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Catalan recursion on externally ordered bases of unit interval positroids

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The Catalan numbers form a sequence that counts over 200 combinatorial objects. A remarkable property of the Catalan numbers, which extends to these objects, is its recursive definition; that is, we can determine the n-th object from previous ones. A matroid is a combinatorial object that generalizes the notion of linear independence with connections to many fields of mathematics. A family of matroids, called unit interval positroids (UIP), are Catalan objects induced by the antiadjacency matrices of unit interval orders. Associated to each UIP is the set of externally ordered bases, which due to Las Vergnas, produces a lattice after adjoining a bottom element. We study the poset of externally ordered UIP bases and the implied Catalan-induced recursion. Explicitly, we describe an algorithm for constructing the lattice of a rank-n UIP from the lattice of lower ranks. Using their inherent combinatorial structure, we define a simple formula to enumerate the bases for a given UIP.

1. Introduction

The Catalan numbers form an infinite sequence that is ubiquitous in mathematics. Its rich history and connection to the enumeration of over 200 combinatorial objects is captured in [Stanley 2015]. Some of these combinatorial objects are triangulations of convex polygons with $n+2$ vertices (Figure 1), binary trees with $n+2$ vertices, plane trees with $n+1$ vertices, ballot sequences of length $2n$, parenthetizations, and Dyck paths of length $2n$.

The modern description of the Catalan numbers stems from Eugène Catalan's interest in the triangulation of polygons problem [Stanley 2015]. Catalan numbers are defined as

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

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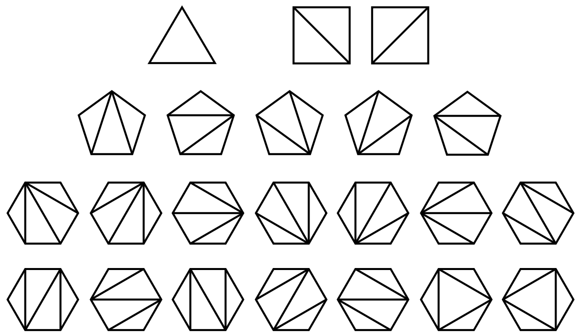

Figure 1. Triangulations of convex polygons with 3, 4, 5, and 6 vertices.

While studying these triangulations, Euler and Goldbach derived [Stanley 2015] the recursive definition

$$C_{n+1} = \sum_{k=0}^{n} C_n C_{n-k}, \quad C_0 = 1.$$

Remarkably, objects enumerated by the Catalan numbers exhibit a similar recursive property. That is, given a Catalan object, there is a rule that describes how to produce an object from the objects that came before it.

We introduce a new Catalan object called the *externally ordered poset of a unit interval positroid* and explore its potential recursion. The elements of this poset are bases of a special matroid called a unit interval positroid (UIP), a Catalan object introduced in [Chavez and Gotti 2018]. The elements are partially ordered by the external order of matroid bases, $\leq_{\text{Ext}}$, as defined in [Las Vergnas 2001]. We denote such a poset as $\text{Ex}(\mathcal{P}_n)$, where $\mathcal{P}_n$ is a rank-n positroid.

Our main results are:

- We introduce an algorithm that generates a partially ordered set of elements from $\text{Ex}(\mathcal{P}_n)$, where $\mathcal{P}_n$ is a certain rank-n UIP we call the trivial UIP (see Section 3).
- We prove that the algorithm produces exactly the bases of the rank-$(n+1)$ trivial UIP (see Theorem 4.4).
- We prove the algorithm induces the desired external ordering on the rank-$(n+1)$ trivial UIP bases (see Theorem 4.5).

That is, we show $\text{Ex}(\mathcal{P}_{n+1})$ can be recursively described in terms of $\text{Ex}(\mathcal{P}_n)$ for the rank-$(n+1)$ and rank-n trivial UIPs. This is the first step towards describing the complete recursion for externally ordered posets of every UIP.

This paper is organized as follows. In Section 2 we provide the background material necessary for this paper. In Section 3 we introduce our algorithm. In Section 4 we present our main theorems. In Section 5 we discuss future work.

2. Background

Matroids. Matroids capture the essence of dependence as we know it in linear algebra and graph theory. Though several definitions of a matroid exist, we utilize the basis definition in our work. We refer to [Oxley 1992] for a deeper study of matroids.

Definition 2.1. A *matroid* $\mathcal{M}$ is a pair $(E, \mathcal{B})$ consisting of a finite set E and a nonempty collection of subsets $\mathcal{B} = \mathcal{B}(\mathcal{M})$ of E that satisfy the following properties:

(B1) $\mathcal{B} \neq \varnothing$.

(B2) (basis exchange axiom) If B_1, $B_2 \in \mathcal{B}$ and $b_1 \in B_1 - B_2$, then there exists an element $b_2 \in B_2 - B_1$ such that $B_1 - \{b_1\} \cup \{b_2\} \in \mathcal{B}$.

An element $B \in \mathcal{B}$ is called a *basis* and $|B|$ is the rank of $\mathcal{M}$. A set $I \subset E$ such that $I \subset B$, for a basis B, is called *independent*. A minimally dependent set C, that is, $C - \{e\}$ is independent for any $e \in C$, is called a *circuit*. The finite set E is called the *ground set*.

For a subset $B = \{b_1 < b_2 < \cdots < b_k\}$ of the ordered set E, we call b_1 the *smallest element of* B and denote it as $\min B$. Similarly, we call b_k the *largest element of* B and denote it as $\max B$.

Example 2.2. Let $E = \{1, 2, 3, 4\}$ be the set of labels of the columns of the matrix

$$A = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

Then the set of bases $\mathcal{B}$ of the matroid over E is the collection of all maximally independent sets of E. That is, $\mathcal{B} = \{12, 13, 14, 23, 24\}$. Considering all the sets of minimally dependent sets of the columns of A, we get that the set of circuits of $\mathcal{M}$ is $\mathcal{C} = \{123, 124, 34\}$. The reader can check that $\mathcal{B}$ satisfies the bases axioms above.

A matroid $\mathcal{M}$ is called *realizable* and denoted by $\mathcal{M}(A)$ if its bases are in bijection with the set of maximally independent columns of some matrix A over some field $\mathbb{F}$. The matroid described in Example 2.2 is realizable.

A matrix A is called *totally nonnegative* if all of its maximal minors are nonnegative. Postnikov [2006] defined a family of realizable matroids whose associated matrices are totally nonnegative.

Definition 2.3. A *positroid* of rank r over $[n]$ is a realizable matroid such that the associated full rank $r \times n$ $\mathbb{R}$-matrix A is totally nonnegative.

The matroid $\mathcal{M}(A)$ given in Example 2.2 is a positroid of rank 2. In fact, $\mathcal{M}(A)$ is a unit interval positroid, which we discuss in the next section.

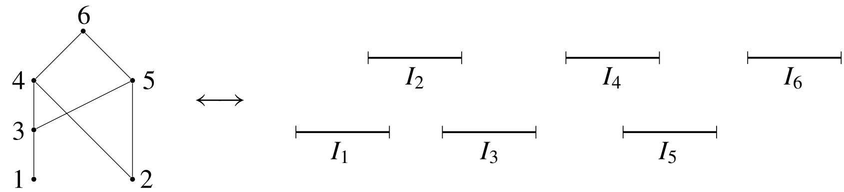

Figure 2. Hasse diagram of a unit interval order with six elements and its unit interval representation.

Unit interval positroids. Recall that a *partially ordered* set $(P, \leq)$, or *poset*, is a set P with a relation $\leq$ that satisfies reflexivity, antisymmetry, and transitivity [Stanley 2012]. We can represent the poset $(P, \leq)$ as a Hasse diagram that shows the elements of P with the cover relations. See Figure 2.

Definition 2.4. A poset P is a *unit interval order* if there exists a bijective map $i \mapsto [q_i, q_i + 1]$ from P to a set $S = \{[q_i, q_i + 1] \mid 1 \leq i \leq n,\ q_i \in \mathbb{R}\}$ of closed unit intervals of the real line such that, for $i, j \in P$, we have $i <_P j$ if and only if $q_i + 1 < q_j$. We then say that S is an *interval representation* of P.

Associated to every unit interval order is an antiadjacency matrix, the key to describing unit interval positroids.

Definition 2.5. For an n-labeled poset P, the *antiadjacency matrix* of P is the $n \times n$ binary matrix $A = (a_{i,j})$ with $a_{i,j} = 0$ if and only if $i <_P j$.

Skandera and Reed [2003] proved that by labeling the unit interval order appropriately, every minor of the corresponding antiadjacency matrix A is nonnegative. That is, the determinant of every submatrix of A is 0 or a positive number. This fact, combined with the following result, leads us to our positroid of interest.

Lemma 2.6 [Postnikov 2006, Lemma 3.9]. *For an $n \times n$ real matrix $A = (a_{i,j})$, consider the $n \times 2n$ matrix $B = \psi(A)$ where*

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n} \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix} \overset{\psi}{\longmapsto} \begin{pmatrix} 1 & \cdots & 0 & 0 & (-1)^{n-1}a_{n,1} & \cdots & (-1)^{n-1}a_{n,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & -a_{2,1} & \cdots & -a_{2,n} \\ 0 & \cdots & 0 & 1 & a_{1,1} & \cdots & a_{1,n} \end{pmatrix}.$$

For each pair (I, J) with $I, J \subseteq [n]$ and $|I| = |J|$, define the set

$$K = K(I, J) = \{n + 1 - k \mid k \in [n] \backslash I\} \cup \{n + j \mid j \in J\}.$$

Then we have $\Delta_{I,J}(A) = \Delta_{[n],K}(B)$.

Lemma 2.6 allows the information from the $n \times n$ antiadjacency matrix to be encoded as the corresponding $n \times 2n$ matrix. Explicitly, Lemma 2.6 associates the determinants of the submatrices of the antiadjacency matrix to the maximal minors of the $n \times 2n$ matrix. The antiadjacency matrix of a properly labeled unit interval order generates a totally nonnegative matrix, that is, a positroid. A positroid on $[2n]$ induced by a unit interval order is a *unit interval positroid* or UIP.

Let $\mathcal{P}$ be a unit interval positroid and C a circuit of $\mathcal{P}$. Assume $e \in C$ is the smallest element of C. Then $C - \{e\}$ is called a *broken circuit*. Bases that do not contain broken circuits are called *atoms*. However, if B is a broken circuit, then there is an element e such that $B \cup \{e\}$ is a circuit and $e = \min\{B \cup \{e\}\}$ (i.e., the smallest element of $B \cup \{e\}$).This notion of broken circuits is captured by the ordering discussed in the next section.

External activities. Las Vergnas [2001] introduced the notion of active orders, a collection of related orders on bases of a matroid using broken circuits. We use only the external order to define our poset of interest, though one can derive the results for other orders from this one.

Definition 2.7. Let M be a matroid on a linearly ordered set E, and let $A \subseteq E$. We say an element $e \in E$ is *M-active* with respect to A if there is a circuit C of M such that $e \in C \subseteq A \cup \{e\}$ and e is the smallest element of C. We denote the set of *M-active* elements with respect to A by $\mathrm{Act}_M(A)$.

To determine the active elements for our positroids we look at those subsets $A \subseteq E$ where A is a basis. Then those elements e are exactly the elements which make or break a circuit.

Definition 2.8. The *external set* of an element A is obtained by setting $\mathrm{Ext}_M(A) = \mathrm{Act}_M(A) \backslash A$.

Las Vergnas defines the *external ordering* of bases A and B, denoted by $A \leq_{\mathrm{Ext}} B$, of a matroid M over an ordered ground set by considering how bases are formed from broken circuits (see [Las Vergnas 2001] for details). We instead use the equivalent statement (2) in the following proposition as the definition of $A \leq_{\mathrm{Ext}} B$. That is, $A \leq_{\mathrm{Ext}} B$ if and only if $A \subseteq B \cup \mathrm{Ext}_M(B)$.

Proposition 2.9 [Las Vergnas 2001, Proposition 3.1]. *Let A, B be two bases of an ordered matroid M. The following properties are equivalent*:

(1) $A \leq_{\mathrm{Ext}} B$.

(2) $A \subseteq B \cup \mathrm{Ext}_M(B)$.

(3) $A \cup \mathrm{Ext}_M(A) \subseteq B \cup \mathrm{Ext}_M(B)$.

(4) *B is the greatest*, *for the lexicographic ordering*, *of all bases of M contained in $A \cup B$.*

Let $\mathcal{P}$ be the unit interval positroid generated by the poset where all elements are incomparable, which we call *trivial*. We utilize the second equivalence of

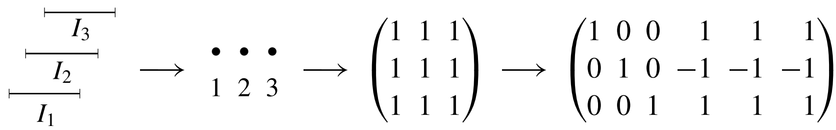

Figure 3. Generating the matrix representation of a trivial UIP of rank 3 from a unit interval order of three elements.

Proposition 2.9 on the trivial UIP in order to determine the externally ordered poset of its bases. For an example of generating the matrix representing the trivial UIP of rank 3, see Figure 3.

Example 2.10. Consider $\mathcal{P}_3$, the trivial UIP of rank 3. We will determine the external ordering on its bases. Using Proposition 2.9, we first determine the active elements for every basis $B \in \mathcal{B}$, that is, $\mathrm{Act}_{\mathcal{P}_3}(\mathcal{B})$. The circuits and bases of $\mathcal{P}_3$ are given by

$$\mathcal{C}_{\mathcal{P}_3} = \{(1,2,3,4), (1,2,3,5), (1,2,3,6), (4,5), (4,6), (5,6)\},$$
$$\mathcal{B}_{\mathcal{P}_3} = \{(1,2,3), (1,2,4), (1,2,5), (1,2,6), (1,3,4), (1,3,5), (1,3,6), (2,3,4), (2,3,5), (2,3,6)\}.$$

Table 1 shows necessary calculations.

e	$c \in \mathcal{C}_{\mathcal{P}}$	$A \cup \{e\}$
1	$(1,2,3,4)$	$(2,3,4) \cup \{1\}$
	$(1,2,3,5)$	$(2,3,5) \cup \{1\}$
	$(1,2,3,6)$	$(2,3,6) \cup \{1\}$
2	none	
3	none	
4	$(4,5)$	$(1,2,5) \cup \{4\}$
		$(1,3,5) \cup \{4\}$
		$(2,3,5) \cup \{4\}$
	$(4,6)$	$(1,2,6) \cup \{4\}$
		$(1,3,6) \cup \{4\}$
		$(2,3,6) \cup \{4\}$
5	$(5,6)$	$(1,2,6) \cup \{5\}$
		$(1,3,6) \cup \{5\}$
		$(2,3,6) \cup \{5\}$
6	none	

A	$\mathrm{Act}_M(A)$	$\mathrm{Ext}_M(A)$
$(1,2,3)$	none	$\varnothing$
$(1,2,4)$	none	$\varnothing$
$(1,2,5)$	4	4
$(1,2,6)$	4,5	4,5
$(1,3,4)$	none	$\varnothing$
$(1,3,5)$	4	4
$(1,3,6)$	4,5	4,5
$(2,3,4)$	1	1
$(2,3,5)$	1,4	1,4
$(2,3,6)$	1,4,5	1,4,5

Table 1. Left: calculating M-active elements. Right: calculating the external set.

$\mathrm{Ex}(\mathcal{P}_2)$	
A	$\mathrm{Ext}_{\mathcal{P}_2}(A)$
(1, 2)	$\varnothing$
(1, 3)	$\varnothing$
(1, 4)	3
(2, 3)	1
(2, 4)	1,3

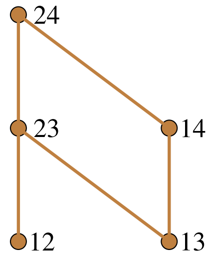

Figure 4. $\mathrm{Ex}(\mathcal{P}_2)$ and the Hasse diagram.

To form the externally ordered poset, we use Proposition 2.9(2) in order to obtain the relations described by Proposition 2.9(1). For example, for bases $A = (1, 2, 3)$ and $B = (2, 3, 4)$, we check if the containment $A \subseteq B \cup \mathrm{Ext}_M(B)$ is satisfied. We can observe that $(1, 2, 3) \subseteq (2, 3, 4) \cup \{1\}$, and thus $A \leq_{\mathrm{Ext}} B$. Continue this calculation for all pairs of bases until all bases are ordered. The resulting object, which is a set of bases ordered by the relation $\leq_{\mathrm{Ext}}$, is the externally ordered poset of the bases of $\mathcal{P}_3$.

Recall the poset of the trivial UIP bases with the external order is denoted as $\mathrm{Ex}(\mathcal{P}_n)$. Continuing with Example 2.10, we now show how to compute $\mathrm{Ex}(\mathcal{P}_2)$. We compare the bases of $\mathcal{P}_2$ pairwise using Proposition 2.9(2). That is, for every $A, B \in \mathcal{B}_{\mathcal{P}_2}$, check if $A \subseteq B \cup \mathrm{Ext}_M(B)$ is satisfied. Given the information from the table of $\mathrm{Ex}(\mathcal{P}_2)$, we can draw the Hasse diagram for $\mathrm{Ex}(\mathcal{P}_2)$ as seen in Figure 4.

3. Algorithm

To prove $\mathrm{Ex}(\mathcal{P}_n)$ can be described recursively we first describe an algorithm that generates a set of elements $\mathcal{S}$ from the set of bases of $\mathcal{P}_n$ and defines an order induced by $\mathrm{Ex}(\mathcal{P}_n)$. Let $\mathcal{B}_{\mathcal{P}_n}$ be the bases of the trivial UIP $\mathcal{P}_n$. The algorithmic steps are named for future reference:

(1) Reinforce: for $B \in \mathcal{B}_{\mathcal{P}_n}$, the set $s \in \mathcal{S}$ is achieved by adding 1 to all elements of B and adjoining $\min B$. The original order is kept.

(2) Build up: for $s \in \mathcal{S}$, if $2n+1 \in s$, then $s' = s \backslash \{2n+1\} \cup \{2(n+1)\}$ is in $\mathcal{S}$ and s covers s'.

(3) Grow spine: for $s \in \mathcal{S}$ such that $2 = \min s$, we have $s' = s \backslash \{2\} \cup \{1\}$ is in $\mathcal{S}$ and s covers s'.

In the next section we prove the set of elements $\mathcal{S}$ is precisely the set of bases of $\mathcal{P}_{n+1}$ and the induced order preserves the external order of the bases of $\mathcal{P}_{n+1}$.

Figure 5 illustrates each step of the algorithm applied to the external poset of the trivial UIP of rank 2.

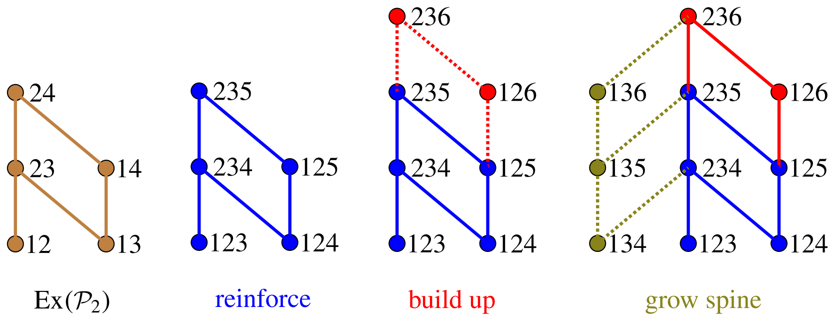

Figure 5. The algorithm applied to the trivial UIP $\mathcal{P}_2$ which produces the final ordered set.

4. Main theorem

Denote the algorithm above by γ; that is, $\gamma(\mathcal{B}_{\mathcal{P}_n})$ is the ordered set of elements produced by γ. To prove our main result, we show $\gamma(\mathcal{B}_{\mathcal{P}_n})$ is precisely the externally ordered poset of the bases of the rank-$(n+1)$ trivial UIP. That is, we show γ produces the set of bases expected and the bases are externally ordered. We first give a simple description of the bases and circuits of the trivial UIP.

Lemma 4.1. *Let $\mathcal{P}$ be the trivial unit interval positroid of rank n and $B \in \mathcal{B}$ a basis of $\mathcal{P}$. Then B satisfies one of the following*:

(1) $B = (1, \dots, n)$.

(2) $B = (1, \dots, \hat{j}, \dots, n) \cup \{j\}$ *for* $j \in [n+1, \dots, 2n]$, *where* $\hat{j}$ *indicates* j *is not included in the basis* B.

Moreover, $|\mathcal{B}| = n^2 + 1$.

Proof. By Lemma 2.6, we know that P is represented by the $n \times 2n$ matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & (-1)^{n-1} & \cdots & (-1)^{n-1} \\ 0 & 1 & \cdots & 0 & \vdots & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 & (-1) & \cdots & (-1) \\ 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \end{pmatrix},$$

where the first n columns form the $n \times n$ identity matrix and the last $[n+1, \dots, 2n]$ columns are equal.

Since the $[n+1, \dots, 2n]$ columns are all the same, any nonsingular maximal submatrix can include at most one of these columns.

The first basis description corresponds to the $n \times n$ identity matrix, which contains no column from $[n+1, \dots, 2n]$.

The second basis description corresponds to replacing one column of the $n \times n$ identity matrix with one column from $[n+1, \dots, 2n]$.

We use our description to enumerate the bases. To form a basis of the second kind, we remove an element from $\{1, 2, \dots, n\}$ and replace it with any of the n elements from $[n+1, \dots, 2n]$. This is done in $\binom{n}{1}n$ ways. Including the basis $\{1, 2, \dots, n\}$, we get

$$\binom{n}{1}n + 1 = n^2 + 1$$

bases, as desired. □

Lemma 4.2. *Let $\mathcal{P}_n$ be the trivial UIP and $\mathcal{C}_{\mathcal{P}_n}$ its set of circuits. Then any $C \in \mathcal{C}_{\mathcal{P}_n}$ must either be $I \cup i$, where $i \in [n+1, 2n]$ or i, j for $i, j \in [n+1, 2n]$ and $i \neq j$.*

Proof. Notice that a circuit corresponds precisely to a minimally singular submatrix of

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & (-1)^{n-1} & \cdots & (-1)^{n-1} \\ 0 & 1 & \cdots & 0 & \vdots & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 & (-1) & \cdots & (-1) \\ 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Thus, any circuit corresponds to the $n \times n$ identity matrix adjoin a column from $[n+1, \dots, 2n]$ or a pair of columns from $[n+1, \dots, 2n]$. Note these are minimally singular since the removal of one column produces either a basis as described in Lemma 4.1 or a single nonzero column which is independent. □

We state the following observation about bases of trivial UIPs, though it is not used in any subsequent theorem.

Lemma 4.3. *Let $B \in \mathcal{B}(\mathcal{P}_n)$, where $\mathcal{P}_n$ is a trivial unit interval positroid. Then $\epsilon(B) = 0$ if and only if B is a minimally ordered basis.*

Proof. Assume $\epsilon(B) = 0$ (i.e., $\mathrm{Ext}(B) = \varnothing$) and there exists $A \in \mathcal{B}$ such that $A \lessdot B$. Then $A \subset B \cup \mathrm{Ext}(B) = B$, which implies $A = B$.

Assume B is minimal and $\mathrm{Ext}(B) \neq \varnothing$. First, note B minimal implies $1 \in B$. If not, let $B' = B \backslash 2 \cup 1$. Then $\mathrm{Ext}(B') = \mathrm{Ext}(B) \backslash \{1\}$, so that $B' \subset B \cup \mathrm{Ext}(B)$, implying $B' \lessdot B$, a contradiction. Since $\mathrm{Ext}(B) \neq \varnothing$, there exists $e = \min C$ for some circuit C of P_n such that $C \subset B \cup \{e\}$. By Lemma 4.2, $C = \{e, j\}$ for some $j \in [n+1, 2n]$ and $j \neq e$. Then $\mathrm{Ext}(B) = \{n+1, \dots, e\}$, and $j \in B$ necessarily. Let $A = B \backslash \{j\} \cup \{e\}$. Then $A \lessdot B$, a contradiction. □

Next we prove γ produces the set of desired bases.

Theorem 4.4. *Let $\mathcal{B}$ be the set of bases of the trivial unit interval positroid $\mathcal{P}_n$ of rank n over ground set $[2n]$. Then, after applying algorithm γ to $\mathcal{B}$, we have $\gamma(\mathcal{B})$ is the set of bases for the trivial unit interval positroid $\mathcal{P}_{n+1}$ of rank $n+1$ over ground set $[2(n+1)]$.*

Proof. Let $\mathcal{B}=\mathcal{B}(\mathcal{P}_{2n})$. We will show that $\gamma(\mathcal{B})$ produces a $B^{'}\in\mathcal{B}(\mathcal{P}_{2(n+1)})$ for every $B\in\mathcal{B}$. Let $B=\{1,\ldots,n\}$. Then $\gamma(B)=\{1,\ldots,n+1\}$, which by Lemma 4.1(1) is a basis of $\mathcal{P}_{2(n+1)}$. Let $B=\{1,2,\ldots,\hat{j},\ldots,n\}$, where $\hat{j}\in[n+1,\ldots,2n]$. Applying step 1 of γ we have that $\gamma(B)=B+1\cup\{\min B\}\in\gamma(\mathcal{B})$. Explicitly, this means

$$\{1,2,3,\ldots,\hat{j}+1,\ldots,n+1\}\in\gamma(\mathcal{B}),$$

where

$$\hat{j}+1\in[n+2,2n+1].$$

Since $[n+2,\ldots,2n+1]\subset[(n+1)+1,\ldots,2(n+1)]$, by Lemma 4.1 we have that $\gamma(B)$ is a basis of $\mathcal{P}_{2(n+1)}$. Moreover, there are n^2 bases created at this step.

For the second step of γ, let $\mathcal{B}_{\max}$ be the set of bases such that

$$\max(\gamma(B))=2n+1.$$

Then for all $B\in\mathcal{B}_{\max}$, we have $B\backslash\{2n+1\}\cup\{2(n+1)\}$. All sets generated prior to this step have maximum value $2n+1$. Thus, all sets generated at this step of γ are new elements of $\gamma(\mathcal{B})$. Every new set generated is formed by choosing a value of $[n]$ and replacing it with $2(n+1)$. Thus each set is a basis and we have n new bases.

For the final step of γ, let $\mathcal{B}_{\min}$ be the set of bases such that $\min(\gamma(B))=2$. Then, for every $B\in\mathcal{B}_{\min}$, we have $B\backslash\{2\}\cup\{1\}\in\gamma(\mathcal{B})$. Note Lemma 4.1 implies $3\in B$ for every $B\in\mathcal{B}_{\min}$. Then for every set generated at this step, it must contain 1, 3 and not 2. By construction, all sets generated by γ prior to the third step of γ have that their first two entries are consecutive. Thus, all sets generated in this third step are new and, by Lemma 4.1, a basis. Thus all sets are bases of $\mathcal{P}_{n+1}$.

To enumerate the bases generated in this final step of γ, let us analyze the number of bases it applies to. By the first step of the algorithm, every basis $B\in\mathcal{B}$ such that $\min B=2$ produces a basis in $\mathcal{B}(\mathcal{P}_{2(n+1)})$ with minimum element 2. Thus, the number of bases produced equals the number of $B\in\mathcal{B}$ such that $\min B=2$, which is n. By the second step of the algorithm, the only basis produced with minimum element 2 is the one that also has maximum element $2n$. Thus in total, there are $n+1$ bases with minimum element 2 produced by γ. Accounting for the all the bases produced at every step of γ, we have that

$$|\gamma(\mathcal{B}(\mathcal{P}_{2(n+1)}))|=n^2+n+n+1+1=(n+1)^2+1,$$

as desired. □

Using the above results, we may now prove our main theorem.

Theorem 4.5. *Let* $\mathcal{B}(\mathcal{P}_n)$ *be the bases of the rank-n trivial UIP on* $[2n]$. *Assume* γ *is defined to be the algorithm described in Section 3. Then* $\gamma(\mathcal{B}(\mathcal{P}_n))$ *is equal to* $\mathrm{Ex}(\mathcal{P}_{n+1})$, *the externally ordered poset of bases of the rank* $n+1$ *trivial UIP on* $[2(n+1)]$.

Proof. By Theorem 4.4, we know $\gamma(\mathcal{B}(\mathcal{P}_n))$ produces the set of bases of $\mathcal{P}_{n+1}$. It is left to show that γ also induces the external order on $\mathcal{B}(\mathcal{P}_{n+1})$, thus producing $\mathrm{Ex}(\mathcal{P}_{n+1})$. Let $A, B \in \mathcal{B}(\mathcal{P}_n)$. We prove this by cases for each step of the algorithm, confirming that the condition $A \subseteq B \cup \mathrm{Ext}(B)$ in Proposition 2.9 is satisfied. Let $\hat{A} = \gamma(A)$ for any $A \in \mathcal{B}(\mathcal{P}_n)$. For the following cases, $\hat{A}$ denotes the basis produced by γ after performing only the first step of the algorithm. Since every basis in $\mathcal{B}(\mathcal{P}_n)$ contains either 1 or 2, we need consider only the following three cases.

<u>Case 1</u>: $1 \in A$ and $1 \in B$. Since $1 \in A$, the only circuits contained in $A \cup \{i\}$ for $i \in [2n]$ are those of the form (i, j) for $i, j \in [n+1, \ldots, 2n]$. Thus $\mathrm{Ext}(A) = \{n+1, \ldots, \max A - 1\}$. Similarly, $\mathrm{Ext}(B) = \{n+1, \ldots, \max B - 1\}$. After applying step 1 of γ, we have $1 \in \hat{A}$ and $1 \in \widehat{B}$. Thus $\mathrm{Ext}(\hat{A}) = \{n+2, \ldots, \max \hat{A} - 1\}$ and $\mathrm{Ext}(\widehat{B}) = \{n+2, \ldots, \max \widehat{B} - 1\}$. Let $a \in \hat{A} \backslash \{1\}$. Then $a = a_i + 1$ for $a_i \in A$. Since $A \leq_{\mathrm{Ext}} B$, it follows that $a_i \in B \cup \{n+1, \ldots, \max B - 1\}$. Thus, $a_i + 1 \in B + 1 \cup \{n+2, \ldots, \max B\}$. Note that $\max \widehat{B} - 1 = \max B$. Therefore, $a \in \widehat{B} \cup \mathrm{Ext}(\widehat{B})$.

<u>Case 2</u>: $1 \in A$ and $1 \notin B$. By definition, if $1 \notin B$ then $1 \in \mathrm{Ext}(B)$ for every B of a trivial UIP. Let $a \in \hat{A} \backslash \{1\}$. Then $a = a_i + 1$ for $a_i \in A$. We know $a_i \in B \cup \{1, n+1, \ldots, \max B - 1\}$. Thus, $a_i + 1 \in B + 1 \cup \{1, n+2, \ldots, \max B\}$. Note that $\max \widehat{B} - 1 = \max B$. Therefore, $a \in \widehat{B} \cup \mathrm{Ext}(\widehat{B})$.

<u>Case 3</u>: $1 \notin A$ or B. By assumption we know that $1 \in \mathrm{Ext}(A)$ and $1 \in \mathrm{Ext}(B)$. Then for $a \in \hat{A} \backslash \{2\}$, we have that $a = a_i + 1$ for $a_i \in A$. Since $a_i \in B \cup \{1, n+1, \ldots, \max(B) - 1\}$, it follows $a_i + 1 \in B + 1 \cup \{1, n+2, \ldots, \max B\}$. Note that $\max \widehat{B} - 1 = \max B$. Therefore, $a \in \widehat{B} \cup \mathrm{Ext}(\widehat{B})$.

For the second step of the algorithm, notice that the only change to a basis B is that the element $2n+1$ is replaced with $2(n+1)$. This means that $\mathrm{Ext}(\widehat{B}) = \mathrm{Ext}(B) \cup \{2n+1\}$. Then the cases to check are exactly those done for step 1 of the algorithm, and they proceed in the same way. Similarly, for the third step, $\mathrm{Ext}(\widehat{B}) = \mathrm{Ext}(B) \backslash \{1\}$ since this step replaces the element 2 with 1. And then the cases proceed as above.

In all cases, the external order of $\mathcal{B}(\mathcal{P}_{n+1})$ is shown to be induced by the external order of $\mathcal{B}(\mathcal{P}_n)$. Thus, $\gamma(\mathcal{B}(\mathcal{P}_n)) = \mathrm{Ex}(\mathcal{P}_{n+1})$ as desired. □

5. Conclusion and future work

The recursive algorithm for the external poset on the bases of trivial UIPs provides a stepping stone to explore and develop either general or specific recursive algorithms for the other UIPs. In Figure 6 we see the trivial UIP and the remaining UIPs of rank $r = 3$. There appears to be some symmetry in the Hasse diagrams, but it is unclear how these five UIPs give rise to the nontrivial UIPs of rank $r = 4$. See

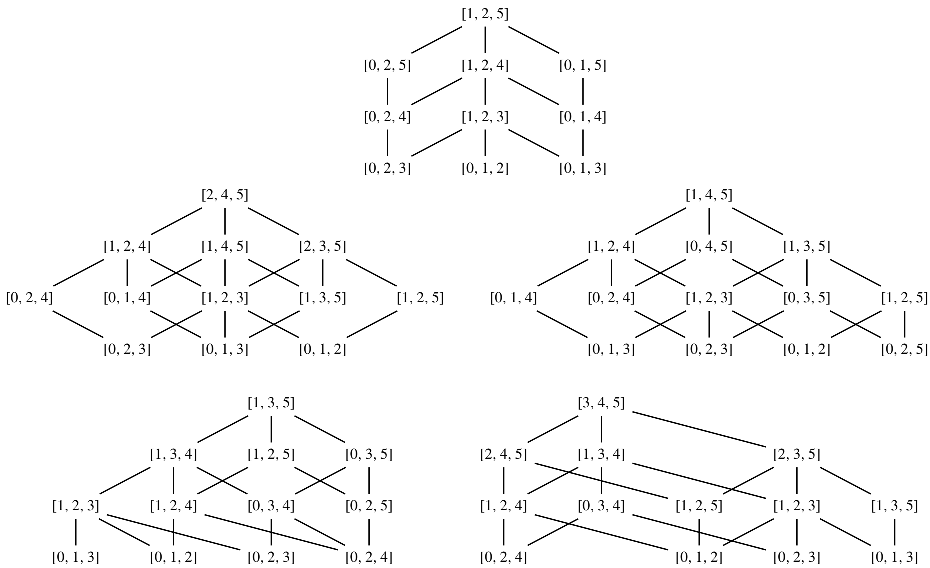

Figure 6. All UIPs with rank $r = 3$.

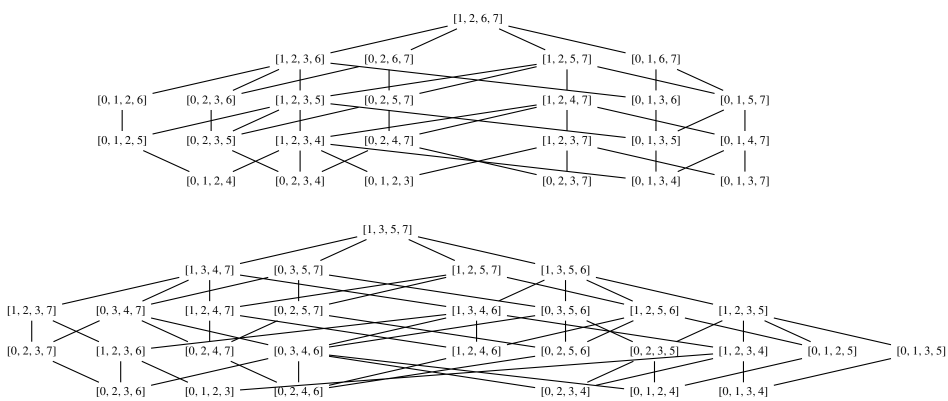

Figure 7. Two of the 14 UIPs with rank $r = 4$.

Figure 7 for examples of externally ordered posets for nontrivial UIPs of rank $r = 4$. As was done for the recursion for the externally ordered poset for trivial UIPs, a potential first step is to find nice descriptions for the bases and circuits of the remaining UIPs.

Another approach to recursively describe the remaining externally ordered posets is to consider matroid minors. One can check if minors of rank-$(n+1)$ UIPs are isomorphic to rank-n UIPs.

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