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DEMAND POINT ESTIMATES IN CAPACITATED MULTI-ITEM DYNAMIC LOT SIZING PROBLEMS WITH UNCERTAIN DEMANDS

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ABSTRACT

Dynamic Lot Sizing problem and its variations has been widely used for the scheduling of the productions and inventories. When demands are uncertain, one can use the mean of historical data or the expected value, which is a point estimate of demand. In addition to the mean, this work considers another point estimate, which is called median. We show that the total backorders, as the result of capacity limitation and uncertain demand, can be lower when median is used instead of the mean. It is shown that for an asymmetric distribution, the total backorder is lower significantly when median is used. Furthermore, when demand follows a symmetric distribution, the total backorder do not differ significantly between the two point estimates.

KEYWORDS: Dynamic Lot-Sizing, Multi-Item, Uncertain Demand, Point Estimate

INTRODUCTION AND LITERATURE REVIEW

Since it was first conceived by [16], DLS problem has remained in the center of attention to manage supply chains. This problem considers the production and inventory levels for multiple periods of planning. In this work, we concentrate on the multi-item extension of this problem, i.e. Multi-item Dynamic Lot Sizing (*MIDLS*), and we consider a variation of this problem that considers the backorders and uncertain demands.

To present the formulation of *MIDLS* problem, we use the same notation as [6]. We show the set of items by $\mathscr{I} = \{1,...,|\mathscr{I}|\}$ which is indexed by i and we show the set of periods by $\mathscr{T} = \{1,2,...,|\mathscr{T}|\}$ which is indexed by t. We assume the unit holding cost, set up cost, production cost, and start up cost of item i in period t is h_i^t, s_i^t, p_i^t, and o_i^t respectively. The demand of item i in period t is d_i^t. In addition, we have four variables in the problem. The production of item i in period t is shown by x_i^t. The inventory level of i at the end of period t is shown by q_i^t. Binary variable $y_i^t = 1$, if there is a production of item i in period t and otherwise, $y_i^t = 0$. Finally, $z_i^t = 1$ when we have a production of i in period t and no production of i in previous period. Given $M^t = \sum_{i\in\mathscr{I}} d_i^t$, we can formulate *MIDLS* problem as follows, which is similar to [1, 10, 2, 6]:

$$\begin{aligned}
\textit{MIDLS}: \quad \min \quad & \sum_{t\in\mathscr{T}}\sum_{i\in\mathscr{I}}(h_i^t q_i^t + p_i^t x_i^t + s_i^t y_i^t) + \sum_{t\in\mathscr{T}\setminus\{1\}}\sum_{i\in\mathscr{I}}(o_i^t z_i^t) && (1)\\
\text{s.t.} \quad & q_i^{t-1} + x_i^t \geq q_i^t + d_i^t \quad \forall i\in\mathscr{I}, \forall t\in\mathscr{T}\setminus\{1\} && (2)\\
& x_i^t \leq M^t y_i^t \quad \forall i\in\mathscr{I}, \forall t\in\mathscr{T} && (3)\\
& z_i^t \geq y_i^t - y_i^{t-1} \quad \forall i\in\mathscr{I}, \forall t\in\mathscr{T}\setminus\{1\} && (4)\\
& x_i^t, q_i^t \geq 0,\ y_i^t \in \{0,1\} \quad \forall i\in\mathscr{I}, \forall t\in\mathscr{T} && (5)\\
& z_i^t \in \{0,1\} \quad \forall i\in\mathscr{I}, \forall t\in\mathscr{T}\setminus\{1\} && (6)
\end{aligned}$$

Objective (1) is set to minimize the total cost of inventories, productions, set ups, and start ups. Constraints (2) are the inventory balance constraints. These constraints guarantee that in each period, the total previous inventory and production for each item is greater than or equal to the demand plus the ending inventory. If there is a production in a period, constraints (3) guarantee the corresponding binary variables will be equal to one. If there is a new production in a given period, constraints (4) force the corresponding z_i^t variable equal to one. Constraints (5)-(6) represent the definitions of variables. Upon solving this problem, variable x_i^t decides on the production of each item in each period; variables q_i^t shows the end of period inventory level; variable y_i^t shows whether there is a production of each item in a period; and, variable z_i^t decides if we have a new start-up for a product in a given period.

In the*MIDLS* problem, demand d_i^t is assumed to be stationary through all periods [16]. When demand is not stationary, i.e. fluctuating and/or uncertain, the problem may render infeasible. This means no solution considering the constraints can be found to satisfy the realized demand. To overcome this issue, one may add backorder variables to the constraints (2). The backorder variables will act as catalysts that absorb the additional demand or backorders. In the literature, backorder has been included in inventory balance constraints [3, 5] or as stochastic variables [7, 15]. It is worth noting that service-level has been widely used to address demand uncertainty, which is beyond the scope of this research. For example, the work of [4] studies the probability of having in-hand inventory as a constraint and [8] connects the backorders and service-levels. More similar studies related to the service-levels can be found in the work of [14, 15, 13].

To solve optimization problems with uncertain parameters, such as the *MIDLS* problem, there are several approaches utilized in the literature. One of the popular approaches is to solve the expected value problem [11]. The expected value problem substitutes the uncertain parameter of the problem with its mean. Note that the mean is a point estimate of the uncertain parameter. Alternatively, one can substitute the uncertain parameter with its median. For an uncertain parameter that has a symmetric distribution, it should not affect the backorders, largely. However, if the distribution is asymmetric, backorders should change more compared to using the median of the uncertain parameters. The comparison of the point estimates median and mean when approximating the uncertain parameters is the focus of this work.

In the next section alternative formulations including backorders are presented. In the later section, we analyze different point estimate solution approaches and we show which solution approach leads to a lower backorder through a statistical analysis. We conclude the research and hints on future directions of this research in the last section.

DLS PROBLEMS WITH UNCERTAIN DEMANDS AND BACKORDERS

When demand fluctuates high and the capacity in production/inventory is limited, constraints (2) may become infeasible. The infeasibility lies in the possibility of having a large enough demand that cannot be satisfied due to the limited capacities. To overcome this problem, we use backorder variables and update inventory balance constraints. In the work of [3, 5, 9, 12], the inventory balance constraints are updated by including backorder variables u_i^t. Assuming r_i^t is the backorder cost, we have the Capacitated MIDLS with Backorder (*CMIDLS-BO*) as follows:

$$\textit{CMIDLS-BO}: \quad \min \sum_{t\in\mathscr{T}}\sum_{i\in\mathscr{I}}(h_i^t q_i^t + p_i^t x_i^t + s_i^t y_i^t + r_i^t u_i^t) + \sum_{t\in\mathscr{T}\setminus\{1\}}\sum_{i\in\mathscr{I}}(o_i^t z_i^t) \tag{7}$$

$$\text{s.t.} \quad q_i^{t-1} + x_i^t + u_i^t \geq q_i^t + d_i^t + u_i^{t-1} \quad \forall i \in \mathscr{I}, \forall t \in \mathscr{T} \tag{8}$$

$$x_i^t \leq M^t y_i^t \quad \forall i \in \mathscr{I}, \forall t \in \mathscr{T} \tag{9}$$

$$z_i^t \geq y_i^t - y_i^{t-1} \quad \forall i \in \mathscr{I}, \forall t \in \mathscr{T}\setminus\{1\} \tag{10}$$

$$\sum_{i\in\mathscr{I}} x_i^t \leq C_P \quad \forall t \in \mathscr{T} \tag{11}$$

$$\sum_{i\in\mathscr{I}} q_i^t \leq C_Q \quad \forall t \in \mathscr{T} \tag{12}$$

$$x_i^t, q_i^t, u_i^t \geq 0,\ y_i^t \in \{0,1\} \quad \forall i \in \mathscr{I}, \forall t \in \mathscr{T} \tag{13}$$

$$z_i^t \in \{0,1\} \quad \forall i \in \mathscr{I}, \forall t \in \mathscr{T}\setminus\{1\} \tag{14}$$

Objective (7) is set to minimize the total cost of inventories, productions, set ups, start ups, and backorders. Constraints (8) are the inventory balance constraints. These constraints guarantee that in each period, the total previous inventory, production, and backorder for each item is equal to the demand plus summation of previous backorder and the ending inventory. The rest of constraints (9)-(14) are similar to (3)-(6), except that capacity constraints (11) and (12) are added and the definition of backorder variables is included in constraints (13).

Next, we use two point estimates, i.e. mean and median, to have two solution approaches to solve the *CMIDLS-BO* problem. We compare the solution approaches in terms of their backorders and we show the median is a better point estimate with demand asymmetrically distributed.

SOLUTION ANALYSIS

Solution Approaches

Before analyzing the solution methods, we first write *CMIDLS-BO* into its compact form. Let's show all variables by λ, the cost coefficients of all variables by C, demands by D, Matrix of left-hand-side of constraints (8) by A, and the feasible space of the rest of constraints by $\mathscr{X}$. The compact form is:

$$\begin{aligned} \mathscr{P}\colon \quad & \min \quad C\lambda \\ & \text{s.t.} \quad A\lambda \geq D \\ & \qquad\;\; \lambda \in \mathscr{X} \end{aligned}$$

Based on the historical data $D^k, \forall k \in \mathscr{K}$, we develop two approaches to solve $\mathscr{P}$ given the uncertain demand D. One approach is based on the mean of historical data $\bar{D}$ and the other approach is based on the median of historical data or $\widehat{D}$. We call the corresponding problems $\mathscr{P}_{\bar{D}}$ and $\mathscr{P}_{\widehat{D}}$. The purpose is to understand which approach results in a lower total backorder through a numerical analysis. This analysis is repeated for demands that have a Normal distribution, a Uniform distribution, and a Poisson distribution. In each case, the total backorder of each approach is computed and compared.

Numerical Analysis

In the numerical analysis, parameters are generated uniformly as integers within a lower bound and an upper bound that is shown by $IU[lb, ub]$. Particularly, $h_i^t \in IU[1,5]$, $p_i^t \in IU[10,20]$, $s_i^t \in IU[20,40]$, $o_i^t \in IU[15,25]$, and $r_i^t \in IU[100,200]$. Note that vector D of demands is available for the past $|\mathscr{K}| = 100$ time horizon. In another word, we have D^k for all $k \in \mathscr{K}$. If demands follow a Uniform distribution, we generate historical demands as $D^k \in [800,1000]$. If demands follow a Normal distribution, we generate historical demands as D^k as a normal random variable with a mean of 900 and standard deviation of 100. If demands follow a Poisson distribution, we generate historical demands as D^k (d_i^{tk}) as a Poisson random variable with a mean of 900 ($\mu = 900$). Note that if we use the mean as the point estimate, we set $D = \bar{D^k}$ and if we use median as the point estimate, we set $D = \widehat{D^k}$.

Additionally, initial inventory and capacity of production and inventory should be calculated. For this purpose, we assume the initial inventories are all zeros, i.e. $q_i^t = 0$. To find C_P and C_Q we first generate an array of $D^k \in [800,1000]$. Then, given γ being a random number between 2 and 3, these capacities are calculated as:

$$C_Q = \frac{1}{\gamma \times |\mathscr{K}|} \sum_{k \in \mathscr{K}} \sum_{i \in \mathscr{I}} d_i^{1k} \tag{15}$$

$$C_P = \frac{1}{|\mathscr{T}| \times |\mathscr{K}|} \sum_{k \in \mathscr{K}} \sum_{t \in \mathscr{T}} \sum_{i \in \mathscr{I}} d_i^{tk} \tag{16}$$

The number of items belongs to $|\mathscr{I}| \in \{10,20,30,40\}$ and the number of periods belongs to $|\mathscr{T}| \in \{40,60,80,100\}$. For every $i \in \mathscr{I}$ and $t \in \mathscr{T}$, we generate 10 problem instances according to the above settings. Note that the average of these instances are presented in this paper. These instances of $\mathscr{P}_{\bar{D}}$ and $\mathscr{P}_{\widehat{D}}$ are implemented in Python 7 and run on computer with 2×2.4 GHz CPU, 4 GB RAM, and 64-bit Windows operating system. Instances are solved using Gurobi 9.0.3 academic solver. The computational time is shown by cpu and the difference between total backorders of problems $\mathscr{P}_{\bar{D}}$ and $\mathscr{P}_{\widehat{D}}$ is shown by Δ. This difference is divided by the minimum backorder among $\bar{U}$ and $\widehat{U}$ to be shown by percentages. We define this measure as:

$$\Delta = \frac{\mathbb{1}(\bar{U} - \widehat{U})}{min\{\mathbb{1}\bar{U}, \mathbb{1}\widehat{U}\}} \tag{17}$$

where $\mathbb{1}$ is a vector of 1's of an appropriate size. Table 1 in the APPENDIX summarizes the numerical analysis of $\mathscr{P}_{\bar{D}}$ and $\mathscr{P}_{\widehat{D}}$ instances, when D is Uniformly distributed. Despite the choice of point estimate, $\bar{D}$ or $\widehat{D}$, all instances are solved in less than 6 seconds with an average being less than 1.5 seconds. The Δ or the percentage of relative difference between $\bar{U}$ or $\widehat{U}$ is sometimes negative and sometimes positive. On average, it is $\Delta = 0.09\%$.

Table 2 summarizes the numerical analysis of $\mathscr{P}_{\bar{D}}$ and $\mathscr{P}_{\widehat{D}}$ instances, when D is Normally distributed. Despite the choice of point estimate, all instances are solved in less than 5 seconds with an average being less than 1.5 seconds. The Δ is sometimes negative and sometimes positive. On average, it is $\Delta = -0.11\%$. These results are similar to Table 1.

Table 3 summarizes the numerical analysis of $\mathscr{P}_{\bar{D}}$ and $\mathscr{P}_{\widehat{D}}$ instances, when D is Poisson distributed. Despite the choice of point estimate, $\bar{D}$ or $\widehat{D}$, all instances are solved in less than 6 seconds with an average being less than 1.5 seconds. Unlike Uniform or Normal data in tables 1 and 2, here Δ is always positive, and it is $\Delta = 1.27\%$, on average. A positive value of Δ means the point estimate median can return lower backorders. In addition, Δ has increased more than tenfold when data is asymmetric (Poisson distribution). This spike in the value of Δ requires further attention. In the following, we discuss whether the value of Δ shows a significant difference between point estimates $\bar{D}$ and $\widehat{D}$.

Significance of Point Estimates

To test the significance of point estimates $\bar{D}$ and $\widehat{D}$, we conduct a hypothesis test to see if the total backorder changes when we use different point estimates. Tables 4-6 investigate this. Every row of these tables corresponds to a combination of $|\mathscr{I}|$ and $|\mathscr{T}|$, in which 100 instances are generated and solved. This means 1,600 instances are solved for the significance study.

In Table 4, parameter D assumes a Uniform distribution. When comparing point estimates of this parameter, i.e. $\bar{D}$ and $\widehat{D}$, we see there is no significant change as the p_{value}'s are very large. The minimum p_{value} is 0.47 which makes the difference between backorders when using different point estimates insignificant. Similar results have been observed when D assumes a Normal distribution in Table 4. We can conclude that the difference between backorders when using different point estimates is insignificant.

Note that both Uniform and Normal distributions are symmetric. When the distribution of demand is asymmetric, such as Poisson distribution, the difference between point estimates becomes more apparent; $\Delta = 1.27\%$. Particularly, Table 6, for a Poisson distribution, shows that p_{value}'s are very small, the maximum being 0.0021. This shows that when D follows a Poisson distribution, we are more than 99.79% confident that the backorders return by mean and median demands differ from each other. Note that the setting of the hypothesis test is for the difference. If we change the setting to the smaller backorders, the p_{value}'s will become even smaller. This means we will have even more confidence that the median point estimate reduces the backorders.

CONCLUSION AND FUTURE RESEARCH

This study investigates the use of two point estimates to reduce the backorders in Capacitated Dynamic Lot Sizing problems with uncertain demands. The two point estimates are the mean and median. It is shown that median can reduce total backorders significantly when demand distribution is asymmetric, i.e. it has a Poisson distribution. However, since service levels are not studied, further analysis is needed to assure demands are satisfied.

One possible future research is to study the effects of the median point estimate using real world data. Moreover, one can investigate more point estimates and more distributions. It is very important to incorporate service levels in the study to assure demands are met at the desired level. Finally, comparison of point estimates and other stochastic optimization approaches can be another future direction of this research.

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APPENDIX: TABLES

Table 1: Instances of $\mathscr{P}_{\bar{D}}$ and $\mathscr{P}_{\widehat{D}}$ problems when D has a Uniform distribution

$\lvert\mathscr{I}\rvert$	$\lvert\mathscr{T}\rvert$	$cpu\,(\mathscr{P}_{\bar{D}})$	$cpu\,(\mathscr{P}_{\widehat{D}})$	Δ%
10	40	0.102	0.106	0.16
	60	0.176	0.173	0.79
	80	0.281	0.277	0.3
	100	0.402	0.383	0.29
20	40	0.33	0.269	0.29
	60	0.575	0.523	0.37
	80	1.004	0.83	-0.18
	100	1.482	1.273	0.12
30	40	0.58	0.513	-0.06
	60	1.447	1.025	-0.1
	80	2.315	1.757	-0.46
	100	3.209	2.68	-0.01
40	40	0.906	0.842	0.27
	60	1.888	1.712	-0.33
	80	3.644	2.851	0.54
	100	5.354	4.229	-0.55
Average		1.481	1.215	0.09

Table 2: Instances of $\mathscr{P}_{\bar{D}}$ and $\mathscr{P}_{\widehat{D}}$ problems when D has a Normal distribution

$\lvert\mathscr{I}\rvert$	$\lvert\mathscr{T}\rvert$	$cpu\,(\mathscr{P}_{\bar{D}})$	$cpu\,(\mathscr{P}_{\widehat{D}})$	Δ%
10	40	0.13	0.106	-0.27
	60	0.188	0.173	-0.15
	80	0.29	0.284	-1.4
	100	0.418	0.393	-1.13
20	40	0.405	0.281	0.52
	60	0.623	0.525	0.15
	80	1.003	0.83	0.2
	100	1.457	1.221	-0.04
30	40	0.585	0.52	-0.03
	60	1.225	1.044	-0.06
	80	1.911	1.638	0.14
	100	3.273	2.407	-0.28
40	40	1.025	0.827	-0.03
	60	2.004	1.632	0.24
	80	3.297	2.704	0.34
	100	4.888	4.177	0.04
Average		1.42	1.173	-0.11

Table 3: Instances of $\mathscr{P}_{\bar{D}}$ and $\mathscr{P}_{\widehat{D}}$ problems when D has a Poisson distribution

$\lvert\mathscr{I}\rvert$	$\lvert\mathscr{T}\rvert$	$cpu\ (\mathscr{P}_{\bar{D}})$	$cpu\ (\mathscr{P}_{\widehat{D}})$	Δ%
10	40	0.096	0.105	0.5
	60	0.188	0.181	1.05
	80	0.288	0.291	1.23
	100	0.432	0.42	1.8
20	40	0.319	0.283	0.72
	60	0.607	0.529	1.41
	80	0.872	0.852	1.5
	100	1.313	1.281	2.02
30	40	0.66	0.527	0.74
	60	1.151	1.024	1.05
	80	1.933	1.761	1.44
	100	2.975	2.627	1.89
40	40	0.913	0.835	0.64
	60	2.087	1.752	1.34
	80	3.61	3.113	1.27
	100	5.165	4.794	1.78
Average		1.413	1.273	1.27

Table 4: p_{value} comparison of instances of $\mathscr{P}_{\bar{D}}$ and $\mathscr{P}_{\widehat{D}}$ problems, when D has a Uniform distribution

$\lvert\mathscr{I}\rvert$	$\lvert\mathscr{T}\rvert$	p_{value}
10	40	0.98
	60	0.77
	80	0.06
	100	0.10
20	40	0.87
	60	0.70
	80	0.48
	100	0.68
30	40	0.31
	60	0.98
	80	1.00
	100	0.71
40	40	0.74
	60	0.70
	80	0.47
	100	0.73

Table 5: p_{value} comparison of instances of $\mathscr{P}_{\bar{D}}$ and $\mathscr{P}_{\widehat{D}}$ problems, when D has a Normal distribution

$\lvert\mathscr{I}\rvert$	$\lvert\mathscr{T}\rvert$	p_{value}
10	40	0.93
	60	0.54
	80	0.95
	100	0.89
20	40	0.89
	60	0.58
	80	0.61
	100	0.71
30	40	0.96
	60	0.72
	80	0.47
	100	0.60
40	40	0.83
	60	0.64
	80	0.93
	100	0.87

Table 6: p_{value} comparison of instances of $\mathscr{P}_{\bar{D}}$ and $\mathscr{P}_{\widehat{D}}$ problems, when D has a Poisson distribution

$\lvert\mathscr{I}\rvert$	$\lvert\mathscr{T}\rvert$	p_{value}
10	40	2.10E-03
	60	7.50E-07
	80	1.63E-05
	100	6.25E-09
20	40	1.39E-06
	60	1.61E-07
	80	2.86E-09
	100	1.09E-12
30	40	2.89E-07
	60	1.25E-13
	80	1.94E-14
	100	6.96E-17
40	40	8.66E-12
	60	2.36E-13
	80	3.41E-16
	100	2.73E-19