

Classification of Frobenius Forms in Five Variables



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1 Introduction

Fix a field k of prime characteristic p . A Frobenius form over k is a homogeneous polynomial in indeterminates $x_1, \dots, x_n$ of the form

$$x_1^{p^e} L_1 + x_2^{p^e} L_2 + \cdots + x_n^{p^e} L_n \quad (1)$$

where each L_i is some linear form and e is a positive integer. Put differently, a Frobenius form is a homogeneous polynomial of degree $p^e + 1$ that is in the ideal generated by $x_1^{p^e}, \dots, x_n^{p^e}$.

Frobenius forms can be compared to quadratic forms: if we allow $e = 0$ in the expression (1) above, we get a quadratic form. Quadratic forms are well-studied in the classical literature. For example, much is known about the geometry of quadric

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hypersurfaces, and, at least over a quadratically closed field, their classification up to linear changes of coordinates is well-known. Both admit a convenient matrix factorization, where the action of changing coordinates behaves similarly.

One special but very interesting case of Frobenius forms are those of degree three (which are necessarily defined over a field of characteristic two). A smooth cubic surface is always Frobenius split, it turns out, *unless* it is defined by a Frobenius form; in particular, non-Frobenius split (smooth) cubic surfaces exist only over fields of characteristic two [8, 5.5]. A detailed examination of non-Frobenius split cubic surfaces of characteristic two, including the non-smooth ones, was undertaken in [10]. In particular, the Frobenius forms of degree three in up to four variables are classified there, up to projective equivalence. In this paper, we extend that classification to arbitrary Frobenius forms in up to five variables. Put differently, we classify the projective equivalence classes of three-dimensional projective hypersurfaces defined by Frobenius forms—a class called *extremal three-folds* in [11]. Section 4 describes this classification in detail, including a particularly “sparse” equation representing each of the seven types of projective equivalence classes of extremal three-folds.

Frobenius forms and the projective schemes they define have various “extreme” properties, both algebraically and geometrically. For example, cubic surfaces defined by Frobenius forms are characterized by the geometric property that they “contain no triangles”—that is, any plane section consisting of three lines *must* contain a point on all three [10, 5.1]. Analogous extremal configurations of linear subvarieties occur more generally for extremal hypersurfaces of higher degree and dimension; see [11, §8]. Algebraically, reduced Frobenius forms can be characterized as those achieving the minimal possible F -pure threshold among reduced forms of the same degree [11, 1.1].

2 Matrix Factorization of Frobenius Forms

Fix a field k of prime characteristic p . Let q denote an integral positive power of p .

The beauty of Frobenius forms is that, like quadratic forms, they admit a matrix factorization: a Frobenius form h in n variables can be written as

$$h = \begin{bmatrix} x_1^{p^e} & x_2^{p^e} & \cdots & x_n^{p^e} \end{bmatrix} A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

for some $n \times n$ matrix A with entries in k . Because $e > 0$, the matrix A representing the Frobenius form h is *unique*.

When $e = 0$, we recover the case of quadratic forms. In the quadratic form case, of course, A is not unique, but we can force uniqueness, for example, by insisting that A be symmetric (when $p \neq 2$).

There is an interesting story of Frobenius forms with “conjugate symmetric” matrices, where conjugacy is defined using a Frobenius automorphism. By definition, a *Hermitian Frobenius form* over k is a Frobenius form whose matrix A satisfies $A^\top = A^{[q]}$ for some $q = p^e$ (in particular, such a matrix is defined over $\mathbb{F}_{q^2}$). This special class of Frobenius forms gives a characteristic p analog of Hermitian forms over the complex numbers, and has been studied, for example, in [1, 9, 15] and [12, §35]. Like quadratic forms over quadratically closed fields, there is exactly one non-degenerate Hermitian Frobenius form, up to projective equivalence, in each dimension [1, 4.1] over $\mathbb{F}_{q^2}$. As we will see, the classification of more general Frobenius forms is more complicated.

To classify Frobenius forms up to projective equivalence, we need to understand how linear changes of coordinates act on them.

Let $g \in GL_n(k)$ be any linear change of coordinates for the polynomial ring $k[x_1, \dots, x_n]$. We represent the action of g on the variables as

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \mapsto g \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

the usual matrix multiplication. In particular, when g acts¹ on the Frobenius form h , we have

$$g \cdot h = g \cdot \left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \right) = \begin{bmatrix} x_1^{p^e} & x_2^{p^e} & \cdots & x_n^{p^e} \end{bmatrix} (g^{[p^e]})^\top A g \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

where $g^{[p^e]}$ denotes the matrix whose entries are the p^e -th powers of the entries of g and $B^\top$ indicates the transpose of a matrix B . Thus, the action of g on the Frobenius form h transforms the matrix A representing h into the matrix $(g^{[p^e]})^\top A g$.

A Frobenius form is said to be **nondegenerate** if it cannot be written, after a linear change of coordinates, in a smaller number of variables, and is otherwise **degenerate**, similarly as for quadratic forms. We define its **embedding dimension** to be the smallest number of variables needed to write f up to linear change of coordinates.

The **rank** of a Frobenius form is defined as the rank of the representing matrix, just as the rank of a quadratic form is the rank of the corresponding symmetric matrix when $p \neq 2$. The rank of a Frobenius form is invariant under changes of

¹ We write “ $g \cdot h$ ” to indicate the group action on polynomials, whereas adjacency, “ gB ,” will indicate the usual matrix multiplication on an $n \times m$ matrix B .

coordinates, since the rank of a matrix is unchanged by multiplication on the left and right by invertible matrices.

The rank of a Frobenius form is equal to the codimension of the singular locus of the corresponding hypersurface [11, 5.3]. This is analogous to the corresponding statement about quadratic forms of non-even characteristic, when we work with the corresponding symmetric matrix.

There is precisely one full rank Frobenius form, up to change of coordinates, over an algebraically closed field, in each fixed degree $p^e + 1$ and embedding dimension n :

Theorem 2.1 ([11, 6.1][2]) *Every full rank Frobenius form over an algebraically closed field k of characteristic $p > 0$ is represented, in suitable linear coordinates, by the diagonal form $x_1^{q+1} + \dots + x_n^{q+1}$, where q is an integral power of p . $\square$*

Put differently, every smooth projective hypersurface defined by a Frobenius form is projectively equivalent to one defined by $x_1^{q+1} + x_2^{q+1} + \dots + x_n^{q+1}$. This is analogous to the situation for quadratic forms over a quadratically closed field of characteristic not two, though the proof is a bit more involved. In particular, we do not have a complete understanding of the situation over non-closed fields, an interesting open problem. We have no counterexample to the speculation that the full rank Frobenius forms may be “diagonalizable” over a perfect field closed under all degree p^e extensions.

In the non-full rank case, there are finitely many projective equivalence classes of Frobenius forms in each fixed degree and embedding dimension [11, 7.4]. Indeed, the number of non-degenerate Frobenius forms of embedding dimension n (of fixed degree) is bounded above by the n -th Fibonacci number. However, the paper [11] stopped short of precisely classifying the Frobenius forms in each dimension, a task essentially completed for Frobenius forms in four variables in [10]. The case of five variables is treated here in Sect. 5. For the statement, see Sect. 4.

3 Quadratic Forms

To complete our story, we recall the classification of quadratic forms:

Proposition 3.1 *Let f be a non-degenerate quadratic form in n variables over a quadratically closed² field. If the characteristic of k is two, then f is projectively equivalent to either*

1. $x_1x_2 + x_3x_4 + \dots + x_{n-2}x_{n-1} + x_n^2$ if n is odd, or
2. $x_1x_2 + x_3x_4 + \dots + x_{n-3}x_{n-2} + x_{n-1}x_n$ if n is even.

² By quadratically closed, we mean that every degree two polynomial over k splits. In characteristic two, this is a stronger assumption than requiring that the field contain the square root of every element.

If the characteristic of k is not two, then f is projectively equivalent to

$$x_1^2 + x_2^2 + \cdots + x_n^2.$$

In particular, over a quadratically closed field, there is exactly one quadratic form in each embedding dimension. $\square$

This classification is well known, but since we could not find a low-tech proof in the modern literature for the case $p = 2$, we include one here. Alternate discussions using more machinery can be found, for example, in [5, I.16], [7, 12.9] or [6, 7.32], which also contain more refined classifications over non-closed fields. A lower-tech proof can be found in the classic book of [4], but for the convenience of the reader we include a direct and concise proof here.³

Proof The only quadratic form in one variable is x_1^2 . Likewise, the two-variable case is trivial: a degree two form in two variables must factor into two linear forms over a quadratically closed field, so in suitable coordinates, the form is either x_1x_2 or x_1^2 (which is degenerate). $\square$

Case of Characteristic Not Two

It is straightforward to check (even without closure assumptions on k) that a suitable choice of linear change of coordinates puts f in the form $\lambda_1x_1^2 + \cdots + \lambda_nx_n^2$, where the λ_i are nonzero (e.g., see [13]). So, if the ground field is quadratically closed, the change of coordinates taking each $x_i \mapsto \frac{1}{\sqrt{\lambda_i}}x_i$ normalizes the form to $x_1^2 + \cdots + x_n^2$.

Case of Characteristic Two

Say that $n \geq 3$. Since f is non-degenerate, it is not the square of a linear form. Thus some square-free term, which we can assume to be x_1x_2 , appears with nonzero coefficient. Scaling, we may assume the coefficient of x_1x_2 is 1.

Now write f in the form

$$L^2 + x_1x_2 + \sum_{j=3}^n a_{1j}x_1x_j + \sum_{j=3}^n a_{2j}x_2x_j + h_1(x_3, \dots, x_n) \quad (2)$$

where L is a (possibly zero) linear form in x_1, x_2 , and h_1 is a quadratic form in $x_3, \dots, x_n$. Apply the linear change of coordinates sending x_2 to $x_2 + \sum_{j=3}^n a_{1j}x_j$, fixing the other variables. This transforms (2) into an expression which can be written

³ While the stated classification of quadratic forms in characteristic two is well known, it appears that an elementary proof is not. In his history of quadratic forms [14], Scharlau laments that Dickson's 1899 work [3], which is elementary but "rather involved," is not better known, stating "However, one must admit, that this paper—like most of Dickson's work—is not very pleasant to read. It is entirely algebraic." We hope the reader will find our straightforward and entirely algebraic proof more pleasant to read.

$$L^2 + x_1x_2 + \sum_{j=3}^n a'_{2j}x_2x_j + h_2(x_3, \dots, x_n), \quad (3)$$

where again h_2 is a quadratic form in $x_3, \dots, x_n$.

With another change of coordinates, we may assume the summand $\sum_{j=3}^n a'_{2j}x_2x_j$ is zero. Indeed, if some a'_{2j} is nonzero, then we can assume $a'_{23} = 1$ after renumbering and scaling if necessary. Now apply the linear transformation sending x_3 to $x_3 + \sum_{j=4}^n a'_{2j}x_j$, fixing the other variables. This transforms the expression (3) into

$$L^2 + (x_1 + x_3)x_2 + h_3(x_3, \dots, x_n), \quad (4)$$

where h_3 is quadratic in $x_3, \dots, x_n$. Next, the coordinate change taking x_1 to $x_1 + x_3$, fixing the other variables, transforms (4) into the form

$$L^2 + x_1x_2 + h_4(x_3, \dots, x_n), \quad (5)$$

where h_4 is quadratic.

Finally, by induction, we separately apply linear changes of coordinates to the quadratic $L^2 + x_1x_2$ in $\{x_1, x_2\}$ and the quadratic h_4 in $\{x_3, \dots, x_n\}$ to put each into the desired form. It is easy to see, then, that their sum has the desired form as well, depending on the parity of n in the stated way. This completes the proof. $\square$

Remark 3.2 Our proof easily adapts to show the well-known basic fact that a quadratic form over an arbitrary field of characteristic two is a sum of binary quadratics in distinct variables (plus a quadratic in one variable if the embedding dimension is odd). Alternatively, our proof adapts to prove Theorem 199 in [4] over any perfect field of characteristic two.

Remark 3.3 Quadratic forms behave like Frobenius forms from the point of view of achieving the minimal F -pure threshold. For a reduced form of degree d , it is proved in [11, 1.1] that the F -pure threshold is at least $\frac{1}{d-1}$, with equality if and only if the form is a Frobenius or quadratic form. Another way in which Frobenius forms and quadratic forms are similar is that the corresponding hypersurfaces both contain many high-dimensional linear subvarieties; see [11, §8].

4 Frobenius Forms in Five Variables: Classification Statements

Fix an algebraically closed field k of positive characteristic p . Let q be an integral positive power of p .

Theorem 4.1 *There are seven projective equivalence classes of Frobenius forms of a fixed degree $q + 1$ and embedding dimension five. Specifically, these are represented by the following forms:*

1. $x_1^{q+1} + x_2^{q+1} + x_3^{q+1} + x_4^{q+1} + x_5^{q+1}$ (rank 5)
2. $x_1^q x_5 + x_2^q x_4 + x_3^{q+1} + x_4^q x_2$ (rank 4)
3. $x_1^q x_5 + x_2^q x_4 + x_3^{q+1} + x_4^q x_1$ (rank 4)
4. $x_1^q x_5 + x_2^q x_4 + x_3^q x_2 + x_4^q x_1$ (rank 4)
5. $x_1^q x_5 + x_2^q x_3 + x_3^q x_2 + x_4^q x_1$ (rank 4)
6. $x_1^q x_5 + x_2^q x_4 + x_3^{q+1}$ (rank 3)
7. $x_1^q x_5 + x_2^q x_4 + x_3^q x_2$ (rank 3)

For completeness, we also describe degenerate Frobenius forms in five variables in the following two theorems. The proofs are the same as in [10], although that source considered only cubic Frobenius forms.

Theorem 4.2 *There are five projective equivalence classes of Frobenius forms of a fixed degree $q + 1$ and embedding dimension four. Specifically, these are represented by the following forms:*

1. $x_1^{q+1} + x_2^{q+1} + x_3^{q+1} + x_4^{q+1}$ (rank 4)
2. $x_1^q x_4 + x_2^{q+1} + x_3^q x_1$ (rank 3)
3. $x_1^q x_4 + x_2^q x_3 + x_3^q x_1$ (rank 3)
4. $x_1^q x_4 + x_2^q x_3 + x_3^q x_2$ (rank 3)
5. $x_1^q x_4 + x_2^q x_3$ (rank 2)

Theorem 4.3 *There are three projective equivalence classes of nondegenerate Frobenius forms in three variables in each fixed degree, represented by precisely one of the following forms:*

1. The diagonal form $x_1^{q+1} + x_2^{q+1} + x_3^{q+1}$ (rank 3)
2. The cuspidal form $x_1^q x_3 + x_2^{q+1}$ (rank 2)
3. The reducible form $x_1^q x_3 + x_2^q x_1$ (rank 2)

Moreover, in two variables, each Frobenius form is projectively equivalent to exactly one of the following:

1. The form $x_1 x_2 (x_1^{q-1} + x_2^{q-1})$, defining $q + 1$ distinct points $0, \infty$ and the $(q - 1)$ -st roots of unity in $\mathbb{P}^1$.
2. The form $x_1^q x_2$, defining the union of a q -fold point and a reduced point.
3. The form x_1^{q+1} , defining a $(q + 1)$ -fold point.

5 Classification of Frobenius Forms in Five Variables: Proofs

In this section, we prove Theorem 4.1.

For this, we recall the method in [11] for showing that there are only finitely many Frobenius forms of any fixed degree and embedding dimension, up to projective equivalence. A key point is that a Frobenius form is equivalent to one represented by a “sparse matrix:”

Theorem 5.1 ([11])⁴ *Fix any algebraically closed field of characteristic $p > 0$. A Frobenius form of embedding dimension n and rank r can be represented by a matrix A with the following properties:*

1. *All rows beyond the r -th are zero.*
2. *All columns beyond the n -th are zero.*
3. *There are exactly r nonzero entries (all of which are 1) occurring in positions $(1\ j_1), (2\ j_2), \dots, (r\ j_r)$, where $j_1 > j_2 > \dots > j_r$.*

In particular, we may assume that all columns of A are zero but for r of them, which are the standard unit basis vectors $e_r, \dots, e_1$ (in that order, and possibly interspersed with zero columns). For example, any full rank Frobenius form is projectively equivalent to a Frobenius form whose matrix is the anti-diagonal matrix, that is, whose columns are $e_n, \dots, e_1$. A Frobenius form whose matrix satisfies the three conditions of Theorem 5.1 will be called a **sparse form**.

Theorem 5.1 implies the following bounds:

Corollary 5.2

1. *The rank of a Frobenius form of embedding dimension n is at least $\frac{n}{2}$.*
2. *The number of non-degenerate $n \times n$ matrices of rank r satisfying the three conditions in Theorem (5.1) is $\binom{r}{n-r}$.*
3. *The number of projective equivalence types of Frobenius forms of rank r and embedding dimension n (and fixed degree) is at most $\binom{r}{n-r}$.*

Proof Choosing a sparse matrix to represent the Frobenius form, we can assume it looks like

$$x_1^{p^e} L_1 + \dots + x_r^{p^e} L_r,$$

where $L_1, \dots, L_r \in \{x_1, \dots, x_n\}$ are **variables** that appear in reverse order, each variable appearing at most once. All the variables $x_{r+1}, \dots, x_n$ must appear in the list $L_1, \dots, L_r$ (otherwise the embedding dimension is less than n). In particular, $n - r \leq r$. This proves (1).

⁴ An older version of [11] contained the statement of Theorem 5.1 explicitly; observe now that it follows easily from Lemma 5.5 in the new version of [11]. The updated version of [11], in fact, classifies Frobenius forms in all dimensions ([11, Thm. 7.1]), so [11] gives a new proof of Theorem 4.1 as well.

For (2), we continue by observing that conditions (1) and (3) of sparseness (Theorem 5.1) force

$$L_1 = x_n, L_2 = x_{n-1}, \dots, L_{n-r} = x_{r+1}.$$

For the remaining linear forms $\{L_{n-r+1}, \dots, L_r\}$, we can choose $2r - n$ variables out of the remaining variables $\{x_1, \dots, x_r\}$. There are

$$\binom{r}{2r-n} = \binom{r}{n-r}$$

such choices. So (3) follows as well. $\square$

It is easy to determine the embedding dimension of a sparse Frobenius form:

Lemma 5.3 *A rank r Frobenius form of the type*

$$x_1^q x_{j_1} + x_2^q x_{j_2} + \dots + x_r^q x_{j_r} \quad (6)$$

has embedding dimension equal to the number of distinct variables appearing in the expression (6). $\square$

Proof Suppose that a Frobenius form f of the type (6) involves n variables and has rank r . If its embedding dimension is not n , then f could be written as a polynomial in $n - 1$ independent linear forms, $y_1, \dots, y_{n-1}$. Without loss of generality, we can assume that the y_i have the following very special property: there is an index j such that every y_i is a *binomial* linear form $x_{\ell_i} + a_i x_j$ for some index $\ell_i \neq j$ and scalar a_i . To see this, write

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} = B \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

where B is an $(n - 1) \times n$ matrix of full rank, and then left-multiply by the inverse in $GL(n - 1)$ of a full rank $(n - 1) \times (n - 1)$ submatrix of B . This replaces $\{y_1, \dots, y_{n-1}\}$ by a set of linear forms spanning the same space and with the desired binomial form.

There are two cases to consider, depending on whether or not $j \in \{1, \dots, r\}$.

If $j \leq r$, then without loss of generality $j = 1$, so that $y_i = a_i x_1 + x_{i+1}$ for each $i = 1, \dots, n - 1$. Now if f is a Frobenius form in $y_1, \dots, y_{n-1}$, then

$$x_1^q x_{j_1} + x_2^q x_{j_2} + \dots + x_r^q x_{j_r} = y_1^q L_1 + y_2^q L_2 + \dots + y_{n-1}^q L_{n-1} \quad (7)$$

for some linear forms L_i in the y_i . Because the only terms on the right side of (7) that involve $x_{r+1}^q, \dots, x_n^q$ come from $y_r^q, \dots, y_{n-1}^q$, respectively, it follows that $L_r = \dots = L_{n-1} = 0$. In this case,

$$f = y_1^q L_1 + y_2^q L_2 + \dots + y_{r-1}^q L_{r-1},$$

so that f has rank less than r , contrary to our hypothesis.

If $j > r$, then without loss of generality $j = n$, so that $y_i = x_i + a_i x_n$ for each $i = 1, \dots, n-1$. Now assuming

$$x_1^q x_{j_1} + x_2^q x_{j_2} + \dots + x_r^q x_{j_r} = y_1^q L_1 + y_2^q L_2 + \dots + y_{n-1}^q L_{n-1} \quad (8)$$

for some linear forms L_i in the y_j , again it follows that $L_{r+1} = \dots = L_{n-1} = 0$ by looking at the x_i^q terms, with $i > r$. Moreover, since the only terms on the right side of (8) that involve $x_1^q, \dots, x_r^q$ come from $y_1^q, \dots, y_r^q$, we see that $L_i = x_{j_i}$ for $i \leq r$. But note that x_n must appear among the variables $x_{j_1}, \dots, x_{j_r}$, since the original form f involves all n variables. So x_n is one of the L_i . This says that x_n is a linear combination of $x_1 + a_1 x_n, \dots, x_{n-1} + a_{n-1} x_n$, a contradiction.

Combining the two cases, we conclude that f is not a form in $y_1, \dots, y_{n-1}$, and so the embedding dimension of f is n . $\square$

Remark 5.4 The total number of projective equivalence classes of Frobenius forms of embedding dimension n is bounded above by the n -th Fibonacci number [11, 7.4].⁵ This follows from Corollary 5.2 (3) simply by adding the bounds for each relevant rank. This upper bound is sharp for $n \leq 4$, but not in general. There are distinct sparse matrices that define equivalent Frobenius forms starting in five variables: $\square$

Example 5.4.1 Consider the Frobenius forms corresponding to the following matrices:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

namely, $x_1^q x_5 + x_2^q x_4 + x_3^q x_2$ and $x_1^q x_5 + x_2^q x_4 + x_3^q x_1$. These are rank three non-degenerate Frobenius forms corresponding to distinct matrices satisfying the three conditions of Theorem 5.1, but they are equivalent by the change of coordinates which swaps $x_1 \leftrightarrow x_2$ and $x_4 \leftrightarrow x_5$, but fixes x_3 . $\square$

⁵ In an updated version of [11], the authors have recently given a precise count of the number of projective equivalence classes with a fixed embedding dimension.

The proof of Theorem 5.1 in [11] relied on noting that a sparse $n \times n$ matrix of rank r will take one of two forms

$$\begin{bmatrix} | & \mathbf{0} & | \\ \mathbf{0} & B & e_1 \\ | & \mathbf{0} & | \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} | & \mathbf{0} & | \\ e_r & B & e_1 \\ | & \mathbf{0} & | \end{bmatrix}$$

where B is a sparse matrix of rank $r - 1$ in the first case, and a sparse matrix of rank $r - 2$ in the second. In light of this, we make the following definition.

Definition 5.4.2 For $n \geq 3$, we say a sparse $n \times n$ matrix of rank $r < n$ has *type a* if its first column is $\mathbf{0}$, and *type b* if its first column is e_r .

In particular, a Frobenius form of embedding dimension n and rank r is equivalent, after a linear change of coordinates, to a polynomial of one of two forms:

- (**type a**) $h = x_1^q x_n + h'(x_2, \dots, x_{n-1})$ where h' is a Frobenius form of rank $r - 1$ and embedding dimension $n - 2$. In this case, B is non-degenerate of rank $r - 1$.
 (**type b**) $h = x_1^q x_n + x_r^q x_1 + h'(x_2, \dots, x_{n-1})$ where h' is a Frobenius form of rank $r - 2$. If h' is non-degenerate in the $n - 2$ variables $x_2, \dots, x_{n-1}$, then B is non-degenerate of rank $r - 2$. Otherwise, h' must be non-degenerate in the $n - 3$ variables $x_2, \dots, \widehat{x_r}, \dots, x_{n-1}$, so that B is degenerate of rank $r - 2$.

The proof of Theorem 4.1 will now follow from the following two propositions.

Proposition 5.5.2 *There are precisely $n - 1$ projective equivalence classes of Frobenius forms of embedding dimension n and rank $n - 1$ (in fixed degree $p^e + 1$).*

The proof of Proposition 5.5.2 uses the following lemma:

Lemma 5.6 *Let h_1 and h_2 be two Frobenius forms of the same degree, both of embedding dimension n and rank $r = n - 1$, and both represented by matrices (say A_1 and A_2) in sparse form. Then h_1 and h_2 are projectively equivalent if and only if A_1 and A_2 are the same type, and their B matrices are projectively equivalent.*

Proof Without loss of generality, by Theorem 5.1 we may assume

$$\begin{aligned} h_1 &= x_1^q x_n + x_2^q x_{j_2} + x_3^q x_{j_3} + \cdots + x_{n-1}^q x_{j_{n-1}}, \text{ and} \\ h_2 &= x_1^q x_n + x_2^q x_{j'_2} + x_3^q x_{j'_3} + \cdots + x_{n-1}^q x_{j'_{n-1}} \end{aligned}$$

where $x_{j_i}, x_{j'_i} \in \{x_1, \dots, x_{n-1}\}$ for each i . In particular, we note that the only term containing x_n in both h_1 and h_2 is $x_1^q x_n$. Now suppose some change of coordinates ϕ sends h_1 to h_2 . The singular locus of both h_1 and h_2 is defined by the vanishing of $x_1, x_2, \dots, x_{n-1}$, so the ideal $\langle x_1, \dots, x_{n-1} \rangle$ is stable under ϕ . In particular, ϕ must send x_n to a linear form involving x_n . Supposing that ϕ maps

$$\begin{aligned} x_1 &\mapsto \lambda_{11}x_1 + \cdots + \lambda_{1,n-1}x_{n-1}, \text{ and} \\ x_n &\mapsto \lambda_{n1}x_1 + \cdots + \lambda_{nn}x_n \quad \text{with } \lambda_{nn} \neq 0, \end{aligned}$$

then ϕ sends h_1 to a polynomial

$$\lambda_{11}^q \lambda_{nn} x_1^q x_n + \cdots + \lambda_{1,n-1}^q \lambda_{nn} x_{n-1}^q x_n + \text{terms that do not involve } x_n.$$

In order for this to be equal to h_2 , we must have $\lambda_{1i} = 0$ for all $i > 1$. So ϕ sends

$$x_1 \mapsto \lambda_{11} x_1$$

where $\lambda_{11} \neq 0$. In particular, if h_1 and h_2 are equivalent, they must be equivalent after modding out by x_1 .

We can now see that if h_1 and h_2 are equivalent, then A_1 and A_2 have the same type. Letting $\sim$ denote equivalence up to change of coordinates, if

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & B & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 1 \\ e_{r-1} & B' & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

then

$$\begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} B' & 0 \\ 0 & 0 \end{bmatrix}$$

which is the same as saying $B \sim B'$. By rank considerations this never happens between type a and type b (see the discussion following Definition 5.4.2). Thus the sparse matrices of projectively equivalent Frobenius forms whose rank is one less than the embedding dimension must have the same type. Furthermore, the argument above also shows that their “B” matrices are projectively equivalent.

Because the converse is obvious, the lemma is proved. $\square$

Proof (Proof of Proposition 5.5.2) Fix q . Let $N(n, r)$ denote the number of projective equivalence classes of Frobenius forms of degree $q + 1$, rank r , and embedding dimension n . We want to show that $N(n, n - 1) = n - 1$. We will induce on n .

One readily verifies that

$$N(1, 0) = 0, N(2, 1) = 1, \text{ and } N(3, 2) = 2.$$

Since type a and type b matrices yield distinct classes for $r = n - 1$, the number of classes $N(n, n - 1)$ is equal to the number of classes of type a plus the number of type b. The discussion of the types following Definition 5.4.2 informs us, therefore, that for $n \geq 4$

$$N(n, n - 1) = N(n - 2, n - 2) + N(n - 2, n - 3) + N(n - 3, n - 3).$$

Recalling that there is only one full rank form in each degree and dimension (Theorem 2.1), it follows that

$$N(n, n-1) = 1 + N(n-2, n-3) + 1 = N(n-2, n-3) + 2.$$

Finally, by induction on n ,

$$N(n, n-1) = (n-3) + 2 = n-1,$$

as desired. $\square$

In light of Proposition 5.5.2, we have now fully classified all Frobenius forms in five variables of ranks four and five. By Corollary 5.2 (1), it only remains to analyze the rank three case. Using Corollary 5.2 (2), we see that there are $\binom{3}{2} = 3$ sparse forms of rank three in five variables, and we have seen in Example 5.4.1 that two of them are projectively equivalent. To complete the classification, it remains to see that the third sparse matrix produces a Frobenius form not equivalent to these. This is accomplished by the following:

Proposition 5.7 *The following rank three Frobenius forms in five variables are not projectively equivalent:*

$$f = x_1^q x_5 + x_2^q x_4 + x_3^{q+1}$$

and

$$g = x_1^q x_5 + x_2^q x_4 + x_3^q x_2.$$

Proof To see this, note that $g \in \langle x_1, x_2 \rangle$. Thus, if f and g are projectively equivalent, there must be some linear forms L_1 and L_2 such that $f \in \langle L_1, L_2 \rangle$. Since any projective change of coordinates must respect the singular locus, any form in x_1, x_2, x_3 must be sent to another form in x_1, x_2, x_3 , as the vanishing of these coordinates defines the singular set of both hypersurfaces. This implies that L_1, L_2 , being the images of x_1 and x_2 under our linear change of coordinates, are forms in x_1, x_2, x_3 .

Now note that $\langle x_3^{q+1}, x_4, x_5 \rangle = \langle f, x_4, x_5 \rangle \subset \langle L_1, L_2, x_4, x_5 \rangle$. In particular, $x_3 \in \sqrt{\langle L_1, L_2, x_4, x_5 \rangle} = \langle L_1, L_2, x_4, x_5 \rangle$, so that $x_3 \in \langle L_1, L_2 \rangle$. Without loss of generality, we may assume $\langle L_1, L_2 \rangle = \langle x_3, L_2 \rangle$, where L_2 is a linear form in x_1 and x_2 , so that $f \in \langle x_3, L_2 \rangle$. Therefore,

$$x_1^q x_5 + x_2^q x_4 \in \langle x_3, L_2 \rangle \quad (9)$$

as well. Considering the image of the expression (9) under the natural quotient map $k[x_1, \dots, x_5] \rightarrow k[x_1, \dots, x_5]/\langle x_3 \rangle$, we see that

$$x_1^q x_5 + x_2^q x_4 \in \langle L_2 \rangle \quad (10)$$

in a polynomial ring in four variables. But the polynomial $x_1^q x_5 + x_2^q x_4$ is irreducible (for example, by Eisenstein's criterion), so we arrive at a contradiction. This contradiction ensures that f and g are not projectively equivalent. $\square$

This completes our classification of Frobenius forms in up to five variables.

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