A Computable Plant-Optimizer Region of Attraction Estimate for Time-distributed Linear Model Predictive Control

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***Abstract*— Time-distributed optimization is a suboptimal implementation strategy for reducing the computational effort required to implement Model Predictive Control (MPC). Time-distributed MPC (TDMPC) methods maintain a running estimate of the solution to an Optimal Control Problem and improve this estimate using a limited number of optimizer iterations during each sampling period. This paper studies closed-loop stability properties of a constrained linear system controlled using TDMPC. A sufficient bound on the number of iterations per sampling period required to enforce asymptotic stability is derived in closed-form with a computable region of attraction estimate in the plant-optimizer space. Conditions under which a user-provided optimizer initialization yields asymptotically stable trajectories are also established. The results of numerical experiments are reported to illustrate theoretical concepts and demonstrate the computation of the plant-optimizer region of attraction estimate.**

I. INTRODUCTION

Model Predictive Control (MPC) is a feedback strategy that computes control inputs by solving an Optimal Control Problem (OCP) over a receding horizon [1]. Implementation of MPC requires the OCP to be solved faster than the sampling period of the controller. However, this may not be possible in applications with limited computing power and/or fast sampling rates. To resolve these issues, a common approach is to approximate the OCP solution by performing a limited number of optimizer iterations, leading to a suboptimal MPC law.

Time-distributed Optimization (TDO) is an approach to suboptimal MPC that maintains a running estimate of the OCP solution and improves this estimate by performing a limited number of optimizer iterations per sampling period. The closed-loop dynamics of a system controlled by Time-distributed MPC (TDMPC) can be described as the interconnection of two dynamical systems representing the plant and optimizer (Figure 1).

Careful consideration must be given to stability when suboptimal MPC is used, since typical guarantees available for optimal MPC control laws do not hold. For example, studies have established conditions under which suboptimal MPC is stabilizing when applied to unconstrained discrete-time [2] and input constrained continuous-time [3] nonlinear systems. The popular Real-Time Iteration (RTI) scheme is studied in [4], which demonstrates that a single Sequential Quadratic Programming (SQP) iteration can be used to drive an unconstrained discrete-time nonlinear system to the origin if the sampling period is sufficiently small. Asymptotic stability of this method is established in [5]. The stability of a generalized TDMPC strategy applied to a discrete-time nonlinear system with state and control constraints is analyzed in [6], which shows that the stability, robustness, and constraint satisfaction properties of the optimal MPC controller can be obtained using a finite number of optimizer iterations. A Lyapunov function for the combined system-optimizer dynamics in a nonlinear model predictive control problem is derived in [7]. For a detailed discussion and comparison of various suboptimal MPC approaches, see [6].

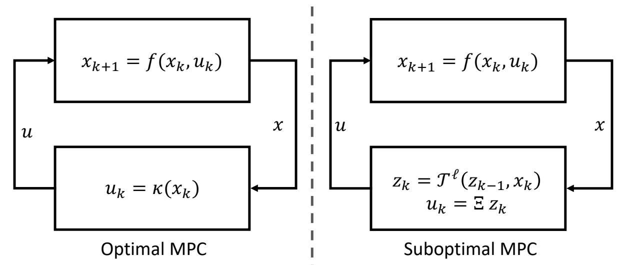

Fig. 1. Optimal MPC can be represented by a static feedback law $\kappa(x)$, while Time-distributed MPC is a dynamic feedback law with an internal state defined by an OCP solution estimate z and dynamics defined by ℓ iterations of an optimization algorithm $\mathcal{T}^\ell$.

Other studies have aimed to provide computable guarantees and complexity certifications for specific TDMPC methods. A bound on the number of optimizer iterations needed to enforce a specified level of suboptimality for input constrained linear MPC is derived in [8]. Similarly, a bound on the number of optimizer iterations needed to stabilize state and input constrained linear MPC controllers is derived in [9]. An invariant set where convergence to the origin is achieved using a single Alternating Direction Method of Multipliers (ADMM) iteration is characterized in [10] for a state and input constrained linear MPC problem. Methods have also been established to compute the exact worst-case bound for the number of active set method [11] and block principal pivoting method [12] iterations needed to solve quadratic programs that arise in linear MPC.

Our recent work [13] analyzed the stability of input constrained linear TDMPC and derived a closed-form expression for the number of iterations needed to satisfy a sufficient condition for asymptotic stability. However, only the existence of a plant-optimizer region of attraction (ROA) is proven in [13] as the analysis relies on input-to-state stability results.

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A framework to determine a plant-optimizer ROA estimate and a Lyapunov function for RTI-type schemes has recently been developed in [7]. However, no methods of explicitly computing the plant-optimizer ROA estimate are provided in [7], since this study addresses a general nonlinear setting.

In this paper, we extend upon the work of [7] and [13] by analyzing the stability of a state and input constrained linear system controlled using TDMPC based on a projected gradient optimization method (PGM). By considering this specific case, a sufficient bound on the number of iterations per sampling period required to enforce asymptotic stability is derived in closed-form with a computable plant-optimizer ROA estimate. These results extend the work in [13], which analyzes the stability of a similar TDMPC approach but does not provide a ROA estimate; and the work in [7], which derives a plant-optimizer ROA estimate but does not explore methods to compute it. Additionally, [7] enforces stability through a maximum sampling period rather than a minimum number of iterations. Practical methods of inner approximating the derived ROA estimate are also discussed. Additionally, conditions under which a user-provided optimizer initialization will yield asymptotically stable trajectories are provided. The results of numerical experiments are reported to illustrate theoretical concepts and demonstrate the computation of the plant-optimizer ROA estimate.

Notation: Let $\mathbb{S}^n_{++}$ and (respectively $\mathbb{S}^n_{+}$) denote the set of symmetric $n \times n$ positive definite (respectively semidefinite) matrices. Let $\mathbb{R}^{n\times m}_{>0}$, $\mathbb{R}^n_{>0}$, and $\mathbb{R}_{>0}$ denote the set of real $n \times m$ matrices, $n \times 1$ vectors, and scalars with strictly positive elements (define $\mathbb{R}^{n\times m}_{\geq 0}$, $\mathbb{R}^n_{\geq 0}$, and $\mathbb{R}_{\geq 0}$ in the corresponding manner). Given $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, then $(x, y) = [x^T \; y^T]^T$. Given $x \in \mathbb{R}^n$ and $W \in \mathbb{S}^n_{+}$, the W-norm of x is $\|x\|_W = \sqrt{x^T W x}$. Given $M \in \mathbb{S}^n_{++}$ and $W \in \mathbb{S}^n_{++}$, then $\lambda^-_W(M)$ and $\lambda^+_W(M)$ denote the minimum and maximum eigenvalues of $\sqrt{W}^{-1} M \sqrt{W}^{-1}$, which satisfy $\lambda^-_W(M)\|x\|^2_W \leq \|x\|^2_M \leq \lambda^+_W(M)\|x\|^2_W$. If a subscript is omitted in any of these cases, then the corresponding matrix is understood to be the identity matrix of appropriate size. Given $\alpha : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, then a sublevel set of α of radius $r > 0$ is represented by $\mathcal{B}_\alpha(r) = \{x \in \mathbb{R}^n \mid \alpha(x) \leq r\}$. Using a slight overloading of notation, if $P \in \mathbb{S}^n_{++}$ then we use $\mathcal{B}_P(r) = \{x \in \mathbb{R}^n \mid \|x\|_P \leq r\}$ to represent a P-norm ball of radius $r > 0$.

II. PROBLEM SETTING

Consider the Linear Time Invariant (LTI) system,

$$x_{k+1} = f(x_k, u_k) = Ax_k + Bu_k, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times m}$. The control objective is to stabilize (1) at the origin subject to constraints

$$x_k \in \mathcal{X}, \; u_k \in \mathcal{U}, \quad \forall k > 0, \tag{2}$$

where $\mathcal{X} \subseteq \mathbb{R}^n$, $\mathcal{U} \subseteq \mathbb{R}^m$ are constraint sets.

We approach this problem using MPC. Consider the following Parametric Optimal Control Problem (POCP)

$$\min_{\xi, z} \quad \|\xi_N\|^2_P + \sum_{i=0}^{N-1} \|\xi_i\|^2_Q + \|\mu_i\|^2_R \tag{3a}$$

$$\text{s.t.} \quad \xi_{i+1} = A\xi_i + B\mu_i, \qquad i = 0, \ldots, N-1, \tag{3b}$$

$$\xi_0 = x, \tag{3c}$$

$$\mu_i \in \mathcal{U}, \qquad i = 0, \ldots, N-1. \tag{3d}$$

where $Q \in \mathbb{R}^{n\times n}, R \in \mathbb{R}^{m\times m}$, and $P \in \mathbb{R}^{n\times n}$ are weighting matrices, $N \in \mathbb{N}$ is the horizon length, $x \in \mathbb{R}^n$ is the measurement (and the parameter), and $\xi = (\xi_0, \ldots, \xi_{N-1})$, $z = (\mu_0, \ldots, \mu_{N-1})$ are the predicted state and control sequences[1].

Remark 1: The OCP in (3) does not directly enforce the state constraint $\xi_k \in \mathcal{X}$. However, in Section III-C it is shown that the state constraint can be implicitly satisfied by deriving an invariant set under the input constrained MPC law generated by (3).

The following assumptions ensure that (3) is well posed and produces a stabilizing MPC feedback law.

Assumption 1: The pair (A, B) is stabilizable, $Q \in \mathbb{S}^n_{++}$, $R \in \mathbb{S}^m_{++}$, and $P \in \mathbb{S}^n_{++}$ satisfies the discrete algebraic Riccati equation (DARE)

$$P = Q + A^T PA - (A^T PB)K,$$

where $K = (R + B^T PB)^{-1}(B^T PA)$ is the linear quadratic regulator (LQR) gain.

Assumption 2: The input constraint set $\mathcal{U}$ is closed, convex, and contains the origin in its interior. The state constraint set $\mathcal{X}$ contains the origin in its interior.

The OCP can be rewritten by algebraically eliminating the state sequence to obtain the condensed form

$$\min_{z \in \mathcal{Z}} \quad J(x, z) = \|(x, z)\|^2_M, \tag{4}$$

where $\mathcal{Z} = \mathcal{U}^N \subseteq \mathbb{R}^{n_z}$ and

$$M = \begin{bmatrix} W & G^T \\ G & H \end{bmatrix}. \tag{5}$$

Expressions for the matrices $H \in \mathbb{S}^{n_z}_{++}$, $W \in \mathbb{S}^n_{++}$, and $G \in \mathbb{R}^{n_z \times n}$ in terms of the data of (3) can be found in [13]. Moreover, note that $W \succeq P \succ 0$ [13, Lemma 1] and $M \succ 0$.

Since $H \succ 0$, the variational inequality

$$Hz + Gx + \mathcal{N}_\mathcal{Z}(z) \ni 0, \tag{6}$$

is necessary and sufficient for optimality of (4), where

$$\mathcal{N}_\mathcal{Z}(z) = \begin{cases} \{y \mid y^T(w - z) \leq 0, \forall w \in \mathcal{Z}\}, & \text{if } z \in \mathcal{Z}, \\ \emptyset & \text{else.} \end{cases}$$

is the normal cone mapping of $\mathcal{Z}$. The solution mapping is

$$S(x) = (H + \mathcal{N}_\mathcal{Z})^{-1}(-Gx), \tag{7}$$

[1] We define $n_z = Nm$ such that $z \in \mathbb{R}^{n_z}$.

where S is a function due to the strong convexity of (4). The optimal MPC feedback policy is then

$$u = \kappa(x) = \Xi S(x), \tag{8}$$

where $\Xi = [I_{m\times m}\ 0\ \dots\ 0] \in \mathbb{R}^{m\times n_z}$ selects the first control input from $S(x)$.

Due to computational limitations, it is often not possible to evaluate $\kappa(x)$ at each sampling instant. Thus, we consider a suboptimal approximation of the optimal control policy, where we introduce a running solution estimate[2] z_k and improve it at each sampling instant using a fixed number of iterations of an optimization algorithm.

If the output of $\ell \in \mathbb{N}$ iterations of an optimization algorithm is denoted by the function $\mathcal{T}^\ell : \mathbb{R}^n \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_z}$, then the closed-loop dynamics under this suboptimal MPC policy can be expressed as

$$z_k = \mathcal{T}^\ell(x_k, z_{k-1}), \tag{9a}$$

$$x_{k+1} = Ax_k + \bar{B}z_k, \tag{9b}$$

where $\bar{B} = B\Xi$. In this paper, we define $\mathcal{T}^\ell$ using the projected gradient method, i.e.,

$$\mathcal{T}(x,z) = \Pi_{\mathcal{Z}}\left[z - \alpha \nabla_z J(x,z)\right], \tag{10a}$$

$$\mathcal{T}^\ell(x,z) = \mathcal{T}(x, \mathcal{T}^{\ell-1}(x,z)), \tag{10b}$$

where $\mathcal{T}^0(x,z) = z$, and $\alpha = 2/(\lambda^+(H) + \lambda^-(H))$ is the optimal step length. Note that the projection operator $\Pi_{\mathcal{Z}}$ can be defined in closed-form if $\mathcal{U}$ is a box.

The objective of this paper is to derive numerically verifiable conditions on ℓ under which (9) is asymptotically stable with a computable ROA estimate in (x,z)-space that satisfies the constraints (2).

III. PROPERTIES OF THE INDIVIDUAL SUBSYSTEMS

The first step in studying (9) is to study the properties of (9a) and (9b) separately. We begin by detailing some properties of the solution mapping and value function.

A. Properties of the Solution Mapping

The solution mapping can be rewritten as

$$S(x) = \mathcal{A}^{-1}(-Gx), \quad \mathcal{A} = H + \mathcal{N}_{\mathcal{Z}}, \tag{11}$$

where the operator $\mathcal{A} : \mathbb{R}^{n_z} \rightrightarrows \mathbb{R}^{n_z}$ has the following properties.

Proposition 1: ([13, Proposition 1]) Given Assumptions 1-2, $\mathcal{A}$ has the following properties:

1) Strong monotonicity: $\langle u - v, y - z\rangle \geq \|y - z\|_H^2, \quad u \in A(y), v \in A(z)$;
2) Co-coercivity of $\mathcal{A}^{-1}$: $\left\langle \mathcal{A}^{-1}u - \mathcal{A}^{-1}v, u - v\right\rangle \geq ||\mathcal{A}^{-1}u - \mathcal{A}^{-1}v||_H^2$;
3) Lipschitz continuity of $\mathcal{A}^{-1}$: $\|\mathcal{A}^{-1}u - \mathcal{A}^{-1}v\|_H \leq ||u - v||_{H^{-1}}$.

[2] See [14] for a survey on running methods for time-varying convex optimization with exogenous parameter updates.

Corollary 1: ([13, Corollary 2]) Let Assumptions 1-2 hold, then $\forall x, y \in \mathbb{R}^n$, the solution mapping satisfies

$$\langle S(x) - S(y), G(x-y)\rangle \leq -\|S(x) - S(y)\|_H^2, \tag{12}$$

and $\|S(x) - S(y)\|_H \leq \|G(x-y)\|_{H^{-1}}$.

B. Properties of the Value Function

The value function

$$V(x) = \min_{z\in\mathcal{Z}}\ J(x,z) = \|(x, S(x))\|_M^2, \tag{13}$$

serves as a Lyapunov function for the closed-loop system under the optimal MPC feedback policy [15], [16].

Lemma 1: ([13, Lemma 2]) Let Assumptions 1-2 hold, then $\forall x \in \mathbb{R}^n$ the value function satisfies

$$\|x\|_P^2 \leq V(x) \leq \|x\|_W^2, \tag{14}$$

hi and $V(x) \leq \|x\|_W^2 - \|S(x)\|_H^2$.

We can also combine the upper and lower bounds on V to bound the solution mapping in terms of the value function.

$$\|S(x)\|_H^2 \leq (\lambda_P^+(W) - 1)V(x). \tag{15}$$

Note that $W \succeq P \implies \lambda_P^+(W) \geq 1$.

The square root of the value function

$$\psi(x) = \sqrt{V(x)}, \tag{16}$$

is also vital for the analysis of the plant system since it is Lipschitz continuous.

Lemma 2: ([13, Lemma 3]) Let Assumptions 1-2 hold then

$$|\psi(x) - \psi(y)| \leq \|x - y\|_W, \quad \forall x, y \in \mathbb{R}^n. \tag{17}$$

C. Plant Subsystem

Given the properties of S, V, and ψ, it is possible to upper bound the evolution of $\psi(x)$ under TDMPC. For any $x \in \mathbb{R}^n$ consider the following ideal and disturbed one step updates,

$$x_+^* = Ax + \bar{B}S(x), \tag{18}$$

$$x_+ = Ax + \bar{B}z = Ax + \bar{B}S(x) + \bar{B}e. \tag{19}$$

where the disturbance term $e = z - S(x)$ represents suboptimality due to incomplete optimization.

To begin, we derive a forward invariant ROA estimate $\Gamma_N \subset \mathcal{X}$ for the optimal closed-loop system (18). To do so, we first derive a ROA estimate $\Gamma_N' \subset \mathbb{R}^n$ of (18) that is not necessarily contained in $\mathcal{X}$. Then, we define Γ_N as a forward invariant set contained in $\Gamma_N' \cap \mathcal{X}$.

Consider the set

$$\Omega = \left\{x \in \mathbb{R}^n \mid \|x\|_P^2 \leq c\right\} \tag{20}$$

where $c > 0$ is chosen to satisfy $\Omega \subset \{x \mid -Kx \in \mathcal{U}\}$. Note that if a terminal constraint of $\xi_N \in \Omega$ was added to the OCP (3), then the N-step backwards reachable set to Ω would be a ROA estimate of (18) [1], [15]. However, the addition of state constraints into the OCP would necessitate the use of a more sophisticated optimization strategy, thereby complicating the analysis of the optimizer subsystem. Thus, we instead use the results of [16] to determine a ROA estimate for (18) under the terminal constraint free MPC law generated by (3).

Theorem 1: ([16, Theorem 1]) Let Assumptions 1-2 hold. Then, $\forall N \in \mathbb{N}$ the optimal closed-loop system (18) is asymptotically stable with a forward invariant region of attraction estimate

$$\Gamma'_N = \{x \in \mathbb{R}^n \mid V(x) \le Nd + c\}, \tag{21}$$

where N is the horizon length, $d = c \cdot \lambda^-(Q)/\lambda^+(P)$, and c is defined in (20).

Proof: The proof follows by showing that Ω and d satisfy [16, Assumption 1-2] and thus [16, Theorem 1] holds. To follow the notation of [16], denote the terminal and stage costs as $F(x) = \|x\|_P^2$ and $l(x,u) = \|x\|_Q^2 + \|u\|_R^2$ respectively.

First, we show that Ω satisfies [16, Assumption 1], that is

$$\min_{u \in \mathcal{U}} [F(Ax+Bu) - F(x) + l(x,u)] \le 0, \ \forall x \in \Omega. \tag{22}$$

Standard stability analysis procedures (see [1], [15]) can be used to show that (22) holds since $-Kx \in \mathcal{U}$ for $x \in \Omega$.

Next, we show that d satisfies [16, Assumption 2], that is

$$\ell(x,u) > d, \ \forall x \notin \Omega, \ \forall u \in \mathcal{U}. \tag{23}$$

Note that $l(x,u) \ge \|x\|_Q^2 \ge \frac{d}{c}\|x\|_P^2, \ \forall x \in \mathbb{R}^n, u \in \mathbb{R}^m$ and that $x \notin \Omega \iff \|x\|_P^2 > c$. So, we have that $\ell(x,u) > d, \forall x \notin \Omega, \forall u \in \mathbb{R}^m$.

Therefore [16, Assumptions 1-2] are satisfied, so Γ'_N is a ROA estimate for (18) by [16, Theorem 1]. ■

Remark 2: The ROA estimate in Theorem 1 can be expanded by weighting the terminal cost [16]. However, we do not explore this strategy in this paper.

Corollary 2: For any $x \in \Gamma'_N$,

$$V(x_+^*) \le \beta^2 V(x) \tag{24}$$

where x_+^* is defined in (18) and $\beta^2 = (1 - \lambda_W^-(Q)) \in (0,1)$.

Proof: As stated in [16, Theorem 1], the standard value function decrease condition

$$V(x_+^*) - V(x) \le -\|x\|_Q^2, \tag{25}$$

holds $\forall x \in \Gamma'_N$. Rearranging this expression and combining it with Lemma 1 gives

$$V(x_+^*) - V(x) \le -\lambda_W^-(Q)\|x\|_W^2 \le -\lambda_W^-(Q)V(x) \tag{26}$$

and thus $V(x_+^*) \le (1 - \lambda_W^-(Q))V(x) = \beta^2 V(x)$. Noting that $W \succeq P \succ Q \succ 0 \implies \lambda_W^-(Q) \in (0,1)$ completes the proof. ■

As a consequence of Corollary 2, any sublevel set of V contained in Γ'_N is forward invariant. Thus, an invariant ROA $\Gamma_N \subset \mathcal{X}$ can be defined by selecting the largest sublevel set of V that is contained in $\Gamma'_N \cap \mathcal{X}$.

Theorem 2: Let Assumptions 1-2 hold. Then, $\forall N \in \mathbb{N}$ the closed-loop optimal MPC system (18) is asymptotically stable with a forward invariant region of attraction estimate

$$\Gamma_N = \{x \in \mathbb{R}^n \mid \psi(x) \le r_\psi\} \subset \Gamma_N \cap \mathcal{X}, \tag{27}$$

where $r_\psi = \min(\sqrt{Nd+c}, c_{\mathcal{X}})$, c, d are defined in Theorem 1, and $c_{\mathcal{X}} > 0$ satisfies $\mathcal{B}_\psi(c_{\mathcal{X}}) \in \mathcal{X}$.

Proof: The result follows from Theorem 1 and Corollary 2. ■

Next, we consider how the growth of $\psi(x)$ is bounded under the arbitrarily suboptimal update in (19).

Lemma 3: Given Assumptions 1-2, $\forall x \in \Gamma_N$, $z \in \mathcal{Z}$,

$$\psi(x_+) \le \beta\psi(x) + \mu\|e\|, \tag{28}$$

where x_+ is defined in (19), $e = z - S(x)$, and $\mu = \|W^{\frac{1}{2}}\bar{B}\|$.

Proof: By Lipschitz continuity of ψ (Lemma 2)

$$|\psi(x_+^*) - \psi(x_+)| \le \|x_+^* - x_+\|_W = \|\bar{B}e\|_W, \tag{29}$$

and thus using (24)

$$\begin{aligned}\psi(x_+) &\le \psi(x_+^*) + |\psi(x_+^*) - \psi(x_+)| \\ &\le \beta\psi(x) + \|\bar{B}e\|_W \\ &\le \beta\psi(x) + \|W^{\frac{1}{2}}\bar{B}\|\|e\|\end{aligned}$$

as claimed. ■

D. Optimizer Subsystem

The analysis of the optimizer subsystem (9a) is based on the merit function

$$\phi(x,z) = \|z - S(x)\|. \tag{30}$$

The following Lemma summarizes the convergence properties of the projected gradient method (PGM).

Lemma 4: ([17, Theorem 3.1]) Let $\mathcal{T}^\ell(x,z)$ represent ℓ iterations of PGM and suppose Assumptions 1-2 hold. Then, for all $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^{n_z}$

$$\phi(x, \mathcal{T}^\ell(x,z)) \le \eta^\ell \phi(x,z), \tag{31}$$

where $\eta = \frac{\lambda^+(H) - \lambda^-(H)}{\lambda^+(H) + \lambda^-(H)} \in (0,1)$.

Similar to the previous section, we now consider how the evolution of $\phi(x,z)$ is influenced by perturbations in x.

Lemma 5: Suppose Assumptions 1-2 hold. Then, for all $x, x' \in \mathbb{R}^n$, $z \in \mathcal{Z}$ and $z' = \mathcal{T}^\ell(x', z)$,

$$\phi(x', z') \le \eta^\ell \phi(x,z) + \eta^\ell b\|G(x'-x)\|_{H^{-1}} \tag{32}$$

where $b = \|H^{-\frac{1}{2}}\|$.

Proof: By virtue of Lemma 4, we obtain

$$\begin{aligned}\phi(x', z') &\le \eta^\ell \phi(x', z) \\ &= \eta^\ell \|z - S(x') + S(x) - S(x)\| \\ &\le \eta^\ell \phi(x,z) + \eta^\ell \|H^{-\frac{1}{2}}(S(x') - S(x))\|_H \\ &\le \eta^\ell \phi(x,z) + \eta^\ell \|H^{-\frac{1}{2}}\|\|G(x'-x)\|_{H^{-1}}\end{aligned}$$

where the last line is by Corollary 1. ■

Next we specialize this result based on the plant dynamics.

Lemma 6: Given Assumptions 1-2 consider any $x \in \mathbb{R}^n$, $z \in \mathcal{Z}$, and let $x_+ = Ax + \bar{B}z$ and $z_+ = \mathcal{T}^\ell(x_+, z)$. Then

$$\phi(x_+, z_+) \le \eta^\ell \omega \phi(x,z) + \eta^\ell \rho \psi(x), \tag{33}$$

where $\omega = 1 + \|H^{-\frac{1}{2}}\|\|H^{-\frac{1}{2}}G\bar{B}\|$, and

$$\rho = \|H^{-\frac{1}{2}}\|\|H^{-\frac{1}{2}}G(A-I)P^{-\frac{1}{2}}\| + \|H^{-\frac{1}{2}}\|\sqrt{\lambda_H^+(G\bar{B})(\lambda_P^+(W)-1)}.$$

Proof: Specializing Lemma 5 to the case where $x' = x^+$, and $z' = z_+$ we obtain

$$\phi(x_+, z_+) \le \eta^\ell \phi(x,z) + \eta^\ell b\|G(x_+ - x)\|_{H^{-1}}.$$

The goal is to obtain a bound in terms of $\psi(x)$ and $\phi(x,z)$. Note that $x_+ - x = (A-I)x + \bar{B}S(x) + \bar{B}e$ and thus

$$\|G(x_+ - x)\|_{H^{-1}} \le \|G(A-I)x\|_{H^{-1}} + \|G\bar{B}S(x)\|_{H^{-1}} + \|G\bar{B}e\|_{H^{-1}}. \quad (34)$$

Proceeding term by term, the first term can be bounded as

$$\begin{aligned}\|G(A-I)x\|_{H^{-1}} &= \|H^{-\frac{1}{2}}G(A-I)P^{-\frac{1}{2}}P^{\frac{1}{2}}x\| \\ &\le \|H^{-\frac{1}{2}}G(A-I)P^{-\frac{1}{2}}\|\|x_k\|_P \\ &\le \|H^{-\frac{1}{2}}G(A-I)P^{-\frac{1}{2}}\|\psi(x)\end{aligned}$$

The second terms can be bounded as

$$\begin{aligned}\|G\bar{B}S(x)\|_{H^{-1}} &= \|H^{-\frac{1}{2}}G\bar{B}S(x)\| \\ &\le \sqrt{\lambda_H^+(G\bar{B})}\ \|S(x)\|_H \\ &\le \sqrt{\lambda_H^+(G\bar{B})(\lambda_P^+(W)-1)}\ \psi(x_k)\end{aligned}$$

where the last step uses (15). Finally,

$$\begin{aligned}\|G\bar{B}e\|_{H^{-1}} &= \|H^{-\frac{1}{2}}G\bar{B}e\| \\ &\le \|H^{-\frac{1}{2}}G\bar{B}\|\phi(x_k, z_k).\end{aligned}$$

Combining and simplifying, we obtain that

$$\phi(x_+, z_+) \le \eta^\ell \omega\phi(x,z) + \eta^\ell \rho\psi(x)$$

where ω and ρ are as defined. ∎

IV. STABILITY OF THE SYSTEM-OPTIMIZER DYNAMICS

In the previous section, we established bounds on the change in $\psi(x)$ and $\phi(x,z)$ subject to arbitrary perturbations in z and x respectively. We now combine these results to analyze the dynamics in (9).

Lemma 7: Consider the closed-loop system in (9) and let Assumptions 1-2 hold. Then, for all $x_k \in \Gamma_N$ and $z_k \in \mathcal{Z}$,

$$\psi(x_{k+1}) \le \beta\psi(x_k) + \mu\phi(x_k, z_k) \quad (35a)$$

$$\phi(x_{k+1}, z_{k+1}) \le \eta^\ell\rho\psi(x_k) + \eta^\ell\omega\phi(x_k, z_k) \quad (35b)$$

Proof: The result follows from Lemmas 3 and 6. ∎

Note that (x_{k+1}, z_{k+1}) must remain in $\Gamma_N \times \mathcal{Z}$ to guarantee that Lemma 7 holds at the following time step. Thus, the following Lemma derives an invariant subset of $\Gamma_N \times \mathcal{Z}$.

Lemma 8: Given Assumptions 1-2, if the number of iterations performed satisfies $\ell > \ell^*$, where

$$\ell^* = \frac{\log(1-\beta) - \log(\mu\rho + \omega(1-\beta))}{\log(\eta)},$$

then the set

$$\Sigma = \{(x,z) \in \Gamma_N \times \mathcal{Z} \mid \psi(x) \le r_\psi,\ \phi(x,z) \le r_\phi\}, \quad (36)$$

is forward invariant under the closed-loop dynamics (9), where r_ψ is defined in Theorem 2 and $r_\phi = (1-\beta)r_\psi/\mu$.

Proof: Consider any $(x_k, z_k) \in \Sigma$. First, we show that $\psi(x_{k+1}) \le r_\psi$ by using Lemma 7 to write

$$\begin{aligned}\psi(x_{k+1}) &\le \beta\psi(x_k) + \mu\phi(x_k, z_k) \\ &\le \beta r_\psi + \mu r_\phi \\ &\le r_\psi\end{aligned}$$

where the last inequality follows from the definition of r_ϕ. Next, we use Lemma 7 to write

$$\begin{aligned}\phi(x_{k+1}, z_{k+1}) &\le \eta^\ell\rho\psi(x_k) + \eta^\ell\omega\phi(x_k, z_k) \\ &\le \eta^\ell\left(\rho r_\psi + \frac{\omega(1-\beta)r_\psi}{\mu}\right) \\ &\le r_\phi\end{aligned}$$

where the last inequality follows from the restriction $\ell > \ell^*$. So we have that $\psi(x_{k+1}) \le r_\psi$ and $\phi(x_{k+1}, z_{k+1}) \le r_\phi$, thus $(x_{k+1}, z_{k+1}) \in \Sigma$. ∎

Since $\Sigma \subset \Gamma_N \times \mathcal{Z}$ is forward invariant, then $(x_0, z_0) \in \Sigma$ implies that the constraints (2) and the inequalities (35) and are satisfied $\forall k \ge 0$. Following the methodology in [7], asymptotic stability of (9) can then be ensured by enforcing that the following *auxiliary system* is asymptotically stable, where

$$\nu_{k+1} = \beta\nu_k + \mu\epsilon_k, \quad (37a)$$

$$\epsilon_{k+1} = \eta^\ell\rho\nu_k + \eta^\ell\omega\epsilon_k. \quad (37b)$$

Note that if $\nu_k = \psi(x_k)$ and $\epsilon_k = \phi(x_k, z_k)$ then $\psi(x_{k+1}) \le \nu_{k+1}$ and $\phi(x_{k+1}, z_{k+1}) \le \epsilon_{k+1}$. Moreover, the auxiliary system can be expressed as a positive LTI system

$$w_{k+1} = A_a w_k, \quad A_a = \begin{bmatrix}\beta & \mu \\ \eta^\ell\rho & \eta^\ell\omega\end{bmatrix}, \quad (38)$$

where $w_k = (\nu_k, \epsilon_k) \in \mathbb{R}^2_{\ge 0}$ and $A_a \in \mathbb{R}^{2\times 2}_{>0}$. The following result pertains to the stability of positive LTI systems.

Theorem 3: ([7, Theorem 27], [18]), A positive discrete-time LTI system

$$w_{k+1} = Aw_k, \quad (39)$$

where $A \in \mathbb{R}^{n_a\times n_a}_{\ge 0}$ and $w \in \mathbb{R}^{n_a}_{\ge 0}$ is asymptotically stable if there exists a strictly positive vector $\hat{w} \in \mathbb{R}^{n_a}_{>0}$ and a strictly positive constant $\hat{d} > 0$ such that

$$\max_{i=1,\dots,n_a} \left[(A^T - I)\hat{w}\right]_i \le -\hat{d}. \quad (40)$$

Moreover, if these conditions are satisfied then $V_l(w_k) = \hat{w}_k{}^T w_k$ is a Lyapunov function for (39) in $\mathbb{R}^{n_a}_{\ge 0}$.

Thus, we adapt the proof in [7, Theorem 28] to prove that the auxiliary dynamics (37) are asymptotically stable subject to a minimum number of iterations.

Theorem 4: The auxiliary dynamics (37) are asymptotically stable if $\ell > \ell^*$. Moreover, $V_l(w) = \hat{w}^T w$ is a

Lyapunov function for (37) in $\mathbb{R}^2_{\geq 0}$, where $\hat{w} = (1, \zeta)$ and $\zeta > 0$ satisfies

$$\frac{\mu}{1-\eta^\ell \omega} < \zeta < \frac{1-\beta}{\eta^\ell \rho}. \tag{41}$$

Proof: To prove the auxiliary dynamics are asymptotically stable, we want to show that $\hat{w} = (1, \zeta)$ satisfies the conditions in Theorem 3 for the system in (37). For this to be true, the following set of inequalities must hold

$$(\beta - 1) + \zeta \eta^\ell \rho < 0, \quad \mu + (\eta^\ell \omega - 1)\zeta < 0, \quad \zeta > 0.$$

This set of inequalities is satisfied if (41) holds. To prove ζ can be chosen to satisfy (41), note that that the left and right sides of (41) are monotonically decreasing and increasing in ℓ, so we can find a bound on ℓ such that

$$\frac{\mu}{1-\eta^\ell \omega} < \frac{1-\beta}{\eta^\ell \rho}.$$

Rearranging this inequality yields the same requirement derived in Lemma 8 that is satisfied for $\ell > \ell^*$. Additionally, note that $\mu/(1-\eta^\ell\omega) > 0$ if $\ell > \ell^*$. Thus, $\zeta > 0$ can be chosen to satisfy (41) and Theorem 3 is satisfied for $\hat{w} = (1, \zeta)$. ■

Having now established conditions for the stability of the auxiliary system, we invoke [7, Theorem 29] to prove that the associated closed-loop dynamics in (9) are asymptotically stable with a known ROA estimate and Lyapunov function.

Theorem 5: Let Assumptions 1-2 hold and $\ell > \ell^*$, then the origin of the plant-optimizer system (9) is asymptotically stable with a forward invariant region of attraction estimate Σ. Moreover, the function

$$\mathcal{V}(x, z) = \psi(x) + \zeta \phi(x, z), \tag{42}$$

is a Lyapunov function in Σ, where Σ is defined in (36) and $\zeta > 0$ is defined in Theorem 4.

Proof: The proof follows directly from [7, Theorem 29] and Theorem 4. Specifically, the proof of [7, Theorem 29] can be used to show that $\exists \alpha_1, \alpha_2 \in \mathcal{K}_\infty$[3] such that $\alpha_1(\|(x,z)\|) \leq \mathcal{V}(x,z) \leq \alpha_2(\|(x,z)\|)$, and that $\exists \alpha_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\mathcal{V}(x_+, z_+) - \mathcal{V}(x, z) \leq -\alpha_3(\|(x,z)\|)$. ■

V. Inner Approximation of the Region of Attraction Estimate

In the previous section it was shown that Σ is a ROA estimate of (9) if $\ell > \ell^*$. But in the context of TDMPC, one cannot directly check if $(x, z) \in \Sigma$ is satisfied since $\psi(x)$ and $\phi(x, z)$ cannot be computed in real-time. Thus, a set of real-time verifiable sufficient conditions satisfying $(x, z) \in \Sigma$ must be established.

[3]The set of $\mathcal{K}_\infty$ functions follows the standard definition in [19].

A. Inner approximation of Γ_N

As detailed in Theorem 2, a constant $c_\mathcal{X} > 0$ satisfying $\mathcal{B}_\psi(c_\mathcal{X}) \subset \mathcal{X}$ must be computed to approximate Γ_N. If $\mathcal{X}$ is a polytope, then the procedure in [20, Proposition S1] can be used to compute the largest $r > 0$ satisfying $\mathcal{B}_P(r) \subset \mathcal{X}$. Then, since $\mathcal{B}_\psi(r) \subset \mathcal{B}_P(r)\ \forall r > 0$, we have that $c_\mathcal{X} = r$ satisfies $\mathcal{B}_\psi(c_\mathcal{X}) \subset \mathcal{X}$.

Once an estimate of r_ψ has been obtained, one can collect a set sample points $\mathcal{S} = \{x_i \mid \psi(x_i) \leq r_\psi,\ i = 1, \ldots, q\}$ using offline computations. Then, the convex hull $\mathcal{H}_N = \mathrm{Conv}(\mathcal{S})$ satisfies $\mathcal{H}_N \subset \Gamma_N$ since Γ_N is convex under Assumption 2 [21, Lemma 5.6.2]. Furthermore, the inclusion $x \in \mathcal{H}_N$ can be easily verified online since $\mathcal{H}_N$ is a polytope. In cases where it may be difficult to compute such sample points offline, one can instead use the sublevel set $\mathcal{B}_W(r_\psi)$ to approximate Γ_N, since $\mathcal{B}_W(r) \subset \mathcal{B}_\psi(r)\ \forall r > 0$. However, this is likely to be a conservative approximation.

B. Upper bound of $\phi(x, z)$

An estimate of r_ϕ can be computed using the formula in Lemma 8 once an estimate of r_ψ is available. Then, one can upper bound $\phi(x, z)$ by noting that

$$\begin{aligned}\phi(x,z) &\leq (\lambda^-(H))^{-\frac{1}{2}} \|z - S(x)\|_H \\ &\leq (\lambda^-(H))^{-\frac{1}{2}} \sqrt{J(x,z) - V(x)} \\ &\leq (\lambda^-(H))^{-\frac{1}{2}} \sqrt{J(x,z) - \|x\|_P^2}\end{aligned} \tag{43}$$

where the second line follows directly from the strong monotonicity of J and optimality of S and the third line follows from the lower bound of V. Then, it is sufficient to check that the right-side of (43) is less than r_ϕ in order to verify that $\phi(x, z) \leq r_\phi$.

VI. Requirements on the Optimizer Initialization

As discussed, the restriction $(x_0, z_0) \in \Sigma$ is sufficient to guarantee asymptotic convergence. However, the initial optimizer state z_0 is in fact the output of the optimizer given some user-provided optimizer initialization $\bar{z}$, i.e. $z_0 = \mathcal{T}^\ell(x_0, \bar{z})$. Thus, in this section we develop guidelines for how $\bar{z}$ may be selected to ensure that $(x_0, z_0) \in \Sigma$.

The following proposition shows that if $\bar{z}$ is chosen to be the optimal solution at some point $\bar{x} \in \mathbb{R}^n$, then there exist a region around $\bar{x}$ where $\bar{z} = S(\bar{x}) \implies (x_0, z_0) \in \Sigma$.

Proposition 2: Let Assumptions 1-2 hold, $\bar{x} \in \mathbb{R}^n$, $\bar{z} = S(\bar{x})$, and $z_0 = \mathcal{T}^\ell(x_0, \bar{z})$. Then, $(x_0, z_0) \in \Sigma$ for all $x_0 \in \mathcal{D}^\ell(\bar{x}) \cap \Gamma_N$, where

$$\mathcal{D}^\ell(\bar{x}) = \{x \in \mathbb{R}^n \mid \|G(x - \bar{x})\|_{H^{-1}} \leq r_\mathcal{D}(\ell)\}, \tag{44}$$

and $r_\mathcal{D}(\ell) = \lambda^-(H)^{\frac{1}{2}} r_\phi / \eta^\ell$.

Proof: Noting that $z_0 = \mathcal{T}^\ell(x_0, S(\bar{x}))$, we use Lemma 4 to write

$$\begin{aligned}\phi(x_0, z_0) &\leq \eta^\ell \|S(\bar{x}) - S(x_0)\| \\ &\leq \frac{\eta^\ell}{\sqrt{\lambda^-(H)}} \|G(\bar{x} - x_0)\|_{H^{-1}} \leq r_\phi\end{aligned}$$

where the second inequality follows from Corollary 1. Thus, the restriction $x_0 \in \mathcal{D}^\ell(\bar{x})$ implies that $\phi(x_0, z_0)$ and thus $(x_0, z_0) \in \Sigma$. ■

Remark 3: By using Proposition 2, it is possible to provide conditions under which a valid optimizer initialization is available for any $x \in \Gamma_N$. For example, one could compute several solution estimates $\bar{z}_i = S(\bar{x}_i)$, $i = 1, \ldots, p$, such that $\cup_{i=1}^{p} D^\ell(\bar{x}_i) \supset \Gamma_N$. Alternatively, one could perform a sufficient number of iterations such that $D^\ell(\bar{x}) \supset \Gamma_N$ for a particular $\bar{x}$.

Remark 4: The set $\mathcal{D}^\ell(0)$ is particularly useful since no effort is needed to compute and store $\bar{z} = S(0) = 0$.

VII. NUMERICAL EXAMPLES

A linear model of an inverted pendulum is used for illustration. The continuous-time model is given by

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}, \quad u = \tau, \quad A_c = \begin{bmatrix} 0 & 1 \\ \frac{3g}{2L} & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ \frac{3}{ML^2} \end{bmatrix}, \quad (45)$$

where θ is the pendulum angle relative to the upright position, τ is a controlled torque, $L = 1$ is the pendulum length, $m = 0.1$ is the pendulum mass, and $g = 9.81$ is the gravitational acceleration. Two cases with different constraints are considered. In case (a), only an input constraint of $\mathcal{U} = [-1, 1]$ is enforced, whereas in case (b) both state and control constraints of $\mathcal{X} = [-\pi/4, \pi/4] \times [-2\pi, 2\pi]$ and $\mathcal{U} = [-1, 1]$ are present. Unless otherwise stated, all results and simulations use a sampling period of $T = 0.1$, a horizon length of $N = 3$, weighting matrices $Q = \text{diag}(1, 1)$ and $R = 1$, a PGM iteration count of $\ell = 117$, and an optimizer initialization of $\bar{z} = 0$. Using this problem data, the iteration bound for stability is $\ell^* = 116.2 < \ell$ and the size parameters of Σ for each case are: (a) $r_\psi = 3.14$ and $r_\phi = 0.0012$, and (b) $r_\psi = 2.73$ and $r_\phi = 0.0011$.

As a comparative tool, we use Υ_N to denote the N-step backwards reachable set to the maximal output admissible set $\mathcal{O}_\infty$ of $x_{k+1} = (A - BK)x_k$ [22]. Thus, Υ_N is a ROA estimate of (1) under an MPC law generated by an OCP similar to (3), but with the addition of a state constraint $\xi_k \in \mathcal{X}$ and a terminal constraint $\xi_N \in \mathcal{O}_\infty$ [1], [15].

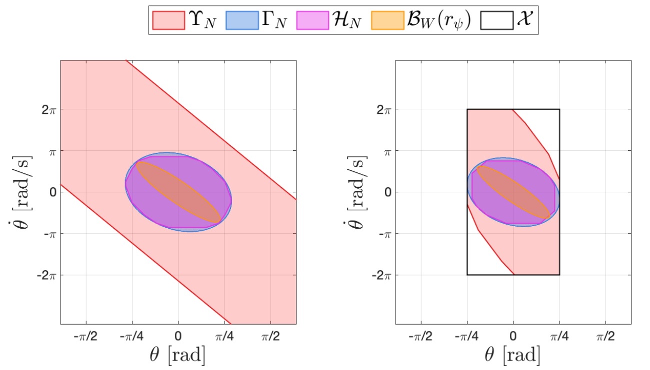
Figure 2 shows a comparison of Γ_N, the inner approximations of Γ_N (see Section V-A), and Υ_N for both cases[4]. In case (a), Γ_N reduces to the ROA estimate Γ'_N in Theorem 1 since no state constraints are present, whereas in case (b), Γ_N is the largest sublevel set of ψ contained in $\mathcal{X}$ as discussed in Theorem 2. The ROA estimate in (b) could be expanded by altering the OCP in a manner that shapes the sublevel sets of ψ to be similar to $\mathcal{X}$. In particular, it is likely that adding the state constraint $\xi_k \in \mathcal{X}$ to the OCP would expand this ROA estimate. However, enforcing state constraints in the OCP would complicate the analysis of the optimizer subsystem as discussed in Section III-C.

Figure 3 shows that $\mathcal{D}^\ell(0) \supset \Gamma_N$ in both cases, therefore initializing TDMPC with $\bar{z} = 0$ will yield asymptotically convergent trajectories for any $x_0 \in \Gamma_N$ by Proposition 2. While this is a convenient result, it is likely a consequence of conservatism in the bound $\ell > \ell^*$ enlarging $\mathcal{D}^\ell(0)$. If instead $\mathcal{D}^\ell(0) \cap \Gamma_N$ was non-trivial, then $\bar{z} = 0$ would only be guaranteed to be a valid optimizer initialization in this intersection.

Figure 4 demonstrates that asymptotically convergent plant-optimizer trajectories are obtained using TDMPC for several $x_0 \in \Gamma_N$. Note that for each trajectory, $x_k \in \Gamma_N$ and $\phi(x_k, z_k) \leq r_\phi \; \forall k \geq 0$ as a consequence of the invariance of Σ (Lemma 8). However, the solution error is generally much less than the required bound of $\phi(x, z) \leq r_\phi = 0.0012$. This suggests that the iteration bound ℓ^* is somewhat conservative.

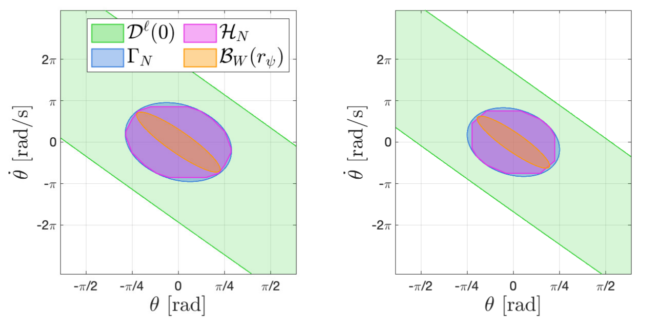
The simulations in Figure 5 show that the system is asymptotically stable for any $\ell \geq 6$, further indicating that the estimated bound of $\ell^* = 116.2$ is conservative. Additionally, for all $\ell \geq 6$ the simulated ROA remains (seemingly) constant and resembles the terminal constrained MPC ROA estimate Υ_N, indicating that the terminal constraint free MPC ROA estimate in Theorem 1 is conservative. However,

[4] Γ_N was plotted using Matlab's contour function since it cannot be computed explicitly. The convex hull approximation was computed by generating a 20×20 uniform grid of sample points around Γ_N.



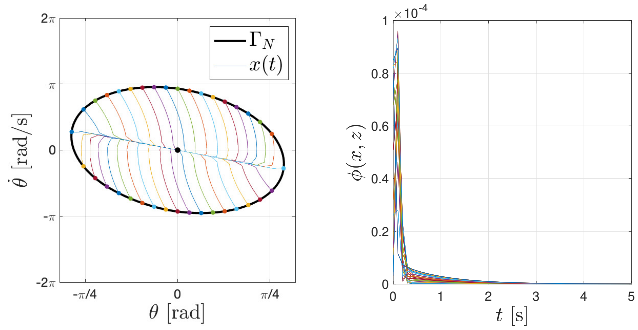
(a) Input Constrained Case (b) State & Input Constrained Case

Fig. 2. A comparison of Υ_N, Γ_N, the convex hull approximation $\mathcal{H}_N$, and the sublevel set approximation $\mathcal{B}_W(r_\psi)$. The ROA estimate Γ_N is smaller than Υ_N due to the lack of terminal constraint in the OCP. The convex hull approximation of Γ_N is quite good (since it can be refined to an arbitrary accuracy), while the sublevel set approximation $\mathcal{B}_W(r_\psi)$ is quite conservative.



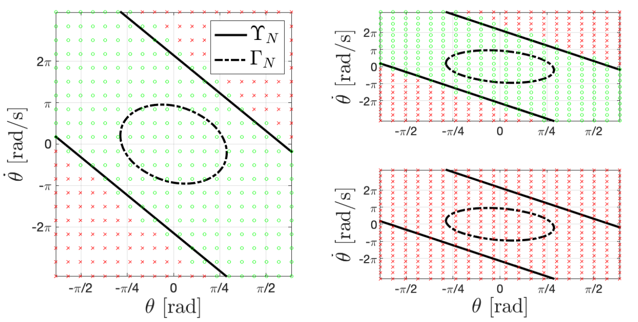
(a) Input Constrained Case (b) State & Input Constrained Case

Fig. 3. In both cases, initializing TDMPC with $\bar{z} = 0$ will yield asymptotically convergent trajectories for any $x_0 \in \Gamma_N$ since $\mathcal{D}^\ell(0) \supset \Gamma_N$ (Proposition 2).



(a) x Trajectories (b) $\phi(x, z)$ Trajectories

Fig. 4. (Input constrained case) Simulations of the closed-loop TDMPC system demonstrate that Γ_N is forward invariant, the solution error $\phi(x, z)$ never exceeds $r_\phi = 0.0012$, and the plant-optimizer trajectories are asymptotically convergent.



(a) $\ell = 117$ (b) $\ell = 6$ (Top), $\ell = 5$ (Bottom)

Fig. 5. (Input constrained case) A comparison of Γ_N and Υ_N to TDMPC simulation data using 5, 6, and 117 iterations per sampling period (green "O"s and red "X"s represent initial conditions resulting in convergent and non-convergent simulated trajectories respectively). For $\ell \geq 6$, the simulations indicate that the origin is asymptotically stable with a ROA resembling Υ_N. However, for $\ell < 6$ the origin is rendered unstable.

when $\ell < 6$ the origin is rendered unstable. While it is expected that the system can be destabilized if too few iterations are performed, it is particularly interesting that using $\ell = 6$ iterations yields a (seemingly) optimal ROA, while $\ell = 5$ yields an unstable equilibrium. However, this observation is consistent with Theorem 5 in the sense that $\exists \ell^* \in [5, 6)$ such that if $\ell > \ell^*$, then a ROA exists for the simulated closed-loop system.

VIII. Conclusions

This paper investigated the closed-loop properties of a state and input constrained linear system controlled using Time-distributed Model Predictive Control. A sufficient bound on the number of iterations per sampling period required to enforce asymptotic stability was derived in closed-form with a computable region of attraction estimate in the plant-optimizer space. Methods of approximating the region of attraction estimate and initializing the optimization procedure were also discussed. Future work will investigate using a weighted terminal cost to expand the region of attraction estimate [16], and implementation strategies capable of enforcing state constraints in the optimal control problem.

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