JUSTIFICATIONS STUDENTS USE WHEN WRITING AN EQUATION DURING A MODELING TASK

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Literature typically describes mathematization, the process of transforming a real-world situation into a mathematical model, in terms of desirable actions and behaviors students exhibit. We attended to STEM undergraduate students' quantitative reasoning as they derived equations. Analysis of the meanings they held for arithmetic operations ($+, -, \cdot, \div$) provided insight into how participants expressed real-world relationships among entities with arithmetic relationships among values. We extend the findings from K-12 literature (e.g., using multiplication to instantiate a rate) to STEM undergraduates and found evidence of new ways of justifying the usage of arithmetic operations (e.g., using multiplication to instantiate an amount).

Keywords: Modeling, Undergraduate Education, Mathematical Representations

Mathematical modeling (hereon called modeling) provides a venue within which to promote STEM education, a stance taken by the US government (Committee on STEM Education, 2018, p. 17; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), and researchers alike. Modeling is typically defined as the cyclical process of taking a real-world phenomenon, transforming that phenomenon into a mathematical representation (called mathematizing), working with that mathematical representation, and interpreting that work back into the real-world. The idea to use modeling activities to enrich STEM education is well established. Modeling helps students develop general competences and attitudes towards creative problem-solving while building feelings of self-reliance and competence, prepare to live and act as informed socially conscious citizens, use math to describe extra-mathematical situations, see a richer more comprehensive picture of mathematics, and acquire, learn, and keep mathematical concepts by providing motivation for and relevance of mathematical studies (Blum & Niss, 1991).

Literature on modeling competencies, and specifically mathematizing, typically frame the research as identifying some sort of "blockage" or that an action associated with a competency was "difficult" to perform (Brahmia, 2014; Galbraith & Stillman, 2006; Jankvist & Niss, 2020; Stillman & Brown, 2014). While these descriptive actions are an important step in fully understanding the complexity of mathematizing, we do not yet know why certain actions, such as writing an equation, is difficult for students. In particular, it is often unclear how or why a participant chooses to represent a real-world relationship with a given arithmetic operation ($+, -, \cdot, \div$). Understanding the justifications students use when writing down their equations will inform facilitator scaffolding of mathematizing and may also provide insight into when and how students ensure their resulting equations are adequate. We target this one aspect of mathematizing in order to make strides towards developing rich, empirically- and theoretically-grounded descriptions of students' mathematical reasoning during mathematizing.

Theoretical Framework

Previous studies have indicated that quantitative reasoning promotes building of models (Ellis, 2007; Ellis, Ozgur, Kulow, Williams, & Amidon, 2012; Mkhatshwa, 2020) and is a lens with which to understand STEM undergraduates' reasoning while mathematizing during a modeling task (Carlson, Larsen, & Lesh, 2003; Czocher & Hardison, 2021; Larson, 2013). Czocher and Hardison (2019, 2021) report findings of a task-based interview with a participant working on a modeling problem. Their retrospective analysis documented how the participant's model changed through the course of the interview by attending to quantities imposed onto the task situation, the relationships among those quantities, mathematical inscriptions, and changes to each of those elements. An interesting implication of this analysis is that the models that students could potentially make during a modeling task depend on (and are constrained by), the quantities the participant imposes onto the situation. Given the arguments and implications of the literature above, it is appropriate to use quantitative reasoning as a lens to study students' reasoning while mathematizing.

In the remainder of this section, we put together the theoretical constructs from quantitative reasoning in order to describe the justifications students used for certain arithmetic operations $(+, -, \cdot, \div)$ when writing an equation during a modeling task. "Quantitative reasoning is the analysis of a situation into a quantitative structure- a network of quantities and quantitative relationships" (Thompson, 1993). A quantity is a mental construct, whose creation is (often) effortful (Thompson, 2011). That effort is characterized by the act of someone conceptualizing an object that has some attribute and intending to measure that attribute. (Thompson & Carlson, 2017, p. 425). This means a quantity is a triple consisting of an object, attribute, and quantification (Thompson, 2011). Quantification in this instance means to conceptualize some object with an attribute that has a measure, and that measure has a proportional relationship with its unit (Thompson, 2011). Further, quantification can be operationalized as the set of operations an individual could enact on the attribute (Hardison, 2019). According to Ellis (2007), length, area, volume, cardinality, speed, temperature, and density are all attributes of some object that can undergo quantification. A quantity is made from an individual's conceptions of objects within the situation, rather than the objects or situations themselves (Ellis, 2007), further, each individual could quantify an attribute using differing sets of operations, therefore a quantity is idiosyncratic to the individual. Because a quantity is idiosyncratic to the individual, quantities are (cognitively) distinct from (mathematical) variables.

A quantitative operation is a conceptual operation where an individual creates a new quantity in relation to one (or more) already created quantities (Ellis, 2007; Thompson, 2011). Enacting a quantitative operation upon two quantities can be thought of as composing or combining two quantities (which could be denoted through arithmetic operations) to yield the new one. Thompson (2011) outlined the arithmetic operations associated with the quantitative operation as seen in his data with students k-12, see **Error! Reference source not found.**1.

Table 1 Arithmetic operations done to construct a new quantity (Thompson, 2011, p. 43).

Structure	Arithmetic Operation to Evaluate the Resultant Quantity
A quantity is the result of an additive combination of two quantities	Addition
A quantity is the result of an additive comparison of two quantities	Subtraction
A quantity is the result of a multiplicative combination of two quantities	Multiplication
A quantity is the result of a multiplicative comparison of two quantities	Division
A quantity is the result of an instantiation of rate	Multiplication
A quantity is the result of a composition of ratios	Multiplication

The left side of Table 1 describes the type of relationships between two quantities that results in a new quantity. In other words, this left-hand side is the schema of action used to apply arithmetic operations to symbols that represent quantities. A schema of action is an organized pattern of thoughts or behaviors (actions) that can be applied to different cognitive objects in different situations (Nunes & Bryant, 2021). For example, in k-12 literature, two schemas of action for addition are: putting together (i.e., quantity is the result of additive combination), and part-whole relations (i.e., a comparison of two quantities) (Nunes & Bryant, 2021). Additionally, a schema of action for multiplicative relationships could be a one-to-many correspondence for quantities with a fixed ratio relationship (i.e., a multiplicative combination of two quantities). There might be other schemas of action participants are using to apply structure to a real-world phenomenon. Structure here is defined to be the schema of action, and the arithmetic operations are the result of the imposition of the schema of action in the mathematical representation.

Research has shown that some schemas of action are not advantageous for combining quantities. As pointed out by Schwartz (1988), the notion of multiplication being a one-to-many correspondence does not work for cases such as (miles/hours * hours = miles), because iterating the relationship between miles and hours "number of hours times" cannot be done. Similarly, Brahmia (2014) argued that students who conceptualize multiplication as repeated addition are not prepared to conceptually understand products of quantities. Additionally, students who see division only as an operation that creates parts from a whole do not have the mental operations available to conceptualize ratio quantities (e.g., density, velocity). That is, students being able to effectively construct quantitates in applied contexts must have strong conceptualizations of multiplication and division that differ from the schema of action of "repeated addition" and "creating parts from a whole" (Brahmia, 2014).

In order to describe the justifications participants hold for the equations they write down, we address the question: *What schema of action do STEM undergraduates use to justify their choices of arithmetic operations while mathematizing during a modeling task?*

Methods

This study was part of a larger study of facilitator scaffolding moves that foster participants' modeling competencies. For the present study, 11 STEM undergraduates participated in sets of individual cognitive task-based interviews through Zoom. Each participant saw at least six tasks over ten sessions. The tasks were modeling problems designed to scaffold participants' modeling activities by attending to quantitative reasoning and appealing to similarities in structures. We report findings from three participants' work on the *Cats and Birds* task (see below). Pattern, Neturo, and Khriss were all enrolled in, or had already taken, differential equations (DE) at the time. Pattern majored in civil engineering while both Neturo and Khriss majored in physics with a minor in mathematics. These three participants were chosen from the larger study to showcase schema of action that differ from those described in previous literature.

> *Cats & Birds.* Cats, our most popular pet, are becoming our most embattled. A national debate has simmered since a 2013 study by the Smithsonian's Migratory Bird Center and the U.S. Fish and Wildlife Service (Raasch, 2013) concluded that cats kill up to 3.7 billion birds and 20.7 billion small mammals annually in the United States. The study blamed feral "unowned" cats but noted that their domestic peers "still cause substantial wildlife mortality." In this problem, we will build a model (step-by-step) that predicts the species' population dynamics, considering the interaction of the two species.

In this task, participants were asked to consider a back-yard habitat where birds are the primary prey for cats. The participants were then asked a series of questions with the aim of having them write an equation for the instantaneous rate of change of the bird and cat population. Here we report the students' work to build a differential equations model dynamics of the bird population due only to cat predation. It is appropriate to study students' mathematization while modeling in dynamic tasks like Cats and Birds because dynamic tasks elicit dynamic reasoning, which is connected to quantitative reasoning (Keene, 2007).

Data were analyzed in four phases. First, the transcripts and written work were segmented according to changes in discussion topic (e.g, the interviewer asked a new question). Second, we identified the quantities the participant imposed onto the task scenario by describing the object, attribute and how the participant exhibited quantification for that attribute according to the quantification criteria developed Czocher and Hardison (2021). In phase three, we noted instances of arithmetic operations used on the quantities the participant constructed and documented the participants' reason (or inferred reason when the participant's reason was not stated) for using that specific arithmetic operation and if the instance is a quantitative operation, that is, the arithmetic operation is done to create a new quantity. In phase four, we compared the reason we inferred the participant was using to the schema of action documented in Thompson (2011) and Nunes and Bryant (2021) and noted if the schema of action matched previous literature or was not present in previous literature. For example, Neturo stated that he used multiplication because one cat will meet many birds. Neturo's explanation was comparable to a one-to-many correspondence as described by Brahmia (2014) and Nunes and Bryant (2021).

Results

We report the schema of action participants employed when expressing arithmetic operations with the quantities they imposed onto the Cats and Birds task as they built their model for the rate of change of the bird population due only to cat predation. According to our theoretical perspective, arithmetic operations are the result of imposing the schema of action in the mathematical representation and thus tell us how the participant was justifying (either explicitly or implicitly) their usage of arithmetic operations. First, we provide an overview of all of the inferred schema of action exhibited by our participants in Table 2. This table reports the quantities the participant combined, the resultant quantity, the inferred schema of action, and if this schema of action was present or not present in previous literature.

Table 2 Schema of action exhibited by all three participants

Quantity 1	Quantity 2	Resultant quantity	Schema of action	In previous literature?
Number of birds at time t	Number of cats at time t	Number of total possible cat-bird interactions that could occur in the back yard habitat.	Multiplication- One-to-many correspondence (Brahmia, 2014; Nunes and Bryant, 2021,)	Yes
Rate of encounters between cats and birds per time interval	Number of total possible cat-bird interactions	Number of encounters per unit time	Multiplication- Instantiate a rate. (Thompson, 2011)	Yes
Percentage of encounters that are realized	Number of total possible cat-bird interactions	Number of encounters that are realized	Multiplication- Subsetting	No
Number of encounters that result in a death	Duration of time	Number of birds that die during Δt	Multiplication- Instantiate an amount.	No
Total accumulated dead birds during a time segment	Total number of observations of the population made during a time segment	Average change in the bird population	Division - Separating into equal parts. (Brahmia, 2014)	Yes
Change in the bird population due to cats	Number of birds that die during Δt	Change in the bird population due to cats per change in time	Division - Evaluate a quantity that is an operand of a quantitative operation. (Thompson, 2011)	Yes
Number of birds present at the beginning of a time segment Δt	Number of birds at the end of a time segment Δt	Change in the bird population due to cats	Subtraction - Additive comparison between two quantities (Thompson, 2011)	Yes

Quantity 1	Quantity 2	Resultant quantity	Schema of action	In previous literature?
Number of lethal encounters between a cat and a bird on day t_1	Number of lethal encounters between a cat and a bird on day t_n	Total accumulated dead birds during a time segment	Addition - Additive combining multiple quantities	No

Overall, our participants did exhibit schema of action described in previous literature. This indicates that schemas of action present in k-12 literature are also exhibited by undergraduate STEM students. We next illustrate the three schemas of action not present in the previous literature.

Schema of action not present in the literature

Additively combining multiple quantities. Khriss was asked how he could model the decrease in the bird population due to cats during some arbitrary passage of time $\Delta\tau$. Khriss initally reponded by writing down the symbol Σ, indicating he was thinking of constructing some form of summation. The interviewer suggested he find the decrease in the bird population due to cats after five days, instead of an arbitrary duration of time. In Khriss' reasoning, the number of birds dying each day was variable. He used $LE(t)$ to represent the number of lethal encounters between a cat and a bird at a specific time t. In response to the prompt, Khriss wrote $\int_0^5 LE(t)dt$, denoting summation. Khriss understood the decrease in bird population during five days as an accumulation of the non-constant values of the number of lethal encountes on a specific day. Khriss used this schema of action again when he was asked to consider the average rate of change of the bird population during some number of days D. In reponce to this, Khriss labeled multiple instances of time that the number of lethal encounters were recorded that occured during the number of days D. He labeld these instances of time $t_1, t_2, t_3, \dots, t_n$. Khriss then added the number of lethal encounters for each day to yeild the total accumulated dead birds during some number of days D. The equation written for this calculation was $\sum_{t=1}^{n} LE(t)$. A snapshot of Khriss's work is provided to showcase how Khirss thought of this sumation (see Figure 1), and below we provide a quote of Khriss explaning his summation.

Khriss: So during a segment of time D, to find an average I guess we would need multiple readings of what the population is throughout that time. I guess we could we use t_1, t_2, t_3, t_n. Yeah, I guess that would be it. Just to make sure what question I'm

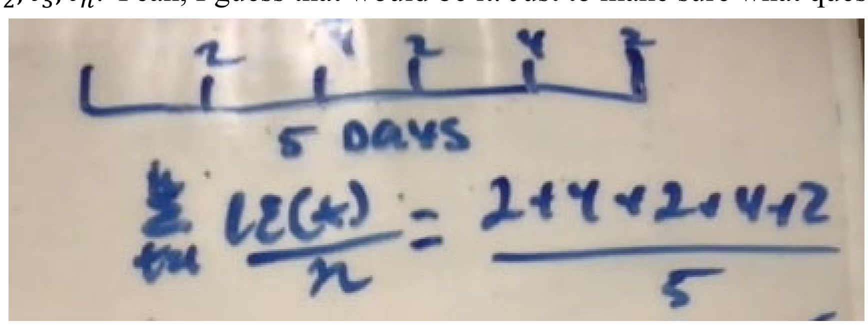

Figure 1 Average change in bird population over five days according to Khriss

answering here, I have lethal encounters at time t adding up all of those for each different t all the way to my last t and dividing that by the number of measurements gives me an average. So that gives me my average number of dead birds within t.

His schema of action for both of these instances was additively combing multiple quantities to arrive at a new quantity. The schema of action described in Thompson (2011) occurs strictly between two quantities. Because Khriss is adding more than one quantity at a time, Khriss' schema of action is different. We infer that Khriss' schema of action is a step up in complexity from the schema described in Thomspon (2011), because Khriss is able to implicetly construct each measure of dead birds between $LE(t_1)$ and $LE(t_n)$ in order to combine an unknown number of quantites additevely.

Multiplication for subsetting. Pattern had previously constructed a quantity that represented the number of total encounters, which is calculated by $B(t) \cdot C(t)$. In order to account for the fact that not all birds encounter all cats, he constructed two new quantities. The first quantity represented the percentage of encounters that are realized, labeled α, and the second quantity represented the number of encounters that are realized, calculated by $\alpha \cdot C(t) \cdot B(t)$. Pattern said that he multiplied α and $C(t) \cdot B(t)$ to take a percentage of the total possible encounters, indicating the schema of action for this instance of multiplication was different than a one-to-many correspondence, and was more like subsetting from a larger amount.

Pattern: Okay. So now that we have a percentage, then you just do $C(t)$ times $B(t)$ times α equals encounters. Because you're going to take… So this is the total possible encounters that could possibly happen if perfect conditions are met for each cat to meet each bird, and then you're going to take a percentage of that total, and that would be your total...

Multiplication for instantiating an amount. Pattern had previously constructed a quantity that represents the number of birds that have died, calculated by $\beta \cdot C(t) \cdot B(t)$. Pattern then constructed a new quantity that representsed the number of birds that die during Δt, labeled D_t, which was calculated by $\left(C(t) \cdot B(t)\right) \cdot \beta \cdot \Delta t$. Pattern multiplied the amount of birds that have died, which he did not view as connected to any specific duration? of time, by some duration of time to yield the number of birds that have died during Δt. The schema of action used for multiplication was combining two quantities, however this was not a one-to-many correspondence and was also different from the subsetting schema of action. We infer that Pattern was multiplying by Δt in order to instantiate the number of birds that have died specifically during Δt. Pattern did not evidence multiplying a rate and a duration of time to yield an amount (per-time rate $\cdot\ \Delta t =$ amount) because Pattern did not associate a duration of time with $\beta \cdot C(t) \cdot B(t)$. For Pattern, the quantity calculated by $\beta \cdot C(t) \cdot B(t)$ was an amount, not a rate. The quote below shows Pattern explaining that multiplying by Δt transforms $\beta \cdot C(t) \cdot B(t)$ into the quantity that represents mortality (the number of birds that die) during Δt,

Pattern: I honestly just know that I have to add [append] this Δt, and it kind of makes sense in my head because this mortality doesn't have a specific time attached to it. It (referring to $C(t) \cdot B(t)$) has already the t, but that's just telling you the amount of cats and birds at that time. It doesn't tell you how much time has passed. So if you multiply all of that by this number (pointing at Δt),… this is time that has passed….It's not encounters anymore, now it's.. What this is telling me the mortality during Δt.

Discussion

Collectively, our participants exhibited schema of action discussed previously in the literature. Patten constructed a quantity by additively comparing between two quantities. Khriss used division when finding the average by separating a quantity into equal parts. Pattern and Khriss used division to evaluate a quantity that is an operand of a quantitative operation. Neturo, Pattern, and Khriss used multiplication to combine to quantities in a one-to-many correspondence fashion. Lastly, Khriss used multiplication to instantiate a rate. All of these schemas of action are documented in previous literature (Brahmia, 2014; Nunes & Bryant, 2021; Thompson, 2011). Our participants also exhibited schema of action not present in the previous literature. Khriss used addition to additively combine multiple quantities instead of just two quantities. Neturo used a minus sign to indicate a negative magnitude for a quantity. Khriss showcased a new schema of action where he simultaneously additively combined multiple quantities in order to arrive at a new quantity. Khriss' schema of action is a step up in complexity from the schema described in Thompson (2011) because Thompson's definition is restricted to combining two quantites and Khriss is implicitly combining an unknown number of quantities. Pattern constructed a quantity that indicated a fraction or percentage and then multiplied by a quantity that represents an amount to gain some fraction of the original quantity. This schema of action does resemble the schema of action described in Thompson (2011) as a multiplicative combination of two quantities, however the schema of action for multiplication was different from a "one-to-many" correspondence described by Brahmia (2014) and Nunes and Bryant (2021). Pattern used multiplication in order to take a subset from a larger whole. Lastly, Pattern exhibited a new schema of action where multiplication was used to combine two quantities in order to instantiate an amount.

Overall, our results document justifications STEM undergraduates used when carrying out arithmetic operations ($+, -, \cdot, \div$) on the quantities while mathematizing that were not previously in the literature. We extended what is known about schema of action present in k-12 literature by confirming the presence of schema of action already described, and by introducing three new schemas of action. We postulate that these new schemas of action were observable because our participants are working on a modeling task focused on a predator-prey relationship. This conjecture is based findings that the models participants make are depend on (and are constrained by), the quantities the participant imposes onto the situation (Czocher & Hardison, 2019, 2021), which inherently impacts the types of relationships the participants were able to express using arithmetic operations. Attending to these new schemas of action reveals how students are expressing real-world relationships among entities with arithmetic relationships among quantities, specifically in a predator-prey context. We speculate additional schema of actions could be observed for arithmetic operations in other task scenarios that call for advanced mathematics. Once we know more about the justifications students use when utilizing arithmetic operations, we can then design task scaffolding moves that directly address, preempt, or build upon, those justifications.

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