Macdonald Operators and Quantum Q-Systems for Classical Types

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To Nicolai Reshetikhin on his 60th birthday

Abstract We propose solutions of the quantum Q-systems of types B_N, C_N, D_N in terms of q-difference operators, generalizing our previous construction for the Q-system of type A. The difference operators are interpreted as q-Whittaker limits of discrete time evolutions of Macdonald-van Diejen type operators. We conjecture that these new operators act as raising and lowering operators for q-Whittaker functions, which are special cases of graded characters of fusion products of KR-modules.

1 Introduction

The characters of tensor products of KR-modules of Yangians, quantum affine algebras, or affine algebras have fermionic formulas, generalizing Bethe's original counting formula of the Bethe eigenstates of the Heisenberg spin chain [Bet31]. The fermionic formulas, in the case of KR-modules of Yangians of the classical Lie algebras $\mathfrak{g} = ABCD$, were conjectured by Kirillov and Reshetikhin [KR87] and further generalized in [HKO+99]. They were proved for the case of any simple Lie algebra $\mathfrak{g}$ in [AK07, DFK08]. In the course of the proof, it becomes clear that there is

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a close connection between solutions of recursion relations known as the Q-systems and the fermionic formulas.

In [KR87, HKO$^+$99], q-analogs of the fermionic characters of tensor products of KR-modules, or graded characters, are also presented. There are several interpretations of the grading of these tensor products, which gives rise to these q-analogues, or graded characters: (1) as linearized energy function for the corresponding generalized inhomogeneous Heisenberg spin chain, (2) as a charge function for the crystal limit of the corresponding quantum affine algebra modules, when it exists, or (3) as the natural grading of the underlying affine algebra [FL99]. The latter definition was used in [AK07, DFK14, Lin19] to prove the q-graded fermionic character formulas. It turns out that there is a close connection between the graded character formulas and q-deformed versions of Q-systems known as the quantum Q-systems.

Quantum Q-systems are defined by quantizing [BZ05] the cluster algebraic [FZ02] structure of the classical Q-systems [Ked08, DFK09]. They are recursion relations for non-commuting variables $\{\mathcal{Q}_{a,k}\}$, with a running over the Dynkin labels of $\mathfrak{g}$ and $k \in \mathbb{Z}$.

Graded characters are Weyl-symmetric functions with coefficients in $\mathbb{Z}_+[q]$. The relation with the quantum Q-system can be schematically described as follows (details can be found in [DFK14]). One can construct a linear functional ϕ from the ordered product of (opposite, with parameter q^{-1}) quantum Q-system solutions $\mathcal{Q}_{a,k}^*$ to the graded characters of the corresponding tensor product:[1]

$$\chi_{\mathbf{n}}(q^{-1}; \mathbf{x}) = \phi \left(\overrightarrow{\prod_{a,k}} (\mathcal{Q}_{a,k}^*)^{n_{a,k}} \right), \tag{1.1}$$

where $\chi_{\mathbf{n}}(q; \mathbf{x})$ is the graded character of the tensor product (or fusion product in the sense of [FL99]) of KR-modules $\otimes_{a,k} \mathrm{KR}_{k\omega_a}^{\otimes n_{a,k}}$, $\mathbf{n} = \{n_{a,k}\}$ denoting the collection of tensor powers, and $\mathbf{x} = (x_1, x_2, \ldots, x_N)$. Here, $\mathrm{KR}_{k\omega_a}$ is a $\mathfrak{g}$-module, the restriction of the KR-module of the (quantum) affine algebra module, which has highest weight $k\omega_a$, $k \in \mathbb{Z}_+$ and ω_a being one of the fundamental weights of $\mathfrak{g}$. The arrow on top of the product sign refers to a specific ordering of the terms [DFK14, Lin19], in which long and short roots play different roles.

Tensoring the tensor product above by an extra factor $\mathrm{KR}_{k'\omega_b}$ corresponds to the insertion of the factor $\mathcal{Q}_{b,k'}^*$ on the right in the product inside the functional, if k' is sufficiently large, so that the product remains ordered.

In the case of $\mathfrak{g} = \mathfrak{sl}_N$ we introduced [DFK18, DFK17] a set of q-difference operators $\mathcal{D}_{b,k'}$, acting on the space of symmetric functions of $\mathbf{x}$ to the right, and representing the insertion on the right of the factor $\mathcal{Q}_{b,k'}^*$ in the linear functional of

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[1] The functional uses only "half-space" solutions $\mathcal{Q}_{a,k}^*$ of the quantum Q-system for $k \in \mathbb{Z}_+$, with a special prescription which amounts to evaluate $\mathcal{Q}_{a,0}^*$ to 1.

Eq. (1.1). The following diagram explains the action by difference operators on the space of symmetric functions:

$$
\begin{array}{ccc}
\displaystyle\overset{\rightarrow}{\prod_{a,k}}(\mathcal{Q}^*_{a,k})^{n_{a,k}} & \xrightarrow{\;\;\mathcal{Q}^*_{b,k'}\;\;} & \left(\displaystyle\overset{\rightarrow}{\prod_{a,k}}(\mathcal{Q}^*_{a,k})^{n_{a,k}}\right)\mathcal{Q}^*_{b,k'} \\
\Big\downarrow{\scriptstyle\phi} & & \Big\downarrow{\scriptstyle\phi} \\
\chi_{\mathbf{n}}(q^{-1};\mathbf{x}) & \xrightarrow{\;\;\mathcal{D}_{b,k'}\;\;} & \chi_{\mathbf{n}+\epsilon_{b,k'}}(q^{-1};\mathbf{x})
\end{array}
$$

The result is an explicit expression for graded characters as the iterated action of q-difference operators $\mathcal{D}_{a,k}$ on the constant 1. The difference operators satisfy the quantum Q-system. We refer to them as the functional representation of the quantum Q-system.

In [DFK18], it was observed that the difference operators $\mathcal{D}_{a,k}$ for $k = 0$ are the $t \to \infty$ limit of the Macdonald difference operators of type A, of which Macdonald polynomials form a set of common eigenfunctions. Thus, in [DFK19], we identified the t-deformation of the A type quantum Q-system as the spherical Double Affine Algebra Hecke (sDAHA) of type A. This is the algebra which underlies Macdonald theory [Mac95, Che05].

In the $t \to \infty$ limit, in which Macdonald polynomials tend to (dual) q-Whittaker functions with parameter q^{-1}, we further identified in [DFK18] the operator representation $\mathcal{D}_{a,k}$ of the quantum Q-system when $k = 1$ (resp. $k = -1$) as the $t \to \infty$ limits of the Kirillov-Noumi [KN99] raising (resp. lowering) operators. These operators, acting on a Macdonald polynomial indexed by some Young diagram, have the effect of adding (resp. subtracting) a column of a boxes to the Young diagram. Equivalently, using the correspondence of the Young diagram λ with a dominant $\mathfrak{gl}_N$-weight, it corresponds to adding or subtracting the fundamental weight ω_a.

As a consequence, in the case where all KR-modules in the tensor product are fundamental modules ($n_{a,k} = 0$ unless $k = 1$), the graded characters are identified as limits of Macdonald polynomials as $t \to \infty$ or $t = 0$ upon changing $q \to q^{-1}$, and can therefore be identified with specialized q-Whittaker functions. See also [LNS$^+$17].

We remark that the above difference operators can be compared to the so-called minuscule monopole operators representing the Coulomb branches of 4D $N = 4$ quiver gauge theories [BFN16] in the particular case of the "Jordan quiver" with a single node, and when the equivariant parameter t is taken to infinity (with the result of imposing that the representation N is trivial).

It is natural to look for the generalization of the functional representation of the quantum Q-systems, corresponding to the affine algebras of types BCD. These are described in terms of the root systems of the finite BCD type. Motivated by

the results in type A, we expect the functional representation of the other type quantum Q-systems to involve the $t \to \infty$ limits of the corresponding BCD type generalized Macdonald operators. In this paper, we present, without proof, a set of such difference operators. Our main Conjecture 3 is that these operators satisfy the relevant quantum Q-systems relations.

The construction of the BCD type difference operators is best understood by thinking of the quantum Q-systems as evolution equations in the discrete time variable $k \in \mathbb{Z}$ for the elements $\mathcal{Q}_{a,k}$. In type A, from the relation to sDAHA, we noted in [DFK19] that the discrete time evolution $k \to k + 1$ is given by the adjoint action of a generator of the $SL_2(\mathbb{Z})$ symmetry of DAHA. The latter is also expressed as the adjoint action of the "Gaussian" function γ^{-1} of the variables $\mathbf{x}$, where

$$\gamma \equiv \gamma(\mathbf{x}) = e^{\sum_{i=1}^{N} \frac{\text{Log}(x_i)}{2\text{Log}(q)}}, \tag{1.2}$$

and such that $\mathcal{D}_{a,k+1} \propto \gamma^{-1}\mathcal{D}_{a,k}\gamma$.

The aim of the present paper is to present constructions of difference operator solutions to the quantum Q-systems of types BCD, such that they coincide, at $k = 0$, with the $t \to \infty$ limit of suitable Macdonald operators, and which have raising/lowering properties at $k = \pm 1$. To this end, we first identify the $t \to \infty$ limits of suitable Macdonald-type operators in types BCD by use of works of Macdonald and van Diejen [Mac01, vD95, vDE11]. Next, we construct their time evolution, by a suitable Gaussian conjugation. Our main result is the Conjectures 3 and 5.1 stating that (1) these operators obey a renormalized version of the quantum Q-systems in types BCD and (2) the operators at times $k = \pm 1$ act as raising/lowering operators on the corresponding q-Whittaker functions.

2   Q-Systems and Quantum Q-Systems

2.1   Weights and Roots

Let $\mathfrak{g}$ be a Lie algebra of classical type, $\mathfrak{g} \in \{A_{N-1}, D_N, B_N, C_N\}$. For each of these, we list in Table 1 the standard data of fundamental weights ω_a, the simple roots α_a, and the conditions on the non-increasing sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N)$ corresponding to dominant weights $\sum_a n_a \omega_a = \sum_i \lambda_i e_i$ ($n_a \in \mathbb{Z}_+$). In Table 1, the set $\{e_a\}_{a=1}^{N}$ is the standard basis of $\mathbb{R}^N$, whereas $\{\hat{e}_a = e_a - \epsilon/N\}_{a=1}^{N-1}$ where ϵ is the sum over all the basis elements.

We also denote by t_a the integers $\frac{2}{||\alpha_a||^2}$, so that $t_a = 1$ for long roots and $t_a = 2$ for the short roots in types BC.

Table 1 Root and weight data for the classical Lie algebras

Algebra	Fundamental weights ω_a	Simple roots α_a	λ
A_{N-1}	$\omega_a = \sum_{i=1}^{a} \hat{e}_i,\ a \in [1, N-1]$	$e_a - e_{a+1},\ a < N$	$\lambda_a \in \mathbb{Z}_+$
B_N	$\omega_a = \begin{cases} \sum_{i=1}^{a} e_i, & a < N; \\ \frac{1}{2}\sum_{i=1}^{N} e_i, & a = N. \end{cases}$	$e_a - e_{a+1},\ a < N$ e_N	$\lambda_a \in \mathbb{Z}_+$ for all a or $\lambda_a \in \mathbb{Z}_+ + \frac{1}{2}$ for all a
C_N	$\omega_a = \sum_{i=1}^{a} e_i,\ a \in [1, N]$	$e_a - e_{a+1},\ a < N$ $2e_N$	$\lambda_a \in \mathbb{Z}_+$
D_N	$\omega_a = \sum_{i=1}^{a} e_i,\ a < N-1$ $\omega_{N-1} = \frac{1}{2}(\omega_{N-2} + e_{N-1} - e_N)$ $\omega_N = \frac{1}{2}(\omega_{N-2} + e_{N-1} + e_N)$	$e_a - e_{a+1},\ a < N$ $e_a + e_{a+1},\ a < N$	$\lambda_a \in \mathbb{Z}$ for all a or $\lambda_a \in \mathbb{Z} + \frac{1}{2}$ for all a, $\lambda_{N-1} \geq |\lambda_N| \geq 0$.

2.2 The Classical Q-Systems for Untwisted Affine $ABCD$

The Q-systems are recursion relations for the variables $\{Q_{a,k}\}$ where a is a label in the Dynkin diagram, and k is any integer. We list the Q-systems associated with types $ABCD$ [KR87, KNS94]. The boundary condition $Q_{0,k} = 0$ is assumed in all cases.

$$\mathfrak{g} = A_{N-1} : \ Q_{a,k+1} Q_{a,k-1} = Q_{a,k}^2 - Q_{a+1,k} Q_{a-1,k}, \qquad (a \in [1, N-1]),$$

$$Q_{N,k} = 1. \tag{2.1}$$

$$\mathfrak{g} = B_N : \ Q_{a,k+1} Q_{a,k-1} = Q_{a,k}^2 - Q_{a+1,k} Q_{a-1,k}, \qquad (a \in [1, N-2]),$$

$$Q_{N-1,k+1} Q_{N-1,k-1} = (Q_{N-1,k})^2 - Q_{N,2k} Q_{N-2,k}, \tag{2.2}$$

$$Q_{N,2k+1} Q_{N,2k-1} = (Q_{N,2k})^2 - (Q_{N-1,k})^2,$$

$$Q_{N,2k+2} Q_{N,2k} = (Q_{N,2k+1})^2 - Q_{N-1,k+1} Q_{N-1,k}.$$

$$\mathfrak{g} = C_N : \ Q_{a,k+1} Q_{a,k-1} = Q_{a,k}^2 - Q_{a+1,k} Q_{a-1,k}, \qquad (a \in [1, N-2]),$$

$$Q_{N-1,2k+1} Q_{N-1,2k-1} = Q_{N-1,2k}^2 - Q_{N-2,2k} Q_{N,k}^2, \tag{2.3}$$

$$Q_{N-1,2k+2} Q_{N-1,2k} = Q_{N-1,2k+1}^2 - Q_{N-2,2k+1} Q_{N,k+1} Q_{N,k}.$$

$$Q_{N,k+1} Q_{N,k-1} = Q_{N,k}^2 - Q_{N-1,2k},$$

$$\mathfrak{g} = D_N : \ Q_{a,k+1} Q_{a,k-1} = Q_{a,k}^2 - Q_{a+1,k} Q_{a-1,k}, \qquad (a \in [1, N-3]),$$

$$Q_{N-2,k+1} Q_{N-2,k-1} = Q_{N-2,k}^2 - Q_{N,k} Q_{N-1,k} Q_{N-3,k}, \tag{2.4}$$

$$Q_{a,k+1}Q_{a,k-1} = Q_{a,k}^2 - Q_{N-2,k} \qquad (a \in \{N-1, N\}).$$

These recursion relations (for $k \geq 1$) were originally observed [KR87] to be relations satisfied by characters of finite-dimensional irreducible Yangian modules.

Each set of the Q-systems is associated with a cluster algebra:
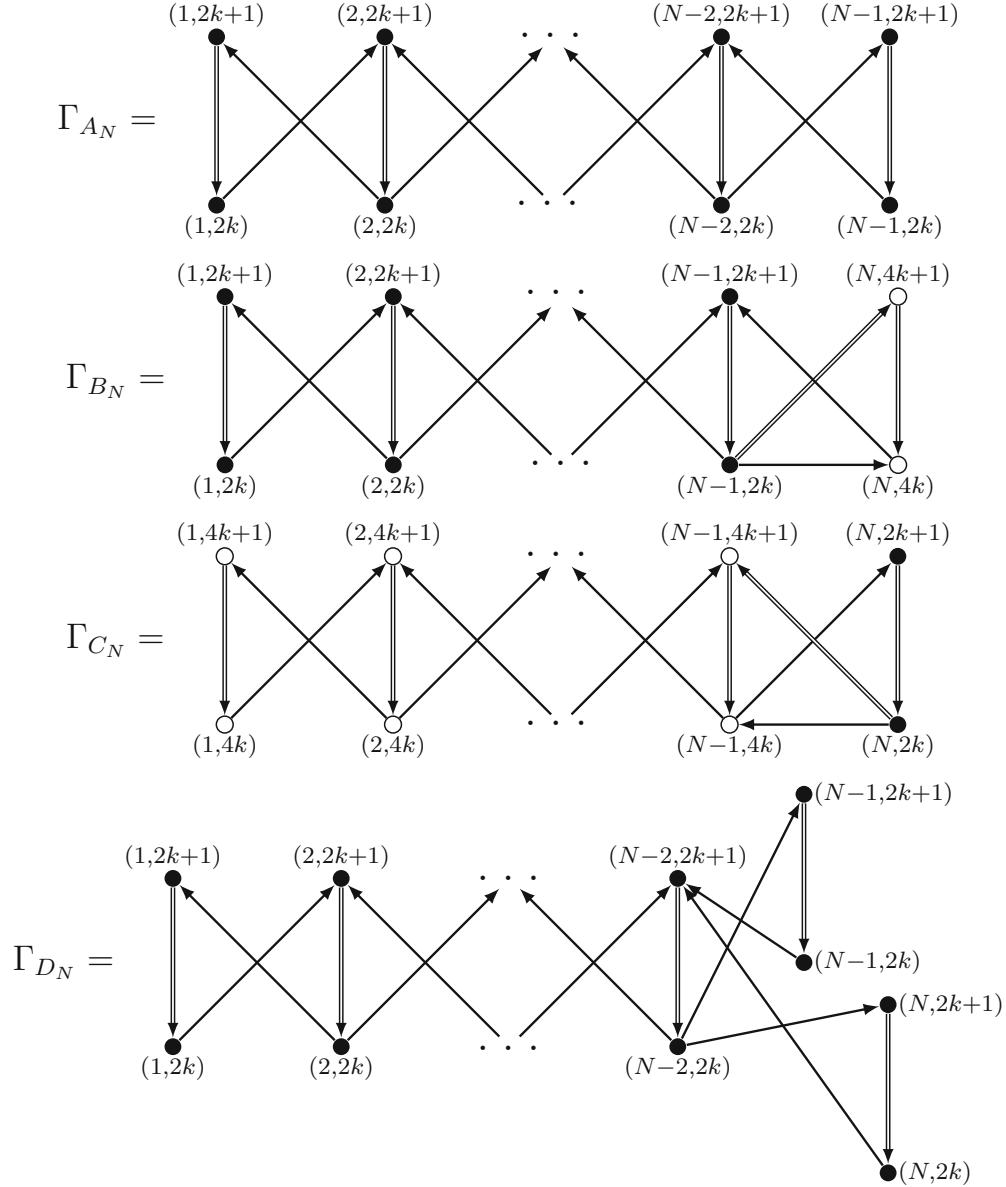
Fig. 1 The quivers for the A_{N-1}, D_N, B_N, C_N Q-system cluster algebras. We have indicated a generic Q-system cluster along the bipartite belt: each vertex labelled (a, k) corresponds to a cluster variable $Q_{a,k}$. Nodes corresponding to short roots are denoted by empty circles

Theorem 2.1 ([Ked08, DFK09]) *For each algebra $\mathfrak{g}$, the variables $\{Q_{a,k} : a \in [1, r], k \in \mathbb{Z}\}$, up to a simple rescaling which eliminates the minus sign on the right hand side, are cluster variables in a corresponding cluster algebra. Each of the Q-system relations is an exchange relation in the cluster algebra.*

The cluster algebras are defined via a $2r \times 2r$ skew-symmetric exchange matrix B, (r being the rank of $\mathfrak{g}$), or quiver Γ as in Fig. 1, which depends only on the Cartan matrix C of $\mathfrak{g}$:

$$B = \left(\begin{array}{c|c} C^T - C & -C^T \\ \hline C & 0 \end{array} \right). \tag{2.5}$$

This exchange matrix, together with the initial cluster variables $\mathbf{X} = (Q_{a,0}; Q_{a,1})_{a=1}^{r}$, defines the cluster algebra. The cluster variables $\{Q_{a,k}\}$ are obtained from a generalized bipartite evolution of the initial cluster $(\mathbf{X}, B)$ [DFK09]. The subset of mutations on the (generalized) bipartite belt which generates all the cluster variables corresponding to the Q-system algebra was given in [DFK09, Theorem 3.6].

2.3 Quantum Q-Systems

One of the advantages of formulating the Q-system relations in terms of cluster algebra mutations is that there is a canonical quantization of the cluster algebra, using the canonical Poisson structure [GSV10] and its quantization [BZ05].

The quantum cluster algebra attached to a non-degenerate, skew-symmetric matrix B is the non-commutative algebra generated by the cluster variables $\mathbf{X} = (X_i)$ at an initial cluster and their inverses, with exchange matrix B, as well as the cluster variables at all mutation equivalent clusters. Within the cluster $(\mathbf{X}, B)$ the cluster variables q-commute according to a skew-symmetric Λ proportional to the inverse of B:

$$X_i X_j = q^{\Lambda_{ij}} X_j X_i.$$

The exchange relations are given by normal-ordering of the classical mutations:

$$X_i' =: \prod_{j:B_{ij}>0} X_j^{B_{ij}} X_i^{-1} : + : \prod_{j:B_{ij}<0} X_j^{-B_{ij}} X_i^{-1} : .$$

Here, given a monomial $\prod X_i^{b_i}$ such that $X_i X_j = q^{a_{i,j}} X_j X_i$, the normal ordered product is

$$: X_1^{b_1} \cdots X_\ell^{b_\ell} := q^{-\frac{1}{2} \sum_{i<j} a_{i,j} b_i b_j} X_1^{b_1} \cdots X_\ell^{b_\ell}.$$

The exchange matrices (2.5) corresponding to the Q-system cluster algebras are skew-symmetric and invertible. We use the associated quantum cluster algebra to define the quantum Q-systems as the quantized exchange relations corresponding to the exchange relations appearing in the classical Q-systems.

Let Λ be a $2r \times 2r$ skew-symmetric matrix, with $\Lambda^T B = -D$, a diagonal integer matrix with negative integer entries. Then

$$\Lambda = \left(\begin{array}{c|c} 0 & \lambda \\ \hline -\lambda^T & \lambda^T - \lambda \end{array} \right), \tag{2.6}$$

where λ is proportional to the inverse Cartan matrix. We use the following normalizations:

$$A_{N-1} : \lambda_{ab} = C_{ab}^{-1} = \min(a, b) - \frac{ab}{N}$$

$$B_N : \lambda_{ab} = 2C_{ab}^{-1} = \begin{cases} 2\min(a, b), & 1 \le a \le N, \ 1 \le b \le N - 1; \\ \min(a, b), & 1 \le a \le N = b, \end{cases}$$

$$C_N : \lambda_{ab} = 2C_{ab}^{-1} = \begin{cases} 2\min(a, b), & 1 \le a \le N - 1, \ 1 \le b \le N; \\ \min(a, b), & 1 \le b \le N = a, \end{cases}$$

$$D_N : \lambda_{ab} = 2C_{ab}^{-1} = \begin{cases} 2\min(a, b), & 1 \le a, b \le N - 2, \\ b, & 1 \le b \le N - 2, \ a \in \{N - 1, N\}, \\ a, & 1 \le a \le N - 2, \ b \in \{N - 1, N\}, \\ \frac{1}{2}N, & a = b \in \{N - 1, N\}, \\ \frac{1}{2}(N - 2), & a \ne b \in \{N - 1, N\}. \end{cases} \tag{2.7}$$

This choice of normalization results, in the cases B_N, C_N, D_N, in the value $\lambda_{a,b} = 2\min(a, b)$ whenever a, b correspond to the part of the Dynkin diagram that forms an A-type chain, that is, $a, b \in [1, N - 1]$ for types BC and $a, b \in [1, N - 2]$ for type D.

The quantum Q-systems are recursion relations for the non-commuting variables $\mathcal{Q}_{a,k}$. These take the form

$$A_{N-1} : \mathcal{Q}_{a,k} \, \mathcal{Q}_{b,p} = q^{\lambda_{a,b}(p-k)} \, \mathcal{Q}_{b,p} \, \mathcal{Q}_{a,k},$$

$$q^{\lambda_{a,a}} \, \mathcal{Q}_{a,k+1} \, \mathcal{Q}_{a,k-1} = \mathcal{Q}_{a,k}^2 - q^{\frac{1}{2}} \, \mathcal{Q}_{a+1,k} \, \mathcal{Q}_{a-1,k}, \quad (a \in [1, N]), \tag{2.8}$$

$$\mathcal{Q}_{0,k} = 1, \qquad \mathcal{Q}_{N+1,k} = 0.$$

$$B_N : \mathcal{Q}_{a,k} \, \mathcal{Q}_{b,p} = q^{p\lambda_{a,b} - k\lambda_{b,a}} \, \mathcal{Q}_{b,p} \, \mathcal{Q}_{a,k},$$

$$q^{2a} \, \mathcal{Q}_{a,k+1} \, \mathcal{Q}_{a,k-1} = \mathcal{Q}_{a,k}^2 - q \, \mathcal{Q}_{a+1,k} \, \mathcal{Q}_{a-1,k} \, \mathcal{Q}_{\alpha-1,k}, \quad (\alpha \in [1, N - 2]),$$

$$q^{2N-2} \, \mathcal{Q}_{N-1,k+1} \, \mathcal{Q}_{N-1,k-1} = (\mathcal{Q}_{N-1,k})^2 - q \, \mathcal{Q}_{N,2k} \, \mathcal{Q}_{N-2,k}, \tag{2.9}$$

$$q^N \, \mathcal{Q}_{N,2k+1} \, \mathcal{Q}_{N,2k-1} = (\mathcal{Q}_{N,2k})^2 - q(\mathcal{Q}_{N-1,k})^2,$$

$$q^N \, \mathcal{Q}_{N,2k+2} \, \mathcal{Q}_{N,2k} = (\mathcal{Q}_{N,2k+1})^2 - q^N \mathcal{Q}_{N-1,k+1} \, \mathcal{Q}_{N-1,n},$$

$$\mathcal{Q}_{0,k} = 1.$$

$$C_N : \mathcal{Q}_{a,k} \, \mathcal{Q}_{b,p} = q^{p\lambda_{a,b} - k\lambda_{b,a}} \, \mathcal{Q}_{b,p} \, \mathcal{Q}_{a,k},$$

$$q^{2a} \mathcal{Q}_{a,k+1} \, \mathcal{Q}_{a,k-1} = \mathcal{Q}_{a,k}^2 - q \, \mathcal{Q}_{a+1,k} \, \mathcal{Q}_{a-1,k}, \qquad (\alpha \in [1, N-2]),$$

$$q^{2N-2} \mathcal{Q}_{N-1,2k+1} \mathcal{Q}_{N-1,2k-1} = \mathcal{Q}_{N-1,2k}^2 - q \mathcal{Q}_{N-2,2k} \mathcal{Q}_{N,k}^2, \tag{2.10}$$

$$q^{2N-2} \mathcal{Q}_{N-1,2k+2} \mathcal{Q}_{N-1,2k} = \mathcal{Q}_{N-1,2k+1}^2 - q^{1+\frac{N}{2}} \, \mathcal{Q}_{N-2,2k+1} \mathcal{Q}_{N,k+1} \mathcal{Q}_{N,k},$$

$$q^N \mathcal{Q}_{N,k+1} \, \mathcal{Q}_{N,k-1} = \mathcal{Q}_{N,k}^2 - q \mathcal{Q}_{N-1,2k},$$

$$\mathcal{Q}_{0,k} = 1.$$

$$D_N : \mathcal{Q}_{a,k} \, \mathcal{Q}_{b,p} = q^{\lambda_{a,b}(p-k)} \, \mathcal{Q}_{b,p} \, \mathcal{Q}_{a,k},$$

$$q^{2a} \mathcal{Q}_{a,k+1} \, \mathcal{Q}_{a,k-1} = \mathcal{Q}_{a,k}^2 - q \, \mathcal{Q}_{a+1,k} \, \mathcal{Q}_{a-1,k} \qquad (a \in [1, N-3]),$$

$$q^{2(N-2)} \mathcal{Q}_{N-2,k+1} \, \mathcal{Q}_{N-2,k-1} = \mathcal{Q}_{N-2,k}^2 - q \, \mathcal{Q}_{N,k} \, \mathcal{Q}_{N-1,k} \, \mathcal{Q}_{N-3,k}, \tag{2.11}$$

$$q^{\frac{N}{2}} \mathcal{Q}_{N-1,k+1} \, \mathcal{Q}_{N-1,k-1} = \mathcal{Q}_{N-1,k}^2 - q \, \mathcal{Q}_{N-2,k},$$

$$q^{\frac{N}{2}} \mathcal{Q}_{N,k+1} \, \mathcal{Q}_{N,k-1} = \mathcal{Q}_{N,k}^2 - q \, \mathcal{Q}_{N-2,k},$$

$$\mathcal{Q}_{0,k} = 1.$$

In each case, the q-commutation relation, i.e. the first equation in each set, holds only for variables within the same cluster, hence the restriction on possible values of the second index. For example, in the case A_{N-1}, the restriction is $|p-k| \leq |a-b| + 1$. In general, the clusters consisting of Q-system solutions only are parameterized by generalized Motzkin paths [Lin19].

Remark 2.2 The general boundary condition in (2.8) the quantum Q-system of type A_{N-1} is $\mathcal{Q}_{N+1,k} = 0$, as opposed to the condition $Q_{N,k} = 1$ for the classical system (2.1), which is compatible but more restrictive. This means that there is an additional set of variables in the center of the quantum cluster algebra which satisfy $\mathcal{Q}_{N,k+1} \mathcal{Q}_{N,k-1} = \mathcal{Q}_{N,k}^2$. This is consistent with the extension of the definition (2.7) of the matrix $\lambda_{a,b}$ in type A to an $N \times N$ matrix using the same formula, so that $\lambda_{a,N} = 0$. The most general solution subject to the boundary conditions has $\mathcal{Q}_{N,k} = \mathcal{Q}_{N,1}^k \mathcal{Q}_{N,0}^{k-1}$, in terms of the two central elements $\mathcal{Q}_{N,1}$ and $\mathcal{Q}_{N,0}$. This more general system can be embedded into a cluster algebra with coefficients.

3 The A_{N-1} Case Solution: Generalized Macdonald Difference Operators and Quantum Determinants

3.1 Renormalized Quantum Q-System

With the choice of boundary condition for the type A quantum Q-system as in Remark 2.2, the quantum Q-system is homogeneous with respect to the grading $\deg(\mathcal{Q}_{a,k}) := ak$ ($a \in [1, N]$). We adjoin an invertible degree operator $\Delta^{1/N}$ to the algebra, such that

$$\Delta \, \mathcal{Q}_{a,k} = q^{ak} \, \mathcal{Q}_{a,k} \, \Delta, \qquad a \in [1, N], k \in \mathbb{Z}.$$

Using $\lambda_{a,a} + \frac{a}{N} = a$ and $\lambda_{a+1,a+1} + \lambda_{a-1,a-1} - 2\lambda_{a,a} = -\frac{2}{N}$, the renormalized variables

$$\widetilde{\mathcal{Q}}_{a,k} = q^{-\frac{1}{2}(k+\frac{N}{2})\lambda_{a,a}} \, \mathcal{Q}_{a,k} \, \Delta^{\frac{a}{N}}, \qquad a \in [1, N], k \in \mathbb{Z}, \tag{3.1}$$

satisfy the *renormalized quantum Q-system*:

$$\begin{aligned} &\widetilde{\mathcal{Q}}_{a,k} \, \widetilde{\mathcal{Q}}_{b,k'} = q^{(k'-k)\min(a,b)} \, \widetilde{\mathcal{Q}}_{b,k'} \, \widetilde{\mathcal{Q}}_{a,k}, \quad |k - k'| \le |a - b| + 1, \\ &q^a \, \widetilde{\mathcal{Q}}_{a,k+1} \, \widetilde{\mathcal{Q}}_{a,k-1} = \widetilde{\mathcal{Q}}_{a,k}^2 - \widetilde{\mathcal{Q}}_{a+1,k} \, \widetilde{\mathcal{Q}}_{a-1,k}, \qquad (a \in [1, N]), \\ &\widetilde{\mathcal{Q}}_{0,k} = 1, \qquad \widetilde{\mathcal{Q}}_{N+1,k} = 0. \end{aligned} \tag{3.2}$$

The choice $\mathcal{Q}_{N,0} = 1$ implies $\widetilde{\mathcal{Q}}_{N,0} = \Delta$. Defining $A = \widetilde{\mathcal{Q}}_{N,1} \widetilde{\mathcal{Q}}_{N,0}^{-1}$, a homogeneous element of degree N, we have $\widetilde{\mathcal{Q}}_{N,k} = A^k \Delta$. The quiver with coefficients corresponding to this cluster algebra is illustrated in Fig. 2.
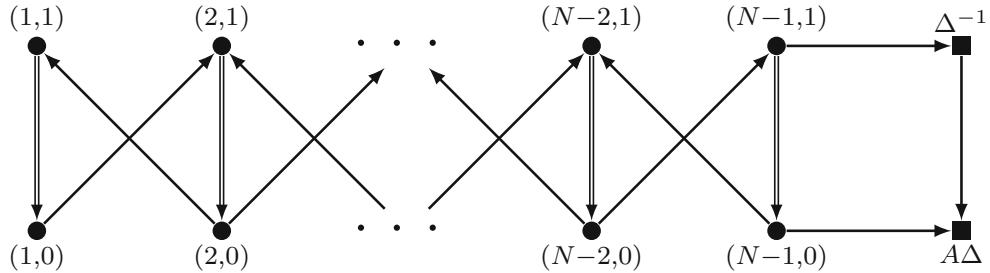
Fig. 2 The type A quantum Q-system quiver corresponding to the initial seed $\{\mathcal{Q}_{a,0}, \mathcal{Q}_{a,1}\}$. Square nodes denote coefficients

3.2 The Quantum Determinant

The exchange relations (3.2) define a quantum determinant: The variables $\widetilde{\mathcal{Q}}_{a,k}$ with $a > 1$ are polynomials in the variables $\{\widetilde{\mathcal{Q}}_{1,k'} : |k' - k| \leq a - 1\}$. Below, we use the notation $\widetilde{\mathcal{Q}}_k := \widetilde{\mathcal{Q}}_{1,k}$. The quantum determinant is best defined in terms of generating functions.

Definition 3.1 Given a set of integers $k_1, \ldots, k_a \in \mathbb{Z}$, define the Hankel matrix

$$(\widetilde{\mathcal{Q}}_{k_i+i-j})_{1 \leq i, j \leq a}. \tag{3.3}$$

The quantum determinant of this matrix, denoted by $|\widetilde{\mathcal{Q}}[k_1, k_2, \ldots, k_a]|_q$, is given by the coefficients of the generating function:

$$\sum_{k_1,\ldots,k_a \in \mathbb{Z}} u_1^{k_1} \cdots u_a^{k_a} \, |\widetilde{\mathcal{Q}}[k_1, \ldots, k_a]|_q = \prod_{1 \leq i < j \leq a} \left(1 - q\frac{u_j}{u_i}\right) \widetilde{\mathcal{Q}}(u_1)\widetilde{\mathcal{Q}}(u_2) \cdots \widetilde{\mathcal{Q}}(u_\alpha),$$

where

$$\widetilde{\mathcal{Q}}(u) := \sum_{k \in \mathbb{Z}} u^k \, \widetilde{\mathcal{Q}}_k.$$

The quantum determinant $|\widetilde{\mathcal{Q}}[k_1, \ldots, k_a]|_q$ is a homogeneous polynomial of degree a in the $\widetilde{\mathcal{Q}}_k$s.

Theorem 3.2 *[DFK17] The solutions $\widetilde{\mathcal{Q}}_{a,k}$ of the system (3.2) with $a \geq 1$ and $k \in \mathbb{Z}$ are the quantum determinants*

$$\widetilde{\mathcal{Q}}_{a,k} = |\widetilde{\mathcal{Q}}[\underbrace{k, k, \ldots, k}_{a \text{ times}}]|_q.$$

3.3 The Functional Representation of the Quantum Q-System

We recall the functional representation of the renormalized quantum Q-system (3.2) $\rho(\widetilde{\mathcal{Q}}_{a,k}) = M_{a,k}$ [DFK18], which act on the space of symmetric functions of N variables $x_1, x_2, \ldots, x_N$.

Theorem 3.3 ([DFK18, DFK17]) *Let Γ_i be the q-shift operator acting on the space of functions in N variables, defined by $\Gamma_i f(x_1, \ldots, x_i, \ldots, x_N) = f(x_1, \ldots, qx_i, \ldots, x_N)$. The q-difference operators*

$$M_{a,k} = \sum_{\substack{I \subset [1,N] \\ |I|=a}} \left(\prod_{i \in I} x_i\right)^k \prod_{\substack{i \in I \\ j \notin I}} \frac{x_i}{x_i - x_j} \prod_{i \in I} \Gamma_i, \quad a \in [1, N], k \in \mathbb{Z} \tag{3.4}$$

satisfy the quantum Q-system relations (3.2).

In this representation,

$$A = x_1 x_2 \cdots x_N, \qquad \Delta = \Gamma_1 \Gamma_2 \cdots \Gamma_N. \tag{3.5}$$

Remark 3.4 For $\mathfrak{sl}_N$-characters, the products A and Δ are taken to be equal to 1. The more general boundary condition here corresponds to $\mathfrak{gl}_N$-characters.

3.4   The Spherical Double Affine Hecke Algebra

The reader will have recognized that the difference operators $M_{a,0}$ in (3.4) are the limit $t \to \infty$ of the (renormalized) Macdonald difference operators in type A. This is the main observation which led to the results of [DFK19], where it is shown that the spherical DAHA [Che05] of type A_{N-1} is the natural t-deformation of the quantum Q-system.

An important observation in [DFK19] is that the evolution in the discrete time variable k is induced by the adjoint action of one of the generators of the $SL_2(\mathbb{Z})$ symmetry of the DAHA.

Theorem 3.5 ([DFK19]) *The discrete time evolution of the operators $M_{a,k}$ is induced by the adjoint action of the Gaussian γ^{-1}, with γ as in (1.2):*

$$M_{a,k} = q^{-ak/2} \gamma^{-k} M_{a,0} \gamma^k. \tag{3.6}$$

The difference operators $\{M_{a,k}\}$, together with the elementary symmetric functions of $\mathbf{x}$, are the image in the functional representation of the generators of the spherical DAHA.

The double affine Hecke algebra and corresponding Macdonald operators are defined for other Lie algebras. In the following sections, we will use this as the inspiration to give conjectures for the functional representations of the quantum Q-systems for the other classical types by following the inverse reasoning: Starting from the appropriate choice of Macdonald difference operators, act with the adjoint action of γ to obtain the discrete time-evolved operators, imitating the contents of Theorem 3.5. The resulting q-difference operators, in the limit $t \to \infty$, are (conjecturally) solutions of renormalized quantum Q-systems for the other classical types.

3.5   Raising and Lowering Operators

The dual q-Whittaker functions are defined as

$$\Pi_\lambda \equiv \Pi_\lambda(q^{-1}; \mathbf{x}) = \lim_{t \to \infty} P_\lambda(q, t; \mathbf{x}),$$

where the P_λ are the Macdonald polynomials for type A_{N-1} [Mac95]. Here, the partition λ corresponds to a dominant integral weight of $\mathfrak{gl}_N$, $\lambda = \sum_{i=1}^N \lambda_i e_i$ in the standard basis of $\mathbb{R}^N$, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$. The polynomials P_λ are common monic eigenfunctions of the Macdonald operators D_a

$$D_a = \sum_{\substack{I \subset [1,N] \\ |I|=a}} \prod_{\substack{i \in I \\ j \notin I}} \frac{t x_i - x_j}{x_i - x_j} \prod_{i \in I} \Gamma_i, \quad a \in [1, N], \tag{3.7}$$

with

$$D_a P_\lambda = t^{-\frac{a(a-1)}{2}} e_a(q^{\lambda_1} t^{N-1}, q^{\lambda_2} t^{N-2}, \ldots, q^{\lambda_N}) P_\lambda,$$

where e_a are the elementary symmetric functions in N variables. Using $\lim_{t \to \infty} t^{-a(N-a)} D_a = M_{a,0}$, this means that the q-Whittaker functions are eigenfunctions of the difference operators $M_{a,0}$:

$$M_{a,0} \Pi_\lambda = q^{(\lambda, \omega_a)} \Pi_\lambda, \quad a \in [1, N], \tag{3.8}$$

where $\omega_a = e_1 + e_2 + \cdots + e_a$ are the fundamental weights of $\mathfrak{gl}_N$.

In [DFK18] we showed that the operators $M_{a;\pm 1}$ are the limit $t \to \infty$ of the raising and lowering operators for Macdonald polynomials constructed by Kirillov and Noumi [KN99]. However, the commutation relations in (3.2) provide an easier proof, which we present here.

Theorem 3.6 *The operators $M_{a,\pm 1}$ act on the polynomials Π_λ as follows:*

$$M_{a,1} \Pi_\lambda = q^{(\lambda, \omega_a)} \Pi_{\lambda + \omega_a}, \tag{3.9}$$

$$M_{a,-1} \Pi_\lambda = q^{(\lambda, \omega_a)} \left(1 - q^{-(\lambda, \alpha_a)}\right) \Pi_{\lambda - \omega_a}, \tag{3.10}$$

where $\alpha_a = e_a - e_{a+1}$ are the simple roots of $\mathfrak{sl}_N$.

Note that if λ is a partition, $\lambda + \omega_a$ is also a partition, and if $\lambda - \omega_a$ is not a partition, the scalar factor $(1 - q^{(\lambda, \alpha_a)})$ vanishes.

Proof Up to scalar multiple, the polynomials Π_λ are uniquely determined by the collection of eigenvalues $q^{(\omega_a, \lambda)}$ corresponding to the diagonal action of the operators $M_{a,0}$, $a = 1, 2, \ldots, N$. The commutation relations in (3.2) are

$$M_{a,0} M_{b,\pm 1} = q^{\pm \min(a,b)} M_{b,0\pm 1} M_{a,0}.$$

Applying this to Π_λ and using (3.8),

$$M_{a,0} M_{b,\pm 1} \Pi_\lambda = q^{(\lambda, \omega_a) \pm \min(a,b)} M_{b,\pm 1} \Pi_\lambda = q^{(\lambda \pm \omega_b, \omega_a)} M_{b,\pm 1} \Pi_\lambda,$$

since $\min(a, b) = (\omega_b, \omega_a)$. That is, $M_{b,\pm 1} \Pi_\lambda$ is an eigenfunction of $M_{a,0}$ with eigenvalue $q^{(\lambda \pm \omega_b, \omega_a)}$. Therefore there exist scalars $c_{\lambda,b}^\pm$ such that $M_{b,\pm 1} \Pi_\lambda = c_{\lambda,b}^\pm \Pi_{\lambda \pm \omega_b}$.

Recall that the Macdonald polynomials P_λ, and therefore Π_λ, have a triangular decomposition with respect to the monomial symmetric functions with leading term m_λ. The following analysis provides the scalar factor $c_{\lambda,b}^+$. When $|x_1| \gg |x_2| \gg \cdots \gg |x_N|$, $\Pi_\lambda = x_1^{\lambda_1} \cdots x_N^{\lambda_N} + \text{lower order}$ and $M_{b,1} = x_1 x_2 \ldots x_b \Gamma_1 \Gamma_2 \cdots \Gamma_b + \text{lower order}$. Thus,

$$M_{b,1} \Pi_\lambda = q^{\lambda_1 + \cdots + \lambda_b} x_1^{\lambda_1 + 1} \cdots x_b^{\lambda_b + 1} x_{b+1}^{\lambda_{b+1}} \cdots x_N^{\lambda_N} + \text{ lower order}$$

$$= q^{(\lambda, \omega_b)} \Pi_{\lambda + \omega_b} + \text{ lower order},$$

and thus $c_{\lambda,b}^+ = q^{(\lambda, \omega_b)}$.

Applying the exchange relation in (3.2) with $k = 0$ to Π_λ, we get

$$q^a M_{a,1} M_{a,-1} \Pi_\lambda = q^a c_{\lambda - \omega_a, a}^+ c_{\lambda, a}^- \Pi_\lambda = (q^{2(\lambda, \omega_a)} - q^{(\lambda, \omega_{a+1} + \omega_{a-1})}) \Pi_\lambda,$$

which shows that $c_{\lambda,a}^- = q^{(\lambda, \omega_a)} - q^{(\lambda, \omega_{a+1} + \omega_{a-1} - \omega_a)}$ and the Theorem follows. □

3.6 Graded Characters in Terms of Difference Operators

The difference operators (3.4) can be used to efficiently generate graded characters (1.1). Theorem 3.3 is a necessary condition for the following theorem:

Theorem 3.7 *([DFK18], Corollary 18): Starting with the trivial character $\chi_0 = 1$, the difference operators (3.4) act consecutively to generate the character of the graded tensor product of KR-modules $\otimes_{a,i} \text{KR}_{i\omega_a}^{\otimes n_{a,i}}$ as follows:*

$$\chi_{\mathbf{n}}(q^{-1}, \mathbf{x}) = q^{-\frac{1}{2} Q(\mathbf{n})} \prod_{a=1}^{N-1} (M_{a,k})^{n_{a,k}} \prod_{a=1}^{N-1} (M_{a,k-1})^{n_{a,k-1}} \cdots \prod_{a=1}^{N-1} (M_{a,1})^{n_{a,1}} 1,$$

(3.11)

where

$$Q(\mathbf{n}) = \sum_{a,b=1}^{N-1} \sum_{i,j \geq 1} n_{a,i} \min(i, j) \min(a, b) n_{b,j} - \sum_{a=1}^{N-1} \sum_{i \geq 1} i \, a \, n_{a,i}.$$

Alternatively, when $k \geq \max\{j : n_{a,j} > 0\}$,

$$D_{a,k}\chi_{\mathbf{n}}(q^{-1}; \mathbf{x}) = q^{(\omega_a, \sum_{b,j} jn_{b,j}\omega_b)}\chi_{\mathbf{n}+\epsilon_{a,k}}(q^{-1}; \mathbf{x}),$$

where we write $\mathbf{n} = \sum_{b,j} n_{b,j}\epsilon_{b,j}$.

4 The Quantum Q-System Conjectures for Types BCD

In order to formulate the conjectures for the functional representation of the quantum Q-systems of types D_N, B_N, C_N, we start with the results of Macdonald and van Diejen. There are N algebraically independent commuting Hamiltonians in the sDAHA of those types. Our goal is to find a set that will be the seed, in the q-Whittaker limit, of the Q-system solutions.

4.1 Macdonald and van Diejen Operators

In [Mac01], Macdonald constructed certain difference operators for types D_N, B_N, C_N, corresponding to minuscule coweights. For these types, there are, respectively, 3, 1, and 1 minuscule coweights, indexed by some of the extremal nodes of the Dynkin diagrams. The corresponding Macdonald operators are

$$D_N: \quad \mathcal{E}_1^{(D_N)} = \sum_{\epsilon=\pm 1} \sum_{i=1}^{N} \prod_{j\neq i} \frac{1 - tx_i^\epsilon x_j}{1 - x_i^\epsilon x_j} \frac{tx_i^\epsilon - x_j}{x_i^\epsilon - x_j}\Gamma_i^{2\epsilon}, \tag{4.1}$$

$$\mathcal{E}_{N-1}^{(D_N)} = \sum_{\substack{\epsilon_1,\ldots,\epsilon_N=\pm 1 \\ \epsilon_1\epsilon_2\cdots\epsilon_N=-1}} \prod_{1\leq i<j\leq N} \frac{1 - tx_i^{\epsilon_i} x_j^{\epsilon_j}}{1 - x_i^{\epsilon_i} x_j^{\epsilon_j}} \prod_{i=1}^{N}\Gamma_i^{\epsilon_i}, \tag{4.2}$$

$$\mathcal{E}_{N}^{(D_N)} = \sum_{\substack{\epsilon_1,\ldots,\epsilon_N=\pm 1 \\ \epsilon_1\epsilon_2\cdots\epsilon_N=1}} \prod_{1\leq i<j\leq N} \frac{1 - tx_i^{\epsilon_i} x_j^{\epsilon_j}}{1 - x_i^{\epsilon_i} x_j^{\epsilon_j}} \prod_{i=1}^{N}\Gamma_i^{\epsilon_i}. \tag{4.3}$$

$$B_N: \quad \mathcal{E}_1^{(B_N)} = \sum_{\epsilon=\pm 1} \sum_{i=1}^{N} \frac{1 - tx_i^\epsilon}{1 - x_i^\epsilon} \prod_{j\neq i} \frac{1 - tx_i^\epsilon x_j}{1 - x_i^\epsilon x_j} \frac{tx_i^\epsilon - x_j}{x_i^\epsilon - x_j}\Gamma_i^{2\epsilon}. \tag{4.4}$$

$$C_N: \quad \mathcal{E}_N^{(C_N)} = \sum_{\epsilon_1,\ldots,\epsilon_N=\pm 1} \prod_{i=1}^{N} \frac{1 - tx_i^{2\epsilon_i}}{1 - x_i^{2\epsilon_i}} \prod_{1\leq i<j\leq N} \frac{1 - tx_i^{\epsilon_i} x_j^{\epsilon_j}}{1 - x_i^{\epsilon_i} x_j^{\epsilon_j}} \prod_{i=1}^{N}\Gamma_i^{\epsilon_i}. \tag{4.5}$$

The Macdonald operator for C_N is of "order" N, namely acts by shifts of N variables in each term, as opposed to the operators $\mathcal{E}_1^{(D_N)}$ and $\mathcal{E}_1^{(B_N)}$, which are linear combinations of shifts of a single variable. A first order difference operator for C_N, $\mathcal{E}_1^{(C_N)}$, can be obtained using the commuting operators constructed in [vD95, vDE11]. We choose the following first order C_N operator, which is a particular linear combination of the identity and the first order van Diejen operator:

$$
\mathcal{E}_1^{(C_N)} = (1 + t^{N+1})\frac{1 - t^N}{1 - t} + \sum_{\epsilon = \pm 1} \sum_{i=1}^{N} \frac{1 - tx_i^{2\epsilon}}{1 - x_i^{2\epsilon}} \frac{1 - tq^2 x_i^{2\epsilon}}{1 - q^2 x_i^{2\epsilon}}
$$

$$
\times \prod_{j \neq i} \frac{1 - tx_i^\epsilon x_j}{1 - x_i^\epsilon x_j} \frac{tx_i^\epsilon - x_j}{x_i^\epsilon - x_j} (\Gamma_i^{2\epsilon} - 1) \tag{4.6}
$$

This particular choice ensures that the corresponding eigenfunctions are the Macdonald polynomials $P_\lambda(\mathbf{x})$ of type C_N with eigenvalues

$$
t^N \hat{e}_1(\{t^{N+1-i} q^{2\lambda_i}\}_{i=1}^N), \qquad \text{where } \hat{e}_1(z_1, z_2, \ldots, z_N) = \sum_{i=1}^{N}(z_i + z_i^{-1}).
$$

In particular, for $P_0(\mathbf{x}) = 1$, we have $\mathcal{E}_1^{(C_N)} 1 = t^N \hat{e}_1(\{t, t^2, \ldots, t^N\}) = (1 + t^{N+1})\frac{1-t^N}{1-t}$.

4.2   The q-Whittaker Limit

Each of the operators $\mathcal{E}_i^{(\mathfrak{g})}$ in Eqs. (4.1)–(4.6) have suitable limits as $t \to \infty$, denoted by

$$
M_{i,0}^{(\mathfrak{g})} := \lim_{t \to \infty} t^{-a_i^{(\mathfrak{g})}} \mathcal{E}_i^{(\mathfrak{g})},
$$

where:

$$
a_1^{(D_N)} = 2(N-1), \quad a_{N-1}^{(D_N)} = a_N^{(D_N)} = \frac{N(N-1)}{2},
$$

$$
a_1^{(B_N)} = 2N - 1,
$$

$$
a_1^{(C_N)} = 2N, \quad a_N^{(C_N)} = \frac{N(N+1)}{2}.
$$

It is also necessary to define the additional operator in type C_N:

$$M_{1,1}^{(C_N)} := \sum_{\epsilon=\pm 1} \sum_{i=1}^{N} \frac{x_i^{2\epsilon}}{x_i^{2\epsilon} - 1} \frac{q^2 x_i^{2\epsilon}}{q^2 x_i^{2\epsilon} - 1} \prod_{j \neq i} \frac{x_i^{\epsilon} x_j}{x_i^{\epsilon} x_j - 1} \frac{x_i^{\epsilon}}{x_i^{\epsilon} - x_j} x_i^{-\epsilon} (x_i^{2\epsilon} \Gamma_i^{2\epsilon} - q^{-2}).$$

(4.7)

4.3 Discrete Time Evolution by Adjoint Action of the Gaussian

By analogy with the A_{N-1} case (see Theorem 3.5), define the Gaussian function for types BCD:

$$\gamma := e^{\sum_{i=1}^{N} \frac{(\mathrm{Log}(x_i))^2}{4\mathrm{Log}(q)}}.$$

(4.8)

Note the slight modification $q \to q^2$ compared to the A type Gaussian (1.2).

The adjoint action of Gaussian function on the difference operators of Sect. 4.2 induces a discrete time evolution:

Definition 4.1 Let $k \in \mathbb{Z}$. The difference operators $M_{a,k}^{(\mathfrak{g})}$ are defined as follows:

$$D_N : \quad M_{1,k}^{(D_N)} := q^{-k} \gamma^{-k} M_{1,0}^{(D_N)} \gamma^k$$

$$M_{N-1,k}^{(D_N)} := q^{-kN/4} \gamma^{-k} M_{N-1,0}^{(D_N)} \gamma^k$$

$$M_{N,k}^{(D_N)} := q^{-kN/4} \gamma^{-k} M_{N,0}^{(D_N)} \gamma^k$$

$$B_N : \quad M_{1,k}^{(B_N)} := q^{-k} \gamma^{-k} M_{1,0}^{(B_N)} \gamma^k$$

$$C_N : \quad M_{1,2k}^{(C_N)} := q^{-2k} \gamma^{-2k} M_{1,0}^{(C_N)} \gamma^{2k}$$

$$M_{1,2k+1}^{(C_N)} := q^{-2k} \gamma^{-2k} M_{1,1}^{(C_N)} \gamma^{2k}$$

$$M_{N,k}^{(C_N)} := q^{-kN/2} \gamma^{-2k} M_{N,0}^{(C_N)} \gamma^{2k}.$$

Note that in type C, the definition for the time evolution of $M_{1,k}^{(C_N)}$ splits into two separate evolutions for even and odd k, involving the operators $M_{1,0}^{(C_N)}$ for even k and $M_{1,1}^{(C_N)}$ of (4.7) for odd k. This is due to the fact that α_1 is a short root in type C. The same parity phenomenon is expected for the short root α_N of B_N.

Using the simple relation $\gamma^{-1}\Gamma_i^2\gamma = qx_i\Gamma_i^2$ for γ as in (4.8), we see that Definition 4.1 results immediately in the following explicit expressions:

$$\text{Type } D_N: \qquad M_{1,k}^{(D_N)} = \sum_{\epsilon=\pm 1}\sum_{i=1}^{N}\prod_{j\neq i}\frac{x_i^\epsilon x_j}{x_i^\epsilon x_j - 1}\frac{x_i^\epsilon}{x_i^\epsilon - x_j}x_i^{k\epsilon}\,\Gamma_i^{2\epsilon}$$

$$M_{N-1,k}^{(D_N)} = \sum_{\substack{\epsilon_1,\ldots,\epsilon_N=\pm 1 \\ \epsilon_1\epsilon_2\cdots\epsilon_N=-1}}\ \prod_{1\leq i<j\leq N}\frac{x_i^{\epsilon_i}x_j^{\epsilon_j}}{x_i^{\epsilon_i}x_j^{\epsilon_j}-1}\prod_{i=1}^{N}x_i^{\frac{k\epsilon_i}{2}}\prod_{i=1}^{N}\Gamma_i^{\epsilon_i} \quad (4.9)$$

$$M_{N,k}^{(D_N)} = \sum_{\substack{\epsilon_1,\ldots,\epsilon_N=\pm 1 \\ \epsilon_1\epsilon_2\cdots\epsilon_N=1}}\ \prod_{1\leq i<j\leq N}\frac{x_i^{\epsilon_i}x_j^{\epsilon_j}}{x_i^{\epsilon_i}x_j^{\epsilon_j}-1}\prod_{i=1}^{N}x_i^{\frac{k\epsilon_i}{2}}\prod_{i=1}^{N}\Gamma_i^{\epsilon_i} \quad (4.10)$$

$$\text{Type } B_N: \qquad M_{1,k}^{(B_N)} = \sum_{\epsilon=\pm 1}\sum_{i=1}^{N}\frac{x_i^\epsilon}{x_i^\epsilon - 1}\prod_{j\neq i}\frac{x_i^\epsilon x_j}{x_i^\epsilon x_j - 1}\frac{x_i^\epsilon}{x_i^\epsilon - x_j}x_i^{k\epsilon}\,\Gamma_i^{2\epsilon}$$

$$\text{Type } C_N: \qquad M_{1,2k}^{(C_N)} = q^{-2k} + \sum_{\epsilon=\pm 1}\sum_{i=1}^{N}\frac{x_i^{2\epsilon}}{x_i^{2\epsilon}-1}\frac{q^2 x_i^{2\epsilon}}{q^2 x_i^{2\epsilon}-1}$$

$$\times\prod_{j\neq i}\frac{x_i^\epsilon x_j}{x_i^\epsilon x_j - 1}\frac{x_i^\epsilon}{x_i^\epsilon - x_j}\,(x_i^{2k\epsilon}\Gamma_i^{2\epsilon} - q^{-2k})$$

$$M_{1,2k-1}^{(C_N)} = \sum_{\epsilon=\pm 1}\sum_{i=1}^{N}\frac{x_i^{2\epsilon}}{x_i^{2\epsilon}-1}\frac{q^2 x_i^{2\epsilon}}{q^2 x_i^{2\epsilon}-1}\prod_{j\neq i}\frac{x_i^\epsilon x_j}{x_i^\epsilon x_j - 1}\frac{x_i^\epsilon}{x_i^\epsilon - x_j}x_i^{-\epsilon}(x_i^{2k\epsilon}\Gamma_i^{2\epsilon} - q^{-2k})$$

$$M_{N,k}^{(C_N)} = \sum_{\epsilon_1,\ldots,\epsilon_N=\pm 1}\prod_{i=1}^{N}\frac{x_i^{2\epsilon_i}}{x_i^{2\epsilon_i}-1}\prod_{1\leq i<j\leq N}\frac{x_i^{\epsilon_i}x_j^{\epsilon_j}}{x_i^{\epsilon_i}x_j^{\epsilon_j}-1}\prod_{i=1}^{N}x_i^{k\epsilon_i}\prod_{i=1}^{N}\Gamma_i^{\epsilon_i}. \quad (4.11)$$

These expressions motivate the main conjectures of this paper.

4.4   Other Macdonald Operators via Quantum Determinants

The list of operators defined in the previous section can be completed to include $M_{a,k}^{(\mathfrak{g})}$ for the Dynkin labels a which are not in the list above, by using quantum determinants.

Comparing the commutation relations and mutation relations of the quantum BCD Q-systems involving only the Dynkin labels $\{1,\ldots,N-1\}$ (types BC) and $\{1,\ldots,N-2\}$ (type D), we see that the relations are the same as in type A of (3.2),

expressed in the variable $M_{a,k}$, up to the change of parameter $q \mapsto q^2$ (which results in a change in the first factor $q^a \mapsto q^{2a}$, and the commutation relations $q^{\min(a,b)} \mapsto q^{2\min(a,b)}$). This observation leads to the following definitions:

Definition 4.2 For types BCD, define the difference operators $M_{a,k}^{(\mathfrak{g})}$ by the following quantum determinants:

$$M_{a,k}^{(\mathfrak{g})} := |\mathbf{M}^{(\mathfrak{g})}[\underbrace{k, k, \ldots, k}_{a \text{ times}}]|_{q^2} = M_{k,k,\ldots,k}^{(\mathfrak{g})} \qquad (a = 1, 2, \ldots, n_{\mathfrak{g}}), \qquad (4.12)$$

where $n_{D_N} = N - 2$, and $n_{B_N} = n_{C_N} = N - 1$. Here the matrix $\mathbf{M}^{(\mathfrak{g})}(\{\mathbf{a}\})$ is obtained by replacing $M_{1,k}$ with $M_{1,k}^{(\mathfrak{g})}$ in the defining expression (3.3), and the quantum determinant has parameter q^2 instead of q in the Vandermonde factor.

By homogeneity of the quantum determinant as a polynomial in the variables $M_{1,k}^{(\mathfrak{g})}$, Definition (4.12) is compatible with the discrete time evolution relations:

$$M_{a,k}^{(D_N)} = q^{-ka} \gamma^{-k} M_{a,0}^{(D_N)} \gamma^k, \qquad (a = 1, 2, \ldots, N - 2);$$

$$M_{a,k}^{(B_N)} = q^{-ka} \gamma^{-k} M_{a,0}^{(B_N)} \gamma^k, \qquad (a = 1, 2, \ldots, N - 1);$$

$$M_{a,2k+\eta}^{(C_N)} = q^{-2ka} \gamma^{-2k} M_{a,\eta}^{(C_N)} \gamma^{2k}, \qquad (a = 1, 2, \ldots, N - 1, \eta = 0, 1),$$

with the Gaussian function γ as in (4.8).

4.5 The Quantum Q-System Conjectures

The main conjecture of this paper is the following:

Conjecture 3 For $G = D_N, B_N, C_N$, the Macdonald operators $M_{a,n}^{(\mathfrak{g})}$ obey the following renormalized versions (we omit the superscript $\mathfrak{g}$ for simplicity) of the quantum Q-systems (2.12–2.10):

Type D_N :
$$M_{a,k} M_{b,p} = q^{\Lambda_{a,b}(p-k)} M_{b,p} M_{a,k}$$

$$q^{2a} M_{a,k+1} M_{a,k-1} = M_{a,k}^2 - M_{a+1,k} M_{a-1,k} \qquad (a \in [1, N-3])$$

$$q^{2(N-2)} M_{N-2,k+1} M_{N-2,k-1} = M_{N-2,k}^2$$

$$-q^{-\frac{(N-2)k}{2}} M_{N,k} M_{N-1,k} M_{N-3,k} \qquad (4.13)$$

$$q^{\frac{N}{2}} M_{N-1,k+1} M_{N-1,k-1} = M_{N-1,k}^2 - q^{\frac{(N-4)k}{2}} M_{N-2,k}$$

$$q^{\frac{N}{2}} M_{N,k+1} M_{N,k-1} = M_{N,k}^2 - q^{\frac{(N-4)k}{2}} M_{N-2,k}$$

Type B_N :
$$M_{a,k} M_{b,p} = q^{\Lambda_{a,b}p - \Lambda_{b,a}k} M_{b,p} M_{a,k}$$

$$q^{2a} M_{a,k+1} M_{a,k-1} = (M_{a,k})^2 - M_{a+1,k} M_{a-1,k} \quad (a \in [1, N-2])$$

$$q^{2N-2} M_{N-1,k+1} M_{N-1,k-1} = (M_{N-1,k})^2 - M_{N,2k} M_{N-2,k} \tag{4.14}$$

$$q^N M_{N,2k+1} M_{N,2k-1} = (M_{N,2k})^2 - q^{-2k}(M_{N-1,k})^2$$

$$q^N M_{N,2k+2} M_{N,2k} = (M_{N,2k+1})^2$$
$$-q^{N-1-(2k+1)} M_{N-1,k+1} M_{N-1,k} \tag{4.15}$$

Type C_N :
$$M_{a,k} M_{b,p} = q^{\Lambda_{a,b}p - \Lambda_{b,a}k} M_{b,p} M_{a,k}$$

$$q^{2a} M_{a,k+1} M_{a,k-1} = M_{a,k}^2 - M_{a+1,k} M_{a-1,k} \quad (a \in [1, N-2])$$

$$q^{2N-2} M_{N-1,2k+1} M_{N-1,2k-1} = M_{N-1,2k}^2 - q^{-Nk} M_{N-2,2k} M_{N,k}^2$$

$$q^{2N-2} M_{N-1,2k+2} M_{N-1,2k} = M_{N-1,2k+1}^2 - q^{-Nk} M_{N-2,2k+1} M_{N,k+1} M_{N,k}$$

$$q^N M_{N,k+1} M_{N,k-1} = M_{N,k}^2 - q^{(N-2)k} M_{N-1,2k} \tag{4.16}$$

In the case of B_N we have not given an *a priori* definition of the order N operators $M_{N,k}$. They can be constructed as a suitable linear combination of the limits at $t \to \infty$ of van Diejen operators [vD95, vDE11], compatible with the quantum Q-system in type B. These operators can be defined also by taking the quantum Q-system as the defining set of equations. First, define $M_{N,2k} = |M(\{k, k, \ldots, k\})|_{q^2}$ (k repeated N times), compatible with Eq. (4.14), in which $M_{N,2k}$ plays the role of the quantum determinant of size N. Then Eq. (4.15) gives information about $M_{N,2k+1}$:

$$(M_{N,2k+1})^2 = q^N M_{N,2k+2} M_{N,2k} + q^{N-1-(2k+1)} M_{N-1,k+1} M_{N-1,k}.$$

This can be used to determine the relevant difference operators. The fact that all the other equations of the system are satisfied is a highly non-trivial check.

These conjectures have been checked numerically up to $N = 6$. We illustrate them in cases of small rank in Appendix A.

Lemma 4.4 *The difference operators $M_{a,k}$ are a functional representation of the quantum Q-system (2.12–2.10) up to the following rescaling:*

D_N :
$$M_{a,k} = q^{\frac{a(a+1)}{2} - a(N+k)} Q_{a,k} \quad (\alpha \in [1, N-2])$$

$$M_{N-1,k} = q^{-\frac{N(N-1)}{4} - \frac{Nk}{4}} Q_{N-1,k}$$

$$M_{N,k} = q^{-\frac{N(N-1)}{4} - \frac{Nk}{4}} Q_{N,k} \tag{4.17}$$

B_N :
$$M_{a,k} = q^{\frac{a^2}{2} - a(N+k)} Q_{a,k} \quad (a \in [1, N-1])$$

$$M_{N,k} = q^{-\frac{N^2}{2} - \frac{Nk}{2}} Q_{N,k} \tag{4.18}$$

$$C_N: \qquad M_{a,k} = q^{\frac{a(a-1)}{2} - a(N+k)} \, \mathcal{Q}_{a,k} \qquad (a \in [1, N-1])$$

$$M_{N,k} = q^{-\frac{N(N+1)}{4} - \frac{Nk}{2}} \, \mathcal{Q}_{N,k} \tag{4.19}$$

Proof By straightforward inspection. □

5 The Raising/Lowering Operator Conjectures for Types BCD

In this section, we present conjectures that extend the result of Theorem 3.6 to types BCD. These involve the action of the difference operators $M_{a,k}$ on the dual q-Whittaker functions:

$$\Pi_\lambda^{(\mathfrak{g})}(q^{-1}; \mathbf{x}) := \lim_{t \to \infty} P_\lambda^{(\mathfrak{g})}(q, t; \mathbf{x}).$$

The general idea is that while the operators $M_{a,0}^{(\mathfrak{g})}$ are all limits of Macdonald operators, for which the q-Whittaker functions $\Pi_\lambda^{(\mathfrak{g})}$ are common eigenfunctions, the operators $M_{a,\pm 1}^{(\mathfrak{g})}$ are simple raising and lowering operators on those eigenfunctions.

Conjecture 5.1 *The operators $M_{a,0}^{(\mathfrak{g})}$ and $M_{a,\pm 1}^{(\mathfrak{g})}$ have the following action on the q-Whittaker functions $\Pi_\lambda^{(\mathfrak{g})}$, valid for all $a \in [1, N]$:*

$$D_N, B_N: \qquad M_{a,0} \, \Pi_\lambda = q^{2t_a(\lambda, \omega_a)} \, \Pi_\lambda$$

$$M_{a,1} \, \Pi_\lambda = q^{2t_a(\lambda, \omega_a)} \, \Pi_{\lambda+\omega_a}$$

$$M_{a,-1} \, \Pi_\lambda = q^{2t_a(\lambda, \omega_a)} \left(1 - q^{-2t_a(\alpha_a, \lambda)}\right) \Pi_{\lambda-\omega_a} \tag{5.1}$$

$$C_N: \qquad M_{a,0} \, \Pi_\lambda = q^{t_a(\omega_a, \lambda)} \, \Pi_\lambda$$

$$M_{a,1} \, \Pi_\lambda = q^{t_a(\omega_a, \lambda)} \, \Pi_{\lambda+\omega_a}$$

$$M_{a,-1} \, \Pi_\lambda = q^{t_a(\omega_a, \lambda)} \left(1 - q^{-t_a(\alpha_a, \lambda)}\right) \Pi_{\lambda-\omega_a}, \tag{5.2}$$

where ω_a and α_a are the fundamental weights and simple roots of the corresponding algebras, $t_a = 2$ for the short roots in types BC, and $t_a = 1$ otherwise.

The eigenvalue equations (the first equation in each set) are a consequence of the Macdonald eigenvalue equations of the finite t case. As in type A, up to a scalar multiple, the raising equations (the second equation in each set) are a consequence

of the quantum commutation relations $M_{a,0}^{(\mathfrak{g})} M_{b,1}^{(\mathfrak{g})} = q^{\lambda_{a,b}^{(\mathfrak{g})}} M_{b,1}^{(\mathfrak{g})} M_{a,0}^{(\mathfrak{g})}$. When applied to the q-Whittaker functions, this gives

$$M_{a,0}^{(\mathfrak{g})}\left(M_{b,1}^{(\mathfrak{g})} \Pi_\lambda^{(\mathfrak{g})}\right) = q^{\lambda_{a,b}^{(\mathfrak{g})}} M_{b,1}^{(\mathfrak{g})} M_{a,0}^{(\mathfrak{g})} \Pi_\lambda^{(\mathfrak{g})} = q^{\epsilon^{(\mathfrak{g})} t_a^{(\mathfrak{g})}(\omega_a^{(\mathfrak{g})}, \lambda + \omega_b^{(\mathfrak{g})})} M_{b,1}^{(\mathfrak{g})} \Pi_\lambda^{(\mathfrak{g})},$$

where we have defined $\epsilon^{(\mathfrak{g})} = 2$ for $\mathfrak{g} = B_N, D_N$ and $\epsilon^{(\mathfrak{g})} = 1$ for $\mathfrak{g} = C_N$, and used the fact that $\lambda_{a,b}^{(\mathfrak{g})} = \epsilon^{(\mathfrak{g})} t_a^{(\mathfrak{g})}(\omega_a^{(\mathfrak{g})}, \omega_b^{(\mathfrak{g})})$. As the simultaneous eigenvalue property for the action of all $M_{a,0}$ determines the eigenvectors uniquely, up to a scalar multiple, we deduce that $M_{b,1}^{(\mathfrak{g})} \Pi_\lambda^{(\mathfrak{g})}$ must be proportional to $\Pi_{\lambda+\omega_b}^{(\mathfrak{g})}$. The conjecture concerns simply the explicit value of the proportionality factor $q^{\epsilon^{(\mathfrak{g})} t_a^{(\mathfrak{g})}(\omega_a^{(\mathfrak{g})}, \lambda)}$, which we checked numerically up to $N = 6$.

Similarly, the action of the lowering operator (third equation in each set) follows from the eigenvalue and raising operator actions and the quantum Q-system equations (4.13–4.16) of Conjecture 3. This is readily seen by applying the exchange relations involving $M_{a,1}$ and $M_{a,-1}$ to Π_λ.

Examples of q-Whittaker functions and raising/lowering difference operators are given in Appendix A.

6 The Graded Character Conjectures for Types BCD

The main Conjecture 3 is a necessary ingredient in proving the following conjecture about the expression of the graded characters in terms of difference operators. Let $I_<$ be the subset labels of the short roots the Dynkin diagram in types BC. Recall that $t_a = 2$ for $a \in I_<$, and $t_a = 1$ otherwise.

Conjecture 6.1 *The graded characters of the tensor products of KR-modules in types BCD can be expressed using the iterated action of the difference operators in types BCD on the polynomial 1:*

$$\chi_{\mathbf{n}}^{(\mathfrak{g})}(q^{-1}; \mathbf{x}) = q^{-\frac{1}{2} Q^{(\mathfrak{g})}(\mathbf{n})} \prod_{\ell=k}^{1} \left(\prod_{a \in I} (M_{a,t_a\ell}^{(\mathfrak{g})})^{n_{a,t_a\ell}} \prod_{a \in I_<} (M_{a,t_a\ell-1}^{(\mathfrak{g})})^{n_{a,t_a\ell-1}} \right) \cdot 1,$$

where k is chosen large enough to cover all the non-zero n's, and

$$Q^{(\mathfrak{g})}(\mathbf{n}) := \sum_{a,b=1}^{N} \sum_{i,j\geq 1} n_{a,i} \frac{\lambda_{a,b}^{(\mathfrak{g})}}{t_a^{(\mathfrak{g})}} \min(t_b^{(\mathfrak{g})} i, t_a^{(\mathfrak{g})} j) n_{b,j} - \sum_{a=1}^{N} \sum_{i\geq 1} i\, \lambda_{a,a}^{(\mathfrak{g})} n_{a,i}.$$

The ordering of the difference operators in Conjecture 6.1 is consistent with [DFK14, Lin19] and is determined by the order of mutations in the bipartite belt of the quantum cluster algebra.

In particular, Conjecture 6.1 allows one to interpret the so-called level 1 graded characters corresponding to only possibly non-zero $n_a = n_{a,1}$, as the limiting Macdonald polynomials (or dual q-Whittaker functions) Π_λ, with the correspondence:

$$\chi_{\{n_a\}}^{(\mathfrak{g})}(q^{-1}, \mathbf{x}) = q^{-\frac{1}{2}Q^{(\mathfrak{g})}(\{n_a\})} \prod_{a \in I_>} \left(M_{a,1}^{(\mathfrak{g})}\right)^{n_a} \prod_{a \in I_<} \left(M_{a,1}^{(\mathfrak{g})}\right)^{n_a} \cdot 1 = \Pi_\lambda^{(\mathfrak{g})}(q^{-1}; \mathbf{x}),$$

(6.1)

where $I_> = I \setminus I_<$ is the set of long root labels, $\lambda = \sum_a n_a \omega_a$, and the quadratic form $Q^{(\mathfrak{g})}$ is

$$Q^{(\mathfrak{g})}(\{n_a\}) = \sum_{a,b=1}^{N} n_a \frac{\lambda_{a,b}^{(\mathfrak{g})}}{t_a^{(\mathfrak{g})}} \min(t_a^{(\mathfrak{g})}, t_b^{(\mathfrak{g})}) n_b - \sum_{a=1}^{N} \lambda_{a,a}^{(\mathfrak{g})} n_a$$

$$= \epsilon^{(\mathfrak{g})} \left\{ \sum_{a,b=1}^{N} n_a (\omega_a^{(\mathfrak{g})}, \omega_b^{(\mathfrak{g})}) \min(t_a^{(\mathfrak{g})}, t_b^{(\mathfrak{g})}) n_b - \sum_{a=1}^{N} t_a^{(\mathfrak{g})} (\omega_a^{(\mathfrak{g})}, \omega_a^{(\mathfrak{g})}) n_a \right\}.$$

Alternatively, for a fixed $\mathbf{n}$, acting with $M_{a,k}^{(\mathfrak{g})}$ such that $t_a k \geq \max\{t_b j : n_{b,j} > 0\}$ on $\chi_{\mathbf{n}}$ gives

$$M_{a,k}^{(\mathfrak{g})} \chi_{\mathbf{n}}^{(\mathfrak{g})}(q^{-1}; \mathbf{x}) = q^{\epsilon^{(\mathfrak{g})} \sum_{j,b}(\omega_a^{(\mathfrak{g})}, \omega_b^{(\mathfrak{g})}) \min(t_b^{(\mathfrak{g})} k, t_a^{(\mathfrak{g})} j) n_{b,j}} \chi_{\mathbf{n}+\epsilon_{a,k}}^{(\mathfrak{g})}(q^{-1}; \mathbf{x}),$$

where, again, $\mathbf{n} = \sum n_{b,j} \epsilon_{b,j}$.

It is a non-trivial exercise to check the compatibility between (6.1) and the raising operator conditions on the second lines of (5.1) and (5.2). Indeed, we note that for a short root label a, we may only act with $M_{a,1}^{(\mathfrak{g})}$ on characters obtained themselves by short root raising operators say $\prod_{b \in I_<}(M_{b,1}^{(\mathfrak{g})})^{n_b} \cdot 1$:

$$M_{a,1}^{(\mathfrak{g})} \chi_{\{n_b\}}^{(\mathfrak{g})}(q^{-1}, \mathbf{x}) = q^{-\frac{1}{2}Q^{(\mathfrak{g})}(\{n_b\})} \prod_{b \in I_<} (M_{b,1}^{(\mathfrak{g})})^{n_b+\delta_{a,b}} \cdot 1$$

$$= q^{\frac{1}{2}(Q^{(\mathfrak{g})}(\{n_b+\delta_{a,b}\}) - Q^{(\mathfrak{g})}(\{n_b\}))} \chi_{\{n_b+\delta_{a,b}\}}^{(\mathfrak{g})}(q^{-1}, \mathbf{x}).$$

We compute

$$Q^{(\mathfrak{g})}(\{n_b + \delta_{a,b}\}) - Q^{(\mathfrak{g})}(\{n_b\}) = \epsilon^{(\mathfrak{g})} \sum_{b \in I_<} n_b \min(t_a^{(\mathfrak{g})}, t_b^{(\mathfrak{g})})(\omega_a^{(\mathfrak{g})}, \omega_b^{(\mathfrak{g})})$$

$$= \epsilon^{(\mathfrak{g})} t_a^{(\mathfrak{g})}(\omega_a^{(\mathfrak{g})}, \sum_b n_b \omega_b^{(\mathfrak{g})}) = \epsilon^{(\mathfrak{g})} t_a^{(\mathfrak{g})}(\omega_a^{(\mathfrak{g})}, \lambda)$$

by use of $t_a^{(\mathfrak{g})} = t_b^{(\mathfrak{g})} = 2$. This is in agreement with (5.1) and (5.2), upon writing $\lambda = \sum n_b \omega_b$. For a long root label a, we have similarly:

$$Q^{(\mathfrak{g})}(\{n_b + \delta_{a,b}\}) - Q^{(\mathfrak{g})}(\{n_b\}) = \epsilon^{(\mathfrak{g})} \sum_{b \in I} n_b \min(t_a^{(\mathfrak{g})}, t_b^{(\mathfrak{g})})(\omega_a^{(\mathfrak{g})}, \omega_b^{(\mathfrak{g})})$$

$$= \epsilon^{(\mathfrak{g})} (\omega_a^{(\mathfrak{g})}, \sum_b n_b \omega_b^{(\mathfrak{g})}) = \epsilon^{(\mathfrak{g})} t_a^{(\mathfrak{g})} (\omega_a, \lambda)$$

by use of $t_a^{(\mathfrak{g})} = 1$. This is again in agreement with (5.1) and (5.2).

This special case is also in agreement with [LNS$^+$17].

7   Conclusion

7.1   *Summary: Macdonald Operators and Quantum Cluster Algebra*

In this paper we have presented two main conjectures about the difference operators $M_{a,k}^{(\mathfrak{g})}$ corresponding to root systems of types BCD.

The first conjecture states that the difference operators $\{M_{a,k}^{(\mathfrak{g})}\}$ satisfy renormalized quantum Q-systems of types BCD. The commuting difference operators $\{M_{a,0}^{(\mathfrak{g})}\}$ can be obtained as the limits $t \to \infty$ of appropriate Macdonald operators for the relevant Lie root systems. The operators $M_{a,k}^{(\mathfrak{g})}$ are their $SL_2(\mathbb{Z})$ "discrete time evolution."

The second conjecture is that the difference operators $M_{a,\pm 1}^{(\mathfrak{g})}$ at times $k = \pm 1$ act as raising and lowering operators on q-Whittaker functions $\Pi_\lambda^{(\mathfrak{g})}$.

The quantum Q-systems are mutations in the corresponding quantum cluster algebras. From this point of view, the sets $S_\pm := \{M_{a,0}^{(\mathfrak{g})}, M_{a,\pm 1}^{(\mathfrak{g})}\}$ are two possible valid initial cluster seeds, and the conjectures state that, as in type A, these are formed of the Macdonald and raising (reps. lowering) operators, at $t \to \infty$. This raises a number of questions regarding cluster variables in general: from preliminary inspection, it appears that *all cluster variables* in the corresponding cluster algebra are difference operators as well, a quite surprising property which goes way beyond the usual Laurent property of quantum cluster algebra, which would only imply that all cluster variables are Laurent polynomials of those in an initial cluster. These other difference operators should tell us something new about Macdonald theory (including in type A).

Finally we note that the second conjecture, if true, gives a perturbative starting point for constructing generalizations of finite t, A-type raising operators of Kirillov and Noumi for other types. Indeed, we expect the finite t operators to have a finite $1/t$ expansion, whose coefficients are themselves difference operators.

Conjecture 5.1 provides the leading order term in this expansion. We will return to this question in a future publication.

7.2 Towards Proving the Conjectures

The second conjecture of this paper offers a possible strategy for proving both conjectures which goes as follows. The second conjecture indeed can be restated as follows: the Macdonald and raising operators $M_{a,0}^{(\mathfrak{g})}$ and $M_{a,1}^{(\mathfrak{g})}$ act on the basis Π_λ of Weyl-symmetric polynomials of **x** as very simple *dual* operators X_a, P_a:

$$M_{a,0}^{(\mathfrak{g})} \Pi_\lambda^{(\mathfrak{g})} = \Pi_\lambda^{(\mathfrak{g})} X_a, \quad M_{a,1}^{(\mathfrak{g})} \Pi_\lambda^{(\mathfrak{g})} = \Pi_\lambda^{(\mathfrak{g})} P_a,$$

where the operators X_a, P_a act to the left on the variables $\Lambda_a = q^{2\lambda_a}$. More precisely the operators X_a acts diagonally on the basis $\Pi_\lambda^{(\mathfrak{g})}$ with eigenvalues $q^{\epsilon_a^{(\mathfrak{g})}(\omega_a^{(\mathfrak{g})}, \lambda)}$, whereas P_a includes a multiplicative shift of Λ variables. As a result, the operators X_a, P_a obey the simple (opposite) commutation relations: $X_a P_b = q^{-\lambda_{a,b}^{(\mathfrak{g})}} P_b X_a$.

Reversing the logic entirely, we may *start from the data* of operators X_a, P_a, and *define* the left action of the operators $M_{a,k}^{(\mathfrak{g})}$ via the (opposite) renormalized quantum Q-systems of this paper. To identify the left action and the right one, we need to construct the Gaussian operator in X_a, P_a variables, and the conjectures will follow from the anti-homomorphism mapping left and right actions. We will pursue this program elsewhere.

7.3 sDAHA, EHA and the t-Deformation of Quantum Q-Systems

In [DFK19], we investigated the natural t-deformation of the A type quantum Q-system provided by the type A sDAHA, also expressed as a quotient of the elliptic Hall algebra (EHA) [Sch12], or the quantum toroidal algebra of $\mathfrak{gl}_1$. There exist sDAHAs for all classical types [Che05], and this would be the natural candidate for a generalization of the A type results. We may use as starting points the (q, t)-Macdonald operators of Sect. 4.1, and their time evolution via the suitable $SL_2(\mathbb{Z})$ action, to derive current algebra relations that will generalize EHA or quantum toroidal algebra relations, ideally giving rise to new interesting algebras.

Appendix A: Examples

This appendix gathers a few inter-connected examples, which give non-trivial checks of the conjectures of this paper. We give here the examples of the B_2 and C_2 cases, that of the A_3 and D_3, and finally the D_4 case. We verify in particular that the symmetries of the Dynkin diagrams are also reflected on our difference operators.

A.1 Weyl-Invariant Schur Functions

Definition A.1 We define the Weyl-invariant Schur functions $s_\lambda^{(\mathfrak{g})}(\mathbf{x})$ for $\mathfrak{g} = A_{N-1}, B_N, C_N, D_N$ to be the characters of the irreducible representations of corresponding weight [Miz03]:[2]

$$s_\lambda^{(A_{N-1})}(\mathbf{x}) = \frac{\det_{1 \leq i,j \leq N}\left(x_i^{N-j+\lambda_j}\right)}{\det_{1 \leq i,j \leq N}\left(x_i^{N-j}\right)} \tag{A.1}$$

$$s_\lambda^{(B_N)}(\mathbf{x}) = \frac{\det_{1 \leq i,j \leq N}\left(x_i^{N-j+\lambda_j+\frac{1}{2}} - \frac{1}{x_i^{N-j+\lambda_j+\frac{1}{2}}}\right)}{\det_{1 \leq i,j \leq N}\left(x_i^{N-j+\frac{1}{2}} - \frac{1}{x_i^{N-j+\frac{1}{2}}}\right)} \tag{A.2}$$

$$s_\lambda^{(C_N)}(\mathbf{x}) = \frac{\det_{1 \leq i,j \leq N}\left(x_i^{N-j+\lambda_j+1} - \frac{1}{x_i^{N-j+\lambda_j+1}}\right)}{\det_{1 \leq i,j \leq N}\left(x_i^{N-j+1} - \frac{1}{x_i^{N-j+1}}\right)} \tag{A.3}$$

$$s_\lambda^{(D_N)}(\mathbf{x}) = \frac{\det_{1 \leq i,j \leq N}\left(x_i^{N-j+\lambda_j} - \frac{1}{x_i^{N-j+\lambda_j}}\right)}{\det_{1 \leq i,j \leq N}\left(x_i^{N-j} + \frac{1}{x_i^{N-j}}\right)} + \frac{\det_{1 \leq i,j \leq N}\left(x_i^{N-j+\lambda_j} + \frac{1}{x_i^{N-j+\lambda_j}}\right)}{\det_{1 \leq i,j \leq N}\left(x_i^{N-j} + \frac{1}{x_i^{N-j}}\right)}. \tag{A.4}$$

The Weyl-invariant Schur functions form a basis of the space of Weyl-invariant (Laurent) polynomials. In the following sections, we write the dual q-Whittaker functions Π_λ in this Schur basis. We drop the superscript $(\mathfrak{g})$ for simplicity.

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[2]In [Miz03], the author restricts the definition to λ's that are actual partitions. Here we include all possible dominant weights.

Remark A.2 Note that Eq. (A.1) for $\mathfrak{gl}_N$ Schur functions gives the $\mathfrak{sl}_N$ Schur functions, upon restriction to $x_1 x_2 \cdots x_N = 1$ and then noting that $s_\lambda = s_{\lambda-\epsilon}$ where $\epsilon = e_1 + e_2 + \cdots + e_N$.

A.2 The B_2 Case

A.2.1 M Operators

$$M_{1,k} = \sum_{\epsilon=\pm 1} x_1^{k\epsilon} \frac{x_1^\epsilon}{x_1^\epsilon - 1} \frac{x_1^\epsilon x_2}{x_1^\epsilon x_2 - 1} \frac{x_1^\epsilon}{x_1^\epsilon - x_2} \Gamma_1^{2\epsilon}$$

$$+ x_2^{k\epsilon} \frac{x_2^\epsilon}{x_2^\epsilon - 1} \frac{x_2^\epsilon x_1}{x_2^\epsilon x_1 - 1} \frac{x_2^\epsilon}{x_2^\epsilon - x_1} \Gamma_2^{2\epsilon}$$

$$M_{2,2k} = q^{-2k} + \sum_{\epsilon_1,\epsilon_2=\pm 1} \frac{x_1^{\epsilon_1}}{x_1^{\epsilon_1} - 1} \frac{x_2^{\epsilon_2}}{x_2^{\epsilon_2} - 1} \frac{x_1^{\epsilon_1} x_2^{\epsilon_2}}{x_1^{\epsilon_1} x_2^{\epsilon_2} - 1} \frac{q^2 x_1^{\epsilon_1} x_2^{\epsilon_2}}{q^2 x_1^{\epsilon_1} x_2^{\epsilon_2} - 1}$$

$$\times (x_1^{k\epsilon_1} x_2^{k\epsilon_2} \Gamma_1^{2\epsilon_1} \Gamma_2^{2\epsilon_2} - q^{-2k})$$

$$M_{2,2k-1} = \sum_{\epsilon_1,\epsilon_2=\pm 1} \frac{x_1^{\epsilon_1}}{x_1^{\epsilon_1} - 1} \frac{x_2^{\epsilon_2}}{x_2^{\epsilon_2} - 1} \frac{x_1^{\epsilon_1} x_2^{\epsilon_2}}{x_1^{\epsilon_1} x_2^{\epsilon_2} - 1} \frac{q^2 x_1^{\epsilon_1} x_2^{\epsilon_2}}{q^2 x_1^{\epsilon_1} x_2^{\epsilon_2} - 1}$$

$$\times (x_1^{\epsilon_1} x_2^{\epsilon_2})^{-\frac{1}{2}} (x_1^{k\epsilon_1} x_2^{k\epsilon_2} \Gamma_1^{2\epsilon_1} \Gamma_2^{2\epsilon_2} - q^{-2k})$$

Here $M_{1,k}$ is given by the k-th iterate conjugation of (4.4) w.r.t. the Gaussian, and $M_{2,2k}$ by the quantum determinant $M_{1,k}^2 - q^2 M_{1,k+1} M_{1,k-1}$.

A.2.2 Dual q-Whittaker Functions

$$\Pi_{0,0} = s_{0,0}$$

$$\Pi_{1,0} = s_{1,0}$$

$$\Pi_{2,0} = s_{2,0} + q^{-2} s_{1,1} + q^{-4} s_{0,0}$$

$$\Pi_{1,1} = s_{1,1} + q^{-2} s_{1,0} + q^{-2} s_{0,0}$$

$$\Pi_{3,0} = s_{3,0} + \frac{1+q^2}{q^4} s_{2,1} + q^{-6} s_{1,1} + \frac{1+q^2+q^4}{q^8} s_{1,0}$$

$$\Pi_{2,1} = s_{2,1} + q^{-2} s_{2,0} + \frac{1+q^2}{q^4} s_{1,1} + \frac{1+q^2}{q^4} s_{1,0} + q^{-6} s_{0,0}$$

$$\Pi_{\frac{1}{2},\frac{1}{2}} = s_{\frac{1}{2},\frac{1}{2}}$$

$$\Pi_{\frac{3}{2},\frac{1}{2}} = s_{\frac{3}{2},\frac{1}{2}} + q^{-2}s_{\frac{1}{2},\frac{1}{2}}$$

$$\Pi_{\frac{5}{2},\frac{1}{2}} = s_{\frac{5}{2},\frac{1}{2}} + q^{-2}s_{\frac{3}{2},\frac{3}{2}} + \frac{1+q^2}{q^4}s_{\frac{3}{2},\frac{1}{2}} + \frac{1+q^2}{q^6}s_{\frac{1}{2},\frac{1}{2}}$$

$$\Pi_{\frac{3}{2},\frac{3}{2}} = s_{\frac{3}{2},\frac{3}{2}} + \frac{1+q^2}{q^4}s_{\frac{3}{2},\frac{1}{2}} + \frac{1+q^2+q^4}{q^6}s_{\frac{1}{2},\frac{1}{2}}$$

A.3 The C_2 Case

A.3.1 The $B_2 \leftrightarrow C_2$ Symmetry

The B_2 case can be mapped onto the C_2 case, by interchanging the roles of the two fundamental weights. More precisely, let us denote by $\omega_a = \omega_a^{(B_2)}$ (resp. $\omega_a' = \omega_a^{(C_2)}$). The variables x_1, x_2 of the B_2 case can be thought of as $x_i = e^{e_i}$, with $e_1 = \omega_1$ and $e_2 = 2\omega_2 - \omega_1$, and similarly for C_2, where $x_i' = e^{e_i'}$, $e_1' = \omega_1'$ and $e_2' = \omega_2' - \omega_1'$. The mapping $(\omega_1, \omega_2) \mapsto (\omega_2', \omega_1')$ sends

$$x_1 = e^{\omega_1} \mapsto e^{\omega_2'} = x_1' x_2', \quad x_2 = e^{2\omega_2 - \omega_1} \mapsto e^{2\omega_1' - \omega_2'} = \frac{x_1'}{x_2'}. \tag{A.5}$$

Similarly, we have

$$\Gamma_1^2 \mapsto \Gamma_1' \Gamma_2', \qquad \Gamma_2^2 \mapsto \Gamma_1' \Gamma_2'^{-1}. \tag{A.6}$$

The map (A.5–A.6) sends the operators $M_{a,k}^{(B_2)} \mapsto M_{3-a,k}^{(C_2)}$ for $a = 1, 2$ and $k \in \mathbb{Z}$, and the Macdonald polynomials $P_{\lambda_1,\lambda_2}^{(B_2)} \mapsto P_{\lambda_1+\lambda_2,\lambda_1-\lambda_2}^{(C_2)}$ and similarly for the q-Whittaker functions, using

$$\lambda = \lambda_1 e_1 + \lambda_2 e_2 = (\lambda_1 - \lambda_2)\omega_1 + 2\lambda_2\omega_2 \mapsto (\lambda_1 - \lambda_2)\omega_2' + 2\lambda_2\omega_1'$$

$$= (\lambda_1 + \lambda_2)e_1' + (\lambda_1 - \lambda_2)e_2'.$$

A.3.2 M Operators

From the definitions in type C,

$$M_{1,2k} = q^{-2k} + \sum_{\epsilon=\pm 1}\left\{\frac{x_1^{2\epsilon}}{x_1^{2\epsilon}-1}\frac{q^2 x_1^{2\epsilon}}{q^2 x_1^{2\epsilon}-1}\frac{x_1^\epsilon x_2}{x_1^\epsilon x_2-1}\frac{x_1^\epsilon}{x_1^\epsilon-x_2}(x_1^{2k\epsilon}\Gamma_1^{2\epsilon}-q^{-2k})\right.$$

$$\left. +\frac{x_2^{2\epsilon}}{x_2^{2\epsilon}-1}\frac{q^2 x_2^{2\epsilon}}{q^2 x_2^{2\epsilon}-1}\frac{x_2^\epsilon x_1}{x_2^\epsilon x_1-1}\frac{x_2^\epsilon}{x_2^\epsilon-x_1}(x_2^{2k\epsilon}\Gamma_2^{2\epsilon}-q^{-2k})\right\},$$

$$M_{1,2k-1} = \sum_{\epsilon=\pm1} \left\{ \frac{x_1^{2\epsilon}}{x_1^{2\epsilon}-1} \frac{q^2 x_1^{2\epsilon}}{q^2 x_1^{2\epsilon}-1} \frac{x_1^\epsilon x_2}{x_1^\epsilon x_2 - 1} \frac{x_1^\epsilon}{x_1^\epsilon - x_2} x_1^{-\epsilon}(x_1^{2k\epsilon}\Gamma_1^{2\epsilon} - q^{-2k}) \right.$$

$$\left. + \frac{x_2^{2\epsilon}}{x_2^{2\epsilon}-1} \frac{q^2 x_2^{2\epsilon}}{q^2 x_2^{2\epsilon}-1} \frac{x_2^\epsilon x_1}{x_2^\epsilon x_1 - 1} \frac{x_2^\epsilon}{x_2^\epsilon - x_1} x_2^{-\epsilon}(x_2^{2k\epsilon}\Gamma_2^{2\epsilon} - q^{-2k}) \right\},$$

$$M_{2,k} = \sum_{\epsilon_1,\epsilon_2=\pm1} \frac{x_1^{2\epsilon_1}}{x_1^{2\epsilon_1}-1} \frac{x_2^{2\epsilon_2}}{x_2^{2\epsilon_2}-1} \frac{x_1^{\epsilon_1}x_2^{\epsilon_2}}{x_1^{\epsilon_1}x_2^{\epsilon_2}-1} (x_1^{k\epsilon_1}x_2^{k\epsilon_2})\,\Gamma_1^{\epsilon_1}\Gamma_2^{\epsilon_2}.$$

The expected symmetry between $M_{1,k}^{(B_2)} \mapsto M_{2,k}^{(C_2)}$ and $M_{2,k}^{(B_2)} \mapsto M_{1,k}^{(C_2)}$ is easily checked.

A.3.3 Dual q-Whittaker Functions

$$\Pi_{0,0} = s_{0,0}$$

$$\Pi_{1,0} = s_{1,0}$$

$$\Pi_{2,0} = s_{2,0} + q^{-2}s_{1,1} + q^{-2}s_{0,0}$$

$$\Pi_{1,1} = s_{1,1}$$

$$\Pi_{3,0} = s_{3,0} + \frac{1+q^2}{q^4}s_{2,1} + \frac{1+q^2+q^4}{q^6}s_{1,0}$$

$$\Pi_{2,1} = s_{2,1} + q^{-2}s_{2,0} + q^{-2}s_{1,0}$$

$$\Pi_{4,0} = s_{4,0} + \frac{1+q^2+q^4}{q^6}s_{3,1} + \frac{1+q^4}{q^8}s_{2,2} + \frac{(1+q^2+q^4)(1+q^4)}{q^{10}}s_{2,0}$$

$$+ \frac{1+2q^2+q^4+q^6}{q^{10}}s_{1,1} + \frac{1+q^4+q^8}{q^{12}}s_{0,0}$$

$$\Pi_{3,1} = s_{3,1} + q^{-2}s_{2,2} + \frac{(1+q^2)}{q^4}s_{2,0} + \frac{1+q^2}{q^4}s_{1,1} + q^{-6}s_{0,0}$$

$$\Pi_{2,2} = s_{2,2} + q^{-2}s_{2,0} + q^{-4}s_{0,0}.$$

The expected symmetry relations between B_2 and C_2 q-Whittaker functions are easily checked, using the explicit expressions for the relevant Schur functions (A.2–A.3).

A.4 The $\mathfrak{sl}_4$ and D_3 Cases

A.4.1 The $\mathfrak{sl}_4 \leftrightarrow D_3$ Symmetry

Compared to the $\mathfrak{gl}_4$ case, the symmetric functions of the case $\mathfrak{sl}_4$ involve the extra condition that $x_1 x_2 x_3 x_4 = 1$, and accordingly $\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 = 1$ (see Remark 3.4). This is implemented by imposing the extra condition $e_1 + e_2 + e_3 + e_4 = 0$ under which:

$$e_1 = \omega_1, \ e_2 = \omega_2 - \omega_1, \ e_3 = \omega_3 - \omega_2, \ e_4 = -\omega_3.$$

Primed variables are used for D_3:

$$e_1' = \omega_1', \ e_2' = \omega_2' + \omega_3' - \omega_1', \ e_3' = \omega_3' - \omega_2'.$$

We use the mapping

$$\omega_1 \mapsto \omega_3', \quad \omega_2 \mapsto \omega_1', \quad \omega_3 \mapsto \omega_1'.$$

This is equivalent to the changes of variables (using $x_i = e^{e_i}$, $x_i' = e^{e_i'}$):

$$x_1 \mapsto \sqrt{x_1' x_2' x_3'}, \quad x_2 \mapsto \sqrt{\frac{x_1'}{x_2' x_3'}}, \quad x_3 \mapsto \sqrt{\frac{x_2'}{x_1' x_3'}}, \quad x_4 \mapsto \sqrt{\frac{x_3'}{x_1' x_2'}}. \tag{A.7}$$

Moreover, to account for our choice of normalization we must also take $q \mapsto q^2$, which results in

$$\Gamma_1 \mapsto \Gamma_1' \Gamma_2' \Gamma_3', \quad \Gamma_2 \mapsto \frac{\Gamma_1'}{\Gamma_2' \Gamma_3'}, \quad \Gamma_3 \mapsto \frac{\Gamma_2'}{\Gamma_1' \Gamma_3'}, \quad \Gamma_4 \mapsto \frac{\Gamma_3'}{\Gamma_1' \Gamma_2'}. \tag{A.8}$$

The above transformations send the $\mathfrak{sl}_4$ Macdonald operators to the D_3 ones, namely $M_{1,k} \mapsto M_{3,k}'$, $M_{2,k} \mapsto M_{1,k}'$ and $M_{3,k} \mapsto M_{2,k}'$. The corresponding mapping of Macdonald polynomials is

$$P^{(\mathfrak{sl}_4)}_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} \mapsto P^{(D_3)}_{\frac{\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4}{2}, \frac{\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4}{2}, \frac{\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4}{2}}. \tag{A.9}$$

A.4.2 M Operators

The $\mathfrak{sl}_4$ operators are

$$M_{1,k}^{(\mathfrak{sl}_4)} = \sum_{i=1}^{4} x_i^k \prod_{j \neq i} \frac{x_i}{x_i - x_j} \Gamma_i$$

$$M_{2,k}^{(\mathfrak{sl}_4)} = \sum_{1 \leq i < j \leq 4} (x_i x_j)^k \prod_{m \neq i,j} \frac{x_m}{x_m - x_i} \frac{x_m}{x_m - x_j} \Gamma_i \Gamma_j$$

$$M_{3,k}^{(\mathfrak{sl}_4)} = \sum_{i=1}^{4} x_i^{-k} \prod_{j \neq i} \frac{x_j}{x_j - x_i} \Gamma_i^{-1},$$

where $x_1 x_2 x_3 x_4 = 1$ and $\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 = 1$.

The D_3 operators are

$$M_{1,k}^{(D_3)} = \sum_{\epsilon = \pm 1} \sum_{i=1}^{3} x_i^k \prod_{j \neq i} \frac{x_i^\epsilon x_j}{x_i^\epsilon x_j - 1} \frac{x_i^\epsilon}{x_i^\epsilon - x_j} \Gamma_i^{2\epsilon}$$

$$M_{2,k}^{(D_3)} = \sum_{\substack{\epsilon_1, \epsilon_2, \epsilon_3 = \pm 1 \\ \epsilon_1 \epsilon_2 \epsilon_3 = -1}} \prod_{i=1}^{3} x_i^{k\epsilon_i} \prod_{1 \leq i < j \leq 3} \frac{x_i^{\epsilon_i} x_j^{\epsilon_j}}{x_i^{\epsilon_i} x_j^{\epsilon_j} - 1} \prod_{i=1}^{3} \Gamma_i^{\epsilon_i}$$

$$M_{3,k}^{(D_3)} = \sum_{\substack{\epsilon_1, \epsilon_2, \epsilon_3 = \pm 1 \\ \epsilon_1 \epsilon_2 \epsilon_3 = 1}} \prod_{i=1}^{3} x_i^{k\epsilon_i} \prod_{1 \leq i < j \leq 3} \frac{x_i^{\epsilon_i} x_j^{\epsilon_j}}{x_i^{\epsilon_i} x_j^{\epsilon_j} - 1} \prod_{i=1}^{3} \Gamma_i^{\epsilon_i}.$$

It is straightforward to see that the change of variables (A.7) and (A.8) map the M operators as follows: $M_{1,k}^{(\mathfrak{sl}_4)} \mapsto M_{3,k}^{(D_3)}$, $M_{2,k}^{(\mathfrak{sl}_4)} \mapsto M_{1,k}^{(D_3)}$, and $M_{3,k}^{(\mathfrak{sl}_4)} \mapsto M_{2,k}^{(D_3)}$.

A.4.3 Dual q-Whittaker Functions

In terms of the Schur functions (A.1), the first few A_3 dual q-Whittaker functions are

$$\Pi_{0,0,0,0} = 1$$

$$\Pi_{1,0,0,0} = s_{1,0,0,0}$$

$$\Pi_{2,0,0,0} = s_{2,0,0,0} + q^{-1} s_{1,1,0,0}$$

$$\Pi_{1,1,0,0} = s_{1,1,0,0}$$

$$\Pi_{3,0,0,0} = s_{3,0,0,0} + \frac{1+q}{q^2} s_{2,1,0,0} + q^{-3} s_{1,1,1,0}$$

$$\Pi_{2,1,0,0} = s_{2,1,0,0} + q^{-1} s_{1,1,1,0}$$

$$\Pi_{1,1,1,0} = s_{1,1,1,0}$$

$$\Pi_{4,0,0,0} = s_{4,0,0,0} + \frac{1+q+q^2}{q^3} s_{3,1,0,0} + \frac{1+q^2}{q^4} s_{2,2,0,0}$$
$$+ \frac{1+q+q^2}{q^5} s_{2,1,1,0} + q^{-6} s_{1,1,1,1}$$

$$\Pi_{3,1,0,0} = s_{3,1,0,0} + q^{-1} s_{2,2,0,0} + \frac{1+q}{q^2} s_{2,1,1,0} + q^{-3} s_{1,1,1,1}$$

$$\Pi_{2,2,0,0} = s_{2,2,0,0} + q^{-1} s_{2,1,1,0} + q^{-2} s_{1,1,1,1}$$

$$\Pi_{2,1,1,0} = s_{2,1,1,0} + q^{-1} s_{1,1,1,1}$$

$$\Pi_{1,1,1,1} = s_{1,1,1,1}.$$

In terms of the Schur functions (A.4), the first few D_3 dual q-Whittaker functions are

$$\Pi_{0,0,0} = 1$$

$$\Pi_{1,0,0} = s_{1,0,0}$$

$$\Pi_{2,0,0} = s_{2,0,0} + q^{-2} s_{1,1,0} + q^{-4} s_{0,0,0}$$

$$\Pi_{1,1,0} = s_{1,1,0} + q^{-2} s_{0,0,0}$$

$$\Pi_{3,0,0} = s_{3,0,0} + \frac{1+q^2}{q^4} s_{2,1,0} + q^{-6} (s_{1,1,1} + s_{1,1,-1}) + \frac{1+q^2+q^4}{q^8} s_{1,0,0}$$

$$\Pi_{2,1,0} = s_{2,1,0} + q^{-2} (s_{1,1,1} + s_{1,1,-1}) + \frac{1+q^2}{q^4} s_{1,0,0}$$

$$\Pi_{1,1,\epsilon} = s_{1,1,\epsilon} + q^{-2} s_{1,0,0}$$

$$\Pi_{\frac{1}{2},\frac{1}{2},\frac{\epsilon}{2}} = s_{\frac{1}{2},\frac{1}{2},\frac{\epsilon}{2}}$$

$$\Pi_{\frac{3}{2},\frac{1}{2},\frac{\epsilon}{2}} = s_{\frac{3}{2},\frac{1}{2},\frac{\epsilon}{2}} + q^{-2} s_{\frac{1}{2},\frac{1}{2},-\frac{\epsilon}{2}}$$

$$\Pi_{\frac{5}{2},\frac{1}{2},\frac{\epsilon}{2}} = s_{\frac{5}{2},\frac{1}{2},\frac{\epsilon}{2}} + q^{-2} s_{\frac{3}{2},\frac{3}{2},\frac{\epsilon}{2}} + \frac{1+q^2}{q^4} s_{\frac{3}{2},\frac{1}{2},-\frac{\epsilon}{2}} + \frac{1+q^2}{q^6} s_{\frac{1}{2},\frac{1}{2},\frac{\epsilon}{2}}$$

$$\Pi_{\frac{3}{2},\frac{3}{2},\frac{\epsilon}{2}} = s_{\frac{3}{2},\frac{3}{2},\frac{\epsilon}{2}} + q^{-2} s_{\frac{3}{2},\frac{1}{2},-\frac{\epsilon}{2}} + \frac{1+q^2}{q^4} s_{\frac{1}{2},\frac{1}{2},\frac{\epsilon}{2}}$$

$$\Pi_{\frac{3}{2},\frac{3}{2},\frac{3\epsilon}{2}} = s_{\frac{3}{2},\frac{3}{2},\frac{3\epsilon}{2}} + \frac{1+q^2}{q^4} s_{\frac{3}{2},\frac{1}{2},\frac{\epsilon}{2}} + q^{-6} s_{\frac{1}{2},\frac{1}{2},-\frac{\epsilon}{2}}.$$

The relations (A.9) are easily checked for these polynomials. First we note that the mapping (A.9) extends to the Weyl-invariant Schur functions as well. We then simply have to check that the coefficients in the D_3 case match those in the $\mathfrak{sl}_4$ case up to $q \to q^2$. For instance, we have $\Pi_{2,1,0,0}^{(\mathfrak{sl}_4)} \mapsto \Pi_{\frac{3}{2},\frac{1}{2},\frac{1}{2}}^{(D_3)}$ as the coefficient of $s_{1,1,1,0}^{(\mathfrak{sl}_4)}$ (resp. $s_{\frac{1}{2},\frac{1}{2},-\frac{\epsilon}{2}}^{(D_3)}$) is q^{-1} (resp. q^{-2}). Similarly the coefficients in $\Pi_{3,0,0,0}^{(\mathfrak{sl}_4)}$ and $\Pi_{\frac{3}{2},\frac{3}{2},\frac{3}{2}}^{(D_3)}$ also agree up to $q \to q^2$.

A.5 The D_4 Case

A.5.1 Symmetries

There are two simple symmetries of the Dynkin diagram which induce symmetries of the Macdonald operators and polynomials:

- The $\mathbb{Z}_2$ automorphism of the Dynkin diagram under which $1 \to 1$, $2 \to 2$ and $3 \leftrightarrow 4$,
- The $\mathbb{Z}_3$ automorphism of the Dynkin diagram under which $1 \to 3 \to 4 \to 1$ and $2 \to 2$.

Recall the relations

$$e_1 = \omega_1, \ e_2 = \omega_2 - \omega_1, \ e_3 = \omega_4 + \omega_3 - \omega_2, \ e_4 = \omega_4 - \omega_3.$$

The $\mathbb{Z}_2$ symmetry under:

$$\omega_1 \mapsto \omega_1, \ \omega_2 \mapsto \omega_2, \ \omega_3 \mapsto \omega_4, \ \omega_4 \mapsto \omega_3$$

induces the transformations

$$x_1 \mapsto x_1, \ x_2 \mapsto x_2, \ x_3 \mapsto x_3, \ x_4 \mapsto \frac{1}{x_4}. \tag{A.10}$$

Similarly, we have

$$\Gamma_1 \mapsto \Gamma_1, \ \Gamma_2 \mapsto \Gamma_2, \ \Gamma_3 \mapsto \Gamma_3, \ \Gamma_4 \mapsto \Gamma_4^{-1}.$$

We will check that under these transformations, we have $M_{1,k} \mapsto M_{1,k}$, $M_{2,k} \mapsto M_{2,k}$, $M_{3,k} \mapsto M_{4,k}$ and $M_{4,k} \mapsto M_{3,k}$ for Macdonald operators, and

$$P_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} \mapsto P_{\lambda_1,\lambda_2,\lambda_3,-\lambda_4}$$

for Macdonald polynomials as well as q-Whittaker functions.

The $\mathbb{Z}_3$ symmetry sends

$$\omega_1 \mapsto \omega_3, \ \omega_2 \mapsto \omega_2, \ \omega_3 \mapsto \omega_4, \ \omega_4 \mapsto \omega_1$$

therefore induces the transformations

$$x_1 \mapsto \sqrt{\frac{x_1 x_2 x_3}{x_4}}, \ x_2 \mapsto \sqrt{\frac{x_1 x_2 x_4}{x_3}}, \ x_3 \mapsto \sqrt{\frac{x_1 x_3 x_4}{x_2}}, \ x_4 \mapsto \sqrt{\frac{x_1}{x_2 x_3 x_4}}. \quad \text{(A.11)}$$

Similarly, we have

$$\Gamma_1^2 \mapsto \frac{\Gamma_1 \Gamma_2 \Gamma_3}{\Gamma_4}, \ \Gamma_2^2 \mapsto \frac{\Gamma_1 \Gamma_2 \Gamma_4}{\Gamma_3}, \ \Gamma_3^2 \mapsto \frac{\Gamma_1 \Gamma_3 \Gamma_4}{\Gamma_2}, \ \Gamma_4^2 \mapsto \frac{\Gamma_1}{\Gamma_2 \Gamma_3 \Gamma_4}.$$

We expect that under these transformations, we have $M_{1,k} \mapsto M_{3,k} \mapsto M_{4,k} \mapsto M_{1,k}$ as well as $M_{2,k} \mapsto M_{2,k}$ for Macdonald operators, and

$$P_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} \mapsto P_{\frac{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4}{2}, \frac{\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4}{2}, \frac{\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4}{2}, \frac{-\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4}{2}}$$

for Macdonald polynomials as well as q-Whittaker functions.

The fact that our operators/polynomials obey these symmetry relations is a highly non-trivial check of our construction.

A.5.2 M Operators

$$M_{1,k} = \sum_{\epsilon = \pm 1} \sum_{i=1}^4 \prod_{j \neq i} \frac{x_i^\epsilon x_j}{x_i^\epsilon x_j - 1} \frac{x_i^\epsilon}{x_i^\epsilon - x_j} x_i^{k\epsilon} \Gamma_i^{2\epsilon}$$

$$M_{2,k} = M_{1,k}^2 - q^2 M_{1,k+1} M_{1,k-1}$$

$$M_{3,k} = \sum_{\substack{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 = \pm 1 \\ \epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 = -1}} \prod_{1 \leq i < j \leq 4} \frac{x_i^{\epsilon_i} x_j^{\epsilon_j}}{x_i^{\epsilon_i} x_j^{\epsilon_j} - 1} \prod_{i=1}^4 x_i^{\frac{k\epsilon_i}{2}} \prod_{i=1}^4 \Gamma_i^{\epsilon_i}$$

$$M_{4,k} = \sum_{\substack{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 = \pm 1 \\ \epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 = 1}} \prod_{1 \leq i < j \leq 4} \frac{x_i^{\epsilon_i} x_j^{\epsilon_j}}{x_i^{\epsilon_i} x_j^{\epsilon_j} - 1} \prod_{i=1}^4 x_i^{\frac{k\epsilon_i}{2}} \prod_{i=1}^N \Gamma_i^{\epsilon_i}.$$

The expected symmetry relations under both $\mathbb{Z}_2$ and $\mathbb{Z}_3$ automorphisms are easily checked. For instance, in the $\mathbb{Z}_2$ case, we see immediately that $M_{3,k}$ and $M_{4,k}$ are interchanged under the transformation $(x_4, \Gamma_4) \to (x_4^{-1}, \Gamma_4^{-1})$ while all other x_i, Γ_i remain unchanged, as the transformation amounts to changing $\epsilon_4 \to -\epsilon_4$ in the summation.

In the general D_N case, the $\mathbb{Z}_2$ symmetry of the Dynkin diagram that interchanges the two end-nodes $N - 1$ and N implies the symmetry under $x_N \mapsto 1/x_N$ while all other x's remain unchanged, together with $\Gamma_N \mapsto \Gamma_N^{-1}$. It is clear that under this transformation, we have $M_{N-1,k}^{(D_N)} \mapsto M_{N,k}^{(D_N)}$ and $M_{N,k}^{(D_N)} \mapsto M_{N-1,k}^{(D_N)}$. Indeed, Eqs. (4.9) and (4.10) are interchanged under the change of summation variable $\epsilon_N \to -\epsilon_N$, which amounts exactly to $x_N \to 1/x_N$ and $\Gamma_N \to \Gamma_N^{-1}$.

A.5.3 Dual q-Whittaker Functions

$$\Pi_{0,0,0,0} = s_{0,0,0,0}$$

$$\Pi_{1,0,0,0} = s_{1,0,0,0}$$

$$\Pi_{2,0,0,0} = s_{2,0,0,0} + q^{-2}s_{1,1,0,0} + q^{-4}s_{0,0,0,0}$$

$$\Pi_{1,1,0,0} = s_{1,1,0,0} + q^{-2}s_{0,0,0,0}$$

$$\Pi_{3,0,0,0} = s_{3,0,0,0} + \frac{1+q^2}{q^4}s_{2,1,0,0} + q^{-6}s_{1,1,1,0} + \frac{1+q^2+q^4}{q^8}s_{1,0,0,0}$$

$$\Pi_{2,1,0,0} = s_{2,1,0,0} + q^{-2}s_{1,1,1,0} + \frac{1+q^2}{q^4}s_{1,0,0,0}$$

$$\Pi_{1,1,1,0} = s_{1,1,1,0} + q^{-2}s_{1,0,0,0}$$

$$\Pi_{4,0,0,0} = s_{4,0,0,0} + \frac{1+q^2+q^4}{q^6}s_{3,1,0,0} + \frac{1+q^4}{q^8}s_{2,2,0,0} + \frac{1+q^2+q^4}{q^{10}}s_{2,1,1,0}$$

$$+ q^{-12}(s_{1,1,1,1} + s_{1,1,1,-1}) + \frac{(1+q^4)(1+q^2+q^4)}{q^{12}}s_{2,0,0,0}$$

$$+ \frac{(1+q^4)(1+q^2+q^4)}{q^{14}}s_{1,1,0,0} + \frac{1+q^4+q^8}{q^{16}}s_{0,0,0,0}$$

$$\Pi_{3,1,0,0} = s_{3,1,0,0} + q^{-2}s_{2,2,0,0} + \frac{1+q^2}{q^4}s_{2,1,1,0} + q^{-6}(s_{1,1,1,1} + s_{1,1,1,-1})$$

$$+ \frac{1+q^2+q^4}{q^6}s_{2,0,0,0} + \frac{1+q^2+2q^4}{q^8}s_{1,1,0,0} + \frac{1+q^4}{q^{10}}s_{0,0,0,0}$$

$$\Pi_{2,2,0,0} = s_{2,2,0,0} + q^{-2}s_{2,1,1,0} + q^{-4}(s_{1,1,1,1} + s_{1,1,1,-1})$$

$$+ q^{-4}s_{2,0,0,0} + \frac{1+q^2+q^4}{q^6}s_{1,1,0,0} + \frac{1+q^4}{q^8}s_{0,0,0,0}$$

$$\Pi_{2,1,1,0} = s_{2,1,1,0} + q^{-2}(s_{1,1,1,1} + s_{1,1,1,-1}) + q^{-2}s_{2,0,0,0}$$

$$+ \frac{1+q^2}{q^4}s_{1,1,0,0} + q^{-6}s_{0,0,0,0}$$

$$\Pi_{1,1,1,\epsilon} = s_{1,1,1,\epsilon} + q^{-2}s_{1,1,0,0} + q^{-4}s_{0,0,0,0}$$

$$\Pi_{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{\epsilon}{2}} = s_{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{\epsilon}{2}}$$

$$\Pi_{\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{\epsilon}{2}} = s_{\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{\epsilon}{2}} + q^{-2}s_{\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{\epsilon}{2}}$$

$$\Pi_{\frac{5}{2},\frac{1}{2},\frac{1}{2},\frac{\epsilon}{2}} = s_{\frac{5}{2},\frac{1}{2},\frac{1}{2},\frac{\epsilon}{2}} + q^{-2}s_{\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{\epsilon}{2}} + \frac{1+q^2}{q^4}s_{\frac{3}{2},\frac{1}{2},\frac{1}{2},-\frac{\epsilon}{2}} + \frac{1+q^2}{q^6}s_{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{\epsilon}{2}}$$

$$\Pi_{\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{\epsilon}{2}} = s_{\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{\epsilon}{2}} + q^{-2}s_{\frac{3}{2},\frac{1}{2},\frac{1}{2},-\frac{\epsilon}{2}} + \frac{1+q^2}{q^4}s_{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{\epsilon}{2}}$$

for $\epsilon = \pm 1$.

The expected symmetry relations under the $\mathbb{Z}_3$ Dynkin automorphism are easily checked using the explicit expressions for Weyl-invariant D-type Schur functions (A.4). As an illustration, the reader can check that $\Pi_{2,0,0,0} \mapsto \Pi_{1,1,1,-1}$, as a consequence of $s_{2,0,0,0} \mapsto s_{1,1,1,-1}$, $s_{1,1,0,0} \mapsto s_{1,1,0,0}$ and $s_{0,0,0,0} \mapsto s_{0,0,0,0}$ under the transformation (A.11). The $\mathbb{Z}_2$ symmetry is simply the covariance of Π under $\epsilon \to -\epsilon$.

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