harary, Covering and packing in graphs. IV. Linear arboricity, Networks 11 (1) (1981) 69–72.
[2] N. Alon, The linear arboricity of graphs, Isr. J. Math. 62 (3) (1988) 311–325.
[3] Noga Alon, Joel H. Spencer, The Probabilistic Method, fourth edition, Wiley Series in Discrete Mathematics and Optimization, John Wiley & Sons, Inc., Hoboken, NJ, 2016.
[4] Adrian Bondy, Beautiful conjectures in graph theory, Eur. J. Comb. 37 (2014) 4–23.
[5] Guantao Chen, Yanli Hao, Guoning Yu, The Linear Arboricity Conjecture on Degenerate Graphs, September 2020.
[6] Genghua Fan, Path decompositions and Gallai's conjecture, J. Comb. Theory, Ser. B 93 (2) (2005) 117–125.
[7] Asaf Ferber, Jacob Fox, Vishesh Jain, Towards the linear arboricity conjecture, J. Comb. Theory, Ser. B 142 (2020) 56–79.
[8] Stefan Glock, Daniela Kühn, Deryk Osthus, Optimal path and cycle decompositions of dense quasirandom graphs, J. Comb. Theory, Ser. B 118 (2016) 88–108.
[9] L. Lovász, On covering of graphs, in: Theory of Graphs, Proc. Colloq., Tihany, 1966, Academic Press, New York, 1968, pp. 231–236.

[10] Colin McDiarmid, Bruce Reed, Linear arboricity of random regular graphs, Random Struct. Algorithms 1 (4) (1990) 443–445.
[11] L. Pyber, Covering the edges of a connected graph by paths, J. Comb. Theory, Ser. B 66 (1) (1996) 152–159.
[12] Douglas B. West, Introduction to Graph Theory, Prentice Hall, Inc., Upper Saddle River, NJ, 1996.
[13] Jian-Liang Wu, On the linear arboricity of planar graphs, J. Graph Theory 31 (2) (1999) 129–134.
[14] Jian-Liang Wu, Yu-Wen Wu, The linear arboricity of planar graphs of maximum degree seven is four, J. Graph Theory 58 (3) (2008) 210–220.