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F-Stable Secondary Representations and Deformation of F-Injectivity

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Dedicated to Professor Nguyen Tu Cuong

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Abstract

We prove that deformation of F-injectivity holds for local rings $(R, \mathfrak{m})$ that admit secondary representations of $H^i_{\mathfrak{m}}(R)$ which are stable under the natural Frobenius action. As a consequence, F-injectivity deforms when $(R, \mathfrak{m})$ is sequentially Cohen–Macaulay (or more generally when all the local cohomology modules $H^i_{\mathfrak{m}}(R)$ have no embedded attached primes). We obtain some additional cases if $R/\mathfrak{m}$ is perfect or if R is $\mathbb{N}$-graded.

Keywords Deformation of F-injectivity · Secondary representations

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1 Introduction

Throughout this article, all rings are commutative, Noetherian, and with multiplicative identity. For rings containing a field of characteristic $p > 0$, the seminal work of Hochster and Huneke on tight closure, and subsequent works of many others, has led to a systematic study of the so-called F-singularities. Roughly speaking, these are singularities that can be defined using the Frobenius endomorphism $F\colon R \to R$, which is the map that raises every element of R to its p-th power. One of the most studied F-singularities is F-injectivity, which is defined in terms of injectivity of the natural Frobenius actions on the local cohomology modules $H^i_{\mathfrak{m}}(R)$. It was first introduced and studied by Fedder in [3].

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We say that a property $\mathcal{P}$ of local rings deforms if, whenever $(R, \mathfrak{m})$ is a local ring and $x \in \mathfrak{m}$ is a nonzerodivisor such that $R/(x)$ satisfies $\mathcal{P}$, then R satisfies $\mathcal{P}$. While this deformation problem for other classical F-singularities has been settled [3, 4, 8, 9], whether F-injectivity deforms or not in general is still an open question. Fedder proved that F-injectivity deforms when R is Cohen–Macaulay [3, Theorem 3.4], and Horiuchi, Miller, Shimomoto proved that F-injectivity deforms either if $R/(x)$ is F-split [5, Theorem 4.13], or if $H^i_{\mathfrak{m}}(R/(x))$ has finite length for all $i \neq \dim(R)$ and $R/\mathfrak{m}$ is perfect [5, Theorem 4.7]. More recently, the second author and Pham [7] extended some of these results by further relaxing the assumptions on $R/(x)$.

In this paper, we consider secondary representations of the local cohomology modules $H^i_{\mathfrak{m}}(R)$ (see Section 2.2 for definitions and basic properties of secondary representations of Artinian modules). It seems natural to ask how the Frobenius action on $H^i_{\mathfrak{m}}(R)$ interacts with a given secondary representation. Our first main result is that F-injectivity deforms when each local cohomology module $H^i_{\mathfrak{m}}(R)$ admits a secondary representation which is stable under the natural Frobenius action (see Definition 3.1 for details).

Theorem A (Theorem 3.4) *Let $(R, \mathfrak{m})$ be a d-dimensional local ring of characteristic $p > 0$ and let $x \in \mathfrak{m}$ be a nonzerodivisor on R. Suppose for each $i \neq d$, $H^i_{\mathfrak{m}}(R)$ has an F-stable secondary representation. If $R/(x)$ is F-injective, then R is F-injective.*

We prove that secondary components that correspond to minimal attached primes of $H^i_{\mathfrak{m}}(R)$ are always F-stable (see Lemma 3.2). As a consequence, F-injectivity deforms when the attached primes of $H^i_{\mathfrak{m}}(R)$ are all minimal (see Corollary 3.5 for a slightly stronger statement). In particular, we obtain the following:

Corollary B (Corollary 3.6) *Let $(R, \mathfrak{m})$ be a d-dimensional sequentially Cohen–Macaulay local ring of characteristic $p > 0$ and let $x \in \mathfrak{m}$ be a nonzerodivisor on R. If $R/(x)$ is F-injective, then R is F-injective.*

We can further relax our assumptions if either the residue field of R is perfect, or if R is $\mathbb{N}$-graded over a field, by only putting conditions on those secondary components of $H^i_{\mathfrak{m}}(R)$ whose attached primes are not equal to $\mathfrak{m}$. We refer to Definition 3.7 for the precise meaning of F$^\circ$-stable secondary representations.

Theorem C (Theorems 3.8 and 3.10) *Let $(R, \mathfrak{m}, k)$ be a d-dimensional local ring of characteristic $p > 0$ that is either local with perfect residue field or $\mathbb{N}$-graded over a field k, and let $x \in \mathfrak{m}$ be a nonzerodivisor on R (homogeneous in the graded case). Suppose for each $i \neq d$, $H^i_{\mathfrak{m}}(R)$ has an F°-stable secondary representation. If $R/(x)$ is F-injective, then R is F-injective.*

2 Preliminaries

2.1 Frobenius Actions on Local Cohomology and F-Injectivity

Let R be a ring of characteristic $p > 0$. A Frobenius action on an R-module W is an additive map $F\colon W \to W$ such that $F(r\eta) = r^p F(\eta)$ for all $r \in R$ and $\eta \in W$.

Let $I = (f_1, \dots, f_n)$ be an ideal of R, then we have the Čech complex

$$C^\bullet(f_1, \dots, f_n; R) := 0 \to R \to \oplus_i R_{f_i} \to \cdots \to R_{f_1 f_2 \cdots f_n} \to 0.$$

Since the Frobenius endomorphism on R induces the Frobenius endomorphism on all localizations of R, it induces a natural Frobenius action on $C^\bullet(f_1, \dots, f_n; R)$, and hence it induces a natural Frobenius action on each $H^i_I(R)$. In particular, there is a natural Frobenius action $F\colon H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R)$ on each local cohomology module of R supported at a maximal ideal $\mathfrak{m}$. A local ring $(R, \mathfrak{m})$ is called F-injective if $F\colon H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R)$ is injective for all i.

2.2 Secondary Representations

We recall some well-known facts on secondary representations that we will use throughout this article. For unexplained facts, or further details, we refer the reader to [1, Section 7.2].

Definition 2.1 Let R be a ring. An R-module W is called secondary if $W \neq 0$ and for each $x \in R$ the multiplication by x map on W is either surjective or nilpotent.

One can easily check that, if W is a secondary R-module, then $\mathfrak{p} = \sqrt{\operatorname{ann}_R(W)}$ is a prime ideal, and $\operatorname{ann}_R(W)$ is $\mathfrak{p}$-primary.

Definition 2.2 Let R be a ring and W be an R-module. A secondary representation of W is an expression of W as a sum of secondary submodules, $W = \sum_{i=1}^t W_i$, where each W_i is called a secondary component of this representation.

A secondary representation of W is called irredundant if the prime ideals $\mathfrak{p}_i = \sqrt{\operatorname{ann}_R(W_i)}$ are all distinct and none of the summands W_i can be removed from the sum. The set $\{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$ is independent of the irredundant secondary representation and is called the set of attached primes of W, denoted by $\operatorname{Att}_R(W)$.

Clearly a secondary module has a unique attached prime. Moreover, over a local ring $(R, \mathfrak{m})$, if a nonzero module W has finite length, then W is secondary with $\operatorname{Att}_R(W) = \{\mathfrak{m}\}$. A key fact is that every Artinian R-module admits an irredundant secondary representation. In particular, all local cohomology modules $H^i_{\mathfrak{m}}(R)$ have an irredundant secondary representation.

Remark 2.3 When $(R, \mathfrak{m})$ is a complete local ring, Matlis duality induces a correspondence between (irredundant) secondary representations of Artinian modules and (irredundant) primary decompositions of Noetherian modules. In particular, if $(R, \mathfrak{m})$ is complete, and S is an n-dimensional regular local ring mapping onto R, then $\operatorname{Att}_R(H^i_{\mathfrak{m}}(R)) = \operatorname{Ass}_R(\operatorname{Ext}^{n-i}_S(R, S))$, as the Matlis dual of $H^i_{\mathfrak{m}}(R)$ is isomorphic to $\operatorname{Ext}^{n-i}_S(R, S)$.

We conclude this section by recalling the definition of surjective element and strictly filter regular element.

Definition 2.4 Let $(R, \mathfrak{m})$ be a local ring of dimension d. An element $x \in \mathfrak{m}$ is called a surjective element if $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in \bigcup_{i=0}^d \operatorname{Att}_R(H^i_{\mathfrak{m}}(R))$, and x is called a strictly filter regular element if $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in (\bigcup_{i=0}^d \operatorname{Att}_R(H^i_{\mathfrak{m}}(R))) \setminus \{\mathfrak{m}\}$.

Remark 2.5 (1) The definition of surjective element we give is not the original one introduced by Horiuchi-Miller-Shimomoto in [5]. However, note that $\mathrm{Ass}_R(R) \subseteq \cup_{i=0}^{\dim(R)} \mathrm{Att}_R(H^i_{\mathfrak{m}}(R))$ by [1, 11.3.9] and thus surjective elements are always nonzerodivisors. Moreover, it follows from the definition that x is a surjective element if and only if $H^i_{\mathfrak{m}}(R) \xrightarrow{\cdot x} H^i_{\mathfrak{m}}(R)$ is surjective for each i. Therefore our definition is equivalent to the original definition of surjective element by [7, Proposition 3.3].
(2) The definition of strictly filter regular element was originally introduced by Cuong-Morales-Nhan in [2]. It is easy to see that x is a strictly filter regular element if and only if $\mathrm{coker}(H^i_{\mathfrak{m}}(R) \xrightarrow{x} H^i_{\mathfrak{m}}(R))$ has finite length for each i.

Surjective elements are important in the study of the deformation problem for F-injectivity. For instance, it was first proved in [5, Theorem 3.7] that if $R/(x)$ is F-injective and x is a surjective element, then R is F-injective (see also [7, Corollary 3.8] or the proof of Theorem 3.4 in the next section). In fact, we do not know any example that $R/(x)$ is F-injective but x is not a surjective element (see Question 4.3).

3 F-Stable Secondary Representation

We introduce the key concept of this article.

Definition 3.1 Let R be a ring of characteristic $p > 0$, and let W be an R-module with a Frobenius action F. We say that W admits an F-stable secondary representation if there exists a secondary representation $W = \sum_{i=1}^{t} W_i$ such that each W_i is F-stable, i.e., $F(W_i) \subseteq W_i$ for all i.

Observe that, even though we are not explicitly asking that the F-stable secondary representation is irredundant, this can always be achieved, whenever such a representation exists. It seems natural to ask when a secondary component of an Artianina module is F-stable, we show this is always the case for secondary components whose attached primes are minimal in the set of all attached primes.

Lemma 3.2 *Let R be a ring of characteristic $p > 0$, and let W be an Artinian R-module with a Frobenius action F. Let $W = \sum_{i=1}^{t} W_i$ be an irredundant secondary representation, with $\mathfrak{p}_i = \sqrt{\mathrm{ann}_R(W_i)}$. If $\mathfrak{p}_i \in \mathrm{MinAtt}_R(W)$, then W_i is F-stable.*

In particular, if $(R, \mathfrak{m})$ is a local ring of characteristic $p > 0$ and dimension d, then $H^d_{\mathfrak{m}}(R)$ has an F-stable secondary representation.

Proof Since $\mathfrak{p}_i \in \mathrm{MinAtt}_R(W)$, we can pick $y \in \cap_{j \neq i} \mathfrak{p}_j$ but $y \notin \mathfrak{p}_i$. Then $yW_i = W_i$ and $y^N W_j = 0$ for all $j \neq i$ and $N \gg 0$. Therefore we have $y^N W = W_i$ for all $N \gg 0$, and thus $F(W_i) = F\left(y^N W_i\right) \subseteq F(y^N W) = y^{pN} F(W) \subseteq y^{pN} W = W_i$.

The last conclusion follows since it is well-known that $\mathrm{Att}_R(H^d_{\mathfrak{m}}(R)) = \{\mathfrak{p} \mid \dim(R/\mathfrak{p}) = d\}$, see [1, Theorem 7.3.2], in particular, $\mathrm{Att}_R(H^d_{\mathfrak{m}}(R)) = \mathrm{MinAtt}_R(H^d_{\mathfrak{m}}(R))$. $\square$

For secondary components whose attached primes are not necessarily minimal, the corresponding secondary components may not be F-stable. However, we do not know whether this can happen when W is a local cohomology module with its natural Frobenius action (see Question 4.1).

Example 3.3 Let $R = \mathbb{F}_p[\![x, y]\!]$ and let $W = \mathbb{F}_p \oplus H^2_{\mathfrak{m}}(R)$. Consider the Frobenius action F on W that sends $(1, 0)$ to $(1, x^{-p}y^{-1})$ and is the natural one on $H^2_{\mathfrak{m}}(R)$. Then F is injective on W, but we claim that $H^2_{\mathfrak{m}}(R)$ is the only proper nontrivial F-stable submodule of W. Indeed, let $0 \neq W'$ be an F-stable submodule of W, it is enough to show that $0 \oplus H^2_{\mathfrak{m}}(R) \subseteq W'$. Choose $a = (b, c) \neq 0$ inside W'. If $c = 0$, then $b \neq 0$. By replacing a with $F(a)$, we can assume that $c \neq 0$. Note that $yF(a) = yF(b, 0) + (0, yF(c)) = (0, yF(c)) \neq 0$ since the action $yF : H^2_{\mathfrak{m}}(R) \to H^2_{\mathfrak{m}}(R)$ is injective. Moreover $H^2_{\mathfrak{m}}(R)$ is simple as an R-module with a Frobenius action, so $0 \oplus H^2_{\mathfrak{m}}(R) \subseteq W'$. Since W is not secondary, this implies that there is no secondary representation of W which is stable with respect to the given Frobenius action (any secondary component with attached prime $\mathfrak{m}$ is not F-stable).

We let $\mathbb{V}(x)$ denote the set of primes of R which contain x. Our first main result is the following.

Theorem 3.4 *Let $(R, \mathfrak{m})$ be a d-dimensional local ring of characteristic $p > 0$ and let $x \in \mathfrak{m}$ be a nonzerodivisor on R. Suppose for each $i \neq d$, $H^i_{\mathfrak{m}}(R)$ admits a secondary representation in which the secondary components whose attached primes belong to $\mathbb{V}(x)$ are F-stable (e.g., $H^i_{\mathfrak{m}}(R)$ has an F-stable secondary representation). If $R/(x)$ is F-injective, then x is a surjective element and R is F-injective.*

Proof We prove by induction on $i \geq -1$ that multiplication by x is surjective on $H^i_{\mathfrak{m}}(R)$ and that $x^{p^e-1}F^e$ is injective on $H^i_{\mathfrak{m}}(R)$ for all $e > 0$. This will conclude the proof, since the first assertion implies x is a surjective element and the second assertion implies F is injective on $H^i_{\mathfrak{m}}(R)$ for all i. The base case $i = -1$ is trivial. Suppose both assertions hold for $i - 1$; we show them for i. Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^{i-1}_{\mathfrak{m}}(R/(x)) \arrow[r] \arrow[d, "F^e"] & H^i_{\mathfrak{m}}(R) \arrow[r, "\cdot x"] \arrow[d, "x^{p^e-1}F^e"] & H^i_{\mathfrak{m}}(R) \arrow[r] \arrow[d, "F^e"] & H^i_{\mathfrak{m}}(R/(x)) \arrow[r] \arrow[d, "F^e"] & \cdots \\
0 \arrow[r] & H^{i-1}_{\mathfrak{m}}(R/(x)) \arrow[r] & H^i_{\mathfrak{m}}(R) \arrow[r, "\cdot x"] & H^i_{\mathfrak{m}}(R) \arrow[r] & H^i_{\mathfrak{m}}(R/(x)) \arrow[r] & \cdots
\end{tikzcd}$$

where injectivity on the left of the rows follows from our inductive hypotheses. Let $u \in \operatorname{soc}(H^i_{\mathfrak{m}}(R)) \cap \ker(x^{p^e-1}F^e)$. Then $xu = 0$, and thus u is the image of an element $v \in H^{i-1}_{\mathfrak{m}}(R/(x))$. Chasing the diagram shows that $F^e(v) = 0$. But since $R/(x)$ is F-injective, F^e is injective on $H^{i-1}_{\mathfrak{m}}(R/(x))$ for all $e > 0$, so $v = 0$ and thus $u = 0$. This shows that $x^{p^e-1}F^e$ is injective on $H^i_{\mathfrak{m}}(R)$ for all $e > 0$.

It remains to show that multiplication by x is surjective on $H^i_{\mathfrak{m}}(R)$. Let $H^i_{\mathfrak{m}}(R) = \sum W_j$ be the secondary representation that satisfies the conditions of the theorem (note that $H^d_{\mathfrak{m}}(R)$ always has an F-stable secondary representation by Lemma 3.2). If there exists $W_j \neq 0$ whose attached prime $\mathfrak{p}_j \in \mathbb{V}(x)$, then it follows from the assumptions that W_j is F-stable. Thus $x^{p^e-1}F^e(W_j) \subseteq x^{p^e-1}W_j = 0$ for all $e \gg 0$ (since $x \in \mathfrak{p}_j = \sqrt{\operatorname{ann}_R(W_j)}$). However, we have proved that $x^{p^e-1}F^e$ is injective on $H^i_{\mathfrak{m}}(R)$ for all $e > 0$, this implies $W_j = 0$ and we arrive at a contradiction. Therefore $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Att}_R\left(H^i_{\mathfrak{m}}(R)\right)$, i.e., multiplication by x is surjective on $H^i_{\mathfrak{m}}(R)$. □

Corollary 3.5 *Let $(R, \mathfrak{m})$ be a d-dimensional local ring of characteristic $p > 0$ and let $x \in \mathfrak{m}$ be a nonzerodivisor on R. Suppose that* $\operatorname{Att}_R(H^i_{\mathfrak{m}}(R)) \cap \mathbb{V}(x) \subseteq \operatorname{MinAtt}_R(H^i_{\mathfrak{m}}(R))$

for all $i \neq d$ (e.g., when each $H^i_{\mathfrak{m}}(R)$ has no embedded attached primes). If $R/(x)$ is F-injective, then x is a surjective element and R is F-injective.

Proof By Lemma 3.2, every irredundant secondary representation of $H^i_{\mathfrak{m}}(R)$ satisfies the assumptions of Theorem 3.4 so the conclusion follows. $\square$

We next exhibit an explicit new class of rings for which deformation of F-injectivity holds. Recall that a finitely generated R-module M is called sequentially Cohen–Macaulay if there exists a finite filtration $0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = M$ such that each M_{i+1}/M_i is Cohen–Macaulay and $\dim(M_i/M_{i-1}) < \dim(M_{i+1}/M_i)$. A local ring $(R, \mathfrak{m})$ is called sequentially Cohen–Macaulay if R is sequentially Cohen–Macaulay as an R-module.

Corollary 3.6 *Let $(R, \mathfrak{m})$ be a d-dimensional sequentially Cohen–Macaulay local ring of characteristic $p > 0$ and let $x \in \mathfrak{m}$ be a nonzerodivisor on R. If $R/(x)$ is F-injective, then x is a surjective element and R is F-injective.*

Proof First we observe that R is sequentially Cohen–Macaulay implies $\widehat{R}$ is sequentially Cohen–Macaulay and whether R is F-injective (and whether x is a surjective element) is unaffected by passing to the completion. Therefore we may assume R is complete and thus R is a homomorphic image of a regular local ring S. By [6, Theorem 1.4], R is sequentially Cohen–Macaulay is equivalent to saying that, for each $0 \leq i \leq d$, $\operatorname{Ext}_S^{\dim(S)-i}(R, S)$ is either zero or Cohen–Macaulay of dimension i. In particular, $\operatorname{Ext}_S^{\dim(S)-i}(R, S)$ has no embedded associated primes and hence by Remark 2.3, $H^i_{\mathfrak{m}}(R)$ has no embedded attached primes for each $0 \leq i \leq d$, that is, $\operatorname{Att}_R(H^i_{\mathfrak{m}}(R)) = \operatorname{MinAtt}_R(H^i_{\mathfrak{m}}(R))$. The conclusion now follows from Corollary 3.5. $\square$

3.1 Results on Local Rings with Perfect Residue Field

If we assume the residue field of $(R, \mathfrak{m})$ is perfect, then we can prove some slight stronger results. The arguments are based on appropriate modifications of the proof of Theorem 3.4, together with some ideas employed in [7, Section 5]. First, we make a modification of the definition of F-stable secondary representation.

Definition 3.7 Let R be a ring of characteristic $p > 0$ and $\mathfrak{m}$ be a maximal ideal of R. Let W be an R-module with a Frobenius action F. We say that W admits an F°-stable secondary representation if there exists a secondary representation $W = \sum_{i=1}^t W_i$ such that W_i is F-stable for all i such that $\operatorname{Att}_R(W_i) \neq \{\mathfrak{m}\}$.

Theorem 3.8 *Let $(R, \mathfrak{m})$ be a d-dimensional local ring of characteristic $p > 0$ with perfect residue field, and let $x \in \mathfrak{m}$ be a nonzerodivisor on R. Suppose for each $i \neq d$, $H^i_{\mathfrak{m}}(R) \neq 0$ admits a secondary representation in which the secondary components whose attached primes belong to $\mathbb{V}(x) \setminus \{\mathfrak{m}\}$ are F-stable (e.g., $H^i_{\mathfrak{m}}(R)$ has an F°-stable secondary representation). If $R/(x)$ is F-injective, then x is a strictly filter regular element and R is F-injective.*

Proof For every i, we let $L_i = \operatorname{coker}(H^i_{\mathfrak{m}}(R) \xrightarrow{x} H^i_{\mathfrak{m}}(R))$. We prove by induction on $i \geq -1$ that L_i has finite length and that the Frobenius action $x^{p^e-1}F^e$ on $H^i_{\mathfrak{m}}(R)$ is injective for all $e > 0$. This will conclude the proof, since the first assertion implies x is a strictly filter regular element and the second assertion implies F is injective on $H^i_{\mathfrak{m}}(R)$ for all i. The initial case $i = -1$ is trivial. Suppose both assertions hold for $i - 1$; we show them for i. Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^{i-1}_{\mathfrak{m}}(R/(x))/L_{i-1} \arrow[r] \arrow[d, "F^e"] & H^i_{\mathfrak{m}}(R) \arrow[r, "\cdot x"] \arrow[d, "x^{p^e-1}F^e"] & H^i_{\mathfrak{m}}(R) \arrow[r] \arrow[d, "F^e"] & H^i_{\mathfrak{m}}(R/(x)) \arrow[r] \arrow[d, "F^e"] & \cdots \\
0 \arrow[r] & H^{i-1}_{\mathfrak{m}}(R/(x))/L_{i-1} \arrow[r] & H^i_{\mathfrak{m}}(R) \arrow[r, "\cdot x"] & H^i_{\mathfrak{m}}(R) \arrow[r] & H^i_{\mathfrak{m}}(R/(x)) \arrow[r] & \cdots
\end{tikzcd}$$

Since L_{i-1} has finite length, F^e is injective on $H^{i-1}_{\mathfrak{m}}(R/(x))$ by assumption, and $R/\mathfrak{m}$ is perfect, we have that F^e induces a bijection on $L_{i-1} \subseteq H^{i-1}_{\mathfrak{m}}(R/(x))$. Thus, F^e induces an injection on $H^{i-1}_{\mathfrak{m}}(R/(x))/L_{i-1}$ for all $e > 0$. Therefore, chasing the diagram above as in the proof of Theorem 3.4 we know that $x^{p^e-1}F^e$ is injective on $H^i_{\mathfrak{m}}(R)$ for all $e > 0$.

It remains to show that L_i has finite length. If there exists $W_j \neq 0$ whose attached prime $\mathfrak{p}_j \in \mathbb{V}(x) \setminus \{\mathfrak{m}\}$, then it follows from the assumptions that W is F-stable (note that $H^d_{\mathfrak{m}}(R)$ always has an F-stable secondary representation by Lemma 3.2). Thus $x^{p^e-1}F^e(W_j) \subseteq x^{p^e-1}W_j = 0$ for all $e \gg 0$ (since $x \in \mathfrak{p}_j = \sqrt{\operatorname{ann}_R(W_j)}$). However, we have proved that $x^{p^e-1}F^e$ is injective on $H^i_{\mathfrak{m}}(R)$ for all $e > 0$, this implies $W_j = 0$ and we arrive at a contradiction. Therefore $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Att}_R(H^i_{\mathfrak{m}}(R)) \setminus \{\mathfrak{m}\}$, i.e., $L_i = \operatorname{coker}(H^i_{\mathfrak{m}}(R) \xrightarrow{x} H^i_{\mathfrak{m}}(R))$ has finite length. $\square$

Corollary 3.9 *Let* $(R, \mathfrak{m})$ *be a d-dimensional local ring of characteristic* $p > 0$ *with perfect residue field, and let* $x \in \mathfrak{m}$ *be a nonzerodivisor on R. Suppose that* $\operatorname{Att}_R(H^i_{\mathfrak{m}}(R)) \cap \mathbb{V}(x) \subseteq \operatorname{MinAtt}_R(H^i_{\mathfrak{m}}(R)) \cup \{\mathfrak{m}\}$ *for all* $i \neq d$. *If* $R/(x)$ *is F-injective, then x is a strictly filter regular element and R is F-injective. In particular, F-injectivity deforms if* $\dim(R/\operatorname{ann}_R(H^i_{\mathfrak{m}}(R))) \leq 1$ *for all* $i \neq d$ *and* $R/\mathfrak{m}$ *is perfect.*

Proof By Lemma 3.2, every irredundant secondary representation of $H^i_{\mathfrak{m}}(R)$ satisfies the assumptions of Theorem 3.8 so the first conclusion follows. To see the second conclusion, it is enough to observe that when $\dim(R/\operatorname{ann}_R(H^i_{\mathfrak{m}}(R))) \leq 1$, we have $\operatorname{Att}_R(H^i_{\mathfrak{m}}(R)) \subseteq \operatorname{MinAtt}_R(H^i_{\mathfrak{m}}(R)) \cup \{\mathfrak{m}\}$. $\square$

3.2 Results on $\mathbb{N}$-Graded Rings

For the rest of this section, we assume that $(R, \mathfrak{m}, k)$ is an $\mathbb{N}$-graded algebra over a field k of characteristic $p > 0$ (k is not necessarily perfect). Given a graded module $W = \bigoplus_j W_j$ and $a \in \mathbb{Z}$, we denote by $W(a)$ the shift of W by a, that is, the graded R-module such that $W(a)_j = W_{a+j}$. In this context, when talking about a Frobenius action F on a graded module W, we insist that $\deg(F(\eta)) = p \cdot \deg(\eta)$ for all homogeneous $\eta \in W$. This is the case for the natural Frobenius action F on the local cohomology modules $H^i_{\mathfrak{m}}(R)$.

The goal of this subsection is to extend Theorem 3.8 in this $\mathbb{N}$-graded setting, by removing the assumption that the residue field k is perfect and by strengthening the conclusion to that x is actually a surjective element.

Theorem 3.10 *Let $(R, \mathfrak{m}, k)$ be a d-dimensional $\mathbb{N}$-graded k-algebra of characteristic $p > 0$ and let $x \in \mathfrak{m}$ be a homogeneous nonzerodivisor on R. Suppose for each $i \neq d$, $H^i_{\mathfrak{m}}(R)$ admits a secondary representation in which the secondary components whose attached primes belong to $\mathbb{V}(x) \smallsetminus \{\mathfrak{m}\}$ are F-stable (e.g., $H^i_{\mathfrak{m}}(R)$ has an F°-stable secondary representation). If $R/(x)$ is F-injective, then x is a surjective element and R is F-injective.*

Proof Let $\deg(x) = t > 0$. We have a graded long exact sequence of local cohomology, induced by the short exact sequence $0 \to R(-t) \xrightarrow{x} R \to R/(x) \to 0$. Moreover, this exact sequence fits in the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^{i-1}_{\mathfrak{m}}(R/(x)) \arrow[r] \arrow[d, "F^e"] & H^i_{\mathfrak{m}}(R)(-t) \arrow[r, "\cdot x"] \arrow[d, "x^{p^e-1}F^e"] & H^i_{\mathfrak{m}}(R) \arrow[r] \arrow[d, "F^e"] & H^i_{\mathfrak{m}}(R/(x)) \arrow[r] \arrow[d, "F^e"] & \cdots \\
\cdots \arrow[r] & H^{i-1}_{\mathfrak{m}}(R/(x)) \arrow[r] & H^i_{\mathfrak{m}}(R)(-t) \arrow[r, "\cdot x"] & H^i_{\mathfrak{m}}(R) \arrow[r] & H^i_{\mathfrak{m}}(R/(x)) \arrow[r] & \cdots
\end{tikzcd}$$

Observe that all the Frobenius actions are compatible with the grading. We show by induction on $i \geq -1$ that the map $H^i_{\mathfrak{m}}(R)(-t) \xrightarrow{x} H^i_{\mathfrak{m}}(R)$ is surjective and that $x^{p^e-1}F^e$ is injective on $H^i_{\mathfrak{m}}(R)(-t)$. This will conclude the proof, since the first assertion implies x is a surjective element and the second assertion implies F is injective on $H^i_{\mathfrak{m}}(R)$ for all i. The base case $i = -1$ is trivial. Suppose both assertions hold for $i - 1$; we show them for i. By the same argument as in the proof of Theorem 3.4, we have that $x^{p^e-1}F^e$ is injective on $H^i_{\mathfrak{m}}(R)(-t)$ for all $e > 0$.

It remains to show that multiplication by x map $H^i_{\mathfrak{m}}(R)(-t) \xrightarrow{x} H^i_{\mathfrak{m}}(R)$ is surjective. Now by the same argument as in the proof of Theorem 3.8, we know that $L_i = \operatorname{coker}(H^i_{\mathfrak{m}}(R)(-t) \xrightarrow{x} H^i_{\mathfrak{m}}(R))$ has finite length (note that we can ignore the graded structure here). Finally, consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & L_i \arrow[r] \arrow[d, "F^e"] & H^i_{\mathfrak{m}}(R/(x)) \arrow[r] \arrow[d, "F^e"] & H^{i+1}_{\mathfrak{m}}(R)(-t) \arrow[r, "x"] \arrow[d, "x^{p^e-1}F^e"] & \cdots \\
0 \arrow[r] & L_i \arrow[r] & H^i_{\mathfrak{m}}(R/(x)) \arrow[r] & H^{i+1}_{\mathfrak{m}}(R)(-t) \arrow[r, "x"] & \cdots
\end{tikzcd}$$

Since F^e is injective on $H^i_{\mathfrak{m}}(R/(x))$ by assumption, it is also injective on L_i. But since the finite length module L_i is graded and the Frobenius action is compatible with the grading (as the action is induced from $H^i_{\mathfrak{m}}(R/(x))$), this forces L_i to be concentrated in degree zero. If $L_i \neq 0$, then $[L_i]_0 \cong [H^i_{\mathfrak{m}}(R)/xH^i_{\mathfrak{m}}(R)(-t)]_0 \neq 0$, in particular $[H^i_{\mathfrak{m}}(R)]_0 \neq 0$. However, this implies the existence of a nonzero element $u \in [H^i_{\mathfrak{m}}(R)(-t)]_t$. Since we have proved that $x^{p^e-1}F^e$ is injective on $H^i_{\mathfrak{m}}(R)(-t)$, this gives a nonzero element $x^{p^e-1}F^e(u)$ in degree $p^e t > 0$ for all $e > 0$, which is a contradiction because $[H^i_{\mathfrak{m}}(R)(-t)]_{\gg 0} = 0$ (here we are using that the Frobenius action $x^{p^e-1}F^e$ is compatible with the grading on $H^i_{\mathfrak{m}}(R)(-t)$, that is, $\deg(x^{p^e-1}F^e(\eta)) = p^e \deg(\eta)$ for all $\eta \in H^i_{\mathfrak{m}}(R)(-t)$). Therefore $L_i = 0$, i.e., the multiplication by x map $H^i_{\mathfrak{m}}(R)(-t) \xrightarrow{x} H^i_{\mathfrak{m}}(R)$ is surjective. □

Corollary 3.11 *Let $(R, \mathfrak{m}, k)$ be a d-dimensional $\mathbb{N}$-graded k-algebra of characteristic $p > 0$ and let $x \in \mathfrak{m}$ be a homogeneous nonzerodivisor on R. Suppose that $\mathrm{Att}_R(H^i_{\mathfrak{m}}(R)) \cap \mathbb{V}(x) \subseteq \mathrm{MinAtt}_R(H^i_{\mathfrak{m}}(R)) \cup \{\mathfrak{m}\}$ for all $i \neq d$ (e.g., x is a strictly filter regular element). If $R/(x)$ is F-injective, then x is a surjective element and R is F-injective.*

Proof By Lemma 3.2, every irredundant secondary representation of $H^i_{\mathfrak{m}}(R)$ satisfies the assumptions of Theorem 3.10 so the conclusion follows. $\square$

4 Ending Questions and Remarks

We end by collecting some questions that arise from the results in this article. Motivated by Definition 3.1 and Theorem 3.4, it is natural to ask the following.

Question 4.1 Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$. If $H^i_{\mathfrak{m}}(R) \neq 0$, does it admit an F-stable secondary representation?

By Theorem 3.4, a positive answer to Question 4.1 implies that F-injectivity deforms.

Question 4.2 Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$. If $H^i_{\mathfrak{m}}(R) \neq 0$, does it admit a secondary representation such that the secondary component with attached prime $\mathfrak{m}$, if not zero, is F-stable?

This is weaker than Question 4.1, but an affirmative answer also implies that F-injectivity deforms. Suppose $R/(x)$ is F-injective, we will show x is a surjective element and thus R is F-injective by [5, Theorem 3.7] (or use the same argument as in Theorem 3.4). In fact, if $x \in \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Att}_R(H^i_{\mathfrak{m}}(R))$, then $x \in \mathfrak{p}R_{\mathfrak{p}} \in \mathrm{Att}_{R_{\mathfrak{p}}}(H^j_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}))$ for some j and $R_{\mathfrak{p}}/xR_{\mathfrak{p}}$ is still F-injective. Now an affirmative answer to Question 4.2 applied to $(R_{\mathfrak{p}}, \mathfrak{p}R_{\mathfrak{p}})$ implies that there exists a nonzero secondary component of $H^j_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}})$ with attached prime $\mathfrak{p}R_{\mathfrak{p}}$ that is F-stable, and we can argue as in the proof of Theorem 3.4 to arrive at a contradiction.

Question 4.3 Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$, and let $x \in \mathfrak{m}$ be a nonzerodivisor on R. If $R/(x)$ is F-injective, is it true that $\mathfrak{m} \notin \mathrm{Att}(H^i_{\mathfrak{m}}(R))$ for all i?

Similar to the discussion above, we point out that an affirmative answer to Question 4.3 also implies that x is a surjective element (and hence implies that F-injectivity deforms): if not, then $x \in \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Att}_R(H^i_{\mathfrak{m}}(R))$, but then $R_{\mathfrak{p}}/xR_{\mathfrak{p}}$ is still F-injective and $\mathfrak{p}R_{\mathfrak{p}} \in \mathrm{Att}_{R_{\mathfrak{p}}}(H^j_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}))$ for some j, which contradicts Question 4.3 for $\big(R_{\mathfrak{p}}, \mathfrak{p}R_{\mathfrak{p}}\big)$.

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