Applied Mathematics & Optimization (2022) 85:4
https://doi.org/10.1007/s00245-022-09868-x

Projective Splitting as a Warped Proximal Algorithm

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Accepted: 11 March 2022

Abstract
We show that the asynchronous block-iterative primal-dual projective splitting framework introduced by P. L. Combettes and J. Eckstein in their 2018 *Math. Program.* paper can be viewed as an instantiation of the recently proposed warped proximal algorithm.

Keywords Warped proximal algorithm · Projective splitting · Primal-dual algorithm · Splitting algorithm · Monotone inclusion · Monotone operator

In [4], the warped proximal algorithm was proposed and its pertinence was illustrated through the ability to unify existing methods such as those of [1, 6, 10, 11], and to design novel flexible algorithms for solving challenging monotone inclusions. Let us state a version of [4,Theorem 4.2].

Proposition 1 *Let* H *be a real Hilbert space, let* $\mathsf{M}\colon\mathsf{H}\to 2^{\mathsf{H}}$ *be a maximally monotone operator such that* $\operatorname{zer}\mathsf{M}\neq\varnothing$, *let* $\mathsf{x}_0\in\mathsf{H}$, *let* $\varepsilon\in\,]0,1[$, *let* $\alpha\in\,]0,+\infty[$, *and let* $\beta\in[\alpha,+\infty[$. *For every* $n\in\mathbb{N}$, *let* $\mathsf{K}_n\colon\mathsf{H}\to\mathsf{H}$ *be* α*-strongly monotone and* β*-Lipschitzian, and let* $\lambda_n\in[\varepsilon,2-\varepsilon]$. *Iterate*

$$
\begin{array}{l}
\textit{for}\ n=0,1,\ldots\\
\left\lfloor
\begin{array}{l}
\textit{take}\ \widetilde{\mathsf{x}}_n\in\mathsf{H}\\
\mathsf{y}_n=(\mathsf{K}_n+\mathsf{M})^{-1}(\mathsf{K}_n\widetilde{\mathsf{x}}_n)\\
\mathsf{y}_n^*=\mathsf{K}_n\widetilde{\mathsf{x}}_n-\mathsf{K}_n\mathsf{y}_n\\
\textit{if}\ \langle\mathsf{x}_n-\mathsf{y}_n\mid\mathsf{y}_n^*\rangle>0\\
\left\lfloor\ \mathsf{x}_{n+1}=\mathsf{x}_n-\dfrac{\lambda_n\langle\mathsf{x}_n-\mathsf{y}_n\mid\mathsf{y}_n^*\rangle}{\|\mathsf{y}_n^*\|^2}\,\mathsf{y}_n^*\right.\\
\textit{else}\\
\left\lfloor\ \mathsf{x}_{n+1}=\mathsf{x}_n.\right.
\end{array}
\right.
\end{array}
\tag{1}
$$

Then the following hold:

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Published online: 13 April 2022

(i) $(\mathbf{x}_n)_{n\in\mathbb{N}}$ *is bounded.*
(ii) $\sum_{n\in\mathbb{N}} \|\mathbf{x}_{n+1} - \mathbf{x}_n\|^2 < +\infty$.
(iii) $(\forall n \in \mathbb{N})\ \langle \mathbf{x}_n - \mathbf{y}_n \mid \mathbf{y}_n^* \rangle \leq \varepsilon^{-1} \|\mathbf{y}_n^*\|\, \|\mathbf{x}_{n+1} - \mathbf{x}_n\|$.
(iv) *Suppose that* $\widetilde{\mathbf{x}}_n - \mathbf{x}_n \to \mathbf{0}$. *Then* $(\mathbf{x}_n)_{n\in\mathbb{N}}$ *converges weakly to a point in* $\operatorname{zer} \mathbf{M}$.

Proof We deduce from [4,Proposition 3.9(i)[d]&(ii)[b]] that (1) is a special case of [4,Eq. (4.5)].

(i): An inspection of the proof of [4,Theorem 4.2] reveals that $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is Fejér monotone with respect to zer $\mathbf{M}$, that is, $(\forall \mathbf{z} \in \operatorname{zer} \mathbf{M})(\forall n \in \mathbb{N})\ \|\mathbf{x}_{n+1} - \mathbf{z}\| \leq \|\mathbf{x}_n - \mathbf{z}\|$. Therefore, the boundedness of $(\mathbf{x}_n)_{n\in\mathbb{N}}$ follows from [3,Proposition 5.4(i)].

(ii): [4,Theorem 4.2(i)].

(iii): [4,Eqs. (4.8), (4.9), and (4.4)].

(iv): Combine [4,Theorem 4.2(ii)] and [4,Remark 4.3]. □

A problem of interest in modern nonlinear analysis is the following (see, e.g., [1, 5–7] and the references therein for discussions on this problem).

Problem 2 Let $(\mathcal{H}_i)_{i\in I}$ and $(\mathcal{G}_k)_{k\in K}$ be finite families of real Hilbert spaces. For every $i \in I$ and every $k \in K$, let $A_i : \mathcal{H}_i \to 2^{\mathcal{H}_i}$ and $B_k : \mathcal{G}_k \to 2^{\mathcal{G}_k}$ be maximally monotone, let $z_i^* \in \mathcal{H}_i$, let $r_k \in \mathcal{G}_k$, and let $L_{k,i} : \mathcal{H}_i \to \mathcal{G}_k$ be linear and bounded. The problem is to

$$\text{find } (\overline{x}_i)_{i\in I} \in \bigtimes_{i\in I} \mathcal{H}_i \text{ and } (\overline{v}_k^*)_{k\in K} \in \bigtimes_{k\in K} \mathcal{G}_k \text{ such that}$$
$$\begin{cases} (\forall i \in I)\ z_i^* - \sum_{k\in K} L_{k,i}^* \overline{v}_k^* \in A_i \overline{x}_i \\ (\forall k \in K)\ \sum_{i\in I} L_{k,i} \overline{x}_i - r_k \in B_k^{-1} \overline{v}_k^*. \end{cases} \tag{2}$$

The set of solutions to (2) is denoted by $\mathbf{Z}$.

The first asynchronous block-iterative algorithm to solve Problem 2 was proposed in [7,Algorithm 12] as an extension of the projective splitting techniques found in [1, 8]. The goal of this short note is to interpret these projective splitting frameworks in simple terms as warped proximal iterations. More precisely, we show that [7,Algorithm 12] can be viewed as an instantiation of (1). To this end, we first derive an abstract weak convergence principle from Proposition 1. (We refer the reader to [3] for background on monotone operator theory and nonlinear analysis.)

Theorem 3 *Let* $\mathbf{H}$ *be a real Hilbert space, let* $\mathbf{A} : \mathbf{H} \to 2^{\mathbf{H}}$ *be a maximally monotone operator, and let* $\mathbf{S} : \mathbf{H} \to \mathbf{H}$ *be a bounded linear operator such that* $\mathbf{S}^* = -\mathbf{S}$. *In addition, let* $\mathbf{x}_0 \in \mathbf{H}$, *let* $\varepsilon \in]0, 1[$, *let* $\alpha \in]0, +\infty[$, *let* $\rho \in [\alpha, +\infty[$, *and for every* $n \in \mathbb{N}$, *let* $\mathbf{F}_n : \mathbf{H} \to \mathbf{H}$ *be* α*-strongly monotone and* ρ*-Lipschitzian, and let* $\lambda_n \in [\varepsilon, 2 - \varepsilon]$. *Iterate*

$$
\begin{array}{l}
\textit{for}\, n = 0, 1, \ldots \\
\left\lfloor
\begin{array}{l}
\textit{take}\, \mathbf{u}_n \in \mathbf{H},\ \mathbf{e}_n^* \in \mathbf{H},\ \textit{and}\, \mathbf{f}_n^* \in \mathbf{H} \\
\mathbf{u}_n^* = \mathbf{F}_n \mathbf{u}_n - \mathbf{S}\mathbf{u}_n + \mathbf{e}_n^* + \mathbf{f}_n^* \\
\mathbf{y}_n = (\mathbf{F}_n + \mathbf{A})^{-1} \mathbf{u}_n^* \\
\mathbf{a}_n^* = \mathbf{u}_n^* - \mathbf{F}_n \mathbf{y}_n \\
\mathbf{y}_n^* = \mathbf{a}_n^* + \mathbf{S}\mathbf{y}_n \\
\pi_n = \langle \mathbf{x}_n \mid \mathbf{y}_n^* \rangle - \langle \mathbf{y}_n \mid \mathbf{a}_n^* \rangle \\
\textit{if}\, \pi_n > 0 \\
\left\lfloor
\begin{array}{l}
\tau_n = \|\mathbf{y}_n^*\|^2 \\
\theta_n = \lambda_n \pi_n / \tau_n \\
\mathbf{x}_{n+1} = \mathbf{x}_n - \theta_n \mathbf{y}_n^*
\end{array}
\right. \\
\textit{else} \\
\left\lfloor\ \mathbf{x}_{n+1} = \mathbf{x}_n . \right.
\end{array}
\right.
\end{array}
\tag{3}
$$

Suppose that $\mathrm{zer}(\mathbf{A} + \mathbf{S}) \neq \varnothing$. *Then the following hold:*

(i) $\sum_{n\in\mathbb{N}} \|\mathbf{x}_{n+1} - \mathbf{x}_n\|^2 < +\infty$.

(ii) *Suppose that* $\mathbf{u}_n - \mathbf{x}_n \to \mathbf{0}$, *that* $\mathbf{e}_n^* \to \mathbf{0}$, *that* $(\mathbf{f}_n^*)_{n\in\mathbb{N}}$ *is bounded, and that there exists* $\delta \in\,]0, 1[$ *such that*

$$
(\forall n \in \mathbb{N}) \quad
\begin{cases}
\langle \mathbf{u}_n - \mathbf{y}_n \mid \mathbf{f}_n^* \rangle \geq -\delta \langle \mathbf{u}_n - \mathbf{y}_n \mid \mathbf{F}_n \mathbf{u}_n - \mathbf{F}_n \mathbf{y}_n \rangle \\
\langle \mathbf{a}_n^* + \mathbf{S}\mathbf{u}_n - \mathbf{e}_n^* \mid \mathbf{f}_n^* \rangle \leq \delta \|\mathbf{a}_n^* + \mathbf{S}\mathbf{u}_n - \mathbf{e}_n^*\|^2 .
\end{cases}
\tag{4}
$$

Then $(\mathbf{x}_n)_{n\in\mathbb{N}}$ *converges weakly to a point in* $\mathrm{zer}(\mathbf{A} + \mathbf{S})$.

Proof Set $\mathbf{M} = \mathbf{A} + \mathbf{S}$ and $(\forall n \in \mathbb{N})$ $\mathbf{K}_n = \mathbf{F}_n - \mathbf{S}$. Then, it follows from [3,Example 20.35 and Corollary 25.5(i)] that $\mathbf{M}$ is maximally monotone with $\mathrm{zer}\,\mathbf{M} \neq \varnothing$. Now take $n \in \mathbb{N}$. We have

$$
\mathbf{K}_n + \mathbf{M} = \mathbf{F}_n + \mathbf{A}. \tag{5}
$$

Since $\mathbf{S}^* = -\mathbf{S}$, we deduce that

$$
\mathbf{K}_n \text{ is } \alpha\text{-strongly monotone and } \beta\text{-Lipschitzian,} \tag{6}
$$

where $\beta = \rho + \|\mathbf{S}\|$. Thus, [3,Corollary 20.28 and Proposition 22.11(ii)] guarantee that there exists $\widetilde{\mathbf{x}}_n \in \mathbf{H}$ such that

$$
\mathbf{u}_n^* = \mathbf{K}_n \widetilde{\mathbf{x}}_n . \tag{7}
$$

Hence, by (3) and (5),

$$\mathbf{y}_n = (\mathbf{K}_n + \mathbf{M})^{-1}(\mathbf{K}_n\widetilde{\mathbf{x}}_n) \quad \text{and} \quad \mathbf{y}_n^* = \mathbf{u}_n^* - \mathbf{F}_n\mathbf{y}_n + \mathbf{S}\mathbf{y}_n = \mathbf{K}_n\widetilde{\mathbf{x}}_n - \mathbf{K}_n\mathbf{y}_n. \tag{8}$$

At the same time, we have $\langle \mathbf{y}_n \mid \mathbf{S}\mathbf{y}_n \rangle = 0$ and it thus results from (3) that

$$\pi_n = \langle \mathbf{x}_n \mid \mathbf{y}_n^* \rangle - \langle \mathbf{y}_n \mid \mathbf{a}_n^* + \mathbf{S}\mathbf{y}_n \rangle = \langle \mathbf{x}_n - \mathbf{y}_n \mid \mathbf{y}_n^* \rangle. \tag{9}$$

Altogether, (3) is a special case of (1).

(i): Proposition 1(ii).

(ii): In the light of Proposition 1(iv), it suffices to verify that $\widetilde{\mathbf{x}}_n - \mathbf{x}_n \to \mathbf{0}$. For every $n \in \mathbb{N}$, since $\mathbf{K}_n + \mathbf{M}$ is maximally monotone [3,Corollary 25.5(i)] and α-strongly monotone, [3,Example 22.7 and Proposition 22.11(ii)] implies that $(\mathbf{K}_n + \mathbf{M})^{-1} \colon \mathbf{H} \to \mathbf{H}$ is $(1/\alpha)$-Lipschitzian. Therefore, we derive from (3), (5), [4,Proposition 3.10(i)], and (6) that $(\forall \mathbf{z} \in \operatorname{zer} \mathbf{M})(\forall n \in \mathbb{N})\ \alpha\|\mathbf{y}_n - \mathbf{z}\| = \alpha\|(\mathbf{K}_n + \mathbf{M})^{-1}\mathbf{u}_n^* - (\mathbf{K}_n + \mathbf{M})^{-1}(\mathbf{K}_n\mathbf{z})\| \leq \|\mathbf{u}_n^* - \mathbf{K}_n\mathbf{z}\| = \|\mathbf{K}_n\mathbf{u}_n - \mathbf{K}_n\mathbf{z} + \mathbf{e}_n^* + \mathbf{f}_n^*\| \leq \|\mathbf{K}_n\mathbf{u}_n - \mathbf{K}_n\mathbf{z}\| + \|\mathbf{e}_n^*\| + \|\mathbf{f}_n^*\| \leq \beta\|\mathbf{u}_n - \mathbf{z}\| + \|\mathbf{e}_n^*\| + \|\mathbf{f}_n^*\|$. Thus, since Proposition 1(i) and our assumption imply that $(\mathbf{u}_n)_{n\in\mathbb{N}}$ is bounded, it follows that $(\mathbf{y}_n)_{n\in\mathbb{N}}$ is bounded. At the same time, for every $n \in \mathbb{N}$, we get from (3) that

$$\mathbf{y}_n^* = \mathbf{F}_n\mathbf{u}_n - \mathbf{F}_n\mathbf{y}_n + \mathbf{e}_n^* + \mathbf{f}_n^* - (\mathbf{S}\mathbf{u}_n - \mathbf{S}\mathbf{y}_n) = \mathbf{K}_n\mathbf{u}_n - \mathbf{K}_n\mathbf{y}_n + \mathbf{e}_n^* + \mathbf{f}_n^* \tag{10}$$

and, thus, from (6) that $\|\mathbf{y}_n^*\| \leq \|\mathbf{K}_n\mathbf{u}_n - \mathbf{K}_n\mathbf{y}_n\| + \|\mathbf{e}_n^*\| + \|\mathbf{f}_n^*\| \leq \beta\|\mathbf{u}_n - \mathbf{y}_n\| + \|\mathbf{e}_n^*\| + \|\mathbf{f}_n^*\|$. Thus, $(\mathbf{y}_n^*)_{n\in\mathbb{N}}$ is bounded, from which, (i), and Proposition 1(iii) we obtain $\overline{\lim}\langle \mathbf{x}_n - \mathbf{y}_n \mid \mathbf{y}_n^* \rangle \leq 0$. In turn, since $\mathbf{x}_n - \mathbf{u}_n \to \mathbf{0}$ and $\mathbf{e}_n^* \to \mathbf{0}$, it results from (10) and (4) that

$$\begin{aligned}
0 &\geq \overline{\lim}\langle \mathbf{x}_n - \mathbf{y}_n \mid \mathbf{y}_n^* \rangle \\
&= \overline{\lim}\ \big(\langle \mathbf{u}_n - \mathbf{y}_n \mid \mathbf{y}_n^* \rangle + \langle \mathbf{x}_n - \mathbf{u}_n \mid \mathbf{y}_n^* \rangle\big) \\
&= \overline{\lim}\langle \mathbf{u}_n - \mathbf{y}_n \mid \mathbf{y}_n^* \rangle \\
&= \overline{\lim}\ \big(\langle \mathbf{u}_n - \mathbf{y}_n \mid \mathbf{F}_n\mathbf{u}_n - \mathbf{F}_n\mathbf{y}_n + \mathbf{e}_n^* + \mathbf{f}_n^* \rangle - \langle \mathbf{u}_n - \mathbf{y}_n \mid \mathbf{S}\mathbf{u}_n - \mathbf{S}\mathbf{y}_n \rangle\big) \\
&= \overline{\lim}\ \big(\langle \mathbf{u}_n - \mathbf{y}_n \mid \mathbf{F}_n\mathbf{u}_n - \mathbf{F}_n\mathbf{y}_n + \mathbf{f}_n^* \rangle + \langle \mathbf{u}_n - \mathbf{y}_n \mid \mathbf{e}_n^* \rangle\big) \\
&\geq \overline{\lim}\ \big((1-\delta)\langle \mathbf{u}_n - \mathbf{y}_n \mid \mathbf{F}_n\mathbf{u}_n - \mathbf{F}_n\mathbf{y}_n \rangle + \langle \mathbf{u}_n - \mathbf{y}_n \mid \mathbf{e}_n^* \rangle\big) \\
&\geq \overline{\lim}\,\alpha(1-\delta)\|\mathbf{u}_n - \mathbf{y}_n\|^2 \\
&\geq \overline{\lim}\,\alpha(1-\delta)\rho^{-2}\|\mathbf{F}_n\mathbf{u}_n - \mathbf{F}_n\mathbf{y}_n\|^2.
\end{aligned} \tag{11}$$

Hence, $\mathbf{F}_n\mathbf{u}_n - \mathbf{F}_n\mathbf{y}_n \to \mathbf{0}$. On the other hand, since $(\mathbf{f}_n^*)_{n\in\mathbb{N}}$ is bounded and since (3) yields $(\mathbf{a}_n^* + \mathbf{S}\mathbf{u}_n - \mathbf{e}_n^*)_{n\in\mathbb{N}} = (\mathbf{F}_n\mathbf{u}_n - \mathbf{F}_n\mathbf{y}_n + \mathbf{f}_n^*)_{n\in\mathbb{N}}$, we derive from (4) that

$$\begin{aligned}\overline{\lim}(1-\delta)\|\mathbf{f}_n^*\|^2 &= \overline{\lim}\ \big(\langle \mathbf{F}_n\mathbf{u}_n - \mathbf{F}_n\mathbf{y}_n \mid \mathbf{f}_n^*\rangle + (1-\delta)\|\mathbf{f}_n^*\|^2\big)\\ &= \overline{\lim}\ \big(\langle \mathbf{F}_n\mathbf{u}_n - \mathbf{F}_n\mathbf{y}_n + \mathbf{f}_n^* \mid \mathbf{f}_n^*\rangle - \delta\|\mathbf{f}_n^*\|^2\big)\\ &\leq \overline{\lim}\ \big(\delta\|\mathbf{F}_n\mathbf{u}_n - \mathbf{F}_n\mathbf{y}_n + \mathbf{f}_n^*\|^2 - \delta\|\mathbf{f}_n^*\|^2\big)\\ &= \overline{\lim}\ \big(\delta\|\mathbf{F}_n\mathbf{u}_n - \mathbf{F}_n\mathbf{y}_n\|^2 + 2\delta\langle \mathbf{F}_n\mathbf{u}_n - \mathbf{F}_n\mathbf{y}_n \mid \mathbf{f}_n^*\rangle\big)\\ &= 0. \end{aligned} \tag{12}$$

Therefore, $\mathbf{f}_n^* \to \mathbf{0}$. Consequently, by (6), (7), and (3), $\alpha\|\widetilde{\mathbf{x}}_n - \mathbf{x}_n\| \leq \|\mathbf{K}_n\widetilde{\mathbf{x}}_n - \mathbf{K}_n\mathbf{x}_n\| = \|\mathbf{K}_n\mathbf{u}_n - \mathbf{K}_n\mathbf{x}_n + \mathbf{e}_n^* + \mathbf{f}_n^*\| \leq \beta\|\mathbf{u}_n - \mathbf{x}_n\| + \|\mathbf{e}_n^*\| + \|\mathbf{f}_n^*\| \to 0$. □

We are now ready to recover [7,Theorem 13]; see also [7,Remark 4] for comments on the error sequences $(e_{i,n})_{n\in\mathbb{N},i\in I_n}$ and $(f_{k,n})_{n\in\mathbb{N},k\in K_n}$ in (15). The reader is referred to [7] for discussions on the features of the algorithm (15). Recall that, given a real Hilbert space $\mathcal{H}$ with identity operator Id, the resolvent of an operator $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ is $J_A = (\mathrm{Id} + A)^{-1}$.

Corollary 4 ([7]) *Consider the setting of Problem 2 and suppose that* $\mathbf{Z} \neq \varnothing$. *Let* $(I_n)_{n\in\mathbb{N}}$ *be nonempty subsets of* I *and* $(K_n)_{n\in\mathbb{N}}$ *be nonempty subsets of* K *such that*

$$I_0 = I, \quad K_0 = K, \quad \textit{and} \quad (\exists\, T \in \mathbb{N})(\forall n \in \mathbb{N}) \ \bigcup_{j=n}^{n+T} I_j = I \ \textit{and} \ \bigcup_{j=n}^{n+T} K_j = K. \tag{13}$$

In addition, let $D \in \mathbb{N}$, *let* $\varepsilon \in \,]0,1[$, *let* $(\lambda_n)_{n\in\mathbb{N}}$ *be in* $[\varepsilon, 2-\varepsilon]$, *and for every* $i \in I$ *and every* $k \in K$, *let* $(c_i(n))_{n\in\mathbb{N}}$ *and* $(d_k(n))_{n\in\mathbb{N}}$ *be in* $\mathbb{N}$ *such that*

$$(\forall n \in \mathbb{N}) \quad n - D \leq c_i(n) \leq n \quad \textit{and} \quad n - D \leq d_k(n) \leq n, \tag{14}$$

let $(\gamma_{i,n})_{n\in\mathbb{N}}$ *and* $(\mu_{k,n})_{n\in\mathbb{N}}$ *be in* $[\varepsilon, 1/\varepsilon]$, *let* $x_{i,0} \in \mathcal{H}_i$, *and let* $v_{k,0}^* \in \mathcal{G}_k$. *Iterate*

$$
\begin{array}{l}
\textit{for}\, n = 0, 1, \ldots \\
\left\lfloor \begin{array}{l}
\textit{for every}\, i \in I_n \\
\left\lfloor \begin{array}{l}
\textit{take}\, e_{i,n} \in \mathcal{H}_i \\
l_{i,n}^* = \sum_{k\in K} L_{k,i}^* v_{k,c_i(n)}^* \\
a_{i,n} = J_{\gamma_{i,c_i(n)} A_i} \left(x_{i,c_i(n)} + \gamma_{i,c_i(n)} (z_i^* - l_{i,n}^*) + e_{i,n} \right) \\
a_{i,n}^* = \gamma_{i,c_i(n)}^{-1} (x_{i,c_i(n)} - a_{i,n} + e_{i,n}) - l_{i,n}^*
\end{array} \right. \\
\textit{for every}\, i \in I \smallsetminus I_n \\
\left\lfloor \begin{array}{l}
a_{i,n} = a_{i,n-1} \\
a_{i,n}^* = a_{i,n-1}^*
\end{array} \right. \\
\textit{for every}\, k \in K_n \\
\left\lfloor \begin{array}{l}
\textit{take}\, f_{k,n} \in \mathcal{G}_k \\
l_{k,n} = \sum_{i\in I} L_{k,i} x_{i,d_k(n)} \\
b_{k,n} = r_k + J_{\mu_{k,d_k(n)} B_k} \left(l_{k,n} + \mu_{k,d_k(n)} v_{k,d_k(n)}^* + f_{k,n} - r_k \right) \\
b_{k,n}^* = v_{k,d_k(n)}^* + \mu_{k,d_k(n)}^{-1} (l_{k,n} - b_{k,n} + f_{k,n}) \\
t_{k,n} = b_{k,n} - \sum_{i\in I} L_{k,i} a_{i,n}
\end{array} \right. \\
\textit{for every}\, k \in K \smallsetminus K_n \\
\left\lfloor \begin{array}{l}
b_{k,n} = b_{k,n-1} \\
b_{k,n}^* = b_{k,n-1}^* \\
t_{k,n} = b_{k,n} - \sum_{i\in I} L_{k,i} a_{i,n}
\end{array} \right. \\
\textit{for every}\, i \in I \\
\left\lfloor t_{i,n}^* = a_{i,n}^* + \sum_{k\in K} L_{k,i}^* b_{k,n}^* \right. \\
\pi_n = \sum_{i\in I} \left(\langle x_{i,n} \mid t_{i,n}^* \rangle - \langle a_{i,n} \mid a_{i,n}^* \rangle \right) + \sum_{k\in K} \left(\langle t_{k,n} \mid v_{k,n}^* \rangle - \langle b_{k,n} \mid b_{k,n}^* \rangle \right) \\
\textit{if}\, \pi_n > 0 \\
\left\lfloor \begin{array}{l}
\tau_n = \sum_{i\in I} \|t_{i,n}^*\|^2 + \sum_{k\in K} \|t_{k,n}\|^2 \\
\theta_n = \lambda_n \pi_n / \tau_n
\end{array} \right. \\
\textit{else} \\
\left\lfloor \theta_n = 0 \right. \\
\textit{for every}\, i \in I \\
\left\lfloor x_{i,n+1} = x_{i,n} - \theta_n t_{i,n}^* \right. \\
\textit{for every}\, k \in K \\
\left\lfloor v_{k,n+1}^* = v_{k,n}^* - \theta_n t_{k,n}. \right.
\end{array} \right.
\end{array}
\tag{15}
$$

In addition, suppose that there exist $\eta \in \,]0, +\infty[$, $\chi \in \,]0, +\infty[$, $\sigma \in \,]0, 1[$, *and* $\zeta \in \,]0, 1[$ *such that*

$$
(\forall n \in \mathbb{N})(\forall i \in I_n) \quad
\begin{cases}
\|e_{i,n}\| \leq \eta \\
\langle x_{i,c_i(n)} - a_{i,n} \mid e_{i,n} \rangle \geq -\sigma \|x_{i,c_i(n)} - a_{i,n}\|^2 \\
\langle e_{i,n} \mid a_{i,n}^* + l_{i,n}^* \rangle \leq \sigma \gamma_{i,c_i(n)} \|a_{i,n}^* + l_{i,n}^*\|^2
\end{cases}
\tag{16}
$$

and that

$$(\forall n \in \mathbb{N})(\forall k \in K_n) \quad \begin{cases} \|f_{k,n}\| \leq \chi \\ \langle l_{k,n} - b_{k,n} \mid f_{k,n} \rangle \geq -\zeta \|l_{k,n} - b_{k,n}\|^2 \\ \langle f_{k,n} \mid b^*_{k,n} - v^*_{k,d_k(n)} \rangle \leq \zeta \mu_{k,d_k(n)} \|b^*_{k,n} - v^*_{k,d_k(n)}\|^2. \end{cases} \tag{17}$$

Then $((x_{i,n})_{i \in I}, (v^*_{k,n})_{k \in K})_{n \in \mathbb{N}}$ *converges weakly to a point in* $\mathbf{Z}$.

Proof Denote by $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{G}}$ the Hilbert direct sums of $(\mathcal{H}_i)_{i \in I}$ and $(\mathcal{G}_k)_{k \in K}$, set $\mathbf{H} = \boldsymbol{\mathcal{H}} \oplus \boldsymbol{\mathcal{G}}$, and define the operators

$$\mathbf{A} \colon \mathbf{H} \to 2^{\mathbf{H}} \colon \big((x_i)_{i \in I}, (v^*_k)_{k \in K}\big) \mapsto \Big(\bigtimes_{i \in I} \big(-z^*_i + A_i x_i\big)\Big) \times \Big(\bigtimes_{k \in K} \big(r_k + B_k^{-1} v^*_k\big)\Big) \tag{18}$$

and

$$\mathbf{S} \colon \mathbf{H} \to \mathbf{H} \colon \big((x_i)_{i \in I}, (v^*_k)_{k \in K}\big) \mapsto \Bigg(\Big(\sum_{k \in K} L^*_{k,i} v^*_k\Big)_{i \in I}, \Big(-\sum_{i \in I} L_{k,i} x_i\Big)_{k \in K}\Bigg). \tag{19}$$

Using the maximal monotonicity of the operators $(A_i)_{i \in I}$ and $(B_k)_{k \in K}$, we deduce from [3,Propositions 20.22 and 20.23] that $\mathbf{A}$ is maximally monotone. In addition, we observe that $\mathbf{S}$ is a bounded linear operator with $\mathbf{S}^* = -\mathbf{S}$. At the same time, it results from (18), (19), and (2) that

$$\operatorname{zer}(\mathbf{A} + \mathbf{S}) = \mathbf{Z} \neq \varnothing. \tag{20}$$

Furthermore, (15) yields

$$\big[\,(\forall i \in I)(\forall n \in \mathbb{N})\ a^*_{i,n} \in -z^*_i + A_i a_{i,n}\,\big] \ \text{ and } \\ \big[\,(\forall k \in K)(\forall n \in \mathbb{N})\ b_{k,n} \in r_k + B_k^{-1} b^*_{k,n}\,\big]. \tag{21}$$

Next, define

$$(\forall k \in K)(\forall n \in \mathbb{N}) \quad \overline{\vartheta}_k(n) = \max\big\{j \in \mathbb{N} \mid j \leq n \text{ and } k \in K_j\big\} \ \text{ and } \\ \vartheta_k(n) = d_k\big(\overline{\vartheta}_k(n)\big), \tag{22}$$

and

$$(\forall i \in I)(\forall n \in \mathbb{N}) \quad \begin{cases} \overline{\ell}_i(n) = \max\big\{j \in \mathbb{N} \mid j \leq n \text{ and } i \in I_j\big\}, \ \ \ell_i(n) = c_i\big(\overline{\ell}_i(n)\big) \\ u^*_{i,n} = \gamma^{-1}_{i,\ell_i(n)} x_{i,\ell_i(n)} - l^*_{i,\overline{\ell}_i(n)} + \gamma^{-1}_{i,\ell_i(n)} e_{i,\overline{\ell}_i(n)} \\ w^*_{i,n} = \sum_{k \in K} L^*_{k,i} v^*_{k,\vartheta_k(n)} - l^*_{i,\overline{\ell}_i(n)}. \end{cases} \tag{23}$$

Then, for every $i \in I$ and every $n \in \mathbb{N}$, it follows from (15) and [3,Proposition 23.17(ii)] that

$$a_{i,n} = a_{i,\overline{\ell}_i(n)} = J_{\gamma_{i,\ell_i(n)} A_i}\big(\gamma_{i,\ell_i(n)}(u^*_{i,n} + z^*_i)\big) = \big(\gamma^{-1}_{i,\ell_i(n)} \mathrm{Id} - z^*_i + A_i\big)^{-1} u^*_{i,n} \tag{24}$$

and, therefore, that

$$a_{i,n}^{*}=a_{i,\overline{\ell}_i(n)}^{*}=u_{i,n}^{*}-\gamma_{i,\ell_i(n)}^{-1}a_{i,\overline{\ell}_i(n)}=u_{i,n}^{*}-\gamma_{i,\ell_i(n)}^{-1}a_{i,n}. \tag{25}$$

Likewise, for every $k\in K$ and every $n\in\mathbb{N}$, upon setting

$$\begin{cases}v_{k,n}=\mu_{k,\vartheta_k(n)}v_{k,\vartheta_k(n)}^{*}+l_{k,\overline{\vartheta}_k(n)}+f_{k,\overline{\vartheta}_k(n)}\\ w_{k,n}=l_{k,\overline{\vartheta}_k(n)}-\sum_{i\in I}L_{k,i}x_{i,\ell_i(n)}\end{cases} \tag{26}$$

as well as invoking (22), we get from (15) and [3,Proposition 23.17(iii)] that

$$b_{k,n}=b_{k,\overline{\vartheta}_k(n)}=J_{\mu_{k,\vartheta_k(n)}B_k(\cdot-r_k)}v_{k,n} \tag{27}$$

and, in turn, from (15) and [3,Proposition 23.20] that

$$\begin{aligned}b_{k,n}^{*}&=b_{k,\overline{\vartheta}_k(n)}^{*} &(28)\\ &=\mu_{k,\vartheta_k(n)}^{-1}\big(v_{k,n}-b_{k,\overline{\vartheta}_k(n)}\big)\\ &=\mu_{k,\vartheta_k(n)}^{-1}(v_{k,n}-b_{k,n}) &(29)\\ &=J_{\mu_{k,\vartheta_k(n)}^{-1}(r_k+B_k^{-1})}\big(\mu_{k,\vartheta_k(n)}^{-1}v_{k,n}\big)\\ &=\big(\mu_{k,\vartheta_k(n)}\mathrm{Id}+r_k+B_k^{-1}\big)^{-1}v_{k,n}. &(30)\end{aligned}$$

Let us set

$$(\forall n\in\mathbb{N})\quad\begin{cases}\mathbf{x}_n=\big((x_{i,n})_{i\in I},(v_{k,n}^{*})_{k\in K}\big),\ \mathbf{u}_n=\big(\big(x_{i,\ell_i(n)}\big)_{i\in I},\big(v_{k,\vartheta_k(n)}^{*}\big)_{k\in K}\big)\\ \mathbf{e}_n^{*}=\big((w_{i,n}^{*})_{i\in I},(w_{k,n})_{k\in K}\big),\ \mathbf{f}_n^{*}=\big(\big(\gamma_{i,\ell_i(n)}^{-1}e_{i,\overline{\ell}_i(n)}\big)_{i\in I},\big(f_{k,\overline{\vartheta}_k(n)}\big)_{k\in K}\big)\\ \mathbf{u}_n^{*}=\big((u_{i,n}^{*})_{i\in I},(v_{k,n})_{k\in K}\big),\ \mathbf{y}_n=\big((a_{i,n})_{i\in I},(b_{k,n}^{*})_{k\in K}\big)\\ \mathbf{a}_n^{*}=\big((a_{i,n}^{*})_{i\in I},(b_{k,n})_{k\in K}\big),\ \mathbf{y}_n^{*}=\big((t_{i,n}^{*})_{i\in I},(t_{k,n})_{k\in K}\big)\\ \mathbf{F}_n\colon\mathbf{H}\to\mathbf{H}\colon\big((x_i)_{i\in I},(v_k^{*})_{k\in K}\big)\mapsto\big(\big(\gamma_{i,\ell_i(n)}^{-1}x_i\big)_{i\in I},\big(\mu_{k,\vartheta_k(n)}v_k^{*}\big)_{k\in K}\big).\end{cases} \tag{31}$$

Then, the operators $(\mathbf{F}_n)_{n\in\mathbb{N}}$ are ε-strongly monotone and $(1/\varepsilon)$-Lipschitzian. For every $n\in\mathbb{N}$, by virtue of (23) and (26), we deduce from (19) that

$$\mathbf{S}\mathbf{u}_n-\mathbf{e}_n^{*}=\Big(\big(l_{i,\overline{\ell}_i(n)}^{*}\big)_{i\in I},\big(-l_{k,\overline{\vartheta}_k(n)}\big)_{k\in K}\Big), \tag{32}$$

which yields

$$\mathbf{u}_n^{*}=\mathbf{F}_n\mathbf{u}_n-\mathbf{S}\mathbf{u}_n+\mathbf{e}_n^{*}+\mathbf{f}_n^{*}. \tag{33}$$

Furthermore, we infer from (24), (30), and (18) that

$$(\forall n\in\mathbb{N})\quad\mathbf{y}_n=(\mathbf{F}_n+\mathbf{A})^{-1}\mathbf{u}_n^{*}. \tag{34}$$

At the same time, (25) and (29) imply that

$$(\forall n \in \mathbb{N}) \quad \mathbf{a}_n^* = \mathbf{u}_n^* - \mathbf{F}_n \mathbf{y}_n, \tag{35}$$

while (31), (15), and (19) guarantee that

$$(\forall n \in \mathbb{N}) \quad \mathbf{y}_n^* = \mathbf{a}_n^* + \mathbf{S}\mathbf{y}_n \quad \text{and} \quad \pi_n = \langle \mathbf{x}_n \mid \mathbf{y}_n^* \rangle - \langle \mathbf{y}_n \mid \mathbf{a}_n^* \rangle. \tag{36}$$

Altogether, it follows from (33)–(36) that (15) is an instantiation of (3). Hence, Theorem 3(i) yields $\sum_{n \in \mathbb{N}} \|\mathbf{x}_{n+1} - \mathbf{x}_n\|^2 < +\infty$. In turn, using (13), (14), (22), and (23), we deduce from [5,Lemma A.3] that, for every $i \in I$ and every $k \in K$, we have $\mathbf{x}_{\ell_i(n)} - \mathbf{x}_n \to \mathbf{0}$ and $\mathbf{x}_{\vartheta_k(n)} - \mathbf{x}_n \to \mathbf{0}$. This and (31) imply that

$$\mathbf{u}_n - \mathbf{x}_n \to \mathbf{0}. \tag{37}$$

Moreover, in view of (15), we deduce from (23) that

$$\begin{aligned}(\forall i \in I) \quad \|w_{i,n}^*\| &\leq \sum_{k \in K} \|L_{k,i}^*\| \, \left\| v_{k,\vartheta_k(n)}^* - v_{k,\ell_i(n)}^* \right\| \\ &\leq \sum_{k \in K} \|L_{k,i}^*\| \, \|\mathbf{x}_{\vartheta_k(n)} - \mathbf{x}_{\ell_i(n)}\| \to 0 \end{aligned} \tag{38}$$

and from (26) that

$$\begin{aligned}(\forall k \in K) \quad \|w_{k,n}\| &\leq \sum_{i \in I} \|L_{k,i}\| \, \|x_{i,\vartheta_k(n)} - x_{i,\ell_i(n)}\| \\ &\leq \sum_{i \in I} \|L_{k,i}\| \, \|\mathbf{x}_{\vartheta_k(n)} - \mathbf{x}_{\ell_i(n)}\| \to 0. \end{aligned} \tag{39}$$

Therefore, $\mathbf{e}_n^* \to \mathbf{0}$. By (16) and (17), $(\mathbf{f}_n^*)_{n \in \mathbb{N}}$ is bounded. In view of (31), (16), (17), and (32), we deduce that

$$\begin{aligned}(\forall n \in \mathbb{N}) \quad \langle \mathbf{u}_n - \mathbf{y}_n \mid \mathbf{f}_n^* \rangle &= \sum_{i \in I} \left\langle x_{i,\ell_i(n)} - a_{i,n} \mid \gamma_{i,\ell_i(n)}^{-1} e_{i,\overline{\ell}_i(n)} \right\rangle \\ &\quad + \sum_{k \in K} \left\langle v_{k,\vartheta_k(n)}^* - b_{k,n}^* \mid f_{k,\overline{\vartheta}_k(n)} \right\rangle \\ &\geq -\sigma \sum_{i \in I} \gamma_{i,\ell_i(n)}^{-1} \|x_{i,\ell_i(n)} - a_{i,n}\|^2 \\ &\quad - \zeta \sum_{k \in K} \mu_{k,\vartheta_k(n)} \|v_{k,\vartheta_k(n)}^* - b_{k,n}^*\|^2 \\ &\geq - \max\{\sigma, \zeta\} \langle \mathbf{u}_n - \mathbf{y}_n \mid \mathbf{F}_n \mathbf{u}_n - \mathbf{F}_n \mathbf{y}_n \rangle \end{aligned} \tag{40}$$

and that

$$\begin{aligned}\langle \mathbf{a}_n^* + \mathbf{S}\mathbf{u}_n - \mathbf{e}_n^* \mid \mathbf{f}_n^* \rangle &= \sum_{i\in I} \left\langle a^*_{i,\overline{\ell}_i(n)} + l^*_{i,\overline{\ell}_i(n)} \mid \gamma^{-1}_{i,\ell_i(n)} e_{i,\overline{\ell}_i(n)} \right\rangle \\ &\quad + \sum_{k\in K} \left\langle b_{k,\overline{\vartheta}_k(n)} - l_{k,\overline{\vartheta}_k(n)} \mid f_{k,\overline{\vartheta}_k(n)} \right\rangle \\ &\leq \sigma \sum_{i\in I} \left\| a^*_{i,\overline{\ell}_i(n)} + l^*_{i,\overline{\ell}_i(n)} \right\|^2 + \zeta \sum_{k\in K} \left\| b_{k,\overline{\vartheta}_k(n)} - l_{k,\overline{\vartheta}_k(n)} \right\|^2 \\ &\leq \max\{\sigma, \zeta\} \|\mathbf{a}_n^* + \mathbf{S}\mathbf{u}_n - \mathbf{e}_n^*\|^2. \end{aligned} \tag{41}$$

Altogether, the conclusion follows from Theorem 3(ii). □

Remark 5 Here are a few comments on Corollary 4.

(i) Using similar arguments, one can show that the asynchronous strongly convergent block-iterative method [7,Algorithm 14] and its special case [2,Eq. (3.10)] can be viewed as instances of [4,Theorem 4.8].

(ii) In the special case of (15) where $I = \{1\}$ and

$$(\forall n \in \mathbb{N}) \quad K_n = K \quad \text{and} \quad \begin{cases} e_{1,n} = 0, & c_1(n) = n \\ (\forall k \in K) \ f_{k,n} = 0, & d_k(n) = n, \end{cases} \tag{42}$$

the connection between [7,Theorem 13] and an instance of the warped proximal algorithm was established in [9,Proposition 19]. Nevertheless, it does not seem possible to prove [7,Theorem 13] in its full generality by using the techniques of [9].

Remark 6 Take $n \in \mathbb{N}$. Then, upon setting

$$\mathbf{H}_n = \left\{ \mathbf{x} \in \mathbf{H} \mid \langle \mathbf{x} - \mathbf{y}_n \mid \mathbf{y}_n^* \rangle \leq 0 \right\} \tag{43}$$

as well as invoking (9) and (31), we deduce that the update step

$$\begin{array}{l} \pi_n = \sum_{i\in I} \left(\langle x_{i,n} \mid t^*_{i,n} \rangle - \langle a_{i,n} \mid a^*_{i,n} \rangle \right) + \sum_{k\in K} \left(\langle t_{k,n} \mid v^*_{k,n} \rangle - \langle b_{k,n} \mid b^*_{k,n} \rangle \right) \\ \text{if } \pi_n > 0 \\ \left\lfloor \begin{array}{l} \tau_n = \sum_{i\in I} \|t^*_{i,n}\|^2 + \sum_{k\in K} \|t_{k,n}\|^2 \\ \theta_n = \lambda_n \pi_n / \tau_n \end{array} \right. \\ \text{else} \\ \lfloor \ \theta_n = 0 \\ \text{for every } i \in I \\ \lfloor \ x_{i,n+1} = x_{i,n} - \theta_n t^*_{i,n} \\ \text{for every } k \in K \\ \lfloor \ v^*_{k,n+1} = v^*_{k,n} - \theta_n t_{k,n} \end{array} \tag{44}$$

of (15) can be rewritten as

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \lambda_n(\mathrm{proj}_{\mathbf{H}_n}\mathbf{x}_n - \mathbf{x}_n), \tag{45}$$

which is the same as that of [7,Algorithm 12]; see [7,Eq. (22)]. Since $\mathbf{S}^* = -\mathbf{S}$, we derive from (36) and (31) that

$$\pi_n = \langle \mathbf{x}_n - \mathbf{y}_n \mid \mathbf{y}_n^* \rangle \quad \text{and} \quad \|\mathbf{y}_n^*\|^2 = \sum_{i\in I} \|t_{i,n}^*\|^2 + \sum_{k\in K} \|t_{k,n}\|^2, \tag{46}$$

from which we obtain the implication $\pi_n > 0 \Rightarrow \tau_n = \|\mathbf{y}_n^*\|^2 > 0$.

Acknowledgements This work is a part of the author's Ph.D. dissertation and it was supported by the National Science Foundation under Grant DMS-1818946. The author thanks his Ph.D. advisor P. L. Combettes for his guidance during this work. The author thanks the anonymous referees for suggestions which helped improve the manuscript.

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