Letter

Negative critical currents in single-channel Josephson junctions

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We argue that negative critical currents arise generically in Josephson junctions formed by single-channel conductors. Specifically, we theoretically study the Josephson coupling between two superconducting leads connected by a one-dimensional conductor in the Coulomb blockade regime. We show that in the clean regime the sign of the critical current alternates with the number of electrons in the normal region. For odd occupancy the critical current is negative even when the number of electrons on the conductor is large.

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The energy of a superconductor–normal metal–superconductor (SNS) junction depends on the order parameter phase difference between the two superconductors. In the limit of weak tunneling between the normal region and the superconductors, this dependence has the form

$$E_J(\chi_1 - \chi_2) = -\frac{\hbar J_c}{2e}\cos(\chi_1 - \chi_2). \tag{1}$$

Here, χ_1 and χ_2 are the order parameter phases in the two superconductors, e is the electron charge, and J_c is the critical current of the junction.

It is possible to prove that in the single-particle approximation the critical current is always positive [1]. Beyond the noninteracting electron approximation, there are no general principles which determine the sign of J_c. Several physical mechanisms of negative currents have been proposed. The sign of the critical current of a superconductor-ferromagnet-superconductor junction is an oscillating function of the magnetization and the length of the ferromagnet (see Refs. [2,3]). Even in the absence of macroscopic magnetization, the critical current can be negative if it is mediated by tunneling through a magnetic impurity [4], a resonant state [5–7], or a quantum dot in the Coulomb blockade regime [8]. Recently, negative critical currents were observed in SNS Josephson junctions with a normal region comprised of a carbon nanotube [9] and semiconductors [10–12].

We consider an SNS junction formed by a one-dimensional (1D) metallic wire in the Coulomb blockade regime. We show that the alternation of the sign of the critical current as a function of the number of electrons in the normal region is a generic property of such systems. Namely, the critical current is positive when the number of electrons is even, and negative if it is odd, even when the number of electrons in the junction is large.

The physical reason for the sign alternation can be traced to the node theorem for electron wave functions in one dimension (see, for example, Ref. [13]). According to this theorem, the number of nodes in the wave function of an energy eigenstate is given by the ordinal number of the energy level counted from the ground state. Indeed, we note that the amplitude of the electron tunneling through an insulating barrier from a single-particle state in the normal region with wave function $\psi_m(x)$ to a state in the lead $i = 1, 2$ with wave function $\phi_k^{(i)}(x)$ may be expressed as [13,14]

$$t_{mk}^{(i)} = \frac{1}{2m^*}\left[\phi_k^{(i)}(x)\partial_x\psi_m(x) - \psi_m(x)\partial_x\phi_k^{(i)}(x)\right]\Big|_{x=x_i}. \tag{2}$$

Here, m^* denotes the electron mass, the x axis is along the wire, and x_i is located inside the tunneling barrier between the wire and lead i; $x_1 = 0$, and $x_2 = L$, where L is the length of the wire. We assume that all electron wave functions can be chosen real. In second-order perturbation theory the single-particle tunneling amplitude between the two leads through the virtual state m in the wire is proportional to the product of derivatives $\partial_x\psi_m(x)$ at the contacts,

$$[\partial_x\psi_m(0)][\partial_x\psi_m(L)] = (-1)^{m+1}|\partial_x\psi_m(0)||\partial_x\psi_m(L)|. \tag{3}$$

The alternation of the sign in this equation is a direct consequence of the node theorem.

To elucidate the mechanism of the critical current sign alternation we consider a system described by the following Hamiltonian,

$$\hat{H} = E_C(\hat{N} - \mathcal{N}_0)^2 + \hat{H}^{1D} + \sum_{i=1,2}\left[\hat{H}_t^{(i)} + \hat{H}_{SC}^{(i)}\right]. \tag{4}$$

In this expression, $\hat{H}_{SC}^{(i)}$ is the Hamiltonian of the ith superconducting lead, and $\hat{H}^{1D}$ is the Hamiltonian of the normal metal region,

$$\hat{H}^{1D} = \sum_{m,\sigma}\xi_m c_{\sigma m}^\dagger c_{\sigma m}. \tag{5}$$

Here, $c_{\sigma m}$ is the annihilation operator of an electron with spin $\sigma = \uparrow, \downarrow$ and the mth single-electron state, whose energy ξ_m

is measured relative to the chemical potential (we will assume that ξ_m is a monotonic function of index m). The mean level spacing for the conductor is $\delta \sim \hbar v_{\rm F}/L$, where $v_{\rm F}$ is the Fermi velocity.

The tunneling Hamiltonian may be expressed in terms of the tunneling matrix elements in Eq. (2) as

$$\hat{H}_t^{(i)} = \sum_{m,k,\sigma} t_{mk}^{(i)} c_{\sigma m}^{\dagger} a_{\sigma k}^{(i)} + \text{H.c.}, \tag{6}$$

where $a_{\sigma k}^{(i)}$ denotes the electron annihilation operator in state k in the superconducting lead i.

To keep the presentation more transparent we treat the electron interactions in the "zero mode" approximation. In doing so we neglect the correlations induced by the electron-electron interactions inside the normal region. This approximation is applicable for small ratios between the electron potential and kinetic energy: $r_s = e^2/\hbar v_F \ll 1$. Furthermore, we assume that the length of the metallic wire L satisfies the inequality $r_s \ln(k_F L) \ll 1$, so that the Tomonaga-Luttinger liquid effects [15–17] may be neglected.

The zero mode interaction is represented by the first term in Eq. (4). Here, $E_C \sim e^2/L \ll \delta$ is a single-electron charging energy, and operator $\hat{N} = \sum_{\sigma m} c_{\sigma m}^{\dagger} c_{\sigma m}$ counts the number of electrons in the 1D conductor. (In the absence of tunneling this number is quantized.) The parameter $\mathcal{N}_0$ is proportional to the gate voltage and controls the number of electrons in the wire. We work in the approximation of weak tunneling and assume that the system is sufficiently far from the charge degeneracy point, so that quantum charge fluctuations are small. In this case the spin in the ground state is 0 for an even number of electrons and $1/2$ for an odd number.

The lowest order of perturbation theory with respect to $H_t^{(i)}$ which yields the dependence of the junction energy on $(\chi_1 - \chi_2)$ is fourth. For simplicity we assume that Δ, the quasiparticle energy gap in superconductors, exceeds both the Coulomb energy and the mean level spacing: $\Delta \gg E_C, \delta$. In this regime, quasiparticles can tunnel from superconductors to the normal metal wire only by pairs, and the part of the pair-tunneling Hamiltonian between superconductor i and the metallic wire may be written in the form

$$\hat{H}_{\rm T}^{(i)} = e^{i\chi_i} \sum_{mn} T_{mn}^{(i)} c_{\uparrow m}^{\dagger} c_{\downarrow n}^{\dagger} + \text{H.c.}, \tag{7}$$

where $T_{mn}^{(i)}$ denotes the tunneling amplitude of a Cooper pair from lead i into the states m and n in the normal region. For low-lying excited states m and n in the wire, satisfying $\Delta \gg |\xi_n|, |\xi_m|$, amplitude $T_{mn}^{(i)}$ can be expressed in terms of the single-particle tunneling amplitudes in Eq. (2) in the form [18,19]

$$T_{mn}^{(i)} = -\sum_k \frac{t_{km}^{(i)} t_{kn}^{(i)} |\langle a_{\uparrow k} a_{\downarrow k}\rangle|}{\epsilon_k}. \tag{8}$$

In this approximate expression, ϵ_k is the quasiparticle energy in state k of the superconductor, and $\langle a_{\uparrow k} a_{\downarrow k}\rangle$ denotes the Cooper pair condensation amplitude. Performing the summation over k in Eq. (8), one obtains the following estimate,

$$|T_{mn}^{(i)}| \propto g^{(i)} \delta, \tag{9}$$

where $g^{(i)}$ is the dimensionless conductance of the ith SN interface.

Once higher-energy degrees of freedom are "integrated out," the effective model Hamiltonian reads

$$\hat{H}_{\rm eff} = E_C(\hat{N} - \mathcal{N}_0)^2 + \hat{H}^{\rm 1D} + \hat{H}_{\rm T}^{(1)} + \hat{H}_{\rm T}^{(2)}. \tag{10}$$

Note that it only contains degrees of freedom in the normal metal.

We evaluate the Josephson coupling energy of the system using second-order perturbation theory in powers of $\hat{H}_{\rm T}^{(1,2)}$,

$$E_{\rm J} = \sum_{mn} \frac{\langle 0|\hat{H}_{\rm T}^{(1)}| \uparrow m, \downarrow n\rangle\langle \uparrow m, \downarrow n|\hat{H}_{\rm T}^{(2)}|0\rangle}{E_0 - E_{mn}} + \text{c.c.}, \tag{11}$$

where $|0\rangle$ is the ground state of the wire, while $|\uparrow m, \downarrow n\rangle$ is the two-particle excited state characterized by the presence (or absence) of two electrons with opposite spins, one on level n, another on level m. The structure of the ground state $|0\rangle$ depends on the parity of N_0. When the number of electrons is even, $N_0 = 2M$, the ground state $|0\rangle$ is a spin singlet, and any $m \geqslant M$ level is empty, while any $0 < m < M$ level is doubly occupied. This ground state remains stable as long as the gate potential $\mathcal{N}_0$ satisfies the following inequalities,

$$-E_C - \xi_M < 2E_C(2M - \mathcal{N}_0) < E_C - \xi_{M-1}. \tag{12}$$

Alternatively, for a fixed gate potential $\mathcal{N}_0$, one can view this relation as a condition on $N_0 = 2M$.

When either of the strict inequalities in Eq. (12) become an equality, the ground state becomes charge degenerate. For example, if $2E_C(2M - \mathcal{N}_0) = -E_C - \xi_M$, the state with $2M$ electrons and a state with $2M + 1$ electrons become degenerate. The extra electron occupies the $m = M$ level. The ground state with odd $N_0 = 2M + 1$ is stable when

$$|2E_C(2M + 1 - \mathcal{N}_0) + \xi_M| < E_C. \tag{13}$$

For odd N_0, the ground state is a spin doublet.

In the following, we will redefine the single-electron index m as follows: $m \to m - M$. That way, all levels with $m < 0$ are doubly occupied, and all levels with $m > 0$ are empty. A single electron resides on the $m = 0$ level for odd N_0, otherwise this level is empty. Furthermore, without loss of generality, we can assume that $\xi_0 = 0$.

Ultimately, the perturbation-theory expression (11) for the Josephson coupling $E_{\rm J}$ can be written as

$$E_{\rm J} = -(E_+ + E_-)\cos(\chi_1 - \chi_2), \tag{14}$$

where energies $E_\pm$ represent two-electron and two-hole contributions

$$E_\pm = \sum_{mn} \frac{2T_{mn}^{(1)} T_{mn}^{(2)} \Theta(\pm m \pm 1/2)\Theta(\pm n \pm P/2)}{4E_C \pm [4E_C(N_0 - \mathcal{N}_0) + \xi_m + \xi_n]}, \tag{15}$$

where the choice of the sign (top/bottom) on the right-hand side is dictated by the sign on the left, and the half-integer terms in the arguments of the Heaviside functions $\Theta(x)$ are introduced to avoid the uncertainty of $\Theta(0)$. Note that in Eq. (15) the summation range for index n depends on the parity P defined as

$$P = (-1)^{N_0}. \tag{16}$$

For odd N_0, this accounts for the Pauli blocking of the $n = 0$ single-electron state by a single electron occupying this state.

Another crucial observation about the sum in Eq. (15) is the sign alternation of the terms being summed. While the denominators are always positive, the numerator signs demonstrate a different pattern,

$$T_{mn}^{(1)}T_{mn}^{(2)} = (-1)^{m+n}\left|T_{mn}^{(1)}T_{mn}^{(2)}\right|. \tag{17}$$

To justify the sign-alternating factor $(-1)^{m+n}$ in this relation one can use Eqs. (2) and (8) to express $T_{mn}^{(i)}$ in terms of the derivatives $\partial_x\psi_m|_{x=0,L}$ and $\partial_x\psi_n|_{x=0,L}$, and then apply the node theorem (3) to the product $T_{mn}^{(1)}T_{mn}^{(2)}$.

So far we did not make any assumptions about the strength of disorder in the wire. Now we apply the formalism to the clean-wire case, $L \gg l$, where l is the elastic electron mean free path. In such a situation, the wave functions are $\psi_m \propto \sin(k_m x)$, where the quantized momentum is $k_m = k_F + \pi m/L$, and the Fermi momentum is equal to $k_F \approx \pi N_0/(2L)$ for $N_0 \gg 1$. The single-particle energies are $\xi_m = m\delta$.

In this regime, $|T_{mn}^{(1)}T_{mn}^{(2)}|$ may be considered independent of m and n, while the sign of the product $T_{mn}^{(1)}T_{mn}^{(2)}$ satisfies Eq. (17). Thus, Eq. (15) for an even number of electrons can be expressed as

$$E_{\pm}^{(\mathrm{even})} = E_0 \sum_{\substack{m\geqslant 0\\ n\geqslant 0}} \frac{(-1)^{m+n}}{\kappa_\pm + m + n}. \tag{18}$$

Here, $E_0 \propto g^{(1)}g^{(2)}\delta$, and the dimensionless offset parameters are

$$\kappa_\pm = \frac{4E_C}{\delta}\left[1 \pm (N_0 - \mathcal{N}_0) + \frac{(1\mp 1)\delta}{4E_C}\right]. \tag{19}$$

Note that $\kappa_\pm > 0$, as ensured by inequalities (12), and $\kappa_+ + \kappa_- \approx 2$. The latter relation means that at least one of the κ's is of order unity, and neither of them exceed 2.

Therefore, we reduce the issue of finding the Josephson coupling to the task of evaluating the sum in Eq. (18). To proceed, we rewrite this double sum in the form

$$\mathcal{S}_\pm = \sum_{n=0}^{+\infty}(-1)^n f(\kappa_\pm + n), \tag{20}$$

where

$$f(y) = \sum_{m=0}^{+\infty}\frac{(-1)^m}{y+m}. \tag{21}$$

Since the right-hand side of Eq. (20) is a sign-alternating series satisfying the Leibniz criterion [sequence $(y+m)^{-1}$ monotonically decreases to zero for growing index m], we conclude that, for positive y, the series is convergent, and function $f(y)$ is finite. Moreover, the Leibniz theorem guarantees that, for positive y, $f(y) > 0$ since the first term in the sum (20) is positive. Additionally, it is easy to prove that $f(y)$ decreases monotonically when $y \to +\infty$. Indeed, the derivative of f,

$$f'(y) = \sum_{\ell=0}^{+\infty}\left[\frac{1}{(y+2\ell+1)^2} - \frac{1}{(y+2\ell)^2}\right], \tag{22}$$

is negative since it is a convergent series of strictly negative terms. Therefore, the series in Eq. (20) also passes the Leibniz test. Furthermore, $\mathcal{S}_\pm > 0$ since $f(\kappa_\pm)$ are both positive. Thus we conclude that for an even number of electrons in the wire the critical current J_c in Eq. (1) is positive.

Let us now consider the situation with an odd number of electrons in the wire. In this case, using Eq. (15) with $P = -1$, we obtain the following expression for $E_\pm$ [see Eq. (14)],

$$E_{\pm}^{(\mathrm{odd})} = -E_0 \sum_{\substack{m\geqslant 0\\ n\geqslant 0}} \frac{(-1)^{m+n}}{(\kappa_\pm \pm 1) + m + n}. \tag{23}$$

This relation is the odd-N_0 counterpart of Eq. (18). The differences between these two expressions, both in terms of the overall sign and the denominator structure, follow from the difference in values of P for even/odd N_0.

The argumentation presented above for the even-N_0 case is trivially applicable for Eq. (23). This allows us to conclude that $E_\pm^{(\mathrm{odd})} < 0$. Thus, for an odd number of electrons in the wire the critical current J_c is negative. In other words, an addition or subtraction of a single electron from the conductor changes the sign of the critical current even in the case where the number of electrons is large.

The reason for the opposite sign of the critical current J_c in the cases of even and odd numbers of electrons in the wire N_0 is related to the fact that the signs of the corresponding alternating series, Eqs. (18) and (23), are determined by the signs of the terms with the smallest energy denominator.

So far we considered the case of a clean metallic wire. In the presence of a disorder potential $V(x)$, the general perturbative expression (15) remains valid. Equation (17), which follows from the node theorem, is valid as well. However, generally speaking, the absolute values of the terms in the alternating series Eq. (15) do not decrease monotonically since the products $|T_{mn}^{(1)}T_{mn}^{(2)}|$ become sample-specific functions of n and m. Therefore, the double sum in Eq. (15) may be dominated by the states whose wave functions have the strongest coupling to the leads, rather than those with the smallest energy denominators. In this case the value of the critical current depends on the realization of disorder.

Interestingly, it is possible to prove that, similarly to the case of noninteracting electrons [1], the Josephson coupling for an even number of electrons in the wire remains positive for any disorder in the wire. Our proof relies on the factorization $T_{mn}^{(1)}T_{mn}^{(2)} = s_n s_m$, where the function $s_n \propto \partial_x\psi_n(0)\partial_x\psi_n(L)$. We also make use of the following integral representation of a fraction, $1/x = \int_0^{+\infty} du e^{-ux}$, valid for any $x > 0$. Applying the latter representation to the denominator in expression (15), one derives

$$E_{\pm}^{(\mathrm{even})} = \int_0^{+\infty} du\, G_\pm^2 e^{-4E_C[1\pm(N_0-\mathcal{N}_0)]u}, \tag{24}$$

where functions $G_\pm = G_\pm(u)$ are defined for $u > 0$ by a convergent series

$$G_\pm = \sum_m (-1)^m e^{\mp u\xi_m}|s_m|\Theta(\pm m \pm 1/2). \tag{25}$$

It is clear from Eq. (24) that $E_\pm^{(\mathrm{even})}$ are both positive for any spectrum and the wave-function structure.

Finally, we would like to make the following observations.

(i) In the case where the number of electrons in the wire is odd and in the presence of a disordered potential the critical

current of the junction has a random sign, and the probability of the negative sign decreases as the strength of the disorder in the wire increases. For example, in the strongly disordered case where the electron mean free path l is shorter than the wire length, $l < L$, the electron wave functions are localized. In this case the distribution of $|T^{(1)}_{mn}T^{(2)}_{mn}|$ in Eq. (17) is exponentially broad. As a result, the double sum for the critical current in Eq. (15) is dominated by contributions of states whose wave functions have the strongest coupling to both leads. Since such states are typically either doubly occupied or empty, the probability of a negative critical current is small.

(ii) Since the results obtained above are based on the one-dimensional node theorem, they are of purely one-dimensional character. In general, in the systems where $E_F \gg \delta$ and in the case where the normal metal stripe has a finite width larger than the Fermi wavelength, the probability for the critical current to be negative decreases when the sample width grows.

(iii) For longer conductors, $L \gg L_{TL}$, the Luttinger liquid effects become significant. In this regime the 0-junction critical current $J_c(L)$ has been extensively studied theoretically [20–22]. In this limit the Luttinger liquid effects change the L dependence of the critical current from the single-particle dependence $1/L$ to a power of L which depends on the value of the interaction constant. As far as a question of the sign of J_c is concerned, we conjecture that the Luttinger liquid effects do not destroy the alternation of the Josephson coupling sign. In particular, at $r_s \gg 1$, where the system is close to an antiferromagnetic Wigner crystal, an extension of the arguments presented in Ref. [6] suggests that the sign of the critical current should oscillate as a function of the number of electrons. We also would like to mention that the sign oscillation of the pair field correlator as a function of distance in a 1D Kondo system has been discussed in Ref. [23].

To summarize, we showed that in clean single-channel Josephson junctions the sign of the critical current alternates with the number of electrons in the normal region, being positive if the number of electrons is even and negative when it is odd.

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