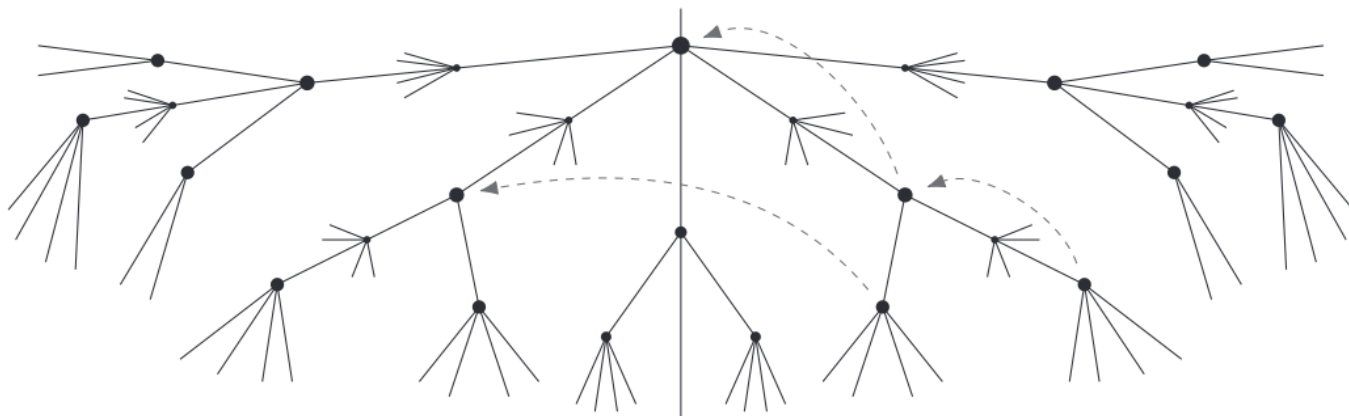


A Survey of Non-Archimedean Dynamics



Robert L. Benedetto

1. Introduction

In complex dynamics, one considers a rational function $f \in \mathbb{C}(z)$ as a map from the Riemann sphere $\mathbb{P}^1(\mathbb{C}) := \mathbb{C} \cup \{\infty\}$ to itself. One then studies the iterates of f , given by

$$f^n := \underbrace{f \circ \cdots \circ f}_{n \text{ times}}$$

as they act on the sphere. A rich theory follows — see, for example, the expositions in [CG93, Mil06] — with famous fractal pictures that exemplify beautiful theorems.

In the past few decades, a younger, parallel theory has developed for the case that we replace $\mathbb{C}$ with p -adic and, more generally, non-archimedean fields. Three principal motivations have driven the non-archimedean theory: seeking dynamical phenomena for comparison and contrast with the complex theory, as in [HY83, RL03]; applying local field results to number-theoretic questions, as in [BR10, Sil07]; and analyzing families of complex dynamical systems, especially at degeneration points, as in [DMF14, Kiw15].

Some of these motivations involve the Laurent series, Puiseux series, and Levi-Civita fields we will discuss in Examples 1.2 and 1.5, while others focus more on the p -adic

fields we will discuss in Examples 1.3 and 1.6. The unifying theme is to work over non-archimedean fields, which we now describe.

1.1. Absolute values. A field is a set equipped with two binary operations $+$ and $\cdot$ satisfying all the usual algebraic axioms; the prototypical example is the field $\mathbb{Q}$ of rational numbers. In analysis, one works with the real line $\mathbb{R}$, a completion of $\mathbb{Q}$, formed by adjoining limits of all Cauchy sequences. To do so, however, one first needs the absolute value function $|\cdot|$, since the definitions of limits and Cauchy-ness of sequences both use the absolute value. But the familiar absolute value is only one example of a real-valued function that satisfies many properties important in analysis proofs, as follows.

Definition 1.1. Let K be a field. An *absolute value* on K is a function $|\cdot| : K \rightarrow \mathbb{R}$ such that for all $x, y \in K$,

- $|x| \geq 0$, with equality if and only if $x = 0$,
- $|xy| = |x| \cdot |y|$,
- $|x + y| \leq |x| + |y|$.

We say that $|\cdot|$ is *non-archimedean* if in fact

$$|x + y| \leq \max\{|x|, |y|\} \quad \text{for all } x, y \in K. \quad (1.1)$$

The trivial absolute value on K is given by $|x| = 1$ for all $x \in K^\times := K \setminus \{0\}$ (and $|0| := 0$ by the first axiom above). We will restrict our attention to *nontrivial* absolute values, i.e., those for which there is some $x \in K$ with $|x| \neq 0, 1$.

The familiar absolute value on $\mathbb{R}$ is an *archimedean* absolute value, because it does not satisfy the non-archimedean triangle inequality (1.1). Our focus in this article will

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concern fields equipped with a non-archimedean (and nontrivial) absolute value.

Example 1.2. Let $\mathbb{C}(T)$ be the field of rational functions with complex coefficients. Let $\text{ord}_0(f)$ denote the order of vanishing of $f \in \mathbb{C}(T)^\times$ at $T = 0$. (If $f(0) \neq 0, \infty$, we set $\text{ord}_0(f) = 0$; if f has a pole at $T = 0$, we set $\text{ord}_0(f) = -\text{ord}_0(1/f)$.) Define $|\cdot|_0 : \mathbb{C}(T) \rightarrow \mathbb{R}$ by

$$|f|_0 = \begin{cases} e^{-\text{ord}_0(f)} & \text{if } f \neq 0, \\ 0 & \text{if } f = 0. \end{cases}$$

It is an exercise to check that $|\cdot|_0$ is a non-archimedean absolute value on $\mathbb{C}(T)$ according to Definition 1.1. If f has a zero at $T = 0$, it is “small” — and the greater the order of the zero, the smaller $|f|_0$ is — and if f has a pole there, it is “big.”

Example 1.3. Fix a prime number $p \geq 2$. The p -adic absolute value on $\mathbb{Q}$ is defined by

$$\left| \frac{r}{s} p^n \right|_p := p^{-n} \quad \text{for } r, s \in \mathbb{Z} \text{ not divisible by } p,$$

with $|0|_p := 0$. Thus, a rational number whose numerator is divisible by p is p -adically small — the higher the power, the smaller the p -adic absolute value — and one whose denominator is divisible by p is p -adically big. Thus, 13 and $389/5$ are very close together 3-adically, because:

$$\left| \frac{389}{5} - 13 \right|_3 = \left| \frac{324}{5} \right|_3 = \left| \frac{4}{5} \cdot 3^4 \right|_3 = \frac{1}{3^4} = \frac{1}{81}.$$

As foreign as $|\cdot|_p$ may seem at first, it is again an exercise to show that it is a non-archimedean absolute value on $\mathbb{Q}$.

1.2. Non-archimedean fields. Given a field K equipped with an absolute value $|\cdot|_v$, we can mimic basic definitions from real analysis. We say that a sequence $\{a_n\}_{n \geq 0} \subseteq K$:

- is *Cauchy* if for every real $\varepsilon > 0$, there exists $N \geq 0$ such that for all $m, n \geq N$, we have $|a_m - a_n|_v < \varepsilon$,
- *converges* to $a \in K$ if for every real $\varepsilon > 0$, there exists $N \geq 0$ such that for all $n \geq N$, we have $|a_n - a|_v < \varepsilon$.

We say that K is *complete* if every Cauchy sequence converges. If K is complete with respect to a non-archimedean absolute value, we say K is a *non-archimedean field*. It is another exercise to prove the following series convergence test that many calculus students wish were true for $\mathbb{R}$.

Proposition 1.4. Let K be a (complete) non-archimedean field with absolute value $|\cdot|_v$. Then for any sequence $\{a_n\}_{n \geq 0} \subseteq K$, we have

$$\sum_{n=0}^{\infty} a_n \text{ converges if and only if } \lim_{n \rightarrow \infty} a_n = 0,$$

where in both cases, convergence is with respect to $|\cdot|_v$.

Even if K is not complete, we can construct a *completion* $K_v \supseteq K$ by adjoining limits of all Cauchy sequences.

In archimedean analysis, one proceeds from $\mathbb{Q}$ to $\mathbb{R}$ by completion, and then to $\mathbb{C}$ by algebraic closure. Curiously, the algebraic closure $\bar{K}_v$ of K_v often fails to be complete, but fortunately we may complete once more, and the resulting field $\mathbb{C}_v$ is, like $\mathbb{C}$, both complete and algebraically closed.

Example 1.5. The completion of $\mathbb{C}(T)$ with respect to $|\cdot|_0$ is the field

$$\mathbb{C}((T)) = \left\{ \sum_{n \geq n_0} a_n T^n \text{ and } a_n \in \mathbb{C} \right\}$$

of formal Laurent series with complex coefficients. (That is, we do not worry about convergence issues, and we allow at worst a pole at $T = 0$, not an essential singularity.) The completion $\mathbb{L}$ of the algebraic closure $\overline{\mathbb{C}((T))}$ is the *Levi-Civita field*. (Elements of $\mathbb{C}((T))$ are called *Puiseux series*, and sometimes elements of $\mathbb{L}$ are informally also called *Puiseux series*.)

The field $\mathbb{L}$ consists of all formal sums

$$\sum_{i \geq 0} a_i T^{q_i}$$

with $a_i \in \mathbb{C}$ and $q_i \in \mathbb{Q}$ for which the sequence $q_0 < q_1 < \dots$ is an unbounded, strictly increasing sequence of rational numbers. We say that $\mathbb{L}$ is infinitely *ramified* over $\mathbb{C}((T))$ because of the introduction of arbitrarily large denominators in the rational powers $T^{m/n}$ that appear.

Example 1.6. The completion of $\mathbb{Q}$ with respect to $|\cdot|_p$ is the field $\mathbb{Q}_p$ of p -adic rationals. Since $\lim_{n \rightarrow \infty} p^n = 0$ with respect to $|\cdot|_p$, any power series in p with integer coefficients will converge in $\mathbb{Q}_p$. Combining this idea with base- p expansion of positive integers, it is not difficult to characterize $\mathbb{Q}_p$ as the set of all formal Laurent series in p with coefficients in the set $\{0, 1, \dots, p-1\}$, i.e.,

$$\mathbb{Q}_p = \left\{ \sum_{n \geq n_0} a_n p^n \text{ and } a_n \in \{0, 1, \dots, p-1\} \right\}.$$

In contrast to the Laurent series of Example 1.5, addition and multiplication in $\mathbb{Q}_p$ involve carrying digits as in base- p arithmetic.

When we pass to the completion $\mathbb{C}_p$ of the algebraic closure $\bar{\mathbb{Q}}_p$, then besides increasing ramification by allowing rational powers $p^{m/n}$ as in Example 1.5, we also see an infinite extension of the *residue field*. Roughly speaking, this means that we go from the original set of coefficients $\{0, 1, \dots, p-1\}$, which may be identified with the p -element field $\mathbb{F}_p$, to an infinite set corresponding to the algebraic closure $\bar{\mathbb{F}}_p$. Unfortunately, there is no explicit description

of the elements of $\mathbb{C}_p$ as there was for $\mathbb{L}$, because the existence of the algebraic closure $\overline{\mathbb{Q}_p}$ relies on the use of non-constructive tools such as the Axiom of Choice.

Throughout this paper, we declare that

$\mathbb{C}_v$ denotes an algebraically closed and complete non-archimedean field with absolute value $|\cdot|_v$

We remark that the set $|\mathbb{C}_v^\times|_v$ of absolute values attained by nonzero elements of $\mathbb{C}_v$ is a dense subgroup of the positive real line $(0, \infty)$ under multiplication, but it is usually not the whole of $(0, \infty)$. Indeed, in Example 1.5, we have $|\mathbb{L}^\times|_0 = e^{\mathbb{Q}}$, and in Example 1.6, we have $|\mathbb{C}_p^\times|_p = p^{\mathbb{Q}}$.

1.3. Non-archimedean disks. For any point $a \in \mathbb{C}_v$ and any positive real number $r > 0$, we define the open and closed disks of radius r centered at a by

$$D(a, r) := \{x \in \mathbb{C}_v : |x - a|_v < r\},$$

$$\overline{D}(a, r) := \{x \in \mathbb{C}_v : |x - a|_v \leq r\},$$

respectively. That is, we define disks according to the metric given by $d(x, y) = |x - y|_v$, which induces a topology on $\mathbb{C}_v$. Observe that if $r \in |\mathbb{C}_v^\times|_v$, then $D(a, r) \subsetneq \overline{D}(a, r)$, in which case we call these two disks *rational open* and *rational closed*, respectively. On the other hand, if $r \notin |\mathbb{C}_v^\times|_v$, then $D(a, r) = \overline{D}(a, r)$, and we call this disk *irrational*. Non-archimedean disks, and the topology on $\mathbb{C}_v$, have several properties not shared by their archimedean cousins, but which are easy to prove:

- Any point of a disk is its center. That is, if $b \in D(a, r)$, then $D(b, r) = D(a, r)$; and if $b \in \overline{D}(a, r)$, then $\overline{D}(b, r) = \overline{D}(a, r)$.
However, the radius of a disk D is well-defined, as $r = \sup\{|x - y|_v : x, y \in D\}$.
- If two disks have nonempty intersection, then one contains the other.
- All disks are both open and closed topologically. But any disk is exactly one of the three types: rational open, rational closed, or irrational.
- $\mathbb{C}_v$ is *not* locally compact.
- $\mathbb{C}_v$ is totally disconnected. That is, the only nonempty connected subsets of $\mathbb{C}_v$ are singletons.

Properties (c), (d), and (e) above follow from the fact that any rational closed disk may be written as a disjoint union of open disks. For example, in the Levi-Civita field $\mathbb{L}$, it is easy to check that

$$\overline{D}(0, 1) = \bigcup_{a \in \mathbb{C}} D(a, 1) \quad (1.2)$$

and that the union in (1.2) is a disjoint union, partitioning the set of Levi-Civita series in $\overline{D}(0, 1)$ according to constant

term. Thus, speaking in very loose terms, $\mathbb{C}_v$ topologically resembles an infinite disjoint union of Cantor sets.

Lest you think that $\mathbb{C}_v$ is all pathology, here is one lovely property of non-archimedean disks. Let $D \subseteq \mathbb{C}_v$ be a disk, and let $f \in \mathbb{C}_v[z]$ be a nonconstant polynomial. Then the image $f(D)$ is also a disk — not just homeomorphic to a disk, but actually *equal* to a disk.

2. Basic Dynamics on $\mathbb{C}_v$

As in complex dynamics, a rational function $f(z) \in \mathbb{C}_v(z)$ maps the projective line $\mathbb{P}^1(\mathbb{C}_v) := \mathbb{C}_v \cup \{\infty\}$ to itself, and we wish to study the iterates f^n . For any point $x \in \mathbb{P}^1(\mathbb{C}_v)$, the sets

$$\text{Orb}_f^+(x) := \{f^n(x) : n \geq 0\},$$

$$\text{Orb}_f^-(x) := \{y \in \mathbb{P}^1(\mathbb{C}_v) :$$

$$f^n(y) = x \text{ for some } n \geq 0\}$$

are called the *forward orbit* and the *backward orbit* of x , respectively.

Writing $f \in \mathbb{C}_v(z)$ as $f = g/h$ for relatively prime polynomials $g, h \in \mathbb{C}_v[z]$, we define the *degree* of f to be

$$\deg f := \max\{\deg g, \deg h\}.$$

It is often convenient to change coordinates on $\mathbb{P}^1(\mathbb{C}_v)$ via a degree-one rational function, i.e., via a Möbius transformation $\theta \in \text{PGL}(2, \mathbb{C}_v)$, given by

$$\theta(z) = \frac{az + b}{cz + d} \quad a, b, c, d \in \mathbb{C}_v, \quad ad - bc \neq 0.$$

If we change coordinates by θ on both the domain and range of the map $f : \mathbb{P}^1(\mathbb{C}_v) \rightarrow \mathbb{P}^1(\mathbb{C}_v)$, then f becomes the conjugated map $\theta \circ f \circ \theta^{-1}$.

Naturally, we are most interested in dynamical phenomena that are not specific to a particular choice of coordinate. For instance, a point $x \in \mathbb{P}^1(\mathbb{C}_v)$ is

- *fixed* by f if $f(x) = x$.
- *periodic* under f if there is some $n \geq 1$ such that $f^n(x) = x$. In that case, we say x has period n ; and if n is the smallest such integer, we say x has *exact* period n .
- *preperiodic* under f if there are integers $n > m \geq 0$ such that $f^n(x) = f^m(x)$, i.e., if some $f^m(x)$ is periodic, or equivalently, if the forward orbit $\text{Orb}_f^+(x)$ is finite.
- *exceptional* if the backward orbit $\text{Orb}_f^-(x)$ is finite.
- a *critical point*, or *ramification point*, if $f'(x) = 0$. (At least for $x, f(x) \neq \infty$.)

All of these properties are coordinate-independent, in the sense that x has the given property for f if and only if $\theta(x)$ has the same property for $\theta \circ f \circ \theta^{-1}$.

If x is periodic under f of exact period n , then the *multiplier* of x , defined to be $\lambda := (f^n)'(x)$, is also

coordinate-independent. Locally near x , we have

$$f^n(z) \approx x + \lambda(z - x). \quad (2.1)$$

If $|\lambda|_v < 1$, we say x is *attracting*, because (2.1) implies that nearby points in $\mathbb{P}^1(\mathbb{C}_v)$ approach x under iteration of f^n . Similarly if $|\lambda|_v > 1$, we say x is *repelling*. If $|\lambda|_v = 1$, we say x is *indifferent*, in which case one can show that f^n is an isometry on a neighborhood of x . This last fact may surprise readers familiar with complex dynamics, where indifferent periodic points exhibit more intricate behavior; but the isometry here is a simple consequence of (2.1) and the non-archimedean triangle inequality (1.1).

3. The Berkovich Line

As observed by Rivera-Letelier in [RL03], the appropriate setting for non-archimedean dynamics is not the classical projective line $\mathbb{P}^1(\mathbb{C}_v)$, but rather the Berkovich projective line $\mathbb{P}_{\text{an}}^1$, which is a topological space that contains $\mathbb{P}^1(\mathbb{C}_v)$ as a dense subset, first described in [Ber90]. The Berkovich line $\mathbb{P}_{\text{an}}^1$ is both compact and path-connected, in stark contrast with $\mathbb{P}^1(\mathbb{C}_v)$.

The precise definition of $\mathbb{P}_{\text{an}}^1$ involves multiplicative seminorms on the polynomial ring $\mathbb{C}_v[z]$ — for details, see Chapter 6 of [Ben19] or Chapters 1–2 of [BR10] — but here we describe the space more intuitively. In particular, there are four types of points in $\mathbb{P}_{\text{an}}^1$:

- The points in the classical projective line $\mathbb{P}^1(\mathbb{C}_v) = \mathbb{C}_v \cup \{\infty\}$ are of Type I.
- Each closed disk $D = \overline{D}(a, r) \subseteq \mathbb{C}_v$ gives a point of $\mathbb{P}_{\text{an}}^1$, which we denote $\zeta(a, r)$. If D is a rational closed disk, then $\zeta(a, r)$ is of Type II; if D is irrational, then $\zeta(a, r)$ is of Type III.
- Type IV points correspond to decreasing chains of disks $D_1 \supsetneq D_2 \supsetneq \cdots$ with empty intersection, a perhaps surprising phenomenon that occurs in some complete non-archimedean fields, including $\mathbb{L}$ and $\mathbb{C}_p$.

Figure 1 is a rough sketch of $\mathbb{P}_{\text{an}}^1$. The line segments you see are truly line segments, i.e., homeomorphic copies of the real interval $[0, 1]$. For example, the points on the line segment running between the two type I points 0 and ∞ are the points $\zeta(0, r)$ of types II and III, for $0 < r < \infty$. These include the type II point $\zeta(0, 1)$, corresponding to the closed unit disk $\overline{D}(0, 1)$, which is called the *Gauss point*. Emanating from the Gauss point, or indeed from any type II point, are infinitely many branches, corresponding to the infinitely many elements of the residue field (plus one extending towards ∞), or equivalently corresponding to the infinitely many open disks in the disjoint union (1.2).

Thus, for example, we may proceed along the line segment from 0 to $\zeta(0, 1)$, which consists of points of the form $\zeta(0, r)$, by increasing r from 0 to 1. Then, because $\overline{D}(0, 1) = \overline{D}(1, 1)$, we may proceed from $\zeta(0, 1) = \zeta(1, 1)$ to

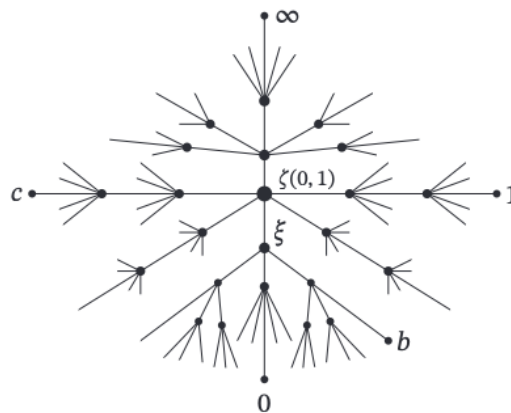


Figure 1. The Berkovich projective line.

the type I point 1 via the line segment consisting of points of the form $\zeta(1, r)$, with r decreasing from 1 back down to 0. Similarly, the type I points $b, c \in \mathbb{C}_p$ in Figure 1 satisfy $0 < |b| < |c| = 1$, and we can travel from one to the other as follows. First, proceed along the line segment from b to $\xi := \zeta(b, |b|) = \zeta(0, |b|)$ via points of the form $\zeta(b, r)$ with $0 < r \leq |b|$; then from there to $\zeta(0, 1) = \zeta(c, 1)$ via points of the form $\zeta(0, r) = \zeta(b, r)$ with $|b| \leq r \leq 1$; and finally to c via points of the form $\zeta(c, r)$, with r decreasing back to 0. The tracing of such paths illustrates, at least conceptually, that $\mathbb{P}_{\text{an}}^1$ is path-connected.

Along any segment, there is a dense collection of type II points, because as noted earlier, $|\mathbb{C}_v^\times|$ is dense in $(0, \infty)$, and each radius $r \in |\mathbb{C}_v^\times|$ gives a type II point. Moreover, as we have just seen, there is infinite branching at each type II point. Thus, $\mathbb{P}_{\text{an}}^1$ has the structure of an infinitely-branched tree: along each segment, there are infinitely many points of branching, each with an infinite number of branches. The type I points, which may be viewed as lying around the outer rim of the figure, sit at the ends of many of the branches. (The type IV points, which we will mostly brush under the rug here, lie at the ends of other branches.)

Figure 1 may appear to conflict with our earlier claim that the set $\mathbb{P}^1(\mathbb{C}_v)$ of Type I points is dense in $\mathbb{P}_{\text{an}}^1$. However, the topology on $\mathbb{P}_{\text{an}}^1$ is weaker than the picture might suggest. In particular, for any Type II point ζ and any open set U containing ζ , all but finitely many of the branches emanating from ζ must be completely contained in U .

Any rational function $f \in \mathbb{C}_v(z)$ has a unique extension from $\mathbb{P}^1(\mathbb{C}_v)$ to a continuous map $f : \mathbb{P}_{\text{an}}^1 \rightarrow \mathbb{P}_{\text{an}}^1$. If f is nonconstant, this map is surjective, preserving types of points. For example, if $f \in \mathbb{C}_v[z]$ is a nonconstant polynomial and ζ is a point of Type II or III corresponding to a disk D , then $f(\zeta)$ is the point of the same type corresponding to the disk $f(D)$. For a fully general description of $f(\zeta)$ for $f \in \mathbb{C}_v(z)$ and $\zeta \in \mathbb{P}_{\text{an}}^1$, see Chapter 7 of [Ben19].

Each open disk $D(a, r) \subseteq \mathbb{C}_v$ has a Berkovich version $D_{\text{an}}(a, r)$ whose set of type I points is precisely $D(a, r)$.

Similarly, a closed disk $\overline{D}(a, r)$ has a Berkovich version $\overline{D}_{\text{an}}(a, r)$. Moreover, we define an open disk in $\mathbb{P}_{\text{an}}^1$ containing ∞ to be the complement of a (Berkovich) closed disk, and a closed disk in $\mathbb{P}_{\text{an}}^1$ containing ∞ to be the complement of a (Berkovich) open disk. Through the rest of this paper, a “disk” will refer to a Berkovich disk in this sense, i.e., a disk in $\mathbb{P}_{\text{an}}^1$, possibly containing the point ∞ .

4. Dynamics on $\mathbb{P}_{\text{an}}^1$

Throughout this section, let $f \in \mathbb{C}_v(z)$ be a rational function of degree $d \geq 2$, and consider its iterates f^n as functions from $\mathbb{P}_{\text{an}}^1$ to itself.

4.1. Periodic points. As in Section 2, points of $\mathbb{P}_{\text{an}}^1$ may or may not be fixed, periodic, or preperiodic under f . Moreover, Berkovich points of types II, III, or IV that are periodic may be classified as indifferent or repelling, as follows. Suppose $\zeta \in \mathbb{P}_{\text{an}}^1 \setminus \mathbb{P}^1(\mathbb{C}_v)$ is periodic of period n . If ζ has a small neighborhood U that maps into itself, then ζ is indifferent; otherwise ζ is repelling, although the reader should be cautioned that even in that case, ζ does not generally “repel” individual nearby points. (On the other hand, periodic points of types II–IV cannot be considered attracting, because it turns out that if $\bigcap_{i \geq 1} f^{ni}(U) = \{\zeta\}$, then ζ must be of type I.) It would take us too far afield to define this classification formally — although in practice it is straightforward — so we will settle for an example.

Example 4.1. Let $f(z) = a_0 + a_1 z + \cdots + a_d z^d$ be a polynomial with $|a_i|_v \leq 1$ for all i . Suppose that $|a_m|_v = 1$ for some $m \geq 1$, with $|a_i|_v < 1$ for $i > m$. Then f fixes the Gauss point $\zeta(0, 1)$, which is indifferent if $m = 1$, but repelling if $m \geq 2$.

Indeed, if $m = 1$, then there is some small enough $r > 1$ such that f maps the open disk $D_{\text{an}}(0, r)$ into itself. On the other hand, if $m \geq 2$, then for any $r > 1$, we have

$$f(D_{\text{an}}(0, r)) \supseteq D_{\text{an}}(0, r^m) \subsetneq D_{\text{an}}(0, r),$$

and hence $\bigcup_{n \geq 0} f^n(D_{\text{an}}(0, r)) = \mathbb{P}_{\text{an}}^1 \setminus \{\infty\}$.

4.2. Fatou and Julia sets. The two qualitatively different behaviors in Example 4.1 inspire the following definition.

Definition 4.2. Let $f \in \mathbb{C}_v(z)$. The *Julia set* of f is the set $\mathcal{J}_{\text{an}, f}$ of points $\zeta \in \mathbb{P}_{\text{an}}^1$ with the following property: for every neighborhood U of ζ , the union of iterates $\bigcup_{n \geq 0} f^n(U)$ omits only finitely many points of $\mathbb{P}_{\text{an}}^1$. Its complement $\mathcal{F}_{\text{an}, f} := \mathbb{P}_{\text{an}}^1 \setminus \mathcal{J}_{\text{an}, f}$ is the *Fatou set* of f .

The properties of the Fatou and Julia sets of a rational function $f \in \mathbb{C}_v(z)$ of degree $d \geq 2$ include the following:

- The Fatou set $\mathcal{F}_{\text{an}, f}$ is open in $\mathbb{P}_{\text{an}}^1$, and the Julia set $\mathcal{J}_{\text{an}, f}$ is closed.
- All attracting and most indifferent periodic points of f lie in the Fatou set. (Certain indifferent periodic points

of type II may be in the Julia set; this can be determined easily.)

- All repelling periodic points of f lie in the Julia set.
- Both sets are invariant under f , meaning that $f^{-1}(\mathcal{J}_{\text{an}, f}) = \mathcal{J}_{\text{an}, f}$ and $f^{-1}(\mathcal{F}_{\text{an}, f}) = \mathcal{F}_{\text{an}, f}$.
- Both sets are nonempty.
- For any Julia point $\zeta \in \mathcal{J}_{\text{an}, f}$, the backward orbit $\text{Orb}_f^-(\zeta)$ is dense in $\mathcal{J}_{\text{an}, f}$.
- The repelling periodic points are dense in $\mathcal{J}_{\text{an}, f}$.

(For proofs of these facts, see, for example, Chapter 8 and Section 12.2 of [Ben19].) The easy determination for indifferent points in property (b), as well as the nonemptiness of the Fatou set in property (e), contrast with the situation in complex dynamics. In addition, the proofs of property (g) and of the nonemptiness of the Julia set in property (e) both require significantly more work than their complex counterparts. Otherwise, however, the above properties exactly match corresponding statements in complex dynamics.

The invariance under f means that non-archimedean Julia sets, like their complex analogues, have a self-similar structure, and most are fractals. On the other hand, as they are subsets of the inherently undrawable space $\mathbb{P}_{\text{an}}^1$, there is no known way to draw meaningful pictures of non-archimedean Julia sets, outside of trivial cases.

4.3. Good reduction. In Example 1.6, we made reference to the residue field of $\mathbb{C}_p$. In general, every non-archimedean field $\mathbb{C}_v$ has a residue field k_v , and our rational function $f(z) \in \mathbb{C}_v(z)$ has a *reduction* $\bar{f}(z) \in k_v(z)$. This is a simple construction; for example, for the Levi-Civita field $\mathbb{C}_v = \mathbb{L}$ of Example 1.5, the residue field is $k_v = \mathbb{C}$, and the reduction $\bar{f}(z) \in \mathbb{C}(z) \cup \{\infty\}$ is obtained simply by setting the parameter T to 0 in all the coefficients of f , after clearing any negative powers of T . However, the reduced map $\bar{f}$ may have smaller degree than $d = \deg f$, either because coefficients of higher powers of z were reduced to 0, or because formerly coprime factors now cancel from numerator and denominator. We say $f \in \mathbb{C}_v(z)$

- has *explicit good reduction* if $\deg \bar{f} = \deg f$,
- has *good reduction* if after some coordinate change $\theta \in \text{PGL}(2, \mathbb{C}_v)$, the map $\theta \circ f \circ \theta^{-1}$ has explicit good reduction, or
- has *bad reduction* otherwise.

Example 4.3. Let $f(z) = \frac{Tz^2 + z}{z^2 + Tz + 1} \in \mathbb{L}(z)$. Then $\bar{f}(z) = \frac{z}{z^2 + 1} \in \mathbb{C}(z)$ has $\deg \bar{f} = 2 = \deg f$, so f has explicit good reduction.

Example 4.4. Let $g(z) = \frac{z^2 + Tz - 1}{Tz^2 + z + 1} \in \mathbb{L}(z)$. Then

$$\bar{g}(z) = \frac{z^2 - 1}{z + 1} = z - 1$$

has $\deg \bar{g} = 1 < 2 = \deg g$, so g does not have explicit good reduction. In fact, g has bad reduction; see Example 4.7 below.

Example 4.5. Let $h(z) = Tz^3 + z + 1 \in \mathbb{L}(z)$. Then $\bar{h}(z) = z + 1$ has smaller degree than h . However, with $\theta(z) = T^{1/2}z$, we have

$$H(z) := \theta \circ h \circ \theta^{-1}(z) = z^3 + z + T^{1/2},$$

so that $\bar{H}(z) = z^3 + z$, which satisfies $\deg \bar{H} = 3 = \deg H$. Thus, H has explicit good reduction, so h has good reduction.

Good reduction corresponds to having a one-point Julia set, as the following result shows.

Theorem 4.6. Let $f \in C_v(z)$ with $\deg f = d \geq 2$. Let $\mathcal{J}_{\text{an},f}$ be the Julia set of f .

- f has explicit good reduction if and only if $\mathcal{J}_{\text{an},f} = \{\zeta(0, 1)\}$.
- f has good reduction if and only if $\mathcal{J}_{\text{an},f} = \{\zeta\}$ consists of a single point. In that case, ζ is of type II, and for any $\theta \in \text{PGL}(2, C_v)$, the map $\theta \circ f \circ \theta^{-1}$ has explicit good reduction if and only if $\theta(\zeta) = \zeta(0, 1)$.
- f has bad reduction if and only if $\mathcal{J}_{\text{an},f}$ is a perfect set.

(A perfect set is a nonempty closed set that has no isolated points. Such a set is necessarily uncountable.) The “trivial cases” referred to at the end of Section 4.2 are precisely the one-point Julia sets of maps of good reduction. The undrawable fractals are the Julia sets of maps of bad reduction.

Example 4.7. Let $g(z) = \frac{z^2 + Tz - 1}{Tz^2 + z + 1} \in \mathbb{L}(z)$ as in Example 4.4. Then $g(-1) = -1$, so the type I point -1 is a fixed point of g . A simple computation shows

$$|g'(-1)|_0 = \left| \frac{-T-1}{T} \right|_0 = e > 1,$$

so that -1 is a repelling fixed point and hence lies in the Julia set $\mathcal{J}_{\text{an},g}$ of g . Thus, $\mathcal{J}_{\text{an},g}$ does not consist solely of a type II point, so by Theorem 4.6.b, the map g cannot have good reduction, as claimed in Example 4.4.

As suggested by equation (1.2) and Section 3, the infinitely many branches of $\mathbb{P}_{\text{an}}^1$ emanating from the Gauss point $\zeta(0, 1)$ — each of which is an open disk in $\mathbb{P}_{\text{an}}^1$ — correspond to the infinitely many points of $\mathbb{P}^1(k_v)$. If f has explicit good reduction, then f maps these branches to one another, and more precisely, it maps them in exactly the way that the reduced map $\bar{f} \in k_v(z)$ maps the corresponding points of $\mathbb{P}^1(k_v)$.

5. Fatou Components

Even when $f \in C_v(z)$ has bad reduction, its Fatou set $\mathcal{F}_{\text{an},f}$ may be partitioned into connected components, although

they may be arranged in a more complicated fashion than in the good reduction case. These Fatou components map onto one another; that is, if U is a connected component of $\mathcal{F}_{\text{an},f}$, then so is $f(U)$. It therefore makes sense to talk about fixed, periodic, or preperiodic Fatou components. If a Fatou component is not preperiodic, we say it is a *wandering domain*.

5.1. Rivera-Letelier’s classification. Rivera-Letelier gave a fundamental classification of dynamics on periodic Fatou component in [RL03]. (More precisely, [RL03] concerned only the p -adic field C_p , but his results could be extended to other non-archimedean fields. For a fully general exposition, see Chapter 9 of [Ben19].) To present his classification, we define a *rational open connected affinoid* in $\mathbb{P}_{\text{an}}^1$ to be a rational open disk with finitely many rational closed disks removed. Such a region is indeed both open and connected, and moreover its boundary is a finite set of type II points.

Theorem 5.1. Let U be a periodic Fatou component of exact period $n \geq 1$. Then exactly one of the following two possibilities holds:

- U is attracting, meaning that there is a type I point $a \in U$ such that for all $\zeta \in U$, we have $\lim_{i \rightarrow \infty} f^{ni}(\zeta) = a$. In this case, $f^n : U \rightarrow U$ is m -to-1 for some $m \geq 2$, and either
 - U is a rational open disk, or
 - U is a domain of Cantor type, as defined below.
- U is indifferent, meaning that $f^n : U \rightarrow U$ is a bijective mapping, and in fact f^n is an isometry on the type I points of U . In this case, U is a rational open connected affinoid.

In part (a)(ii) above, a *domain of Cantor type* is a nested union $U = \bigcup_{i \geq 0} V_i$ of rational open connected affinoids with $V_0 \subsetneq V_1 \subsetneq \dots$, such that the boundary of U is homeomorphic to a Cantor set.

Example 5.2. Any quadratic polynomial is conjugate to one of the form $f(z) = z^2 + az$. (Complex dynamicists traditionally use $z^2 + c$, but for fields of characteristic 2, that form does not cover all possibilities.)

If $|a|_v \leq 1$, then f has explicit good reduction, so that its Fatou components are the open disks $D_{\text{an}}(x, 1)$ for $|x|_v \leq 1$, as well as the disk $D_\infty := \mathbb{P}_{\text{an}}^1 \setminus \bar{D}_{\text{an}}(0, 1)$ containing ∞ . This disk D_∞ is an attracting fixed component, as it maps 2-to-1 to itself, and $\lim_{i \rightarrow \infty} f^i(\zeta) = \infty$ for all $\zeta \in D_\infty$. Some of the other disks $D_{\text{an}}(x, 1)$ are periodic, and such a disk of period n is indifferent if $|(f^n)'(x)|_v = 1$, or attracting if $|(f^n)'(x)|_v < 1$. Other disks are preperiodic but not periodic, and depending on the residue field k_v , others may be wandering. (In particular, if $C_v = C_p$, then all of these components are preperiodic, but if $C_v = \mathbb{L}$, then some of them are wandering domains.)

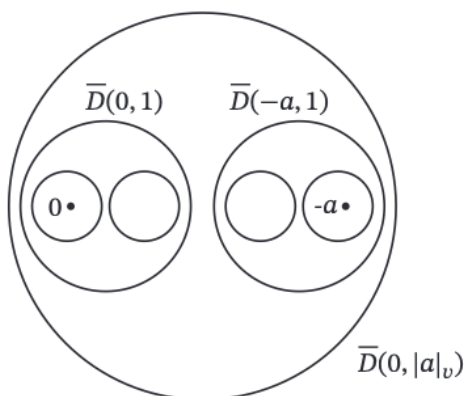


Figure 2. V_0 , V_1 , and V_2 in Example 5.2.

On the other hand, if $|a|_v > 1$, then f has bad reduction, and the Julia set is a Cantor set consisting entirely of type I points. The Fatou set consists of a single attracting fixed component U of Cantor type, and all points of $U = \mathcal{F}_{\text{an},f}$ are attracted to ∞ under iteration. More precisely, define V_0 to be the rational open disk that is the complement of the closed disk $\overline{D}_{\text{an}}(0, |a|_v)$, and iteratively define $V_{i+1} := f^{-1}(V_i) \supsetneq V_i$. It can be shown that each resulting rational open connected affinoid V_i is the complement of 2^i closed disks, each of radius $|a|_v^{1-i}$. Thus, $U := \bigcup_{i \geq 0} V_i$ is the entire Fatou set, and the Julia set $\mathcal{J}_{\text{an},f}$ is contained in two disks, then contained in four smaller disks, then contained in eight still smaller disks, and so on. See Figure 2, showing the (type I versions of) the disk that is the complement of V_0 , the two disks forming the complement of V_1 , and the four disks forming the complement of V_2 . Since the radii of the disks decrease to 0, we are left with a Cantor set for $\mathcal{J}_{\text{an},f}$, and a domain of Cantor type for its complement.

Example 5.3. Let $g(z) = bz^3 + z^2$ with $0 < |b|_v < 1$. Similar to the $|a|_v > 1$ case of Example 5.2, we may define V_0 to be the complement of $\overline{D}_{\text{an}}(0, |b|_v^{-1})$ and then iteratively define $V_{i+1} := f^{-1}(V_i) \supsetneq V_i$. The number of disks in the complements of the rational open connected affinoids V_i again increase, and their radii decrease, but in more complicated fashion. This is because the complement of V_1 consists of two disks of different sizes, the larger of which maps 2-to-1 onto $\overline{D}_{\text{an}}(0, |b|_v^{-1})$, and the smaller of which maps 1-to-1 onto the same disk. (See Figure 3.) In particular, not all of the radii approach zero, so although $U_\infty := \bigcup_{i \geq 0} V_i$ is again an attracting fixed Fatou component of Cantor type, it is not the whole Fatou set.

Indeed, g maps the closed unit disk $\overline{D}_{\text{an}}(0, 1)$ into itself, and on this disk we have $g(z) \approx z^2$, so that g behaves much like the good reduction case of Example 5.2 on the disk. In particular, the open unit disk $U_0 := D_{\text{an}}(0, 1)$ is an attracting fixed Fatou component, with $g^n(\zeta) \rightarrow 0$ for all $\zeta \in U_0$. Its boundary point is the Gauss point $\zeta(0, 1)$, which is a repelling fixed point of g and hence lies in the Julia set $\mathcal{J}_{\text{an},g}$.

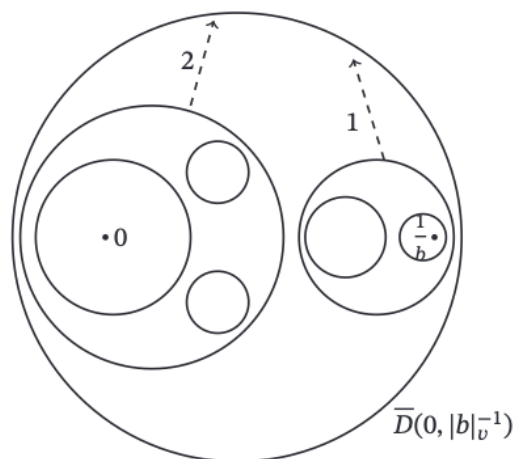


Figure 3. V_0 , V_1 , and V_2 in Example 5.3.

The other open disks of the form $D_{\text{an}}(x, 1)$ with $|x|_v = 1$ are also Fatou components, again mapping to one another as they would for the good reduction case of Example 5.2.

5.2. Comparisons and contrasts with complex dynamics. Much of the foregoing theory of the dynamics of non-archimedean Fatou components is similar in outline to that of complex dynamics. Indeed, Sullivan's classification [Sul83] shows periodic complex Fatou components have one of a short list of possible behaviors and structures. Each may be described as either attracting (with every point in the component approaching an attracting periodic point under iteration) or indifferent. However, there are crucial differences in the details, including the following:

- Complex attracting components include parabolic domains, where the limit point of the attraction lies on the boundary of the component. No such phenomenon occurs in the non-archimedean setting.
- Complex indifferent domains are homeomorphic either to a disk or an annulus, with dynamics conjugate to rotation by an irrational multiple of 2π . A non-archimedean indifferent domain can be a disk with any finite number of subdisks removed; and although there may be local conjugacies to a simple normal form, such conjugacies do not extend to the whole Fatou component.
- Every periodic cycle of complex attracting components contains a critical point. In the non-archimedean case, this statement may fail if the characteristic p of the residue field k_v satisfies $2 \leq p \leq d$, where d is the degree of the map.
- A complex rational function $f \in \mathbb{C}(z)$ has only finitely many periodic cycles of Fatou components. But in non-archimedean dynamics, $f \in \mathbb{C}_v(z)$ may have infinitely many such periodic component cycles. On the other hand, there can

only be finitely many cycles of periodic attracting components that are *not disks*. The same appears to be true about periodic indifferent components, but there is still no complete proof that there can only be finitely many cycles of periodic indifferent components that are not disks.

- A complex indifferent component contains at most one periodic point, necessarily of the same period as the component itself. A non-archimedean indifferent component usually contains infinitely many periodic points of type I, most of which are of much longer period than the component. In fact, over some fields like the p -adic field $\mathbb{C}_p$, any indifferent component *necessarily* contains infinitely many periodic points of type I.

Another contrast is that (again by work of Sullivan [Sul85]), complex rational functions have no wandering domains. However, as noted in Examples 5.2 and 5.3, wandering domains are actually quite common in non-archimedean dynamics when working over the Levi-Civita field $\mathbb{L}$. In particular, for certain fields including $\mathbb{L}$ — for which the residue field is not algebraic over a finite field — there are always wandering domains attached to *any* repelling periodic type II point, just like the wandering domains attached to the Gauss point in Examples 5.2 and 5.3.

So the real question about non-archimedean wandering domains is whether they can occur in other circumstances. The answer is: it depends, but we don't yet know the full details. That is, there are some known constructions for some choices of the field $\mathbb{C}_v$, and there are also some No Wandering Domains theorems assuming mild hypotheses. For more details, see Chapter 11 of [Ben19].

6. The Equilibrium Measure

The results described in Section 5 concern dynamics on non-archimedean Fatou sets; we now turn to the Julia sets. In 2004, three different pairs of mathematicians announced the existence of an important measure on $\mathbb{P}_{\text{an}}^1$ that is supported exactly on the Julia set $\mathcal{J}_{\text{an},f}$ of a non-archimedean rational function $f \in \mathbb{C}_v(z)$, as described in [BR10, CLT09, FRL10]. Analogous measures had been constructed in complex dynamics [Bro65, FLM83, Lju83] using classical (complex) potential theory, and the non-archimedean constructions use potential theory on the Berkovich line $\mathbb{P}_{\text{an}}^1$, developed in [BR10, FJ04].

The central object of non-archimedean potential theory is a *Laplacian operator* Δ that inputs certain real-valued functions on $\mathbb{P}_{\text{an}}^1$ and outputs signed Borel measures on $\mathbb{P}_{\text{an}}^1$. Although we will not make any attempt to define Δ formally, we can give two simple examples:

$$\Delta(\zeta \mapsto \log |\zeta|_v) = [0] - [\infty]$$

and

$$\Delta(\zeta \mapsto \log \max\{1, |\zeta|_v\}) = [\zeta(0, 1)] - [\infty],$$

where $[a]$ denotes the delta measure at the point a . With the operator Δ in hand, one can construct a function G_f that satisfies a certain invariance property under our map $f \in \mathbb{C}_v(z)$, and then take the Laplacian of G_f . After cancelling a certain auxiliary factor, the result is a measure μ_f on $\mathbb{P}_{\text{an}}^1$, called the *equilibrium measure* for f , which has the following properties:

- μ_f is a probability measure, i.e., it takes real values in $[0, 1]$, with $\mu_f(\mathbb{P}_{\text{an}}^1) = 1$.
- μ_f is invariant under f , i.e., for any Borel measurable set $V \subseteq \mathbb{P}_{\text{an}}^1$, we have $\mu_f(f^{-1}(V)) = \mu_f(V)$.
- The support of μ_f is precisely $\mathcal{J}_{\text{an},f}$.
- μ_f is mixing for the map $f : \mathcal{J}_{\text{an},f} \rightarrow \mathcal{J}_{\text{an},f}$.

Property (d), proven in Section 3.3 of [FRL10], implies that f is ergodic with respect to μ_f . (*Mixing* and *ergodic* are two properties that formally describe the intuitive idea that f acts chaotically on $\mathcal{J}_{\text{an},f}$, the support of the measure.)

The first main application of the equilibrium measure in [BR10, CLT09, FRL10] was to prove certain number-theoretic results known as *equidistribution theorems*. Such statements concern a rational function f that is defined over a global field like $\mathbb{Q}$, or more generally over a “product formula” field like $\mathbb{C}(T)$, for which there are many different absolute values whose (infinite) product is the trivial absolute value. A product formula field allows the construction of height functions in the sense of Weil, including the *canonical height* h_f associated with the dynamics of f . (See Section 3.4 of [Sil07] for a discussion of canonical heights in this context.) In that setting, the dynamical equidistribution results of [BR10, CLT09, FRL10] say that the Galois conjugates of points of small canonical height must accumulate on the associated Berkovich Julia sets. More precisely, they say that the appropriately weighted sums of delta measures at those Galois conjugates converge, in the weak- $*$ sense, to the equilibrium measure μ_f .

Since 2004, when those equidistribution results were first announced, the equilibrium measure has been used to study other aspects of the dynamics of $f \in \mathbb{C}_v(z)$, mirroring the use of such measures in complex dynamics. When f has good reduction, its equilibrium measure is simply a delta mass at the one point of the Julia set; but when f has bad reduction, μ_f is far more intricate, like its complex analogue.

Once equipped with the equilibrium measure μ_f , one can define and study the entropy of f , its Lyapunov exponent, and more. Similar sorts of measures can also be constructed on moduli spaces of rational maps, with support on the *bifurcation locus*, where the dynamics of a map in the family undergoes a qualitative change. Non-archimedean dynamics today thus continues the themes

that spawned the subject: exhibiting a broad class of dynamical systems for analysis, enabling the study of degenerations of complex dynamical systems, and providing essential local tools for number-theoretic questions arising in arithmetic dynamics.

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