Minimal Gap in the Spectrum of the Sierpiński Gasket

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This paper studies the size of the minimal gap between any two consecutive eigenvalues in the Dirichlet and in the Neumann spectrum of the standard Laplace operator on the Sierpiński gasket. The main result shows the remarkable fact that this minimal gap is achieved and coincides with the spectral gap. The Dirichlet case is more challenging and requires some key observations in the behavior of the dynamical system that describes the spectrum.

1 Introduction

The standard Laplacian on the Sierpiński gasket (SG) is a non-negative self-adjoint operator with a pure point spectrum, also called discrete [19, Theorem VII.10], which consists of countable many non-negative eigenvalues with finite multiplicity and only accumulation point at infinity. The study of this spectrum goes back to works in physics by Rammal and Tolouse [18, 20], where it was observed that suitable series of eigenvalues in the finite level approximations of SG produced an orbit of a particular dynamical system. This phenomenon was named in [11] *spectral decimation*. There, Fukushima and Shima were able to describe the complete spectrum of the Laplacian on SG by tracing back re-normalized orbits of the quadratic polynomial describing the aforementioned dynamical system.

Having a complete description of the spectrum led to extensive research, see for example, [14, 16, 22, 24, 25], dealing with the study of its properties. Some of

them are in stark contrast to more classical settings: the Weyl ratio does not have a limit and in particular the eigenvalue counting function $N(x)$ fails Weyl and Berry's conjecture $N(x) \sim c_d \mathcal{H}^d(SG)x^{d/2} + o(x^{d/2})$ with d the Hausdorff dimension and $\mathcal{H}^d$ the d-dimensional Hausdorff measure of SG, see [16] and [24, Example 5.1] for precise asymptotics. The oscillations that prevent the existence of a Weyl ratio are partly due to the existence of large gaps in the spectrum, meaning that if $\lambda_{(1)} \leq \lambda_{(2)} \leq \ldots$ denote the eigenvalues of the Laplacian, then $\limsup_n \lambda_{(n+1)}/\lambda_{(n)} > 1$, c.f. [12, 13, 22, 26].

As it turns out, the existence of these gaps in the spectrum of SG is equivalent to the fact that the Julia set describing the spectrum is totally disconnected [13, Theorem 2], a property shared by certain classes of fractals, see [13, 25] and the references therein. On the other hand, the presence of exponentially large gaps has some advantageous consequences: for instance, they are responsible for a "better than usual" convergence of the Fourier series analogue of a function in $L^2(SG)$, see [22, Theorem 1]. Also, they provide a natural candidate for dyadic intervals as described in Section 2.2, whose properties may be useful in the study of estimates involving eigenfunctions.

Most of the existent work has focused on the structure of large gaps [7, 12, 22, 26], whereas smaller ones have eluded further investigation. The study of small gaps in the spectrum of the Laplacian can become fairly challenging [3, 5, 9] and the question addressed in the present paper investigates what happens on SG: is it possible to provide a *uniform lower bound* for the small gaps in the spectrum of the Laplacian?

Theorem 3.1 provides a positive and optimal answer: any two consequent eigenvalues in the Dirichlet or in the Neumann spectrum of the Laplacian on SG are separated *at least* by the spectral gap. In other words, the first and the second eigenvalues are the closest within the whole spectrum. Although the existence of *a* lower bound might possibly be derived from an abstract argumentation, proving that the spectral gap *is in fact the smallest* spacing between Dirichlet eigenvalues requires a couple of rather non-trivial observations. These constitute the core of Section 3 and have been established after a careful analysis of the inverse function describing the spectrum of the finite level approximations. Note also that the size of the spacings do *not* appear in an straightforward increasing order: numerical computations show for example that the third gap is actually smaller than the second, and the seventh gap is smaller than the third, c.f. Table 1.

The paper is structured as follows: Section 2 reviews the construction of the Dirichlet and the Neumann spectrum of the Laplacian along with some facts about the large gaps that were not available in the literature in this form. The main result, Theorem 3.1, is presented and proved for both spectra in Section 3. Especially, the proof

of the Dirichlet case relies on several key properties of the dynamical system and the iterative construction of the spectrum. Possible directions for future investigation are briefly outlined in Section 4.

2   Preliminaries and Useful Facts

For each $m \geq 0$, let V_m denote the vertex set of the finite n-level approximation of the Sierpiński gasket (SG in the sequel) as depicted in Figure 1, and let Δ_m denote the associated graph Laplacian

$$\Delta_m u(p) := \sum_{q \overset{m}{\sim} p} (u(q) - u(p)), \tag{1}$$

with either Dirichlet or Neumann boundary conditions.

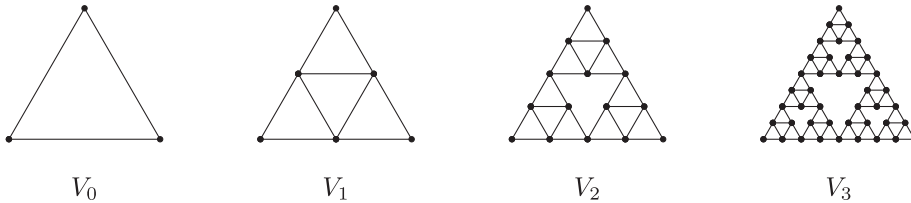V_0 $\qquad\qquad$ V_1 $\qquad\qquad$ V_2 $\qquad\qquad$ V_3

Fig. 1. Graph approximations of the Sierpiński gasket.

The standard Laplacian on SG, denoted by Δ in throughout the paper, is expressible as a limit of suitably renormalized graph Laplacians

$$\Delta u(x) = \lim_{m \to \infty} 5^m \Delta_m u(x),$$

see for example, [16], where the limit converges uniformly in x for a dense set of continuous functions in the domain of Δ. Further details concerning the precise construction and properties can be found in the books [15, 23]. This section reviews how the Dirichlet and Neumann spectrum of Δ_m describe that of Δ and records several useful facts about its structure.

2.1   Eigenvalues

Fukushima and Shima described in [11] all Dirichlet eigenvalues of the Laplacian on SG by means of a genealogical tree as in Figure 2; a similar one can be made for the Neumann case. In this way, any eigenvalue λ of Δ is related to a unique sequence of

ancestors $\{\lambda_m\}_{m \geq j}$ starting at a particular generation $j \geq 1$, where λ_m is an eigenvalue of Δ_m. The starting generation j is called the *generation of birth* of λ. First ancestors (who start a lineage) may only take the values 2, 5, or 6 and to refer to that particular value one speaks of λ as being an i-series eigenvalue, where $i = \lambda_j \in \{2, 5, 6\}$.
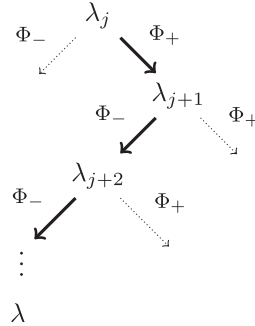
Fig. 2. Genealogy tree picture for an eigenvalue λ born at level j.

Fukushima–Shima's main result [11, Theorem 5.1] states that all Dirichlet eigenvalues λ of Δ satisfy

$$\lambda = \lim_{m \to \infty} 5^m \lambda_m \tag{2}$$

and the same is true for the Neumann ones. As Rammal and Toulouse observed in [20], the genealogy of an eigenvalue arises from the inverse functions of the quadratic polynomial $R(z) = z(5 - z)$ given by

$$\Phi_\pm(z) = \frac{1}{2}\big(5 \pm \sqrt{25 - 4z}\big). \tag{3}$$

Each generation of an eigenvalue λ is thus related to the immediate next by

$$\lambda_m = \lambda_{m+1}(5 - \lambda_{m+1}). \tag{4}$$

Once the first ancestor is born, its successor is determined by a choice of one of the inverse functions Φ_- or Φ_+ and the same for all subsequent generations. The genealogy of an eigenvalue λ with generation of birth $j \geq 1$ is thus described through a sequence $\{\lambda_{j+k}\}_{k \geq 0}$ with $\lambda_j := i \in \{2, 5, 6\}$ and

$$\lambda_{j+k} = \Phi_{w|_k}(\lambda_j) = \Phi_{w|_k}(i) := \Phi_{w_k} \circ \Phi_{w_{k-1}} \circ \cdots \circ \Phi_{w_1}(i) \tag{5}$$

for any $k \geq 1$, where $w := \ldots w_2 w_1 \in \{-, +\}^{\mathbb{N}}$. Such an eigenvalue is generically called an *i-series*. In view of (2) and (5) it also admits the expression

$$\lambda = 5^j \lim_{k \to \infty} 5^k \Phi_{w|_k}(i), \tag{6}$$

which will play a fundamental role in the proof of Theorem 3.1. Due to the properties of $\Phi_{\pm}$, c.f. Section 3, in order for the limit (6) to exist, there must be a specific generation $\ell \geq j$, called the *generation of fixation* of λ, after which all descendants are obtained via Φ_-. In other words, for any $m \geq \ell$,

$$\lambda_m = \Phi_-^{(m-\ell)}(\lambda_\ell) = \Phi_-^{(m-\ell)} \Phi_{w|_{\ell-1}}(i), \tag{7}$$

where $\Phi_-^{(n)}$ denotes the n-th composition of Φ_- with itself and $\Phi_{w|_0} = \mathrm{id}$. Note that $w_{\ell-1} = +$ as long as $\ell \geq 2$.

Example 2.1.   The lowest 5-series and 6-series Dirichlet eigenvalues have generation of birth 1, respectively 2, and generation of fixation 2, respectively 4. They admit the limit representation

$$\lim_{k \to \infty} 5^{k+1} \Phi_{w|_k}(5) =: \lambda_0^{(5)}, \quad \text{and} \quad \lim_{k \to \infty} 5^{k+2} \Phi_{w|_k}(6) =: \lambda_1^{(6)},$$

where in the first case $w_k = -$ for all $k \geq 1$ and in the second $w_1 = +$ and $w_k = -$ for all other $k \geq 2$. The notation used for these eigenvalues follows [7, Section 3] and depends on the word w in a non-trivial way, which is not discussed here. In view of (6), the lowest 5-series with generation of birth $j \geq 1$ satisfies

$$\lambda = \lim_{k \to \infty} 5^{j+k} \Phi_{w|_k}(5) = 5^{j-1} \lim_{k \to \infty} 5^{k+1} \Phi_-^{(k)}(5) = 5^{j-1} \lambda_0^{(5)},$$

while the lowest 6-series with generation of birth $j \geq 2$ is

$$\lambda = \lim_{k \to \infty} 5^{j+k} \Phi_{w|_k}(6) = 5^{j-2} \lim_{k \to \infty} 5^{k+2} \Phi_-^{(k-1)} \Phi_+(6) = 5^{j-2} \lambda_1^{(6)}.$$

2.2 Large gaps

The existence of large gaps in both the Dirichlet and the Neumann spectrum of the Laplacian has been extensively studied in the literature, see for example, [13, 22, 26]. This section gives a brief account of that phenomenon in the case of the Dirichlet

spectrum and records some observations of interest that had not appeared in this form yet.

Recall that Δ is a non-negative self-adjoint operator with discrete spectrum whose ordered eigenvalues will be denoted by $0 \leq \lambda_{(1)} \leq \lambda_{(2)} \leq \lambda_{(3)} \leq \ldots$ including multiplicity (the notation should not be confused with that of the eigenvalues of a finite level used in the previous section). Two of the most prominent recurrent gaps in the spectrum, described in [22, Theorem 1] and [12, Theorem 5.1], occur between the eigenvalues

$$\lambda_{(N_m - N_{m-1})} < \lambda_{(N_m)} < \lambda_{(N_m+1)}, \tag{8}$$

where $N_m = \frac{1}{2}(3^{m+1} - 3)$, $m \geq 1$, coincides with the size of the spectrum of Δ_m. The eigenvalues in (8) correspond to the lowest 5-series, the lowest 6-series, and the second lowest 5-series with generation of birth $j = m$ computed as in Example 2.1. More precisely,

$$\frac{\lambda_{(N_m)}}{\lambda_{(N_m - N_{m-1})}} = \frac{\lambda_1^{(6)}}{5\lambda_0^{(5)}} \approx 2.425 \qquad \text{and} \qquad \frac{\lambda_{(N_m+1)}}{\lambda_{(N_m)}} = \frac{5\lambda_1^{(5)}}{\lambda_1^{(6)}} \approx 1.271, \tag{9}$$

c.f. [12, Theorem 5.1]. There are more fractals for which similar statements hold, see for example, [10, 13] and the references therein.

2.3   Dyadic intervals

In applications like [22], the largest gaps from (9) can be used to decompose the positive real line into the analogue of classical "dyadic intervals." For any $m \geq 2$, these may be defined as

$$B_m := \begin{cases} [0, \lambda_{(N_2)}), & m = 2, \\ [\lambda_{(N_{m-1})}, \lambda_{(N_m)}), & m \geq 3, \end{cases} \tag{10}$$

see Figure 3.

Fig. 3. Eigenvalues in dyadic intervals are separated by the largest gaps (dotted).

As a consequence of (9), the distance between any two intervals increases with their index; the distance is in fact comparable to the magnitude of the first eigenvalue in the larger one.

Proposition 2.2.   Let $B_m, B_{m'}$ be intervals with $2 \leq m < m'$. Then,

$$\min\{|\lambda - \lambda'| : \ \lambda \in B_m, \lambda' \in B_{m'}\} > \frac{1}{2}\lambda_{(N_{m'-1})}.$$

Proof.   Let $g_0 := \frac{\lambda_1^{(6)}}{5\lambda_0^{(5)}} > 2$. By virtue of (9) and since $\lambda_{(N_{m'})} = 5^{m'-m}\lambda_{(N_m)}$ by construction,

$$\lambda_{(N_{m'-1})} - \lambda_{(N_m - N_{m-1})} = \lambda_{(N_{m'-1})}(1 - g_0^{-1} 5^{m-1-m'}) \geq \lambda_{(N_{m'-1})}(1 - g_0^{-1}) > \frac{1}{2}\lambda_{(N_{m'-1})},$$

where the last inequality holds because $g_0 > 2$.   ∎

The last observation in this paragraph refers to the fact that the gaps are actually so large that the sum of any two eigenvalues, which may belong to the same or to different intervals, always remains within the larger interval.

Proposition 2.3.   If $\lambda \in B_m$, $\lambda' \in B_{m'}$ are eigenvalues with $2 \leq m \leq m'$, then $\lambda + \lambda' \in B_{m'}$.

Proof.   Setting again $g_0 := \frac{\lambda_1^{(6)}}{5\lambda_0^{(5)}} > 2$, it follows from (10) and (9) that

$$\lambda_{(N_{m'-1})} < \lambda + \lambda' \leq \lambda_{(N_m - N_{m-1})} + \lambda_{(N_{m'} - N_{m'-1})}$$
$$= g_0^{-1}(\lambda_{(N_m)} + \lambda_{(N_m')}) \leq 2g_0^{-1}\lambda_{(N_{m'})} < \lambda_{(N_{m'})}.$$
   ∎

3   Small Gaps

The main result of the present paper establishes another remarkable property of both the Dirichlet and the Neumann spectrum of Δ: the minimal spacing between any two distinct eigenvalues *equals* the corresponding spectral gap and hence provides an optimal lower bound for the size of small gaps in the spectrum. The description of the eigenvalues as limits in (2) already suggests that the proof will rely in a careful analysis of the corresponding spectrum of the finite Laplacian Δ_m.

3.1 Minimal eigenvalue spacing

To state the main result precisely, let $\lambda_0^{(2)}$ and $\lambda_0^{(5)}$ denote the lowest 2-series and 5-series Dirichlet eigenvalue, and $\lambda_1^{(6)}$ the lowest non-zero Neumann eigenvalue of Δ. As limits of the form (6) they admit the expression

$$\lambda_0^{(2)} = \lim_{k\to\infty} 5^{k+1}\Phi_-^{(k)}(2), \quad \lambda_0^{(5)} = \lim_{k\to\infty} 5^{k+1}\Phi_-^{(k)}(5), \quad \lambda_1^{(6)} = \lim_{k\to\infty} 5^{k+2}\Phi_-^{(k-1)}\Phi_+(6) \qquad (11)$$

and also satisfy $\lambda_0^{(2)} < \lambda_0^{(5)} < \lambda_1^{(6)}$, c.f. [12, Theorem 5.1], see also Example 2.1. The following theorem provides the minimal eigenvalue spacing in both the Dirichlet and the Neumann spectrum; the Dirichlet case will be significantly more involved.

Theorem 3.1. The spacing between any two distinct eigenvalues in the (Dirichlet, or Neumann) spectrum of Δ is bounded below by the corresponding spectral gap. Precisely,

$$\min\{|\lambda - \lambda'|\colon \lambda \neq \lambda' \text{ Dirichlet eigenvalues of } \Delta\} = \lambda_0^{(5)} - \lambda_0^{(2)} \qquad (12)$$

and

$$\min\{|\lambda - \lambda'|\colon \lambda \neq \lambda' \text{ Neumann eigenvalues of } \Delta\} = \lambda_1^{(6)}. \qquad (13)$$

Proof of Theorem 3.1. We prove the result in the Dirichlet case, the Neumann case follows similarly. Let λ, λ' be two distinct Dirichlet eigenvalues with generations of birth $j, j' \geq 1$, respectively, generations of fixation $\ell \geq j, \ell' \geq j'$, and associated sequences $\{\lambda_{j+k}\}_{k\geq 0}$ and $\{\lambda'_{j'+k}\}_{k\geq 0}$. Without loss of generality, we may assume $\ell \leq \ell'$ and $\lambda_\ell \leq \lambda'_{\ell'}$. By construction, c.f. (7), for any $m \geq \ell'$, we have $\lambda_m, \lambda'_m \in \Phi_-^{(m-\ell')}(A_{\ell'}^D \backslash \{6\})$, where $A_{\ell'}^D$ denotes the Dirichlet spectrum of $\Delta_{\ell'}$. By virtue of Lemma 3.13,

$$5^m \lambda_m - 5^m \lambda'_m \geq 5^m \min\{|\lambda - \lambda'|\colon \lambda, \lambda' \in \Phi_-^{(m-\ell')}(A_{\ell'}^D \backslash \{6\})\}$$

$$\geq 5^m \Phi_-^{(m-1)}(5) - 5^m \Phi_-^{(m-1)}(2).$$

In view of (11), letting $m \to \infty$ yields (12). For the Neumann case, replace Lemma 3.13 by Lemma 3.15 noticing that $\Phi_+(6) = 3$, and use the fact that zero is a Neumann eigenvalue. ∎
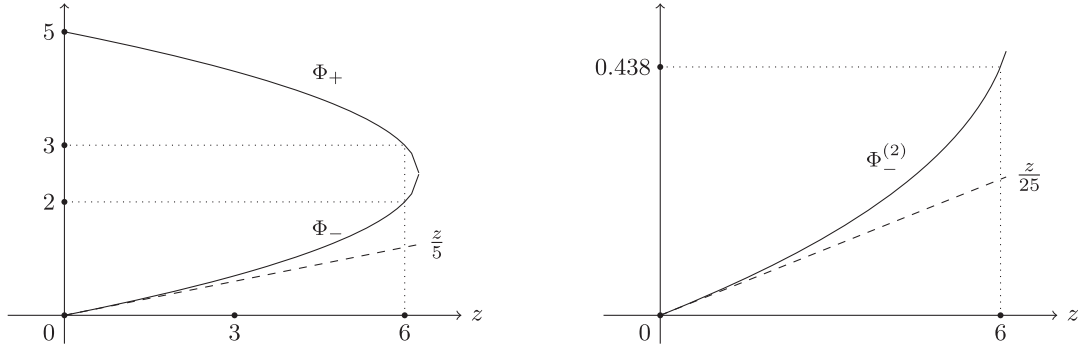
3.2   Key properties of the inverse functions

Recall from (3) that the functions $\Phi_\pm$ describing the spectrum of Δ are given by

$$\Phi_\pm(z) = \frac{1}{2}\left(5 \pm \sqrt{25 - 4z}\right)$$

and correspond to the inverse functions of the polynomial $R(z) = z(5 - z)$. Their graphs, displayed in Figure 4, provide a fairly good insight of the following properties.

Fig. 4. Inverse functions Φ_-, Φ_+, and $\Phi_-^{(2)}$. Note the rescaled y-axis in the latter.

Proposition 3.2.   For any $0 \le z \le 6$,

(i) $0 \le \Phi_-(z) \le 2$, $3 \le \Phi_+(z) \le 5$ and

$$\Phi_-(0) = 0, \quad \Phi_-(6) = 2, \quad \Phi_+(0) = 5 \quad \text{and} \quad \Phi_+(6) = 3.$$

(ii) $\Phi_+(z) - \Phi_-(z) = \sqrt{25 - 4z}$ and in particular $\Phi_-(z) < \Phi_+(z)$.

Lemma 3.3.   For any $0 \le x \le y \le 6$,

$$\Phi_-(y) - \Phi_-(x) = \Phi_+(x) - \Phi_+(y) = \frac{1}{2}(\sqrt{25 - 4x} - \sqrt{25 - 4y}).$$

A direct computation also shows that both Φ_- and the composition $\Phi_-^{(2)}$ have positive first derivative and are thus strictly convex. They also turn to be strongly convex, although only the former property will be used. The strict convexity of Φ_- and especially that of $\Phi_-^{(2)}$ will play a fundamental role in proving the key lemmas in Section 3.

Lemma 3.4. The functions Φ_- and $\Phi_-^{(2)}$ are strictly convex on $[0,6]$. In addition, for any $0 \le w \le x \le y \le z \le 6$,

(i) $\Phi_-(x) - \Phi_-(w) \le \Phi_-(z) - \Phi_-(y)$ for $x - w \le z - y$.

(ii) $\Phi_-(x) - \Phi_-(w) \le x - w$.

Proof.   To prove (i), the mean value theorem yields

$$\Phi_-(x) - \Phi_-(w) = \Phi_-'(\xi_{wx})(x - w) \qquad w \le \xi_{wx} \le x$$

and the same for y, z instead of w, x. Since $\xi_{wx} \le x \le y \le \xi_{yz}$ and $\Phi_-'(\xi) = (25 - 4\xi)^{-1/2}$, we have $\Phi_-'(\xi_{wx}) \le \Phi_-'(\xi_{yz})$, hence

$$\Phi_-(x) - \Phi_-(w) = \Phi_-'(\xi_{wx})(x - w) \le \Phi_-'(\xi_{yz})(z - y) = \Phi_-(z) - \Phi_-(y).$$

Part (ii) follows because $\Phi_-'([0,6]) \subset (0,1)$. ∎

Remark 1.   All items in the previous lemma hold with strict inequality when $0 \le w < x < y < z \le 6$.

3.3   Small gaps at finite level

This section analyzes the Dirichlet and Neumann spectrum of the finite graph Laplacian Δ_m. The types of computations in both cases are of the same nature; however, the Dirichlet case is strikingly less straightforward and requires a delicate analysis of the inverse function Φ_-.

In terms of general notation, $\Phi_\pm^{(m)}$ denotes the m-th concatenation of $\Phi_\pm$; for completeness $\Phi_\pm^{(0)} := \mathrm{id}$.

3.3.1   *Dirichlet spectrum*

Following [11, Theorem 3.1] and [14, Proposition 5.1], the Dirichlet spectrum of Δ_m can be described recursively as

$$A_0^D := \emptyset \qquad A_1^D := \{2, 5\} \qquad A_2^D := \Phi_\pm(A_1) \cup \{5, 6\}$$

$$A_m^D := \Phi_\pm(A_{m-1}^D \setminus \{6\}) \cup \{3, 5, 6\}, \quad m \ge 3. \tag{14}$$

The first step is to determine the difference between the first two eigenvalues in A_m^D.

Lemma 3.5.   For any $m \geq 1$, the first gap in the Dirichlet spectrum of Δ_m equals

$$\Phi_-^{(m-1)}(5) - \Phi_-^{(m-1)}(2).$$

Proof.   We prove by induction that

$$\min(A_m^D) = \Phi_-^{(m-1)}(2) \qquad \text{and} \qquad \min(A_m^D \setminus \{\Phi_-^{(m-1)}(2)\}) = \Phi_-^{(m-1)}(5).$$

Note that, due to the construction of A_m^D, the first three cases need to be considered separately. To ease the notation, we write A_m without superscript in what follows.

- Case $m = 1$. Clear by direct inspection since $\Phi_-^{(0)}(2) = 2$ and $\Phi_-^{(0)}(5) = 5$.
- Case $m = 2$. Again by direct inspection, Proposition 3.2 and Lemma 3.4 allow to describe elements of A_2 in increasing order as

$$A_2 = \{\Phi_-(2), \Phi_-(5), \Phi_+(5), \Phi_+(2), 5, 6\}.$$

- Case $m = 3$. By construction, $A_3 = \Phi_-(A_2 \setminus \{6\}) \cup \Phi_+(A_1 \setminus \{6\}) \cup \{3, 5, 6\}$. Proposition 3.2 and Lemma 3.4 now yield

$$\min(A_3) = \min(\Phi_-(A_2)) = \Phi_-(\min A_2) = \Phi_- \Phi_-(2).$$

Analogously, the second smallest element is $\Phi_-\big(\min(A_2 \setminus \{\Phi_-(2)\})\big) = \Phi_-^{(2)}(5)$.
- General case $m > 3$. Using the hypothesis of induction, the same arguments as before apply so that

$$\min(A_{m+1}) = \min(\Phi_-(A_m)) = \Phi_-(\min A_m) = \Phi_- \Phi_-^{(m-1)}(2) = \Phi_-^{(m)}(2) \quad (15)$$

and for the second smallest element $\Phi_-\big(\min(A_m \setminus \{\Phi_-^{(m-1)}(2)\})\big) = \Phi_-^{(m)}(5)$.

∎

An immediate consequence of (15) and the fact that Φ_+ is monotone decreasing is that $\Phi_+\big(\min(A_m^D \setminus \{5, 6\})\big) = \max\big(\Phi_+(A_m^D \setminus \{5, 6\})\big)$. This provides the largest 2-series eigenvalue in A_m^D, which will be relevant in the proof of the main Lemma 3.12.

Corollary 3.6.   For any $m \geq 2$, $\max(A_m^D \setminus \{5, 6\}) = \Phi_+ \Phi_-^{(m-2)}(2)$.

Remark 2.   The eigenvalue $\Phi_+ \Phi_-^{(m-2)}(2)$ is the closest to the eigenvalue 5 in A_m^D.

The next two technical observations are fundamental in the quest of finding the exact minimal gap within a subset of the Dirichlet spectrum of Δ_m that will be relevant for Theorem 3.1. The first one concerns the ratio between the second smallest eigenvalue at level m and the spectral gap at level $m+1$: this ratio gets larger as the level increases.

Lemma 3.7.   For any $m \geq 1$,

$$\frac{\Phi_-^{(m-1)}(5)}{\Phi_-^{(m)}(5) - \Phi_-^{(m)}(2)} < \frac{\Phi_-^{(m)}(5)}{\Phi_-^{(m+1)}(5) - \Phi_-^{(m+1)}(2)}. \tag{16}$$

Proof.   Let $m \geq 1$. Multiplying and dividing by $\Phi_-^{(m)}(5)$ the left hand side of (16), the mean value theorem and the strict convexity of Φ_- yield

$$\frac{\Phi_-^{(m-1)}(5)}{\Phi_-^{(m)}(5) - \Phi_-^{(m)}(2)} < \frac{\Phi_-^{(m-1)}(5)}{\Phi_-^{(m)}(5)} \Phi_-'(\Phi_-^{(m)}(5)) \frac{\Phi_-^{(m)}(5)}{\Phi_-^{(m+1)}(5) - \Phi_-^{(m+1)}(2)}.$$

It remains to show that

$$\frac{\Phi_-^{(m-1)}(5)}{\Phi_-^{(m)}(5)} \Phi_-'(\Phi_-^{(m)}(5)) < 1. \tag{17}$$

Using the relation (4) with $\lambda_{m-1} = \Phi_-^{(m-1)}(5)$ and $\lambda_m = \Phi_-^{(m)}(5)$, and the explicit expression of the derivative Φ_-', condition (17) is equivalent to

$$\frac{5 - \Phi_-^{(m)}(5)}{\sqrt{25 - 4\Phi_-^{(m)}(5)}} < 1 \quad \Leftrightarrow \quad (5 - \Phi_-^{(m)}(5))^2 < 25 - 4\Phi_-^{(m)}(5) \quad \Leftrightarrow \quad \Phi_-^{(m)}(5) < 6,$$

which is clearly true for any $m \geq 1$.   ∎

The second observation relies on the former and pertains the fact that the ratio between the smallest eigenvalue at level m and the spectral gap at level $m + 2$ becomes larger as the level increases. The reason for comparing between "two-level steps" instead of consecutive levels is not obvious; this insight was gained after a thorough numerical analysis, which also hints that the ratio grows *slower* as the level increases.

Lemma 3.8.   For any $m \geq 1$,

$$\frac{\Phi_-^{(m)}(2)}{\Phi_-^{(m+2)}(5) - \Phi_-^{(m+2)}(2)} > \frac{\Phi_-^{(m-1)}(2)}{\Phi_-^{(m+1)}(5) - \Phi_-^{(m+1)}(2)}. \tag{18}$$

Proof.   Let $m \geq 1$. Adding and subtracting $\Phi_-^{(m+1)}(5)$ in the numerator of the left hand side of (18), applying the mean value theorem and since $\Phi_-^{(m)}(5) > \Phi_-^{(m+1)}(5)$ we obtain

$$\begin{aligned}
\frac{\Phi_-^{(m)}(2)}{\Phi_-^{(m+2)}(5) - \Phi_-^{(m+2)}(2)} &= \frac{\Phi_-^{(m)}(2) - \Phi_-^{(m+1)}(5)}{\Phi_-^{(m+2)}(5) - \Phi_-^{(m+2)}(2)} + \frac{\Phi_-^{(m+1)}(5)}{\Phi_-^{(m+2)}(5) - \Phi_-^{(m+2)}(2)} \\
&> \frac{\Phi_-^{(m-1)}(2) - \Phi_-^{(m)}(5)}{\Phi_-^{(m+1)}(5) - \Phi_-^{(m+1)}(2)} + \frac{\Phi_-^{(m+1)}(5)}{\Phi_-^{(m+2)}(5) - \Phi_-^{(m+2)}(2)},
\end{aligned} \tag{19}$$

where the inequality follows from strict convexity . Reordering terms we may write (19) as

$$\frac{\Phi_-^{(m-1)}(2)}{\Phi_-^{(m+1)}(5) - \Phi_-^{(m+1)}(2)} + \left(\frac{\Phi_-^{(m+1)}(5)}{\Phi_-^{(m+2)}(5) - \Phi_-^{(m+2)}(2)} - \frac{\Phi_-^{(m)}(5)}{\Phi_-^{(m+1)}(5) - \Phi_-^{(m+1)}(2)} \right).$$

By virtue of Lemma 3.9, the quantity in brackets is strictly positive, hence (19) implies

$$\frac{\Phi_-^{(m)}(2)}{\Phi_-^{(m+2)}(5) - \Phi_-^{(m+2)}(2)} > \frac{\Phi_-^{(m-1)}(2)}{\Phi_-^{(m+1)}(5) - \Phi_-^{(m+1)}(2)}$$

as we wanted to prove.   ■

The previous observations are crucial to obtain the main lemma used in the proof of Theorem 3.1; it determines the minimal spacing between any consecutive eigenvalues in level $m + k$ with generation of fixation $\ell \leq m + 1$. The reason why it is enough to focus on those is that, if an eigenvalue λ_{m+k} has a higher level of fixation, it is always possible to find a suitable level $m' > m$, where its successor $\lambda_{m'+k-1}$ will have generation of fixation $\ell = m' + 1$.

The complete argument proceeds by a double induction on m and k starting at level $m = 3$, $k = 2$. Starting at $k = 2$ turns to be key and the proof of this initial case is rather non-trivial, in particular in view of the following observation that is discussed at the end of this section.

Remark 3.   Taking on account all Dirichlet eigenvalues at level m, that is, $k = 0$, yields

$$\min\{|\lambda - \lambda'|: \lambda, \lambda' \in A_m^D, \lambda \neq \lambda'\} = \Phi_-^{(m-1)}(2),$$

which is *strictly smaller* than (20). One can also see that this minimal spacing occurs between the eigenvalues $\lambda = 5$ and $\lambda' = \Phi_+\Phi_-^{(m-2)}(2)$. In the case $k = 1$, the corresponding minimum is again *strictly smaller* than (20); there is strong numerical evidence that it corresponds to the spacing between $\lambda = \Phi_-(5)$ and $\lambda' = \Phi_-\big(\Phi_+\Phi_-^{(m-2)}(2)\big)$.

Lemma 3.9.   For any $m \geq 3$,

$$\min\{|\lambda' - \lambda|: \ \lambda, \lambda' \in \Phi_-^{(2)}\big(A_m^D\backslash\{6\}\big), \ \lambda \neq \lambda'\} = \Phi_-^{(m+1)}(5) - \Phi_-^{(m+1)}(2). \tag{20}$$

The minimum is attained for $\lambda = \Phi_-^{(m+1)}(2)$ and $\lambda' = \Phi_-^{(m+1)}(5)$.

Proof.   To ease the notation, we write A_m without superscript.

- Case $m = 3$. We start by describing explicitly the elements of the set $A_3\backslash\{6\}$ in increasing order

$$\{\Phi_-^{(2)}(2), \Phi_-^{(2)}(5), \Phi_-\Phi_+(5), \Phi_-\Phi_+(2), \Phi_-(5), 3, \Phi_+(5), \Phi_+^{(2)}(2), \Phi_+^{(2)}(5),$$
$$\Phi_+\Phi_-(5), \Phi_+\Phi_-(2), 5\}.$$

  The size of the spacing between any two consecutive eigenvalues $\Phi_-^{(2)}(\lambda)$, $\Phi_-^{(2)}(\lambda')$ with $\lambda, \lambda' \in A_3\backslash\{6\}$ can be explicitly computed and compared to the size of the Dirichlet spectral gap at level $m = 5$. By virtue of Lemma 3.6 that first gap is $g_1 := \Phi_-^{(4)}(5) - \Phi_-^{(4)}(2)$. Let now denote $g_2, \ldots, g_{11}$ the values of the gaps between consecutive eigenvalues of Δ_5 in $\Phi_-^{(2)}\big(A_3\backslash\{6\}\big)$. Table 1 shows the *explicit difference* (computed here using Python, values approximated to four digits) between the size of each of these gaps and size of the spectral (first) gap. All non-trivial differences are strictly positive, hence g_1 is the smallest gap.

Table 1   Differences in size between gaps g_i and the first gap g_1

i	1	2	3	4	5	6	7	8	9	10	11
$g_i - g_1$	0	0.0164	0.0061	7.9131	0.0758	0.0303	0.0039	0.0149	0.0395	0.0108	0.0005

- $(m) \Rightarrow (m+1)$. Let $\lambda, \lambda' \in \Phi_-^{(2)}(A_{m+1}\backslash\{6\})$ with $\lambda < \lambda'$ and $\lambda = \Phi_-(\mu)$, $\lambda' = \Phi_-(\mu')$ for some $\mu < \mu'$ in $\Phi_-(A_{m+1}\backslash\{6\})$. Since by construction $A_{m+1}\backslash\{6\} = \Phi_\pm(A_m) \cup \{3, 5\}$, we have that

$$\mu, \mu' \in \Phi_-^{(2)}(A_m\backslash\{6\}) \cup \{\Phi_-(3)\} \cup \Phi_-\Phi_+(A_m\backslash\{6\}) \cup \{\Phi_-(5)\}.$$

We analyze the possible situations for the gap $\lambda' - \lambda$ based on the subsets μ and μ' belong.

(a) If $\mu, \mu' \in \Phi_-^{(2)}(A_m\backslash\{6\})$, the strict convexity of Φ_- and the induction hypothesis yield

$$\lambda' - \lambda = \Phi_-(\mu') - \Phi_-(\mu) \geq \Phi_-'(\xi_{\Phi_-^{(m+1)}(2),\Phi_-^{(m+1)}(5)})(\mu' - \mu)$$

$$\geq \Phi_-'(\xi_{\Phi_-^{(m+1)}(2),\Phi_-^{(m+1)}(5)})\big(\Phi_-^{(m+1)}(5) - \Phi_-^{(m+1)}(2)\big) = \Phi_-^{(m+2)}(5) - \Phi_-^{(m+2)}(2).$$

(b) If $\mu, \mu' \in \Phi_-\Phi_+(A_m\backslash\{6\})$, then $\mu = \Phi_-\Phi_+(\nu)$ and $\mu' = \Phi_-\Phi_+(\nu')$ for some $\nu' < \nu$ belonging to $A_m\backslash\{6\}$. In particular, c.f. Lemma 3.4, $\Phi_+(\nu) > \Phi_-(\nu)$, which together with the strict convexity of $\Phi_-^{(2)}$ and Lemma 3.3 yields

$$\lambda' - \lambda = \Phi_-^{(2)}\Phi_+(\nu') - \Phi_-^{(2)}\Phi_+(\nu) = \Phi_-^{(2)'}(\xi_{\Phi_+(\nu'),\Phi_+(\nu)})(\Phi_+(\nu') - \Phi_+(\nu))$$

$$> \Phi_-^{(2)'}(\xi_{\Phi_-(\nu'),\Phi_-(\nu')})(\Phi_+(\nu') - \Phi_+(\nu)) = \Phi_-^{(2)'}(\xi_{\Phi_-(\nu'),\Phi_-(\nu')})(\Phi_-(\nu') - \Phi_-(\nu))$$

$$= \Phi_-^{(3)}(\nu') - \Phi_-^{(3)}(\nu).$$

Since $\Phi_-^{(3)}(\nu'), \Phi_-^{(3)}(\nu) \in \Phi_-^{(2)}(A_m\backslash\{6\})$, the previous case (a) applies and hence $\lambda' - \lambda > \Phi_-^{(m+2)}(5) - \Phi_-^{(m+2)}(2)$.

(c) If $\mu = \Phi_-(3)$ or $\mu' = \Phi_-(3)$, we show that $\mu' = \Phi_-\Phi_+(5)$ or $\mu = \Phi_-^{(2)}(5)$, respectively, whence $\lambda, \lambda' \in \Phi_-^{(2)}(A_3\backslash\{6\})$ and the claim follows from the induction start $m = 3$. Indeed, since $3 = \Phi_+(6)$, its closest eigenvalues in $A_{m+1}\backslash\{6\}$ are $\Phi_+(5)$ and $\Phi_-(5)$.

(d) If $\mu' = \Phi_-(5)$, Corollary 3.7 implies $\mu = \Phi_-\Phi_+\Phi_-^{(m-1)}(2)$ so that $\lambda = \Phi_-^{(2)}\Phi_+\Phi_-^{(m-1)}(2)$ and $\lambda' = \Phi_-^{(2)}(5)$. Using the strict convexity of $\Phi_-^{(2)}$ and the explicit expression of the derivative $\Phi_-^{(2)'}(z) = \Phi_-'(\Phi_-(z))\Phi_-'(z)$, we

obtain

$$
\begin{aligned}
\lambda' - \lambda &= \Phi_-^{(2)'}(\xi_{\Phi_+\Phi_-^{(m-1)}(2),5})\big(5 - \Phi_+\Phi_-^{(m-1)}(2)\big) \\
&= \Phi_-^{(2)'}(\xi_{\Phi_+\Phi_-^{(m-1)}(2),5})\Phi_-^{(m)}(2) \\
&> \Phi_-^{(2)'}(\Phi_+\Phi_-^{(m-1)}(2))\Phi_-^{(m)}(2) \\
&= \Phi_-'(\Phi_-\Phi_+\Phi_-^{(m-1)}(2))\Phi_-'(\Phi_+\Phi_-^{(m-1)}(2)))\Phi_-^{(m)}(2) \\
&= \frac{\Phi_-^{(m)}(2)}{\sqrt{25 - 4\Phi_-\Phi_+\Phi_-^{(m-1)}(2)}\sqrt{25 - 4\Phi_+\Phi_-^{(m-1)}(2))}}.
\end{aligned}
$$

Multiplying and dividing the latter by $\Phi_-^{(m+2)}(5) - \Phi_-^{(m+2)}(2)$ it follows that

$$
\lambda' - \lambda > \frac{\Phi_-^{(m)}(2)/\big(\Phi_-^{(m+2)}(5) - \Phi_-^{(m+2)}(2)\big)}{\sqrt{25 - 4\Phi_-\Phi_+\Phi_-^{(m-1)}(2)}\sqrt{25 - 4\Phi_+\Phi_-^{(m-1)}(2))}}\big(\Phi_-^{(m+2)}(5) - \Phi_-^{(m+2)}(2)\big). \tag{21}
$$

Note now that $m > 3$, hence $\Phi_-\Phi_+\Phi_-^{(m-1)}(2) > \Phi_-\Phi_+\Phi_-^{(2)}(2)$ and $\Phi_+\Phi_-^{(m-1)}(2) > \Phi_+\Phi_-^{(2)}(2)$. Together with Lemma 3.10, the right hand side of (21) is bounded below by

$$
\frac{\Phi_-^{(3)}(2)/\big(\Phi_-^{(5)}(5) - \Phi_-^{(5)}(2)\big)}{\sqrt{25 - 4\Phi_-\Phi_+\Phi_-^{(2)}(2)}\sqrt{25 - 4\Phi_+\Phi_-^{(2)}(2)}}\big(\Phi_-^{(m+2)}(5) - \Phi_-^{(m+2)}(2)\big)
$$

and an explicit computation reveals that

$$
\frac{\Phi_-^{(3)}(2)/\big(\Phi_-^{(5)}(5) - \Phi_-^{(5)}(2)\big)}{\sqrt{25 - 4\Phi_-\Phi_+\Phi_-^{(2)}(2)}\sqrt{25 - 4\Phi_+\Phi_-^{(2)}(2)}} > 1
$$

whence $\lambda' - \lambda > \Phi_-^{(m+2)}(5) - \Phi_-^{(m+2)}(2)$ as we wanted to prove. ∎

Lemma 3.10.   For any $m \geq 3$ and $k \geq 2$,

$$
\min\{|\lambda' - \lambda|: \ \lambda, \lambda' \in \Phi_-^{(k)}\big(A_m^D\backslash\{6\}\big)\} = \Phi_-^{(m+k-1)}(5) - \Phi_-^{(m+k-1)}(2). \tag{22}
$$

The minimum is attained for $\lambda = \Phi_-^{(m+k-1)}(2)$ and $\lambda' = \Phi_-^{(m+k-1)}(5)$.

Proof.   To ease the notation, we write A_m without superscript. Consider first $m = 3$ and apply induction on k:

- Case $k = 2$. See Lemma 3.12.
- $(k) \Rightarrow (k + 1)$. Let $\lambda < \lambda'$ belong to $\Phi_-^{(k+1)}(A_3 \backslash \{6\})$. Then, $\lambda = \Phi_-(\mu)$ and $\lambda' = \Phi_-(\mu')$ for some $\mu, \mu' \in \Phi_-^{(k)}(A_3 \backslash \{6\})$ with $\mu < \mu'$ and $\mu \geq \Phi_-^{(k+2)}(2)$, $\mu' \geq \Phi_-^{(k+2)}(5)$. Applying the mean value theorem, the strict convexity of Φ_-, and the hypothesis of induction yields

$$\lambda' - \lambda = \Phi_-(\xi_{\mu,\mu'}))(\mu' - \mu)$$
$$\geq \Phi_-(\xi_{\Phi_-^{(k+2)}(2), \Phi_-^{(k+2)}(5)}))(\Phi_-^{(k+2)}(5) - \Phi_-^{(k+2)}(2)) = \Phi_-^{(k+3)}(5) - \Phi_-^{(k+3)}(2).$$

Now we perform induction over the parameter m: assuming that the claim is true for m and any $k \geq 2$, we prove that it also holds for (fixed) $m + 1$ and any $k \geq 2$.

- Case $k = 2$. See Lemma 3.12.
- $(k) \Rightarrow (k + 1)$. Verbatim to the case $m = 3$ substituting 2 by m.   ∎

As pointed out in Remark 3.11, including all Dirichlet eigenvalues of Δ_m provides a bound that is *strictly lower* than (20). Indeed, Corollary 3.7, Proposition 3.2(i), and Lemma 3.3 yield

$$5 - \Phi_+ \Phi_-^{(m-2)}(2) = \Phi_+(5) - \Phi_+ \Phi_-^{(m-2)}(2) = \Phi_-^{(m-1)}(2),$$

which is *strictly smaller* than $\Phi_-^{(m-1)}(5) - \Phi_-^{(m-1)}(2)$: for $m = 1$ this is clear ($2 < 5 - 2$) and in general, using the fact that $\Phi_-^{(m)}(0) = 0$ for any $m \geq 1$, strict convexity and induction yield

$$\frac{\Phi_-^{(m)}(2)}{\Phi_-^{(m)}(5) - \Phi_-^{(m)}(2)} = \frac{\Phi_-^{(m)}(2) - \Phi_-^{(m)}(0)}{\Phi_-^{(m)}(5) - \Phi_-^{(m)}(2)} < \frac{\Phi_-^{(m-1)}(2) - \Phi_-^{(m-1)}(0)}{\Phi_-^{(m-1)}(5) - \Phi_-^{(m-1)}(2)} < 1.$$

3.3.2   *Neumann spectrum*

From a general point of view, the description of the Neumann spectrum of Δ_m is similar the Dirichlet one; however, changing from Dirichlet to Neumann makes the question at hand much easier to tackle. In particular, the absence of 2-series eigenvalues happens to be especially advantageous. We recall for example, from [14,Proposition 5.5] that the

Neumann spectrum can be described recursively as

$$A_0^N := \{0, 6\} \qquad A_1^N := \{0, 3, 6\} \qquad A_m^N := \Phi_\pm(A_{m-1}^N \backslash \{6\}) \cup \{3, 6\} \quad m \geq 2.$$

Again, note that in this case there are no eigenvalues starting from 2, whereas the eigenvalue 5 does appear in all A_m^N with $m \geq 2$ since $\Phi_+(0) = 5$.

Lemma 3.11.   For any $m \geq 1$, the Neumann spectral gap of Δ_m equals $\Phi_-^{(m-1)}(3)$.

Proof.   By similar arguments as to those in Lemma 3.6 one proves by induction that

$$\min(A_m^N \backslash \{0\}) = \Phi_-^{(m-1)}(3).$$

∎

Obtaining the minimal spacing between Neumann eigenvalues of Δ_m is significantly easier than in the Dirichlet case.

Lemma 3.12.   For any $m \geq 1$,

$$\min\{|\lambda - \lambda'|: \lambda, \lambda' \in A_m^N, \lambda \neq \lambda'\} = \Phi_-^{(m-1)}(3).$$

The minimum is attained for $\lambda = 0$ and $\lambda' = \Phi_-^{(m-1)}(3)$.

Proof.   To ease the notation, we write A_m^N without superscript. The first two cases are treated separately and induction starts at $m = 3$:

- Case $m = 1$. By direct inspection, since $A_1 = \{0, 3, 6\}$ the minimum is 3.
- Case $m = 2$. Writing explicitly $A_2 = \{0, \Phi_-(3), 3, \Phi_+(3), 5, 6\}$ and applying Proposition 3.2 and Lemma 3.3 gives the bounds $3 - \Phi_-(3) > 1 > \Phi_-(3)$ and $5 - \Phi_+(3) = \Phi_-(3)$. In addition, a direct computation shows $\Phi_+(3) - 3 \approx 1.303 > \Phi_-(3) \approx 0.6972$.
- Case $m = 3$. By construction, the elements of A_3 in increasing order are

$$\Phi_-\big(A_2 \backslash \{6\}\big) \cup \{3\} \cup \Phi_+\big(A_2 \backslash \{6\}\big) \cup \{6\}.$$

Given a pair of consecutive eigenvalues (λ, λ') with $\lambda < \lambda'$: if both belong to $\Phi_-(A_2 \backslash \{6\})$, then there exist $\mu', \mu \in A_2 \backslash \{6\}$ with $\mu' - \mu \geq \Phi_-(3)$ from the

previous case so that Lemma 3.4(i) yields

$$\lambda' - \lambda = \Phi_-(\mu) - \Phi_-(\mu') \geq \Phi_-\Phi_-(3) = \Phi_-^{(2)}(3).$$

The same is valid for $\lambda, \lambda' \in \Phi_+(A_2\backslash\{6\})$ by Lemma 3.3. If $\lambda = 3 = \Phi_+(6)$, then $\lambda' = \Phi_+(5)$ and Lemma 3.3 yields $\lambda' - \lambda = (\sqrt{5} - 1)/2 \approx 0.618 > \Phi_-^{(2)}(3) \approx 0.0351$. If $\lambda' = 3$, then $\lambda = \Phi_-(5)$ and Proposition 3.2 yields $\lambda' - \lambda > 1$.

- Induction. Writing again $A_{m+1} = \Phi_-(A_m\backslash\{6\}) \cup \{3\} \cup \Phi_+(A_m\backslash\{6\}) \cup \{6\}$ the hypothesis of induction yields

$$\Phi_-^{(m-1)}(3) \leq \mu' - \mu \qquad \text{for any } \mu < \mu' \text{ in } A_m\backslash\{6\}.$$

If the pair of consecutive eigenvalues (λ, λ') has both $\lambda, \lambda' \in \Phi_-(A_m\backslash\{6\})$, there are $\mu \geq 0$ and $\mu' \geq \Phi_-^{(m-1)}(3)$ such that

$$\lambda' - \lambda = \Phi_-(\mu) - \Phi_-(\mu') \geq \Phi_-\Phi_-^{(m-1)}(3) = \Phi_-^{(m)}(3),$$

where the inequality follows from Lemma 3.4. The same applies to $\lambda, \lambda' \in \Phi_+(A_m\backslash\{6\})$ by virtue of Lemma 3.3. Further, since $3 = \Phi_+(6)$, its closest eigenvalues in A_{m+1} are $\Phi_-(5)$ and $\Phi_+(5)$. These gaps are included in the case $m = 3$, and bounded below by $\Phi_-^{(m)}(3)$ by the strict convexity of Φ_-. Finally, we also note that $\min\{6 - z : z \in \Phi_+(A_m\backslash\{6\})\} = 6 - 5 = 1 > \Phi_-^{(m)}(3)$, hence the proof is complete. ■

4   Conclusion and Final Remarks

Laplace operators on fractals and their spectrum are a recurrent object of study in the physics literature [1, 6, 8, 17, 20] and in particular play a fundamental role in the study of Schrödinger and wave equations [2, 4]. The present paper investigates the size of the smallest possible gap between any two consecutive Dirichlet or Neumann eigenvalues of the Laplacian on the SG, which turns out to coincide with the corresponding spectral gap, meaning the distance between the first two eigenvalues; see Theorem 3.1. The proof relies on the properties of the dynamical system that describes the spectrum and it opens the possibility that a similar result is true for a class of fractals whose spectrum enjoys similar properties. One possible starting point would be to consider fractals whose Laplacian admits spectral decimation such as those described in [21], and to find

general conditions for the associated inverse functions that would produce the desired outcome.

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