Electronic Journal of Qualitative Theory of Differential Equations
2021, No. **60**, 1–9; https://doi.org/10.14232/ejqtde.2021.1.60 www.math.u-szeged.hu/ejqtde/

Existence of positive solutions for a fractional compartment system

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Received 4 May 2021, appeared 5 August 2021
Communicated by Nickolai Kosmatov

Abstract. In this article, we investigate the existence of positive solutions of a boundary value problem for a system of fractional differential equations. The resilience of a fractional compartment system is also studied to demonstrate the application of the result.

Keywords: boundary value problem, Green's function, positive solution, fractional compartment model.

2020 Mathematics Subject Classification: 34B18, 34B15, 34B60.

1 Introduction

In this paper, we consider the boundary value problem (BVP) consisting of a system of n fractional order compartment models

$$u_i' + a_i D_{0+}^{\alpha_i} u_i = f_i(u_1 \dots, u_n, t), \qquad 0 < t < 1, \tag{1.1}$$

and the boundary conditions (BCs)

$$u_i(0) = b_i u_i(1), \qquad i = 1, \dots, n, \tag{1.2}$$

where $0 < \alpha_i < 1$ and $D_{0+}^{\alpha_i} u_i$ denotes the α_i-th left Riemann–Liouville fractional derivative of u_i defined by

$$\left(D_{0+}^{\alpha_i} u_i\right)(t) = \frac{1}{\Gamma(1-\alpha_i)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha_i} u_i(s) ds,$$

provided the right-hand side exists with Γ being the Gamma function. We further assume that for any $i = 1, \dots, n$,

(H1) $a_i > 0$, $b_i > 0$, $f_i \in C(\mathbb{R}^n \times [0,1])$, and $f_i(0, \dots, 0, t) \not\equiv 0$ on $[0,1]$.

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Fractional differential equations have been an active research area for decades and attracted extensive attention from scholars in both applied and theoretic fields. Due to the superior capability of capturing long term memory and/or long range interaction, fractional models have been successfully developed to investigate problems on fractal porous media, social media networks, epidemiology, finance, control, etc. Those models were further generalized and studied both analytically and numerically. The reader is referred to [1–11, 13–16] and references therein for some recent advances.

This paper is mainly motivated by the study of a fractional compartment system for a bike share system. In [7], the station inventory, i.e., the number of bikes at a station, is modeled by

$$y_i' = q_i(t) - \omega_i(t)y_i - \Theta_i(t)c_i^{-\beta_i}D_{0+}^{1-\beta_i}\left(\frac{y_i}{\Theta_i}\right), \qquad t > 0,\ i = 1,\dots,n. \tag{1.3}$$

The resilience of station inventory, i.e., the capability that the station inventory will restore to certain level without extra interference, was further studied in [10, 16] by converting the resilience of Eq. (1.3) to a special case of BVP (1.1), (1.2) with $n = 1$ (the scalar case). Intuitively, it is more sensible to investigate BVP (1.1), (1.2) with $n > 1$ as the interactions among multiple stations are inevitable. From the practical perspective, we are particularly interested in finding conditions that guarantee the existence of positive solutions of BVP (1.1), (1.2). However, the extension from scalar to system is not trivial and it will require new auxiliary results to study the existence of positive solutions of the resulting system.

In this paper, a framework consisting of an appropriate Banach space and the associated operator will be proposed so that the fixed point theory can be applied to study the existence of positive solutions of BVP (1.1), (1.2). This framework will also be applicable to other fixed point theorems. Our result will be further applied to establish the sufficient conditions for the resilience of a fractional bike share inventory model. These conditions will provide guidance for the development of operational policy. Therefore, our work will make contributions in both theoretic and application aspects.

The paper is organized as follows: After this introduction, the main theoretic result and its proof will be presented in Section 2. The resilience of a bike share model will then be considered in Section 3 to demonstrate the application of our result.

2 Main results

We first introduce some needed notations and definitions. For any $x = (x_1,\dots,x_n) \in \mathbb{R}^n$, let $\|x\|_1 = \sum_{i=1}^n |x_i|$ and

$$K_r = \{x \in \mathbb{R}^n \ :\ \|x\|_1 \le r,\ x_i \ge 0,\ i = 1,\dots,n\}. \tag{2.1}$$

For any $u = (u_1,\dots,u_n) \in \Pi_{i=1}^n C[0,1]$, let $\|u\| = \max_{t\in[0,1]} \sum_{i=1}^n |u_i(t)|$. By a solution of BVP (1.1), (1.2), we mean a vector-valued function $u \in \Pi_{i=1}^n C[0,1]$ that satisfies (1.1) and (1.2). Furthermore, u is said to be a positive solution of BVP (1.1), (1.2) if $u_i(t) \ge 0$, $i = 1,\dots,n$, and $\|u\| > 0$.

Let $E_\alpha(t)$ be the Mittag-Leffler function defined by

$$E_\alpha(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(n\alpha+1)}$$

and $\Lambda_i(t)$ be defined by

$$\Lambda_i(t) = E_{1-\alpha_i}(-a_i t^{1-\alpha_i}), \qquad i = 1, \dots, n. \tag{2.2}$$

Throughout this paper, we assume

(H2) $b_i \Lambda_i(1) < 1$, $i = 1, \dots, n$.

Define

$$\overline{G_i} = \max_{t \in [0,1]} \left\{ \frac{b_i \Lambda_i(t)}{1 - b_i \Lambda_i(1)}, \frac{b_i \Lambda_i(t) \Lambda_i(1-t)}{1 - b_i \Lambda_i(1)} + 1 \right\} \tag{2.3}$$

and

$$\underline{G_i} = \min_{t \in [0,1]} \left\{ \frac{\Lambda_i(t)}{1 - b_i \Lambda_i(1)}, \frac{b_i \Lambda_i(t) \Lambda_i(1-t)}{1 - b_i \Lambda_i(1)} \right\}, \qquad i = 1, \dots, n. \tag{2.4}$$

Then we have the following result.

Theorem 2.1. *Let K_r and $\overline{G_i}$, $i = 1, \dots, n$, be defined in (2.1) and (2.3), respectively. Assume that (H1) and (H2) hold and that there exist $r > 0$ and $\eta_i > 0$, $i = 1, \dots, n$, such that*

(a) $\sum_{i=1}^{n} \overline{G_i} \eta_i \le r$; and

(b) for any $t \in [0,1]$ and $x \in K_r$, $0 \le f_i(x,t) \le \eta_i$, $i = 1, \dots, n$.

Then BVP (1.1), (1.2) has at least one positive solution u with $\|u\| \le r$.

The following lemma plays an important role in the proof of Theorem 2.1.

Lemma 2.2. *Assume (H2) holds. For $i = 1, \dots, n$, let Λ_i, $\overline{G_i}$, $\underline{G_i}$ be defined by (2.2), (2.3), (2.4), respectively, and*

$$G_i(t,s) = \begin{cases} \dfrac{b_i \Lambda_i(t) \Lambda_i(1-s)}{1 - b_i \Lambda_i(1)} + \Lambda_i(t-s), & 0 \le s \le t \le 1, \\[2ex] \dfrac{b_i \Lambda_i(t) \Lambda_i(1-s)}{1 - b_i \Lambda_i(1)}, & 0 \le t < s \le 1. \end{cases} \tag{2.5}$$

Then $G_i(t,s)$ is the Green's function for the scalar BVP

$$\begin{aligned} u_i' + a_i D_{0+}^{\alpha_i} u_i &= 0, \qquad 0 < t < 1, \\ u_i(0) &= b_i u(1), \end{aligned}$$

and satisfies

$$0 < \underline{G_i} \le G_i \le \overline{G_i}, \qquad i = 1, \dots, n. \tag{2.6}$$

Proof. Let $\mathbb{R}_+ := [0, \infty)$. By [7, Lemma 3.1], for any $h \in C(\mathbb{R}_+, \mathbb{R})$ and $i = 1, \dots, n$, the equation

$$u_i' + a_i D_{0+}^{\alpha_i} u_i = h(t), \qquad t > 0$$

has a unique solution given by

$$u_i(t) = \int_0^t \Lambda_i(t-s)h(s)ds + u_i(0)\Lambda_i(t).$$

Then BC (1.2) implies

$$u_i(0) = \int_0^1 \frac{b_i\Lambda_i(1-s)}{1-b_i\Lambda_i(1)}h(s)ds.$$

Hence by (2.5) we have

$$\begin{aligned} u_i(t) &= \int_0^t \Lambda_i(t-s)h(s)ds + \left(\int_0^1 \frac{b_i\Lambda_i(1-s)}{1-b_i\Lambda_i(1)}h(s)ds\right)\Lambda_i(t) \\ &= \int_0^1 G_i(t,s)h(s)ds. \end{aligned} \tag{2.7}$$

It is notable that when $\alpha_i \in (0,1)$, we have $\Lambda_i'(t) \le 0$ on $(0,\infty)$, $\Lambda_i(0)=1$, $\lim_{t\to\infty}\Lambda_i(t)=0$, and $0<\Lambda_i(t)<1$; see for example [12]. Then by (2.5), for any $t\in[0,1]$,

$$\frac{\partial G_i}{\partial s} \ge 0 \quad \text{on } (0,t)\cup(t,1).$$

Hence

$$\begin{aligned} \frac{\Lambda_i(t)}{1-b_i\Lambda_i(1)} &\le G_i(t,s) \le \frac{b_i\Lambda_i(t)\Lambda_i(1-t)}{1-b_i\Lambda_i(1)}+1, \quad 0\le s\le t, \\ \frac{b_i\Lambda_i(t)\Lambda_i(1-t)}{1-b_i\Lambda_i(1)} &\le G_i(t,s) \le \frac{b_i\Lambda_i(t)}{1-b_i\Lambda_i(1)}, \quad t<s\le 1. \end{aligned}$$

Therefore, (2.6) holds. □

Remark 2.3. It is clear that G_i defined by (2.5) is discontinuous at $t=s$. However, by (2.7), u_i is continuous on $[0,1]$ when $h\in C[0,1]$, $i=1,\dots,n$.

With Lemma 2.2, we are able to construct a needed operator on an appropriate Banach space. In the sequel, we choose the Banach space $X = \Pi_{i=1}^n C[0,1]$ with the norm $\|u\| = \max_{t\in[0,1]}\sum_{i=1}^n |u_i(t)|$, where $u=(u_1(t),\dots,u_n(t))\in X$. Define an operator $T:\ X\to X$ by

$$(Tu)_i(t) = \int_0^1 G_i(t,s)f_i(u_1(s),\dots,u_n(s),s)ds, \qquad t\in[0,1],\ i=1,\dots,n, \tag{2.8}$$

where G_i is defined by (2.5). By Lemma 2.2, it is easy to see that u is a solution of BVP (1.1), (1.2) if and only if u is a fixed point of T.

Proof of Theorem 2.1. First of all, it is obvious that $(0,\dots,0)$ is not a fixed point of T. By Remark 2.3, we have $T(X)\subset X$. We need to prove that $T: X\to X$ is a compact operator. For any $u,v\in X$, $t\in[0,1]$, and $i=1,\dots,n$,

$$\begin{aligned} |(Tu)_i(t)-(Tv)_i(t)| &= \left|\int_0^1 G_i(t,s)f_i(u_1(s),\dots,u_n(s),s)ds - \int_0^1 G_i(t,s)f_i(v_1(s),\dots,v_n(s),s)ds\right| \\ &\le \overline{G_i}\max_{s\in[0,1]}|f_i(u_1(s),\dots,u_n(s),s)-f_i(v_1(s),\dots,v_n(s),s)|. \end{aligned}$$

Hence T is continuous by the continuity of f_i, $i=1,\dots,n$.

Let $\Omega = \{u \in X : \|u\| \le B\}$. For any $u \in \Omega$, $t \in [0,1]$, and $i = 1, \dots, n$,

$$\begin{aligned}|(Tu)_i(t)| &= \left| \int_0^1 G_i(t,s) f_i(u_1(s), \dots, u_n(s), s) ds \right| \\ &\le \overline{G_i} \max_{v \in \Omega, s \in [0,1]} |f_i(v_1(s), \dots, v_n(s), s)|.\end{aligned}$$

Hence T is uniformly bounded. For any $0 \le t_1 < t_2 \le 1$, by (2.7),

$$\begin{aligned}&|(Tu)_i(t_1) - (Tu)_i(t_2)| \\ &\quad= \left| \int_0^1 G_i(t_1,s) f_i(u_1(s), \dots, u_n(s), s) ds - \int_0^1 G_i(t_2,s) f_i(u_1(s), \dots, u_n(s), s) ds \right| \\ &\quad= \left| \int_0^{t_1} \Lambda_i(t_1 - s) f_i(u_1(s), \dots, u_n(s), s) ds \right. \\ &\qquad+ \left(\int_0^1 \frac{b_i \Lambda_i(1-s)}{1 - b_i \Lambda_i(1)} f_i(u_1(s), \dots, u_n(s), s) ds \right) \Lambda_i(t_1) \\ &\qquad- \int_0^{t_2} \Lambda_i(t_2 - s) f_i(u_1(s), \dots, u_n(s), s) ds \\ &\qquad- \left. \left(\int_0^1 \frac{b_i \Lambda_i(1-s)}{1 - b_i \Lambda_i(1)} f_i(u_1(s), \dots, u_n(s), s) ds \right) \Lambda_i(t_2) \right| \\ &\quad\le \int_0^{t_1} |\Lambda_i(t_1 - s) - \Lambda_i(t_2 - s)| |f_i(u_1(s), \dots, u_n(s), s)| ds \\ &\qquad+ \int_{t_1}^{t_2} |\Lambda_i(t_2 - s)| |f_i(u_1(s), \dots, u_n(s), s)| ds \\ &\qquad+ \int_0^1 \left| \frac{b_i \Lambda_i(1-s)}{1 - b_i \Lambda_i(1)} \right| |f_i(u_1(s), \dots, u_n(s), s)| ds |\Lambda_i(t_1) - \Lambda_i(t_2)| \\ &\quad\le \max_{v \in \Omega, s \in [0,1]} (|f_i(v_1(s), \dots, v_n(s), s)| |\Lambda_i(t_1 - s) - \Lambda_i(t_2 - s)|) \\ &\qquad+ |t_1 - t_2| \max_{v \in \Omega, s \in [0,1]} |f_i(v_1(s), \dots, v_n(s), s)| \\ &\qquad+ |\Lambda_i(t_1) - \Lambda_i(t_2)| \max_{v \in \Omega, s \in [0,1]} \left(\left| \frac{b_i \Lambda_i(1-s)}{1 - b_i \Lambda_i(1)} \right| |f_i(v_1(s), \dots, v_n(s), s)| \right).\end{aligned}$$

Then T is equicontinuous on Ω since Λ_i is uniformly continuous on $[0,1]$. By Arzelà–Ascoli Theorem, we can prove T is a compact operator.

Let K_r be defined by (2.1) and $K \subset X$ be defined by

$$K = \{u \in X : u(t) \in K_r, \ t \in [0,1]\}.$$

It is easy to see that K is a nonempty, closed, bounded, and convex subset of X. We claim that $T(K) \subset K$.

In fact, by (2.8), for any $u \in K$ and $i = 1, \dots, n$,

$$\begin{aligned}|(Tu)_i(t)| &= \left| \int_0^1 G_i(t,s) f_i(u_1(s), \dots, u_n(s), s) ds \right| \\ &\le \int_0^1 \overline{G_i} f_i(u_1(s), \dots, u_n(s), s) ds \le \int_0^1 \overline{G_i} \eta_i ds \le \overline{G_i} \eta_i.\end{aligned}$$

Then we have

$$\sum_{i=1}^n |(Tu)_i(t)| \le \sum_{i=1}^n \overline{G_i} \eta_i \le r.$$

So $\|Tu\| \leq r$. Moreover, it is easy to see that $(Tu)_i(t) \geq 0$ on $[0,1]$, $i = 1,\dots,n$. Hence $T(K) \subset K$.

Therefore by the Schauder Fixed-Point Theorem [17, Theorem 2.A], T has a fixed point $u \in K$. □

Remark 2.4. It is notable that the Banach space $(X, \|\cdot\|)$ and the operator T defined by (2.8) form a general framework to study the existence of solutions for BVP (1.1), (1.2). Other fixed point theorems can also be applied to obtain more results on the existence and/or uniqueness of solution or positive solutions, see for example [4, 9, 15, 17].

3 Resilience of a bike share inventory model

In this section, we consider the resilience of a bike share inventory model involving multiple stations. We first revisit the inventory model proposed in [7]. Let $y_i(t)$ be the inventory at time t at Station i, $i = 1,\dots,n$. Then y_i satisfies

$$y_i' = q_i(t) - \omega_i(t)y_i - \Theta_i(t)c_i^{-\beta_i} D_{0+}^{1-\beta_i}\left(\frac{y_i}{\Theta_i}\right), \quad t > 0,\ i = 1,\dots,n. \tag{3.1}$$

For $i = 1,\dots,n$,

- $q_i(t)$ represents the arrival flux at a station;
- $\omega_i(t)y_i$ represents a Markov removal process that is independent of the history;
- $\beta_i \in (0,1)$ is a parameter relating to the bike waiting time distribution at a station; and
- $\Theta_i(t)c_i^{-\beta_i} D_{0+}^{1-\beta_i}\left(\frac{y_i}{\Theta_i}\right)$ represents a non-Markov removal process that relates to the bike waiting time at a station with

$$\Theta_i(t) = \exp\left(-\int_0^t \omega(s)ds\right).$$

All the terms above are nonnegative. The reader is referred to [7] for the details of the terms.

To reflect the interactions among stations, we will extend Eq. (3.1) by modifying the arrival flux term q_i. Assume the total number of bikes in the entire bike share system is a constant Y. Clearly $\left(Y - \sum_{j=1}^n y_j\right)$ represents the total number of bikes in use at time t. Let $p_i(y_i,t) \in [0,1]$ be the return rate of in-use bikes to Station i at time t with

$$p_i(y_i,t) \geq 0, \qquad \sum_{i=1}^n p_i(y_i,t) \leq 1, \qquad t > 0,\ i = 1,\dots,n. \tag{3.2}$$

Then the inventory y_i satisfies

$$y_i' = \left(Y - \sum_{j=1}^n y_j\right) p_i(y_i,t) - \omega_i(t)y_i - \Theta_i(t)c_i^{-\beta_i} D_{0+}^{1-\beta_i}\left(\frac{y_i}{\Theta_i}\right), \qquad t > 0,\ i = 1,\dots,n. \tag{3.3}$$

If the inventory will restore at some time $\tau_1 > 0$, then y_i must satisfy

$$y_i(0) = y_i(\tau_1), \qquad i = 1,\dots,n. \tag{3.4}$$

Therefore, the resilience problem can be described by BVP (3.3), (3.4).

Remark 3.1. Since Eq. (3.3) models the rate of changes of y_i at Station i, we assume the units of both p_i and ω_i in (3.3) are $1/[\text{unit of time}]$ so that the units on both sides of the equation are consistent.

The following result is obtained by applying Theorem 2.1.

Theorem 3.2. *Let K_r and $\overline{G_i}$ be defined by (2.1) and (2.3), respectively. If for any $x = (x_1, \dots, x_n) \in K_Y$ and $t \in [0,1]$, the return rates p_i, $i = 1, \dots, n$, satisfy*

$$\sum_{i=1}^{n} \overline{G_i} \frac{\tau_1 p_i(\Theta_i(\tau_1 t)x_i, t)}{\Theta_i(\tau_1 t)} \leq 1, \tag{3.5}$$

then BVP (3.3), (3.4) has at least one positive solution y with $\|y\| \leq Y$.

Proof. By an idea similar to [7], i.e., making a change of variables and rescaling $[0, \tau_1]$ to $[0,1]$, BVP (3.3), (3.4) can be converted to BVP (1.1), (1.2) with $u_i(t) = y_i(\tau_1 t)/\Theta_i(\tau_1 t)$, $\alpha_i = 1 - \beta_i$, $a_i = c_i^{-\beta_i}$, $b_i = \Theta_i(\tau_1)$, and

$$f_i(u_1, \dots, u_n, t) = \frac{\tau_1 p_i(\Theta_i(\tau_1 t)u_i, t)}{\Theta_i(\tau_1 t)} \left(Y - \sum_{j=1}^{n} (\Theta_j(\tau_1 t)u_j) \right), \qquad i = 1, \dots, n. \tag{3.6}$$

Let K_Y be defined by (2.1) with $r = Y$. By (3.2) and (3.6), it is easy to see that for any $x \in K_Y$ and $i = 1, \dots, n$, we have $f_i(x,t) \geq 0$ and

$$\begin{aligned} f_i(x_1, \dots, x_n, t) &= \frac{\tau_1 p_i(\Theta_i(\tau_1 t)x_i, t)}{\Theta_i(\tau_1 t)} \left(Y - \sum_{j=1}^{n} (\Theta_j(\tau_1 t)x_j) \right) \\ &\leq \frac{\tau_1 p_i(\Theta_i(\tau_1 t)x_i, t)}{\Theta_i(\tau_1 t)} Y. \end{aligned}$$

Therefore, all the conditions of Theorem 2.1 are satisfied. The conclusion then follows immediately from Theorem 2.1. □

Remark 3.3. Based on our assumption, the return rates p_i, $i = 1, \dots, n$, depend on both time t and current station inventory. Theorem 3.2 shows that it is feasible to manage the station inventory by adjusting the return rates based on real-time status at each station. Therefore, new operational policies may be developed based on Theorem 3.2 by monitoring the return rates so that (3.5) is satisfied all the time.

Acknowledgement

M. Wang's research in this paper is supported by the National Science Foundation under Grant No. 1830489.

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