Intersections in subvarieties of $\mathbb{G}_{\mathrm{m}}^{l}$ and applications to lacunary polynomials

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Abstract. We investigate intersections of a given subvariety X of $\mathbb{G}_{\mathrm{m}}^{l}$ with cosets of 1-parameter subtori, on interpreting the context in terms of S-unit points over function fields. On adopting a function field version of a method introduced recently by the second author, extending to arbitrary dimensions previous work of the first and third author, we prove that when the number of intersections is substantially higher than expected, one can classify the relevant subtori. As a consequence, we obtain a classification of the cosets of subtori such that there are many multiple intersections with X. This also allows a new proof of a conjecture of Erdős and Rényi on lacunary polynomials. We finally show how the methods yield results in the realm of Unlikely Intersections in $\mathbb{G}_{\mathrm{m}}^{l}$, and in the last section, reinterpret some of the results in terms of Vojta's conjecture with truncated counting functions.

1 Introduction

In the recent paper [11], the second author succeeded in extending to the case of arbitrary dimensions certain results on integral points obtained by the first and third author in the case of surfaces. Such results concern estimates for the greatest common divisor of rational functions evaluated at S-unit points. For instance, the paper [4] proves an upper bound for $\gcd(u-1, v-1)$ for multiplicatively independent S-units u, v (e.g., proving it is $\ll_\epsilon \max(|u|,|v|)^\epsilon$ when $u, v \in \mathbb{Z}$), whereas [11] in particular extends this kind of estimate to arbitrary pairs of expressions in S-units (under natural necessary conditions).

The results in [4] were later formulated also in a version for function fields, e.g. in [5], [6], [7], [8]. In these articles the Schmidt Subspace Theorem, crucial in the numerical case, was replaced by an argument using Wronskians. This varied context (and proof) not only allowed for sharper estimates but also admitted several applications not possible in the former case of number fields.

In view of this, we have thought of formulating as well the improvements of [11] in the case of function fields, and the present paper is a first output of this programme.

We have decided to start just by testing the 'simple' case of S-units in $\mathbb{C}(t)$ with respect to $S = \{0, \infty\}$; in this case, the group of S-units consists exactly of the monomials ct^m, $c \in \mathbb{C}^*, m \in \mathbb{Z}$. Basic as it looks, this case already contains problems whose solution appears far from obvious. For instance, here is an illustrative

Question: *Given two coprime polynomials $P, Q \in \mathbb{C}[x_1, \ldots, x_l]$, for which S-unit points $x_i = a_i t^{m_i}$ do they acquire a common factor of 'substantial' size?*

We shall consider issues of this type. Similar questions had been asked by Schinzel long ago, and treated e.g. in the third author's appendix to [14] and in the paper [3] by Bombieri, Masser and the third author; however, these works considered mainly only the case of *fixed* coefficients a_i, which, as we shall see, is an important limitation for some applications. Instead, the present methods are not affected by the variation of coefficients.[1]

Note that a substitution of the relevant shape, i.e. $x_i \mapsto a_i t^{m_i}$, means just that we are restricting our regular functions on $\mathbb{G}_m^l$ to a certain 1-parameter algebraic coset (or subgroup if $a_i = 1$) of $\mathbb{G}_m^l$. Thus the issue fits into the context of multiplicative tori and their algebraic subgroups, which explains (part of) the title. [2]

Among our results, we provide for instance a complete classification of the cases when a given (but arbitrary) 1-parameter coset (of an algebraic subgroup) meets a given subvariety X of $\mathbb{G}_m^l$, of codimension 2, in 'many' points. As alluded to above, our conclusions have the advantage of being *uniform* in certain data; more precisely, the classification essentially depends neither on the involved *coefficients*, nor on the 'size' of the algebraic subgroup, but merely on the *degree of the variety and the ambient dimension*. See Proposition 1.1 (which is the main tool for all the results) and Theorem 1.3.

These results may be applied to study when a polynomial in several variables without multiple factors obtains several multiple roots after a substitution of the above type (i.e. along a 1-parameter coset); see Corollary 1.4.

We shall then notice two types of applications.

• The first one, given in §3, concerns lacunary polynomials (also called *fewnomials*, by which we mean polynomials with a bounded number of terms - but arbitrary coefficients and degrees of the terms). As an instance we shall offer a completely new proof (with respect to Schinzel's original one) of a former conjecture of Erdős and Rényi predicting lacunarity of $g(t)$, when it is known that $g(t)^2$ is lacunary (see Corollary 3.1).

• The second application concerns so-called Unlikely Intersections in algebraic tori. Here we shall recover a result of [3], which also shall allow a quantitative improvement of

[1] The paper [3] contains also certain uniform results, obtained by a method different from the present one, and anyway not made explicit.

[2] This interpretation also suggests the shape of possible analogues in the context of abelian varieties.

some of the former assertions.

In the last section, we reinterpret Corollary 1.4 as a certain case of Vojta's conjecture with truncated counting functions, and derive an analogous result in Nevanlinna theory.

In this paper we have limited the techniques to the case of constant coefficients and S-units of special shape. But the methods surely extend so as to remove both these restrictions, which does not look possible with the other known methods in this context. We hope to return to these topics in a future paper.

We now introduce just a bit of notation and give the main statements. Other statements are deferred to the respective sections.

1.1 Notation

If $\mathbf{v} = (b_1, \ldots, b_l) \in \mathbb{Z}^l$ we shall write $\mathbf{x}^{\mathbf{v}} := x_1^{b_1} \cdots x_l^{b_l}$.

For a subscheme $Y \subset \mathbb{G}_{\mathrm{m}}^l$ we let $\deg Y$ be the degree of the closure of Y in the natural embedding $\mathbb{G}_{\mathrm{m}}^l \hookrightarrow \mathbb{P}_l$. We agree that the degree of the empty subscheme is 0. For instance, if T is a subtorus of codimension 1 defined by $x_1^{a_1} \cdots x_l^{a_l} = 1$, then its degree is the maximum between the sums of the absolute values of the positive ones, or the negative ones, of the a_i.

Also, if X, Y are subvarieties of $\mathbb{G}_{\mathrm{m}}^l$, then in $\deg(X \cap Y)$ we shall count only those components contained in $\mathbb{G}_{\mathrm{m}}^l$.

It is worth noticing that an automorphism of $\mathbb{G}_{\mathrm{m}}^l$, given by monoidal changes of coordinates (as in [2]) may change the degree; we shall often remark this in the text when relevant.

For a positive 0-cycle $C = \sum h_i \mathbf{a}_i$, where $h_i \in \mathbb{N}$, $\mathbf{a}_i \in \mathbb{G}_{\mathrm{m}}^l$, we let

$$\rho(C) = \sum \max(0, h_i - 1).$$

This of course expresses the multiplicities occurring in C.

For a subtorus $H \subset \mathbb{G}_{\mathrm{m}}^l$, we shall usually denote by $\mathbf{a}H$ the translate of H by $\mathbf{a} \in \mathbb{G}_{\mathrm{m}}^l$. For a regular function P on $\mathbb{G}_{\mathrm{m}}^l$ we shall denote by $P_{\mathbf{a}H}$ its restriction to $\mathbf{a}H$. If H is given a parametrisation $x_i = \prod_{j=1}^d t_j^{a_{ij}}$ (which always exists), we shall understand the restriction as obtained just by substitution into P (clearing, if we want, the denominators which arise, which are monomials). For instance, if $H = U$ has dimension 1, and if $\mathbf{a} = (a_1, \ldots, a_l)$, then $\mathbf{a}U$ may be parametrized as $x_i = a_i t^{m_i}$, and by $P_{\mathbf{a}U}$ we mean $P(a_1 t^{m_1}, \ldots, a_l t^{m_l}) \in \mathbb{C}[t^{\pm 1}]$; if we want a polynomial we just multiply by a power of t, which is immaterial in this context. (See also Remark 1.5 (i).)

In the sequel by 'subtorus' we shall refer to the ambient space $\mathbb{G}_{\mathrm{m}}^l$ unless different explicit mention is made.

1.2 Some statements

The following proposition embodies the essentials of the method and is the basis for the subsequent applications. It basically asserts that if a 1-parameter coset $\mathbf{a}U$ meets a given subvariety $X \subset \mathbb{G}_{\mathrm{m}}^l$, of codimension 2, in too many points, then U must be contained in a (certain) prescribed *finite* union of proper subtori of $\mathbb{G}_{\mathrm{m}}^l$. Then going on to each such subtorus the dimension is decreased, which often allows for an induction process.

PROPOSITION 1.1. *Let P_1, P_2 be coprime polynomials in $\mathbb{C}[x_1, \ldots, x_l]$ of respective degrees d_1, d_2, and denote by X the subscheme of $\mathbb{G}_{\mathrm{m}}^l$ defined by $P_1 = P_2 = 0$. Further, let $d \geq \max(l\sqrt{2d_1d_2}, d_1 + d_2)$ be a positive integer.*

Then, for each subtorus U of dimension 1 and for each $\mathbf{a} \in \mathbb{G}_{\mathrm{m}}^l$, either

(i) There exists a subtorus H of codimension 1 with $\deg H \leq d$ and $U \subset H$, or

(ii) $\deg(X \cap \mathbf{a}U) \leq \frac{4l^2d_1d_2}{d} \deg U$.

REMARK 1.2. (i) The strength of the result lies in its uniformity, in that the alternatives and the estimate (*ii*) does not depend on the coefficients of P_1, P_2 nor on $\mathbf{a}$. Note especially that the number multiplying $\deg U$ on the right of (*ii*) tends to 0 as $d \to \infty$ (for fixed X).

(ii) Similar inequalities may be obtained, with the same proof, by using partial degrees in place of total degree.

The following theorem is our first application, where again the dependence of the bounds (i.e. *not* on coefficients) is a crucial point:

THEOREM 1.3. *Let P_1, P_2 be coprime polynomials in $\mathbb{C}[x_1, \ldots, x_l]$ and denote by X the subscheme of $\mathbb{G}_{\mathrm{m}}^l$ defined by $P_1 = P_2 = 0$. Let $0 < \epsilon < 1$. There exists a finite set Φ of subtori, each of degree $\leq c_1\epsilon^{-c_2}$, where $c_1, c_2 > 0$ are positive numbers depending only on l and $\deg X$, with the following property.*

For every translate $\mathbf{a}U$ of a subtorus U of dimension 1, if $\deg(X \cap \mathbf{a}U) \geq \epsilon \deg U$, then there exists $H \in \Phi$ with $U \subset H$ and $\dim(X \cap \mathbf{a}H) \geq \dim H - 1$.

Note that when $\dim(X \cap \mathbf{a}H) \geq \dim H - 1$, indeed we cannot hope that $\deg(X \cap \mathbf{a}U)$ is small, in view of Bezout's theorem (and since $U \subset H$). So the conclusion in a sense is the correct one.

Here is a first corollary, to classify the cases where restriction along a 1-parameter subgroup or coset produces many more multiple roots than expected (in turn an application of this to lacunary polynomials will be stated and proved later in §3).

COROLLARY 1.4. *Let $P \in \mathbb{C}[x_1, \ldots, x_l]$ have no non-monomial multiple factors and let $0 < \epsilon < 1$. There exists a finite set Φ of proper subtori, each of degree $\leq c_1\epsilon^{-c_2}$, where $c_1, c_2 > 0$ are positive numbers depending only on l and $\deg P$, with the following property.*

For every translate $\mathbf{a}U$ *of a subtorus* U *of dimension* 1*, if* $P|_{\mathbf{a}U}$ *has* $\geq \epsilon \deg U$ *nonzero multiple roots (where a root of multiplicity* m *counts* $m-1$*), then there exists* $H \in \Phi$ *with* $U \subset H$ *and* $P|_{\mathbf{a}H}$ *having a multiple factor vanishing somewhere on* $\mathbf{a}H$*.*

Remark 1.5. (i) Here by $P_{\mathbf{a}U}$ we mean $P(a_1t^{m_1}, \ldots, a_lt^{m_l}) \in \mathbb{C}[t^{\pm 1}]$. Suppose this is not identically zero. Then we remark that an easy Vandermonde argument shows that the multiplicity of any nonzero root in this Laurent polynomial is bounded in terms only of $l, \deg P$ (*Hajos lemma*, see e.g. [14]). Taking this into account, on changing ϵ by a factor depending only on $l, \deg P$ we can then simply count the number of multiple roots, i.e. counting them with multiplicity 1, obtaining an equivalent statement.

(ii) Note that this entails that for a 'generic' subtorus U, in the sense of not being contained in a proper finite union of subtori of $\mathbb{G}_{\mathrm{m}}^l$ depending only on $l, \deg P$ and ϵ, any restriction $P_{\mathbf{a}U}$ has a number of multiple roots bounded by $\epsilon \deg U$. For instance one can take $P = x_1 + \ldots + x_l$, and then we obtain that:

For integers $0 \leq m_1 < \ldots < m_l$ *and nonzero* $a_1, , \ldots, a_l$*, if* $a_1t^{m_1} + \ldots + a_lt^{m_l}$ *has more than* $\epsilon \cdot m_l$ *nonzero multiple roots, then* $m_1, \ldots, m_l$ *satisfy a nontrivial linear equation with coefficients bounded by a function of* l, ϵ *only.*

Here as above it is relevant that the coefficients $a_1, \ldots, a_l$ do not affect the estimates.

(iii) The paper [1] gives a similar conclusion (with a different formulation) assuming only that P_U has *at least one* (nonzero) multiple root. At first sight this may look much stronger; however the point is that the translation by $\mathbf{a}$ is absent in the latter statement. Indeed, the proofs in [1] also use the height of the coefficients of P, and translation affects this feature in the case $\mathbf{a}$ has large height.

This aspect greatly reduces the potential applications of [1] to the general theory of lacunary polynomials in case the coefficients may be arbitrary; see also §3 below. (Note also that by varying coefficients one can obtain several multiple roots without any special conclusion.)

(iv) Alternatively, the conclusion of Corollary 1.4 leads also to: *There exists a proper Zariski closed subset* $Z \subset \mathbb{G}_{\mathrm{m}}^l$*, depending only on* $\epsilon, l,$ *and* $\deg P$*, such that for every translate* $\mathbf{a}U$ *of a subtorus* U *of dimension* 1*, if* $P|_{\mathbf{a}U}$ *has* $\geq \epsilon \deg U$ *nonzero multiple roots then* $\mathbf{a}U \subset Z$*.* This statement can be derived from Corollary 1.4 as follows. From the corollary, it suffices to show that if $H (\in \Phi)$ is a fixed proper subtorus of $\mathbb{G}_{\mathrm{m}}^l$, then non-Zariski density holds for the union of the cosets $\mathbf{a}H$ such that $P|_{\mathbf{a}H}$ has a multiple factor vanishing somewhere on $\mathbf{a}H$. After an automorphism of $\mathbb{G}_{\mathrm{m}}^l$ and clearing monomial factors, it suffices to consider the case where H is defined by $x_1 = \cdots = x_{l'} = 1$ and $P \in \mathbb{C}[x_1, \ldots, x_l]$ has no multiple factors. Then we need to show that the set

$$\{(a_1, \ldots, a_{l'}) \in \mathbb{G}_{\mathrm{m}}^{l'}(\mathbb{C}) \mid P(a_1, \ldots, a_{l'}, x_{l'+1}, \ldots, x_l) \in \mathbb{C}[x_{l'+1}, \ldots, x_l] \text{ has a multiple factor}\}$$

is not Zariski dense in $\mathbb{G}_{\mathrm{m}}^{l'}$. This is well known, and geometrically follows from the general fact (see [10, Th. 9.7.7], [17, Tag 0578]) that if X and Y are schemes, $f : X \to Y$ is a morphism of finite type, Y is irreducible, and the generic fiber of f is geometrically reduced, then the fiber X_y is geometrically reduced for all y in some nonempty open subset $U \subset Y$. (Apply this to the closed subscheme defined by P and the projection onto the first l' coordinates.)

2 Main arguments

The main argument, introduced in the paper [11] in the number field case, appears in the following proof. As mentioned above, a Wronskian replaces the use of the Subspace Theorem.

Proof of Proposition 1.1 . Let us consider the complex vector space V_d of polynomials in $\mathbb{C}[x_1, \dots, x_l]$ of degree $\leq d$ and lying in the ideal (of X) generated by P_1, P_2. We first estimate $k := \dim V_d$. If Pol_h is the space of complex polynomials of degree $\leq h$, we have a linear map

$$Pol_{d-d_1} \oplus Pol_{d-d_2} \to V_d, \qquad (Q_1, Q_2) \mapsto Q_1P_1 + Q_2P_2.$$

If (Q_1, Q_2) is in the kernel of this map, then $Q_1 = BP_2, Q_2 = -BP_1$ for some polynomial B of degree $\leq d - d_1 - d_2$, and conversely, hence the kernel has dimension $\binom{d-d_1-d_2+l}{l}$.

We deduce that the image has dimension $\geq \binom{d-d_1+l}{l} + \binom{d-d_2+l}{l} - \binom{d-d_1-d_2+l}{l}$, which is then a lower bound for $k = \dim V_d$:

$$k = \dim V_d \geq \binom{d-d_1+l}{l} + \binom{d-d_2+l}{l} - \binom{d-d_1-d_2+l}{l}. \tag{2.1}$$

Let $f_1, \dots, f_k$ be a basis for V_d and let $t \to t^{\mathbf{m}} := (t^{m_1}, \dots, t^{m_l})$ be a parametrisation of U, for coprime integers $m_1, \dots, m_l$. In particular, $\max |m_i| \leq \deg U \leq 2 \max |m_i|$.

For a polynomial P in $x_1, \dots, x_l$ we denote for this proof $\tilde{P} = P(\mathbf{a}t^{\mathbf{m}}) \in \mathbb{C}[t, t^{-1}]$ and we let $W = W(\tilde{f}_1, \dots, \tilde{f}_k)$ be the Wronskian of the $\tilde{f}_i$, so that $W = \det(\tilde{f}_i^{(j)})$ is the determinant whose rows are the first k derivatives of $(\tilde{f}_1, \dots, \tilde{f}_k)$ with respect to t.

Let us consider the monomials of degree $\leq d$ (in $x_1, \dots, x_l$) evaluated at $t^{\mathbf{m}}$; we shall obtain powers $t^{a_1}, \dots, t^{a_r}$, ordered by $a_1 < a_2 < \dots < a_r$, where r is at most the number of monomials of degree $\leq d$, i.e. at most $\binom{d+l}{l}$. Note that if r is strictly less than this number, then we fall into the alternative (i) of the proposition, so let us assume that $r = \binom{d+l}{l}$.

The Laurent polynomials $\tilde{f}_i$ will be linear combinations of these powers of t. If $W = 0$ then the $\tilde{f}_i$ are linearly dependent over $\mathbb{C}$, and this implies that two monomials of degree $\leq d$ take equal values at $t^{\mathbf{m}}$; again, we fall into (i), and so let us assume $W \neq 0$.

On changing t into t^{-1} if needed, we may assume that $|a_1| \leq |a_r|$.

For $\xi \in \mathbb{P}_1$ let us denote by v_ξ the order function at ξ of a rational function of t.

Let us estimate $v_0(W)$. By column operations we can suppose that the $\tilde{f}_i$ have distinct valuations at 0; these valuations will lie among the a_i. Then it is easy to see that

$$v_0(W) \geq a_1 + a_2 + \ldots + a_k - \binom{k}{2}. \tag{2.2}$$

By a similar argument we see that

$$v_\infty(W) \geq -(a_r + a_{r-1} + \ldots + a_{r-k+1}) + \binom{k}{2}. \tag{2.3}$$

Let now $\xi \in \mathbb{G}_{\mathrm{m}}$ and let $m = m_\xi$ be the minimum of the orders of $\tilde{P}_1, \tilde{P}_2$ at ξ. By column operations we may assume that $\tilde{f}_i$ vanishes at ξ with order at least $m+i-1$. But then W vanishes at ξ of order at least km_ξ.

Note that since all the f_i lie in the ideal generated by P_1, P_2, this number shall be positive for ξ in $X \cap \mathbf{a}U$, and actually ξ is counted in $\deg(X \cap \mathbf{a}U)$ with weight given by m_ξ.

By the product formula, and since $W \neq 0$, we have $\sum_{\xi \in \mathbb{P}_1} v_\xi(W) = 0$. In view of the above, this yields

$$k \sum_{\xi \in \mathbb{G}_{\mathrm{m}}} m_\xi \leq (a_r + a_{r-1} + \ldots + a_{r-k+1}) - (a_1 + a_2 + \ldots + a_k).$$

In turn, letting $s = r - k$, the right-hand side equals

$$(a_{k+1} + \cdots + a_r) - (a_1 + \cdots + a_s) = (a_{k+1} - a_1) + (a_{k+2} - a_2) + \cdots + (a_{k+s} - a_s),$$

which does not exceed $2s|a_r| \leq 2sd \max|m_i| \leq 2sd \deg U$.

Let us estimate $s = r - k$. By equation (2.1) and since $r = \binom{d+l}{l}$, we have $s \leq \binom{d+l}{l} - \binom{d-d_1+l}{l} - \binom{d-d_2+l}{l} + \binom{d-d_1-d_2+l}{l} = g(d+l) - g(d+l-d_2)$, where $g(x) = \binom{x}{l} - \binom{x-d_1}{l} = \binom{x-1}{l-1} + \ldots + \binom{x-d_1}{l-1}$. So, $g(x) - g(x-d_2) = \sum_{i=1}^{d_1} \left(\binom{x-i}{l-1} - \binom{x-d_2-i}{l-1}\right) = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} \binom{x-i-j}{l-2}$. Hence in particular $s \leq d_1 d_2 \binom{d+l-2}{l-2}$ and therefore

$$\frac{s}{r} \leq \frac{l(l-1)d_1 d_2}{(d+l)(d+l-1)} \leq \frac{l^2 d_1 d_2}{d^2}.$$

In particular, $s/r \leq 1/2$ by our assumptions. Hence we finally obtain

$$\deg(X \cap \mathbf{a}U) \leq \frac{2ds}{r-s} \deg U \leq \frac{4ds}{r} \deg U \leq \frac{4l^2 d_1 d_2}{d} \deg U, \tag{2.4}$$

as required. □

Proof of Theorem 1.3. We argue by induction on l. The statement being empty for $l = 1$, we now assume $l \geq 2$.

We apply Proposition 1.1 with $d = [\frac{4l^2 \deg P_1 \deg P_2}{\epsilon}]+1$. If indeed $\deg(X \cap \mathbf{a}U) \geq \epsilon \deg U$, the conclusion (i) of the proposition delivers a subtorus H of codimension 1 with $\deg H \leq d$ and $U \subset H$.

We start by putting all such subtori, which form a finite set, inside our set Φ. Note that the degree of such tori satisfies a bound of the stated shape.

A first case now occurs if X has codimension ≤ 1 inside $\mathbf{a}H$, i.e. if $\dim(X \cap \mathbf{a}H) \geq \dim H - 1$. Then we fall into a case of the conclusion and are done.

Suppose on the contrary that X has codimension 2 inside $\mathbf{a}H$. In this case we reparametrize H, i.e. we may identify H with the subtorus $x_l = 1$ of $\mathbb{G}_{\mathrm{m}}^l$, after a suitable automorphism of $\mathbb{G}_{\mathrm{m}}^l$ expressed by substitutions $x_i \to \mathbf{x}^{\mathbf{v}_i}$, where the $\mathbf{v}_i$ are integer vectors in $\mathbb{Z}^l$ such that the matrix of the $\mathbf{v}_i$ is in $GL_l(\mathbb{Z})$ (see [2] or [20]).

The degree of H is essentially the height of $\mathbf{v}_l$; also, it is well known that the height of some such matrix can be bounded by $c_3 d^l$ (and even by $c_3 d$ but we do not need this), where c_3 depends only on l.

From now on we view all varieties and subtori inside H, which is identified with $\mathbb{G}_{\mathrm{m}}^{l-1}$. However this identification changes the degrees that we are using. Let us indicate with a subscript $\deg_H$ the new degrees (they depend actually not only on H but on the matrix which we have chosen). The new degrees are expressed in terms of multiplication by the said matrix, so that they will be bounded above by the old degrees multiplied by $c_4 d^{c_5}$, where c_4, c_5 depend only on l. A converse inequality is also true, on using the matrix inverse to the previous one.

Inside this new ambient space H, we also replace X with $X' := \mathbf{a}^{-1} X \cap H$, so that X' will be again defined (in $H \cong \mathbb{G}_{\mathrm{m}}^{l-1}$) by two polynomial equations (obtained on substituting $x_i \to \mathbf{x}^{\mathbf{v}_i}$ into the former ones and then setting $x_l = 1$). By the present assumption X' will again have codimension ≥ 2 in $\mathbb{G}_{\mathrm{m}}^{l-1}$ (so that the two polynomials are coprime). Also, since $\deg(X \cap \mathbf{a}U) \geq \epsilon \deg U$, we shall have $\deg(X' \cap U) \geq \epsilon' \deg_H U$, where the degree of U is now computed inside H and where $\epsilon' \geq c_6 \epsilon^{c_7}$ where $c_6, c_7 > 0$ depend only on $l, \deg X$. This estimate immediately follows from the above observations.

It suffices now to apply induction to this situation. It is again immediately checked that the dependence of the bounds for degrees remains of the stated shape, and the new subtori can be put inside the set Φ.

Note that each H delivers a new finite set Φ_H, but since there are only finitely many H involved, depending only on d, the final set Φ can be taken as the union of the respective Φ_H. □

REMARK 2.1. The proof may be turned into a completely explicit estimate for the involved quantities. We have omitted these calculations, also in order to reduce the length of the paper, and since the involved functions c_1, c_2 appear to be of iterated exponential type, so of very fast growth.

Note that the set Φ may depend on X, but is contained in a finite set of subtori depending only on $l, \epsilon, \deg X$.

Proof of Corollary 1.4. We proceed by induction on l, the assertion being clear for $l = 1$; we then assume $l > 1$ and the statement true up to $l - 1$.

As before, let U be parameterised as $t \mapsto t^{\mathbf{m}} := (t^{m_1}, \ldots, t^{m_l})$ for coprime integers m_i, observing that $\max |m_i| \leq \deg U \leq 2 \max |m_i|$.

We set $Q(\mathbf{x}) = \sum_{i=1}^{l} m_i x_i P_i(\mathbf{x})$, where for this proof we abbreviate $F_i = \frac{\partial}{\partial x_i} F$.

Note first that $t \frac{\mathrm{d}}{\mathrm{d}t} P(\mathbf{t}^{\mathbf{m}}) = Q(\mathbf{t}^{\mathbf{m}})$.

We may assume at the outset that P has no monomial factors, which are irrelevant as to the statement. Now assume first that P and Q are not coprime, and so have a common irreducible factor R not divisible by any x_i. Then we may write $P = RS$, where R and S are coprime since P is assumed to have no multiple factors. Differentiating, we obtain $Q = R(\sum_i m_i x_i S_i) + (\sum_i m_i x_i R_i) S$. We deduce that R divides $\sum_i m_i x_i R_i$. However, the latter polynomial has degree at most $\deg R$. Then for some constant c we must have

$$\sum_{i=1}^{l} m_i x_i R_i = cR.$$

Let $\mathbf{x}^{\mathbf{b}}$ be a monomial appearing in R, with coefficient $\mu \neq 0$. Then $\mathbf{x}^{\mathbf{b}}$ appears in $x_i R_i$ with coefficient μb_i. We infer that the sum $\sum_i m_i b_i$ is constant for all monomials appearing in R. Since R by assumption contains at least two monomials, we deduce a linear relation $\sum_{i=1}^{l} m_i b'_i = 0$ with integer coefficients b'_i not all zero and bounded in absolute value by $2 \deg R \leq 2 \deg P$. This means that U is contained in the finite union of proper subtori H of $\mathbb{G}_{\mathrm{m}}^{l}$ of codimension 1 and degree bounded by $2 \deg P$. In this case we can parametrize each such H (as in the previous proofs) and reduce to dimension $l - 1$, concluding the proof by induction.

Hence let us now assume that P and Q are coprime.

That $P_{\mathbf{a}U}$ has at least $\epsilon \deg U$ multiple nonzero roots (counted as in the statement) amounts to $P(\mathbf{a} t^{\mathbf{m}})$ having at least that number of nonzero multiple roots. Differentiating with respect to t we see that the scheme X defined by P and Q intersects $\mathbf{a}U$ in a scheme of degree $\geq \epsilon \deg U$. But now we may apply Theorem 1.3, obtaining the existence of the finite set Φ as in that statement.

In particular, there is an $H \in \Phi$ such that $U \subset H$ (and $\dim(X \cap \mathbf{a}H) \geq \dim H - 1$, but this this last condition will not be exploited). We consider the polynomial $P_{\mathbf{a}H}$; by parametrizing $\mathbf{a}H$ we can view $P_{\mathbf{a}H}$ as a polynomial in $l' = \dim H < l$ variables. (Here after the parametrisation we may obtain *Laurent* polynomials rather than polynomials. However, the denominator here is immaterial, and we can replace the Laurent polynomial by its numerator). Now, either $P_{\mathbf{a}H}$ has a multiple factor vanishing somewhere on $\mathbf{a}H$ (i.e. a non-monomial multiple factor, once we view $P_{\mathbf{a}H}$ as a polynomial in l' variables), and we are done, or we may apply induction.

This concludes the proof.

□

It will be noted that the variety X in the proof (to which Theorem 1.3 is applied) depends on the torus U (indeed, Q depends on U). It is the uniformity provided by the previous constructions which allows this kind of approach to work. (This is related to Remark 1.5 (iii) above; indeed, the methods of the paper [1] would not fit with this approach since the estimates therein depend on the height of the coefficients.)

3 An application to lacunary polynomials

We now illustrate an application to lacunary polynomials, obtaining in particular a completely new proof of what was a conjecture of Erdős and (independently) Rényi, first confirmed by Schinzel [13]:

Corollary 3.1. *Let $g(t) \in \mathbb{C}[t]$ be such that $g^2(t)$ has at most l terms. Then the number of terms of $g(t)$ is bounded by a function of l alone.*

Schnizel's result covered the case $g(t)^d$ with arbitrary fixed d, and the same holds for the present method; we have limited to $d = 2$ for simplicity.

Proof. We shall prove by induction on l the following statement

Let $P(x_1, \ldots, x_l) \in \mathbb{C}[x_1, x_1^{-1}, \ldots, x_l, x_l^{-1}]$ be a Laurent polynomial having no non-monomial multiple factor. Then if for some 1-dimensional sub-torus $U \subset \mathbb{G}_{\mathrm{m}}^l$ and a point $\mathbf{a} \in \mathbb{G}_{\mathrm{m}}^l$, the Laurent polynomial $P_{\mathbf{a}U}(t)$ is a perfect square $g(t)^2$, up to a power of t, then the number of terms of $g(t)$ is bounded as a function of l and $\deg P$ only (where $\deg P$ is defined to be the degree of the numerator of P).

Clearly, our Corollary 3.1 immediately follows from the above statement by taking $P(x_1, \ldots, x_l) = x_1 + \cdots + x_l$.

To prove our statement above, we use induction on l, observing that the statement is trivial for $l = 1$.

We observe also that, writing $P(\mathbf{x}) = \sum p_{\mathbf{m}} \mathbf{x}^{\mathbf{m}}$, we can suppose that the differences of the exponent vectors $\mathbf{m} - \mathbf{m}' \in \{-\deg P, \dots, \deg P\}^l$ appearing with non-zero coefficients $p_{\mathbf{m}}, p_{\mathbf{m}'}$ generate the $\mathbb{Q}$-vector space $\mathbb{Q}^l$. Indeed, if this were not the case, after multiplying P by an appropriate monomial, we could apply the induction argument, reducing to the same statement in fewer variables.

Let Φ be the set of codimension one subtori of degree at most $\deg P$. If U is a 1-dimensional subtorus and two monomials $\mathbf{x}^{\mathbf{m}}$ of P restrict to the same monomial on U, then clearly $U \subset H$ for some $H \in \Phi$. Then under the above independence assumption on the exponents of P, for a suitable positive number c, depending only on l and $\deg P$, the lower bound

$$\deg_{\mathbf{a}U} P_{\mathbf{a}U} > c \deg U, \tag{3.5}$$

holds for all 1-dimensional subtori U such that $U \not\subset H$ for $H \in \Phi$. Here, $\deg_{\mathbf{a}U} P_{\mathbf{a}U}$ denotes the degree of the subscheme P defines on $\mathbf{a}U$ (equivalently, the degree of the numerator of the Laurent polynomial $P_{\mathbf{a}U}$ after removing an appropriate power of t).

We shall apply Corollary 1.4 with $\epsilon = \frac{c}{2}$. With this choice of ϵ, we enlarge Φ by adding to it the finite set of subtori of $\mathbb{G}_{\mathrm{m}}^l$ provided by the conclusions of Corollary 1.4.

Since $f(t) = P_{\mathbf{a}U}$ is assumed to be a square (up to a power of t), $P_{\mathbf{a}U}$ has at least $(\deg_{\mathbf{a}U} P_{\mathbf{a}U})/2$ (nonzero) multiple roots. Hence if (3.5) holds, the conclusion of Corollary 1.4 delivers a subtorus $H \in \Phi$ such that $U \subset H$ and $P_{\mathbf{a}H}$ has a multiple factor vanishing somewhere on $\mathbf{a}H$. In any case, $U \subset H$ for some $H \in \Phi$, of degree bounded as a function of l and $\deg P$.

Let us reparameterise H by new variables, say, $z_1, \dots, z_r$, $r = \dim H < l$, so that $z_i = \mathbf{x}^{\mathbf{v}_i}$, where the $\mathbf{v}_i$ are certain linearly independent integer vectors. In these coordinates the z_i will become monomials in t under the substitution $x_i \mapsto a_i t^{m_i}$ (i.e. restricting to $\mathbf{a}U$); let us write $z_i \mapsto b_i t^{n_i}$. Also, $P_{\mathbf{a}H}$ will become a Laurent polynomial in $z_1, \dots, z_r$ (the denominator being immaterial since it is well defined and nonzero on $\mathbb{G}_{\mathrm{m}}^l$), and let $Q(z_1, \dots, z_r)$ be the numerator. Let $Q = Q_1^2 Q_2$ be a factorisation, where Q_2 is squarefree.

If Q_2 is a constant times a monomial, we are done since $P(\mathbf{a}t^{\mathbf{m}})$ is then obtained by the substitution $z_i = b_i t^{n_i}$ inside Q_1^2 (up to a power of t), and the degree of the various Q_1 which may appear is bounded as a function of $l, \deg P$, since each torus in Φ has degree bounded in terms of $l, \deg P$ only.

Otherwise, letting V be the subtorus of H parameterized by $z_i = t^{n_i}$, we have that $(Q_2)_{\mathbf{b}V}$ will be a square, and we apply the induction argument. □

Remark 3.2. It will be noted how Corollary 1.4 allows in fact conclusions which go beyond the last statement; i.e., in a sense, for given l, we may parameterise completely the identities of the shape $a_1 t^{m_1} + a_2 t^{m_2} + \dots + a_l t^{m_l} = g(t)^2 h(t)$ such that the degree of

$h(t)$ is 'small'. It seems not obvious to us if/how the known previous proofs of Corollary 3.1 may be extended in this sense.

As to quantitative bounds, Schinzel's first proof produced doubly exponential ones for the number of terms of $g(t)$ as a function of l. See [15] for a quantitative strengthening, following however similar principles. It is not likely that the present approach would lead to a better estimate, but our purpose here is mainly to show how the above general results include this sort of conclusion as an application. (See also the paper [21] for extensions to arbitrary compositions $h(g(t))$ in place of $g(t)^2$ and see further [9] for more general algebraic relations with lacunary polynomials, being treated with methods completely different both from Schinzel's and from the present ones; these last do not directly lead to these more general cases, however have other advantages.)

4 An application to Unlikely Intersections in tori

In the paper [3] of Bombieri, Masser and the third author, in particular the following situation was analysed. Let X be a subvariety of $\mathbb{G}_{\mathrm{m}}^l$, of dimension r say. If we take a subtorus H of dimension s and a translate $\mathbf{a}H$, we expect that $\dim(X \cap \mathbf{a}H)$ will not exceed $r+s-l$. Because of this, the subvarieties of X of positive dimension $\geq r+s-l+1$ and contained in such an intersection, were called *anomalous*. Also, those not contained in any anomalous subvariety of larger dimension were called *maximal anomalous*.

Theorem 1 of [3] gives a kind of general complete description of the maximal anomalous subvarieties, in particular proving that :

The maximal anomalous subvarieties correspond to intersections $X \cap \mathbf{a}H$ where only finitely many subtori H are involved. [3]

In [3] it was also remarked (subsequent to the statement) that *the degrees of these H can be bounded by (effective) functions of l and* $\deg X$ *only*; this more precise uniformity assertion, important in the present context, was not explicitly justified in the paper [3] but it was remarked that it followed naturally (and easily) as a byproduct from the structure of the proofs therein.

In this section we indicate how, in the case when X has codimension 2, such a result may also be achieved using the above theorems, hence in a completely different way with respect to [3]. [4]

[3]For given H, it is not difficult to see that the maximal anomalous subvarieties are obtained *algebraically*, i.e. precisely for $\mathbf{a}$ running in a certain constructible subvariety of X; see [3] for details.

[4]Probably this alternative treatment can be given by the same method in the case of arbitrary codimension.

In §4.2 we shall also remark how this leads to a quantitative improvement of Theorem 1.3, where the needed lower bound $\epsilon \deg U$ is replaced (for large $\deg U$) by $(\deg U)^{1-l^{-1}}$, which is a weaker requirement for large enough $\deg U$, hence for all but finitely many subtori U.

4.1 A sketch of the argument

Since the result in question has already been proved in [3] we shall give merely a sketch of the argument.

Suppose then that X has dimension $l-2$ and $Y \subset X \cap \mathbf{a}H$ is a(n irreducible) subvariety of X of positive dimension $\geq s-1$, where $s = \dim H$, and suppose Y is maximal with these properties. We may assume that $\dim Y = s-1$.

We may replace H with $\mathbb{G}_{\mathrm{m}}^s$ after a parametrisation, similar to what has been done previously several times. Then the ambient dimension is now s and the new variety is $\mathbf{a}^{-1}X \cap H$, which now has a component Y of positive dimension $s-1$. Intersecting Y with a 'general' subtorus U of H of dimension 1, by Bezout's theorem we shall obtain at least $\deg U$ points in a projective closure $\mathbb{P}_s$ of $\mathbb{G}_{\mathrm{m}}^s$. If U is 'very general', at least, say, half of these points will in fact lie in $\mathbb{G}_{\mathrm{m}}^s$.

But then we may apply Theorem 1.3 to X and U, with $\epsilon = 1/2$, and we deduce that there are a finite number of subtori of $\mathbb{G}_{\mathrm{m}}^l$, of degree bounded by $c_3(l, \deg X)$, whose union contains U. Since however U is practically arbitrary in H, this entails that H itself is contained in the union of the said subtori, concluding the argument.

4.2 A quantitative improvement

Finally, we show how such a result, whose proof we have just sketched (so it is indeed independent of the results in [3]) leads to the quantitative improvement of Theorem 1.3 mentioned in the opening remarks of this section. More precisely, we sketch the proof of the following

THEOREM 4.1. *Let P_1, P_2 be coprime polynomials in $\mathbb{C}[x_1, \ldots, x_l]$ and denote by X the subscheme of $\mathbb{G}_{\mathrm{m}}^l$ defined by $P_1 = P_2 = 0$. There exists a finite set Φ of subtori, each of degree $\leq c_1$, where c_1 is a positive number depending only on l and* $\deg X$*, with the following property.*

For every translate $\mathbf{a}U$ of a subtorus U of dimension 1, if $\deg(X \cap \mathbf{a}U) \geq (\deg U)^{1-l^{-1}}$, *then there exists $H \in \Phi$ with $U \subset H$ and* $\dim(X \cap \mathbf{a}H) \geq \dim H - 1$.

PROOF. Let us note that the result is very easy for bounded $\deg U$, since such tori are finite in number. Hence in the sequel we may assume that $\deg U$ is large enough in terms of l and $\deg X$.

By easy Geometry of Numbers (applied to the lattice corresponding to U) we can find a subtorus U' of dimension 2 containing U and of degree $\deg U' \ll (\deg U)^{\frac{l-2}{l-1}}$ (see for instance [22], Ch. 1). Here the implied constant depends only on l.

If $X \cap \mathbf{a}U'$ is finite then by Bezout's theorem we obtain $\deg(X \cap \mathbf{a}U') \le \deg X \cdot \deg U'$, whereas $\deg(X \cap \mathbf{a}U') \ge \deg(X \cap \mathbf{a}U) \ge (\deg U)^{1-l^{-1}}$ by assumption.

Since $\frac{l-2}{l-1} < 1 - \frac{1}{l}$, and since we are assuming that $\deg U$ is large, we obtain a contradiction, and hence we deduce that $X \cap \mathbf{a}U'$ has positive dimension.

But then the intersection $X \cap \mathbf{a}U'$ is anomalous (by definition) and the conclusion of the previous subsection applies. On going to the maximal anomalous intersections, it also follows that U', and therefore U, is contained in a finite union of proper subtori H of $\mathbb{G}_{\mathrm{m}}^l$ of degree bounded only in terms of $l, \deg X$. It also follows that $\dim(X \cap \mathbf{a}H) \ge \dim H - 1$. $\square$

5 A relation to Vojta's conjecture with truncated counting functions

Silverman [16] has explained in detail the connection between the gcd estimates of [4] and [11] and Vojta's conjecture applied to blowups of projective varieties. Proposition 1.1, a function field analogue of these estimates, admits a similar interpretation. We now show that the multiplicity result of Corollary 1.4 can be reinterpreted precisely as a function field version of certain cases of Vojta's conjecture involving truncated counting functions, and that it yields an analogous inequality in Nevanlinna theory which is closely related to a general inequality of Noguchi, Winkelmann, and Yamanoi [12].

We first state the relevant version of Vojta's conjecture over number fields [19, Conjecture 2.3] and refer to [19] for the definitions of the involved quantities (which will be explicitly defined below in our restricted function field setting).

Conjecture 5.1 (Vojta's conjecture with truncated counting functions). *Let X be a smooth projective variety over a number field k, D a normal crossings divisor on X, K_X a canonical divisor on X, A an ample divisor on X, and $\epsilon > 0$. Then there exists a proper Zariski closed subset $Z \subset X$ such that for all points $Q \in X(k) \setminus Z$,*

$$N_S^{(1)}(D, Q) \ge h_{K_X+D}(Q) - \epsilon h_A(Q). \tag{5.6}$$

We now consider the analogous conjecture over the function field $k = \mathbb{C}(t)$ with $X = \mathbb{P}_l$, and begin by explaining the terms in (5.6) in this context.

Let $Q = [x_0 : \cdots : x_l] \in \mathbb{P}_l(\mathbb{C}(t))$. Recall that the height of Q is defined by

$$h(Q) = -\sum_{\xi \in \mathbb{P}_1(\mathbb{C})} \min\{v_\xi(x_0), \dots, v_\xi(x_l)\},$$

where, as before, v_ξ denotes the order of a rational function of t at ξ. If D is a hypersurface in $\mathbb{P}_l$ defined by a homogeneous polynomial $P \in \mathbb{C}(t)[x_0, \dots, x_l]$, $\deg P = d$, and $\xi \in \mathbb{P}_1(\mathbb{C})$, then define

$$\lambda_{D,\xi}(Q) = v_\xi(P(Q)) - d \min_{0 \le j \le l}\{v_\xi(x_j)\},$$
$$h_D(Q) = \sum_{\xi \in \mathbb{P}_1(\mathbb{C})} \lambda_{D,\xi}(Q) = dh(Q),$$

where the last equality follows from the product formula. For a subset $S \subset \mathbb{P}_1(\mathbb{C})$ we define the proximity function and counting function associated to D, respectively, by

$$m_S(D, Q) = \sum_{\xi \in S} \lambda_{D,\xi}(Q),$$
$$N_S(D, Q) = \sum_{\xi \in \mathbb{P}_1(\mathbb{C}) \setminus S} \lambda_{D,\xi}(Q).$$

We define the n-truncated counting function associated to S and D by

$$N_S^{(n)}(D, Q) = \sum_{\xi \in \mathbb{P}_1(\mathbb{C}) \setminus S} \min\{\lambda_{D,\xi}(Q), n\}.$$

As we have done throughout, we now restrict to the case $S = \{0, \infty\}$. Let $P \in \mathbb{C}[x_1, \dots, x_l] \subset \mathbb{C}(t)[x_1, \dots, x_l]$ be nonconstant and let D_P be the hypersurface in $\mathbb{P}_l$ defined by (the homogenization of) P. Let $H_0, \dots, H_l$ be the coordinate hyperplanes of $\mathbb{P}_l$ and let $D = H_0 + H_1 + \cdots H_l + D_P$. Let $U \subset \mathbb{G}_{\mathrm{m}}^l$ be a subtorus of dimension 1 parametrized by $(t^{m_1}, \dots, t^{m_l})$, $\mathbf{a} \in \mathbb{G}_{\mathrm{m}}^l(\mathbb{C})$, and $Q = Q_{\mathbf{a}U} = [1 : a_1 t^{m_1} : \cdots : a_l t^{m_l}]$ be the projective point in $\mathbb{P}_l(\mathbb{C}(t))$ associated to $\mathbf{a}U$. From the form of Q and the definitions, we have

$$N_S^{(1)}(D, Q) = N_S^{(1)}(D_P, Q)$$

(in the language of integral points, Q is S-integral with respect to $H_0, \dots, H_l$). Note also that since $K_{\mathbb{P}_l} = -(H_0 + \cdots + H_l)$,

$$h_{K_{\mathbb{P}_l} + D}(Q) = h_{D_P}(Q).$$

Then assuming that D is a normal crossings divisor on $\mathbb{P}_l$, inequality (5.6) of Vojta's conjecture (over function fields) gives

$$h_{D_P}(Q) - N_S^{(1)}(D_P, Q) = m_S(D_P, Q) + N_S(D_P, Q) - N_S^{(1)}(D_P, Q) \le \epsilon h(Q),$$

as long as Q lies outside of some proper closed subset $Z \subset \mathbb{P}_l$. Equivalently, we can split this into the two inequalities

$$\begin{aligned} m_S(D_P, Q) &\leq \epsilon h(Q), \\ N_S(D_P, Q) - N_S^{(1)}(D_P, Q) &\leq \epsilon h(Q). \end{aligned} \tag{5.7}$$

The first inequality is known to hold in this context over both function fields and number fields (assuming that D is a normal crossings divisor, or at least that some general position condition is satisfied). For the second inequality, we find the easy formula

$$N_S(D_P, Q_{\mathbf{a}U}) - N_S^{(1)}(D_P, Q_{\mathbf{a}U}) = \sum_{\xi \in \mathbb{G}_{\mathrm{m}}} \max\{v_\xi(P|_{\mathbf{a}U}) - 1, 0\}.$$

Thus, the left-hand side of (5.7) counts the nonzero multiple roots of $P|_{\mathbf{a}U}$ as in Corollary 1.4. Since $h(Q_{\mathbf{a}U}) = \deg U$, Corollary 1.4 (combined with the exceptional set Z of Remark 1.5(iv)) can be viewed as giving precisely a certain case of Vojta's conjecture with truncated counting functions, namely inequality (5.7) for points $Q \in \mathbb{P}_l(\mathbb{C}(t))$ with S-unit coordinates (excluding points in some proper closed subset). Note that the normal crossings condition on D turns out to be unnecessary for (5.7), except for the requirement that D_P must be reduced.

We now turn to analogous inequalities in Nevanlinna theory (see [18] for the analogies with Diophantine approximation and the definitions and notation). Consider the corresponding analytic parametrization of $\mathbf{a}U$ given by $f = f_{\mathbf{a}U} : \mathbb{C} \to \mathbb{G}_{\mathrm{m}}^l \subset \mathbb{P}_l$, $f(z) = [1 : a_1 e^{m_1 z} : \cdots : a_l e^{m_l z}]$. Since e^z is an entire function without zeros, it follows from Nevanlinna's Second Main Theorem that for any $\xi \in \mathbb{G}_{\mathrm{m}}$,

$$m_{e^z}(\xi, r) \leq_{\mathrm{exc}} \epsilon T_{e^z}(r),$$

where the subscript exc means that the inequality holds for all $r > 0$ outside a set of finite Lebesgue measure. From Nevanlinna's First Main Theorem,

$$T_{e^z}(r) = m_{e^z}(\xi, r) + N_{e^z}(\xi, r) + O(1),$$

and therefore, for any $\epsilon > 0$,

$$(1-\epsilon) T_{e^z}(r) + O(1) \leq_{\mathrm{exc}} N_{e^z}(\xi, r) \leq T_{e^z}(r) + O(1). \tag{5.8}$$

As before, let $P \in \mathbb{C}[x_1, \ldots, x_l]$ be nonconstant and squarefree and let D_P be the hypersurface in $\mathbb{P}_l$ defined by the homogenization of P. Since $P(a_1 e^{m_1 z}, \ldots, a_l e^{m_l z}) = P|_{\mathbf{a}U}(e^z)$, we have

$$N_{f_{\mathbf{a}U}}(D_P, r) - N_{f_{\mathbf{a}U}}^{(1)}(D_P, r) = \sum_{\xi \in \mathbb{G}_{\mathrm{m}}} \max\{v_\xi(P|_{\mathbf{a}U}) - 1, 0\} N_{e^z}(\xi, r) + O(1). \tag{5.9}$$

Moreover, it is elementary that

$$T_{f_{\mathbf{a}U}}(r) = (\deg U)T_{e^z}(r) + O(1),$$

and, explicitly, $T_{e^z}(r) = \frac{r}{\pi}$. Then in view of (5.8) and (5.9), we find an analogue in Nevanlinna theory of the inequality (5.7). Corollary 1.4 and Remark 1.5(iv) imply that for any $\epsilon > 0$, there exists a proper Zariski closed subset $Z \subset \mathbb{G}_{\mathrm{m}}^l$ such that if $f_{\mathbf{a}U}(\mathbb{C}) \not\subset Z$, then

$$N_{f_{\mathbf{a}U}}(D_P, r) - N^{(1)}_{f_{\mathbf{a}U}}(D_P, r) \le \epsilon T_{f_{\mathbf{a}U}}(r) + O(1).$$

In summary, we have seen that the following three inequalities are essentially equivalent:

$$\sum_{\xi \in \mathbb{G}_{\mathrm{m}}} \max\{v_\xi(P|_{\mathbf{a}U}) - 1, 0\} \le \epsilon \deg U$$
$$N_S(D_P, Q_{\mathbf{a}U}) - N_S^{(1)}(D_P, Q_{\mathbf{a}U}) \le \epsilon h(Q_{\mathbf{a}U})$$
$$N_{f_{\mathbf{a}U}}(D_P, r) - N^{(1)}_{f_{\mathbf{a}U}}(D_P, r) \le \epsilon T_{f_{\mathbf{a}U}}(r).$$

The last two inequalities, however, admit different generalizations in a broader context, as we now discuss. The methods used here are likely to lead to versions of the second inequality for other characteristic 0 function fields, where S is an arbitrary finite set of places and $Q_{\mathbf{a}U}$ is replaced by any S-integral point of $\mathbb{G}_{\mathrm{m}}^l$. On the other hand, analogous methods in Nevanlinna theory are likely to lead to a version of the third inequality where $f_{\mathbf{a}U}$ is replaced by an arbitrary (nonconstant) holomorphic map $f : \mathbb{C} \to \mathbb{G}_{\mathrm{m}}^l$. In this direction, Noguchi, Winkelmann, and Yamanoi [12, Main Theorem (iii)] showed that if A is a semiabelian variety, $f : \mathbb{C} \to A$ is a holomorphic map with Zariski dense image, and D is an effective reduced divisor on A, then

$$N_f(D, r) - N_f^{(1)}(D, r) \le_{\mathrm{exc}} \epsilon T_{\bar{D}, f}(r),$$

where $\bar{D}$ is the closure of D in a suitable compactification $\bar{A}$ of A. Since their result assumes that f has Zariski dense image, their results do not apply to the situation considered in this section. In general, the methods here (adapted to Nevanlinna theory) may be expected to lead to a refinement of Noguchi-Winkelmann-Yamanoi's result in the case $A = \mathbb{G}_{\mathrm{m}}^l$.

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