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LARGE N LIMIT OF THE $O(N)$ LINEAR SIGMA MODEL VIA STOCHASTIC QUANTIZATION

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This article studies large N limits of a coupled system of N interacting Φ^4 equations posed over $\mathbb{T}^d$ for $d = 2$, known as the $O(N)$ linear sigma model. Uniform in N bounds on the dynamics are established, allowing us to show convergence to a mean-field singular SPDE, also proved to be globally well posed. Moreover, we show tightness of the invariant measures in the large N limit. For large enough mass, they converge to the (massive) Gaussian free field, the unique invariant measure of the mean-field dynamics, at a rate of order $1/\sqrt{N}$ with respect to the Wasserstein distance. We also consider fluctuations and obtain tightness results for certain $O(N)$ invariant observables, along with an exact description of the limiting correlations.

CONTENTS

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1. Introduction. In this paper, we consider the following system of equations on the d-dimensional torus $\mathbb{T}^d$ for $d=2$:

$$\mathscr{L}\Phi_i = -\frac{1}{N}\sum_{j=1}^N \Phi_j^2\Phi_i + \xi_i, \quad \Phi_i(0)=\phi_i, \tag{1.1}$$

where $\mathscr{L} = \partial_t - \Delta + m$ with $m \geq 0$, $N \in \mathbb{N}$, and $i \in \{1,\dots,N\}$. The collection $(\xi_i)_{i=1}^N$ consists of N independent space-time white noises on a stochastic basis, that is, $(\Omega, \mathcal{F}, \mathbf{P})$ with a filtration, and $(\phi_i)_{i=1}^N$ are random initial datum independent of $(\xi_i)_{i=1}^N$. In $d=2$, the system (1.1) requires renormalization, and the formal product $\Phi_j^2\Phi_i$ will be interpreted as the Wick product $:\Phi_j^2\Phi_i:$ whose definition is postponed to Section 2.

This system arises as the stochastic quantization of the following N-component generalization of the Φ_d^4 model, given by the (formal) measure

$$\mathrm{d}\nu^N(\Phi) \stackrel{\mathrm{def}}{=} \frac{1}{C_N}\exp\left(-\int_{\mathbb{T}^d}\sum_{j=1}^N|\nabla\Phi_j|^2 + m\sum_{j=1}^N\Phi_j^2 + \frac{1}{2N}\left(\sum_{j=1}^N\Phi_j^2\right)^2\mathrm{d}x\right)\mathcal{D}\Phi \tag{1.2}$$

over $\mathbb{R}^N$ valued fields $\Phi = (\Phi_1, \Phi_2, \dots, \Phi_N)$ and C_N is a normalization constant. In $d=2$, the interaction should be Wick renormalized $:(\sum_{j=1}^N\Phi_j^2)^2:$ for the measure to make sense. This is also referred to as the $O(N)$ linear sigma model, since this formal measure is invariant under a rotation of the N components of Φ. [1] This symmetry will play an important role throughout the paper.

Our focus in this article is on the asymptotic behavior as $N\to\infty$ of the system (1.1) and its invariant measures (1.2) as well as observables which preserve the $O(N)$ symmetry. Note that a factor $1/N$ has been introduced in front of the nonlinearity (resp., the quartic term in the measure), and heuristically, this compensates the sum of N terms so that one could hope to obtain an interesting limit as $N\to\infty$. The study of physically meaningful quantities associated with a quantum field theory model such as (1.2) as $N\to\infty$ is generally referred to as a *large N problem*; see Section 1.1 where we introduce more background, references in physics and mathematics, and different approaches to this problem. To the best of our knowledge, the present article provides the first rigorous results on large N problems in the formulation of stochastic quantization.

In Theorem 1.1 below, we study the $N\to\infty$ limit of each component in the Wick renormalized version of (1.1) in $d=2$ (cf. (2.1) below), and show that a suitable mean-field singular SPDE governs the limiting dynamics. Before giving the statement, let us first comment on the notion of solution used. Recall that the well posedness of (1.1) in the case $N=1$ and $d=2$ (i.e., the dynamical Φ_2^4 model) is now well developed: two classical works being [2] where martingale solutions are constructed and [20] where strong solutions are addressed, as well as the more recent approach to global well posedness in [53]. These results can be generalized to the vector case (with fixed $N>1$) without much extra effort. As in [20] and [53], the solutions are defined by the decomposition $\Phi_i = Z_i + Y_i$, where

$$\mathscr{L}Z_i = \xi_i, \tag{1.3}$$

$$\mathscr{L}Y_i = -\frac{1}{N}\sum_{j=1}^N\left(Y_j^2Y_i + Y_j^2Z_i + 2Y_jY_iZ_j + 2Y_j :Z_iZ_j: + :Z_j^2: Y_i + :Z_iZ_j^2:\right) \tag{1.4}$$

[1] The word "linear" here only means that the target space $\mathbb{R}^N$ is a linear space. "Nonlinear" sigma models on the other hand refers to similar models where the target space is subject to certain nonlinear constraints, for example, Φ takes value in a sphere in $\mathbb{R}^N$ or more generally in a manifold.

and $:Z_i Z_j:$, $:Z_i Z_j^2:$ are Wick renormalized products (see Section 2). For the uninitiated reader, note that (1.4) arises by inserting the decomposition of Φ_i into (1.1) and reinterpreting the ill-defined products $Z_i Z_j$ and $Z_i Z_j^2$ that appear.

The mean-field SPDE formally associated to (1.1) takes the form

$$\mathscr{L}\Psi_i = -\mathbf{E}[\Psi_i^2]\Psi_i + \xi_i, \quad \Psi_i(0) = \psi_i. \tag{1.5}$$

On the formal level this equation arises naturally: assuming the initial conditions $\{\phi_i\}_{i=1}^N$ are exchangeable,[2] the components $\{\Phi_i\}_{i=1}^N$ will have identical laws, so that replacing the empirical average $\frac{1}{N}\sum_{j=1}^N \Phi_j^2$ in (1.1) by its mean and relabeling Φ as Ψ leads us to (1.5). In two space dimensions, (1.5) is a singular SPDE where the ill-defined nonlinearity depends on the law of the solution and similar to (1.1), it also requires a renormalization. Postponing for the moment a more complete discussion of this point, we now state our first main result.

THEOREM 1.1 (Large N limit of the dynamics for $d = 2$). *Let $\{(\phi_i^N, \psi_i)\}_{i=1}^N$ be random initial datum with components in $\mathbf{C}^{-\kappa}$ for some small $\kappa > 0$ and all moments finite, where $\mathbf{C}^{-\kappa}$ denotes the Besov space introduced in Section A. Assume that for each $i \in \mathbb{N}$, ϕ_i^N converges to ψ_i in $L^p(\Omega; \mathbf{C}^{-\kappa})$ for all $p > 1$, $\frac{1}{N}\sum_{i=1}^N \|\phi_i^N - \psi_i\|_{\mathbf{C}^{-\kappa}}^p \to^{\mathbf{P}} 0$ and $(\psi_i)_i$ are i.i.d. Here, $\to^{\mathbf{P}} 0$ means the convergence in probability.*

Then for each component i and all $T > 0$, the solution Φ_i^N defined by (1.3)–(1.4) with initial datum ϕ_i^N converges in probability to Ψ_i in $C([0, T], \mathbf{C}^{-1}(\mathbb{T}^2))$ as $N \to \infty$, where Ψ_i is the unique solution to the mean-field SPDE formally described by

$$\mathscr{L}\Psi_i = -\mathbf{E}[\Psi_i^2 - Z_i^2]\Psi_i + \xi_i, \quad \Psi_i(0) = \psi_i, \tag{1.6}$$

and Z_i is the stationary solution to (1.3). Furthermore, under the additional hypothesis that $(\phi_i^N, \psi_i)_{i=1}^N$ are exchangeable, for each $t > 0$ it holds that

$$\lim_{N\to\infty} \mathbf{E}\|\Phi_i^N(t) - \Psi_i(t)\|_{L^2(\mathbb{T}^2)}^2 = 0. \tag{1.7}$$

In Section 4, we actually prove this convergence result under more general conditions for initial data (see Assumption 4.1). Along the way to Theorem 1.1, we prove new uniform in N bounds through suitable energy estimates on the remainder equation (1.4). We are inspired in part by the approach in [53], but subtleties arise as we track carefully the dependence of the bounds on N. Indeed, the natural approach (e.g., [53] for dynamical Φ_2^4 model) to obtain global in time bounds for fixed N is to exploit the damping effect from $Y_j^2 Y_i$. However, the extra factor $1/N$ before the nonlinear terms makes this effect weaker as N becomes large. In fact, the moral is that we cannot exploit the strong damping effect at the level of a fixed component Y_i, rather we are forced to consider aggregate quantities, and ultimately we focus on the empirical average of the L^2-norm (squared) instead of the L^p-norm, $p > 2$; cf. Lemma 2.3 and Remark 2.6. This is natural on one hand due to the coupling of the components, but also for the slightly more subtle point that we ought to respect the structure of the mean-field SPDE (1.6), for which the damping effect seems to hold only in the mean square sense, not at the path-by-path level.

In this direction, we now discuss a bit more the solution theory for the mean-field SPDE (1.6). While the notion of solution we use is again via the Da-Prato–Debussche trick, the well-posedness theory for (1.5) requires more care than for Φ_2^4 since we cannot proceed by

[2]This means that the sequence of random variables $(\phi_1, \dots, \phi_N)$ has the same joint probability distribution as $(\phi_{\pi(1)}, \dots, \phi_{\pi(N)})$ for any permutation π.

pathwise arguments alone. In fact, similar to (1.3)–(1.4), we understand (1.6) via the decomposition $\Psi_i = Z_i + X_i$ with X_i satisfying

$$\mathscr{L}X_i = -\big(\mathbf{E}[X_j^2]X_i + \mathbf{E}[X_j^2]Z_i + 2\mathbf{E}[X_jZ_j]X_i + 2\mathbf{E}[X_jZ_j]Z_i\big). \tag{1.8}$$

Here, we actually introduce an independent copy (X_j, Z_j) of (X_i, Z_i), which turns out to be useful for both the local and global well-posedness of (1.6). Indeed, one point is that the term $\mathbf{E}[X_jZ_j]Z_i$ in (1.8) cannot be understood in a classical sense; however, we can view it as a conditional expectation $\mathbf{E}[X_jZ_jZ_i|Z_i]$ and use properties of the Wick product Z_iZ_j to give a meaning to this; cf. Lemma 3.1. Furthermore, to obtain global bounds, using this independent copy allows us to approach the a priori estimates for (1.8) much like the uniform in N bounds for (1.4). Indeed, after taking expectation, $\mathbf{E}[X_j^2]X_i$ in (1.8) also plays the role of the damping mechanism, which helps us to obtain uniform bounds on the mean-squared L^2-norm of X_i; cf. Lemma 3.3.

Theorem 1.1 can be viewed as a *mean-field* limit result in the context of singular SPDE systems. Our proof is indeed inspired by certain mean-field limit techniques, and we combine them with a priori estimates that are specific to our model; see the discussion above Theorem 4.1 for a more detailed discussion on this strategy. We will provide more background discussion on mean-field limits below in Section 1.2. By a classical coupling argument, this result also yields a propagation of chaos type statement: if the initial condition is asymptotically chaotic (i.e., independent components as $N \to \infty$), then although the Φ-system is interacting, as $N \to \infty$ the limiting system becomes decoupled ([41], Definition 3, Definition 5).

The second part of this paper (Section 5) is concerned with equilibrium theories, namely stationary solutions, invariant measures and large N convergence. For $N = 1$, the long-time behavior of the solutions was investigated in [56] and [68]. In the vector valued setting, by lattice approximation (see [32, 37, 73]), strong Feller property in [38] and irreducibility in [40], it can be shown that ν^N is the unique invariant measure to (1.1) and the law of $\Phi_i(t)$ converges to ν^N as $t \to \infty$. Our goal then is to study the large N limit of the $O(N)$ linear sigma model ν^N. Our second main result yields the convergence of the unique invariant measure ν^N of (1.1) to the invariant measure of (1.6), provided the mass is sufficiently large.

To state the result, consider the projection onto the ith component,

$$\Pi_i : \mathcal{S}'(\mathbb{T}^d)^N \to \mathcal{S}'(\mathbb{T}^d), \qquad \Pi_i(\Phi) \stackrel{\text{def}}{=} \Phi_i. \tag{1.9}$$

Noting that ν^N is a measure on $\mathcal{S}'(\mathbb{T}^d)^N$, we define the marginal law $\nu^{N,i} \stackrel{\text{def}}{=} \nu^N \circ \Pi_i^{-1}$. Furthermore, consider

$$\Pi^{(k)} : \mathcal{S}'(\mathbb{T}^d)^N \to \mathcal{S}'(\mathbb{T}^d)^k, \qquad \Pi^{(k)}(\Phi) = (\Phi_i)_{1\le i\le k} \tag{1.10}$$

and define the marginal law of the first k components by $\nu_k^N \stackrel{\text{def}}{=} \nu^N \circ (\Pi^{(k)})^{-1}$.

THEOREM 1.2 (Large N limit of the invariant measures). *There exists $m_0 > 0$ such that the following results hold*:

- *For $m \ge 0$, the Gaussian free field $\nu \stackrel{\text{def}}{=} \mathcal{N}(0, \frac{1}{2}(m-\Delta)^{-1})$ is an invariant measure for* (1.6).
- *For $m \ge 0$, the sequence of probability measures $(\nu^{N,i})_{N\ge1}$ are tight on $\mathbf{C}^{-\kappa}$ for $\kappa > 0$.*
- *For $m \ge m_0$, the Gaussian free field $\mathcal{N}(0, \frac{1}{2}(m-\Delta)^{-1})$ is the unique invariant measure to equation* (1.6).
- *For $m \ge m_0$, $\nu^{N,i}$ converges to ν and ν_k^N converges to $\nu \times \cdots \times \nu$, as $N \to \infty$. Furthermore, $\mathbb{W}_2(\nu^{N,i}, \nu) \lesssim N^{-\frac{1}{2}}$.*

These statements will follow from Theorem 5.9, Theorem 5.4 and Theorem 5.11. Here, $\mathbb{W}_2(\nu_1,\nu_2)$ is the $\mathbf{C}^{-\kappa}$-Wasserstein distance defined in (5.12) before Theorem 5.11. The Gaussian free field limit is expected (at a heuristic level) by physicists, for example, [70] and also in mathematical physics [45].[3] Our result, Theorem 1.2, provides a precise justification provided $m \geq m_0$, with the convergence rate $N^{-\frac{1}{2}}$ (which is expected to be optimal; see, for instance, [42], Remark 4) in terms of Wasserstein distance. The large m assumption could also be formulated as a small nonlinearity assumption; see Remark 5.12.

Note that the study of ergodicity properties of the dynamic (1.6) is nontrivial. In fact, the dynamic for Ψ depends on the law of Ψ itself, so the associated semigroup is generally nonlinear (see Section 5.1). As a result, the general ergodic theory for Markov process (see, e.g., [21], [39], [38]) could not be directly applied here. Instead, we prove the solutions to (1.6) converge to the limit directly as time goes to infinity, which requires $m \geq m_0$.

We now comment on our approach to the fourth part of Theorem 1.2. It would be natural to try and use Theorem 1.1 together with the tightness result from the second part of Theorem 1.2 to derive the convergence of $\nu^{N,i}$ to ν directly (see, e.g., [37]). However, it is not clear to the authors how to implement this strategy in the present setting. Indeed, to apply Theorem 1.1, it is important that each component ψ_i of the initial data is independent of each other. However, we are not able to deduce that an arbitrary limit point ν^* has this property. If we use $P_t^*\nu^*$ to denote the marginal distribution of the solution to (1.6) starting from the initial distribution ν^*, we cannot write $P_t^*\nu^*$ as $\int (P_t^*\delta_\psi)\nu^*(\mathrm{d}\psi)$ due to the lack of linearity, which makes it difficult to overcome the assumption of independence. Alternatively, we follow the idea in [32] and construct a jointly stationary process (Φ,Ψ) whose components satisfy (1.1) and (1.6), respectively. In this case $\Psi = Z$, since the Gaussian free field gives the unique invariant measure to (1.6). We then establish the convergence of $\nu^{N,i}$ to ν by deriving suitable uniform estimates on the stationary process.

Our next result is concerned with observables in the stationary setting. In QFT models with continuous symmetries, physically interesting quantities involve more than just a component of the field itself but also quantities composed by the fields, which preserve the symmetries, called invariant observables. These acquire the same interest in SPDE (a natural example being the gauge invariant observables, e.g., [58], Section 2.4). In the present setting of (1.1), a natural quantity that is invariant under $O(N)$-rotation is the "length" of Φ; another being the quartic interaction in (1.2). We thus consider the following two $O(N)$ invariant observables: for $\Phi \backsimeq \nu^N$,

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N} :\Phi_i^2: , \qquad \frac{1}{N} :\Big(\sum_{i=1}^{N} \Phi_i^2\Big)^2: . \tag{1.11}$$

Here, the precise definition is given in Section 6.1. One could consider more general renormalized polynomials of $\sum_i \Phi_i^2$ but we choose to focus on the above two in this article. We establish the large N tightness of these observables as random fields in suitable Besov spaces by using iteration to derive improved uniform estimates in the stationary case.

Note that the physics literature usually considers integrated quantities, that is, partition function of correlations of these observables. Our SPDE approach allows us to study these observables as random fields with precise regularity as $N \to \infty$, which is new.

Moreover, we investigate the *nontrivial statistics* of the large N limit of the $O(N)$ invariant observables. We show that although for large enough m the invariant measure of Φ_i

[3] In [70], it was written that "If one now looks at vacuum expectation values of individual Φ fields, all diagrams vanish like $1/N$ (at least), except for the free-field terms." In the Introduction of [45], it was mentioned that "the $1/N$ expansion predicts that the theory is close to Gaussian as N becomes large enough," but this reference did not intend to prove this statement (see Section 1.1 below).

converges as $N \to \infty$ to the invariant measure of Z_i that is, Gaussian free field, the limits of the observables (1.11) have different laws than those if Φ_i in (1.11) were replaced by Z_i:

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} :Z_i^2: , \qquad \frac{1}{N} :\Big(\sum_{i=1}^{N} Z_i^2\Big)^2: . \tag{1.12}$$

THEOREM 1.3. *Suppose that $\Phi \simeq \nu^N$. For m large enough, the following result holds for any $\kappa > 0$:*

- $\frac{1}{\sqrt{N}} \sum_{i=1}^{N} :\Phi_i^2:$ *is tight in* $B_{2,2}^{-2\kappa}$.
- $\frac{1}{N} :(\sum_{i=1}^{N} \Phi_i^2)^2:$ *is tight in* $B_{1,1}^{-3\kappa}$.
- *The Fourier transform of the two point correlation function of* $\frac{1}{\sqrt{N}} \sum_{i=1}^{N} :\Phi_i^2:$ *in the limit as* $N \to \infty$ *is given by the explicit formula* $2\widehat{C^2}/(1+2\widehat{C^2})$, *where* $C = \frac{1}{2}(m-\Delta)^{-1}$ *and* $\widehat{C}$ *is the Fourier transform*; *moreover*, $\mathbf{E}\frac{1}{N} :(\sum_{i=1}^{N} \Phi_i^2)^2:$ *converges as* $N \to \infty$ *to the explicit formula given by* $-4\sum_{k\in\mathbb{Z}^2} \widehat{C^2}(k)^2/(1+2\widehat{C^2}(k))$. (*In particular, the limiting laws of the observables* (1.11) *are different from those of* (1.12)).

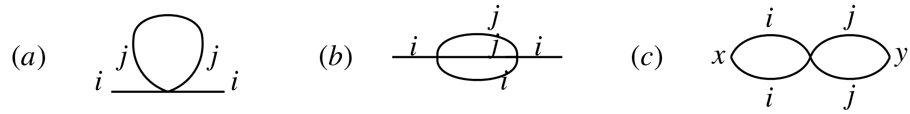
These results are proved in Theorem 6.3 and Theorem 6.5. The last statement on correlation formulas of the observables are known—first heuristically by physicists who expressed these formulas in terms of the sum of "bubble" diagrams, and then derived in [45], equation (15), using constructive field theory techniques such as "chessboard estimates." Our new proofs of these correlation formulas using PDE methods are quite simple and straightforward once all the a priori estimates are available. We expect that these methods can be applied to study more $O(N)$ invariant observables and higher order correlations; we will pursue these in future work. We also mention that all the results in Theorem 1.1-Theorem 1.3 hold for $d = 1$ (see [59] for more details.)

Let us also mention the three-dimensional construction of local solutions [11, 33, 35], global solutions [1, 31, 50, 52], as well as a priori bounds in fractional dimension $d < 4$ by [13], though we focus on $d = 2$ in this paper. It would also be interesting to see if our methodology could be used to study limits of other singular SPDE systems as dimensionality of the target space tends to infinity, such as coupled dynamical Φ^4_3,[4] coupled KPZ systems [26], random loops in N dimensional manifolds [9, 16, 36, 54] and the Yang–Mills model [12] with $N-$ dimensional Lie groups (or abelian case [58] with Higgs field generalized to value in $\mathbb{C}^N$). These are, of course, left to further work.

1.1. *Large N problem in QFT*: *Background and motivation*. Large N methods (or "$1/N$ expansions") in theoretical physics are ubiquitous and are generally applied to models where dimensionality of the target space is large. It was first used in [62] for spin models, and then developed in quantum field theories (QFT), which was pioneered by [70] (Φ^4 type and Fermionic models), [30] (Fermionic models), [65] (Yang–Mills model), and the idea was soon popularized and extended to many other systems; see [8] for an edited comprehensive collection of articles on large N as applied to a wide spectrum of problems in quantum field theory and statistical mechanics; see also the review articles [71], [18], Chapter 8, and [51] for summaries of the progress. Loosely speaking, in terms of our model (1.2), the ordinary QFT perturbative calculation of for instance a two-point correlation of Φ_i is given by sum of Feynman graphs with two external legs and degree-4 internal vertices, each vertex carrying two

[4]In fact, we have obtained some partial results for coupled dynamical Φ^4_3, such as convergence of invariant measures to the Gaussian free field.

distinct summation variables and a factor $1/N$ that represents the interaction $\frac{1}{N}\sum_{i,j}\Phi_i^2\Phi_j^2$, such as (a), (b) below:



Heuristically, graph (a) is of order $\frac{1}{N}\sum_j \approx O(1)$ and graph (b) is of order $\frac{1}{N^2}\sum_j \approx O(\frac{1}{N})$. The philosophy of [70] is that graphs with "self-loops" such as (a) get *canceled* by Wick renormalization, and all other graphs with internal vertices including (b) are at least of order $O(1/N)$, and thus vanish, so the theory would be asymptotically Gaussian free field, which is what we prove in Theorem 1.2. On the other hand, for observables such as $\frac{1}{\sqrt{N}}\sum_{i=1}^N :\Phi_i^2:$, two-point correlation at x, y may have $O(1)$ contributions as shown in graph (c),[5] which is the heuristic behind the existence of a nontrivial correlation structure for such observables as in Theorem 1.3. The "$1/N$ expansion" is a reorganization of the series in the parameter $1/N$, with each term typically being a (formal) sum of infinitely many orders of the ordinary perturbation theory. Besides directly examining the perturbation theory, alternative (and more systematic) methodologies of analyzing such expansion were discovered in physics, for instance, a method via "dual" field [19], [51], Section 2, via Schwinger–Dyson equations [63], or via stochastic quantization (with references below).

Rigorous study of large N in mathematical physics was initiated by Kupiainen [44–46]. The literature most related to the present article is [45], which studied the QFT in continuum in $d = 2$ given by (1.2), and proved that the $1/N$ expansion of the pressure (i.e., vacuum energy or log of partition per area) is asymptotic, and each order in this expansion can be described by sums of infinitely many Feynman diagrams of certain types. Borel summability of $1/N$ expansion of Schwinger functions for this model was discussed in [7].

In [46], Kupiainen also proved that on the lattice with fixed lattice spacing, the large N expansion of correlation functions of the N-component nonlinear sigma model (which simplifies to "spherical model" as $N \to \infty$) is asymptotic above the spherical model criticality; asymptoticity was later extended to Borel summability by [25]. Large N limit and expansion for the Yang–Mills model has also been rigorously studied; see [48] (also [5]) for convergence of Wilson loop observables to master field in the continuum plane, and [14] (resp., [15]) for computation of correlations of the Wilson loops in the large N limit (resp., $1/N$ expansion), which relates to string theory.

Large N problems in the stochastic quantization formalism have also been discussed in the physics literature, for instance, [3, 4], [22], Section 8. [51], Section 5.1, is close to our setting; it makes an "ansatz" that $\frac{1}{N}\sum_{j=1}^N \Phi_j^2$ in (1.1) would self-average in the large N limit to a constant; our present paper justifies this ansatz and in the nonequilibrium setting generalizes it.

In summary, the study of large N problems in QFT is motivated by the following properties (among others). The first property is simplification or solvability as $N \to \infty$. This is the motivation ever since the earliest literature [62] as aforementioned: the model studied therein becomes a simplified, solvable model as $N \to \infty$ known as the Berlin–Kac spherical model. In our setting, this simplification or solvability heuristics are reflected by the Gaussian free field asymptotic as well as the rigorous derivation of exact formula (which would not be possible for finite N) for certain correlation of observables in Theorem 1.2 and Theorem 1.3. Another property is that when N is large, $1/N$ serves as a natural perturbation parameter in QFT models, as already discussed above. Of course, this went much farther than just simplifying things later when applied to more sophisticated models like gauge theory, for

[5] but there are infinitely many $O(1)$ graphs

which $1/N$ expansions led to the discovery of so-called gauge-string duality as mentioned above.

1.2. *Mean-field limits.* As mentioned above, the proof of our main theorems borrows some ingredients from mean-field limit theory (MFT). To the best of our knowledge, the study of mean-field problems originated from McKean [49]. Typically, a mean-field problem is concerned with a system of N particles interacting with each other, which is often modeled by a system of stochastic *ordinary* differential equations, for instance, driven by independent Brownian motions. A prototype of such systems has the form $\mathrm{d}X_i = \frac{1}{N}\sum_j f(X_i, X_j)\,\mathrm{d}t + \mathrm{d}B_i$ (see, for instance, the classical reference by Sznitman [64], Section I(1)), and in the $N \to \infty$ limit one could obtain decoupled SDEs each interacting with the law of itself: $\mathrm{d}Y_i = \int f(Y_i, y)\mu(\mathrm{d}y)\,\mathrm{d}t + \mathrm{d}B_i$ where $\mu(\mathrm{d}y)$ is the law of Y_i. So just as in QFT the motivation of MFT is also a simplification of an N-body system to a one-body equation, which interacts with itself, that is, the system is factorized.

In simple situations, the interaction f is assumed to be "nice," for instance, globally Lipschitz ([49]); much of the literature aims to prove such limits under more general assumptions on the interaction; see [64] for a survey.[6] Our Theorem 1.1 can be viewed as a result of this flavor, in an SPDE setting, and in fact the starting point of our proof is indeed close in spirit to [64], Section I(1), where one subtracts X_i from Y_i to cancel the noise and then bound a suitable norm of the difference.

We note that mean-field limits are studied under much broader frameworks or scopes of applications, such as mean-field limit in the context of rough paths (e.g., [6, 10, 17]), mean-field games (e.g., survey [47]), quantum dynamics (e.g., [24] and references therein). We do not intend to have a comprehensive list, but rather refer to survey articles [28, 41] and the book [61] besides [64], Chapter 8.

The study of mean-field limit for SPDE systems also has precursors; see, for instance, the book [43], Chapter 9, or [23]. However, these results make strong assumptions on the interactions of the SPDE systems such as linear growth and globally Lipschitz drift, and certainly do not cover the singular regime where renormalization is required as in our case.

1.3. *Structure of the paper.* This paper is organized as follows. Sections 2–4 are devoted to the proof of Theorem 1.1. First, in Section 2.1 we recall the definition of the renormalization for Z_i, which satisfies the linear equation (1.3). Then a uniform in N estimate for the average of the L^2-norm of Y_i, the solutions to equation (1.4), is derived in Section 2.2. Local well-posedness to equation (1.6) is proved in Section 3.1. Global well-posedness to equation (1.6) is proved in Section 3.2 by combining a uniform L^p-estimate with Schauder theory. The difference estimate for $\Phi_i - \Psi_i$ is given in Section 4, which gives the proof of Theorem 1.1.

Section 5 is concerned with the proof of Theorem 1.2. In Section 5.1, uniqueness of invariant measures to (1.6) for large m is proved. The convergence of invariant measures from $\nu^{N,i}$ to the Gaussian free field ν is shown in Section 5.2 by comparing the stationary solutions (Φ_i, Z_i).

Section 6 mainly concentrates on the observables and the nontriviality of the statistics of the observables. Section 6.1 is devoted to the study of the observables and the proof of first two parts of Theorem 1.3. We derive an L^p-estimate of Y_i in Section 6.2 and prove the third part of Theorem 1.3 in Section 6.3.

[6]In the context of SDE systems, one also considers the empirical measures of the particle configurations, and aims to show their convergence as $N \to \infty$ to the McKean–Vlasov PDEs, which are typically deterministic. Note that in this paper we do not consider the "analogue" of McKean–Vlasov PDE (which would be infinite dimensional) in the context of our model.

Finally, in Appendix A, we collect the notation and useful lemmas used throughout the paper. In Appendix B, we give the proof of global well-posedness of equation (1.4). In Appendix C, the application of Dyson–Schwinger equations has been derived, which is useful in studying the limiting law of the observables. In Appendix D, we give the proof of Step 7 in the proof of Theorem 4.1.

Notation. Throughout the paper, we use the notation $a \lesssim b$ if there exists a constant $c > 0$ such that $a \le cb$, and we write $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$. Given a Banach space E with a norm $\|\cdot\|_E$ and $T > 0$, we write $C_T E = C([0,T]; E)$ for the space of continuous functions from $[0,T]$ to E, equipped with the supremum norm $\|f\|_{C_T E} = \sup_{t\in[0,T]} \|f(t)\|_E$. For $p \in [1,\infty]$, we write $L^p_T E = L^p([0,T]; E)$ for the space of L^p-integrable functions from $[0,T]$ to E, equipped with the usual L^p-norm. Let $\mathcal{S}'$ be the space of distributions on $\mathbb{T}^d$.

2. Uniform in N bounds on the dynamical linear sigma model. In this section, we obtain new estimates on the Wick renormalized version of (1.1), given by

$$\mathscr{L}\Phi_i = -\frac{1}{N}\sum_{j=1}^{N} :\Phi_j^2\Phi_i: \, + \xi_i, \quad \Phi_i(0) = \phi_i. \tag{2.1}$$

The notion of solution to (2.1) is the same as in [20] and [53], where the case $N = 1$ is treated. For a fixed N, these well-posedness results are easy to generalize to the present setting, so we only give the statement here and refer the reader to Appendix B for the proof. Our primary goal in this section is rather to obtain bounds, which are stable with respect to the number of components N, which we will send to infinity in Section 4.

As is well known, it is natural to consider initial datum to (2.1) belonging to a negative Hölder space with exponent just below zero. We will be slightly more general and consider random initial datum of the form $\phi_i = z_i + y_i$ satisfying $\mathbf{E}\|z_i\|^p_{\mathbf{C}^{-\kappa}} \lesssim 1$ for $\kappa > 0$ small enough and every $p > 1$, and $\mathbf{E}\|y_i\|^2_{L^2} \lesssim 1$, where the implicit constants are independent of i, N.

The notion of solution to (2.1) is based on the now classical trick of Da-Prato and Debussche; cf. [20]. Namely, we say that Φ_i is a solution to (2.1) provided the decomposition $\Phi_i = Z_i + Y_i$ holds, where Z_i is a solution to the linear SPDE,

$$\mathscr{L}Z_i = \xi_i, \quad Z_i(0) = z_i, \tag{2.2}$$

and Y_i is a weak solution to the remainder equation

$$\begin{aligned}\mathscr{L}Y_i &= -\frac{1}{N}\sum_{j=1}^{N}\big(Y_j^2 Y_i + Y_j^2 Z_i + 2Y_jY_iZ_j + 2Y_j :Z_iZ_j: + :Z_j^2: Y_i + :Z_iZ_j^2:\big),\\ Y_i(0) &= y_i.\end{aligned} \tag{2.3}$$

The notation $:Z_iZ_j:$, $:Z_j^2:$ and $:Z_iZ_j^2:$ denotes a renormalized product of Wick type, which will be defined in Section 2.1 below.

2.1. *Renormalization.* To define the renormalized products appearing in (2.3), it is convenient to make a further splitting of Z_i relative to the corresponding stationary solution to (2.2), which we will denote by $\tilde{Z}_i$. For $\tilde{Z}_i$, these products have a canonical definition that we now recall. Namely, let $\xi_{i,\varepsilon}$ be a space-time mollification of ξ_i defined on $\mathbb{R}\times\mathbb{T}^2$ and let $\tilde{Z}_{i,\varepsilon}$ be the stationary solution to $\mathscr{L}\tilde{Z}_{i,\varepsilon} = \xi_{i,\varepsilon}$. For convenience, we assume that all the noises are mollified with a common bump function. In particular, $\tilde{Z}_{i,\varepsilon}$ are i.i.d. mean zero Gaussian. For

$k \geq 1$ and $i_1, \dots, i_k \in \{1, \dots, N\}$, we then write $:\tilde{Z}_{i_1} \cdots \tilde{Z}_{i_k}:$ as the limit of $:\tilde{Z}_{i_1,\varepsilon} \cdots \tilde{Z}_{i_k,\varepsilon}:$ as $\varepsilon \to 0$. Here, $:\tilde{Z}_{i_1,\varepsilon} \cdots \tilde{Z}_{i_k,\varepsilon}:$ is the canonical Wick product, which in particular is mean zero. More precisely,

$$
\begin{aligned}
:\tilde{Z}_i \tilde{Z}_j: &= \begin{cases} \lim_{\varepsilon \to 0} (\tilde{Z}_{i,\varepsilon}^2 - a_\varepsilon) & (i = j), \\ \lim_{\varepsilon \to 0} \tilde{Z}_{i,\varepsilon} \tilde{Z}_{j,\varepsilon} & (i \neq j), \end{cases} \\
:\tilde{Z}_i \tilde{Z}_j^2: &= \begin{cases} \lim_{\varepsilon \to 0} (\tilde{Z}_{i,\varepsilon}^3 - 3a_\varepsilon \tilde{Z}_{i,\varepsilon}) & (i = j), \\ \lim_{\varepsilon \to 0} (\tilde{Z}_{i,\varepsilon} \tilde{Z}_{j,\varepsilon}^2 - a_\varepsilon \tilde{Z}_{i,\varepsilon}) & (i \neq j), \end{cases}
\end{aligned}
\tag{2.4}
$$

where $a_\varepsilon = \mathbf{E}[\tilde{Z}_{i,\varepsilon}^2(0,0)]$ is a diverging constant independent of i and the limits are understood in $C_T \mathbf{C}^{-\kappa}$ for $\kappa > 0$. (see [53], Section 5, for more details).

We now define the Wick products for Z_i by combining the above with the smoothing properties of the heat semigroup S_t associated with $\mathscr{L}$. Defining $\tilde{z}_i \stackrel{\text{def}}{=} z_i - \tilde{Z}_i(0)$, we have the decomposition

$$
Z_i = \tilde{Z}_i + S_t \tilde{z}_i.
$$

We then overload notation and define the Wick products of Z_i by the binomial formula[7] namely

$$
\begin{aligned}
:Z_j^2: &= :\tilde{Z}_j^2: + 2S_t \tilde{z}_j \tilde{Z}_j + (S_t \tilde{z}_j)^2, \\
:Z_j^3: &= :\tilde{Z}_j^3: + 3S_t \tilde{z}_j :\tilde{Z}_j^2: + 3(S_t \tilde{z}_j)^2 \tilde{Z}_j + (S_t \tilde{z}_j)^3,
\end{aligned}
$$

and for $i \neq j$

$$
\begin{aligned}
:Z_i Z_j: &= :\tilde{Z}_i \tilde{Z}_j: + S_t \tilde{z}_i \tilde{Z}_j + S_t \tilde{z}_j \tilde{Z}_i + S_t \tilde{z}_i S_t \tilde{z}_j, \\
:Z_i Z_j^2: &= :\tilde{Z}_i \tilde{Z}_j^2: + S_t \tilde{z}_i :\tilde{Z}_j^2: + 2S_t \tilde{z}_j :\tilde{Z}_i \tilde{Z}_j: + 2S_t \tilde{z}_i S_t \tilde{z}_j \tilde{Z}_j + (S_t \tilde{z}_j)^2 \tilde{Z}_i + S_t \tilde{z}_i (S_t \tilde{z}_j)^2.
\end{aligned}
$$

We caution the reader that this definition is noncanonical, in the sense that these renormalized products are not necessarily mean zero. By the calculation in [53], Corollary 3, (see also [57], Lemma 3.5), we have the following estimate.

LEMMA 2.1. *For each $\kappa' > \kappa > 0$ and all $p \geq 1$, we have the following bounds*:

$$
\begin{aligned}
\mathbf{E}\|\tilde{Z}_i\|_{C_T \mathbf{C}^{-\kappa}}^p + \mathbf{E}\|Z_i\|_{C_T \mathbf{C}^{-\kappa}}^p &\lesssim 1, \\
\mathbf{E}\| :\tilde{Z}_i \tilde{Z}_j: \|_{C_T \mathbf{C}^{-\kappa}}^p + \mathbf{E}\| :\tilde{Z}_i \tilde{Z}_j^2: \|_{C_T \mathbf{C}^{-\kappa}}^p &\lesssim 1, \\
\mathbf{E}\Big(\sup_{t \in [0,T]} t^{\kappa'} \| :Z_i Z_j: \|_{\mathbf{C}^{-\kappa}} \Big)^p + \mathbf{E}\Big(\sup_{t \in [0,T]} t^{2\kappa'} \| :Z_i Z_j^2: \|_{\mathbf{C}^{-\kappa}} \Big)^p &\lesssim 1.
\end{aligned}
$$

Furthermore, the proportional constants in the inequalities are independent of i, j, N.

By Lemma 2.1, there exists a measurable $\Omega_0 \subset \Omega$ with $\mathbf{P}(\Omega_0) = 1$ such that for $\omega \in \Omega_0$ and every i, j,

$$
\|Z_i\|_{C_T \mathbf{C}^{-\kappa}} + \sup_{t \in [0,T]} t^{\kappa'} \| :Z_i Z_j: \|_{\mathbf{C}^{-\kappa}} + \sup_{t \in [0,T]} t^{2\kappa'} \| :Z_i Z_j^2: \|_{\mathbf{C}^{-\kappa}} < \infty.
$$

[7]This definition is in line with [53], (5.42), which first considers a linear solution with 0 initial condition rather than a stationary solution as here.

In the following, we always consider $\omega \in \Omega_0$. With the above choice of renormalization, classical arguments from [20] can be used to obtain local existence and uniqueness to equation (2.3) by a pathwise fixed-point argument. This solution can also be shown to be global, as a simple consequence of a much stronger result, Lemma 2.3, which will be established in detail below. Since the well-posedness arguments for solving equation (2.3) with a fixed number of components is essentially known, we relegate the proof to Appendix B and only state the result here.

LEMMA 2.2. *For each N, there exist unique global solutions (Y_i) to equation (2.3) such that for $1 \le i \le N$, $Y_i \in C_T L^2 \cap L^4_T L^4 \cap L^2_T H^1$.*

2.2. *Uniform in N estimate.* We now turn to our uniform in N bounds on equation (2.3) and note that Y_i itself depends on N, but we omit this throughout. In the following lemma, we show that the empirical averages of the L^2 norms of Y_i can be controlled pathwise in terms of averages of the $C_T\mathbf{C}^{-\kappa}$ norms of Z_i, $:Z_iZ_j:$ and $:Z_i^2Z_j:$ discussed in Lemma 2.1.

LEMMA 2.3. *Let $s \in [2\kappa, \frac{1}{4})$. There exists a universal constant C such that*

$$\begin{aligned}(2.5)\qquad &\frac{1}{N}\sup_{t\in[0,T]}\sum_{j=1}^N \|Y_j\|_{L^2}^2 + \frac{1}{N}\sum_{j=1}^N \|\nabla Y_j\|_{L^2_T L^2}^2 + \left\|\frac{1}{N}\sum_{i=1}^N Y_i^2\right\|_{L^2_T L^2}^2\\ &\quad \le C\int_0^T R_N \,\mathrm{d}t + \frac{1}{N}\sum_{j=1}^N \|y_j\|_{L^2}^2,\end{aligned}$$

where

$$\begin{aligned}(2.6)\qquad R_N :=& 1 + \left(\frac{1}{N}\sum_{j=1}^N \|Z_j\|_{\mathbf{C}^{-s}}^2\right)^{\frac{2}{1-s}} + \left(\frac{1}{N}\sum_{j=1}^N \| :Z_j^2: \|_{\mathbf{C}^{-s}}\right)^{\frac{4}{2-s}}\\ &+ \left(\frac{1}{N^2}\sum_{i,j=1}^N \| :Z_jZ_i: \|_{\mathbf{C}^{-s}}^2\right)^{\frac{2}{2-s}} + \left(\frac{1}{N^2}\sum_{i,j=1}^N \| :Z_j^2Z_i: \|_{\mathbf{C}^{-s}}^2\right).\end{aligned}$$

PROOF. The proof is based on an energy estimate. In Step 1, we establish the energy identity (2.9), which identifies the coercive quantities and involves three types of terms on the right-hand side. These are labeled I_N^1, I_N^2 and I_N^3, which are respectively linear, quadratic and cubic in Y. In Steps 2–4, we estimate each of these quantities, proceeding in order of difficulty, in terms of the coercive terms and the quantities R_N^i for $i = 1, 2, 3$ defined below. The main ingredient is Lemma A.5, restated here: for $s \in (0, 1)$,

$$(2.7)\qquad |\langle g, f\rangle| \lesssim \big(\|\nabla g\|_{L^1}^s \|g\|_{L^1}^{1-s} + \|g\|_{L^1}\big)\|f\|_{\mathbf{C}^{-s}}.$$

The final output of Steps 1–4 is that for some universal constant C it holds

$$\begin{aligned}(2.8)\qquad &\frac{1}{N}\sum_{i=1}^N \frac{\mathrm{d}}{\mathrm{d}t}\|Y_i\|_{L^2}^2 + \frac{1}{N}\sum_{i=1}^N \|\nabla Y_i\|_{L^2}^2 + \frac{1}{N^2}\left\|\sum_{i=1}^N Y_i^2\right\|_{L^2}^2 + \frac{m}{N}\sum_{j=1}^N \|Y_j\|_{L^2}^2\\ &\quad\le C\frac{R_N^1}{N} + C\big(R_N^2 + R_N^3\big)\frac{1}{N}\sum_{i=1}^N \|Y_i\|_{L^2}^2,\end{aligned}$$

where R_N^i for $i = 1, 2, 3$ are defined in (2.11), (2.14), (2.18) below. Noting that by Hölder's inequality

$$\frac{1}{N}\sum_{i=1}^N \|Y_i\|_{L^2}^2 = \frac{1}{N}\left\|\sum_{i=1}^N Y_i^2\right\|_{L^1} \le \frac{1}{N}\left\|\sum_{i=1}^N Y_i^2\right\|_{L^2},$$

the estimate (2.5) follows from Young's inequality with exponents $(2, 2)$ and an integration over $[0, T]$. The condition $s \in [2\kappa, \frac{1}{4})$ ensures that R_N is integrable near the origin; cf. Lemma 2.1.

STEP 1 (Energy balance)

In this step, we justify the energy identity

$$\begin{aligned}(2.9)\qquad &\frac{1}{2}\sum_{i=1}^N \frac{\mathrm{d}}{\mathrm{d}t}\|Y_i\|_{L^2}^2 + \sum_{i=1}^N \|\nabla Y_i\|_{L^2}^2 + m\sum_{i=1}^N \|Y_i\|_{L^2}^2 + \left\|\frac{1}{\sqrt{N}}\sum_{i=1}^N Y_i^2\right\|_{L^2}^2 \\ &\quad = I_N^1 + I_N^2 + I_N^3,\end{aligned}$$

where the quantities I_N^i for $i = 1, 2, 3$ are defined by

$$I_N^1 \stackrel{\text{def}}{=} -\frac{1}{N}\sum_{i,j=1}^N \langle Y_i, :Z_j^2 Z_i:\rangle,$$

$$I_N^2 \stackrel{\text{def}}{=} -\frac{1}{N}\sum_{i,j=1}^N 2\langle Y_i Y_j, :Z_j Z_i:\rangle + \langle Y_i^2, :Z_j^2:\rangle,$$

$$I_N^3 \stackrel{\text{def}}{=} -\frac{1}{N}\sum_{i,j=1}^N 3\langle Y_i^2 Y_j, Z_j\rangle.$$

Notice that I_N^1, I_N^2 and I_N^3 are linear, quadratic and cubic in Y, respectively. Formally, the identity (2.9) follows from testing (2.3) by Y_i, integrating by parts, summing over $i = 1, \dots, N$, and using symmetry with respect i and j. Since Y_i is not sufficiently smooth in the time variable, some care is required to make this fully rigorous, and we direct the reader to [53], Proposition 6.8, for more details.

STEP 2 (Estimates for I_N^1)

In this step, we show there is a universal constant C such that

$$(2.10)\qquad I_N^1 \le \frac{1}{4}\sum_{i=1}^N \|\nabla Y_i\|_{L^2}^2 + \sum_{i=1}^N \|Y_i\|_{L^2}^2 + CR_N^1,$$

where

$$(2.11)\qquad R_N^1 \stackrel{\text{def}}{=} \sum_{i=1}^N \left\|\frac{1}{N}\sum_{j=1}^N :Z_j^2 Z_i:\right\|_{\mathbf{C}^{-s}}^2.$$

To establish (2.10), we apply (2.7) with Y_i playing the role of g and $\frac{1}{N}\sum_{j=1}^N :Z_j^2 Z_i:$ playing the role of f to find

$$(2.12)\qquad I_N^1 \lesssim \sum_{i=1}^N \bigl(\|Y_i\|_{L^1}^{1-s}\|\nabla Y_i\|_{L^1}^s + \|Y_i\|_{L^1}\bigr)\left\|\frac{1}{N}\sum_{j=1}^N :Z_j^2 Z_i:\right\|_{\mathbf{C}^{-s}}.$$

We now use Young's inequality with exponents $(\frac{2}{1-s}, \frac{2}{s}, 2)$ for the first term and $(2, 2)$ for the second term and the embedding of L^2 into L^1 to obtain (2.10).

STEP 3 (Estimates for I_N^2)
In this step, we show there is a universal constant C such that

$$I_N^2 \le \frac{1}{4}\sum_{i=1}^N \|\nabla Y_i\|_{L^2}^2 + C\big(1+R_N^2\big)\Big(\sum_{i=1}^N \|Y_i\|_{L^2}^2\Big), \tag{2.13}$$

where

$$\begin{aligned} R_N^2 &\stackrel{\text{def}}{=} \Big(\frac{1}{N^2}\sum_{i,j=1}^N \|:Z_jZ_i:\|_{\mathbf{C}^{-s}}^2\Big)^{\frac{1}{2-s}} + \Big\|\frac{1}{N}\sum_{j=1}^N :Z_j^2:\Big\|_{\mathbf{C}^{-s}}^{\frac{2}{2-s}} \\ &\stackrel{\text{def}}{=} (\widetilde{R_N^2})^{\frac{1}{2-s}} + (\overline{R_N^2})^{\frac{2}{2-s}}. \end{aligned} \tag{2.14}$$

Applying (2.7) with Y_iY_j playing the role of g and $:Z_jZ_i:$ playing the role of f followed by Hölder's inequality in L^2, the product rule and symmetry with respect to i, j we find

$$\begin{aligned} &\frac{1}{N}\sum_{i,j=1}^N \langle Y_iY_j, :Z_jZ_i:\rangle \\ &\lesssim \frac{1}{N}\sum_{i,j=1}^N \big(\|Y_iY_j\|_{L^1}^{1-s}\|\nabla(Y_iY_j)\|_{L^1}^s + \|Y_iY_j\|_{L^1}\big)\|:Z_jZ_i:\|_{\mathbf{C}^{-s}} \\ &\lesssim \frac{1}{N}\sum_{i,j=1}^N \big(\|Y_j\|_{L^2}\|Y_i\|_{L^2}^{1-s}\|\nabla Y_i\|_{L^2}^s + \|Y_i\|_{L^2}\|Y_j\|_{L^2}\big)\|:Z_jZ_i:\|_{\mathbf{C}^{-s}} \\ &\lesssim \Big(\sum_{j=1}^N \|Y_j\|_{L^2}^2\Big)^{\frac{1}{2}}\Big(\sum_{i=1}^N \|Y_i\|_{L^2}^{2(1-s)}\|\nabla Y_i\|_{L^2}^{2s}\Big)^{\frac{1}{2}}(\widetilde{R_N^2})^{\frac{1}{2}} + \Big(\sum_{i=1}^N \|Y_i\|_{L^2}^2\Big)(\widetilde{R_N^2})^{\frac{1}{2}} \\ &\lesssim \Big(\sum_{j=1}^N \|Y_j\|_{L^2}^2\Big)^{1-\frac{s}{2}}\Big(\sum_{i=1}^N \|\nabla Y_i\|_{L^2}^2\Big)^{\frac{s}{2}}(\widetilde{R_N^2})^{\frac{1}{2}} + \Big(\sum_{i=1}^N \|Y_i\|_{L^2}^2\Big)(\widetilde{R_N^2})^{\frac{1}{2}}, \end{aligned} \tag{2.15}$$

where we used Hölder's inequality for the summation in i, j with exponents $(2, 2)$ followed by Hölder's inequality for the summation in i with exponents $(\frac{1}{1-s}, \frac{1}{s})$. Finally, applying (2.7) with Y_i^2 playing the role of g and $\frac{1}{N}\sum_{j=1}^N :Z_j^2:$ playing the role of f, we find

$$\begin{aligned} \frac{1}{N}\sum_{i,j=1}^N \langle Y_i^2, :Z_j^2:\rangle &\lesssim \sum_{i=1}^N \big(\|Y_i^2\|_{L^1}^{1-s}\|Y_i\nabla Y_i\|_{L^1}^s + \|Y_i^2\|_{L^1}\big)\overline{R_N^2} \\ &\lesssim \sum_{i=1}^N \big(\|Y_i\|_{L^2}^{2-s}\|\nabla Y_i\|_{L^2}^s + \|Y_i\|_{L^2}^2\big)\overline{R_N^2} \\ &\lesssim \Big[\Big(\sum_{i=1}^N \|Y_i\|_{L^2}^2\Big)^{1-\frac{s}{2}}\Big(\sum_{i=1}^N \|\nabla Y_i\|_{L^2}^2\Big)^{\frac{s}{2}} + \sum_{i=1}^N \|Y_i\|_{L^2}^2\Big]\overline{R_N^2}, \end{aligned} \tag{2.16}$$

where we used Hölder's inequality for the summation in i with exponents $(\frac{2}{2-s}, \frac{2}{s})$. The inequality (2.13) now follows from (2.15)–(2.16) by Young's inequality with exponents $(\frac{2}{2-s}, \frac{2}{s})$.

STEP 4 (Estimates for I_N^3: cubic terms in Y)
In this step, we show there exists a universal constant C such that

$$I_N^3 \le \frac{1}{4}\left(\sum_{i=1}^N \|\nabla Y_i\|_{L^2}^2 + \left\|\frac{1}{\sqrt{N}}\sum_{i=1}^N Y_i^2\right\|_{L^2}^2\right) + C\left(1+R_N^3\right)\left(\sum_{i=1}^N \|Y_i\|_{L^2}^2\right), \tag{2.17}$$

where

$$R_N^3 \stackrel{\text{def}}{=} \left(\frac{1}{N}\sum_{j=1}^N \|Z_j\|_{\mathbf{C}^{-s}}^2\right)^{\frac{1}{1-s}} = \left(\frac{1}{N}\mathscr{Z}_N\right)^{\frac{1}{1-s}} \quad \text{with } \mathscr{Z}_N \stackrel{\text{def}}{=} \sum_{j=1}^N \|Z_j\|_{\mathbf{C}^{-s}}^2. \tag{2.18}$$

Appealing again to (2.7), we find

$$\begin{aligned}
I_N^3 &\lesssim \frac{1}{N}\sum_{j=1}^N\left(\left\|\sum_{i=1}^N Y_i^2 Y_j\right\|_{L^1}^{1-s}\left\|\nabla\left(\sum_{i=1}^N Y_i^2 Y_j\right)\right\|_{L^1}^{s} + \left\|\sum_{i=1}^N Y_i^2 Y_j\right\|_{L^1}\right)\|Z_j\|_{\mathbf{C}^{-s}} \\
&\lesssim \frac{1}{N}\left(\sum_{j=1}^N\left\|\sum_{i=1}^N Y_i^2 Y_j\right\|_{L^1}^{2(1-s)}\left\|\nabla\left(\sum_{i=1}^N Y_i^2 Y_j\right)\right\|_{L^1}^{2s}\right)^{\frac{1}{2}}\mathscr{Z}_N^{\frac{1}{2}} \\
&\quad + \frac{1}{N}\left(\sum_{j=1}^N\left\|\sum_{i=1}^N Y_i^2 Y_j\right\|_{L^1}^{2}\right)^{\frac{1}{2}}\mathscr{Z}_N^{\frac{1}{2}}.
\end{aligned} \tag{2.19}$$

By Hölder's inequality, it holds that

$$\left\|\sum_{i=1}^N Y_i^2 Y_j\right\|_{L^1} \lesssim \left\|\sum_{i=1}^N Y_i^2\right\|_{L^2}\|Y_j\|_{L^2}. \tag{2.20}$$

Furthermore, we find that

$$\begin{aligned}
\left\|\nabla\left(\sum_{i=1}^N Y_i^2 Y_j\right)\right\|_{L^1} &\lesssim \left\|\sum_{i=1}^N Y_i^2 \nabla Y_j\right\|_{L^1} + \left\|\sum_{i=1}^N \nabla Y_i Y_i Y_j\right\|_{L^1} \\
&\lesssim \left\|\sum_{i=1}^N Y_i^2\right\|_{L^2}\|\nabla Y_j\|_{L^2} + \left(\sum_{i=1}^N \|\nabla Y_i\|_{L^2}^2\right)^{1/2}\left\|\sum_{i=1}^N Y_i^2 Y_j^2\right\|_{L^1}^{1/2}.
\end{aligned} \tag{2.21}$$

Hence, we find that

$$\begin{aligned}
&\sum_{j=1}^N\left\|\sum_{i=1}^N Y_i^2 Y_j\right\|_{L^1}^{2(1-s)}\left\|\nabla\left(\sum_{i=1}^N Y_i^2 Y_j\right)\right\|_{L^1}^{2s} \\
&\quad\lesssim \left\|\sum_{i=1}^N Y_i^2\right\|_{L^2}^2\left(\sum_{j=1}^N \|Y_j\|_{L^2}^{2(1-s)}\|\nabla Y_j\|_{L^2}^{2s}\right) \\
&\qquad + \left\|\sum_{i=1}^N Y_i^2\right\|_{L^2}^{2(1-s)}\left(\sum_{i=1}^N \|\nabla Y_i\|_{L^2}^2\right)^{s}\left(\sum_{j=1}^N\left\|\sum_{i=1}^N Y_i^2 Y_j^2\right\|_{L^1}^{s}\|Y_j\|_{L^2}^{2(1-s)}\right) \\
&\quad\lesssim \left\|\sum_{i=1}^N Y_i^2\right\|_{L^2}^2\left(\sum_{j=1}^N \|Y_j\|_{L^2}^2\right)^{1-s}\left(\sum_{j=1}^N \|\nabla Y_j\|_{L^2}^2\right)^{s} \\
&\qquad + \left\|\sum_{i=1}^N Y_i^2\right\|_{L^2}^{2(1-s)}\left(\sum_{i=1}^N \|\nabla Y_i\|_{L^2}^2\right)^{s}\left(\sum_{j=1}^N\left\|\sum_{i=1}^N Y_i^2 Y_j^2\right\|_{L^1}\right)^{s}\left(\sum_{j=1}^N \|Y_j\|_{L^2}^2\right)^{1-s}
\end{aligned}$$

$$\lesssim \left\|\sum_{i=1}^N Y_i^2\right\|_{L^2}^2 \left(\sum_{j=1}^N \|Y_j\|_{L^2}^2\right)^{1-s} \left(\sum_{j=1}^N \|\nabla Y_j\|_{L^2}^2\right)^s.$$

Inserting this into (2.19), taking the square root and using (2.20) we find

$$\begin{aligned} I_N^3 &\lesssim \frac{1}{N}\left\|\sum_{i=1}^N Y_i^2\right\|_{L^2} \left(\sum_{j=1}^N \|Y_j\|_{L^2}^2\right)^{\frac{1-s}{2}} \left(\sum_{j=1}^N \|\nabla Y_j\|_{L^2}^2\right)^{\frac{s}{2}} \mathscr{Z}_N^{\frac12} \\ &\quad + \frac{1}{N}\left\|\sum_{i=1}^N Y_i^2\right\|_{L^2} \left(\sum_{j=1}^N \|Y_j\|_{L^2}^2\right)^{\frac12} \mathscr{Z}_N^{\frac12}. \end{aligned} \tag{2.22}$$

Applying Young's inequality with exponent $(2, \frac{2}{1-s}, \frac{2}{s})$ we arrive at (2.17). $\square$

COROLLARY 2.4. *Let $q \geq 1$, $s \in [2\kappa, \frac{2}{q+1})$. There exists a universal constant C such that*

$$\sup_{t\in[0,T]}\left(\frac1N\sum_{j=1}^N \|Y_j\|_{L^2}^2\right)^q + \int_0^T \left(\frac1N\sum_{j=1}^N \|Y_j\|_{L^2}^2\right)^{q-1}\left[\frac1N\sum_{j=1}^N\|\nabla Y_j\|_{L^2}^2 + \left\|\frac1N\sum_{i=1}^N Y_i^2\right\|_{L^2}^2\right]\mathrm{d}t$$

$$\leq C\int_0^T R_N^{\frac{q+1}{2}}\,\mathrm{d}t + \left(\frac1N\sum_{j=1}^N\|y_j\|_{L^2}^2\right)^q,$$

with R_N introduced in Lemma 2.3.

PROOF. Set $V = \frac1N\sum_{i=1}^N \|Y_i\|_{L^2}^2$ and $G = \frac1N\sum_{j=1}^N\|\nabla Y_j\|_{L^2}^2 + \|\frac1N\sum_{i=1}^N Y_i^2\|_{L^2}^2$. By (2.8) in the proof of Lemma 2.3, we deduce for $q \geq 1$,

$$\frac{\mathrm{d}}{\mathrm{d}t}V^q + GV^{q-1} \leq CR_N V^{q-1} \leq CR_N^{\frac{q+1}{2}} + \frac12 V^{q+1}.$$

Note that $G \geq V^2$ since $\|\sum_{i=1}^N Y_i^2\|_{L^1} = \sum_{i=1}^N \|Y_i\|_{L^2}^2$, which implies the result. $\square$

LEMMA 2.5. *Let $s \in [2\kappa, 1/4)$. There exists a universal constant C such that*

$$\begin{aligned} &\sup_{t\in[0,T]}\sum_{j=1}^N\|Y_j\|_{L^2}^2 + \sum_{j=1}^N\|\nabla Y_j\|_{L_T^2L^2}^2 + \frac1N\left\|\sum_{i=1}^N Y_i^2\right\|_{L_T^2L^2}^2 \\ &\quad \leq C\left(\|R_N^0\|_{L_T^1} + \sum_{j=1}^N\|y_j\|_{L^2}^2\right)\exp\left\{\int_0^T (1+R_N^2+R_N^3)\,\mathrm{d}t\right\}, \end{aligned} \tag{2.23}$$

where R_N^2, R_N^3 given in the proof of Lemma 2.3 and

$$R_N^0 = \frac{1}{N^2}\sum_{i=1}^N\left\|\sum_{j=1}^N \Lambda^{-s}(:Z_j^2Z_i:)\right\|_{L^2}^2. \tag{2.24}$$

PROOF. The proof is almost the same as Lemma 2.3. We appeal to Steps 1, 3 and 4 of Lemma 2.3 and only modify Step 2. To estimate I_N^1, we write

$$\begin{aligned} I_N^1 &\leq \frac1N\sum_{i=1}^N\|\Lambda^s Y_i\|_{L^2}\left\|\sum_{j=1}^N\Lambda^{-s}(:Z_j^2Z_i:)\right\|_{L^2} \\ &\leq \frac18\sum_{i=1}^N\|Y_i\|_{L^2}^2 + \frac18\sum_{i=1}^N\|\nabla Y_i\|_{L^2}^2 + \frac{C}{N^2}\sum_{i=1}^N\left\|\sum_{j=1}^N\Lambda^{-s}(:Z_j^2Z_i:)\right\|_{L^2}^2, \end{aligned} \tag{2.25}$$

where, in the last step, we applied Young's inequality for products, and then interpolation. Combining (2.13), and (2.17) (2.16) with (2.25) and inserting these inequalities into the energy identity (2.9), we obtain

$$\begin{aligned}&\sum_{j=1}^N \frac{\mathrm{d}}{\mathrm{d}t}\|Y_j\|_{L^2}^2+\sum_{j=1}^N\|\nabla Y_j\|_{L^2}^2+\frac{1}{N}\Bigg\|\sum_{i=1}^N Y_i^2\Bigg\|_{L^2}^2+m\sum_{j=1}^N\|Y_j\|_{L^2}^2\\&\quad\leq CR_N^0+\sum_{j=1}^N\|Y_j\|_{L^2}^2C\big(1+R_N^2+R_N^3\big).\end{aligned} \tag{2.26}$$

The estimate (2.23) now follows from Gronwall's inequality. $\square$

REMARK 2.6. For the estimate of $\frac{1}{N}\sum_{i=1}^N\|Y_i\|_{L^2}^2$ in Lemma 2.3, the dissipation term $\|\frac{1}{N}\sum_{i=1}^N Y_i^2\|_{L^2}^2$ could be used to avoid Gronwall's lemma. However, for $\sum_{i=1}^N\|Y_i\|_{L^2}^2$ or $\frac{1}{N}\sum_{i=1}^N\|Y_i\|_{L^p}^p$ for $p>2$ it is less clear how to exploit the corresponding dissipation term and we need to use Gronwall's inequality to derive a uniform estimate. Since the R_N^2, R_N^3 appear in the exponential, this makes it unclear how to obtain moment estimates directly.

3. Global solvability of the mean-field SPDE. In this section, we develop a solution theory for the mean-field SPDE (1.6), the renormalized version of the formal equation (1.5). In two dimensions, this is a singular SPDE where the ill-defined nonlinearity depends on the law of the solution. As a result, we cannot proceed via pathwise arguments alone as in [20] and [53] and we need to develop a few new tricks for both the local and global well-posedness.

We begin by explaining our assumptions on the initial data and our notion of solution to (1.6). The initial datum ψ_i decompose as $\psi_i=z_i+\eta_i$, where $\mathbf{E}\|z_i\|_{\mathbf{C}^{-\kappa}}^p\lesssim 1$ for $\kappa>0$ and every $p>1$, and $\mathbf{E}\|\eta_i\|_{L^4}^4<\infty$ (except for Lemma 3.4 which is an L^p estimate). We define Ψ_i to be a solution to the renormalized, mean-field SPDE (1.6) starting from ψ_i provided that $\Psi_i=Z_i+X_i$ holds, where Z_i is the solution to (2.2) with $Z_i(0)=z_i$ as in Section 2 and X_i is a random process satisfying

$$\mathscr{L}X_i=-\mu_{X_i}(X_i+Z_i),\qquad X_i(0)=\eta_i. \tag{3.1}$$

Here, μ_{X_i} depends on the law of X_i and is defined as

$$\mu_{X_i}\stackrel{\text{def}}{=}\mathbf{E}\big[X_i^2\big]+2\mathbf{E}[X_iZ_i]+\mathbf{E}\big[:Z_i^2:\big].$$

In the following, we write μ for μ_{X_i} for simplicity.

We now comment on the meaning of the nonlinearity in equation (3.1). Recall from Section 2.1 that $Z_i\in\mathbf{C}^{-\kappa}$ (Lemma 2.1), while $\mathbf{E}[:Z_i^2:]=\mathbf{E}[(S_t\tilde{z}_i)^2]$ with $\tilde{z}_i=z_i-\tilde{Z}_i(0)$, so by Schauder theory we expect that X_i is Hölder continuous. Hence, we anticipate that $\mathbf{E}[X_i^2]$ is a well-defined function, while $\mathbf{E}[X_iZ_i]$ is a distribution satisfying for $t>0$ and $\beta>\kappa$,

$$\big\|\mathbf{E}[X_iZ_i](t)\big\|_{\mathbf{C}^{-\kappa}}\lesssim\mathbf{E}\big[\|X_i(t)\|_{\mathbf{C}^\beta}\|Z_i(t)\|_{\mathbf{C}^{-\kappa}}\big].$$

We immediately find that all terms in $\mu(X_i+Z_i)$ are classically defined in the sense of distributions except for $\mathbf{E}[X_iZ_i]Z_i$, which requires more care and a suitable probabilistic argument. The idea used to overcome this difficulty, which is repeated in different ways throughout the section, is to view the expectation μ as coming from a suitable independent copy of (X_i,Z_i). To avoid notational confusion, we now comment further on our convention throughout this section. We consider equation (1.6) for a *fixed* i and when we write (η_j,z_j) for $j\neq i$ we mean an independent copy of (η_i,z_i), and we then write (Z_j,X_j) for the solution driven by white noise ξ_j, which is independent of ξ_i, from initial data (z_j,η_j).

3.1. *Local well-posedness.*

LEMMA 3.1. *For $p \in [1, \infty]$ and $0 < \kappa < s$, it holds*

$$\|Z_i \mathbf{E}[Z_i X_i]\|_{B^{-\kappa}_{p,\infty}} \lesssim \big(\mathbf{E}\|X_i\|^2_{B^s_{p,\infty}}\big)^{\frac{1}{2}} \big(\mathbf{E}\big[\| :Z_i Z_j: \|^2_{\mathbf{C}^{-\kappa}} \mid Z_i\big]\big)^{\frac{1}{2}}. \tag{3.2}$$

Here, the conditional expectation is on the σ-algebra generated by the stochastic process Z_i.

PROOF. Letting (X_j, Z_j) be an independent copy of (X_i, Z_i) we have

$$Z_i \mathbf{E}[Z_i X_i] = Z_i \mathbf{E}[Z_j X_j] = \mathbf{E}[:Z_i Z_j: X_j \mid Z_i].$$

We then use Jensen's inequality to find

$$\|Z_i \mathbf{E}[Z_j X_j]\|_{B^{-\kappa}_{p,\infty}} \le \mathbf{E}\big[\| :Z_i Z_j: X_j\|_{B^{-\kappa}_{p,\infty}} \mid Z_i\big] \le \mathbf{E}\big[\| :Z_i Z_j: \|_{\mathbf{C}^{-\kappa}} \|X_j\|_{B^s_{p,\infty}} \mid Z_i\big],$$

where we used Lemma A.3 in the last line. The claim now follows from conditional Hölder's inequality and the independence of X_j from Z_i. □

We now apply the above result to obtain a local well-posedness result for (3.1), which yields in turn a local well-posedness result for (1.6).

LEMMA 3.2. *There exists $T^* > 0$ small enough such that (3.1) has a unique mild solution $X_i \in L^2(\Omega; C_{T^*} L^4 \cap C((0, T^*]; \mathbf{C}^\beta))$ and for $\beta > 3\kappa$ small enough, $\gamma = \beta + \frac{1}{2}$, one has*

$$\mathbf{E}\Big[\sup_{t \in [0,T^*]} t^\gamma \|X_i\|^2_{\mathbf{C}^\beta}\Big] \le 1.$$

PROOF. For $T > 0$, define the ball

$$\mathcal{B}_T \stackrel{\text{def}}{=} \Big\{X_i \in L^2\big(\Omega, C\big((0, T]; \mathbf{C}^\beta\big)\big) \mid \mathbf{E}\Big[\sup_{t \in [0,T]} t^\gamma \|X_i\|^2_{\mathbf{C}^\beta}\Big] \le 1, \quad X_i(0) = \eta_i\Big\}.$$

Here, we endow the space $C((0, T]; \mathbf{C}^\beta)$ with norm $(\sup_{t \in [0,T]} t^\gamma \|f(t)\|^2_{\mathbf{C}^\beta})^{1/2}$.

For $X_i \in \mathcal{B}_T$, define $\mathcal{M}_T X_i : (0, T] \mapsto \mathbf{C}^\beta$ via

$$\mathcal{M}_T X_i(t) := \int_0^t S_{t-s} \mathbf{E}\big[X_i^2 + 2X_i Z_i + :Z_i^2: \big](X_i + Z_i)\,\mathrm{d}s + S_t \eta_i.$$

Using Lemma A.4 and Lemma A.3 noting $\beta > \kappa$, we find that

$$\begin{aligned}
&\|\mathcal{M}_T X_i(t) - S_t \eta_i\|_{\mathbf{C}^\beta} \\
&\quad\lesssim \int_0^t (t-s)^{-\frac{\beta+\kappa}{2}} \|\mathbf{E}[X_i Z_i] Z_i\|_{\mathbf{C}^{-\kappa}}\,\mathrm{d}s \\
&\qquad + \int_0^t \big(\|\mathbf{E}X_i^2\|_{\mathbf{C}^\beta} + \|\mathbf{E} :Z_i^2: \|_{\mathbf{C}^\beta} + (t-s)^{-\frac{\beta+\kappa}{2}} \|\mathbf{E}[X_i Z_i]\|_{\mathbf{C}^{-\kappa}}\big) \|X_i\|_{\mathbf{C}^\beta}\,\mathrm{d}s \\
&\qquad + \int_0^t (t-s)^{-\frac{\beta+\kappa}{2}} \big(\|\mathbf{E}X_i^2\|_{\mathbf{C}^\beta} + \|\mathbf{E} :Z_i^2: \|_{\mathbf{C}^\beta}\big) \|Z_i\|_{\mathbf{C}^{-\kappa}}\,\mathrm{d}s \stackrel{\text{def}}{=} \sum_{i=1}^3 J_i(t).
\end{aligned}$$

We start by applying Lemma 3.1 to obtain the pathwise bound

$$\begin{aligned}
J_1(t) &\lesssim \int_0^t (t-s)^{-\frac{\beta+\kappa}{2}} \big(\mathbf{E}\|X_i\|^2_{\mathbf{C}^\beta}\big)^{\frac{1}{2}} \big(\mathbf{E}\big[\| :Z_i Z_j: \|^2_{\mathbf{C}^{-\kappa}} \mid Z_i\big]\big)^{\frac{1}{2}}\,\mathrm{d}s \\
&\lesssim \Big(\mathbf{E}\Big[\sup_{r \in [0,t]} \big(r^{\kappa'} \| :Z_i Z_j: (r)\|_{\mathbf{C}^{-\kappa}}\big)^2 \mid Z_i\Big]\Big)^{\frac{1}{2}} \int_0^t (t-s)^{-\frac{\beta+\kappa}{2}} s^{-\frac{\gamma+2\kappa'}{2}}\,\mathrm{d}s,
\end{aligned}$$

for $\kappa' > \kappa > 0$, which is integrable provided $\beta < 2-\kappa$ and $\gamma < 2-2\kappa'$. We may now apply Lemma 2.1 to find that

$$\mathbf{E}|J_1(t)|^2 \lesssim \mathbf{E}\Big[\sup_{r\in[0,t]}\big(r^{\kappa'}\|:Z_iZ_j:(r)\|_{\mathbf{C}^{-\kappa}}\big)^2\Big]t^{2-(\beta+\kappa+2\kappa'+\gamma)} \lesssim t^{2-(\beta+\kappa+2\kappa'+\gamma)}.$$

Before estimating $J_2(t)$ and $J_3(t)$, we make three observations. First, note that by Jensen's inequality and Lemma A.3 it holds

$$\|\mathbf{E}X_i^2\|_{\mathbf{C}^\beta} \lesssim \mathbf{E}\big[\|X_i\|_{\mathbf{C}^\beta}^2\big] \lesssim s^{-\gamma}.$$

Furthermore, using again Lemma A.4 we find

$$\|\mathbf{E}:Z_i^2:\|_{\mathbf{C}^\beta} \le \mathbf{E}\big\|(S_t\tilde{z}_i)^2\big\|_{\mathbf{C}^\beta} \le \mathbf{E}\|S_t\tilde{z}_i\|_{\mathbf{C}^\beta}^2 \lesssim s^{-(\beta+\kappa)}.$$

Finally, note that by Lemma A.3

$$\|\mathbf{E}[X_iZ_i]\|_{\mathbf{C}^{-\kappa}} \lesssim \mathbf{E}\big[\|X_i\|_{\mathbf{C}^\beta}\|Z_i\|_{\mathbf{C}^{-\kappa}}\big] \lesssim \big(\mathbf{E}\big[\|X_i\|_{\mathbf{C}^\beta}^2\big]\big)^{\frac{1}{2}}\big(\mathbf{E}\big[\|Z_i\|_{\mathbf{C}^{-\kappa}}^2\big]\big)^{\frac{1}{2}} \lesssim s^{-\frac{\gamma}{2}}.$$

Inserting these three bounds, we find the inequalities

$$J_2(t) \lesssim \int_0^t \big(s^{-\gamma} + s^{-(\beta+\kappa)} + (t-s)^{-\frac{\beta+\kappa}{2}}s^{-\frac{\gamma}{2}}\big)\|X_i\|_{\mathbf{C}^\beta}\,\mathrm{d}s,$$

$$J_3(t) \lesssim \int_0^t (t-s)^{-\frac{\beta+\kappa}{2}}\big(s^{-\gamma} + s^{-(\beta+\kappa)}\big)\|Z_i\|_{\mathbf{C}^{-\kappa}}\,\mathrm{d}s.$$

Squaring and taking expectation, we find

$$\mathbf{E}|J_2(t)|^2 \lesssim t^{-\gamma}\big(t^{2-2\gamma} + t^{2-2(\beta+\kappa)} + t^{2-(\beta+\kappa+\gamma)}\big),$$

$$\mathbf{E}|J_3(t)|^2 \lesssim \big(t^{2-(\beta+\kappa+2\gamma)} + t^{2-3(\beta+\kappa)}\big).$$

Under our assumption on β these are all bounded by $t^{-\gamma}$. Finally, note that by Lemma A.4 and the embedding $L^4 \subset \mathbf{C}^{-\frac{1}{2}}$, we obtain

$$\|S_t\eta\|_{\mathbf{C}^\beta} \lesssim t^{-\frac{1+2\beta}{4}}\|\eta\|_{L^4}. \tag{3.3}$$

Combining the above estimates, we can find T^* small enough to have

$$\mathbf{E}\Big[\sup_{t\in[0,T^*]} t^\gamma\|\mathcal{M}_{T^*}X(t)\|_{\mathbf{C}^\beta}^2\Big] \le 1,$$

which implies that for T^* small enough $\mathcal{M}_{T^*}$ maps $\mathcal{B}_{T^*}$ into itself. The contraction property follows similarly. Now the local existence and uniqueness in $L^2(\Omega; C((0,T],\mathbf{C}^\beta))$ follows. Furthermore, we know $\int_0^t S_{t-s}\mu(X(s)+Z(s))\,\mathrm{d}s$ is continuous in $\mathbf{C}^\beta$ and $S_t\eta_i \in C_TL^4$. The result follows. $\square$

3.2. *Global well-posedness.* We now extend our local solution to a global solution through a series of a priori bounds, starting with a uniform in time on the $L^2(\Omega; L^2)$ norm of X_i together with an $L^2(\Omega; L_T^2H^1)$ bound.

LEMMA 3.3. *There exists a universal constant C such that*

$$\begin{aligned}\sup_{t\in[0,T]} \mathbf{E}\|X_i\|_{L^2}^2 + \mathbf{E}\|\nabla X_i\|_{L_T^2L^2}^2 + \|\mathbf{E}X_i^2\|_{L_T^2L^2}^2 + m\mathbf{E}\|X_i\|_{L_T^2L^2}^2 \\ \le C\int_0^T R\,\mathrm{d}t + \mathbf{E}\|\eta_i\|_{L^2}^2,\end{aligned} \tag{3.4}$$

where, for $i \neq j$ we define

$$R \stackrel{\text{def}}{=} 1 + \big(\mathbf{E}\|Z_i\|_{\mathbf{C}^{-s}}^2\big)^{\frac{2}{1-s}} + \mathbf{E}\|:Z_j^2Z_i:\|_{\mathbf{C}^{-s}}^2 + C\big(\mathbf{E}\|:Z_jZ_i:\|_{\mathbf{C}^{-s}}^2\big)^2 + C\big(\mathbf{E}\|:Z_i^2:\|_{\mathbf{C}^{-s}}\big)^4.$$

PROOF. The proof is similar in spirit to the proof of Lemma 2.3, proceeding by energy estimates.

STEP 1 (Expected energy balance)

In this step, we establish the following identity:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\|X_i\|_{L^2}^2+\mathbf{E}\|\nabla X_i\|_{L^2}^2+\|\mathbf{E}X_i^2\|_{L^2}^2+m\mathbf{E}\|X_i\|_{L^2}^2=I^1+I^2+I^3, \tag{3.5}$$

where

$$\begin{aligned} I^1 &\stackrel{\text{def}}{=}\mathbf{E}\langle X_i,\ :Z_iZ_j^2:\ \rangle,\\ I^2 &\stackrel{\text{def}}{=}\mathbf{E}\langle X_i^2,\ :Z_j^2:\ \rangle+2\mathbf{E}\langle X_iX_j, Z_iZ_j\rangle,\\ I^3 &\stackrel{\text{def}}{=}3\mathbf{E}\langle X_i^2X_j, Z_j\rangle. \end{aligned} \tag{3.6}$$

Testing (3.1) with X_i, integrating by parts and using that X_i, X_iZ_i and $:Z_i^2:$ are respectively equal in law to X_j, X_jZ_j and $:Z_j^2:$ we find

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|X_i\|_{L^2}^2+\|\nabla X_i\|_{L^2}^2+m\|X_i\|_{L^2}^2+\|X_i^2\mathbf{E}X_i^2\|_{L^1}\\ &\quad=-\langle X_i, Z_i\mathbf{E}(\,:Z_j^2:\,)\rangle-\langle X_i^2,\mathbf{E}(\,:Z_j^2:\,)\rangle-2\langle X_i, Z_i\mathbf{E}(X_jZ_j)\rangle\\ &\qquad-2\langle X_i^2,\mathbf{E}(X_jZ_j)\rangle-\langle X_i\mathbf{E}(X_j^2), Z_i\rangle. \end{aligned}$$

Taking expectation on both sides, using independence, and the fact that $X_i^2X_jZ_j$ has the same law as $X_j^2X_iZ_i$ we obtain (3.5).

STEP 2 (Estimates for I^1)

In this step, we show there is a universal constant C such that

$$I^1\le\frac{1}{4}\big(\|\mathbf{E}X_i^2\|_{L^2}^2+\mathbf{E}\|\nabla X_i\|_{L^2}^2\big)+C\big(1+\mathbf{E}\|\,:Z_iZ_j^2:\,\|_{\mathbf{C}^{-s}}^2\big). \tag{3.7}$$

To prove the claim, we apply (2.7) to have

$$I^1\lesssim\mathbf{E}\big[\big(\|X_i\|_{L^1}^{1-s}\|\nabla X_i\|_{L^1}^{s}+\|X_i\|_{L^1}\big)\|\,:Z_j^2Z_i:\,\|_{\mathbf{C}^{-s}}\big].$$

Hence, (3.7) follows from the inequality $\mathbf{E}\|X_i\|_{L^1}^2\le\|\mathbf{E}X_i^2\|_{L^2}$ and Young's inequality with exponents $(\frac{2}{1-s},\frac{2}{s},2)$ and $(2,2)$.

STEP 3 (Estimates for I^2)

In this step, we show that there is a universal constant C such that

$$\begin{aligned} I^2\le{}&\frac{1}{4}\big(\mathbf{E}\|\nabla X_i\|_{L^2}^2+\|\mathbf{E}X_i^2\|_{L^2}^2\big)+C+C\big(\mathbf{E}\|\,:Z_jZ_i:\,\|_{\mathbf{C}^{-s}}^2\big)^2\\ &+C\big(\mathbf{E}\|\,:Z_j^2:\,\|_{\mathbf{C}^{-s}}\big)^4. \end{aligned} \tag{3.8}$$

Using again (2.7), Young's inequality, Hölder's inequality and the independence of X_i and X_j we obtain

$$\begin{aligned} \mathbf{E}\langle X_iX_j,\ :Z_jZ_i:\ \rangle &\lesssim\mathbf{E}\big(\|X_iX_j\|_{L^1}+\|\nabla X_iX_j\|_{L^1}+\|X_i\nabla X_j\|_{L^1}\big)\|\,:Z_jZ_i:\,\|_{\mathbf{C}^{-s}}\\ &\lesssim\mathbf{E}\big(\|X_i\|_{L^2}\|X_j\|_{L^2}+\|\nabla X_i\|_{L^2}\|X_j\|_{L^2}\big)\|\,:Z_jZ_i:\,\|_{\mathbf{C}^{-s}}\\ &\lesssim\big(\mathbf{E}\|X_i\|_{L^2}^2\mathbf{E}\|X_j\|_{L^2}^2+\mathbf{E}\|\nabla X_i\|_{L^2}^2\mathbf{E}\|X_j\|_{L^2}^2\big)^{1/2}\big(\mathbf{E}\|\,:Z_jZ_i:\,\|_{\mathbf{C}^{-s}}^2\big)^{1/2}\\ &\lesssim\big(\|\mathbf{E}X_i^2\|_{L^1}^2+\mathbf{E}\|\nabla X_i\|_{L^2}^2\|\mathbf{E}X_j^2\|_{L^1}\big)^{1/2}\big(\mathbf{E}\|\,:Z_jZ_i:\,\|_{\mathbf{C}^{-s}}^2\big)^{1/2}. \end{aligned} \tag{3.9}$$

Similarly, using this time independence of X_i^2 and $:Z_j^2:$ we obtain

$$\begin{aligned}\mathbf{E}\langle X_i^2, :Z_j^2:\rangle &\lesssim \mathbf{E}\big(\|X_i^2\|_{L^1}+\|\nabla X_i\|_{L^2}\|X_i\|_{L^2}\big)\|:Z_j^2:\|_{\mathbf{C}^{-s}} \\ &= \big(\|\mathbf{E}X_i^2\|_{L^1}+\mathbf{E}\|\nabla X_i\|_{L^2}\|X_i\|_{L^2}\big)\mathbf{E}\|:Z_j^2:\|_{\mathbf{C}^{-s}} \\ &\lesssim \big(\|\mathbf{E}X_i^2\|_{L^1}+(\mathbf{E}\|\nabla X_i\|_{L^2}^2)^{1/2}\|\mathbf{E}X_i^2\|_{L^1}^{1/2}\big)\mathbf{E}\|:Z_j^2:\|_{\mathbf{C}^{-s}}.\end{aligned} \tag{3.10}$$

To obtain (3.8), we use Young's inequality with exponents $(2,2)$ and $(2,4,4)$ for both (3.9) (3.10).

STEP 4 (Estimates for I^3)
In this step, we show there is a universal constant C such that

$$I^3 \le \frac{1}{4}\big(\mathbf{E}\|\nabla X_i\|_{L^2}^2+\|\mathbf{E}X_i^2\|_{L^2}^2\big)+C\big((\mathbf{E}\|Z_j\|_{\mathbf{C}^{-s}}^2)^{\frac{2}{1-s}}+1\big). \tag{3.11}$$

To this end, we write

$$\begin{aligned}I^3 &\lesssim \mathbf{E}\big(\|X_i^2X_j\|_{L^1}^{1-s}\|\nabla(X_i^2X_j)\|_{L^1}^s+\|X_i^2X_j\|_{L^1}\big)\|Z_j\|_{\mathbf{C}^{-s}} \\ &\lesssim (\mathbf{E}\|X_i^2X_j\|_{L^1}\|Z_j\|_{\mathbf{C}^{-s}})^{1-s}(\mathbf{E}\|\nabla(X_i^2X_j)\|_{L^1}\|Z_j\|_{\mathbf{C}^{-s}})^s+\mathbf{E}\|X_i^2X_j\|_{L^1}\|Z_j\|_{\mathbf{C}^{-s}}.\end{aligned}$$

By independence and Hölder's inequality, it holds that

$$\begin{aligned}\mathbf{E}\|X_i^2X_j\|_{L^1}\|Z_j\|_{\mathbf{C}^{-s}} &\lesssim \|\mathbf{E}X_i^2\mathbf{E}[|X_j|\|Z_j\|_{\mathbf{C}^{-s}}]\|_{L^1} \\ &\lesssim \|\mathbf{E}X_i^2\|_{L^2}\|(\mathbf{E}X_j^2)^{1/2}(\mathbf{E}\|Z_j\|_{\mathbf{C}^{-s}}^2)^{1/2}\|_{L^2} \\ &\lesssim \|\mathbf{E}X_i^2\|_{L^2}(\mathbf{E}\|X_j\|_{L^2}^2)^{\frac{1}{2}}(\mathbf{E}\|Z_j\|_{\mathbf{C}^{-s}}^2)^{1/2},\end{aligned} \tag{3.12}$$

where we used that $\|(\mathbf{E}X_j^2)^{1/2}\|_{L^2}=\|\mathbf{E}X_j^2\|_{L^1}^{\frac{1}{2}}=(\mathbf{E}\|X_j\|_{L^2}^2)^{\frac{1}{2}}$. Furthermore,

$$\begin{aligned}\mathbf{E}\|X_i^2\nabla X_j\|_{L^1}\|Z_j\|_{\mathbf{C}^{-s}} &= \|\mathbf{E}X_i^2\mathbf{E}(|\nabla X_j|\|Z_j\|_{\mathbf{C}^{-s}})\|_{L^1} \\ &\le \|\mathbf{E}X_i^2\|_{L^2}\|\mathbf{E}(|\nabla X_j|\|Z_j\|_{\mathbf{C}^{-s}})\|_{L^2} \\ &\le \|\mathbf{E}X_i^2\|_{L^2}\|(\mathbf{E}|\nabla X_j|^2)^{\frac{1}{2}}(\mathbf{E}\|Z_j\|_{\mathbf{C}^{-s}}^2)^{\frac{1}{2}}\|_{L^2} \\ &\lesssim \|\mathbf{E}X_i^2\|_{L^2}(\mathbf{E}\|\nabla X_j\|_{L^2}^2)^{1/2}(\mathbf{E}\|Z_j\|_{\mathbf{C}^{-s}}^2)^{\frac{1}{2}}.\end{aligned} \tag{3.13}$$

Similarly, note that

$$\begin{aligned}\mathbf{E}\|X_iX_j\nabla X_i\|_{L^1}\|Z_j\|_{\mathbf{C}^{-s}} &\lesssim \mathbf{E}\|X_iX_j\|_{L^2}\big(\|\nabla X_i\|_{L^2}\|Z_j\|_{\mathbf{C}^{-s}}\big) \\ &\lesssim (\mathbf{E}\|X_iX_j\|_{L^2}^2)^{1/2}(\mathbf{E}\|\nabla X_i\|_{L^2}^2\mathbf{E}\|Z_j\|_{\mathbf{C}^{-s}}^2)^{1/2} \\ &\lesssim \|\mathbf{E}X_i^2\|_{L^2}(\mathbf{E}\|\nabla X_i\|_{L^2}^2)^{1/2}(\mathbf{E}\|Z_j\|_{\mathbf{C}^{-s}}^2)^{1/2}.\end{aligned} \tag{3.14}$$

Combining the above estimate, we arrive at

$$\begin{aligned}I^3 &\lesssim \|\mathbf{E}X_i^2\|_{L^2}(\mathbf{E}\|\nabla X_i\|_{L^2}^2)^{\frac{s}{2}}(\mathbf{E}\|X_i\|_{L^2}^2)^{\frac{1-s}{2}}(\mathbf{E}\|Z_j\|_{\mathbf{C}^{-s}}^2)^{\frac{1}{2}} \\ &\quad+\|\mathbf{E}X_i^2\|_{L^2}(\mathbf{E}\|X_j\|_{L^2}^2)^{\frac{1}{2}}(\mathbf{E}\|Z_j\|_{\mathbf{C}^{-s}}^2)^{\frac{1}{2}} \\ &\lesssim \|\mathbf{E}X_i^2\|_{L^2}^{\frac{3-s}{2}}(\mathbf{E}\|\nabla X_i\|_{L^2}^2)^{\frac{s}{2}}(\mathbf{E}\|Z_j\|_{\mathbf{C}^{-s}}^2)^{\frac{1}{2}}+\|\mathbf{E}X_i^2\|_{L^2}^{\frac{3}{2}}(\mathbf{E}\|Z_j\|_{\mathbf{C}^{-s}}^2)^{\frac{1}{2}}.\end{aligned} \tag{3.15}$$

Applying Young's inequality with exponents $(\frac{4}{3-s},\frac{2}{s},\frac{4}{1-s})$, we arrive at (3.11). $\square$

In Section 4, we will study the large N limit of (2.1) by comparing the dynamics of each component to the corresponding mean-field evolution. To control the equation for the difference, we will need a stronger control on X_i than the L^2 type bound obtained above. In the following lemma, we show that L^p bounds can be propagated in time, which will turn out to be a necessary ingredient in Section 4.

LEMMA 3.4. *Let $p>2$ and assume that $\mathbf{E}\|\eta_i\|_{L^p}^p \lesssim 1$. Then we have*

$$\sup_{t\in[0,T]} \mathbf{E}\|X_i\|_{L^p}^p + \mathbf{E}\||X_i|^{\frac{p-2}{2}}\nabla X_i\|_{L_T^2L^2}^2 + \|\mathbf{E}|X_i|^p\mathbf{E}X_i^2\|_{L_T^1L^1} \lesssim 1,$$

where the implicit constant is independent of i.

PROOF. Given $p>2$, we fix $s>0$ sufficiently small such that $sp<\frac{1}{2}$ and $\frac{2}{p}+s<1$. We will perform an L^p estimate: integrating (3.1) against $|X_i|^{p-2}X_i$ we get

$$\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\|X_i\|_{L^p}^p + (p-1)\mathbf{E}\||X_i|^{p-2}|\nabla X_i|^2\|_{L^1} + \mathbf{E}\||X_i|^pX_j^2\|_{L^1} + m\mathbf{E}\|X_i\|_{L^p}^p \\
&\quad = -2\mathbf{E}\langle\mathbf{E}[X_jZ_j],|X_i|^p\rangle - \mathbf{E}\langle\mathbf{E}[X_j^2]|X_i|^{p-2}X_i,Z_i\rangle - 2\mathbf{E}\langle\mathbf{E}[X_jZ_j]Z_i,|X_i|^{p-2}X_i\rangle \\
&\qquad + \mathbf{E}\langle\mathbf{E}[:Z_j^2:],|X_i|^p\rangle + \mathbf{E}\langle\mathbf{E}[:Z_j^2:]X_i|X_i|^{p-2},Z_i\rangle =: \sum_{k=1}^5 I_k.
\end{aligned} \tag{3.16}$$

Set $D \stackrel{\text{def}}{=} \|X_i^{p-2}|\nabla X_i|^2\|_{L^1}$ and $A \stackrel{\text{def}}{=} \|X_i^pX_j^2\|_{L^1}$. We claim that there is some R so that

$$I_k \le \frac{1}{10}\mathbf{E}A + \frac{1}{10}\mathbf{E}D + (\mathbf{E}[\|X_i^p\|_{L^1}] + C)R, \quad \text{with} \int_0^T R \lesssim 1. \tag{3.17}$$

STEP 1 (Estimate of I_1)

Using Lemma A.5, we have

$$I_1 \lesssim \mathbf{E}[\|X_jX_i^p\|_{L^1}^{1-s}\|\nabla(X_j|X_i|^p)\|_{L^1}^s\|Z_j\|_{\mathbf{C}^{-s}}] + \mathbf{E}[\|X_jX_i^p\|_{L^1}\|Z_j\|_{\mathbf{C}^{-s}}] =: I_1^{(1)} + I_1^{(2)}.$$

Using

$$\|X_jX_i^p\|_{L^1} \lesssim A^{1/2}\|X_i^p\|_{L^1}^{1/2} \tag{3.18}$$

and independence, one has

$$I_1^{(2)} \le \mathbf{E}[A^{1/2}\|X_i^p\|_{L^1}^{1/2}\|Z_j\|_{\mathbf{C}^{-s}}] \le \frac{1}{10}\mathbf{E}A + C\mathbf{E}\|X_i^p\|_{L^1}\mathbf{E}\|Z_j\|_{\mathbf{C}^{-s}}^2.$$

Regarding $I_1^{(1)}$, using Hölder inequality and then Gagliardo–Nirenberg with $(s,q,r,\alpha) = (0,4,2,\frac{1}{2})$,

$$\begin{aligned}
\|\nabla(X_j|X_i|^p)\|_{L^1} &\le \|\nabla X_j\|_{L^2}\||X_i|^{\frac{p}{2}}\|_{L^4}^2 + 2\||X_i|^{\frac{p}{2}}X_j\|_{L^2}\|\nabla|X_i|^{\frac{p}{2}}\|_{L^2} \\
&\lesssim \||X_i|^{\frac{p}{2}}\|_{H^1}\||X_i|^{\frac{p}{2}}\|_{L^2}\|X_j\|_{H^1} + \sqrt{AD}.
\end{aligned}$$

Since $\||X_i|^{\frac{p}{2}}\|_{H^1} \lesssim D^{\frac{1}{2}} + \||X_i|^{\frac{p}{2}}\|_{L^2}$, together with (3.18) one has

$$\begin{aligned}
I_1^{(1)} &\lesssim \mathbf{E}[A^{\frac{1-s}{2}}\|X_i^p\|_{L^1}^{\frac{1-s}{2}}(D^{\frac{s}{2}}\|X_i^p\|_{L^1}^{\frac{s}{2}}\|X_j\|_{H^1}^s + \|X_i^p\|_{L^1}^s\|X_j\|_{H^1}^s + A^{\frac{s}{2}}D^{\frac{s}{2}})\|Z_j\|_{\mathbf{C}^{-s}}] \\
&\le \frac{1}{10}\mathbf{E}A + \frac{1}{10}\mathbf{E}D + C\mathbf{E}\|X_i^p\|_{L^1}(\mathbf{E}\|X_j\|_{H^1}^{2s}\|Z_j\|_{\mathbf{C}^{-s}}^2 \\
&\qquad + \mathbf{E}\|X_j\|_{H^1}^{\frac{2s}{1+s}}\|Z_j\|_{\mathbf{C}^{-s}}^{\frac{2}{1+s}} + \mathbf{E}\|Z_j\|_{\mathbf{C}^{-s}}^{2/(1-s)}),
\end{aligned}$$

where in the last inequality we used independence and Young's inequality for products with exponents $(\frac{2}{1-s}, 2, \frac{2}{s})$ for the first and third term, and exponents $(\frac{2}{1-s}, \frac{2}{1+s})$ for the second term. Therefore, invoking Lemma 3.3 to deduce

$$\int_0^T \big(\mathbf{E}\|X_j\|_{H^1}^{2s}\|Z_j\|_{\mathbf{C}^{-s}}^2 + \mathbf{E}\|X_j\|_{H^1}^{\frac{2s}{1+s}}\|Z_j\|_{\mathbf{C}^{-s}}^{\frac{2}{1+s}}\big)\,\mathrm{d}s \lesssim 1,$$

which implies a bound of the form (3.17).

STEP 2 (Estimates for I_2)
For the second term on the right-hand side of (3.16), we use Lemma A.2 to have

$$\begin{aligned} I_2 &= \mathbf{E}\langle \Lambda^s(X_j^2 X_i|X_i|^{p-2}), \Lambda^{-s}Z_i\rangle \\ &\lesssim \mathbf{E}\big[\|\Lambda^s(X_j^2)\|_{L^p}\|X_i^{p-1}\|_{L^{\frac{p}{p-1}}}\|\Lambda^{-s}Z_i\|_{L^\infty}\big] \\ &\quad + \mathbf{E}\big[\|X_j^2\|_{L^p}\|\Lambda^s(X_i|X_i|^{p-2})\|_{L^{\frac{p}{p-1}}}\|\Lambda^{-s}Z_i\|_{L^\infty}\big] \\ &=: I_2^{(1)} + I_2^{(2)}. \end{aligned}$$

Using independence, $\|\Lambda^s(X_j^2)\|_{L^p} \lesssim \|\Lambda^s X_j\|_{L^{2p}}\|X_j\|_{L^{2p}}$ by (A.1), and Lemma A.1(iii) with $sp < 1$,

$$I_2^{(1)} \lesssim \mathbf{E}\big[\|X_i\|_{L^p}^{p-1}\|\Lambda^{-s}Z_i\|_{L^\infty}\big]\mathbf{E}\big[\|\Lambda^s(X_j^2)\|_{L^p}\big] \lesssim \big(\mathbf{E}[\|X_i\|_{L^p}^p]+1\big)\mathbf{E}\big[\|X_j\|_{H^1}^2\big].$$

Regarding $I_2^{(2)}$, by the interpolation Lemma A.2 followed by Hölder's inequality,

$$\begin{aligned} \|\Lambda^s(X_i|X_i|^{p-2})\|_{L^{\frac{p}{p-1}}} &\lesssim \|\nabla(X_i|X_i|^{p-2})\|_{L^{\frac{p}{p-1}}}^s\|X_i^{p-1}\|_{L^{\frac{p}{p-1}}}^{1-s} + \|X_i^{p-1}\|_{L^{\frac{p}{p-1}}} \\ &\lesssim D^{s/2}\|X_i^p\|_{L^1}^{(1-s)\frac{p-1}{p}+\frac{s(p-2)}{2p}} + \|X_i^p\|_{L^1}^{\frac{p-1}{p}}. \end{aligned} \tag{3.19}$$

By (A.2) with $(q,s,\alpha,r) = (2p,0,\beta,2)$ with $\beta \stackrel{\text{def}}{=} 1-\frac{1}{p}$, and then Hölder inequality

$$\mathbf{E}\|X_j^2\|_{L^p} \lesssim \mathbf{E}\big[\|X_j\|_{H^1}^{2\beta}\|X_j\|_{L^2}^{2(1-\beta)}\big] \lesssim \big(\mathbf{E}\|X_j\|_{H^1}^2\big)^\beta\big(\mathbf{E}\|X_j\|_{L^2}^2\big)^{1-\beta}. \tag{3.20}$$

Recall from Lemma 3.3 that $\mathbf{E}[\|X_j\|_{L^2}^2] \lesssim 1$. With (3.19)–(3.20), using again independence, and Hölder's inequality with exponents $(\frac{2}{s}, \frac{2}{2-s})$, together with Lemma 2.1, we obtain

$$\begin{aligned} I_2^{(2)} &\lesssim \big(\mathbf{E}\|X_j\|_{H^1}^2\big)^\beta(\mathbf{E}D)^{\frac{s}{2}}\mathbf{E}\big[\|X_i^p\|_{L^1}^\eta\|\Lambda^{-s}Z_i\|_{L^\infty}^{\frac{2}{2-s}}\big]^{1-\frac{s}{2}} \\ &\quad + \mathbf{E}\big[\|X_j\|_{H^1}^2\big]^\beta\big(\mathbf{E}\|X_i^p\|_{L^1}\mathbf{E}\|X_j\|_{L^2}^2+1\big), \end{aligned}$$

where $\eta \stackrel{\text{def}}{=} (1-\frac{1}{p}-\frac{s}{2})\frac{2}{2-s}$ and clearly $\eta < 1$. The first term on the right-hand side can be bounded by, using Young's inequality with exponents $(\frac{2}{s}, \frac{2}{2-s})$ and then with exponents $(\frac{1}{\eta}, \frac{1}{1-\eta})$,

$$\begin{aligned} &\frac{1}{10}\mathbf{E}[D] + C\mathbf{E}\big[\|X_i^p\|_{L^1}^\eta\|\Lambda^{-s}Z_i\|_{L^\infty}^{\frac{2}{2-s}}\big]\mathbf{E}\big[\|X_j\|_{H^1}^2\big]^{\frac{2\beta}{2-s}} \\ &\leq \frac{1}{10}\mathbf{E}[D] + C\big(\mathbf{E}\|X_i^p\|_{L^1} + \mathbf{E}\|\Lambda^{-s}Z_i\|_{L^\infty}^{\frac{2}{2-s}\frac{1}{1-\eta}}\big)\mathbf{E}\big[\|X_j\|_{H^1}^2\big]^{\frac{2\beta}{2-s}}. \end{aligned}$$

By Lemma A.1 and Lemma 2.1, we easily find $\mathbf{E}\|\Lambda^{-s}Z_i\|_{L^\infty}^q \lesssim 1$ for every $q \geq 1$. Using Lemma 2.1, and Lemma 3.3 noting that $2\beta/(2-s) < 1$ by our smallness assumption on $s > 0$, we obtain a bound of the form (3.17) for I_2.

STEP 3 (Estimate of I_3-I_5)
Using Lemma A.2, we obtain

$$\begin{aligned}I_3 &= \mathbf{E}\langle \Lambda^s(X_j X_i |X_i|^{p-2}), \Lambda^{-s}(:Z_i Z_j:)\rangle \\ &\lesssim \mathbf{E}\big[\|\Lambda^{-s}(:Z_i Z_j:)\|_{L^\infty} \|\Lambda^s X_j\|_{L^p} \|X_i^p\|_{L^1}^{\frac{p-1}{p}}\big] \\ &\quad + \mathbf{E}\big[\|\Lambda^{-s}(:Z_i Z_j:)\|_{L^\infty} \|X_j\|_{L^p} \|\Lambda^s(X_i|X_i|^{p-2})\|_{L^{\frac{p}{p-1}}}\big] =: I_3^{(1)} + I_3^{(2)}.\end{aligned}$$

For $I_3^{(1)}$, we use Sobolev embedding $H^1 \subset H^s_p$ and Young's inequality and independence to have

$$I_3^{(1)} \lesssim \mathbf{E}\big[\|\Lambda^{-s}(:Z_i Z_j:)\|^p_{L^\infty}\big] + \mathbf{E}\big[\|X_j\|_{H^1}^{\frac{p}{p-1}}\big]\mathbf{E}\big[\|X_i^p\|_{L^1}\big].$$

For $I_3^{(2)}$, we plug in (3.19):

$$I_3^{(2)} \lesssim \mathbf{E}\big[\big(D^{s/2}\|X_i^p\|_{L^1}^{(2p-2-sp)/(2p)} + \|X_i^p\|_{L^1}^{\frac{p-1}{p}}\big)\|X_j\|_{L^p}\|\Lambda^{-s}(:Z_i Z_j:)\|_{L^\infty}\big].$$

Using Young's inequality with $(\frac{2}{s}, \frac{2p}{2p-2-sp}, p)$ and $(\frac{p}{p-1}, p)$, and Sobolev embedding,

$$I_3^{(2)} \le \frac{1}{10}\mathbf{E}D + C\mathbf{E}\|X_i^p\|_{L^1}\mathbf{E}\|X_j\|_{H^1}^{\frac{2p}{2p-2-sp}} + \mathbf{E}\|X_j\|_{H^1}^{\frac{p}{p-1}}\mathbf{E}\|X_i^p\|_{L^1} + C\mathbf{E}\|\Lambda^{-s}(:Z_i Z_j:)\|^p_{L^\infty}.$$

For $s > 0$ small enough $\frac{2}{p} + s < 1$ so that $\frac{2p}{2p-2-sp} < 2$, so Lemma 3.3 applies.

By (3.19), we have for $\epsilon > 0$ small enough

$$\begin{aligned}&I_4 + I_5 \\ &\lesssim \|\mathbf{E}[:Z_j^2:]\|_{L^\infty}\mathbf{E}[\|X_i\|^p_{L^p}] + \|\mathbf{E}[:Z_j^2:]\|_{\mathbf{C}^{s+\epsilon}}\mathbf{E}\big[\|\Lambda^s(X_i|X_i|^{p-2})\|_{L^1}\|\Lambda^{-s}Z_i\|_{L^\infty}\big] \\ &\lesssim \|\mathbf{E}[:Z_j^2:]\|_{L^\infty}\mathbf{E}[\|X_i\|^p_{L^p}] + \|\mathbf{E}[:Z_j^2:]\|_{\mathbf{C}^{s+\epsilon}}\mathbf{E}\big[D^{s/2}\|X_i^p\|_{L^1}^{(1-s)\frac{p-1}{p}+\frac{s(p-2)}{2p}}\|\Lambda^{-s}Z_i\|_{L^\infty}\big] \\ &\quad + \|\mathbf{E}[:Z_j^2:]\|_{\mathbf{C}^{s+\epsilon}}\mathbf{E}\big[\|X_i^p\|_{L^1}^{\frac{p-1}{p}}\|\Lambda^{-s}Z_i\|_{L^\infty}\big] \\ &\lesssim \mathbf{E}[D]^{s/2}\mathbf{E}\big[\|X_i^p\|^\eta_{L^1}\|\Lambda^{-s}Z_i\|_{L^\infty}^{\frac{2}{2-s}}\big]^{1-s/2}\|\mathbf{E}[:Z_j^2:]\|_{\mathbf{C}^{s+\epsilon}} \\ &\quad + \|\mathbf{E}[:Z_j^2:]\|_{\mathbf{C}^{s+\epsilon}}\big(\mathbf{E}[\|X_i\|^p_{L^p}] + 1\big) \\ &\le \frac{1}{10}\mathbf{E}[D] + C\mathbf{E}[\|X_i^p\|_{L^1}] + C\|\mathbf{E}[:Z_j^2:]\|_{\mathbf{C}^{s+\epsilon}}^{\frac{2q}{2-s}}\mathbf{E}\big[\|\Lambda^{-s}Z_i\|_{L^\infty}^{\frac{2q}{2-s}}\big] \\ &\quad + C\|\mathbf{E}[:Z_j^2:]\|_{\mathbf{C}^{s+\epsilon}}\big(\mathbf{E}[\|X_i\|^p_{L^p}] + 1\big),\end{aligned}$$

for $q = 1/(1-\eta)$. Combining all the above estimates and using Gronwall's inequality, we obtain the claimed bound. □

We now conclude this section by combining our energy estimates with Schauder theory to obtain a global Hölder bound on X_i.

LEMMA 3.5. *Assume that $\mathbf{E}\|\eta_i\|^4_{L^4} \lesssim 1$. For $\beta > \kappa$ sufficiently small, $\gamma = \beta + \frac{1}{2}$, we have*

$$\mathbf{E}\Big[\sup_{t\in[0,T]} t^\gamma \|X_i\|^2_{\mathbf{C}^\beta}\Big] \lesssim 1.$$

PROOF. Recall that X_i satisfies the mild formulation of (3.1), which we write using our independent copy (X_j, Z_j) as

$$X_i(t) = S_t\eta_i + \int_0^t S_{t-s}\mathbf{E}\big[X_j^2 + 2X_jZ_j + :Z_j^2:\big](X_i + Z_i)\,\mathrm{d}s.$$

We start by applying the Schauder estimate, Lemma A.4, with δ playing the role $\beta+1$, $\beta+\kappa$ and $\beta+\kappa+\frac{2}{3}$, respectively, to bound

$$\begin{aligned}
&\|X_i(t) - S_t\eta_i\|_{\mathbf{C}^\beta}\\
&\quad\lesssim \int_0^t (t-s)^{-\frac{\beta+1}{2}}\|\mathbf{E}\big[X_j^2\big]X_i\|_{\mathbf{C}^{-1}}\,\mathrm{d}s\\
&\qquad + \int_0^t (t-s)^{-\frac{\beta+\kappa}{2}}\|\mathbf{E}\big[:Z_j^2:\big](X_i+Z_i)\|_{\mathbf{C}^{-\kappa}}\,\mathrm{d}s\\
&\qquad + \int_0^t (t-s)^{-\frac{\beta+2/3+\kappa}{2}}\|\mathbf{E}[X_jZ_j]Z_i\|_{\mathbf{C}^{-\kappa-\frac{2}{3}}}\,\mathrm{d}s\\
&\qquad + \int_0^t (t-s)^{-\frac{\beta+2/3+\kappa}{2}}\big(\|\mathbf{E}[X_jZ_j]X_i\|_{\mathbf{C}^{-\kappa-\frac{2}{3}}} + \|\mathbf{E}\big[X_j^2\big]Z_i\|_{\mathbf{C}^{-\kappa-\frac{2}{3}}}\big)\,\mathrm{d}s \stackrel{\text{def}}{=} \sum_{i=1}^4 J_i.
\end{aligned}$$

To estimate J_1, first recall Lemma 3.4 implies that

$$\sup_{t\in[0,T]}\|\mathbf{E}X_j^2\|_{L^2}^2 = \sup_{t\in[0,T]}\|\mathbf{E}X_i^2\|_{L^2}^2 \lesssim \sup_{t\in[0,T]}\mathbf{E}\|X_i\|_{L^4}^4 \lesssim 1, \tag{3.21}$$

which can be combined with the Sobolev embedding $L^2 \hookrightarrow \mathbf{C}^{-1}$ in $d=2$ corresponding to Lemma A.1 with $\alpha = 0$, $p_1 = q_1 = 2$ and $p_2 = q_2 = \infty$ to find

$$J_1 \lesssim \int_0^t (t-s)^{-\frac{\beta+1}{2}}\|\mathbf{E}\big[X_j^2\big]\|_{L^2}\|X_i\|_{\mathbf{C}^\beta}\,\mathrm{d}s \lesssim \int_0^t (t-s)^{-\frac{\beta+1}{2}}\|X_i\|_{\mathbf{C}^\beta}\,\mathrm{d}s.$$

We now turn to J_2 and apply Lemma A.3 to find for $\beta > \kappa$ and $\kappa' > \kappa$

$$\begin{aligned}
J_2 &\lesssim \int_0^t (t-s)^{-\frac{\beta+\kappa}{2}}\big[\|\mathbf{E}\big[:Z_j^2:\big]\|_{\mathbf{C}^{-\kappa}}\|X_i\|_{\mathbf{C}^\beta} + \|\mathbf{E}\big[:Z_j^2:\big]\|_{\mathbf{C}^{2\kappa}}\|Z_i\|_{\mathbf{C}^{-\kappa}}\,\mathrm{d}s\\
&\lesssim \int_0^t (t-s)^{-\frac{\beta+\kappa}{2}}s^{-\frac{\kappa'}{2}}\|X_i\|_{\mathbf{C}^\beta}\,\mathrm{d}s + \|Z_i\|_{C_T\mathbf{C}^{-\kappa}}.
\end{aligned}$$

We now turn to J_3 and J_4 and use the Besov embedding $B^{-\kappa}_{3,\infty} \hookrightarrow \mathbf{C}^{-\kappa-\frac{2}{3}}$ in $d=2$ in Lemma A.1. Let us begin with J_3, which is simpler. Using Lemma 3.4 and Lemma A.1 and Lemma A.2, we have

$$\begin{aligned}
\int_0^T \|\mathbf{E}\big[X_i^2\big]\|_{B^{2\kappa}_{3,\infty}}^2\,\mathrm{d}s &\lesssim \int_0^T \big[\mathbf{E}\|X_i^2\|_{B^{2\kappa}_{3,\infty}}\big]^2\,\mathrm{d}s \lesssim \int_0^T \big[\mathbf{E}\|\Lambda^{2\kappa}(X_i^2)\|_{L^3}\big]^2\,\mathrm{d}s\\
&\lesssim \int_0^T \big[\mathbf{E}\|\Lambda(X_i^2)\|_{L^{4/3}}\big]^2\,\mathrm{d}s\\
&\lesssim \int_0^T \big[\mathbf{E}\|X_i\|_{H^1}\|X_i\|_{L^4}\big]^2\,\mathrm{d}s\\
&\lesssim \int_0^T \mathbf{E}\big[\|X_i\|_{H^1}^2\big]\mathbf{E}\big[\|X_i\|_{L^4}^2\big]\,\mathrm{d}s \lesssim 1,
\end{aligned} \tag{3.22}$$

where we used (A.1) in the fourth inequality and Hölder inequality in the fifth inequality. Note that by Hölder's inequality in time which exponents $(\frac{3}{2}, 3)$ and taking into account that

$\frac{3}{4}(\beta+\frac{2}{3}+\kappa)<1$ for β small enough and using Lemma A.3 we find

$$\begin{aligned} J_3 &\lesssim \int_0^T \|\mathbf{E}[X_j Z_j]Z_i\|^3_{B^{-\kappa}_{3,\infty}}\,\mathrm{d}s \\ &\lesssim \int_0^T (\mathbf{E}[\|X_j\|^2_{B^{2\kappa}_{3,\infty}}])^{\frac{3}{2}}(\mathbf{E}[\| :Z_iZ_j: \|^2_{\mathbf{C}^{-\kappa}} \mid Z_i])^{\frac{3}{2}}\,\mathrm{d}s \qquad (3.23)\\ &\lesssim 1+\int_0^T (\mathbf{E}[\| :Z_iZ_j: \|^2_{\mathbf{C}^{-\kappa}} \mid Z_i])^6\,\mathrm{d}s. \end{aligned}$$

Here, we used Lemma 3.1 and (3.22). Finally, we turn to J_4. By Lemma A.3 and Lemma A.2, we deduce

$$\|X_i\|_{B^{2\kappa}_{3,\infty}} \lesssim \|X_i\|_{B^{2\kappa}_{\frac{4}{1+2\kappa},\infty}} \lesssim \|X_i\|^{2\kappa}_{B^1_{2,\infty}}\|X_i\|^{1-2\kappa}_{B^0_{4,\infty}} \lesssim \|X_i\|^{2\kappa}_{H^1}\|X_i\|^{1-2\kappa}_{L^4},$$

which implies that

$$\begin{aligned} \int_0^T \|\mathbf{E}(X_iZ_i)\|^3_{B^{-\kappa}_{3,\infty}}\,\mathrm{d}s &\lesssim \int_0^T \mathbf{E}[\|X_i\|^3_{B^{2\kappa}_{3,\infty}}\|Z_i\|^3_{\mathbf{C}^{-\kappa}}] \\ &\lesssim \int_0^T \mathbf{E}[\|X_i\|^{6\kappa}_{H^1}\|X_i\|^{3(1-2\kappa)}_{L^4}\|Z_i\|^3_{\mathbf{C}^{-\kappa}}] \qquad (3.24)\\ &\lesssim \int_0^T \mathbf{E}[\|X_i\|^2_{H^1}]+\int_0^T \mathbf{E}[\|X_i\|^4_{L^4}]+\int_0^T \mathbf{E}[\|Z_i\|^l_{\mathbf{C}^{-\kappa}}] \lesssim 1, \end{aligned}$$

for some $l>1$. This combined with (3.22) and Lemma A.3 implies that

$$\begin{aligned} J_4 &\lesssim \int_0^t (t-s)^{-\frac{\beta+2/3+\kappa}{2}}(\|\mathbf{E}[X_jZ_j]\|_{B^{-\kappa}_{3,\infty}}\|X_i\|_{\mathbf{C}^\beta}+\|\mathbf{E}[X_j^2]\|_{B^{2\kappa}_{3,\infty}}\|Z_i\|_{\mathbf{C}^{-\kappa}})\,\mathrm{d}s \\ &\lesssim \int_0^t (t-s)^{-\frac{\beta+2/3+\kappa}{2}}\|\mathbf{E}[X_jZ_j]\|_{B^{-\kappa}_{3,\infty}}\|X_i\|_{\mathbf{C}^\beta}\,\mathrm{d}s+\|Z_i\|_{C_T\mathbf{C}^{-\kappa}} \end{aligned}$$

For $S_t\eta_i$, we use (3.3) to have the desired bound. Combining the above estimates, using (3.24), Hölder's inequality and Gronwall's inequality, the result follows. □

Combining the local well-posedness result and the uniform estimate Lemma 3.5 we conclude the following result.

THEOREM 3.6. *For given Z_i as the solution to (2.2) and $\mathbf{E}\|\eta_i\|^4_{L^4}\lesssim 1$, there exists a unique solution $X_i\in L^2(\Omega; C((0,T];\mathbf{C}^\beta)\cap C_TL^4)$ to (3.1) such that*

$$\mathbf{E}\Big[\sup_{t\in[0,T]} t^\gamma\|X_i\|^2_{\mathbf{C}^\beta}\Big]+\sup_{t\in[0,T]}\mathbf{E}\|X_i\|^4_{L^4}+\mathbf{E}\|X_i\|^2_{L^2_TH^1}\lesssim 1.$$

In particular, for every $\psi_i\in\mathbf{C}^{-\kappa}$ with $\mathbf{E}\|\psi_i\|^p_{\mathbf{C}^{-\kappa}}\lesssim 1$, $p>1$, there exists a unique solution $\Psi_i\in L^2(\Omega; C_T\mathbf{C}^{-\kappa})$ to (1.6) such that

$$\mathbf{E}\Big[\sup_{t\in[0,T]} t^\gamma\|\Psi_i-Z_i\|^2_{\mathbf{C}^\beta}\Big]\lesssim 1,$$

for $\beta>3\kappa>0$ small enough and $\gamma=\frac{1}{2}+\beta$.

4. Large N limit of the dynamics. In this section, we study the large N behavior of a fixed component Φ_i^N satisfying (2.1) with initial condition $\phi_i^N = y_i^N + z_i^N$. Namely, under suitable assumptions on the initial conditions, we show that as $N \to \infty$, the component converges to the corresponding solution Ψ_i to (1.6) with initial condition $\psi_i = \eta_i + z_i$. Recall that by definition, $\Phi_i^N = Y_i^N + Z_i^N$, where Y_i^N satisfies (2.3) and Z_i^N satisfies (2.2) with initial conditions y_i^N and z_i^N, respectively. Similarly, $\Psi_i = X_i + Z_i$, where X_i satisfies (3.1) and Z_i satisfies (2.2) with initial conditions y_i and η_i, respectively. We now define

$$v_i^N \stackrel{\text{def}}{=} Y_i^N - X_i.$$

For future reference, we note that in light of the decomposition (cf. Section 2.1 for the definition of $\tilde{Z}_i$),

$$Z_i^N = \tilde{Z}_i + S_t(z_i^N - \tilde{Z}_i(0)), \qquad Z_i = \tilde{Z}_i + S_t(z_i - \tilde{Z}_i(0)),$$

it follows that

$$\Phi_i^N - \Psi_i = Y_i^N - X_i + Z_i^N - Z_i = v_i^N + S_t(z_i^N - z_i).$$

Hence, our main task is to study v_i^N and this will occupy the bulk of the proof. We now give our assumptions on the initial conditions.

ASSUMPTION 4.1. Suppose the following assumptions:

- The random variables $\{(z_i, \eta_i)\}_{i=1}^N$ are i.i.d.
- For every $p > 1$, and every i,

$$\mathbf{E}[\|z_i^N - z_i\|_{\mathbf{C}^{-\kappa}}^p] \to 0, \qquad \mathbf{E}[\|y_i^N - \eta_i\|_{L^2}^2] \to 0, \quad \text{as } N \to \infty,$$

$$\frac{1}{N}\sum_{i=1}^N \|z_i^N - z_i\|_{\mathbf{C}^{-\kappa}}^p \to^{\mathbf{P}} 0, \qquad \frac{1}{N}\sum_{i=1}^N \|y_i^N - \eta_i\|_{L^2}^2 \to^{\mathbf{P}} 0, \quad \text{as } N \to \infty,$$

  where $\to^{\mathbf{P}}$ means the convergence in probability.
- For some $q > 1$, $p_0 > 4/(1-4\kappa)$, and every $p > 1$,

$$\mathbf{E}[\|z_i^N\|_{\mathbf{C}^{-\kappa}}^p + \|z_i\|_{\mathbf{C}^{-\kappa}}^p] \lesssim 1, \qquad \mathbf{E}\|\eta_i\|_{L^{p_0}}^{p_0} \lesssim 1, \qquad \mathbf{E}\left[\frac{1}{N}\sum_{i=1}^N \|y_i^N\|_{L^2}^2\right]^q \lesssim 1,$$

  where the implicit constant is independent of i, N.

The following theorem is our main convergence result, which in particular implies Theorem 1.1. The proof is inspired by mean-field theory for *SDE* systems such as Sznitman's article [64], which as the general philosophy starts by directly subtracting the two dynamics and thereby canceling the white noises, and then controls the difference. To this end, we establish energy estimates for the difference v_i^N below; cf. (4.5) from Step 1. The key to the proof is that for the terms collected in I_1^N, I_2^N below we interpolate with $\mathbf{C}^{-s}$ and $B_{1,1}^s$ spaces and leverage various a priori estimates obtained in the previous sections; but for terms collected in I_3^N, which are suitably *centered*, we interpolate with *Hilbert spaces* and invoke the following fact (4.1), which in certain sense gives us a crucial "factor of $1/N$":

Recall that for mean-zero independent random variables $U_1, \dots, U_N$ taking values in a Hilbert space H, we have

$$\mathbf{E}\left\|\sum_{i=1}^N U_i\right\|_H^2 = \mathbf{E}\sum_{i=1}^N \|U_i\|_H^2. \tag{4.1}$$

This simple fact is important for us since the square of the sum on the left-hand side of (4.1) appears to have "N^2 terms" but under expectation it's only a sum of N terms, in certain sense giving us a "factor of $1/N$."

THEOREM 4.1. *If the initial datum $(z_i^N, y_i^N, z_i, \eta_i)_i$ satisfy Assumption 4.1, then for every i and every $T > 0$, $\|v_i^N\|_{C_T L^2}$ converges to zero in probability, as $N \to \infty$. Moreover, under the additional hypothesis that $(z_i^N, y_i^N, z_i, \eta_i)_{i=1}^N$ are exchangeable, for all $t > 0$ it holds*

$$\lim_{N\to\infty} \mathbf{E}\|\Phi_i^N(t) - \Psi_i(t)\|_{L^2}^2 = 0. \tag{4.2}$$

PROOF. The proof has a similar flavor to the Lemma 2.3, and in fact we will continue to use the notation R_N^i for $i = 1, 2, 3$ for the same quantities. One additional ingredient required is the following instance of the Gagliardo–Nirenberg inequality (a special case of Lemma A.2):

$$\|g\|_{L^4} \le C\|g\|_{H^1}^{1/2}\|g\|_{L^2}^{1/2}. \tag{4.3}$$

In the proof, we omit the superscript N and simply write v_i for v_i^N throughout. Furthermore, in Steps 1–5 we work under the simplifying assumption that $z_i^N = z_i$, so that also $Z_i^N = Z_i$. In Step 7, we sketch the argument in the more general case.

STEP 1 (Energy balance)

In this step, we justify the following energy identity:

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^N \|v_i\|_{L^2}^2 + \sum_{i=1}^N \|\nabla v_i\|_{L^2}^2 + m\sum_{i=1}^N \|v_i\|_{L^2}^2 \\ &\qquad + \frac{1}{N}\sum_{i,j=1}^N \|Y_j v_i\|_{L^2}^2 + \frac{1}{N}\Big\|\sum_{j=1}^N X_j v_j\Big\|_{L^2}^2 = \sum_{k=1}^3 I_k^N, \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} I_1^N &\stackrel{\text{def}}{=} -\frac{1}{N}\sum_{i,j=1}^N \big(2\langle v_i v_j, :Z_j Z_i: \rangle + \langle v_i^2, :Z_j^2: \rangle + 2\langle v_i^2 Y_j, Z_j\rangle\big), \\ I_2^N &\stackrel{\text{def}}{=} -\frac{1}{N}\sum_{i,j=1}^N \langle v_i v_j, \big(X_i Y_j + (3X_j + Y_j)Z_i\big)\rangle, \\ I_3^N &\stackrel{\text{def}}{=} -\frac{1}{N}\sum_{i,j=1}^N \langle\big[:Z_j^2: - \mathbf{E}:Z_j^2: + X_j(X_j + 2Z_j) \\ &\qquad - \mathbf{E}X_j(X_j + 2Z_j)\big](X_i + Z_i), v_i\rangle. \end{aligned} \tag{4.5}$$

In the definition of I_3^N, to have a compact formula, we slightly abuse notation for the contribution of the diagonal part $i = j$, where we understand $Z_i Z_j$ to be $:Z_i^2:$ and $:Z_j^2: Z_i$ to be $:Z_i^3:$.

We now turn the justification of this identity, and for the convenience of the reader, we write the equations for Y_i and X_i side-by-side as

$$\mathscr{L}Y_i = -\frac{1}{N}\sum_{j=1}^N \big(Y_j^2 Y_i + Y_j^2 Z_i + 2Y_j Z_j Y_i + 2Y_j :Z_i Z_j: + Y_i :Z_j^2: + :Z_i Z_j^2: \big), \tag{4.6}$$

$$\begin{aligned}\mathscr{L} X_i = -\frac{1}{N}\sum_{j=1}^N \big(&\mathbf{E}(X_j^2)X_i + \mathbf{E}(X_j^2)Z_i + 2\mathbf{E}(X_jZ_j)X_i + 2\mathbf{E}(X_jZ_j)Z_i + X_i\mathbf{E}:Z_j^2: \\ &+ Z_i\mathbf{E}:Z_j^2:\big),\end{aligned} \tag{4.7}$$

where we used that X_j and X_i are equal in law. We now compare each of the first 4 terms in (4.6) to the corresponding terms in (4.7). Note first that

$$\begin{aligned}Y_j^2Y_i - \mathbf{E}(X_j^2)X_i &= Y_j^2Y_i - X_j^2X_i + (X_j^2 - \mathbf{E}(X_j^2))X_i \\ &= Y_j^2v_i + v_j(Y_j+X_j)X_i + (X_j^2 - \mathbf{E}(X_j^2))X_i.\end{aligned}$$

Similarly, we find

$$\begin{aligned}(Y_j^2 - \mathbf{E}(X_j^2))Z_i &= v_j(Y_j+X_j)Z_i + (X_j^2 - \mathbf{E}(X_j^2))Z_i, \\ 2Y_jZ_jY_i - 2\mathbf{E}(X_jZ_j)X_i &= 2(v_iY_j + v_jX_i)Z_j + 2(X_jZ_j - \mathbf{E}(X_jZ_j))X_i, \\ 2Y_j:Z_iZ_j: - 2\mathbf{E}(X_jZ_j)Z_i &= 2v_j:Z_iZ_j: + 2(X_j:Z_iZ_j: - \mathbf{E}(X_jZ_j)Z_i), \\ Y_i:Z_j^2: &= v_i:Z_j^2: + X_i:Z_j^2: .\end{aligned}$$

Taking the difference of (4.6) and (4.7), using the identities above, multiplying by v_i, integrating by parts, and summing over i leads to (4.5). Indeed, notice that each equality gives a sum of two pieces, one with a factor of v and one without any factor of v, but with a recentering. The terms which have a factor of v lead to I_1^N and I_2^N, except for $Y_j^2v_i$ and $v_jX_jX_i$, which lead to the two coercive quantities on the left-hand side of (4.5). The terms which have been recentered lead to I_3^N.

STEP 2 (Estimates for I_1^N)

In this step, we show there is a universal constant C such that

$$\begin{aligned}I_1^N \le \frac{1}{8}\Bigg(\sum_{i=1}^N \|\nabla v_i\|_{L^2}^2 + \frac{1}{N}\sum_{i,j=1}^N \|Y_jv_i\|_{L^2}^2\Bigg) \\ + C(1 + R_N^2 + R_N^3 + R_N^5)\sum_{i=1}^N \|v_i\|_{L^2}^2,\end{aligned} \tag{4.8}$$

where R_N^2 and R_N^3 are defined in terms of Z in the same way as in (2.14) and (2.18) and

$$R_N^5 \stackrel{\text{def}}{=} \Bigg(1 + \frac{1}{N}\sum_{j=1}^N \|\nabla Y_j\|_{L^2}^2\Bigg)^s \Bigg(\frac{1}{N}\sum_{i=1}^N \|Z_i\|_{\mathbf{C}^{-s}}^2\Bigg),$$

with $1 > s \ge 2\kappa$ and s small enough. Indeed, (4.8) follows from arguments identical to the ones leading to (2.13) and (2.17) in Lemma 2.3, but with a different labelling of the integrands, which we now explain. There are three contributions to I_1^N and each can be treated separately. For the contribution of $v_iv_j:Z_jZ_i:$ we argue exactly as for (2.15) but with v_iv_j in place of Y_iY_j. For the contribution of $v_i^2:Z_j^2:$, we argue exactly as in (2.16), but with v_i^2 in place of Y_i^2. This leads to the inequality

$$\begin{aligned}\frac{1}{N}\sum_{i,j=1}^N \big(2\langle v_iv_j, :Z_iZ_j:\rangle + \langle v_i^2, :Z_j^2:\rangle\big) \\ \le \frac{1}{16}\Bigg(\sum_{i=1}^N \|\nabla v_i\|_{L^2}^2\Bigg) + C(1 + R_N^2)\sum_{i=1}^N \|v_i\|_{L^2}^2.\end{aligned} \tag{4.9}$$

Finally, for the contribution of $v_i^2 Y_j Z_j$ the argument is similar as for (2.17). This leads to the estimate

$$\begin{aligned}
&-\frac{1}{N}\sum_{j=1}^N\Big\langle\sum_{i=1}^N v_i^2 Y_j, Z_j\Big\rangle\\
&\lesssim \frac{1}{N}\sum_{j=1}^N\bigg(\Big\|\sum_{i=1}^N v_i^2 Y_j\Big\|_{L^1}^{1-s}\Big\|\nabla\Big(\sum_{i=1}^N v_i^2 Y_j\Big)\Big\|_{L^1}^{s}+\Big\|\sum_{i=1}^N v_i^2 Y_j\Big\|_{L^1}\bigg)\|Z_j\|_{\mathbf{C}^{-s}}\\
&\lesssim \frac{1}{N}\bigg(\sum_{j=1}^N\Big\|\sum_{i=1}^N v_i^2 Y_j\Big\|_{L^1}^{2(1-s)}\Big\|\nabla\Big(\sum_{i=1}^N v_i^2 Y_j\Big)\Big\|_{L^1}^{2s}\bigg)^{1/2}\mathscr{Z}_N^{\frac12}\\
&\quad+\frac{1}{N}\bigg(\sum_{j=1}^N\Big\|\sum_{i=1}^N v_i^2 Y_j\Big\|_{L^1}^{2}\bigg)^{1/2}\mathscr{Z}_N^{\frac12},
\end{aligned}\tag{4.10}$$

where $\mathscr{Z}_N \stackrel{\text{def}}{=} \sum_{j=1}^N \|Z_j\|_{\mathbf{C}^{-s}}^2$ as in (2.18). By Hölder's inequality, it holds that

$$\Big\|\sum_{i=1}^N v_i^2 Y_j\Big\|_{L^1}\lesssim\Big(\sum_{i=1}^N\|v_iY_j\|_{L^2}^2\Big)^{1/2}\Big(\sum_{i=1}^N\|v_i\|_{L^2}^2\Big)^{1/2}.\tag{4.11}$$

Furthermore, we find that

$$\begin{aligned}
\Big\|\nabla\Big(\sum_{i=1}^N v_i^2 Y_j\Big)\Big\|_{L^1}&\lesssim\Big\|\sum_{i=1}^N v_i^2\nabla Y_j\Big\|_{L^1}+\Big\|\sum_{i=1}^N\nabla v_i v_i Y_j\Big\|_{L^1}\\
&\lesssim\sum_{i=1}^N\|v_i\|_{L^4}^2\|\nabla Y_j\|_{L^2}+\Big(\sum_{i=1}^N\|\nabla v_i\|_{L^2}^2\Big)^{1/2}\Big(\sum_{i=1}^N\|v_iY_j\|_{L^2}^2\Big)^{1/2}\\
&\lesssim\Big(\sum_{i=1}^N\|v_i\|_{H^1}^2\Big)^{1/2}\Big(\sum_{i=1}^N\|v_i\|_{L^2}^2\Big)^{1/2}\|\nabla Y_j\|_{L^2}\\
&\quad+\Big(\sum_{i=1}^N\|\nabla v_i\|_{L^2}^2\Big)^{1/2}\Big(\sum_{i=1}^N\|v_iY_j\|_{L^2}^2\Big)^{1/2},
\end{aligned}$$

where we used (4.3) in the last step. Hence, we find that

$$\begin{aligned}
&\sum_{j=1}^N\Big\|\sum_{i=1}^N v_i^2 Y_j\Big\|_{L^1}^{2(1-s)}\Big\|\nabla\Big(\sum_{i=1}^N v_i^2 Y_j\Big)\Big\|_{L^1}^{2s}\\
&\lesssim\Big(\sum_{i,j=1}^N\|v_iY_j\|_{L^2}^2\Big)^{1-s}\Big(\sum_{i=1}^N\|v_i\|_{H^1}^2\Big)^{s}\Big(\sum_{i=1}^N\|v_i\|_{L^2}^2\Big)\Big(\sum_{j=1}^N\|\nabla Y_j\|_{L^2}^2\Big)^{s}\\
&\quad+\Big(\sum_{i=1}^N\|\nabla v_i\|_{L^2}^2\Big)^{s}\Big(\sum_{i=1}^N\|v_i\|_{L^2}^2\Big)^{1-s}\Big(\sum_{i,j=1}^N\|v_iY_j\|_{L^2}^2\Big).
\end{aligned}$$

Inserting this into (4.10), taking the square root and using (4.11) and Young's inequality with exponent $(\frac{2}{1-s}, \frac{2}{s}, 2)$, we arrive at

$$\begin{aligned}-\frac{1}{N}\sum_{i,j=1}^{N}\langle 2v_i^2Y_j, Z_j\rangle \le \frac{1}{16}\Bigg(\sum_{i=1}^{N}\|\nabla v_i\|_{L^2}^2 + \frac{1}{N}\sum_{i,j=1}^{N}\|Y_jv_i\|_{L^2}^2\Bigg) \\ + C(1+R_N^3+R_N^5)\sum_{i=1}^{N}\|v_i\|_{L^2}^2.\end{aligned} \tag{4.12}$$

Combining (4.9) and (4.12) and recalling the definition of I_1^N we obtain (4.8).

STEP 3 (Estimates for I_2^N)

In this step, we show there is a universal constant C such that

$$\begin{aligned}I_2^N \le \frac{1}{4}\Bigg(\sum_{i=1}^{N}\|\nabla v_i\|_{L^2}^2 + \frac{1}{N}\sum_{i,j=1}^{N}\|v_iY_j\|_{L^2}^2 + \Bigg\|\frac{1}{\sqrt{N}}\sum_{j=1}^{N}v_jX_j\Bigg\|_{L^2}^2\Bigg) \\ + C\Bigg(1+R_N^3+R_N^4+R_N^5+R_N^6+\Bigg(\frac{1}{N}\sum_{i=1}^{N}\|X_i\|_{L^4}^2\Bigg)^2\Bigg)\Bigg(\sum_{i=1}^{N}\|v_i\|_{L^2}^2\Bigg),\end{aligned} \tag{4.13}$$

where R_N^3 is defined as in (2.18) and R_N^4 and R_N^6 are defined by

$$\begin{aligned}R_N^4 &\stackrel{\text{def}}{=} \Bigg(1+\frac{1}{N}\sum_{j=1}^{N}\|X_j\|_{L^4}^2\Bigg)^{\frac{2s}{2-s}}\Bigg(\frac{1}{N}\sum_{i=1}^{N}\|Z_i\|_{\mathbf{C}^{-s}}^2\Bigg)^{\frac{2}{2-s}} \\ &\quad + \Bigg(\frac{1}{N}\sum_{j=1}^{N}\|\nabla X_j\|_{L^2}^2\Bigg)^{s}\Bigg(\frac{1}{N}\sum_{i=1}^{N}\|Z_i\|_{\mathbf{C}^{-s}}^2\Bigg), \\ R_N^6 &\stackrel{\text{def}}{=} \Bigg(\frac{1}{N}\sum_{j=1}^{N}\|Y_j\|_{L^4}^2\Bigg)^{\frac{s}{1-s}}R_N^3.\end{aligned}$$

We break I_2^N into the separate contributions where v_iv_j multiplies X_iY_j, X_jZ_i and Y_jZ_i, respectively. For the first contribution, Cauchy–Schwarz's inequality yields

$$\begin{aligned}&\frac{1}{N}\sum_{i,j=1}^{N}\int v_iv_jX_iY_j\,\mathrm{d}x \\ &\le \frac{1}{8}\frac{1}{N}\sum_{i,j=1}^{N}\int v_i^2Y_j^2\,\mathrm{d}x + C\frac{1}{N}\sum_{i,j=1}^{N}\int v_j^2X_i^2\,\mathrm{d}x \\ &\le \frac{1}{8}\frac{1}{N}\sum_{i,j=1}^{N}\int v_i^2Y_j^2\,\mathrm{d}x + C\Bigg(\sum_{i=1}^{N}\|v_i\|_{L^4}^2\Bigg)\Bigg(\frac{1}{N}\sum_{i=1}^{N}\|X_i\|_{L^4}^2\Bigg) \\ &\le \frac{1}{8}\frac{1}{N}\sum_{i,j=1}^{N}\|v_iY_j\|_{L^2}^2 + C\Bigg(\sum_{i=1}^{N}\|v_i\|_{H^1}^2\Bigg)^{1/2}\Bigg(\sum_{i=1}^{N}\|v_i\|_{L^2}^2\Bigg)^{1/2}\Bigg(\frac{1}{N}\sum_{i=1}^{N}\|X_i\|_{L^4}^2\Bigg),\end{aligned}$$

where we used (4.3). Using Young's inequality with exponents $(2,2)$ leads to the last contribution to (4.13). The remaining contributions to I_2^N are more involved to estimate. Our next claim is that

$$
\frac{3}{N}\sum_{i=1}^{N}\Big\langle v_i\sum_{j=1}^{N}v_jX_j,\,Z_i\Big\rangle\le\frac{1}{8}\bigg(\sum_{i=1}^{N}\|\nabla v_i\|_{L^2}^2+\Big\|\frac{1}{\sqrt{N}}\sum_{j=1}^{N}v_jX_j\Big\|_{L^2}^2\bigg)
+C(1+R_N^3+R_N^4)\sum_{i=1}^{N}\|v_i\|_{L^2}^2. \tag{4.14}
$$

The basic setup is the same as the bound leading to (2.19) via the inequality (2.7) with $v_i\sum_{j=1}^N v_jX_j$ playing the role of g and Z_i playing the role of f, followed by an application of the Cauchy–Schwarz inequality for the summation in i. The left-hand side of (4.14) is then bounded by

$$
\sum_{s'\in\{s,0\}}\frac{1}{\sqrt{N}}\bigg(\sum_{i=1}^{N}\Big\|v_i\sum_{j=1}^{N}v_jX_j\Big\|_{L^1}^{2(1-s')}\Big\|\nabla\Big(v_i\sum_{j=1}^{N}v_jX_j\Big)\Big\|_{L^1}^{2s'}\bigg)^{\frac12}\Big(\frac{1}{N}\mathscr{Z}_N\Big)^{\frac12}. \tag{4.15}
$$

Using Hölder's inequality in the form $\|v_i\sum_{j=1}^N v_jX_j\|_{L^1}\le\|v_i\|_{L^2}\|\sum_{j=1}^N v_jX_j\|_{L^2}$ together with

$$
\begin{aligned}
&\Big\|\nabla\Big(v_i\sum_{j=1}^{N}v_jX_j\Big)\Big\|_{L^1}\\
&\le\|\nabla v_i\|_{L^2}\Big\|\sum_{j=1}^{N}v_jX_j\Big\|_{L^2}+\|v_i\|_{L^2}^{1/2}\|v_i\|_{H^1}^{1/2}\bigg(\sum_{j=1}^{N}\|\nabla v_j\|_{L^2}^2\bigg)^{\frac12}\bigg(\sum_{j=1}^{N}\|X_j\|_{L^4}^2\bigg)^{\frac12}\\
&\quad+\|v_i\|_{L^2}^{1/2}\|v_i\|_{H^1}^{1/2}\bigg(\sum_{j=1}^{N}\|v_j\|_{L^2}^2\bigg)^{1/4}\bigg(\sum_{j=1}^{N}\|v_j\|_{H^1}^2\bigg)^{1/4}\bigg(\sum_{j=1}^{N}\|\nabla X_j\|_{L^2}^2\bigg)^{1/2},
\end{aligned}
$$

where we used (4.3), and inserting this into (4.15) and applying Hölder's inequality for the summation in i together with

$$
\bigg(\sum_{i=1}^{N}\|v_i\|_{L^2}^{2(1-s)}\|v_i\|_{L^2}^{s}\|v_i\|_{H^1}^{s}\bigg)^{\frac12}\le\bigg(\sum_{i=1}^{N}\|v_i\|_{L^2}^2\bigg)^{\frac{2-s}{4}}\bigg(\sum_{i=1}^{N}\|v_i\|_{H^1}^2\bigg)^{\frac{s}{4}},
$$

we obtain a majorization by

$$
\begin{aligned}
&\Big\|\frac{1}{\sqrt{N}}\sum_{j=1}^{N}v_jX_j\Big\|_{L^2}\bigg(\sum_{i=1}^{N}\|v_i\|_{L^2}^2\bigg)^{\frac{1-s}{2}}\bigg(\sum_{i=1}^{N}\|v_i\|_{H^1}^2\bigg)^{\frac{s}{2}}\Big(\frac{1}{N}\mathscr{Z}_N\Big)^{1/2}\\
&\quad+\Big\|\frac{1}{\sqrt{N}}\sum_{j=1}^{N}v_jX_j\Big\|_{L^2}^{1-s}\bigg(\sum_{i=1}^{N}\|v_i\|_{L^2}^2\bigg)^{\frac12-\frac{s}{4}}\bigg(\sum_{i=1}^{N}\|v_i\|_{H^1}^2\bigg)^{\frac{3s}{4}}\bigg(\frac{1}{N}\sum_{j=1}^{N}\|X_j\|_{L^4}^2\bigg)^{\frac{s}{2}}\Big(\frac{1}{N}\mathscr{Z}_N\Big)^{1/2}\\
&\quad+\Big\|\frac{1}{\sqrt{N}}\sum_{j=1}^{N}v_jX_j\Big\|_{L^2}^{1-s}\bigg(\sum_{i=1}^{N}\|v_i\|_{L^2}^2\bigg)^{\frac12}\bigg(\sum_{i=1}^{N}\|v_i\|_{H^1}^2\bigg)^{\frac{s}{2}}\bigg(\frac{1}{N}\sum_{j=1}^{N}\|\nabla X_j\|_{L^2}^2\bigg)^{\frac{s}{2}}\Big(\frac{1}{N}\mathscr{Z}_N\Big)^{1/2}.
\end{aligned}
$$

Finally, we apply Young's inequality with exponents $(2,\frac{2}{1-s},\frac{2}{s})$ for the first term, $(\frac{2}{1-s},\frac{4}{2-s},\frac{4}{3s})$ for the second term, and $(\frac{2}{1-s},2,\frac{2}{s})$ for the third term, which leads to (4.14).

Similar to (4.14), we now claim that

$$\frac{1}{N}\sum_{i=1}^{N}\Big\langle\sum_{j=1}^{N}v_iv_jY_j, Z_i\Big\rangle \le \frac{1}{8}\Big(\sum_{i=1}^{N}\|\nabla v_i\|_{L^2}^2+\frac{1}{N}\sum_{i,j=1}^{N}\|v_iY_j\|_{L^2}^2\Big) + C\Big(1+\sum_{k\in\{3,5,6\}}R_N^k\Big)\sum_{i=1}^{N}\|v_i\|_{L^2}^2. \tag{4.16}$$

The basic setup is again similar to the bound leading to (2.19) via the inequality (2.7) with $\sum_{j=1}^{N}v_iv_jY_j$ playing the role of g and Z_i playing the role of f, followed by an application of the Cauchy–Schwarz inequality for the summation in i. The left-hand side of (4.16) is then bounded by

$$\sum_{s'\in\{s,0\}}\frac{1}{\sqrt{N}}\Big(\sum_{i=1}^{N}\Big\|\sum_{j=1}^{N}v_iv_jY_j\Big\|_{L^1}^{2(1-s')}\Big\|\sum_{j=1}^{N}\nabla(v_iv_jY_j)\Big\|_{L^1}^{2s'}\Big)^{\frac12}\Big(\frac{1}{N}\mathscr{Z}_N\Big)^{\frac12}. \tag{4.17}$$

By Hölder's inequality and the Cauchy–Schwarz inequality, we find

$$\Big\|\sum_{j=1}^{N}v_iv_jY_j\Big\|_{L^1}\le\Big(\sum_{j=1}^{N}\|v_iY_j\|_{L^2}^2\Big)^{1/2}\Big(\sum_{j=1}^{N}\|v_j\|_{L^2}^2\Big)^{1/2},$$

together with (4.3) to have

$$\begin{aligned}\Big\|\sum_{j=1}^{N}\nabla(v_iv_jY_j)\Big\|_{L^1}\le{}&\Big(\sum_{j=1}^{N}\|v_iY_j\|_{L^2}^2\Big)^{1/2}\Big(\sum_{j=1}^{N}\|\nabla v_j\|_{L^2}^2\Big)^{1/2}\\&+\|\nabla v_i\|_{L^2}\Big(\sum_{j=1}^{N}\|v_j\|_{L^4}^2\Big)^{1/2}\Big(\sum_{j=1}^{N}\|Y_j\|_{L^4}^2\Big)^{1/2}\\&+\|v_i\|_{L^4}\Big(\sum_{j=1}^{N}\|v_j\|_{L^4}^2\Big)^{1/2}\Big(\sum_{j=1}^{N}\|\nabla Y_j\|_{L^2}^2\Big)^{1/2}.\end{aligned}$$

Inserting this into (4.17) and applying Hölder's inequality for the summation in i with exponents $(\frac{1}{1-s},\frac{1}{s})$ leads to a majorization by

$$\begin{aligned}&\Big(\frac{1}{N}\sum_{i,j=1}^{N}\|v_iY_j\|_{L^2}^2\Big)^{\frac12}\Big(\sum_{i=1}^{N}\|v_i\|_{L^2}^2\Big)^{\frac{1-s}{2}}\Big(\sum_{i=1}^{N}\|\nabla v_i\|_{L^2}^2\Big)^{\frac{s}{2}}\Big(\frac{1}{N}\mathscr{Z}_N\Big)^{1/2}\\&+\Big(\frac{1}{N}\sum_{i,j=1}^{N}\|v_iY_j\|_{L^2}^2\Big)^{\frac{1-s}{2}}\Big(\sum_{j=1}^{N}\|v_j\|_{L^2}^2\Big)^{\frac{1-s}{2}}\Big(\sum_{j=1}^{N}\|v_j\|_{L^4}^2\Big)^{\frac{s}{2}}\\&\times\Big(\sum_{i=1}^{N}\|\nabla v_i\|_{L^2}^2\Big)^{\frac{s}{2}}\Big(\frac{1}{N}\sum_{j=1}^{N}\|Y_j\|_{L^4}^2\Big)^{\frac{s}{2}}\Big(\frac{1}{N}\mathscr{Z}_N\Big)^{\frac12}\\&+\Big(\frac{1}{N}\sum_{i,j=1}^{N}\|v_iY_j\|_{L^2}^2\Big)^{\frac{1-s}{2}}\Big(\sum_{i=1}^{N}\|v_i\|_{H^1}^2\Big)^{\frac{s}{2}}\Big(\sum_{j=1}^{N}\|v_j\|_{L^2}^2\Big)^{\frac12}\\&\times\Big(\frac{1}{N}\sum_{j=1}^{N}\|\nabla Y_j\|_{L^2}^2\Big)^{\frac{s}{2}}\Big(\frac{1}{N}\mathscr{Z}_N\Big)^{1/2}.\end{aligned}$$

Note that for the third term, we also took advantage of (4.3). We now apply Young's inequality with exponents $(2, \frac{2}{1-s}, \frac{2}{s})$ for the first term, $(\frac{2}{1-s}, \frac{2}{1-s}, \frac{2}{s}, \frac{2}{s})$ for the second term, and $(\frac{2}{1-s}, \frac{2}{s}, 2)$ for the third term, which leads to (4.16). Finally, combining (4.14) and (4.16) we obtain (4.13).

STEP 4 (Law of large numbers type bounds: estimates for I_3^N)

For I_3^N, we obtain a bound in expectation in the spirit of the law of large numbers in a Hilbert space to generate cancellations. To this end, we define

$$G_j \stackrel{\text{def}}{=} (X_j^2 - \mathbf{E}X_j^2) + 2(X_j Z_j - \mathbf{E}X_j Z_j) + (:Z_j^2: - \mathbf{E} :Z_j^2:) \stackrel{\text{def}}{=} G_j^{(1)} + G_j^{(2)} + G_j^{(3)}.$$

We show there is a universal constant C such that

$$\begin{aligned} I_3^N \le C(\bar{R}_N + \bar{R}'_N) + \frac{1}{8}\frac{1}{N}\Big\|\sum_{i=1}^N X_i v_i\Big\|_{L^2}^2 + \frac{1}{4}\sum_{i=1}^N \|v_i\|_{H^1}^2 \\ + C\Big(\sum_{i=1}^N \|v_i\|_{L^2}^2\Big)\Big[R_N^7 + 1 + \frac{1}{N}\sum_{i=1}^N \big(\|X_i\|_{L^4}^{4/(1-2s)} + \|\Lambda^s X_i\|_{L^4}^4\big)\Big], \end{aligned} \tag{4.18}$$

with

$$\begin{aligned} \bar{R}_N &\stackrel{\text{def}}{=} \frac{1}{N}\Big\|\sum_{j=1}^N G_j^{(1)}\Big\|_{H^s}^2 + \sum_{k\in\{2,3\}} \frac{1}{N}\Big\|\sum_{j=1}^N G_j^{(k)}\Big\|_{H^{-s}}^2, \\ \bar{R}'_N &\stackrel{\text{def}}{=} \sum_{k\in\{2,3\}} \frac{1}{N^2}\sum_{i=1}^N \Big\|\sum_{j=1}^N G_j^{(k)} Z_i\Big\|_{H^{-s}}^2, \\ R_N^7 &\stackrel{\text{def}}{=} \Big(\frac{1}{N}\sum_{i=1}^N \|\Lambda^{-s} Z_i\|_{L^\infty}^2\Big)^{\frac{1}{1-s}}. \end{aligned}$$

We write $I_3^N = \sum_{k=1}^3 (I_{3,k}^N + J_{3,k}^N)$ with

$$I_{3,k}^N \stackrel{\text{def}}{=} \frac{1}{N}\sum_{i=1}^N \Big\langle \sum_{j=1}^N G_j^{(k)} X_i, v_i\Big\rangle, \qquad J_{3,k}^N \stackrel{\text{def}}{=} \frac{1}{N}\sum_{i=1}^N \Big\langle \sum_{j=1}^N G_j^{(k)} Z_i, v_i\Big\rangle.$$

We consider each term separately: For $I_{3,1}^N$, we have the following:

$$I_{3,1}^N \le \frac{1}{N}\Big\|\sum_{j=1}^N G_j^{(1)}\Big\|_{L^2}\Big\|\sum_{i=1}^N X_i v_i\Big\|_{L^2} \le C\frac{1}{N}\Big\|\sum_{j=1}^N G_j^{(1)}\Big\|_{L^2}^2 + \frac{1}{8}\frac{1}{N}\Big\|\sum_{i=1}^N X_i v_i\Big\|_{L^2}^2.$$

For $J_{3,1}^N$, we use (A.1), the interpolation Lemma A.2 and Young's inequality to obtain

$$\begin{aligned} J_{3,1}^N &= \frac{1}{N}\sum_{i=1}^N \Big\langle \Lambda^s\Big(\sum_{j=1}^N G_j^{(1)} v_i\Big), \Lambda^{-s} Z_i\Big\rangle \\ &\le \frac{1}{N}\sum_{i=1}^N \Big[\Big\|\Lambda^s \sum_{j=1}^N G_j^{(1)}\Big\|_{L^2}\|v_i\|_{L^2} + \Big\|\sum_{j=1}^N G_j^{(1)}\Big\|_{L^2}\|\Lambda^s v_i\|_{L^2}\Big]\|\Lambda^{-s} Z_i\|_{L^\infty} \\ &\le \frac{C}{N}\Big\|\sum_{j=1}^N G_j^{(1)}\Big\|_{H^s}^2 + \frac{1}{20}\sum_{i=1}^N \|v_i\|_{H^1}^2 + C\sum_{s'\in\{0,s\}}\Big(\sum_{i=1}^N \|v_i\|_{L^2}^2\Big)\Big(\frac{1}{N}\sum_{i=1}^N \|\Lambda^{-s} Z_i\|_{L^\infty}^2\Big)^{\frac{1}{1-s'}}. \end{aligned}$$

For $I_{3,2}^N$, we have

$$
\begin{aligned}
I_{3,2}^N &= \frac{1}{N}\sum_{i=1}^N \left\langle \Lambda^{-s}\sum_{j=1}^N G_j^{(2)}, \Lambda^s(X_i v_i)\right\rangle \\
&\le \frac{C}{N}\left\|\sum_{j=1}^N G_j^{(2)}\right\|_{H^{-s}}^2 + \frac{C}{N}\left(\sum_{i=1}^N \|\Lambda^s(X_i v_i)\|_{L^2}\right)^2 \\
&\le \frac{C}{N}\left\|\sum_{j=1}^N G_j^{(2)}\right\|_{H^{-s}}^2 + \frac{1}{20}\sum_{i=1}^N \|v_i\|_{H^1}^2 \\
&\quad + C\left(\sum_{i=1}^N \|v_i\|_{L^2}^2\right)\left(\frac{1}{N}\sum_{i=1}^N (\|X_i\|_{L^4}^{4/(1-2s)} + \|\Lambda^s X_i\|_{L^4}^4)\right),
\end{aligned}
$$

where we used (A.1), (4.3) to have

$$
\begin{aligned}
&\frac{1}{N}\left(\sum_{i=1}^N \|\Lambda^s(X_i v_i)\|_{L^2}\right)^2 \\
&\lesssim \frac{1}{N}\left(\sum_{i=1}^N \|\Lambda^s X_i\|_{L^4}\|v_i\|_{L^4} + \|\Lambda^s v_i\|_{L^4}\|X_i\|_{L^4}\right)^2 \qquad (4.19) \\
&\lesssim \frac{1}{N}\left(\sum_{i=1}^N \|v_i\|_{H^1}^{1/2}\|v_i\|_{L^2}^{1/2}\|\Lambda^s X_i\|_{L^4} + \sum_{i=1}^N \|v_i\|_{H^1}^{\frac{1}{2}+s}\|v_i\|_{L^2}^{\frac{1}{2}-s}\|X_i\|_{L^4}\right)^2,
\end{aligned}
$$

followed by Hölder inequality with exponents $(4, 4, 2)$, $(\frac{4}{1+2s}, \frac{4}{1-2s}, 2)$, Young's inequality and finally Jensen's inequalities for the terms with X_i in the last inequality. For $J_{3,2}^N$, we have

$$
J_{3,2}^N \lesssim \frac{1}{N^2}\sum_{i=1}^N \left\|\sum_{j=1}^N G_j^{(2)} Z_i\right\|_{H^{-1}}^2 + \frac{1}{20}\sum_{i=1}^N \|v_i\|_{H^1}^2.
$$

For $I_{3,3}^N$, we have

$$
\begin{aligned}
I_{3,3}^N &\lesssim \frac{1}{N}\left\|\sum_{j=1}^N G_j^{(3)}\right\|_{H^{-s}}^2 + \frac{1}{N}\left(\sum_{i=1}^N \|\Lambda^s(X_i v_i)\|_{L^2}\right)^2 \\
&\lesssim \frac{1}{N}\left\|\sum_{j=1}^N G_j^{(3)}\right\|_{H^{-s}}^2 + \frac{1}{20}\sum_{i=1}^N \|v_i\|_{H^1}^2 \\
&\quad + \left(\sum_{i=1}^N \|v_i\|_{L^2}^2\right)\left(\frac{1}{N}\sum_{i=1}^N (\|X_i\|_{L^4}^{4/(1-2s)} + \|\Lambda^s X_i\|_{L^4}^4)\right),
\end{aligned}
$$

where we used (4.19) in the last inequality. For the last term, we have

$$
J_{3,3}^N \lesssim \frac{1}{N^2}\sum_{i=1}^N \left\|\sum_{j=1}^N G_j^{(3)} Z_i\right\|_{H^{-1}}^2 + \frac{1}{20}\sum_{i=1}^N \|v_i\|_{H^1}^2.
$$

Combining all the estimates for $I_{3,k}^N$ and $J_{3,k}^N$, we arrive at (4.18). In the following, we calculate $\mathbf{E}\|\bar{R}_N\|_{L_T^1} + \mathbf{E}\|\bar{R}'_N\|_{L_T^1}$. To this end, we recall the general fact (4.1) for centered independent Hilbert space-valued random variables. Applying (4.1), we obtain

$$\mathbf{E}\|\bar{R}_N\|_{L^1_T} \lesssim \mathbf{E}\|G_1^{(1)}\|^2_{L^2_T H^s} + \mathbf{E}\|G_1^{(2)}\|^2_{L^2_T H^{-s}} + \mathbf{E}\|G_1^{(3)}\|^2_{L^2_T H^{-s}}.$$

It is obvious that $\mathbf{E}\|G_1^{(3)}\|^2_{L^2_T H^{-s}} \lesssim 1$. By Lemma 3.4, we know

$$\begin{aligned} \mathbf{E}\|G_1^{(1)}\|^2_{L^2_T H^s} &\lesssim \int_0^T \mathbf{E}\big(\|X_1 \nabla X_1\|^2_{L^2} + \|X_1\|^4_{L^4}\big)\,\mathrm{d}t \lesssim 1, \\ \mathbf{E}\|G_1^{(2)}\|^2_{L^2_T H^{-s}} &\lesssim \int_0^T \mathbf{E}\big[\|Z_1\|^2_{\mathbf{C}^{-s/2}} \|X_1\|^2_{H^s}\big]\,\mathrm{d}t \\ &\lesssim \int_0^T \big(\mathbf{E}\big[\|X_1\|^2_{H^1} + \|X_1\|^4_{L^4}\big] + 1\big)\,\mathrm{d}t \lesssim 1, \end{aligned} \tag{4.20}$$

where we used Lemma A.2 and Lemma A.3. Therefore, $\mathbf{E}\|\bar{R}_N\|_{L^1_T} \lesssim 1$. For $\bar{R}'_N$, we have

$$\begin{aligned} \mathbf{E}\frac{1}{N^2}\sum_{i=1}^N \Big\|\sum_{j=1}^N G_j^{(2)} Z_i\Big\|^2_{H^{-s}} &= \frac{1}{N^2}\sum_{i,j,\ell=1}^N \mathbf{E}\langle G_j^{(2)} Z_i, G_\ell^{(2)} Z_i\rangle_{H^{-s}} \\ &= \frac{1}{N^2}\Big[\sum_{i=j=\ell} + 2\sum_{i=j\neq\ell} + \sum_{\ell=j\neq i}\Big] \\ &\lesssim \mathbf{E}\|X_1 :Z_1^2:\|^2_{H^{-s}} + \mathbf{E}\|X_1 :Z_1 Z_2:\|^2_{H^{-s}}, \end{aligned}$$

where we used independence to have $\sum_{i\neq j\neq \ell} = 0$. Similarly, we have

$$\mathbf{E}\frac{1}{N^2}\sum_{i=1}^N \Big\|\sum_{j=1}^N G_j^{(3)} Z_i\Big\|^2_{H^{-s}} \lesssim \mathbf{E}\|:Z_1^3:\|^2_{H^{-s}} + \mathbf{E}\|:Z_1^2 Z_2:\|^2_{H^{-s}}.$$

Combining the above two estimates and using Lemma 2.1 and the same argument as in (4.20) with Z_1 replaced by $:Z_1Z_2:$ and $:Z_1^2:$, we obtain $\mathbf{E}\|\bar{R}'_N\|_{L^1_T} \lesssim 1$.

STEP 5 (Convergence of v_i to zero in $L^2(\Omega)$)

We now combine our estimates and conclude the proof of (4.2). Namely, we insert the estimates (4.8) and (4.13) into (4.5) and also appeal to our bounds from Step 4 to obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^N \|v_i\|^2_{L^2} \\ &\leq C\big(\bar{R}_N + \bar{R}'_N\big) \\ &\quad + C\left(1 + \sum_{i=2}^7 R_N^i + \frac{1}{N}\sum_{i=1}^N \big(\|X_i\|^{4/(1-2s)}_{L^4} + \|\Lambda^s X_i\|^4_{L^4}\big)\right)\sum_{i=1}^N \|v_i\|^2_{L^2}, \end{aligned} \tag{4.21}$$

where $\bar{R}_N + \bar{R}'_N$ is uniformly bounded in $L^1(\Omega \times [0,T])$. Furthermore, by Lemma 2.1, Lemma 2.3, Lemma 3.4 and (4.3), we deduce also that R_N^i is uniformly bounded in $L^1(\Omega \times [0,T])$ for each $i = 2, \dots, 7$. By the Gagliardo–Nirenberg inequality in Lemma A.2, we have for $s \geq 2\kappa$, $r > 4$, $\frac{1}{4} = \frac{s}{2} + \frac{1-s}{r}$, $\|\Lambda^s X_i\|^4_{L^4} \lesssim \|X_i\|^{4s}_{H^1}\|X_i\|^{4(1-s)}_{L^r}$, which combined with Lemma 3.4 implies that

$$\mathbf{E}\int_0^T \frac{1}{N}\sum_{i=1}^N \big(\|X_i\|^{\frac{4}{1-2s}}_{L^4} + \|\Lambda^s X_i\|^4_{L^4}\big)\,\mathrm{d}t \lesssim \mathbf{E}\int_0^T \frac{1}{N}\sum_{i=1}^N \big(\|X_i\|^2_{H^1} + \|X_i\|^{\frac{4}{1-2s}}_{L^{\frac{4}{1-2s}}} + 1\big)\,\mathrm{d}t < \infty.$$

We now divide (4.21) by N and use the above observations together with Gronwall's inequality. Note that in light of Assumption 4.1, it holds that

$$\frac{1}{N}\sum_{i=1}^{N}\|v_i(0)\|_{L^2}^2 \to^{\mathbf{P}} 0. \tag{4.22}$$

It now follows that

$$\begin{aligned}&\sup_{t\in[0,T]}\frac{1}{N}\sum_{i=1}^{N}\|v_i\|_{L^2}^2+\frac{1}{N}\sum_{i=1}^{N}\|v_i\|_{L_T^2H^1}^2\\&\quad+\frac{1}{N^2}\sum_{i,j=1}^{N}\|Y_jv_i\|_{L_T^2L^2}^2+\frac{1}{N^2}\Big\|\sum_j X_jv_j\Big\|_{L_T^2L^2}^2\end{aligned}\tag{4.23}$$

converges to zero in probability by Lemma 4.2 below. We now upgrade this from convergence in probability to convergence in $L^1(\Omega)$ by bounding higher moments and applying Vitali's convergence theorem. Only in this part we use the condition that the initial conditions $(z_i^N, y_i^N, z_i, \eta_i)_{i=1}^N$ are exchangeable, which implies that the law of $v_i(t)$ and $v_j(t)$, $i \neq j$ are the same.

Indeed, first note that $\sup_{t\in[0,T]}\frac{1}{N}\sum_{i=1}^N\|Y_i\|_{L^2}^2$ is uniformly bounded in $L^q(\Omega)$ for q in Assumption 4.1 by Lemma 2.3. Additionally, by Lemma 3.4, Jensen's inequality and the fact that X_i and X_j are identically distributed (which follows from the i.i.d. hypothesis in Assumption 4.1), it holds

$$\sup_{N\geq 1}\sup_{t\in[0,T]}\mathbf{E}\bigg(\frac{1}{N}\sum_{i=1}^{N}\|X_i(t)\|_{L^2}^2\bigg)^2\leq \sup_{t\in[0,T]}\mathbf{E}\|X_1(t)\|_{L^2}^4<\infty.$$

Notice that at this stage we are appealing to the assumption $\mathbf{E}\|\eta_i\|_{L^{p_0}}^{p_0}\lesssim 1$ in order to meet the hypotheses of Lemma 3.4 and deduce the final step above. Hence, by the triangle inequality we find that

$$\sup_{N\geq 1}\sup_{t\in[0,T]}\mathbf{E}\bigg(\frac{1}{N}\sum_{i=1}^{N}\|v_i(t)\|_{L^2}^2\bigg)^q<\infty,$$

which implies the following convergence upgrade: $\frac{1}{N}\sum_{i=1}^N\|v_i(t)\|_{L^2}^2$ converges to zero in $L^1(\Omega)$ for each $t\in[0,T]$. Finally, we appeal once more to the first bullet point in Assumption 4.1, which is designed to ensure that v_i and v_j have the same law. As a consequence, we can now pass from empirical averages to components in light of

$$\mathbf{E}\|v_i(t)\|_{L^2}^2=\frac{1}{N}\sum_{i=1}^{N}\mathbf{E}\|v_i(t)\|_{L^2}^2\to 0. \tag{4.24}$$

STEP 6 (Convergence as a stochastic process)

The proof is largely the same as above, except that we do not estimate v_i by an average over i as in (4.24), since a supremum over time would not commute with a sum over i. Instead we deduce the following bound:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|v_i\|_{L^2}^2+\frac{1}{2}\|v_i\|_{H^1}^2\\&\quad\leq C\bigg(\frac{\bar{R}_N}{N}+\tilde{\bar{R}}_N'\bigg)+\frac{1}{N}\sum_{i=1}^{N}\|v_i\|_{H^1}^2+\frac{1}{N^2}\Big\|\sum_j X_jv_j\Big\|_{L^2}^2\end{aligned}\tag{4.25}$$

$$
\begin{aligned}
&+ C\bigg(1 + \tilde{R}_N^{21} + \sum_{k\in\{3,5\}} R_N^k + \sum_{k\in\{3,4,7\}} \tilde{R}_N^k + \frac{1}{N}\sum_{j=1}^N \|X_j\|_{L^4}^4 \\
&+ \big(\|X_i\|_{L^4}^{4/(1-2s)} + \|\Lambda^s X_i\|_{L^4}^4\big)\bigg)\|v_i\|_{L^2}^2 \\
&+ C\big(1 + \tilde{R}_N^{22} + \tilde{R}_N^3 + \tilde{R}_N^5 + \tilde{R}_N^6 + \|X_i\|_{L^4}^4\big)\frac{1}{N}\sum_{j=1}^N \|v_j\|_{L^2}^2,
\end{aligned}
\tag{4.26}
$$

where all the "tilde R-terms" are defined analogously to their "untilde" counterparts with slight tweaks:

$$
\tilde{\tilde{R}}_N' \stackrel{\text{def}}{=} \frac{1}{N^2}\bigg\|\sum_{j=1}^N G_j^2 Z_i\bigg\|_{H^{-s}}^2 + \frac{1}{N^2}\bigg\|\sum_{j=1}^N G_j^3 Z_i\bigg\|_{H^{-s}}^2,
$$

$$
\tilde{R}_N^{21} \stackrel{\text{def}}{=} \frac{1}{N}\sum_{j=1}^N \|:Z_j^2:\|_{\mathbf{C}^{-s}}^{2/(2-s)}, \qquad \tilde{R}_N^{22} \stackrel{\text{def}}{=} \frac{1}{N}\sum_{j=1}^N \|:Z_i Z_j:\|_{\mathbf{C}^{-s}}^2,
$$

$$
\tilde{R}_N^3 \stackrel{\text{def}}{=} \|Z_i\|_{\mathbf{C}^{-s}}^{2/(1-s)}, \qquad \tilde{R}_N^5 \stackrel{\text{def}}{=} \bigg(1 + \frac{1}{N}\sum_{j=1}^N \|\nabla Y_j\|_{L^2}^2\bigg)^{\frac{s}{1-s}} \|Z_i\|_{\mathbf{C}^{-s}}^{\frac{2}{1-s}},
$$

$$
\begin{aligned}
\tilde{R}_N^4 \stackrel{\text{def}}{=} &\bigg(1 + \frac{1}{N}\sum_{j=1}^N \|X_j\|_{H^1}^2\bigg)^{\frac{2s}{2-s}} \big(\|Z_i\|_{\mathbf{C}^{-s}}^{\frac{4}{2-s}} + 1\big) \\
&+ \bigg(\frac{1}{N}\mathscr{Z}_N\bigg)^{\frac{2}{2-s}} \|X_i\|_{H^1}^{\frac{4s}{2-s}} + \bigg(\frac{1}{N}\mathscr{Z}_N\bigg)^2 \|X_i\|_{L^4}^2,
\end{aligned}
$$

$$
\tilde{R}_N^6 \stackrel{\text{def}}{=} \bigg(\frac{1}{N}\sum_{j=1}^N \|Y_j\|_{L^4}^2\bigg)^{\frac{s}{1-s}} \|Z_i\|_{\mathbf{C}^{-s}}^{\frac{2}{1-s}}, \qquad \tilde{R}_N^7 \stackrel{\text{def}}{=} \|\Lambda^{-s} Z_i\|_{L^\infty}^{2/(1-s)},
$$

where $\mathscr{Z}_N$ is as in (2.18). In fact, all of the terms are similar as above except the following two terms:

$$
-\frac{1}{N}\sum_{j=1}^N \int X_i X_j v_j v_i \,\mathrm{d}x - \frac{2}{N}\sum_{j=1}^N \langle X_i v_j v_i, Z_j\rangle := J_1 + J_2.
$$

The term J_1 is treated differently than above, since without the sum over i we could not move it to the left-hand side as a coercive quantity. We have

$$
\begin{aligned}
J_1 &\le \frac{1}{N}\sum_{j=1}^N \int v_j^2 X_i^2 \,\mathrm{d}x + \frac{1}{N}\sum_{j=1}^N \int v_i^2 X_j^2 \,\mathrm{d}x \\
&\le C\bigg(\frac{1}{N}\sum_{j=1}^N \|v_j\|_{L^4}^2\bigg)\|X_i\|_{L^4}^2 + C\|v_i\|_{L^4}^2 \frac{1}{N}\sum_{j=1}^N \|X_j\|_{L^4}^2 \\
&\le C\bigg(\frac{1}{N}\sum_{j=1}^N \|v_j\|_{H^1}^2\bigg)^{1/2} \bigg(\frac{1}{N}\sum_{j=1}^N \|v_j\|_{L^2}^2\bigg)^{1/2} \|X_i\|_{L^4}^2 \\
&\quad + \frac{1}{8}\|v_i\|_{H^1}^2 + C\|v_i\|_{L^2}^2 \bigg(\frac{1}{N}\sum_{j=1}^N \|X_j\|_{L^4}^2\bigg)^2,
\end{aligned}
$$

which by Young's inequality deduce one contribution to (4.26). For the second term, we have

$$J_2 \lesssim \sum_{s' \in \{0,s\}} \left(\frac{1}{N} \sum_{j=1}^{N} \|v_i v_j X_i\|_{L^1}^{2(1-s')} \|\nabla(v_i v_j X_i)\|_{L^1}^{2s'} \right)^{\frac{1}{2}} \left(\frac{1}{N} \mathscr{Z}_N \right)^{\frac{1}{2}}. \tag{4.27}$$

Using Hölder's inequality in the form $\|v_i v_j X_i\|_{L^1} \leq \|v_i\|_{L^2} \|v_j X_i\|_{L^2}$ together with the bound for the first term in J_1, we obtain the estimate for $s' = 0$ in J_2, which corresponds to the last term in $\tilde{R}_N^4$. Moreover, we have

$$\begin{aligned}\|\nabla(v_i v_j X_i)\|_{L^1} \leq \|\nabla v_i\|_{L^2} \|v_j X_i\|_{L^2} + \|v_i\|_{L^2}^{1/2} \|v_i\|_{H^1}^{1/2} \|\nabla v_j\|_{L^2} \|X_i\|_{L^4} \\ + \|v_i\|_{L^2}^{1/2} \|v_i\|_{H^1}^{1/2} \|v_j\|_{L^2}^{1/2} \|v_j\|_{H^1}^{1/2} \|\nabla X_i\|_{L^2},\end{aligned}$$

and inserting this into the term $s' = s$ in (4.27) and applying Hölder's inequality for the summation in j leads to the following:

$$\begin{aligned}&\left(\frac{1}{N} \sum_{j=1}^{N} \|v_j X_i\|_{L^2}^2 \right)^{1/2} \|v_i\|_{L^2}^{1-s} \|v_i\|_{H^1}^{s} \left(\frac{1}{N} \mathscr{Z}_N \right)^{1/2} \\ &+ \left(\frac{1}{N} \sum_{j=1}^{N} \|v_j X_i\|_{L^2}^{2(1-s)} \|v_j\|_{H^1}^{2s} \right)^{1/2} \|v_i\|_{L^2}^{1-s/2} \|v_i\|_{H^1}^{s/2} \|X_i\|_{L^4}^{s} \left(\frac{1}{N} \mathscr{Z}_N \right)^{1/2} \\ &+ \left(\frac{1}{N} \sum_{j=1}^{N} \|v_j X_i\|_{L^2}^{2(1-s)} \|v_j\|_{H^1}^{s} \|v_j\|_{L^2}^{s} \right)^{1/2} \|\nabla X_i\|_{L^2}^{s} \|v_i\|_{L^2}^{1-\frac{s}{2}} \|v_i\|_{H^1}^{\frac{s}{2}} \left(\frac{1}{N} \mathscr{Z}_N \right)^{1/2}.\end{aligned}$$

Finally, we apply Young's inequality and obtain the contribution of $\tilde{R}_N^4$ in the estimate (4.26).

Using the fact that (4.23) converges to zero in probability, we deduce the $L^1(0, T)$ norm of (4.26) and the right-hand side of (4.25) converges to zero in probability. Then by Gronwall's inequality and Lemma 4.2 imply $\sup_{t \in [0,T]} \|v_i(t)\|_{L^2}^2 \to 0$ in probability, as $N \to \infty$. In this step, we see that we do not use the condition that the initial conditions $(z_i^N, y_i^N, z_i, \eta_i)_{i=1}^N$ are exchangeable.

STEP 7 (General initial datum) To this end, define $u_i \stackrel{\text{def}}{=} S_t(z_i^N - z_i)$ and note that we have the following extra terms:

$$\begin{aligned}\bar{I}^N := -\frac{1}{N} \sum_{i,j=1}^{N} [\langle Y_j^2, v_i u_i \rangle + 2\langle Y_j Y_i u_j, v_i \rangle + 2\langle Y_j v_i, :Z_i^N Z_j^N: - :Z_i Z_j: \rangle \\ + \langle Y_i v_i, :Z_j^{N,2}: - :Z_j^2: \rangle + \langle v_i, :Z_i^N Z_j^{N,2}: - :Z_i Z_j^2: \rangle].\end{aligned}$$

These terms could also be estimated similarly as that for I_1^N and I_2^N by using $\|u_i\|_{\mathbf{C}^s} \lesssim t^{-(s+\kappa)/2} \|z_i^N - z_i\|_{\mathbf{C}^{-\kappa}}$. Since the proof follows a similar line of argument as in Steps 2–3, we place the details in Appendix D. □

We recall the following result from probability theory, used in Steps 4 and 6 of the above lemma, which can be deduced with elementary arguments.

LEMMA 4.2. *Let $\{U_N\}_{N=1}^\infty$ be a nonnegative sequence of* 1*d random variables converging to zero in probability. Let $\{V_N\}_{N=1}^\infty$ be a nonnegative sequence of random variables with tight laws. Then the sequence $\{U_N V_N\}_{N=1}^\infty$ converges to zero in probability.*

5. Invariant measure and observables. We now study the invariant measure for the equation

$$\mathscr{L}\Psi = -\mathbf{E}[\Psi^2 - Z^2]\Psi + \xi, \tag{5.1}$$

with $\mathbf{E}[\Psi^2 - Z^2] = \mathbf{E}[X^2] + 2\mathbf{E}[XZ]$ for $X = \Psi - Z$ and ξ space-time white noise. Here, since we are only interested in the stationary setting in this section, we overload the notation in the previous sections and simply write Z for the stationary solution to the linear equation

$$\mathscr{L}Z = \xi, \tag{5.2}$$

and we consider the decomposition (slightly different from Section 3) $X \stackrel{\text{def}}{=} \Psi - Z$, so that

$$\mathscr{L}X = -\mathbf{E}[X^2 + 2XZ](X + Z), \quad X(0) = \Psi(0) - Z(0). \tag{5.3}$$

For the case that $m = 0$, we restrict the solutions Ψ and Z satisfying $\langle \Psi, 1\rangle = \langle Z, 1\rangle = 0$.

By Theorem 3.6, for every initial data

$$\Psi(0) = \psi \in \mathbf{C}^{-\kappa}$$

with $\mathbf{E}\|\psi\|^p_{\mathbf{C}^{-\kappa}} \lesssim 1$ there exists a unique global solution Ψ to (5.1). We immediately find that Z is a stationary solution to (5.1). This follows since the unique solution to (5.3) starting from zero is identically zero. Furthermore, we define a semigroup $P_t^*\nu$ to denote the law of $\Psi(t)$ with the initial condition distributed according to a measure ν. By uniqueness of the solutions to (5.1), we have $P_t^* = P_{t-s}^* P_s^*$ for $t \geq s \geq 0$. By direct probabilistic calculation, we easily obtain the following result, which implies that the implicit constant in Lemma 2.1 is independent of m.

LEMMA 5.1. *For $\kappa' > \kappa > 0$ and $p \geq 1$, it holds that*

$$\sup_{m\geq 0} \mathbf{E}\big[\|Z_i\|^p_{C_T\mathbf{C}^{-\kappa}}\big] + \sup_{m\geq 0} \mathbf{E}\big[\| :Z_iZ_j: \|^p_{C_T\mathbf{C}^{-\kappa}}\big] + \sup_{m\geq 0} \mathbf{E}\big[\| :Z_iZ_j^2: \|^p_{C_T\mathbf{C}^{-\kappa}}\big] \lesssim 1,$$

where the proportional constants are independent of i, j, N.

PROOF. By a standard technique (cf. [34]), it is sufficient to calculate

$$\mathbf{E}\big|\Delta_q Z_i(t)\big|^2 \lesssim \sum_{k\in\mathbb{Z}^2} \int_{-\infty}^t \theta(2^{-q}k)^2 \big|e^{-2(t-s)(|k|^2+m)}\big|\,\mathrm{d}s \lesssim \sum_{k\in\mathbb{Z}^2} 2^{q\kappa} \frac{1}{|k|^\kappa(|k|^2+m)},$$

where Δ_q is a Littlewood–Paley block and θ is the Fourier multiplier associated with Δ_q. From here, we see the bound is independent of m. Other terms can be bounded in a similar way. □

For R_N^0 defined in (2.24) with Z_i stationary, we have the following result.

LEMMA 5.2. *For every $q \geq 1$, it holds that*

$$\mathbf{E}\big[(R_N^0)^q\big] \lesssim 1. \tag{5.4}$$

PROOF. Since we will have several similar calculations in the sequel, we first demonstrate such calculation in the case $q = 1$. We have

$$\mathbf{E}\frac{1}{N^2}\sum_{i=1}^N \bigg\| \sum_{j=1}^N \Lambda^{-s}\big(:Z_j^2 Z_i: \big)\bigg\|^2_{L^2} = \frac{1}{N^2}\sum_{i,j_1,j_2=1}^N \mathbf{E}\langle \Lambda^{-s} :Z_{j_1}^2 Z_i: , \Lambda^{-s} :Z_{j_2}^2 Z_i: \rangle.$$

We have 3 summation indices and a factor $1/N^2$. The contribution to the sum from the cases $j_1 = i$ or $j_2 = i$ or $j_1 = j_2$ is bounded by a constant in light of Lemma 5.1. If i, j_1, j_2 are all different, by independence and the fact that Wick products are mean zero, the terms are zero.

For general $q \geq 1$, by Gaussian hypercontractivity and the fact that R_N^0 is a random variable with finite Wiener chaos decomposition, we have

$$\mathbf{E}\big[(R_N^0)^q\big] \lesssim \mathbf{E}\big[(R_N^0)^2\big]^{q/2}.$$

For the case that $q = 2$, we write it as

$$\frac{1}{N^4} \sum_{\substack{i_1, i_2, j_k = 1 \\ k=1\dots4}}^{N} \mathbf{E}\big\langle \Lambda^{-s} :Z_{j_1}^2 Z_{i_1}: , \Lambda^{-s} :Z_{j_2}^2 Z_{i_1}: \big\rangle \big\langle \Lambda^{-s} :Z_{j_3}^2 Z_{i_2}: , \Lambda^{-s} :Z_{j_4}^2 Z_{i_2}: \big\rangle.$$

We have 6 indices $i_1, i_2, j_k, k = 1, \dots, 4$ summing from 1 to N and an overall factor $1/N^4$. Using again Lemma 5.1, we reduce the problem to the cases where five or six of the indices are different. However, in these two cases, by independence the expectation is zero, so (5.4) follows. $\square$

5.1. *Uniqueness of the invariant measures.* We now turn to the question of uniqueness for the invariant measure of (5.1). Since the nonlinearity in the SPDE (5.1) involves the law of the solution, the associated semigroup P_t^* is generally nonlinear, that is,

$$P_t^* \nu \neq \int \big(P_t^* \delta_\psi\big) \nu(\mathrm{d}\psi),$$

for a nontrivial distribution ν (see, e.g., [69]). As a result, its unclear if the general ergodic theory for Markov processes (see, e.g., [21], [39]) can be applied directly in our setting. Fortunately, (5.3) has a strong damping property in the mean-square sense, which comes to our rescue and allows us to proceed directly by a priori estimates.

LEMMA 5.3. *There exists $C_0 > 0$ such that for all*

$$m \geq 2C_0\big(\mathbf{E}\| :Z_2 Z_1: \|_{\mathbf{C}^{-s}}^2 + \big(\mathbf{E}\|Z_1\|_{\mathbf{C}^{-s}}^2\big)^{\frac{1}{1-s}} + 1\big) := m_0,$$

there exists a universal C with the following property: *for every solution Ψ to* (5.1) *with* $\Psi(0) \in \mathbf{C}^{-\kappa}$,

$$\sup_{t \geq 1} e^{\frac{mt}{2}} \mathbf{E}\big\|\Psi(t) - Z(t)\big\|_{L^2}^2 \leq C. \tag{5.5}$$

PROOF. The proof relies heavily on several computations performed in Lemma 3.3 where we used slightly different notation, so we will write X_i instead of X and Z_i instead of Z for the remainder of this proof. Revisiting the first step of Lemma 3.3 where we established (3.5), we find that I^1 defined in (3.6) and the first contribution to I^2 defined in (3.6) vanishes in light of $\mathbf{E}(:Z_j^2:) = 0$. It follows that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\|X_i\|_{L^2}^2 + \mathbf{E}\|\nabla X_i\|_{L^2}^2 + m\mathbf{E}\|X_i\|_{L^2}^2 + \|\mathbf{E}X_i^2\|_{L^2}^2 \\
&\qquad = -2\mathbf{E}\langle X_i X_j, Z_i Z_j\rangle - 3\mathbf{E}\big\langle X_i X_j^2, Z_i\big\rangle.
\end{aligned}$$

Furthermore, in light of (3.9) and (3.15), we obtain

$$\mathbf{E}\langle X_i X_j, Z_i Z_j\rangle \lesssim \big(\|\mathbf{E}X_i^2\|_{L^1}^2 + \mathbf{E}\|\nabla X_i\|_{L^2}^2 \|\mathbf{E}X_j^2\|_{L^1}\big)^{1/2} \big(\mathbf{E}\| :Z_j Z_i: \|_{\mathbf{C}^{-s}}^2\big)^{\frac{1}{2}}$$

$$\begin{aligned}
\mathbf{E}\big\langle X_i X_j^2, Z_i\big\rangle &\lesssim \|\mathbf{E}X_i^2\|_{L^2} \big(\mathbf{E}\|\nabla X_i\|_{L^2}^2\big)^{\frac{s}{2}} \big(\mathbf{E}\|X_i\|_{L^2}^2\big)^{\frac{1-s}{2}} \big(\mathbf{E}\|Z_i\|_{\mathbf{C}^{-s}}^2\big)^{\frac{1}{2}} \\
&\quad + \|\mathbf{E}X_i^2\|_{L^2} \big(\mathbf{E}\|X_j\|_{L^2}^2\big)^{\frac{1}{2}} \big(\mathbf{E}\|Z_i\|_{\mathbf{C}^{-s}}^2\big)^{\frac{1}{2}}.
\end{aligned}$$

We will use these estimates in two different ways. On one hand, using Young's inequality with respective exponents $(2, 2)$ and $(2, \frac{2}{s}, \frac{2}{1-s})$ followed by Lemma 5.1, we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\|X_i\|_{L^2}^2 + \frac{1}{2}\mathbf{E}\|\nabla X_i\|_{L^2}^2 + m\mathbf{E}\|X_i\|_{L^2}^2 + \|\mathbf{E}X_i^2\|_{L^2}^2 \lesssim 1. \tag{5.6}$$

As a consequence, noting that $\|\mathbf{E}X_i^2\|_{L^2}^2 \geq (\mathbf{E}\|X_i\|_{L^2}^2)^2$, applying Lemma A.6 it holds

$$\sup_{t>0}(t \wedge 1)\mathbf{E}\big[\|X_i(t)\|_{L^2}^2\big] \lesssim 1, \tag{5.7}$$

where the implicit constant is independent of the initial data. On the other hand, Young's inequality also yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\|X_i\|_{L^2}^2 + m\mathbf{E}\|X_i\|_{L^2}^2 \leq C_0\big(\mathbf{E}\| :Z_2Z_1: \|_{\mathbf{C}^{-s}}^2 + \big(\mathbf{E}\|Z_1\|_{\mathbf{C}^{-s}}^2\big)^{\frac{1}{1-s}} + 1\big)\mathbf{E}\|X_i\|_{L^2}^2. \tag{5.8}$$

Applying Gronwall's inequality over $[1, t]$ leads to

$$e^{(m-\frac{m_0}{2})t}\mathbf{E}\|X_i(t)\|_{L^2}^2 \lesssim \mathbf{E}\|X_i(1)\|_{L^2}^2,$$

so choosing $m \geq m_0$, using (5.7), and taking the supremum over $t \geq 1$, we arrive at (5.5). $\square$

We now apply the above result to show that for sufficiently large mass, the unique invariant measure to (5.1) is Gaussian. To this end, define the $\mathbf{C}^{-1}$-Wasserstein distance

$$\mathbb{W}'_p(\nu_1, \nu_2) := \inf_{\pi\in\mathscr{C}(\nu_1,\nu_2)}\bigg(\int \|\phi - \psi\|_{\mathbf{C}^{-1}}^p \pi(\mathrm{d}\phi, \mathrm{d}\psi)\bigg)^{1/p},$$

where $\mathscr{C}(\nu_1, \nu_2)$ denotes the collection of all couplings of ν_1, ν_2 satisfying $\int \|\phi\|_{\mathbf{C}^{-1}}^p \nu_i(\mathrm{d}\phi) < \infty$ for $i = 1, 2$.

THEOREM 5.4. *For m_0 as in Lemma 5.3 and $m \geq m_0$ the unique invariant measure to (5.1) supported on $\mathbf{C}^{-\kappa}$ is $\mathcal{N}(0, \frac{1}{2}(-\Delta + m)^{-1})$, the law of the Gaussian free field.*

PROOF. Recall that Z is a stationary solution to (5.1). Indeed, by definition, $\Psi = X + Z$, where X solves (5.3). However, since $X(0) = 0$, the identically zero process is the unique solution to (5.3). Hence, the law of Z, which we now denote by ν, is invariant under P_t^*. We now claim that for $m \geq m_0$, this is the only invariant measure supported on $\mathbf{C}^{-\kappa}$. Indeed, let ν_1 be another such measure, then modifying the stochastic basis if needed, we may assume there exists $\psi \in \mathbf{C}^{-\kappa}$ on it such that $\psi \sim \nu_1$. By similar arguments as in Theorem 3.6, we may construct a solution Ψ to (5.1) with $\Psi(0) = \psi$. By invariance of ν_1 and ν and the embedding $L^2 \hookrightarrow \mathbf{C}^{-1}$ (cf. Lemma A.1), it follows that

$$\mathbb{W}'_2(\nu, \nu_1)^2 = \mathbb{W}'_2\big(P_t^*\nu, P_t^*\nu_1\big)^2 \leq \mathbf{E}\|\Psi(t) - Z(t)\|_{\mathbf{C}^{-1}}^2 \lesssim e^{-\frac{mt}{2}},$$

for $t \geq 1$ by Lemma 5.3. Letting $t \to \infty$, we obtain $\nu = \nu_1$. $\square$

REMARK 5.5. Note that for the limiting equation $\mathscr{L}\Psi = -\mu\Psi + \xi$, if we assume that μ is simply a constant, it has a Gaussian invariant measure $\mathcal{N}(0, \frac{1}{2}(-\Delta + m + \mu)^{-1})$. Assuming $\Psi \sim \mathcal{N}(0, \frac{1}{2}(-\Delta + m + \mu)^{-1})$ and $Z \sim \mathcal{N}(0, \frac{1}{2}(-\Delta + m)^{-1})$, the self-consistent condition $\mathbf{E}[\Psi^2 - Z^2] = \mu$ then yields

$$\frac{1}{2}\sum_{k\in\mathbb{Z}^2}\bigg(\frac{1}{|k|^2 + m + \mu} - \frac{1}{|k|^2 + m}\bigg) = \mu$$

and for $\mu + m \geq 0$ we only have one solution $\mu = 0$, since the left-hand side is monotonically decreasing in μ.

REMARK 5.6. Changing the renormalization constant in (5.1) will alter the mass of the Gaussian invariant measure. For instance, if we change the renormalization constant in (5.1) to $\mathbf{E}Z_{i,\varepsilon}^2(0,0)$ with Z_i by the stationary solution to $(\partial_t - (\Delta - a))Z_i = \xi_i$ with $a > 0$, one invariant measure is Gaussian $\bar{\nu} \stackrel{\text{def}}{=} \mathcal{N}(0, \frac{1}{2}(-\Delta + m + \mu_0)^{-1})$ with μ_0 satisfying

$$\frac{1}{2}\sum_{k\in\mathbb{Z}^2}\left(\frac{1}{|k|^2+m+\mu_0} - \frac{1}{|k|^2+a}\right) = \mu_0.$$

Moreover, by the same proof of Lemma 5.3 and Theorem 5.4, for $m + \mu_0$ large enough, $\bar{\nu}$ is the unique invariant measure. Indeed, let $\Psi = \bar{X} + \bar{Z}$ with $\bar{Z}$ the stationary solution to $\mathscr{L}\bar{Z} = -\mu_0\bar{Z} + \xi$, then $\bar{X}$ satisfies the following equation:

$$\mathscr{L}\bar{X} = -\mu_0\bar{X} - \mathbf{E}[\bar{X}^2 + 2\bar{X}\bar{Z}](\bar{X} + \bar{Z}),$$

which is the same case as (3.1) with m replaced by $m + \mu_0$.

5.2. *Convergence of the invariant measures.* As a consequence of Lemma 2.2, the solutions $(\Phi_i)_{1\le i\le N}$ to (2.1) form a Markov process on $(\mathbf{C}^{-\kappa})^N$, which by strong Feller property in [38] and irreducibility in [40], will turn out to admit a unique invariant measure, henceforth denoted by ν^N. Our goal in this section is to study the large N behavior of ν^N and show that for sufficiently large mass, as $N \to \infty$, its marginals are simply products of the Gaussian invariant measure for Ψ identified in Theorem 5.4. For this, we rely heavily on the computations from Section 2.2 for the remainder Y, but we leverage these estimates with consequences of stationarity. To this end, it will be convenient to have a stationary coupling of the linear and nonlinear dynamics (2.2) and (2.1), respectively, which is the focus of the following lemma.

LEMMA 5.7. *There exists a unique invariant measure ν^N on $(\mathbf{C}^{-\kappa})^N$ to (2.1). Furthermore, there exists a stationary process $(\Phi_i^N, Z_i)_{1\le i\le N}$ such that the components Φ_i^N, Z_i are stationary solutions to (2.1) and (2.2), respectively. Moreover, $\mathbf{E}\|\Phi_i^N(0) - Z_i(0)\|_{H^1}^2 \lesssim 1$ and for every $q > 1$,*

$$\mathbf{E}\left(\frac{1}{N}\sum_{i=1}^N \|\Phi_i^N(0) - Z_i(0)\|_{L^2}^2\right)^q \lesssim 1. \tag{5.9}$$

PROOF. In the proof, we fix N. Let Φ_i and Z_i be solutions to (2.1) and (2.2), respectively. By the general results of [38], Section 2, it follows that $(\Phi_i, Z_i)_{1\le i\le N}$ is a Markov process on $(\mathbf{C}^{-\kappa})^{2N}$, and we denote by $(P_t^N)_{t\ge0}$ the associated Markov semigroup. To derive the desired structural properties about the limiting measure, we will follow the Krylov–Bogoliubov construction with a specific choice of initial condition that allows to exploit Lemma 2.3. Namely, we denote by Φ_i the solution to (2.1) starting from the stationary solution $\tilde{Z}_i(0)$, so that the process $Y_i = \Phi_i - Z_i$ starts from the origin. Using Lemma 2.3 and Corollary 2.4 with $y_j = 0$ together with Lemma 5.1 to obtain a uniform bound on $\mathbf{E}R_N$ with R_N defined in (2.6), we find for every $T \ge 1$,

$$\int_0^T \mathbf{E}\left(\frac{1}{N}\sum_{i=1}^N \|Y_i(t)\|_{H^1}^2\right) \mathrm{d}t \lesssim T, \tag{5.10}$$

$$\mathbf{E}\int_0^T \left(\frac{1}{N}\sum_{i=1}^N \|Y_i(t)\|_{L^2}^2\right)^q \mathrm{d}t \lesssim T, \tag{5.11}$$

where the implicit constant is independent of T and for $m=0$ we used the Poincaré inequality. By (5.10), we obtain

$$\int_0^T \mathbf{E}\bigg(\frac{1}{N}\sum_{i=1}^N \|\Phi_i(t)\|^2_{\mathbf{C}^{-\kappa/2}}\bigg) + \int_0^T \mathbf{E}\bigg(\frac{1}{N}\sum_{i=1}^N \|Z_i(t)\|^2_{\mathbf{C}^{-\kappa/2}}\bigg) \lesssim T.$$

Defining $R_t^N := \frac{1}{t}\int_0^t P_r^N \,\mathrm{d}r$, the above estimates and the compactness of the embedding $\mathbf{C}^{-\kappa/2} \hookrightarrow \mathbf{C}^{-\kappa}$ imply the induced laws of $\{R_t^N\}_{t\geq 0}$ started from $(\tilde{Z}(0), \tilde{Z}(0))$ are tight on $(\mathbf{C}^{-\kappa})^{2N}$. Furthermore, by the continuity with respect to initial data, it is easy to check that $(P_t^N)_{t\geq 0}$ is Feller on $(\mathbf{C}^{-\kappa})^{2N}$. By the Krylov–Bogoliubov existence theorem (see [21], Corollary 3.1.2), these laws converge weakly in $(\mathbf{C}^{-\kappa})^{2N}$ along a subsequence $t_k \to \infty$ to an invariant measure π_N for $(P_t^N)_{t\geq 0}$. The desired stationary process $(\Phi_i^N, Z_i)_{1\leq i\leq N}$ is defined to be the unique solution to (2.1) and (2.2) obtained by sampling the initial datum $(\phi_i, z_i)_i$ from π_N. By (5.10), we deduce

$$\begin{aligned}
\mathbf{E}^{\pi_N}\|\Phi_i(0)-Z_i(0)\|^2_{H^1} &= \mathbf{E}^{\pi_N}\sup_{\varphi}\langle \Phi_i(0)-Z_i(0),\varphi\rangle^2 \\
&= \mathbf{E}\sup_{\varphi}\lim_{T\to\infty}\bigg[\frac{1}{T}\int_0^T \langle Y_i(t),\varphi\rangle\,\mathrm{d}t\bigg]^2 \\
&\leq \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathbf{E}\|Y_i(t)\|^2_{H^1}\,\mathrm{d}t \lesssim 1,
\end{aligned}$$

where $\sup_\varphi$ is over smooth functions φ with $\|\varphi\|_{H^{-1}} \leq 1$. Similarly using (5.11), (5.9) follows. Finally, we project onto the first component and consider the map $\bar{\Pi}_1 : \mathcal{S}'(\mathbb{T}^2)^{2N} \to \mathcal{S}'(\mathbb{T}^2)^N$ defined through $\bar{\Pi}_1(\Phi, Z) = \Phi$. Letting $\nu^N = \pi_N \circ \bar{\Pi}_1^{-1}$ yields an invariant measure to (2.1), and uniqueness follows from the general results of strong Feller property in [38], Theorem 3.2, and irreducibility in [40], Theorem 1.4. □

REMARK 5.8. Using a lattice approximation (see, e.g., [32], [37, 73]), one can show that the measure $\nu^N(\mathrm{d}\Phi)$ indeed has the form (1.2) (with Wick renormalization).

The next step is to study tightness of the marginal laws of ν^N over $\mathcal{S}'(\mathbb{T}^2)^N$. To this end, consider the projection $\Pi_i : \mathcal{S}'(\mathbb{T}^2)^N \to \mathcal{S}'(\mathbb{T}^2)$ defined by $\Pi_i(\Phi) = \Phi_i$ and let $\nu^{N,i} \stackrel{\text{def}}{=} \nu^N \circ \Pi_i^{-1}$ be the marginal law of the ith component. Furthermore, for $k \leq N$, define the map $\Pi^{(k)} : \mathcal{S}'(\mathbb{T}^2)^N \to \mathcal{S}'(\mathbb{T}^2)^k$ via $\Pi^{(k)}(\Phi) = (\Phi_i)_{1\leq i\leq k}$ and the let $\nu_k^N \stackrel{\text{def}}{=} \nu^N \circ (\Pi^{(k)})^{-1}$ be the marginal law of the first k components. We have the following result.

THEOREM 5.9. *$\{\nu^{N,i}\}_{N\geq 1}$ form a tight set of probability measures on $\mathbf{C}^{-\kappa}$ for $\kappa > 0$.*

PROOF. Let $(\Phi_i^N, Z_i)_{1\leq i\leq N}$ be the jointly stationary solutions to (2.1) and (2.2) constructed in Lemma 5.7. To prove the result, in light of the compact embedding of $\mathbf{C}^{-\kappa/2} \hookrightarrow \mathbf{C}^{-\kappa}$ and the stationarity of Φ_i^N, it suffices to show that the second moment of $\|\Phi_i^N(0)\|_{\mathbf{C}^{-\kappa/2}}$ is bounded uniformly in N. To this end, let $Y_i^N = \Phi_i^N - Z_i$, which is also stationary and note that

$$\begin{aligned}
\mathbf{E}\|\Phi_i^N(0)\|^2_{\mathbf{C}^{-\kappa/2}} &= \frac{2}{T}\int_{T/2}^T \mathbf{E}\|\Phi_i^N(s)\|^2_{\mathbf{C}^{-\kappa/2}}\,\mathrm{d}s \\
&\leq \frac{4}{T}\int_{T/2}^T \mathbf{E}\|Z_i(s)\|^2_{\mathbf{C}^{-\kappa/2}}\,\mathrm{d}s + \frac{4}{T}\int_{T/2}^T \mathbf{E}\|Y_i^N(s)\|^2_{H^1}\,\mathrm{d}s.
\end{aligned}$$

The first term is controlled by Lemma 5.1. For the second term, symmetry yields Y_i^N and Y_j^N are identical in law, which combined with Lemma 2.3 implies that

$$\frac{2}{T}\int_{T/2}^{T}\mathbf{E}\|Y_i^N(s)\|_{H^1}^2\,\mathrm{d}s=\frac{2}{T}\int_{T/2}^{T}\frac{1}{N}\sum_{i=1}^{N}\mathbf{E}\|Y_i^N(s)\|_{H^1}^2\,\mathrm{d}s\le\frac{C}{T}\mathbf{E}\Big[\int_0^T R_N\,\mathrm{d}t\Big]\le C,$$

where we used that by stationarity $\sum_{i=1}^N\mathbf{E}\|Y_i^N(T)\|_{L^2}^2=\sum_{i=1}^N\mathbf{E}\|Y_i^N(T/2)\|_{L^2}^2$, with both being finite in view of Lemma 5.7. For $m=0$, we also used the Poincaré inequality. □

REMARK 5.10. It is reasonable to expect that any limiting measure obtained in Theorem 5.9 is an invariant measure for (1.6) assuming only $m\ge 0$. However, this cannot be directly deduced from our main result in Section 4 because we do not know a priori that any limiting measure of ν^N is a product measure. This is problematic because the initial conditions for each component of Ψ_i are assumed to be independent in Section 4. Nevertheless, we can prove below that this is indeed true if m is large.

In the following, we prove the convergence of the measure to the unique invariant measure by using the estimate in Lemma 2.5, which requires m large enough.

Define the $\mathbf{C}^{-\kappa}$-Wasserstein distance

$$\mathbb{W}_2(\nu_1,\nu_2):=\inf_{\pi\in\mathscr{C}(\nu_1,\nu_2)}\Big(\int\|\phi-\psi\|_{\mathbf{C}^{-\kappa}}^2\pi(\mathrm{d}\phi,\mathrm{d}\psi)\Big)^{1/2},\tag{5.12}$$

where $\mathscr{C}(\nu_1,\nu_2)$ denotes the set of all couplings of ν_1, ν_2 satisfying $\int\|\phi\|_{\mathbf{C}^{-\kappa}}^2\nu_i(\mathrm{d}\phi)<\infty$ for $i=1,2$.

THEOREM 5.11. *Let $\nu=\mathcal{N}(0,\frac{1}{2}(m-\Delta)^{-1})$. There exist $C_0>0$ such that for all $m\ge m_1$ where*

$$m_1\stackrel{\text{def}}{=}C_0\big(\mathbf{E}\|Z_1\|_{\mathbf{C}^{-s}}^{\frac{2}{2-s}}+\mathbf{E}\|:Z_1^2:\|_{\mathbf{C}^{-s}}^{\frac{2}{2-s}}+E\|:Z_2Z_1:\|_{\mathbf{C}^{-s}}^2+1\big)$$

one has

$$\mathbb{W}_2(\nu^{N,i},\nu)\le CN^{-\frac{1}{2}}.\tag{5.13}$$

Furthermore, ν_k^N converges to $\nu\times\cdots\times\nu$, as $N\to\infty$.

PROOF. By Lemma 5.7, we may construct a stationary coupling (Φ_i^N,Z_i) of ν_N and ν whose components satisfy (1.1) and (2.2), respectively. The stationarity of the joint law of (Φ_i^N,Z_i) implies that also $Y_i^N=\Phi_i^N-Z_i$ is stationary. In the following, we freely omit the time argument of expectations of stationary quantities. We now claim that

$$\mathbf{E}\|Y_i^N\|_{H^1}^2\le CN^{-1},\tag{5.14}$$

which implies (5.13) by definition of the Wasserstein metric and the embedding $H^1\hookrightarrow\mathbf{C}^{-\kappa}$ in $d=2$; cf. Lemma A.1. To ease notation, we write $Y_i=Y_i^N$ in the following. By (2.26) combined with the stationarity of $(Y_j)_j$ and $(Z_j)_j$, we find

$$\sum_{j=1}^N\mathbf{E}\|\nabla Y_j\|_{L^2}^2+m\sum_{j=1}^N\mathbf{E}\|Y_j\|_{L^2}^2+\frac{1}{N}\mathbf{E}\Big\|\sum_{i=1}^N Y_i^2\Big\|_{L^2}^2$$
$$\le C\mathbf{E}R_N^0+\mathbf{E}\Big(\sum_{j=1}^N\|Y_j\|_{L^2}^2(D_N+D_N^1)\Big),$$

where R_N^0 is defined in (2.24) and

$$D_N = C\left(\frac{1}{N}\sum_{j=1}^{N}\|Z_j\|_{\mathbf{C}^{-s}}^{\frac{2}{2-s}} + \frac{1}{N}\sum_{j=1}^{N}\|:Z_j^2:\|_{\mathbf{C}^{-s}}^{\frac{2}{2-s}} + 1\right), \tag{5.15}$$

$$D_N^1 = C\left(\frac{1}{N^2}\sum_{i,j=1}^{N}\|:Z_jZ_i:\|_{\mathbf{C}^{-s}}^2\right). \tag{5.16}$$

Setting $A \stackrel{\text{def}}{=} \mathbf{E}D_N$ and $A_1 \stackrel{\text{def}}{=} \mathbf{E}\|:Z_2Z_1:\|_{\mathbf{C}^{-s}}^2$ we may recenter D_N and D_N^1 above and divide by N to obtain

$$\begin{aligned}
&\frac{1}{N}\sum_{j=1}^{N}\mathbf{E}\|\nabla Y_j(t)\|_{L^2}^2 + (m-A-A_1)\frac{1}{N}\sum_{j=1}^{N}\mathbf{E}\|Y_j(t)\|_{L^2}^2 + \frac{1}{N^2}\mathbf{E}\left\|\sum_{i=1}^{N}Y_i^2(t)\right\|_{L^2}^2 \\
&\quad \le C\frac{1}{N}\mathbf{E}R_N^0 + \frac{1}{N}\mathbf{E}\left(\sum_{j=1}^{N}\|Y_j(t)\|_{L^2}^2\big(|D_N-A| + |D_N^1 - A_1|\big)\right) \\
&\quad \le C\frac{1}{N}\mathbf{E}R_N^0 + \frac{1}{2}\frac{1}{N^2}\mathbf{E}\left(\sum_{j=1}^{N}\|Y_j\|_{L^2}^2\right)^2 + \mathbf{E}|D_N - A|^2 + \mathbf{E}|D_N^1 - A_1|^2.
\end{aligned}$$

For $m \ge A + A_1 + 1$, using that Y_i and Y_j are equal in law, we obtain

$$\begin{aligned}
\mathbf{E}\|Y_i\|_{H^1}^2 &\le \frac{1}{N}\sum_{j=1}^{N}\mathbf{E}\|\nabla Y_j(t)\|_{L^2}^2 + (m-A-A_1)\frac{1}{N}\sum_{j=1}^{N}\mathbf{E}\|Y_j\|_{L^2}^2 \\
&\le C\frac{1}{N}\mathbf{E}R_N^0 + \mathbf{E}|D_N-A|^2 + \mathbf{E}|D_N^1 - A_1|^2.
\end{aligned} \tag{5.17}$$

Using independence, we find

$$\mathbf{E}|D_N(t) - A|^2 \le \frac{1}{N}\mathrm{Var}\big(\|Z_1\|_{\mathbf{C}^{-s}}^{\frac{2}{2-s}} + \|:Z_1^2:\|_{\mathbf{C}^{-s}}^{\frac{2}{2-s}}\big) \le \frac{C}{N}. \tag{5.18}$$

To estimate D_N^1, we write $M_{i,j} = \|:Z_jZ_i:\|_{\mathbf{C}^{-s}}^2 - A_1$ for $i \neq j$ and $M_{i,i} = \|:Z_i^2:\|_{\mathbf{C}^{-s}}^2 - A_1$ and have

$$\begin{aligned}
\mathbf{E}\left(\frac{1}{N^2}\sum_{i,j=1}^{N}M_{i,j}\right)^2 &\le \mathbf{E}\left(\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j\neq i}M_{i,j} + \frac{1}{N^2}\sum_{i=1}^{N}M_{i,i}\right)^2 \\
&\le \frac{2}{N^4}\sum_{i_1\neq j_1, i\neq j}\mathbf{E}(M_{i,j}M_{i_1,j_1}) + \frac{2}{N^2}\mathbf{E}(M_{1,1}^2) \\
&\lesssim \frac{1}{N} + \frac{2}{N^2}\mathbf{E}(M_{1,1}^2) \lesssim \frac{1}{N},
\end{aligned}$$

where we used that for the case that (i, j, i_1, j_1) are different, $\mathbf{E}(M_{i,j}M_{i_1,j_1}) = \mathbf{E}M_{i,j} \times \mathbf{E}M_{i_1,j_1} = 0$.

Then we have

$$\mathbf{E}|D_N^1 - A_1|^2 \lesssim \frac{1}{N}. \tag{5.19}$$

Inserting the estimates (5.18), and (5.19) into (5.17) and using (5.4), we obtain (5.14), completing the proof. □

REMARK 5.12. Instead of assuming m large, one could alternatively consider arbitrary $m > 0$ and assume small nonlinearity. Namely, we could consider a nonlinearity $-\frac{\lambda}{N}\sum_{j=1}^N :\Phi_j^2\Phi_i:$ instead of that of (2.1), and $-\lambda\mathbf{E}[\Psi^2 - Z^2]\Psi$ instead of that of (5.1), for $\lambda > 0$. By tracing the proofs of Lemma 5.3 and Theorem 5.11, we can easily see that given any $m > 0$, there exists a constant $\lambda_0 > 0$, so that the statements of Lemma 5.3 and Theorem 5.11 hold for any $\lambda \in (0, \lambda_0)$.

REMARK 5.13. Following Remark 5.6, with a change of renormalization constant therein, we can write $\Phi_i = \bar{Y}_i + \bar{Z}_i$ with $\bar{Z}_i$ the stationary solution to $\mathscr{L}\bar{Z}_i = -\mu_0\bar{Z}_i + \xi_i$. Then $\bar{Y}_i$ satisfies

$$\mathscr{L}\bar{Y}_i = -\mu_0\bar{Y}_i - \frac{1}{N}\sum_{j=1}^N(\bar{Y}_j^2\bar{Y}_i + \bar{Y}_j^2\bar{Z}_i + 2\bar{Y}_j\bar{Y}_i\bar{Z}_j + 2\bar{Y}_j :\bar{Z}_i\bar{Z}_j: + :\bar{Z}_j^2: \bar{Y}_i + :\bar{Z}_i\bar{Z}_j^2:)$$
$$- \frac{2\mu_0}{N}(\bar{Y}_i + \bar{Z}_i),$$

which is the same case as (2.3) with m replaced by $m + \mu_0$ and an extra term $\frac{2\mu_0}{N}(\bar{Y}_i + \bar{Z}_i)$. Here, the Wick product of $\bar{Z}_j$ is defined similarly as in Section 2.1. By the same proof of Theorem 5.11, we know for $m + \mu_0$ large enough, $\nu^{N,i}$ (renormalized as in Remark 5.6) converges to $\bar{\nu}$ and the other results in Theorem 5.11 also hold in this case.

6. Observables and their nontriviality.

6.1. *Observables.* In quantum field theories with symmetries, quantities that are invariant under action of the symmetry group are of particular interest; examples of such quantities in the SPDE setting include gauge invariant observables, for example, [58], Section 2.4. The model we study here exhibits $O(N)$ rotation symmetry and formally functions of the squared "norm" $\sum_i \Phi_i^2$ are quantities that are $O(N)$ invariant. Of course, such observables need to be suitably renormalized to be well defined and suitably scaled by factors of N to have nontrivial limit as $N \to \infty$.

In this section, we study the following two observables:

$$\frac{1}{N^{1/2}}\sum_{i=1}^N :\Phi_i^2: , \qquad \frac{1}{N} :\left(\sum_{i=1}^N \Phi_i^2\right)^2: , \tag{6.1}$$

with $\Phi = (\Phi_i)_{1\le i\le N} \sim \nu^N$ for the invariant measure ν^N to (2.1) given in Lemma 5.7. In this section, we omit the superscript N for simplicity. These are defined as follows. By Lemma 5.7, we decompose $\Phi_i = Y_i + Z_i$ with (Y_i, Z_i) stationary. With this, we define

$$\frac{1}{\sqrt{N}}\sum_{i=1}^N :\Phi_i^2: \stackrel{\text{def}}{=} \frac{1}{\sqrt{N}}\sum_{i=1}^N(Y_i^2 + 2Y_iZ_i + :Z_i^2:), \tag{6.2}$$

$$\frac{1}{N} :\left(\sum_{i=1}^N \Phi_i^2\right)^2: \stackrel{\text{def}}{=} \frac{1}{N}\sum_{i,j=1}^N (Y_i^2Y_j^2 + 4Y_i^2Y_jZ_j + 2Y_i^2 :Z_j^2: \tag{6.3}$$

$$+ :Z_i^2Z_j^2: + 4Y_i :Z_iZ_j^2: + 4Y_iY_j :Z_iZ_j:). \tag{6.4}$$

Here, the Wick products are canonically defined as in (2.4) with $a_\varepsilon = \mathbf{E}[Z_{i,\varepsilon}^2(0,0)]$, in particular,

$$:Z_i^2Z_j^2: = \begin{cases} \lim_{\varepsilon\to 0}(Z_{i,\varepsilon}^4 - 6a_\varepsilon Z_{i,\varepsilon}^2 + 3a_\varepsilon^2) & (i = j), \\ \lim_{\varepsilon\to 0}(Z_{i,\varepsilon}^2 - a_\varepsilon)(Z_{j,\varepsilon}^2 - a_\varepsilon) & (i \neq j). \end{cases} \tag{6.5}$$

REMARK 6.1. One could also define (6.1) in $L^p(\nu^N)$ directly without using the decomposition $\Phi_i = Y_i + Z_i$. In fact, by similar argument as in [27] or [60], Section 8.6, one can show that ν^N is absolutely continuous with respect to the corresponding Gaussian free field $\tilde{\nu}$ with a density in $L^p(\tilde{\nu})$ for $p \in (1, \infty)$. Since (6.1) with each Φ_i replaced by Z_i can be defined via $L^p(\tilde{\nu})$ limit of mollification, using argument along the line of [57], Lemma 3.6, we know that (6.1) can be also defined as $L^p(\nu^N)$ limit of mollification (essentially Hölder inequality), and they have the same law as the right-hand side of (6.2) and (6.3), (6.4).

In this section, we also consider Y_i, Z_i as stationary process with Z_i as the stationary solution of (5.2) and Y_i as the solution of (2.3).

LEMMA 6.2. *There exists an m_0 such that for $m \geq m_0$ and $q \geq 1$*

$$\mathbf{E}\left[\left(\sum_{i=1}^N \|Y_i\|_{L^2}^2\right)^q\right] + \mathbf{E}\left[\left(\sum_{i=1}^N \|Y_i\|_{L^2}^2 + 1\right)^q \left(\sum_{i=1}^N \|\nabla Y_i\|_{L^2}^2\right)\right] \lesssim 1, \tag{6.6}$$

$$\mathbf{E}\left[\left(\sum_{i=1}^N \|Y_i\|_{L^2}^2 + 1\right)^q \left\|\sum_{i=1}^N Y_i^2\right\|_{L^2}^2\right] \lesssim 1, \tag{6.7}$$

where the implicit constant is independent of N.

PROOF. First, we observe that (6.7) may be quickly deduced from (6.6) with the help of the inequality

$$\left\|\sum_{i=1}^N Y_i^2\right\|_{L^2}^2 \lesssim \left(\sum_{i=1}^N \|Y_i\|_{H^1}^2\right)\left(\sum_{i=1}^N \|Y_i\|_{L^2}^2\right). \tag{6.8}$$

To obtain (6.8), note first that

$$\left\|\sum_{i=1}^N Y_i^2\right\|_{L^2}^2 = \sum_{i,j=1}^N \|Y_i Y_j\|_{L^2}^2.$$

Furthermore, by Hölder's inequality, (4.3) and Young's inequality,

$$\begin{aligned}\|Y_i Y_j\|_{L^2}^2 \leq \|Y_i\|_{L^4}^2 \|Y_j\|_{L^4}^2 &\lesssim \|Y_i\|_{H^1} \|Y_j\|_{L^2} \|Y_j\|_{H^1} \|Y_i\|_{L^2} \\ &\lesssim \|Y_i\|_{H^1}^2 \|Y_j\|_{L^2}^2 + \|Y_i\|_{H^1}^2 \|Y_j\|_{L^2}^2.\end{aligned}$$

Summing both sides over i, j and using symmetry with respect to the roles of i and j, we obtain (6.8). The remainder of the proof is devoted to (6.6).

To shorten the expressions that follow, we introduce the quantities $F \stackrel{\text{def}}{=} \sum_{i=1}^N \|\nabla Y_i\|_{L^2}^2 + \frac{1}{N}\|\sum_{i=1}^N Y_i^2\|_{L^2}^2$ and $U \stackrel{\text{def}}{=} \sum_{i=1}^N \|Y_i\|_{L^2}^2$. Note that F and U are stationary, so we will freely omit the time argument below. Our starting point is the key inequality (2.26), which may be recast in terms of U and F as

$$\frac{\mathrm{d}}{\mathrm{d}t} U + F + mU \leq C R_N^0 + C(D_N + D_N^1)U,$$

where D_N and D_N^1 are introduced in (5.15) and (5.16). Muliplying the above by U^{q-1} we find that for $q \geq 1$ it holds

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t} U^q + U^{q-1}F + mU^q \leq C R_N^0 U^{q-1} + C(D_N + D_N^1)U^q.$$

As in the proof of Theorem 5.11, we now define $A \stackrel{\text{def}}{=} \mathbf{E}(D_N)$ and $A_1 \stackrel{\text{def}}{=} \mathbf{E}\| :Z_1 Z_2: \|^2_{\mathbf{C}^{-s}}$. Subtract the mean from $D_N + D_N^1$ and take expectation on both sides to find

$$\begin{aligned}
&\mathbf{E}\big[U^{q-1}F\big] + (m - A - A_1)\mathbf{E}\big[U^q\big] \\
&\quad \le C\mathbf{E}\big[R_N^0 U^{q-1}\big] + C\mathbf{E}\big[(D_N + D_N^1 - A - A_1)U^q\big]. \\
&\quad \le C\|R_N^0\|_{L^q(\Omega)}\big(\mathbf{E}U^q\big)^{\frac{q-1}{q}} + C\|D_N - A + D_N^1 - A_1\|_{L^{q+1}(\Omega)}\big(\mathbf{E}U^{q+1}\big)^{\frac{q}{q+1}} \\
&\quad \le C\big(\mathbf{E}U^q\big)^{\frac{q-1}{q}} + CN^{-\frac{1}{2}}\big(\mathbf{E}U^{q+1}\big)^{\frac{q}{q+1}},
\end{aligned}$$

where we used $\mathbf{E}U^q(t) = \mathbf{E}U^q(0)$ in the first inequality and we used a Gaussian hypercontractivity upgrade of (5.18) and (5.19) in the last line. Using Young's inequality, we may absorb the first term to the left and obtain

$$\mathbf{E}\big[U^{q-1}F\big] + (m - A - A_1 - 1)\mathbf{E}\big[U^q\big] \le C + CN^{-\frac{1}{2}}\big(\mathbf{E}U^{q+1}\big)^{\frac{q}{q+1}}. \tag{6.9}$$

The strategy now is to first use the dissipative quantity on the left-hand side of (6.9) to obtain $\mathbf{E}(U^q) \le CN^{\frac{q-1}{2}}$, and then use the massive term on the left-hand side of (6.9) to iteratively decrease the power of N and eventually arrive at $\mathbf{E}(U^q) \le C$. Once this is established, plugging the bound back into (6.9) completes the proof.

Indeed, first observe that $F \ge N^{-1}U^2$ so that $\mathbf{E}(U^{q-1}F) \ge N^{-1}\mathbf{E}(U^{q+1})$. Hence, Young's inequality with exponents $(q+1, \frac{q+1}{q})$ leads to $\mathbf{E}(U^q) \le CN^{\frac{q-1}{2}}$. Defining $A_q \stackrel{\text{def}}{=} \mathbf{E}U^q$ and discarding the dissipative term, (6.9) implies

$$A_q \lesssim A_{q+1}^{\frac{q}{q+1}} N^{-1/2} + 1. \tag{6.10}$$

We have $A_q \lesssim N^{\frac{q-1}{2}}$, which gives

$$A_q \lesssim N^{\frac{q^2}{2(q+1)} - \frac{1}{2}} + 1.$$

Substituting into (6.10) and use induction we have for $n \ge 1$,

$$A_q \lesssim N^{a_{n,q}} + 1, \tag{6.11}$$

with $a_{n,q} = \frac{q(q+n-1)}{2(q+n)} - \frac{q}{2}(\sum_{k=1}^{n-1}\frac{1}{q+k}) - \frac{1}{2}$. Here, $\sum_{k=1}^{0} = 0$ and the proportional constant may depend on n. In fact, we could check (6.11) by

$$A_q \lesssim \big(N^{a_{n,q+1}}\big)^{\frac{q}{q+1}} N^{-\frac{1}{2}} + 1 \lesssim N^{a_{n+1,q}} + 1.$$

For fixed $q \ge 1$, we could always find n large enough such that $a_{n,q} < 0$, which implies that $A_q \lesssim 1$ and the result follows. Here, we emphasize that even if the proportional constant depends on n, which is a fixed constant for given q, the bound for A_q depends on q and is independent of N as required. □

THEOREM 6.3. *Let $\Phi = (\Phi_i)_{1\le i\le N} \sim \nu^N$ and m be given as in Lemma 6.2, then the laws of $\frac{1}{\sqrt{N}}\sum_{i=1}^N :\Phi_i^2:$ are tight on $B_{2,2}^{-2\kappa}$, and the laws of $\frac{1}{N} :(\sum_{i=1}^N \Phi_i^2)^2:$ are tight on $B_{1,1}^{-3\kappa}$.*

PROOF. Note that the first term $\frac{1}{\sqrt{N}}\sum_{i=1}^N Y_i^2$ on the right-hand side of (6.2) converges to zero in $L^2(\Omega; L^2)$ as an immediate consequence of (6.7); so we can actually prove a stronger result than stated, namely the subsequential limits can be identified with those of the last two terms in (6.2). We will now show that the other two quantities induce tight laws on $B^{-3\kappa}_{2,2}$, which implies the first part of the theorem. The second sum in (6.2) can be estimated using Lemma A.3 and Lemma A.1 to find for $s \in (\kappa, 2\kappa)$,

$$\begin{aligned}
\Big\| \frac{1}{\sqrt{N}} \sum_{i=1}^N Y_i Z_i \Big\|_{B^{-s}_{2,2}} &\lesssim \frac{1}{\sqrt{N}} \sum_{i=1}^N \|Y_i\|_{B^s_{2,2}} \|Z_i\|_{\mathbf{C}^{-\kappa}} \\
&\lesssim \sum_{i=1}^N \|Y_i\|^2_{B^s_{2,2}} + \frac{1}{N} \sum_{i=1}^N \|Z_i\|^2_{\mathbf{C}^{-\kappa}} \\
&\lesssim \sum_{i=1}^N \|Y_i\|^{2s}_{H^1} \|Y_i\|^{2(1-s)}_{L^2} + \frac{1}{N} \sum_{i=1}^N \|Z_i\|^2_{\mathbf{C}^{-\kappa}} \\
&\lesssim \sum_{i=1}^N \|Y_i\|^2_{H^1} + \sum_{i=1}^N \|Y_i\|^2_{L^2} + \frac{1}{N} \sum_{i=1}^N \|Z_i\|^2_{\mathbf{C}^{-\kappa}},
\end{aligned}$$

which is bounded in expectation by a constant using Lemma 6.2 for $q = 1$ and Lemma 5.1. For the third sum in (6.2), we use independence to find for $s \in (\kappa, 2\kappa)$,

$$\mathbf{E} \Big\| \frac{1}{\sqrt{N}} \sum_{i=1}^N :Z_i^2: \Big\|^2_{B^{-s}_{2,2}} = \mathbf{E} \frac{1}{N} \Big\langle \Lambda^{-s} \sum_{i=1}^N :Z_i^2: , \Lambda^{-s} \sum_{i=1}^N :Z_i^2: \Big\rangle = \mathbf{E} \| :Z_i^2: \|^2_{B^{-s}_{2,2}} \lesssim 1.$$

By the triangle inequality and the embedding $L^2 \hookrightarrow B^{-\kappa}_{2,2}$ we find that $\frac{1}{\sqrt{N}} \sum_{i=1}^N :\Phi_i^2:$ is bounded in $L^1(\Omega; B^{-s}_{2,2})$. In light of the compact embedding $B^{-s}_{2,2} \subset B^{-2\kappa}_{2,2}$, the tightness claim follows.

For the second observable, we will also show a stronger result: the subsequential limits can be identified with those of the last three terms in (6.4). We start with the first 3 terms in (6.3), which will be shown to converge to zero in $L^1(\Omega; B^{-2s}_{1,1})$ for $s > \kappa$. For the first term of (6.3), we use Lemma 6.2 to obtain

$$\mathbf{E} \Big\| \frac{1}{N} \sum_{i,j=1}^N Y_i^2 Y_j^2 \Big\|_{L^1} = \frac{1}{N} \mathbf{E} \Big\| \sum_{i=1}^N Y_i^2 \Big\|^2_{L^2} \lesssim \frac{1}{N},$$

so this term converges to zero in $L^1(\Omega; L^1)$. For the second term of (6.3), using (2.22) of Lemma 2.3 with φZ_j in place of Z_j we obtain

$$\begin{aligned}
&\sup_{\|\varphi\|_{\mathbf{C}^{2s}} \le 1} \Big| \frac{1}{N} \sum_{i,j=1}^N \langle Y_i^2 Y_j Z_j, \varphi \rangle \Big| \\
&\lesssim \frac{1}{N} \Big\| \sum_{i=1}^N Y_i^2 \Big\|_{L^2} \Big(\sum_{j=1}^N \|Y_j\|^2_{L^2} \Big)^{\frac{1-s}{2}} \Big(\sum_{j=1}^N \|\nabla Y_j\|^2_{L^2} \Big)^{\frac{s}{2}} \Big(\sum_{j=1}^N \|Z_j\|^2_{\mathbf{C}^{-s}} \Big)^{1/2} \\
&\quad + \frac{1}{N} \Big\| \sum_{i=1}^N Y_i^2 \Big\|_{L^2} \Big(\sum_{j=1}^N \|Y_j\|^2_{L^2} \Big)^{\frac{1}{2}} \Big(\sum_{j=1}^N \|Z_j\|^2_{\mathbf{C}^{-s}} \Big)^{1/2}.
\end{aligned}$$

Hence, using Young's inequality we find for $\delta > 0$ small enough

$$\Big\| \frac{1}{N} \sum_{i,j=1}^N Y_i^2 Y_j Z_j \Big\|_{B^{-2s}_{1,1}} \le N^{-\delta} \Big\| \sum_{i=1}^N Y_i^2 \Big\|^2_{L^2} + N^{-\delta} \Big(\sum_{j=1}^N \|\nabla Y_j\|^2_{L^2} \Big)$$

$$+N^{-\delta}\Big(\sum_{j=1}^N \|Y_j\|_{L^2}^2\Big)\Big(\frac{1}{N}\sum_{j=1}^N \|Z_j\|_{\mathbf{C}^{-s}}^2+1\Big)^{\frac{1}{1-s}}.$$

Both terms above converge to zero in $L^1(\Omega)$ as a consequence of (6.6), (6.7) and Lemma 5.1. For the third term in (6.3), a calculation similar to (2.16) using Lemma A.3 with Y_i replaced by φY_i with $\varphi\in\mathbf{C}^{2s}$ yields

$$\begin{aligned}
&\sup_{\|\varphi\|_{\mathbf{C}^{2s}}\le 1}\Big|\frac{1}{N}\sum_{i,j=1}^N \langle Y_i^2 :Z_j^2: ,\varphi\rangle\Big| \\
&\lesssim \frac{1}{N}\sum_{i=1}^N \|\Lambda^s(Y_i^2)\|_{L^2}\Big\|\sum_{j=1}^N \Lambda^{-s}(:Z_j^2:)\Big\|_{L^2} \\
&\lesssim \Big(\sum_{i=1}^N \|\nabla Y_i\|_{L^2}^{1+s}\|Y_i\|_{L^2}^{1-s}+\|Y_i^2\|_{L^2}\Big)\Big\|\frac{1}{N}\sum_{j=1}^N \Lambda^{-s}(:Z_j^2:)\Big\|_{L^2} \qquad (6.12)\\
&\lesssim \Big(\sum_{i=1}^N \|Y_i\|_{H^1}^2\Big)^{\frac{1+s}{2}}\Big(\sum_{i=1}^N \|Y_i\|_{L^2}^2\Big)^{\frac{1-s}{2}}\Big\|\frac{1}{N}\sum_{j=1}^N \Lambda^{-s}(:Z_j^2:)\Big\|_{L^2},
\end{aligned}$$

where we used (A.1) and Lemma A.2 to have

$$\|\Lambda^s(Y_i^2)\|_{L^2}\lesssim \|\Lambda^s Y_i\|_{L^4}\|Y_i\|_{L^4}\lesssim \|\nabla Y_i\|_{L^2}^{1+s}\|Y_i\|_{L^2}^{1-s}+\|Y_i\|_{L^4}^2.$$

The first part of the product in (6.12) is bounded in $L^1(\Omega)$ by (6.6). For the second part of the product, we use independence to obtain

$$\mathbf{E}\Big\|\frac{1}{N}\sum_{j=1}^N \Lambda^{-s}(:Z_j^2:)\Big\|_{L^2}^2\lesssim \frac{1}{N^2}\sum_{j=1}^N \mathbf{E}\|\Lambda^{-s}(:Z_j^2:)\|_{L^2}^2\lesssim \frac{1}{N},$$

so together we find $\mathbf{E}\|\frac{1}{N}\sum_{i,j} Y_i^2 :Z_j^2: \|_{B^{-2s}_{1,1}}$ converges to 0.

We now turn to terms in (6.4) and derive suitable moment bounds. For the first of these terms, we have

$$\begin{aligned}
\mathbf{E}\Big\|\frac{1}{N}\sum_{i,j=1}^N :Z_i^2Z_j^2:\Big\|_{B^{-\kappa}_{2,2}}^2&=\mathbf{E}\frac{1}{N^2}\Big\langle \Lambda^{-\kappa}\sum_{i,j=1}^N :Z_i^2Z_j^2: ,\Lambda^{-\kappa}\sum_{i,j=1}^N :Z_i^2Z_j^2:\Big\rangle\\
&\lesssim \mathbf{E}\| :Z_1^2Z_2^2: \|_{B^{-\kappa}_{2,2}}^2+\frac{1}{N}\mathbf{E}\| :Z_1^4: \|_{B^{-\kappa}_{2,2}}^2\lesssim 1.
\end{aligned}$$

For the next term, using (2.25) with φY_i in place of Y_i we find

$$\begin{aligned}
&\sup_{\|\varphi\|_{\mathbf{C}^{2s}}\le 1}\Big|\frac{1}{N}\sum_{i,j=1}^N \langle Y_i :Z_iZ_j^2: ,\varphi\rangle\Big|\\
&\lesssim \Big(\sum_{i=1}^N \|\Lambda^s Y_i\|_{L^2}^2\Big)^{1/2}\Big(\frac{1}{N^2}\sum_{i=1}^N\Big\|\sum_{j=1}^N \Lambda^{-s}(:Z_j^2Z_i:)\Big\|_{L^2}^2\Big)^{1/2}\\
&\lesssim \Big(\sum_{i=1}^N \|Y_i\|_{H^1}^2\Big)^{s/2}\Big(\sum_{i=1}^N \|Y_i\|_{L^2}^2\Big)^{(1-s)/2}\Big(\frac{1}{N^2}\sum_{i=1}^N\Big\|\sum_{j=1}^N \Lambda^{-s}(:Z_j^2Z_i:)\Big\|_{L^2}^2\Big)^{1/2}.
\end{aligned}$$

Using (5.4) and Lemma 6.2, we deduce for some p satisfying $sp < 2$,

$$\mathbf{E}\Big\| \frac{1}{N} \sum_{i,j=1}^{N} Y_i ,\ {:}Z_i Z_j^2{:} \ \Big\|_{B_{1,1}^{-2s}}^{p} \lesssim 1.$$

For the last term, we argue similar to (2.13) but ${:}Z_i Z_j{:}$ replace by $\varphi\, {:}Z_i Z_j{:}$ for $\varphi \in \mathbf{C}^s$ to deduce boundedness in $L^1(\Omega; B_{1,1}^{-2s})$.

Combining the above observations with the triangle inequality, we find that the second observable is uniformly bounded in probability as a $B_{1,1}^{-2s}$ valued random variable. By compactness of the embedding of $B_{1,1}^{-2s}$ into $B_{1,1}^{-2s-\delta}$ for $\delta > 0$, we obtain the result. □

6.2. *L^p-estimate.* In light of Lemma 6.2 and the Sobolev embedding theorem, it follows that for each component i, $\mathbf{E}\|Y_i\|_{L^p}^2 \lesssim 1$, $p > 1$. Our goal now in this subsection is to upgrade from the second moment to higher moments of the L^p norm. We do so by revisiting the energy estimates for the PDE (2.3). Since we work with a fixed component rather than an aggregate quantity, these bounds come with a price: the estimate is no longer uniform in N. Nonetheless, the power of N that appears is ultimately small enough for a successful application of the estimate in Lemma 6.6, en route to Theorem 6.5.

LEMMA 6.4. *Let m as in Lemma 6.2 and $p > 2$. For each component i, it holds*

$$\mathbf{E}\|Y_i\|_{L^p}^p + \mathbf{E}\big\|Y_i^{p-2}|\nabla Y_i|^2\big\|_{L^1} + \frac{1}{N}\sum_{j=1}^{N} \mathbf{E}\big\|Y_i^p Y_j^2\big\|_{L^1} \lesssim N^{p/2},$$

where the implicit constant is independent of N and i.

PROOF. Fix a component i. Given $p > 2$, let $s > 0$ be a small number to be selected (depending on p) in the final step of the proof. We will perform an L^p estimate: integrating (2.3) against $|Y_i|^{p-2}Y_i$ we obtain

$$\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|Y_i\|_{L^p}^p + (p-1)\big\||Y_i|^{p-2}|\nabla Y_i|^2\big\|_{L^1} + \frac{1}{N}\sum_{j=1}^{N}\big\||Y_i|^p Y_j^2\big\|_{L^1} + m\|Y_i\|_{L^p}^p \\
&= -2\Big\langle \frac{1}{N}\sum_{j=1}^{N} Y_j Z_j, |Y_i|^p \Big\rangle - \Big\langle \frac{1}{N}\sum_{j=1}^{N} Y_j^2 |Y_i|^{p-2} Y_i, Z_i \Big\rangle \\
&\quad - 2\Big\langle \frac{1}{N}\sum_{j=1}^{N} Y_j \ {:}Z_j Z_i{:} \ , |Y_i|^{p-2} Y_i \Big\rangle \\
&\quad + \Big\langle \frac{1}{N}\sum_{j=1}^{N} {:}Z_j^2{:} \ , |Y_i|^p \Big\rangle + \Big\langle \frac{1}{N}\sum_{j=1}^{N} {:}Z_i Z_j^2{:} \ , Y_i |Y_i|^{p-2} \Big\rangle =: \sum_{k=1}^{5} I_k.
\end{aligned} \tag{6.13}$$

Define $D \stackrel{\text{def}}{=} \|Y_i^{p-2}|\nabla Y_i|^2\|_{L^1}$ and for each j, let $A_j \stackrel{\text{def}}{=} \|Y_i^p Y_j^2\|_{L^1}$.

STEP 1 (Estimate of I_1) In this step, we show that

$$I_1 \le \frac{1}{10}\frac{1}{N}\sum_{j=1}^{N} A_j + \frac{1}{10} D + C\|Y_i\|_{L^p}^p F, \tag{6.14}$$

where

$$F \stackrel{\text{def}}{=} \frac{1}{N}\sum_{j=1}^{N} \|Y_j\|_{H^1}^{2s} \|Z_j\|_{\mathbf{C}^{-s}}^2 + \frac{1}{N}\sum_{j=1}^{N} \|Z_j\|_{\mathbf{C}^{-s}}^{2/(1-s)} + 1.$$

In the following, we prove (6.14). By Lemma A.5, we have

$$I_1 \lesssim \frac{1}{N}\sum_{j=1}^{N}\big[\|Y_j|Y_i|^p\|_{L^1}\|Z_j\|_{\mathbf{C}^{-s}}\big] + \frac{1}{N}\sum_{j=1}^{N}\big[\|Y_j|Y_i|^p\|_{L^1}^{1-s}\|\nabla(Y_j|Y_i|^p)\|_{L^1}^{s}\|Z_j\|_{\mathbf{C}^{-s}}\big]$$
$$=: I_1^{(1)} + I_1^{(2)}.$$

To estimate $I_1^{(1)}$, notice that the Cauchy–Schwarz inequality yields

$$\|Y_j|Y_i|^p\|_{L^1} \le \|Y_j|Y_i|^{\frac{p}{2}}\|_{L^2}\||Y_i|^{\frac{p}{2}}\|_{L^2} = A_j^{1/2}\|Y_i\|_{L^p}^{\frac{p}{2}}. \tag{6.15}$$

Combining this with Young's inequality, we obtain

$$I_1^{(1)} \le \frac{1}{N}\sum_{j=1}^{N}\big[A_j^{1/2}\|Y_i\|_{L^p}^{\frac{p}{2}}\|Z_j\|_{\mathbf{C}^{-s}}\big] \le \frac{1}{10}\frac{1}{N}\sum_{j=1}^{N}A_j + C\|Y_i\|_{L^p}^{p}\frac{1}{N}\sum_{j=1}^{N}\|Z_j\|_{\mathbf{C}^{-s}}^{2}.$$

To estimate $I_1^{(2)}$, note first that $\nabla(Y_j|Y_i|^p) = \nabla Y_j(|Y_i|^{\frac{p}{2}})^2 + 2Y_j|Y_i|^{\frac{p}{2}}\nabla|Y_i|^{\frac{p}{2}}$. Hence, using Hölder's inequality followed by Gagliardo–Nirenberg (A.2) with $(s,q,r,\alpha) = (0,4,2,\frac{1}{2})$,

$$\begin{aligned}\|\nabla(Y_j|Y_i|^p)\|_{L^1} &\le \|\nabla Y_j\|_{L^2}\||Y_i|^{\frac{p}{2}}\|_{L^4}^2 + 2\|Y_j|Y_i|^{\frac{p}{2}}\|_{L^2}\|\nabla|Y_i|^{\frac{p}{2}}\|_{L^2}\\ &\lesssim \||Y_i|^{\frac{p}{2}}\|_{H^1}\||Y_i|^{\frac{p}{2}}\|_{L^2}\|Y_j\|_{H^1} + \sqrt{A_jD}.\end{aligned}$$

Since $\||Y_i|^{\frac{p}{2}}\|_{H^1} \lesssim D^{\frac{1}{2}} + \||Y_i|^{\frac{p}{2}}\|_{L^2}$, using (6.15) again and $\||Y_i|^{\frac{p}{2}}\|_{L^2} = \||Y_i|^p\|_{L^1}^{1/2}$ yields

$$\begin{aligned}I_1^{(2)} &\lesssim \frac{1}{N}\sum_{j=1}^{N}\big[A_j^{\frac{1-s}{2}}\||Y_i|^p\|_{L^1}^{\frac{1-s}{2}}\big(D^{\frac{s}{2}}\||Y_i|^p\|_{L^1}^{\frac{s}{2}}\|Y_j\|_{H^1}^{s} + \||Y_i|^p\|_{L^1}^{s}\|Y_j\|_{H^1}^{s} + A_j^{\frac{s}{2}}D^{\frac{s}{2}}\big)\|Z_j\|_{\mathbf{C}^{-s}}\big]\\
&\le \frac{1}{10}\frac{1}{N}\sum_{j=1}^{N}A_j + \frac{1}{10}D + C\|Y_i\|_{L^p}^{p}\Bigg(\frac{1}{N}\sum_{j=1}^{N}\|Y_j\|_{H^1}^{2s}\|Z_j\|_{\mathbf{C}^{-s}}^{2}\\
&\quad + \frac{1}{N}\sum_{j=1}^{N}\|Y_j\|_{H^1}^{\frac{2s}{1+s}}\|Z_j\|_{\mathbf{C}^{-s}}^{\frac{2}{1+s}} + \frac{1}{N}\sum_{j=1}^{N}\|Z_j\|_{\mathbf{C}^{-s}}^{2/(1-s)}\Bigg)\\
&\le \frac{1}{10}\frac{1}{N}\sum_{j=1}^{N}A_j + \frac{1}{10}D + C\|Y_i\|_{L^p}^{p}F,\end{aligned}$$

where the second inequality follows from three applications of Young's inequality. We view the summand in first term as $A_j^{\frac{1-s}{2}}D^{\frac{s}{2}}(\|Y_i\|_{L^p}^{\frac{p}{2}}\|Y_j\|_{H^1}^{s}\|Z_j\|_{\mathbf{C}^{-s}})$ and use exponents $(\frac{2}{1-s},\frac{2}{s},2)$. We view the second summand as $A_j^{\frac{1-s}{2}}(\|Y_i\|_{L^p}^{\frac{p(1+s)}{2}}\|Y_j\|_{H^1}^{s}\|Z_j\|_{\mathbf{C}^{-s}})$ and we use exponents $(\frac{2}{1-s},\frac{2}{1+s})$. Finally, the third summand is viewed as $A_j^{\frac{1}{2}}D^{\frac{s}{2}}(\|Y_i\|_{L^p}^{\frac{p(1-s)}{2}}\|Z_j\|_{\mathbf{C}^{-s}})$ and we use exponents $(2,\frac{2}{s},\frac{2}{1-s})$.

STEP 2 (Estimates for I_2) In this step, we show that

$$I_2 \le \frac{1}{10}\Bigg\|\frac{1}{N}\sum_{j=1}^{N}Y_j^2Y_i^p\Bigg\|_{L^1} + \frac{1}{10}D + C\|Y_i\|_{L^p}^{p} + CF_1, \tag{6.16}$$

where

$$F_1 \stackrel{\text{def}}{=} \Big\|\frac{1}{N}\sum_{j=1}^N Y_j^2\Big\|_{L^1}\|Z_i\|_{\mathbf{C}^{-s}}^p + \Big\|\frac{1}{N}\sum_{j=1}^N Y_j^2\Big\|_{L^1}\Big(\frac{1}{N}\sum_{j=1}^N\|Y_j\|_{H^1}^2\Big)^{\frac{sp}{1-s}}\|Z_i\|_{\mathbf{C}^{-s}}^{\frac{p}{1-s}} + 1.$$

To prove (6.16), by Lemma A.5 one has

$$\begin{aligned} I_2 &= \frac{1}{N}\sum_{j=1}^N \big\langle (Y_j^2 Y_i |Y_i|^{p-2}), Z_i\big\rangle \\ &\lesssim \Bigg(\Big\|\frac{1}{N}\sum_{j=1}^N Y_j^2 Y_i|Y_i|^{p-2}\Big\|_{L^1} \\ &\quad + \Big\|\frac{1}{N}\sum_{j=1}^N \nabla\big(Y_j^2 Y_i|Y_i|^{p-2}\big)\Big\|_{L^1}^s \Big\|\frac{1}{N}\sum_{j=1}^N Y_j^2 Y_i|Y_i|^{p-2}\Big\|_{L^1}^{1-s}\Bigg)\|Z_i\|_{\mathbf{C}^{-s}}. \end{aligned}$$

Using the triangle inequality, writing $Y_j^2|Y_i|^{p-1} = (Y_j^2|Y_i|^p)^{\frac{p-1}{p}}(Y_j^2)^{\frac{1}{p}}$ and using Hölder's inequality,

$$\Big\|\frac{1}{N}\sum_{j=1}^N Y_j^2 Y_i|Y_i|^{p-2}\Big\|_{L^1} \lesssim \Big\|\frac{1}{N}\sum_{j=1}^N Y_j^2|Y_i|^p\Big\|_{L^1}^{\frac{p-1}{p}}\Big\|\frac{1}{N}\sum_{j=1}^N Y_j^2\Big\|_{L^1}^{\frac{1}{p}}.$$

Another application of Hölder's inequality gives

$$\|\nabla(Y_i|Y_i|^{p-2})\|_{L^{\frac{p}{p-1}}} \lesssim \||\nabla Y_i|Y_i^{\frac{p-2}{2}}\|_{L^2}\||Y_i|^{\frac{p-2}{2}}\|_{L^{\frac{2p}{p-2}}} = D^{1/2}\||Y_i|^p\|_{L^1}^{\frac{p-2}{2p}}. \tag{6.17}$$

For $r > 2$, using Hölder's inequality and the Sobolev embedding $H^1 \subset L^{\frac{2r}{r-2}}$ we have

$$\begin{aligned} &\Big\|\frac{1}{N}\sum_{j=1}^N \nabla\big(Y_j^2 Y_i|Y_i|^{p-2}\big)\Big\|_{L^1} \\ &\quad\lesssim \frac{1}{N}\sum_{j=1}^N \|\nabla(Y_i|Y_i|^{p-2})\|_{L^{\frac{p}{p-1}}}\|Y_j^2\|_{L^p} + \|\nabla Y_j\|_{L^2}\|Y_j\|_{L^{\frac{2r}{r-2}}}\|Y_i|Y_i|^{p-2}\|_{L^r} \\ &\quad\lesssim \frac{1}{N}\sum_{j=1}^N D^{1/2}\||Y_i|^p\|_{L^1}^{\frac{p-2}{2p}}\|Y_j^2\|_{L^p} + \||Y_i|^{\frac{p}{2}}\|_{H^1}^{2(1-\frac{1}{p})}\frac{1}{N}\sum_{j=1}^N\|Y_j\|_{H^1}^2 \\ &\quad\lesssim \Big(\frac{1}{N}\sum_{j=1}^N\|Y_j\|_{H^1}^2\Big)\big(\||Y_i|^{p/2}\|_{H^1}^{2(1-\frac{1}{p})} + D^{1/2}\||Y_i|^p\|_{L^1}^{\frac{p-2}{2p}}\big), \end{aligned}$$

where in the second inequality we used that $\|Y_i|Y_i|^{p-2}\|_{L^r} = \||Y_i|^{\frac{p}{2}}\|_{L^q}^{2(1-\frac{1}{p})}$ for $q = \frac{2r(p-1)}{p}$ and again the Sobolev embedding theorem. Combining the above estimates, we deduce

$$\begin{aligned} I_2 &\lesssim \Big\|\frac{1}{N}\sum_{j=1}^N Y_j^2|Y_i|^p\Big\|_{L^1}^{\frac{p-1}{p}}\Big\|\frac{1}{N}\sum_{j=1}^N Y_j^2\Big\|_{L^1}^{\frac{1}{p}}\|Z_i\|_{\mathbf{C}^{-s}} \\ &\quad + \Big\|\frac{1}{N}\sum_{j=1}^N Y_j^2|Y_i|^p\Big\|_{L^1}^{\frac{(p-1)(1-s)}{p}}\Big\|\frac{1}{N}\sum_{j=1}^N Y_j^2\Big\|_{L^1}^{\frac{1-s}{p}}\Big(\frac{1}{N}\sum_{j=1}^N\|Y_j\|_{H^1}^2\Big)^s \end{aligned}$$

$$
\begin{aligned}
&\times \big(\||Y_i|^{\frac{p}{2}}\|_{H^1}^{2(1-\frac{1}{p})} + D^{1/2}\||Y_i|^p\|_{L^1}^{\frac{p-2}{2p}}\big)^s\|Z_i\|_{\mathbf{C}^{-s}} \\
&\lesssim \frac{1}{10}\bigg\|\frac{1}{N}\sum_{j=1}^N Y_j^2|Y_i|^p\bigg\|_{L^1} + \frac{1}{10}D + \bigg\|\frac{1}{N}\sum_{j=1}^N Y_j^2\bigg\|_{L^1}\|Z_i\|_{\mathbf{C}^{-s}}^p \\
&\quad + \bigg\|\frac{1}{N}\sum_{j=1}^N Y_j^2\bigg\|_{L^1}\bigg(\frac{1}{N}\sum_{j=1}^N \|Y_j\|_{H^1}^2\bigg)^{\frac{sp}{1-s}}\|Z_i\|_{\mathbf{C}^{-s}}^{\frac{p}{1-s}}. + C\|Y_i\|_{L^p}^p + 1,
\end{aligned}
$$

where we applied Cauchy's inequality with exponents $(\frac{p}{p-1}, p)$ for the first term and $(\frac{p}{(p-1)(1-s)}, \frac{p}{1-s}, \frac{1}{s})$ for the second term.

STEP 3 (Estimate of I_3-I_5)
In this step, we show that

$$
\sum_{k=3}^5 I_k \le \frac{1}{10}D + \frac{1}{10}\frac{1}{N}\sum_{j=1}^N \|Y_j^2|Y_i|^p\|_{L^1} + C\|Y_i\|_{L^p}^p F_2 + CF_3, \tag{6.18}
$$

where

$$
\begin{aligned}
F_2 &\stackrel{\text{def}}{=} \bigg\|\frac{1}{N}\sum_{j=1}^N :Z_j^2:\bigg\|_{\mathbf{C}^{-s}}^{\frac{1}{1-s/2}} + 1, \\
F_3 &\stackrel{\text{def}}{=} \bigg(\frac{1}{N}\sum_{j=1}^N \|:Z_iZ_j:\|_{\mathbf{C}^{-s}}^2\bigg)^{\frac{p}{2}} + \bigg(\frac{1}{N}\sum_{j=1}^N \|:Z_iZ_j:\|_{\mathbf{C}^{-s}}^2\bigg)^{\frac{p}{2(1-s)}}\bigg(\frac{1}{N}\sum_{j=1}^N \|Y_j\|_{H^1}^2\bigg)^{\frac{ps}{2(1-s)}} \\
&\quad + \bigg\|\frac{1}{N}\sum_{j=1}^N :Z_iZ_j^2:\bigg\|_{\mathbf{C}^{-s}}^p + 1.
\end{aligned}
$$

To prove (6.18), we apply Lemma A.5 to find

$$
\begin{aligned}
I_3 &= \frac{1}{N}\sum_{j=1}^N \langle Y_jY_i|Y_i|^{p-2}, :Z_iZ_j:\rangle \\
&\lesssim \frac{1}{N}\sum_{j=1}^N \big(\|Y_jY_i|Y_i|^{p-2}\|_{L^1} + \|Y_jY_i|Y_i|^{p-2}\|_{L^1}^{1-s}\|\nabla(Y_jY_i|Y_i|^{p-2})\|_{L^1}^s\big)\|:Z_iZ_j:\|_{\mathbf{C}^{-s}}.
\end{aligned}
$$

By the Cauchy–Schwarz inequality,

$$
\|Y_jY_i|Y_i|^{p-2}\|_{L^1} = \|(Y_j|Y_i|^{\frac{p}{2}})|Y_i|^{\frac{p}{2}-1}\|_{L^1} \lesssim \|Y_j^2|Y_i|^p\|_{L^1}^{\frac{1}{2}}\||Y_i|^{p-2}\|_{L^1}^{\frac{1}{2}},
$$

and by Hölder's inequality and the Sobolev embedding theorem,

$$
\begin{aligned}
\|\nabla(Y_jY_i|Y_i|^{p-2})\|_{L^1} &\lesssim \|\nabla Y_j\|_{L^2}\||Y_i|^{p-1}\|_{L^2} + \|\nabla Y_i|Y_i|^{p-2}\|_{L^{\frac{p}{p-1}}}\|Y_j\|_{L^p} \\
&\lesssim \|\nabla Y_j\|_{L^2}\||Y_i|^{\frac{p}{2}}\|_{L^{\frac{4(p-1)}{p}}}^{2(1-\frac{1}{p})} + D^{1/2}\||Y_i|^p\|_{L^1}^{\frac{p-2}{2p}}\|Y_j\|_{L^p}, \\
&\lesssim \|\nabla Y_j\|_{L^2}\||Y_i|^{\frac{p}{2}}\|_{H^1}^{2(1-\frac{1}{p})} + D^{1/2}\||Y_i|^p\|_{L^1}^{\frac{p-2}{2p}}\|Y_j\|_{L^p},
\end{aligned}
$$

where we used (6.17). Combining these observations and applying the Cauchy–Schwarz inequality for the summation we find

$$
\begin{aligned}
I_3 &\lesssim \Big(\frac{1}{N}\sum_{j=1}^N \|Y_j^2|Y_i|^p\|_{L^1}\Big)^{1/2} \||Y_i|^{p-2}\|_{L^1}^{\frac12}\Big(\frac{1}{N}\sum_{j=1}^N \|:Z_iZ_j:\|_{\mathbf{C}^{-s}}^2\Big)^{1/2}\\
&\quad+\frac{1}{N}\sum_{j=1}^N \|\nabla Y_j\|_{L^2}^s \||Y_i|^{\frac p2}\|_{H^1}^{2s(1-\frac1p)} \|Y_j^2|Y_i|^p\|_{L^1}^{\frac{1-s}{2}} \||Y_i|^{p-2}\|_{L^1}^{\frac{1-s}{2}} \|:Z_iZ_j:\|_{\mathbf{C}^{-s}}\\
&\quad+\frac{1}{N}\sum_{j=1}^N D^{\frac s2}\||Y_i|^p\|_{L^1}^{\frac{s(p-2)}{2p}} \|Y_j\|_{L^p}^s \|Y_j^2|Y_i|^p\|_{L^1}^{\frac{1-s}{2}} \||Y_i|^{p-2}\|_{L^1}^{\frac{1-s}{2}} \|:Z_iZ_j:\|_{\mathbf{C}^{-s}}\\
&:=\sum_{k=1}^3 I_{3k}.
\end{aligned}
$$

To estimate I_{31}, we use Young's inequality with exponents $(p,2,\frac{2p}{p-2})$ together with the embedding $L^p\hookrightarrow L^{p-2}$, and this leads to the 1 in F_2 and first term in F_3. To estimate I_{32}, we use Hölder's inequality for the summation with exponents $(2,\frac2s,\frac{2}{1-s})$ to find

$$
\begin{aligned}
I_{32} &\lesssim \||Y_i|^{\frac p2}\|_{H^1}^{2s(1-\frac1p)} \||Y_i|^{p-2}\|_{L^1}^{\frac{1-s}{2}}\\
&\quad\times\Big(\frac{1}{N}\sum_{j=1}^N \|\nabla Y_j\|_{L^2}^2\Big)^{s/2}\Big(\frac{1}{N}\sum_{j=1}^N \|Y_j^2|Y_i|^p\|_{L^1}\Big)^{\frac{1-s}{2}}\Big(\frac{1}{N}\sum_{j=1}^N \|:Z_iZ_j:\|_{\mathbf{C}^{-s}}^2\Big)^{1/2}\\
&\leq \frac{1}{40}\||Y_i|^{\frac p2}\|_{H^1}^2+\frac{1}{30}\frac{1}{N}\sum_{j=1}^N\|Y_j^2Y_i^p\|_{L^1}+1\\
&\quad+\|Y_i\|_{L^p}^{p-2}\Big(\frac{1}{N}\sum_{j=1}^N \|\nabla Y_j\|_{L^2}^2\Big)^{s/(1-s)}\Big(\frac{1}{N}\sum_{j=1}^N \|:Z_iZ_j:\|_{\mathbf{C}^{-s}}^2\Big)^{1/(1-s)},
\end{aligned}
$$

where we used Young's inequality with exponents $(\frac{p}{(p-1)s},\frac ps,\frac{2}{1-s},\frac{2}{1-s})$ in the last step. The estimate for I_{33} uses Hölder's inequality with the same exponents, followed by the Sobolev embedding theorem to yield

$$
\begin{aligned}
I_{33} &\lesssim D^{\frac s2}\||Y_i|^p\|_{L^1}^{\frac{s(p-2)}{2p}} \||Y_i|^{p-2}\|_{L^1}^{\frac{1-s}{2}}\\
&\quad\times\Big(\frac{1}{N}\sum_{j=1}^N \|Y_j\|_{H^1}^2\Big)^{s/2}\Big(\frac{1}{N}\sum_{j=1}^N \|Y_j^2|Y_i|^p\|_{L^1}\Big)^{\frac{1-s}{2}}\Big(\frac{1}{N}\sum_{j=1}^N \|:Z_iZ_j:\|_{\mathbf{C}^{-s}}^2\Big)^{1/2}\\
&\leq \frac{1}{40}D+\frac{1}{30}\frac{1}{N}\sum_{j=1}^N\|Y_j^2Y_i^p\|_{L^1}+\|Y_i\|_{L^p}^{p-2}\\
&\quad+\|Y_i\|_{L^p}^{p-2}\Big(\frac{1}{N}\sum_{j=1}^N \|Y_j\|_{H^1}^2\Big)^{s/(1-s)}\Big(\frac{1}{N}\sum_{j=1}^N \|:Z_iZ_j:\|_{\mathbf{C}^{-s}}^2\Big)^{1/(1-s)},
\end{aligned}
$$

where we used Young's inequality with exponents $(\frac2s,\frac2s,\frac{2}{1-s},\frac{2}{1-s})$ in the last step. Combining the above estimates, (6.18) follows for I_3 by Young's inequality with exponents $(\frac{p}{p-2},\frac p2)$.

We now turn to I_4 and note that $\|\nabla|Y_i|^p\|_{L^1} \lesssim \|\nabla|Y_i|^{\frac{p}{2}}\|_{L^2}\||Y_i|^{\frac{p}{2}}\|_{L^2} \lesssim D^{1/2}\||Y_i|^p\|_{L^1}^{1/2}$, so that

$$\begin{aligned} I_4 &= \left\langle \frac{1}{N}\sum_{j=1}^N :Z_j^2: , |Y_i|^p \right\rangle \\ &\lesssim \left\| \frac{1}{N}\sum_{j=1}^N :Z_j^2: \right\|_{\mathbf{C}^{-s}} \big(\||Y_i|^p\|_{L^1} + \||Y_i|^p\|_{L^1}^{1-s}\|\nabla|Y_i|^p\|_{L^1}^s\big) \\ &\lesssim \left\| \frac{1}{N}\sum_{j=1}^N :Z_j^2: \right\|_{\mathbf{C}^{-s}} \big(\|Y_i\|_{L^p}^p + \|Y_i\|_{L^p}^{p(1-s/2)} D^{s/2}\big) \\ &\leq \frac{1}{40} D + C\left(\left\| \frac{1}{N}\sum_{j=1}^N :Z_j^2: \right\|_{\mathbf{C}^{-s}}^{\frac{1}{1-s/2}} + 1\right)\|Y_i\|_{L^p}^p, \end{aligned}$$

by Young's inequality with exponents $(\frac{2}{s}, \frac{2}{2-s})$, which implies (6.18) for I_4. Finally, we estimate I_5 and note

$$\begin{aligned} I_5 &= \left\langle \frac{1}{N}\sum_{j=1}^N :Z_i Z_j^2: , |Y_i|^{p-2}Y_i \right\rangle \\ &\lesssim \left\| \frac{1}{N}\sum_{j=1}^N :Z_i Z_j^2: \right\|_{\mathbf{C}^{-s}} \big(\||Y_i|^{p-1}\|_{L^1} + \||Y_i|^{p-1}\|_{L^1}^{1-s}\|\nabla(|Y_i|^{p-2}Y_i)\|_{L^1}^s\big) \\ &\lesssim \left\| \frac{1}{N}\sum_{j=1}^N :Z_i Z_j^2: \right\|_{\mathbf{C}^{-s}} \big(\|Y_i\|_{L^{p-1}}^{p-1} + \|Y_i\|_{L^{p-1}}^{(p-1)(1-s)} D^{\frac{s}{2}}\|Y_i\|_{L^{p-2}}^{\frac{(p-2)s}{2}}\big), \end{aligned}$$

which gives (6.18) for I_5 by Young's inequality.

STEP 4 (Conclusion) We now insert the inequalities (6.14), (6.16) and (6.18) into the right-hand side of (6.13) to obtain

$$\begin{aligned} &\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|Y_i\|_{L^p}^p + \frac{1}{2}\||Y_i|^{p-2}|\nabla Y_i|^2\|_{L^1} + \frac{1}{2N}\sum_{j=1}^N \||Y_i|^p Y_j^2\|_{L^1} + m\|Y_i\|_{L^p}^p \\ &\quad\leq C(F + F_2)\|Y_i\|_{L^p}^p + CF_1 + CF_3, \end{aligned} \tag{6.19}$$

where F and F_1, F_2, F_3 are introduced in (6.14) and (6.16), (6.18). First, note that by Hölder's inequality and Young's inequality, it holds

$$\begin{aligned} \|Y_i\|_{L^p}^p(F + F_2) &\leq \|Y_i\|_{L^{p+2}}^p(F + F_2) \\ &\leq \big(N^{-\frac{p}{p+2}}\|Y_i\|_{L^{p+2}}^p\big)\big(N^{\frac{p}{p+2}}(F + F_2)\big) \\ &\leq \frac{1}{4N}\|Y_i\|_{L^{p+2}}^{p+2} + CN^{\frac{p}{2}}(F + F_2)^{\frac{p+2}{2}}, \end{aligned} \tag{6.20}$$

and by choosing s sufficiently small depending on p, we may apply Lemma 6.2 and Lemma 2.1 to obtain

$$\mathbf{E}(F + F_2)^{\frac{p+2}{2}} + \mathbf{E}(F_1 + F_3) \lesssim 1.$$

By a similar argument as in the proof of Lemma 5.7, we first obtain $\mathbf{E}\|Y_i(0)\|_{L^p}^p \lesssim C(N)$. In fact, we choose the solution $\tilde{\Phi}_i$ to equation (2.1) starting from the stationary solution

$\tilde{Z}_i(0)$, so that the process $\tilde{Y}_i = \tilde{\Phi}_i - Z_i$ starts from the origin. Using (6.19) and (5.10), (5.11), Lemma 2.1, we find for $T \geq 1$,

$$\int_0^T \mathbf{E}\|\tilde{Y}_i(t)\|_{L^p}^p \lesssim T,$$

which implies that $\mathbf{E}\|Y_i(0)\|_{L^p}^p \lesssim C(N)$ by similar argument as in the proof of Lemma 5.7.

Taking expectation on both sides of (6.19) and using stationarity of $(Y_j)_j$, we find

$$\begin{aligned} &m\mathbf{E}\|Y_i\|_{L^p}^p + \frac{1}{2}\mathbf{E}\big\||Y_i|^{p-2}|\nabla Y_i|^2\big\|_{L^1} + \frac{1}{4N}\sum_{j=1}^N \mathbf{E}\big\||Y_i|^p Y_j^2\big\|_{L^1} \\ &\leq N^{\frac{p}{2}}\mathbf{E}(F+F_2)^{\frac{p+2}{2}} + C\mathbf{E}(F_1+F_3), \end{aligned}$$

which completes the proof. □

6.3. *Correlations of observables.* Now we turn to study the statistical property of the limiting observable, namely, we show that the limiting observables have *nontrivial* laws, in the sense that although Φ_i converges to the (trivial) stationary solution Z_i (and $:\Phi_i^2: \to :Z_i^2:$ as $N \to \infty$ for each i), the observables do not converge to the ones with Φ_i replaced by Z_i. We then write for shorthand

$$\mathbf{\Phi}^2 \stackrel{\text{def}}{=} \sum_{i=1}^N \Phi_i^2, \qquad :\mathbf{\Phi}^2: \stackrel{\text{def}}{=} \sum_{i=1}^N :\Phi_i^2:, \qquad \mathbf{Z}^2 \stackrel{\text{def}}{=} \sum_{i=1}^N Z_i^2, \qquad :\mathbf{Z}^2: \stackrel{\text{def}}{=} \sum_{i=1}^N :Z_i^2: \tag{6.21}$$

The two observables in (6.1) can be then written as $\frac{1}{\sqrt{N}} :\mathbf{\Phi}^2:$ and $\frac{1}{N} :(\mathbf{\Phi}^2)^2:$. We are in the same setting as in Section 6.1, that is, we decompose $\Phi_i = Y_i + Z_i$ with (Y_i, Z_i) stationary and we also consider Y_i, Z_i as stationary process with Z_i as the stationary solution of (5.2) and Y_i as the solution of (2.3).

To state such "nontriviality" results, we consider the correlation function

$$G_N(x-z) = \mathbf{E}\Big[\frac{1}{\sqrt{N}} :\mathbf{\Phi}^2: (x)\frac{1}{\sqrt{N}} :\mathbf{\Phi}^2: (z)\Big].$$

This precisely means the following. For a smooth function g and a distribution F (such as $:\mathbf{\Phi}^2:$), we write $F(g) \stackrel{\text{def}}{=} \langle F, g\rangle$. Then for a smooth test function f the above correlation function G_N is understood as

$$G_N(f) \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0}\int \mathbf{E}\Big[\frac{1}{\sqrt{N}} :\mathbf{\Phi}^2: (\rho_x^\varepsilon)\frac{1}{\sqrt{N}} :\mathbf{\Phi}^2: (\rho_z^\varepsilon)\Big] f(x-z)\,\mathrm{d}z$$

for some mollifier ρ with $\rho_x^\varepsilon(z) = \varepsilon^{-2}\rho(\frac{z-x}{\varepsilon})$ and $\rho_x^\varepsilon \to \delta_x$ as $\varepsilon \to 0$. Here, by translation invariance of ν^N, $G_N(f)$ does not depend on x. We define the Fourier transform of G_N as $\widehat{G}_N(k) = G_N(e_{-k})$ with $k \in \mathbb{Z}^2$, with $\{e_k\}$ the Fourier basis. For comparison, we first note that

$$\lim_{\varepsilon\to 0}\mathbf{E}\Big[\frac{1}{\sqrt{N}} :\mathbf{Z}^2: (\rho_x^\varepsilon)\frac{1}{\sqrt{N}} :\mathbf{Z}^2: (\rho_z^\varepsilon)\Big] = 2C(x-z)^2$$

for any N and $x, z \in \mathbb{T}^2$, where $C = \frac{1}{2}(m-\Delta)^{-1}$, which follows from definition of $:\mathbf{Z}^2:$ and Wick's theorem. Also, $\mathbf{E} :(\mathbf{Z}^2)^2: = 0$ for any N.

We denote $\widehat{f}$ the Fourier transform of a function f.

THEOREM 6.5. *Let m as in Lemma 6.2. It holds that*

$$\lim_{N\to\infty}\widehat{G_N}=2\widehat{C^2}/(1+2\widehat{C^2}),$$

$$\lim_{N\to\infty}\lim_{\varepsilon\to 0}\mathbf{E}\frac{1}{N}\langle :(\mathbf{\Phi}^2)^2: , \rho_x^\varepsilon\rangle=-4\sum_{k\in\mathbb{Z}^2}\widehat{C^2}(k)^2/(1+2\widehat{C^2}(k)). \tag{6.22}$$

In particular, in view of the discussion above the theorem, as $N\to\infty$ the limiting law of $\frac{1}{\sqrt{N}}:\mathbf{\Phi}^2:$ and $\frac{1}{N}:(\mathbf{\Phi}^2)^2:$ are different from that of $\frac{1}{\sqrt{N}}:\mathbf{Z}^2:$ and $\frac{1}{N}:(\mathbf{Z}^2)^2:$.

PROOF. Integration by parts formula gives us the following identities (see Appendix C):

$$-\frac{1}{4N}\lim_{\varepsilon\to 0}\mathbf{E}\langle :(\mathbf{\Phi}^2)^2: , \rho_x^\varepsilon\rangle=\frac{(N+2)}{2N}\sum_{k\in\mathbb{Z}^2}\widehat{C^2}(k)\widehat{G_N}(k)+R_N, \tag{6.23}$$

$$\left(\frac{1}{2}+\frac{N+2}{N}\widehat{C^2}\right)\widehat{G_N}=\widehat{CC_N}+\widehat{Q_N}/N \tag{6.24}$$

with

$$\begin{aligned}
C_N(f)&=\lim_{\varepsilon\to 0}\int\mathbf{E}\big[\Phi_1(\rho_x^\varepsilon)\Phi_1(\rho_z^\varepsilon)\big]f(x-z)\,\mathrm{d}z,\\
R_N&=-\frac{1}{N^2}\lim_{\varepsilon\to 0}\int C(x-z_1)C(x-z_2)\\
&\quad\times\mathbf{E}\big[:\Phi_1\mathbf{\Phi}^2:\ (\rho_{z_1}^\varepsilon):\Phi_1\mathbf{\Phi}^2:\ (\rho_{z_2}^\varepsilon):\mathbf{\Phi}^2:\ (\rho_x^\varepsilon)\big]\,\mathrm{d}z_1\,\mathrm{d}z_2,\\
Q_N&=-2\lim_{\varepsilon\to 0}\int C(x-y)C(x-z)\mathbf{E}\big[:\Phi_1\mathbf{\Phi}^2:\ (\rho_y^\varepsilon)\,\Phi_1(\rho_z^\varepsilon)\big]\,\mathrm{d}y\\
&\quad+\lim_{\varepsilon\to 0}\frac{2}{N}\int C(x-z_1)C(x-z_2)\\
&\quad\times\mathbf{E}\big[:\Phi_1\mathbf{\Phi}^2:\ (\rho_{z_1}^\varepsilon):\Phi_1\mathbf{\Phi}^2:\ (\rho_{z_2}^\varepsilon):\mathbf{\Phi}^2:\ (\rho_z^\varepsilon)\big]\,\mathrm{d}z_1\,\mathrm{d}z_2\\
&\stackrel{\text{def}}{=}Q_1^N+Q_2^N.
\end{aligned}$$

We first use (6.24) and we know $\widehat{C_N}\to\widehat{C}$ as $N\to\infty$ by using Lemma 6.2 and

$$\begin{aligned}
\widehat{C_N}(k)&=c\mathbf{E}\int Y_1(x)Y_1(z)e_{-k}(x-z)\,\mathrm{d}z\,\mathrm{d}x+c\mathbf{E}\int Y_1(x)\langle Z_1,e_{-k}(x-\cdot)\rangle\,\mathrm{d}x\\
&\quad+c\mathbf{E}\int Y_1(z)\langle Z_1,e_{-k}(\cdot-z)\rangle\,\mathrm{d}z+\widehat{C}(k),
\end{aligned}$$

for constant c. Since C is positive definite, $\widehat{C}$ is a positive function, and so is $\widehat{C^2}$; this allows us to divide both sides by $\frac{1}{2}+\widehat{C^2}$ in (6.24). In Lemmas 6.8 and 6.9 below, we show that $\widehat{Q_N}(k)/N$ vanishes in the limit for every k. We therefore obtain (6.22).

By Lemma 6.9, R_N converges weakly to 0 as $N\to\infty$. However, R_N is independent of x as a consequence of spatial translation invariance, so we also obtain pointwise convergence of R_N to zero. By Lemma 6.7, $\widehat{CC_N}(k)$ is uniformly bounded by $c(m+|k|^2)^{-1+\kappa}$ for $c>0$ and small $\kappa>0$. By the dominated convergence theorem, we have

$$\frac{(N+2)}{2N}\sum_{k\in\mathbb{Z}^2}\widehat{C^2}(k)\frac{\widehat{CC_N}(k)}{(\frac{1}{2}+\frac{N+2}{N}\widehat{C^2}(k))}\to\sum_{k\in\mathbb{Z}^2}\frac{\widehat{C^2}(k)\cdot\widehat{C^2}(k)}{1+2\widehat{C^2}(k)},\quad\text{as }N\to\infty.$$

By Lemmas 6.8 and 6.9 and the dominated convergence theorem, we find

$$\frac{(N+2)}{2N}\sum_{k\in\mathbb{Z}^2}\widehat{C^2}(k)\frac{\widehat{Q_N}(k)}{N(\frac{1}{2}+\frac{N+2}{N}\widehat{C^2}(k))}\to 0,\quad \text{as } N\to\infty,$$

which combined with Lemma 6.9 and (6.23), (6.24) implies that

$$\lim_{N\to\infty}\mathbf{E}\frac{1}{4N}\langle :(\mathbf{\Phi}^2)^2: , \rho_x^{\varepsilon}\rangle = -\sum_{k\in\mathbb{Z}^2}\widehat{C^2}(k)\cdot\widehat{C^2}(k)/\big(1+2\widehat{C^2}(k)\big)$$

where the sum is over integers (i.e., Fourier variables). This is nonzero, showing that the limiting law of $\frac{1}{N}:(\mathbf{\Phi}^2)^2:$ is different from that of $\frac{1}{N}:(\mathbf{Z}^2)^2:$. $\square$

We will use Lemmas 6.2 and 6.4 to control the remainder terms from integration by parts formula. In fact, all the remainder terms will be controlled by the following terms:

$$\mathbf{E}A_1^{\ell_1}A_2^{\ell_2}A_3^{\ell_3},$$

with $\ell_i\geq 0$, where (for $s>3\kappa>0$ small enough)

$$\begin{aligned}
A_1 &:= \|Y_1\|_{L^2}+\|Z_1\|_{\mathbf{C}^{-\frac{\kappa}{3}}},\\
A_2 &:= \Big\|\sum_{i=1}^N Y_i^2\Big\|_{L^1}+\sum_{i=1}^N\|Y_i\|_{H^\kappa}\|Z_i\|_{\mathbf{C}^{-\frac{\kappa}{2}}}+\Big\|\sum_{i=1}^N :Z_i^2:\Big\|_{H^{-\frac{\kappa}{2}}},\\
A_3 &:= \Big\|Y_1\sum_{i=1}^N Y_i^2\Big\|_{L^{1+s}}+\Big\|\Lambda^{-s}\Big(Z_1\sum_{i=1}^N Y_i^2\Big)\Big\|_{L^{1+s}}\\
&\quad+\Big\|\Lambda^{-s}\Big(Y_1\sum_{i=1}^N Y_iZ_i\Big)\Big\|_{L^{1+s}}+\Big\|\Lambda^{-s}\Big(Z_1\sum_{i=1}^N Y_iZ_i\Big)\Big\|_{L^{1+s}}\\
&\quad+\Big\|\Lambda^{-s}\Big(Y_1\sum_{i=1}^N :Z_i^2:\Big)\Big\|_{L^{1+s}}+\Big\|\sum_{i=1}^N :Z_1Z_i^2:\Big\|_{H^{-s}}:=\sum_{i=1}^6\bar{A}_{3i}.
\end{aligned}$$

LEMMA 6.6. *For m as in Lemma 6.2 and for each $\ell_i\geq 0$ with $\ell_2\frac{\kappa}{2}+3\ell_3 s<1$, it holds*

$$\mathbf{E}A_1^{\ell_1}A_2^{\ell_2}A_3^{\ell_3}\lesssim N^{(\ell_2+\ell_3)/2},$$

where the implicit constant is independent of N.

PROOF. By Lemma 6.2 and Lemma 2.1, it follows that $\mathbf{E}A_1^{\ell_1}\lesssim 1$ for every $\ell_1\geq 0$. Using the interpolation Lemma A.2 followed by Hölder's inequality with exponents $(\frac{2}{\kappa},\frac{2}{1-\kappa},2)$, we find

$$\begin{aligned}
A_2 &\lesssim \sum_{i=1}^N\|Y_i\|_{L^2}^2+\Big(\sum_{i=1}^N\|Y_i\|_{L^2}^2\Big)^{\frac{1-\kappa}{2}}\Big(\sum_{i=1}^N\|Y_i\|_{H^1}^2\Big)^{\frac{\kappa}{2}}\Big(\sum_{i=1}^N\|Z_i\|^2_{\mathbf{C}^{-\frac{\kappa}{2}}}\Big)^{\frac{1}{2}}+\Big\|\sum_{i=1}^N :Z_i^2:\Big\|_{H^{-\frac{\kappa}{2}}}\\
&:= A_{21}+A_{22}+A_{23}.
\end{aligned}$$

By Lemma 6.2 and Lemma 2.1, it follows that for all $\ell_2\geq 0$ with $\frac{\kappa}{2}\ell_2<1$,

$$\mathbf{E}A_{21}^{\ell_2}\lesssim 1,\quad \mathbf{E}A_{22}^{\ell_2}+\mathbf{E}A_{23}^{\ell_2}\lesssim N^{\frac{\ell_2}{2}},$$

where we used (4.1) and Gaussian hypercontractivity for the bound of A_{23}. It remains to consider A_3. Starting with $\bar{A}_{31}$, let $p \in (1,2)$ and $q > 1$ satisfy $\frac{1}{q} + \frac{1}{p} = \frac{1}{1+s}$, then use Hölder's inequality and interpolate to obtain

$$\bar{A}_{31} \lesssim \|Y_1\|_{L^q} \left\| \sum_{i=1}^N Y_i^2 \right\|_{L^p} \lesssim \|Y_1\|_{L^q} \left\| \sum_{i=1}^N Y_i^2 \right\|_{L^2}^{2-\frac{2}{p}} \left\| \sum_{i=1}^N Y_i^2 \right\|_{L^1}^{\frac{2}{p}-1} := A_{31}.$$

Given $\ell_3 \geq 0$, choosing p close enough to 1 and $q > \ell_3$ to ensure $\ell_3(2 - \frac{2}{p})\frac{q}{q-\ell_3} < 2$, we may use Lemma 6.4 and Lemma 6.2 and Hölder's inequality to obtain the bound

$$\mathbf{E} A_{31}^{\ell_3} \lesssim N^{\frac{\ell_3}{2}}.$$

The next three terms can be estimated by using Lemma A.3 and Lemma A.2. Specifically, we use that for $s \in (0,1)$ it holds

$$\|\Lambda^{-s}(fg)\|_{L^{1+s}} \lesssim \|f\|_{B^s_{1+s,1}} \|g\|_{\mathbf{C}^{-s+\kappa}} \lesssim \big(\|f\|_{H^{2s}} \wedge \|\Lambda^{3s} f\|_{L^1}\big) \|g\|_{\mathbf{C}^{-s+\kappa}},$$

for $s > 3\kappa > 0$ small enough.

$$\bar{A}_{32} \lesssim \|Z_1\|_{\mathbf{C}^{-s+\kappa}} \left\| \Lambda^{3s} \sum_{i=1}^N Y_i^2 \right\|_{L^1} \lesssim \|Z_1\|_{\mathbf{C}^{-s}} \left(\sum_{i=1}^N \|Y_i\|_{H^1}^2 \right)^{3s} \left(\sum_{i=1}^N \|Y_i\|_{L^2}^2 \right)^{1-3s} := A_{32},$$

$$\begin{aligned}
\bar{A}_{33} &\lesssim \sum_{i=1}^N \|Z_i\|_{\mathbf{C}^{-s+\kappa}} \|Y_i\|_{H^{3s}} \|Y_1\|_{H^{3s}} \\
&\lesssim \left(\sum_{i=1}^N \|Z_i\|_{\mathbf{C}^{-s+\kappa}}^2 \right)^{1/2} \|Y_1\|_{H^1}^{3s} \|Y_1\|_{L^2}^{1-3s} \left(\sum_{i=1}^N \|Y_i\|_{H^1}^2 \right)^{3s/2} \left(\sum_{i=1}^N \|Y_i\|_{L^2}^2 \right)^{\frac{1-3s}{2}} := A_{33},
\end{aligned}$$

$$\begin{aligned}
\bar{A}_{34} &\lesssim \sum_{i=1}^N \| :Z_1 Z_i: \|_{\mathbf{C}^{-s+\kappa}} \|Y_i\|_{H^{2s}} \\
&\lesssim \left(\sum_{i=1}^N \| :Z_1 Z_i: \|_{\mathbf{C}^{-s+\kappa}}^2 \right)^{1/2} \left(\sum_{i=1}^N \|Y_i\|_{H^1}^2 \right)^{s} \left(\sum_{i=1}^N \|Y_i\|_{L^2}^2 \right)^{\frac{1-2s}{2}} := A_{34},
\end{aligned}$$

and

$$\begin{aligned}
\bar{A}_{35} &\lesssim \left\| \sum_{i=1}^N :Z_i^2: Y_1 \right\|_{B^{-s}_{1+s,1}} \lesssim \left\| \sum_{i=1}^N :Z_i^2: \right\|_{H^{-s+\kappa}} \|Y_1\|_{H^{3s}} \\
&\lesssim \left\| \sum_{i=1}^N :Z_i^2: \right\|_{H^{-s+\kappa}} \|Y_1\|_{H^1}^{3s} \|Y_1\|_{L^2}^{1-3s} := A_{35}.
\end{aligned}$$

By Lemma 6.2, we deduce that for $3\ell_3 s < 1$,

$$\mathbf{E} A_{3i}^{\ell_3} \lesssim N^{\frac{\ell_3}{2}},$$

with $i = 2, \dots, 5$. The last term is given as

$$A_{36} := \left\| \sum_{i=1}^N :Z_1 Z_i^2: \right\|_{H^{-s}},$$

which by similar argument as in the proof of (5.4) implies that $\mathbf{E}A_{36}^{\ell_3} \lesssim N^{\frac{\ell_3}{2}}$, Combining the above estimates and Hölder's inequality, we obtain

$$\mathbf{E}A_1^{\ell_1}A_2^{\ell_2}A_3^{\ell_3} \lesssim \sum_{(i_1,i_2,i_3)} \mathbf{E}A_{1i_1}^{\ell_1}A_{2i_2}^{\ell_2}A_{3i_3}^{\ell_3} \lesssim N^{(\ell_2+\ell_3)/2}.$$

□

In the following proof, we use c to denote positive constant, which may change from line to line.

LEMMA 6.7. *It holds that*

$$\big|\widehat{CC_N}(k)\big| \lesssim \big(1+|k|^2\big)^{-1+\kappa},$$

for $\kappa > 0$, *where the proportional constant is independent of* k.

PROOF. By translation invariance, we know for some positive constant $c > 0$,

$$\begin{aligned}
\widehat{CC_N}(k) &= \lim_{\varepsilon\to 0}\mathbf{E}\int C(x-z)\Phi_1\big(\rho_x^\varepsilon\big)\Phi_1\big(\rho_z^\varepsilon\big)e_{-k}(x-z)\,\mathrm{d}z \\
&= c\lim_{\varepsilon\to 0}\mathbf{E}\int C(x-z)\Phi_1\big(\rho_x^\varepsilon\big)\Phi_1\big(\rho_z^\varepsilon\big)e_{-k}(x-z)\,\mathrm{d}z\,\mathrm{d}x \\
&= c\mathbf{E}\big\langle\Phi_1(m-\Delta)^{-1}(\Phi_1 e_k), e_k\big\rangle \\
&= c\mathbf{E}\sum_{k_1\in\mathbb{Z}^2}\widehat{\Phi_1}(k_1)\big(m+|k-k_1|^2\big)^{-1}\widehat{\Phi_1}(-k_1) \\
&\lesssim \Big[\mathbf{E}\Big(\sum_{k_1\in\mathbb{Z}^2}\big|\widehat{\Phi_1}(k_1)\big|^2\big(m+|k-k_1|^2\big)^{-1}\Big)\Big]^{\frac{1}{2}} \\
&\quad\times\Big[\mathbf{E}\Big(\sum_{k_1\in\mathbb{Z}^2}\big|\widehat{\Phi_1}(-k_1)\big|^2\big(m+|k-k_1|^2\big)^{-1}\Big)\Big]^{\frac{1}{2}} \\
&\stackrel{\text{def}}{=} I_1^{\frac{1}{2}}I_2^{\frac{1}{2}}.
\end{aligned}$$

I_1 is bounded by

$$\mathbf{E}\sum_{k_1\in\mathbb{Z}^2}\big|\widehat{Y_1}(k_1)\big|^2\big(m+|k-k_1|^2\big)^{-1} + \mathbf{E}\sum_{k_1\in\mathbb{Z}^2}\big|\widehat{Z_1}(k_1)\big|^2\big(m+|k-k_1|^2\big)^{-1}.$$

Using Lemma 6.2, we know

$$\mathbf{E}\big|\widehat{Y_1}(k_1)\big|^2 \lesssim \big(1+|k_1|\big)^{-2}\mathbf{E}\|Y_1\|_{H^1}^2 \lesssim \big(1+|k_1|\big)^{-2}.$$

Using translation invariance, we find

$$\mathbf{E}\sum_{k_1\in\mathbb{Z}^2}\big|\widehat{Z_1}(k_1)\big|^2\big(m+|k-k_1|^2\big)^{-1} = c\widehat{C^2}(k).$$

Now the desired bound for I_1 follows from Lemma A.7. Similarly, we deduce the required bound for I_2 and the result follows. □

LEMMA 6.8. *It holds that for every* $k\in\mathbb{Z}^2$,

$$\big|\widehat{Q_1^N}(k)\big| \lesssim N^{1/2}\big(1+|k|\big)^{-\kappa/2},$$

for every $0<\kappa<\frac{1}{6}$, *where the proportional constant is independent of* N *and* k.

PROOF. We use translation invariance property to write the Fourier transform of Q_1^N as

$$\widehat{Q_1^N}(-k)=\int Q_1^N e_k(x-z)\,\mathrm{d}z=c\int Q_1^N e_k(x-z)\,\mathrm{d}x\,\mathrm{d}z$$
$$=c\mathbf{E}\big[\big\langle(m-\Delta)^{-1}\big(:\Phi_1\Phi_i^2:\big)\cdot(m-\Delta)^{-1}[\Phi_1\cdot e_{-k}],e_{-k}\big\rangle\big],$$

which by Lemma A.3 can be bounded by

$$\big(1+|k|\big)^{-\kappa}\mathbf{E}\Bigg[\Bigg\|(m-\Delta)^{-1}\Bigg((Y_1+Z_1)\sum_{i=1}^N\big(Y_i^2+2Y_iZ_i+:Z_i^2:\big)\Bigg)\Bigg\|_{\mathbf{C}^\kappa}$$
$$\times\big\|(m-\Delta)^{-1}\big[(Y_1+Z_1)e_{-k}\big]\big\|_{\mathbf{C}^\kappa}\Bigg],$$

where we used $\|e_k\|_{B_{1,1}^{-\kappa}}\lesssim(1+|k|)^{-\kappa}$, which can be checked by direct calculation. By Besov embedding Lemma A.1 and elliptic Schauder estimate (see, e.g., [67], Theorem 6.5), we know that for $\frac{1}{2}>s>3\kappa>0$ small enough

$$\big\|(m-\Delta)^{-1}f\big\|_{\mathbf{C}^\kappa}\lesssim\|f\|_{L^{1+s}}\wedge\big\|\Lambda^{-s}f\big\|_{L^{1+s}}\wedge\|f\|_{H^{-s}}\wedge\|f\|_{\mathbf{C}^{-\kappa/3}}. \tag{6.25}$$

(6.25) implies that the above term could be controlled by $\mathbf{E}[A_3A_1]\|e_k\|_{\mathbf{C}^{\kappa/2}}$, which by Lemma 6.6 can be bounded by $N^{\frac{1}{2}}(1+|k|)^{-\kappa/2}$. □

LEMMA 6.9. *For every $k\in\mathbb{Z}^2$,*

$$\big|\widehat{Q_2^N}(k)\big|\lesssim N^{1/2}\big(1+|k|\big)^{-\frac{\kappa}{2}},\quad\big|\widehat{R^N}(k)\big|\lesssim N^{-1/2}\big(1+|k|\big)^{\frac{\kappa}{2}},$$

for $0<\kappa<\frac{2}{37}$, where the proportional constant is independent of N and k.

PROOF. We write the Fourier transform of Q_2^N as

$$\widehat{Q_2^N}(-k)=\int Q_2^N e_k(x-z)\,\mathrm{d}x=c\int Q_2^N e_k(x-z)\,\mathrm{d}x\,\mathrm{d}z$$
$$=\frac{c}{N}\mathbf{E}\Bigg\langle\Bigg(\sum_{i=1}^N(m-\Delta)^{-1}\Phi_1\Phi_i^2\Bigg)^2,e_{-k}\Bigg\rangle\Bigg\langle\sum_{i=1}^N:\Phi_i^2:,e_k\Bigg\rangle,$$

which can be bounded by

$$\frac{c}{N}\mathbf{E}\Bigg[\Bigg\|(m-\Delta)^{-1}\Bigg((Y_1+Z_1)\sum_{i=1}^N\big(Y_i^2+2Y_iZ_i+:Z_i^2:\big)\Bigg)\Bigg\|_{\mathbf{C}^\kappa}^2\|e_{-k}\|_{B_{1,1}^{-\kappa}}$$
$$\times\Bigg|\Bigg\langle\Bigg(\sum_{i=1}^N\big(Y_i^2+2Y_iZ_i+:Z_i^2:\big)\Bigg),e_{-k}\Bigg\rangle\Bigg|\Bigg].$$

Using (6.25) for the first line and Lemma A.3 for the second line and $\|e_k\|_{B_{1,1}^{-\kappa}}\lesssim(1+|k|)^{-\kappa}$, the above term can be estimated by $\frac{1}{N}\mathbf{E}[A_3^2A_2](|k|+1)^{-\kappa}\|e_{-k}\|_{B_{2,1}^{\frac{\kappa}{2}}}$ for $s>3\kappa$, $6s+\frac{\kappa}{2}<1$, which by Lemma 6.6 can be bounded by $N^{\frac{1}{2}}(|k|+1)^{-\frac{\kappa}{2}}$.

Similarly, we write

$$\widehat{R^N}(k)=\frac{c}{N^2}\mathbf{E}\Bigg\langle\Bigg(\sum_{i=1}^N(m-\Delta)^{-1}\Phi_1\Phi_i^2\Bigg)^2\Bigg(\sum_{i=1}^N:\Phi_i^2:\Bigg),e_k\Bigg\rangle,$$

which by Lemma A.3 can be bounded by

$$\frac{c(1+|k|)^{\frac{\kappa}{2}}}{N^2}\mathbf{E}\left\|(m-\Delta)^{-1}\left((Y_1+Z_1)\sum_{i=1}^N(Y_i^2+2Y_iZ_i+:Z_i^2:)\right)\right\|_{\mathbf{C}^\kappa}^2 A_2,$$

for $s>3\kappa>0$ and $6s+\frac{\kappa}{2}<1$. By (6.25), we know that the above term could be controlled by $\frac{1}{N^2}\mathbf{E}[A_3^2A_2](1+|k|)^{\frac{\kappa}{2}}$, which by Lemma 6.6 can be bounded by $N^{-\frac{1}{2}}(1+|k|)^{\frac{\kappa}{2}}$. □

APPENDIX A: NOTATION AND BESOV SPACES

We use $(\Delta_i)_{i\geq -1}$ to denote the Littlewood–Paley blocks for a dyadic partition of unity. Besov spaces on the torus with general indices $\alpha\in\mathbb{R}$, $p,q\in[1,\infty]$ are defined as the completion of C^∞ with respect to the norm

$$\|u\|_{B^\alpha_{p,q}}:=\left(\sum_{j\geq -1}(2^{j\alpha}\|\Delta_j u\|^q_{L^p})\right)^{1/q},$$

and the Hölder–Besov space $\mathbf{C}^\alpha$ is given by $\mathbf{C}^\alpha=B^\alpha_{\infty,\infty}$. We will often write $\|\cdot\|_{\mathbf{C}^\alpha}$ instead of $\|\cdot\|_{B^\alpha_{\infty,\infty}}$.

Set $\Lambda=(1-\Delta)^{\frac{1}{2}}$. For $s\geq 0$, $p\in[1,+\infty]$, we use H^s_p to denote the subspace of L^p, consisting of all f, which can be written in the form $f=\Lambda^{-s}g$, $g\in L^p$ and the H^s_p norm of f is defined to be the L^p norm of g, that is, $\|f\|_{H^s_p}:=\|\Lambda^s f\|_{L^p}$. For $s<0$, $p\in(1,\infty)$, H^s_p is the dual space of H^{-s}_q with $\frac{1}{p}+\frac{1}{q}=1$. Set $H^s:=H^s_2$.

The following embedding results will be frequently used (e.g., [66]).

LEMMA A.1. (i) *Let* $1\leq p_1\leq p_2\leq\infty$ *and* $1\leq q_1\leq q_2\leq\infty$, *and let* $\alpha\in\mathbb{R}$. *Then* $B^\alpha_{p_1,q_1}\subset B^{\alpha-d(1/p_1-1/p_2)}_{p_2,q_2}$ (*cf.* [33], *Lemma* A.2).

(ii) *Let* $s\in\mathbb{R}$, $1<p<\infty$, $\epsilon>0$. *Then* $H^s_2=B^s_{2,2}$, *and* $B^s_{p,1}\subset H^s_p\subset B^s_{p,\infty}\subset B^{s-\epsilon}_{p,1}$ (*cf.* [66], *Theorem* 4.6.1).

(iii) *Let* $1\leq p_1\leq p_2<\infty$ *and let* $\alpha\in\mathbb{R}$. *Then* $H^\alpha_{p_1}\subset H^{\alpha-d(1/p_1-1/p_2)}_{p_2}$.

Here, $\subset$ *means continuous and dense embedding.*

We recall the following interpolation inequality and multiplicative inequality for the elements in H^s_p.

LEMMA A.2. (i) *Suppose that* $s\in(0,1)$ *and* $p\in(1,\infty)$. *Then for* $f\in H^1_p$,

$$\|f\|_{H^s_p}\lesssim\|f\|^{1-s}_{L^p}\|f\|^s_{H^1_p}$$

(*cf.* [66], *Theorem* 4.3.1).

(ii) *Suppose that* $s>0$ *and* $p\in[1,\infty)$. *It holds that*

$$\|\Lambda^s(fg)\|_{L^p}\lesssim\|f\|_{L^{p_1}}\|\Lambda^s g\|_{L^{p_2}}+\|g\|_{L^{p_3}}\|\Lambda^s f\|_{L^{p_4}}, \tag{A.1}$$

with $p_i\in(1,\infty]$, $i=1,\dots,4$ *such that*

$$\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4}$$

(*cf. see* [29], *Theorem* 1).

(iii) (*Gagliardo–Nirenberg inequality*) *For* $s\in[0,1)$, $\alpha\in(0,1)$, $r\geq 1$,

$$\|u\|_{H^s_q}\lesssim\|u\|^\alpha_{H^1}\|u\|^{1-\alpha}_{L^r} \tag{A.2}$$

with $\frac{1}{q}=\frac{s}{d}+\alpha(\frac{1}{2}-\frac{1}{d})+\frac{1-\alpha}{r}$.

LEMMA A.3. (i) *Let* $\alpha, \beta \in \mathbb{R}$ *and* $p, p_1, p_2, q \in [1, \infty]$ *be such that* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. *The bilinear map* $(u, v) \mapsto uv$ *extends to a continuous map from* $B^\alpha_{p_1,q} \times B^\beta_{p_2,q}$ *to* $B^{\alpha\wedge\beta}_{p,q}$ *if* $\alpha + \beta > 0$ (*cf.* [53], *Corollary* 2).

(ii) (*Duality.*) *Let* $\alpha \in (0, 1)$, $p, q \in [1, \infty]$, p' *and* q' *be their conjugate exponents*, *respectively. Then the mapping* $(u, v) \mapsto \langle u, v\rangle = \int uv \,\mathrm{d}x$ *extends to a continuous bilinear form on* $B^\alpha_{p,q} \times B^{-\alpha}_{p',q'}$, *and one has* $|\langle u, v\rangle| \lesssim \|u\|_{B^\alpha_{p,q}} \|v\|_{B^{-\alpha}_{p',q'}}$ (*cf.* [53], *Proposition* 7).

We recall the following smoothing effect of the heat flow $S_t = e^{t(\Delta-m)}$, $m \geq 0$ (e.g., [33], Lemma A.7, [53], Proposition 5).

LEMMA A.4. *Let* $u \in B^\alpha_{p,q}$ *for some* $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$. *Then for every* $\delta \geq 0$ *and* $t \in [0, T]$,

$$\|S_t u\|_{B^{\alpha+\delta}_{p,q}} \lesssim t^{-\delta/2} \|u\|_{B^\alpha_{p,q}},$$

where the proportionality constant is independent of t.

LEMMA A.5. *For* $s \in (0, 1)$,

$$\left|\langle g, f\rangle\right| \lesssim \left(\|\nabla g\|^s_{L^1} \|g\|^{1-s}_{L^1} + \|g\|_{L^1}\right) \|f\|_{\mathbf{C}^{-s}}.$$

PROOF. This follows from Lemma A.3, which states that $\langle g, f\rangle$ is a continuous bilinear form on $B^s_{1,1} \times \mathbf{C}^{-s}$, together with [53], Proposition 8, which states that $\|g\|_{B^s_{1,1}} \lesssim \|\nabla g\|^s_{L^1} \|g\|^{1-s}_{L^1} + \|g\|_{L^1}$. $\square$

We also recall the following comparison test result, which has been proved in [68], Lemma 3.8.

LEMMA A.6. *Let* $f : [0, T] \to [0, \infty)$ *differentiable such that for every* $t \in [0, T]$,

$$\frac{\mathrm{d}f}{\mathrm{d}t} + c_1 f^2 \leq c_2.$$

Then for $t > 0$,

$$f(t) \leq \left(t^{-1} \frac{2}{c_1}\right) \vee \left(\frac{2c_2}{c_1}\right)^{\frac{1}{2}}.$$

We recall the following result for sum from [72], Lemma 3.10.

LEMMA A.7. *Let* $0 < l, r < d$, $l + r - d > 0$. *Then it holds*

$$\sum_{k_1 \in \mathbb{Z}^d} (1 + |k_1|)^{-l} (1 + |k - k_1|)^{-r} \lesssim (1 + |k|)^{d-l-r}.$$

APPENDIX B: PROOF OF LEMMA 2.2

PROOF. For initial value $y_i \in \mathbf{C}^\beta(\mathbb{T}^2)$, $\beta \in (1, 2)$, we could use a similar argument as in [53], Theorem 6.1, to obtain global solutions (Y_i) to (2.3) with each $Y_i \in C_T\mathbf{C}^\beta$. In fact, we use mild solutions and a fixed-point argument to obtain unique local solutions. Furthermore, for fixed N we obtain a global in time L^p-estimate, $p > 1$, which gives the required global solutions.

Moreover, for general initial data $y_i \in L^2$, we consider smooth approximation (y_i^ε) to initial data y_i. For (y_i^ε), we construct solutions $Y_i^\varepsilon \in C_T\mathbf{C}^\beta$ by the above argument. For Y_i^ε, we could do the uniform estimate as in Lemma 2.3 and obtain

$$\frac{1}{N}\sup_{t\in[0,T]}\sum_{j=1}^N \|Y_j^\varepsilon\|_{L^2}^2 + \frac{1}{N}\sum_{j=1}^N \|\nabla Y_j^\varepsilon\|_{L^2(0,T;L^2)}^2 + \left\|\frac{1}{N}\sum_{i=1}^N (Y_i^\varepsilon)^2\right\|_{L^2(0,T;L^2)}^2 \leq C,$$

where C is independent of ε. By standard compactness argument, we deduce that there exist a sequence $\{\varepsilon_k\}$ and $Y_i \in L_T^\infty L^2 \cap L_T^2 H^1 \cap L_T^4 L^4$ such that $Y_i^{\varepsilon_k} \to Y$ in $L_T^2 H^\delta \cap C_T H^{-1}$, $\delta < 1$. Furthermore, by a similar argument as in the proof of [55], Theorem 4.3, we obtain $Y_i \in C_T L^2 \cap L_T^4 L^4 \cap L_T^2 H^1$. For the uniqueness part, we could do similar estimate as I_1^N and I_2^N for the difference v_i in Section 4. From the estimates (4.8), (4.13) and (4.18) in Section 4 the regularity for Y_i is enough for the uniqueness. □

APPENDIX C: CONSEQUENCES OF DYSON–SCHWINGER EQUATIONS

Dyson–Schwinger equations are relations between correlation functions of different orders. Here, we derive the identities (6.23) and (6.24) using Dyson–Schwinger equations; these are essentially in [45] (equations (7), (8) therein), but since we are in a slightly different setting, we give some details here to be self-contained. They are consequences of integration by parts formula (e.g., [32], Theorem 6.7, for the Φ^4 model), In the case of the N-component Φ^4 model (i.e., linear sigma model), for a fixed N, $\Phi \sim \nu^N$ and writing $\mathbf{\Phi}^2 \stackrel{\text{def}}{=} \sum_{i=1}^N \Phi_i^2$ as shorthand, it is easy to derive the following integration by parts formula:

$$\mathbf{E}\big(D_{\rho_{1,x}^\varepsilon} F(\Phi)\big) = 2\mathbf{E}\big(\langle \Phi_1, (m-\Delta_x)\rho_x^\varepsilon\rangle F(\Phi)\big) + \frac{2}{N}\mathbf{E}\big(F(\Phi)\langle :\Phi_1\mathbf{\Phi}^2: , \rho_x^\varepsilon\rangle\big),$$

where $D_{\rho_{1,x}^\varepsilon} F(\Phi)$ denotes the Fréchet derivative along $\rho_{1,x}^\varepsilon \stackrel{\text{def}}{=} (\rho_x^\varepsilon, 0, \dots, 0)$ (namely varying only Φ_1 in the direction ρ_x^ε). In terms of Green's function $C(x-y) = \frac{1}{2}(m-\Delta)^{-1}(x-y)$, we can also write it as

$$\begin{aligned}\int C(x-z)&\mathbf{E}\big(D_{\rho_{1,z}^\varepsilon} F(\Phi)\big)\,\mathrm{d}z \\ &= \mathbf{E}\big(\langle \Phi_1, \rho_x^\varepsilon\rangle F(\Phi)\big) + \frac{2}{N}\int C(x-z)\mathbf{E}\big(F(\Phi)\langle :\Phi_1\mathbf{\Phi}^2: , \rho_z^\varepsilon\rangle\big)\,\mathrm{d}z.\end{aligned} \tag{C.1}$$

Here, we apply (C.1) to prove (6.23). Taking $F(\Phi) = \langle :\Phi_1\mathbf{\Phi}^2: , \rho_x^\varepsilon\rangle$, one has

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{2}{N}\int C(x-z)\mathbf{E}\big(\langle :\Phi_1\mathbf{\Phi}^2: , \rho_x^\varepsilon\rangle\langle :\Phi_1\mathbf{\Phi}^2: , \rho_z^\varepsilon\rangle\big)\,\mathrm{d}z \\ &\quad= \lim_{\varepsilon\to 0}\left[\int \frac{N+2}{N} C(x-z)\mathbf{E}\big(\langle :\mathbf{\Phi}^2: \rho_z^\varepsilon, \rho_x^\varepsilon\rangle\big)\,\mathrm{d}z - \mathbf{E}\big(\langle \Phi_1, \rho_x^\varepsilon\rangle\langle :\Phi_1\mathbf{\Phi}^2: , \rho_x^\varepsilon\rangle\big)\right] \\ &\quad= -\lim_{\varepsilon\to 0}\frac{1}{N}\mathbf{E}\langle :(\mathbf{\Phi}^2)^2: , \rho_x^\varepsilon\rangle\end{aligned}$$

using the definition of Wick products and the symmetry (i.e., exchangeability of $(\Phi_i)_i$) and taking limit in $\mathbf{C}^{-\kappa}$. [8]

[8]For instance, in the first step, recalling the precise definition of $:\Phi_1\mathbf{\Phi}^2:$, the derivative $D_{\rho_{1,z}^\varepsilon} F(\Phi)$ gives $3:\Phi_1^2: \rho_z^\varepsilon + \sum_{i=2}^N :\Phi_i^2: \rho_z^\varepsilon$ which by exchangeability can be rewritten as $\frac{N+2}{N} :\mathbf{\Phi}^2: \rho_z^\varepsilon$ inside expectation. The last step follows similarly; or it could be viewed as an N dimensional generalization of the well-known relation $H_{n+1}(x) = xH_n(x) - H_n'(x)$ for Hermite polynomials H_n.

Next, taking $F = \langle :\mathbf{\Phi}^2: , \rho_x^\varepsilon\rangle\langle :\Phi_1\mathbf{\Phi}^2: , \rho_z^\varepsilon\rangle$, one has

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{2}{N}\int C(x-z_1)\mathbf{E}\big(\langle :\mathbf{\Phi}^2: , \rho_x^\varepsilon\rangle\langle :\Phi_1\mathbf{\Phi}^2: , \rho_{z_1}^\varepsilon\rangle\langle :\Phi_1\mathbf{\Phi}^2: , \rho_z^\varepsilon\rangle\big)\,\mathrm{d}z_1\\
&\quad=\lim_{\varepsilon\to 0}\frac{N+2}{N}\int C(x-z_1)\mathbf{E}\big(\langle :\mathbf{\Phi}^2: , \rho_x^\varepsilon\rangle\langle :\mathbf{\Phi}^2: , \rho_{z_1}^\varepsilon\rho_z^\varepsilon\rangle\big)\,\mathrm{d}z_1\\
&\qquad+\lim_{\varepsilon\to 0}\Big[2\int C(x-z_1)\mathbf{E}\big(\langle\Phi_1\rho_{z_1}^\varepsilon, \rho_x^\varepsilon\rangle\langle :\Phi_1\mathbf{\Phi}^2: , \rho_z^\varepsilon\rangle\big)\,\mathrm{d}z_1\\
&\qquad-\mathbf{E}\big(\langle\Phi_1, \rho_x^\varepsilon\rangle\langle :\mathbf{\Phi}^2: , \rho_x^\varepsilon\rangle\langle :\Phi_1\mathbf{\Phi}^2: , \rho_z^\varepsilon\rangle\big)\Big]\\
&\quad=\lim_{\varepsilon\to 0}\frac{N+2}{N}C(x-z)\mathbf{E}\big[\langle :\mathbf{\Phi}^2: , \rho_x^\varepsilon\rangle\langle :\mathbf{\Phi}^2: , \rho_z^\varepsilon\rangle\big]-\lim_{\varepsilon\to 0}\mathbf{E}\big(\langle :\Phi_1\mathbf{\Phi}^2: , \rho_x^\varepsilon\rangle\langle :\Phi_1\mathbf{\Phi}^2: , \rho_z^\varepsilon\rangle\big),
\end{aligned}
$$

where the limit is understood in $\mathbf{C}^{-\kappa}$ and we again used symmetry under expectation. From the above two correlation identities, we cancel out the sixth-order correlation term and then obtain (6.23). Note that we also use Fourier transform to have

$$
\lim_{\varepsilon\to 0}\int\frac{N+2}{N^2}C(x-z)^2\mathbf{E}\big[\langle :\mathbf{\Phi}^2: , \rho_x^\varepsilon\rangle\langle :\mathbf{\Phi}^2: , \rho_z^\varepsilon\rangle\big]\,\mathrm{d}z=\sum_{k\in\mathbb{Z}^2}\frac{N+2}{N}\widehat{C^2}(k)\widehat{G_N}(k).
$$

Next, take $F(\Phi) = \int C(x-y)\langle :\Phi_1\mathbf{\Phi}^2: , \rho_y^\varepsilon\rangle\langle :\mathbf{\Phi}^2: , \rho_z^\varepsilon\rangle\,\mathrm{d}y$. We have, by (C.1),

$$
\begin{aligned}
&\frac{2}{N}\int C(x-y_1)C(x-y_2)\mathbf{E}\big(\langle :\Phi_1\mathbf{\Phi}^2: , \rho_{y_1}^\varepsilon\rangle\langle :\Phi_1\mathbf{\Phi}^2: , \rho_{y_2}^\varepsilon\rangle\langle :\mathbf{\Phi}^2: , \rho_z^\varepsilon\rangle\big)\,\mathrm{d}y_1\,\mathrm{d}y_2\\
&\quad=2\mathbf{E}\big(\langle\Phi_1, C_x^\varepsilon\rho_z^\varepsilon\rangle\langle :\Phi_1\mathbf{\Phi}^2: , C_x^\varepsilon\rangle\big)+\frac{N+2}{N}\mathbf{E}\big(\langle :\mathbf{\Phi}^2: , \rho_z^\varepsilon\rangle\langle :\mathbf{\Phi}^2: , (C_x^\varepsilon)^2\rangle\big)\\
&\qquad-\int C(x-y)\mathbf{E}\big(\langle\Phi_1, \rho_x^\varepsilon\rangle\langle :\Phi_1\mathbf{\Phi}^2: , \rho_y^\varepsilon\rangle\langle :\mathbf{\Phi}^2: , \rho_z^\varepsilon\rangle\big)\,\mathrm{d}y,
\end{aligned}
$$

with $C_x^\varepsilon(\cdot)\stackrel{\text{def}}{=}(C*\rho_0^\varepsilon)(x-\cdot)$. Note that as $\varepsilon\to 0$, the limit of the left-hand side is precisely Q_2^N and the limit of the first term on the right-hand side is just $-Q_1^N$. The limit of the Fourier transform of the second term on the right-hand side equals

$$
(N+2)\widehat{C^2}\widehat{G_N}.
$$

To deal with the last term above, taking $F(\Phi) = \langle\Phi_1, \rho_x^\varepsilon\rangle\langle :\mathbf{\Phi}^2: , \rho_z^\varepsilon\rangle$ and applying (C.1), one has

$$
\begin{aligned}
&2\int C(x-y)\mathbf{E}\big(\langle\Phi_1, \rho_x^\varepsilon\rangle\langle :\Phi_1\mathbf{\Phi}^2: , \rho_y^\varepsilon\rangle\langle :\mathbf{\Phi}^2: , \rho_z^\varepsilon\rangle\big)\,\mathrm{d}y\\
&\quad=N\int C(x-y)\langle\rho_x^\varepsilon, \rho_y^\varepsilon\rangle\,\mathrm{d}y\,\mathbf{E}\langle :\mathbf{\Phi}^2: , \rho_z^\varepsilon\rangle+2N\int C(x-z_1)\mathbf{E}\big(\langle\Phi_1, \rho_x^\varepsilon\rangle\langle\Phi_1, \rho_{z_1}^\varepsilon\rho_z^\varepsilon\rangle\big)\,\mathrm{d}z_1\\
&\qquad-N\mathbf{E}\big[\langle\Phi_1, \rho_x^\varepsilon\rangle^2\langle :\mathbf{\Phi}^2: , \rho_z^\varepsilon\rangle\big].
\end{aligned}
$$

Letting $\varepsilon\to 0$, we deduce the limit of the Fourier transform of the right-hand side is

$$
-N\widehat{G_N}+2N\widehat{CC_N}.
$$

We then obtain (6.24).

APPENDIX D: PROOF OF STEP 7 IN THE PROOF OF THEOREM 4.1

We write $\bar{I}^N$ as $\sum_{i=1}^3 \bar{I}_i^N$:

$$\bar{I}_1^N \stackrel{\text{def}}{=} -\frac{1}{N}\sum_{i,j=1}^N \big[\langle Y_j^2, v_i u_i\rangle + 2\langle Y_j Y_i u_j, v_i\rangle\big],$$

$$\bar{I}_{21}^N \stackrel{\text{def}}{=} -\frac{1}{N}\sum_{i,j=1}^N 2\langle Y_j v_i, \ {:}Z_i^N Z_j^N{:} - {:}Z_i Z_j{:}\ \rangle,$$

$$\bar{I}_{22}^N \stackrel{\text{def}}{=} -\frac{1}{N}\sum_{i,j=1}^N \langle Y_i v_i, \ {:}Z_j^{N,2}{:} - {:}Z_j^2{:}\ \rangle,$$

$$\bar{I}_3^N \stackrel{\text{def}}{=} -\frac{1}{N}\sum_{i,j=1}^N \langle v_i, \ {:}Z_i^N Z_j^{N,2}{:} - {:}Z_i Z_j^2{:}\ \rangle.$$

In the following, we estimate each term and show that for $\delta > 0$ small,

$$\begin{aligned} |\bar{I}^N| &\lesssim \delta\Bigg(\sum_{i=1}^N \|\nabla v_i\|_{L^2}^2 + \frac{1}{N}\sum_{i,j=1}^N \|Y_j v_i\|_{L^2}^2\Bigg) + \sum_{i=1}^N \|v_i\|_{L^2}^2 \\ &\quad + \Bigg(\sum_{i=1}^N \|u_i\|_{L^\infty}^2\Bigg)\Bigg(\frac{1}{N}\sum_{j=1}^N \|Y_j\|_{L^2}^2\Bigg) + \tilde{R}_N, \end{aligned} \tag{D.1}$$

with

$$\begin{aligned} \tilde{R}_N &\stackrel{\text{def}}{=} \frac{1}{N}\sum_{i,j=1}^N \|\ {:}Z_i^N Z_j^{N,2}{:} - {:}Z_i Z_j^2{:}\ \|_{\mathbf{C}^{-s}}^2 \\ &\quad + \Bigg(\frac{1}{N}\sum_{j=1}^N \|Y_j\|_{H^1}^2\Bigg)^s \mathfrak{Z}_N + \mathfrak{Z}_N + \Bigg(\frac{1}{N}\sum_{i=1}^N \|Y_i\|_{H^1}^2\Bigg)^s \Bigg(\frac{1}{N}\sum_{i=1}^N \|Y_i\|_{L^2}^2\Bigg)^{1-s} \bar{\mathfrak{Z}}_N, \end{aligned}$$

with $\bar{\mathfrak{Z}}_N$ and $\mathfrak{Z}_N$ introduced in (D.2) below.

For $\bar{I}_1^N$, we use Young's inequality to have

$$\begin{aligned} |\bar{I}_1^N| &\lesssim \delta\frac{1}{N}\sum_{i,j=1}^N \int Y_j^2 v_i^2 + \frac{1}{N}\sum_{i,j=1}^N \int Y_j^2 u_i^2 \\ &\lesssim \delta\frac{1}{N}\sum_{i,j=1}^N \int Y_j^2 v_i^2 + \Bigg(\frac{1}{N}\sum_{j=1}^N \|Y_j\|_{L^2}^2\Bigg)\Bigg(\sum_{i=1}^N \|u_i\|_{L^\infty}^2\Bigg), \end{aligned}$$

which gives the first contribution to (D.1).

For $\bar{I}_3^N$ we use Lemma A.3, Lemma A.2 and Young's inequality to have

$$\begin{aligned} |\bar{I}_3^N| &\lesssim \frac{1}{N}\sum_{i,j=1}^N \|v_i\|_{H^{2s}} \|\ {:}Z_i^N Z_j^{N,2}{:} - {:}Z_i Z_j^2{:}\ \|_{\mathbf{C}^{-s}} \\ &\lesssim \delta\sum_{i=1}^N \|\nabla v_i\|_{L^2}^2 + \sum_{i=1}^N \|v_i\|_{L^2}^2 + \frac{1}{N}\sum_{i,j=1}^N \|\ {:}Z_i^N Z_j^{N,2}{:} - {:}Z_i Z_j^2{:}\ \|_{\mathbf{C}^{-s}}^2, \end{aligned}$$

which gives the second contribution to (D.1).

For $\bar{I}_{2k}^N$, we set

$$\text{(D.2)}\qquad \mathfrak{Z}_N \stackrel{\text{def}}{=} \frac{1}{N}\sum_{i,j=1}^N \| :Z_i^N Z_j^N: - :Z_i Z_j: \|_{\mathbf{C}^{-s}}^2, \qquad \bar{\mathfrak{Z}}_N \stackrel{\text{def}}{=} \sum_{j=1}^N \| :Z_j^{N,2}: - :Z_j^2: \|_{\mathbf{C}^{-s}}^2.$$

We use Lemma A.5 and Hölder's inequality to have

$$\begin{aligned}
|\bar{I}_{21}^N| &\lesssim \frac{1}{N}\sum_{i,j=1}^N \big(\|\nabla(Y_j v_i)\|_{L^1}^s \|Y_j v_i\|_{L^1}^{1-s} + \|Y_j v_i\|_{L^1}\big)\| :Z_i^N Z_j^N: - :Z_i Z_j: \|_{\mathbf{C}^{-s}} \\
&\lesssim \delta\left(\frac{1}{N}\sum_{j=1}^N \|v_i Y_j\|_{L^2}^2\right) + \mathfrak{Z}_N \\
&\quad + \left(\sum_{i=1}^N \|v_i\|_{L^2}^2\right)^{s/2}\left(\frac{1}{N}\sum_{j=1}^N \|\nabla Y_j\|_{L^2}^2\right)^{s/2}\left(\frac{1}{N}\sum_{i,j=1}^N \|v_i Y_j\|_{L^2}^2\right)^{\frac{1-s}{2}} \mathfrak{Z}_N^{1/2} \\
&\quad + \left(\sum_{i=1}^N \|\nabla v_i\|_{L^2}^2\right)^{s/2}\left(\frac{1}{N}\sum_{j=1}^N \|Y_j\|_{L^2}^2\right)^{s/2}\left(\frac{1}{N}\sum_{i,j=1}^N \|v_i Y_j\|_{L^2}^2\right)^{\frac{1-s}{2}} \mathfrak{Z}_N^{1/2},
\end{aligned}$$

which by Young's inequality gives

$$\begin{aligned}
|\bar{I}_{21}^N| \lesssim{}& \delta\left(\frac{1}{N}\sum_{i,j=1}^N \|v_i Y_j\|_{L^2}^2\right) + \delta\left(\sum_{i=1}^N \|\nabla v_i\|_{L^2}^2\right) \\
&+ \left(\sum_{i=1}^N \|v_i\|_{L^2}^2\right) + \left(\frac{1}{N}\sum_{j=1}^N \|Y_j\|_{H^1}^2\right)^s \mathfrak{Z}_N + \mathfrak{Z}_N.
\end{aligned}$$

Similarly, we deduce

$$\begin{aligned}
|\bar{I}_{22}^N| &\lesssim \frac{1}{N}\sum_{i,j=1}^N \big(\|\nabla(Y_i v_i)\|_{L^1}^s \|Y_i v_i\|_{L^1}^{1-s} + \|Y_i v_i\|_{L^1}\big)\| :Z_j^{N,2}: - :Z_j^2: \|_{\mathbf{C}^{-s}} \\
&\lesssim \left(\sum_{i=1}^N \|v_i\|_{L^2}^2\right) + \bar{\mathfrak{Z}}_N\left(\frac{1}{N}\sum_{i=1}^N \|Y_i\|_{L^2}^2\right) \\
&\quad + \left(\sum_{i=1}^N \|v_i\|_{L^2}^2\right)^{1/2}\left(\frac{1}{N}\sum_{i=1}^N \|\nabla Y_i\|_{L^2}^2\right)^{s/2}\left(\frac{1}{N}\sum_{i=1}^N \|Y_i\|_{L^2}^2\right)^{\frac{1-s}{2}} \bar{\mathfrak{Z}}_N^{1/2} \\
&\quad + \left(\sum_{i=1}^N \|\nabla v_i\|_{L^2}^2\right)^{s/2}\left(\frac{1}{N}\sum_{i=1}^N \|Y_i\|_{L^2}^2\right)^{1/2}\left(\sum_{i=1}^N \|v_i\|_{L^2}^2\right)^{\frac{1-s}{2}} \bar{\mathfrak{Z}}_N^{1/2},
\end{aligned}$$

which implies

$$|\bar{I}_{22}^N| \lesssim \delta\left(\sum_{i=1}^N \|\nabla v_i\|_{L^2}^2\right) + \left(\sum_{i=1}^N \|v_i\|_{L^2}^2\right) + \left(\frac{1}{N}\sum_{i=1}^N \|Y_i\|_{H^1}^2\right)^s \left(\frac{1}{N}\sum_{i=1}^N \|Y_i\|_{L^2}^2\right)^{1-s} \bar{\mathfrak{Z}}_N.$$

Thus we deduce (D.1). By Assumption 4.1 and Lemma 2.1, it is easy to find $\frac{1}{N}\|\tilde{R}_N\|_{L_T^1} \to 0$ in $L^1(\Omega)$. The result follows by similar argument as Step 5 and Step 6.

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SUPPLEMENTARY MATERIAL

Supplement to "Large N limit of the $O(N)$ linear sigma model via stochastic quantization" (DOI: 10.1214/21-AOP1531SUPP; .pdf). Supplementary information.

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